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The Optimal Projection Equations for Reduced-Order, Discrete-Time Modeling, Estimation, and Control

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The optimal projection equations derived previously for reduced-order, continuous-time modeling, estimation, and control are developed for the discrete-time case. The design equations are presented in a concise, unified manner to facilitate their accessibility for the development of numerical algorithms for practical applications. As in the continuous-time case, the standard Kalman filter and linear-quadratic-Gaussian results are immediately obtained as special cases of the estimation and control results.

Nomenclature

A, B, C	$= n \times n, n \times m, \ell \times n$ matrices
A_m, B_m, C_m	$= n_m \times n_m, n_m \times m, \ell \times n_m$ matrices
A_e, B_e, C_e, D_e	$= n_e \times n_e, n_e \times \ell, p \times n_e, p \times \ell$ matrices
A_c, B_c, C_c, D_c	$= n_c \times n_c, n_c \times \ell, m \times n_c, m \times \ell$ matrices
E_i	= matrix with unity in the (i, i) position and zeros elsewhere
$\boldsymbol{E}$	= expected value
I_r	$= r \times r$ identity matrix
k	= discrete-time index $1, 2, 3, \ldots$
L	$= p \times n$ matrix
$n, m, \ell, n_m, n_e, n_c, p$	= positive integers, $1 \le n_m, n_e, n_c \le n$
R, N, R_2	$= \ell \times \ell, p \times p, m \times m$ positive-definite matrices
R_1	$= n \times n$ nonnegative-definite matrix
R_{12}	$= n \times m$ matrix such that $R_1 - R_{12} R_2^{-1} R_{12}^T$ is nonnegative definite
$\boldsymbol{R}, \boldsymbol{R}^{r \times s}$	= real numbers, $r \times s$ real matrices
$\tilde{R}$	$= \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$
$\operatorname{tr} Z$	= trace of square matrix Z
u, y, y_e	$= m$-, ℓ-, p-dimensional vectors
V	$= m \times m$ positive-definite covariance of w
V_1	$= n \times n$ nonnegative-definite covariance of w_1
V_2	$= \ell \times \ell$ positive-definite covariance of w_2
V_{12}	$= n \times \ell$ cross-covariance of w_1, w_2
w, w_1, w_2	$= m$-, n-, ℓ-dimensional zero-mean discrete-time white noise processes
x, x_m, x_e, x_c	$= n$-, n_m-, n_e-, n_c-dimensional vectors
$Z_{(i,j)}$	$= (i, j)$ element of matrix Z
Z^T	= transpose of vector or matrix Z
Z^{-T}	$= (Z^T)^{-1}$ or $(Z^{-1})^T$
$\Pi_i(\Psi)$	$= \Psi E_i \Psi^{-1}$ (unit-rank eigenprojection[47])
$\rho(Z)$	= rank of matrix Z

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I. Introduction

IN a recent series of papers[1-11] it has been shown that the first-order necessary conditions for quadratically optimal, continuous-time, reduced-order modeling, estimation, and control can be transformed into coupled systems of two, three, and four matrix equations, respectively. This coupling, due to the presence of an oblique projection (idempotent matrix), arises as a rigorous consequence of optimality, hence suggesting the name *optimal projection*. For the estimation and control problems, this formulation provides a direct generalization of classical steady-state Kalman filter and linear-quadratic-Gaussian (LQG) control theory. In the full-order case the projection becomes the identity matrix, the additional two modified Lyapunov equations drop out, and the remaining modified Riccati equations become the usual Riccati equations.

Coupling via the optimal projection supports the view that sequential reduced-order design procedures consisting of either 1) model reduction followed by estimator (controller) design or 2) estimator (controller) design followed by estimator (controller) reduction are generally not optimal. Furthermore, for the control problem the coupled structure of the equations yields the further insight that in the reduced-order case there is no longer separation between the operations of state estimation and state-estimate feedback, i.e., the certainty equivalence principle breaks down.

For practical applications, the optimal projection equations permit the development of alternative numerical algorithms that operate through successive iteration of the optimal projection[4,6] rather than by gradient search techniques.[12] By recognizing that each local extremalcorresponds to n_r possible choices out of n rank-1 eigenprojections of the product of a pair of pseudogramians, it is possible to efficiently identify the global minimum.[10] This idea is philosophically similar to Skelton's component-cost analysis.[13-15]

The purpose of the present paper is to develop the optimal projection equations for reduced-order modeling, estimation, and control in the *discrete-time* case. Since the underlying theory has been discussed previously,[1-11] the presentation herein is geared toward a clear and concise statement of the main results to facilitate numerical developments and practical application. For example, by expressing the optimal projection in terms of eigenprojections, a variety of novel algorithms are immediately suggested. For illustrative purposes we apply the results on reduced-order state estimation to a third-order problem to obtain reduced-order estimators and the results on reduced-order dynamic compensation to a tenth-order problem to obtain reduced-order controllers.

Because of the discrete-time setting it is now possible to permit static feedthrough gains in the estimator and controller designs. As previously noted,[7] nonsingular control weighting and measurement noise in the continuous-time case permit only a purely dynamic (strictly proper) controller. Note that this is precisely the case in continuous-time LQG theory, which always yields a strictly proper feedback controller. The static gains in the discrete-time estimator problem permit simultaneous, unified treatment of nondynamic least-squares estimation along with dynamic (Kalman filter-type) estimation.

The references include a representative sampling of papers on quadratically optimal reduced-order modeling,[13-25] estimation,[25-33] and control,[34-46] along with closely related approaches. For emphasis on the discrete-time problem, see Refs. 18, 30, 41, 42, 44, and 45.

II. Problem Statement and Main Results

We now state the reduced-order modeling, estimation, and control problems. The object of the model-reduction problem is to determine a model of reduced state-space dimension whose steady-state response to white noise inputs (or, equivalently, impulse response) best approximates, in a quadratic (least-squares) sense, the response of a given high-order system. In the reduction process the order of the reduced model is fixed and the optimization is performed over the model parameters.

Reduced-Order Modeling Problem

Given the model

$$x(k+1) = Ax(k) + Bw(k) \tag{1}$$

$$y(k) = Cx(k) \tag{2}$$

design a reduced-order model

$$x_m(k+1) = A_m x_m(k) + B_m w(k) \tag{3}$$

$$y_m(k) = C_m x(k) \tag{4}$$

which minimizes the model-reduction criterion

$$J_m(A_m, B_m, C_m) \triangleq \lim_{k\to\infty} E[y_m(k) - y(k)]^T R[y_m(k) - y(k)] \tag{5}$$

The goal of the reduced-order state-estimation problem is to design an estimator of given order which yields quadratically optimal (least squares) estimates of specified linear combinations Lx of states x. In practice, the order of the estimator may be determined by implementation constraints, such as real-time computing capability. Note that the feedthrough term D_e permits the utilization of a *static* least-squares estimator in conjunction with the *dynamic* estimator (A_e, B_e, C_e).

Reduced-Order State-Estimation Problem

Given the observed system

$$x(k+1) = Ax(k) + w_1(k) \tag{6}$$

$$y(k) = Cx(k) + w_2(k) \tag{7}$$

design a reduced-order state estimator

$$x_e(k+1) = A_e x_e(k) + B_e y(k) \tag{8}$$

$$y_e(k) = C_e x_e(k) + D_e y(k) \tag{9}$$

which minimizes the state-estimation criterion

$$J_e(A_e, B_e, C_e, D_e) \triangleq \lim_{k\to\infty} E[y_e(k) - Lx(k)]^T N[y_e(k) - Lx(k)] \tag{10}$$

For the fixed-order dynamic-compensation problem, a static feedthrough term is included, i.e., the controller may be nonstrictly proper.

Reduced-Order Dynamic-Compensation Problem

Given the controlled system

$$x(k+1) = Ax(k) + Bu(k) + w_1(k) \tag{11}$$

$$y(k) = Cx(k) + w_2(k) \tag{12}$$

design a reduced-order dynamic compensator

$$x_c(k+1) = A_c x_c(k) + B_c y(k) \tag{13}$$

$$u(k) = C_c x_c(k) + D_c y(k) \tag{14}$$

which minimizes the dynamic-compensation criterion

$$J_c(A_c, B_c, C_c, D_c) \triangleq \lim_{k\to\infty} E\big[x(k)^T R_1 x(k) + 2x(k)^T R_{12} u(k) + u(k)^T R_2 u(k)\big] \tag{15}$$

To guarantee that J_m, J_e, and J_c are finite and independent of initial conditions, consideration is restricted to the following (open) sets. [A triple $(\hat{A}, \hat{B}, \hat{C})$ is *minimal* if $(\hat{A}, \hat{B})$ is controllable and $(\hat{A}, \hat{C})$ is observable.]

$$\mathcal{S}_m \triangleq \{(A_m, B_m, C_m):\ A_m \text{ is stable and } (A_m, B_m, C_m) \text{ is minimal}\}$$

$$\mathcal{S}_e \triangleq \{(A_e, B_e, C_e, D_e):\ A_e \text{ is stable and } (A_e, B_e, C_e) \text{ is minimal}\}$$

$$\mathcal{S}_c \triangleq \left\{(A_c, B_c, C_c, D_c):\ \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \text{ is stable and } (A_c, B_c, C_c) \text{ is minimal}\right\}$$

Let n_r generically denote n_m, n_e, and n_c. The following factorization lemma will be needed for the main results.

Lemma 2.1. Let $\tau \in \boldsymbol{R}^{n\times n}$. Then

$$\tau^2 = \tau \tag{16}$$

$$\rho(\tau) = n_r \tag{17}$$

if, and only if, there exist $G, \Gamma \in \boldsymbol{R}^{n_r\times n}$ such that

$$G^T\Gamma = \tau \tag{18}$$

$$\Gamma G^T = I_{n_r} \tag{19}$$

Furthermore, G and Γ are unique to a change of basis in $\boldsymbol{R}^{n_r}$.

Proof. Sufficiency is obvious. To prove necessity, first note that due to Eq. (16) the eigenvalues of τ are either 0 or 1. Further, it is easy to see that τ has a diagonal Jordan canonical form. Hence, the result follows from

$$\tau = S\begin{bmatrix} I_{n_r} & 0 \\ 0 & 0 \end{bmatrix} S^{-1} = G^T\Gamma$$

where $G = [\phi^T \quad 0]S^T$, $\Gamma = [\phi^{-1} \quad 0]S^{-1}$, and $\phi \in \boldsymbol{R}^{n_r\times n_r}$. ∎

For convenience, call G and Γ satisfying Eqs. (18) and (19) a *projective factorization* of τ. Furthermore, for $n\times n$ nonnegative-definite matrices (i.e., symmetric matrices with nonnegative eigenvalues) $\mathcal{Q}$ and $\mathcal{P}$, define the set of *contragredi-*

ently diagonalizing transformations

$$\mathscr{D}(\mathscr{Q},\mathscr{P})$$
$$\triangleq \{\Psi \in \boldsymbol{R}^{n\times n}: \Psi^{-1}\mathscr{Q}\Psi^{-T} \text{ and } \Psi^T\mathscr{P}\Psi \text{ are diagonal}\}$$

It follows from Ref. 48, p. 123, Theorem 6.2.5, that $\mathscr{D}(\mathscr{Q},\mathscr{P})$ is always nonempty. This set does not, however, have a unique element since basis rearrangements and sign transpositions may be incorporated into Ψ. Further nonuniqueness arises if $\mathscr{Q}\mathscr{P}$ has repeated eigenvalues.

Theorem 2.1. Suppose A is stable and $(A_m, B_m, C_m) \in \mathscr{S}_m$ solves the reduced-order modeling problem. Then there exist $n\times n$ nonnegative-definite matrices $\hat{Q}$ and $\hat{P}$ such that A_m, B_m, and C_m are given by

$$A_m = \Gamma A G^T \tag{20}$$

$$B_m = \Gamma B \tag{21}$$

$$C_m = CG^T \tag{22}$$

and such that $\hat{Q}$ and $\hat{P}$ satisfy

$$\hat{Q} = A\tau\hat{Q}\tau^T A^T + BVB^T \tag{23}$$

$$\hat{P} = A^T\tau^T\hat{P}\tau A + C^T RC \tag{24}$$

where

$$\tau \triangleq \sum_{i=1}^{n_m} \Pi_i(\Psi) \tag{25}$$

for some $\Psi \in \mathscr{D}(\hat{Q},\hat{P})$ such that $(\Psi^{-1}\hat{Q}\hat{P}\Psi)_{(i,i)} \neq 0$, $i = 1,\ldots,n_m$, and some projective factorization G,Γ of τ. Furthermore, the minimal cost is given by

$$J_m(A_m, B_m, C_m) = \mathrm{tr}\left[\left(W_c - \tau\hat{Q}\tau^T\right)C^T RC\right] \tag{26}$$

where W_c is the unique (nonnegative-definite) solution of

$$W_c = AW_cA^T + BVB^T$$

For convenience in stating the estimator result, define the notation

$$\Sigma_Q \triangleq \left(AQC^T + V_{12}\right)\tilde{V}_2^{-1}\left(AQC^T + V_{12}\right)^T$$
$$\Sigma_P \triangleq (L - D_eC)^T N(L - D_eC)$$
$$A_Q \triangleq A - \left(AQC^T + V_{12}\right)\tilde{V}_2^{-1}C$$
$$\tilde{V}_2 \triangleq V_2 + CQC^T$$
$$\tau_\perp \triangleq I_n - \tau$$

Theorem 2.2. Suppose A is stable and $(A_e, B_e, C_e, D_e) \in \mathscr{S}_e$ solves the reduced-order state-estimation problem. Then there exist $n\times n$ nonnegative-definite matrices Q, $\hat{Q}$, and $\hat{P}$ such that A_e, B_e, C_e, and D_e are given by

$$A_e = \Gamma\left[A - \left(AQC^T + V_{12}\right)\tilde{V}_2^{-1}C\right]G^T \tag{27}$$

$$B_e = \Gamma\left(AQC^T + V_{12}\right)\tilde{V}_2^{-1} \tag{28}$$

$$C_e = (L - D_eC)G^T \tag{29}$$

$$D_e = LQC^T\tilde{V}_2^{-1} \tag{30}$$

and such that Q, $\hat{Q}$, and $\hat{P}$ satisfy

$$Q = AQA^T - \left(AQC^T + V_{12}\right)\tilde{V}_2^{-1}\left(AQC^T + V_{12}\right)^T + V_1 + \tau_\perp\hat{Q}\tau_\perp^T \tag{31}$$

$$\hat{Q} = A\tau\hat{Q}\tau^T A^T + \Sigma_Q \tag{32}$$

$$\hat{P} = A_Q^T\tau^T\hat{P}\tau A_Q + \Sigma_P \tag{33}$$

where

$$\tau \triangleq \sum_{i=1}^{n_e} \Pi_i(\Psi) \tag{34}$$

for some $\Psi \in \mathscr{D}(\hat{Q},\hat{P})$ such that $(\Psi^{-1}\hat{Q}\hat{P}\psi)_{(i,i)} \neq 0$, $i = 1,\ldots,n_e$, and some projective factorization G,Γ of τ. Furthermore, the minimal cost is given by

$$J_e(A_e, B_e, C_e, D_e) = \mathrm{tr}\left[\left(LQL^T - D_e\tilde{V}_2D_e^T\right)N\right] \tag{35}$$

For the control result, define the additional notation

$$\Sigma_Q' \triangleq \left(AQC^T + V_{12} + BD_c\tilde{V}_2\right)\tilde{V}_2^{-1}\left(AQC^T + V_{12} + BD_c\tilde{V}_2\right)^T$$
$$\Sigma_P' \triangleq \left(B^TPA + R_{12}^T + \tilde{R}_2D_cC\right)^T\tilde{R}_2^{-1}\left(B^TPA + R_{12}^T + \tilde{R}_2D_cC\right)$$
$$A_P \triangleq A - B\tilde{R}_2^{-1}\left(B^TPA + R_{12}^T\right)$$
$$\tilde{R}_2 \triangleq R_2 + B^TPB$$
$$M \triangleq \begin{bmatrix} I_n \\ D_cC\end{bmatrix}, \quad \hat{M} \triangleq \begin{bmatrix} I_n \\ -\tilde{R}_2^{-1}\left(B^TPA + R_{12}^T\right)\end{bmatrix}$$

Theorem 2.3. Suppose $(A_c, B_c, C_c, D_c) \in \mathscr{S}_c$ solves the reduced-order dynamic-compensation problem. Then there exist $n\times n$ nonnegative-definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ such that A_c, B_c, C_c, and D_c are given by

$$A_c = \Gamma\left[A - \left(AQC^T + V_{12}\right)\tilde{V}_2^{-1}C - B\tilde{R}_2^{-1}\left(B^TPA + R_{12}^T\right) - BD_cC\right]G^T \tag{36}$$

$$B_c = \Gamma\left[\left(AQC^T + V_{12}\right)\tilde{V}_2^{-1} + BD_c\right] \tag{37}$$

$$C_c = -\left[\tilde{R}_2^{-1}\left(B^TPA + R_{12}^T\right) + D_cC\right]G^T \tag{38}$$

$$D_c = -\tilde{R}_2^{-1}\left(B^TPAQC^T + R_{12}^TQC^T + B^TPV_{12}\right)\tilde{V}_2^{-1} \tag{39}$$

and such that Q, P, $\hat{Q}$, and $\hat{P}$ satisfy

$$Q = AQA^T - \left(AQC^T + V_{12}\right)\tilde{V}_2^{-1}\left(AQC^T + V_{12}\right)^T + V_1 + \tau_\perp\hat{Q}\tau_\perp^T \tag{40}$$

$$P = A^TPA - \left(B^TPA + R_{12}^T\right)^T\tilde{R}_2^{-1}\left(B^TPA + R_{12}^T\right) + R_1 + \tau_\perp^T\hat{P}\tau_\perp \tag{41}$$

$$\hat{Q} = A_P\tau\hat{Q}\tau^TA_P^T + \Sigma_Q' \tag{42}$$

$$\hat{P} = A_Q^T\tau^T\hat{P}\tau A_Q + \Sigma_P' \tag{43}$$

where

$$\tau \triangleq \sum_{i=1}^{n_c} \Pi_i(\Psi) \tag{44}$$

for some $\Psi \in \mathscr{D}(\hat{Q},\hat{P})$ such that $(\Psi^{-1}\hat{Q}\hat{P}\Psi)_{(i,i)} \neq 0$, $i = 1,\ldots,n_c$, and some projective factorization G,Γ of τ. Furthermore, the minimal cost is given by

$$J_c(A_c, B_c, C_c, D_c) = \mathrm{tr}\left[\left(MQM^T + \hat{M}\tau\hat{Q}\tau^T\hat{M}^T\right)\tilde{R}\right] \quad (45)$$

Remark 2.1. To specialize the estimation and control results to the strictly proper (no-feedthrough) case, merely ignore Eqs. (30) and (39) and set $D_e = 0$ and $D_c = 0$ wherever they appear.

Remark 2.2. In the full-order cases $n_e = n$ and $n_c = n$ in Theorems 2.2 and 2.3, the projection τ becomes the identity and Eqs. (32), (33), (42), and (43) play no role. In this case G and Γ are also the identity. Specializing further to the purely dynamic case $D_e = 0$, $D_c = 0$ as in the previous remark yields the standard Kalman filter and LQG results.

Remark 2.3. As previously noted,[7,10,11] the indeterminacy in specifying the projective factorization G, Γ satisfying Eqs. (18) and (19) corresponds to nothing more than an arbitrary choice of internal state-space basis for the design systems (A_m, B_m, C_m), (A_e, B_e, C_e), and (A_c, B_c, C_c).

Remark 2.4. Since $\hat{Q}$ and $\hat{P}$ are balanced by means of the transformation $\Psi \in \mathscr{D}(\hat{Q}, \hat{P})$, it follows that $\Psi^{-1}\hat{Q}\hat{P}\Psi$ is diagonal. Hence, $\hat{Q}\hat{P}$ is semisimple and thus $\Pi_i(\Psi)$ is a rank-1 eigenprojection of $\hat{Q}\hat{P}$. (A semisimple matrix possesses a diagonal Jordan form.[47,48]) Although the optimal projection τ is characterized in Eqs. (25), (34), and (44) as the sum of rank-1 eigenprojections of $\hat{Q}\hat{P}$, because of the nonuniqueness in $\mathscr{D}(\hat{Q}, \hat{P})$, the theorems do not specify *which* eigenprojections actually comprise τ. From analytical examples[10] it can be seen that each of the $\binom{n}{n_r}$ possible projections may correspond to a local extremal in the optimization problem.

Remark 2.5. The proofs of Theorems 2.1-2.3 are similar to the continuous-time results and, hence, have been omitted. To help the reader reconstruct the lengthy manipulations, the key details differing from the continuous-time case are pointed out. For the control problem, an $(n + n_c) \times (n + n_c)$ discrete-time algebraic Lyapunov equation is obtained for the steady-state covariance of the closed-loop system. Regarding this equation as a side constraint, the Lagrange multiplier technique is used to compute stationarity conditions that yield explicit expressions for A_c, B_c, C_c, and D_c. The projection arises when these expressions are substituted into the original augmented Lyapunov equation and its dual. The interesting aspect is that the explicit gain expressions and the definition of the optimal projection arise in the *reverse* order as compared to the continuous-time derivation. Similar remarks apply to the reduced-order modeling and estimation problems.

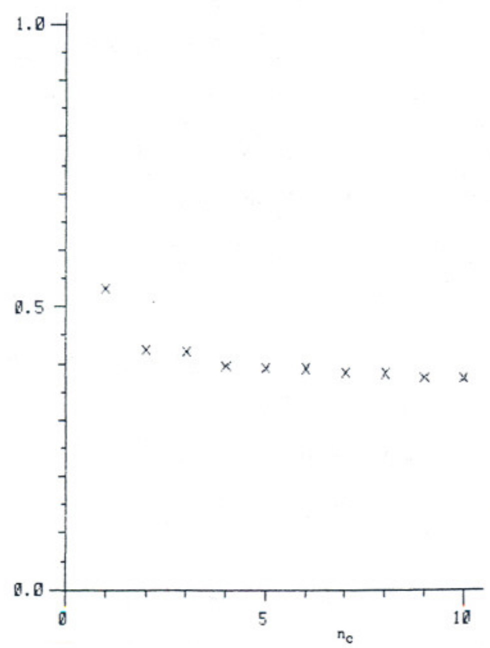

Fig. 1 Root-mean-square performance vs controller order for five-mode beam example.

III. Examples

As an application of Theorem 2.2 on reduced-order state estimation, the stirred-tank example from Ref. 36, pp. 107, 473, and 531, is considered. Ignoring the undisturbed volume state, the remaining states are the incremental tank concentration and variations in the feed concentrations. The problem data are as follows:

$$A = \begin{bmatrix} 0.9048 & 0.06702 & 0.02262 \\ 0 & 0.8825 & 0 \\ 0 & 0 & 0.9048 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0], \; V_2 = 10^{-6}, \; V_{12} = 0, \; N = L = I_3$$

$$V_1 = \begin{bmatrix} 5.399 \times 10^{-6} & 8.015 \times 10^{-5} & 8.762 \times 10^{-5} \\ 8.015 \times 10^{-5} & 2.212 \times 10^{-3} & 0 \\ 8.762 \times 10^{-5} & 0 & 7.251 \times 10^{-3} \end{bmatrix}$$

The standard Kalman filter result is

$$A_e = \begin{bmatrix} -0.7959 & 0.06702 & 0.2262 \\ -8.205 & 0.8825 & 0 \\ -10.29 & 0 & 0.9048 \end{bmatrix}$$

$$B_e = \begin{bmatrix} 1.701 \\ 8.205 \\ 10.29 \end{bmatrix}$$

$$C_e = I_3, \quad D_e = 0$$

with performance $J_e = 0.0358515$.

Permitting nonzero feedthrough D_e, yields

$$A_e = \begin{bmatrix} 0.09885 & 0.1061 & -0.0167 \\ -0.6851 & 0.02176 & 0.09015 \\ 0.1503 & 0.2652 & 0.8707 \end{bmatrix}, \; B_e = \begin{bmatrix} -13.34 \\ -10.08 \\ 2.237 \end{bmatrix}$$

$$C_e = \begin{bmatrix} -0.006261 & -0.007282 & 0.0007445 \\ 0.5716 & -0.0002166 & 0.007202 \\ 0.6822 & -0.006021 & -0.005440 \end{bmatrix},$$

$$D_e = \begin{bmatrix} 0.9068 \\ 9.298 \\ 11.37 \end{bmatrix}$$

where the (improved) performance is $J_e = 0.032401049$.

For the reduced-order results, an algorithm for solving all three equations (31-33) is described briefly. Begin by setting $\tau = I_n$ and solving Eqs. (31-33) for the "full-order" values of Q, $\hat{Q}$, and $\hat{P}$. Choose n_e eigenprojections of $\hat{Q}\hat{P}$ in diagonalizing coordinates and iterate the modified Lyapunov equations (32) and (33) until convergence of τ, $\hat{Q}$, and $\hat{P}$ is obtained. Return to Eq. (31) and solve for Q with $V_1 + \tau_\perp \hat{Q}\tau_\perp^T$ as the new nonhomogeneous term in the Riccati equation. Repeat the above steps until convergence is reached.

In applying this algorithm to the present example, the eigenprojections were chosen for convenience in accordance with the largest eigenvalues of $\hat{Q}\hat{P}$. The results indicate attainment of the global minimum. For the optimal second-order filter, the gains are given by

$$A_e = \begin{bmatrix} 0.09898 & -0.1137 \\ 0.6632 & -0.02285 \end{bmatrix}, \quad B_e = \begin{bmatrix} -13.33 \\ 9.762 \end{bmatrix}$$

$$C_e = \begin{bmatrix} -0.006259 & 0.007796 \\ 0.5716 & -0.0002737 \\ 0.6823 & 0.006121 \end{bmatrix}, \quad D_e = \begin{bmatrix} 0.9068 \\ 9.298 \\ 11.37 \end{bmatrix}$$

with $J_e = 0.032401094$, and, for the first-order filter,

$$A_e = 0.1498, \quad B_e = -14.82$$

$$C_e = \begin{bmatrix} -0.001661 \\ 0.5748 \\ 0.6897 \end{bmatrix}, \quad D_e = \begin{bmatrix} 0.9749 \\ 9.489 \\ 11.65 \end{bmatrix}$$

with $J_e = 0.03240418$. Convergence to this accuracy was obtained with 7 iterations of Eqs. (31-33) for the second-order filter and 10 iterations for the first-order filter. Note that the performance degrades only slightly with reduced order, and the static gain term gives the first-order filter better performance than the standard (full-order) Kalman filter.

To illustrate Theorem 2.3 for designing reduced-order dynamic compensators, consider a simply supported beam with two colocated sensor/actuator pairs. Assuming the beam has length 2 and that the sensor/actuator pairs are placed at coordinates $a = 55/172$ and $b = 46/43$, a continuous-time model of the following form is obtained:

$$\dot{x} = \hat{A}x + \hat{B}u + \hat{w}_1 \qquad y = \hat{C}x + \hat{w}_2$$

where, retaining the first five modes,

$$\hat{A} = \text{block-diag}\left(\begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\zeta\omega_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -\omega_5^2 & -2\zeta\omega_5 \end{bmatrix}\right)$$

$$\omega_i = i^2; \quad i = 1, \ldots, 5, \quad \zeta = 0.005$$

$$\hat{B}_{(i,1)} = 0.5\left(1 + (-1)^i\right)\sin(i\pi b/2), \quad i = 1, \ldots, 10$$

$$\hat{B}_{(i,2)} = -0.5\left(1 + (-1)^i\right)\sin(i\pi a/2), \quad i = 1, \ldots, 10$$

$$\hat{C} = \hat{B}^T$$

The intensities $\hat{V}_1$ and $\hat{V}_2$ of $\hat{w}_1$ and $\hat{w}_2$ are chosen to be

$$\hat{V}_1 = 0.1 I_{10}, \quad \hat{V}_2 = 0.01 I_2$$

and it is assumed that $\hat{w}_1$ and $\hat{w}_2$ are uncorrelated. For the continuous-time cost

$$J = \lim_{t \to \infty} \boldsymbol{E}\left[x^T \hat{R}_1 x + 2x^T \hat{R}_{12} u + u^T \hat{R}_2 u\right]$$

set

$$\hat{R}_1 = \text{block-diag}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1/\omega_1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & 1/\omega_5 \end{bmatrix}\right)$$

$$\hat{R}_{12} = 0, \quad \hat{R}_2 = I_2$$

To convert to the discrete-time problem with discretization interval h, let[36,49]

$$A = e^{\hat{A}h}, \quad B = \int_0^h e^{\hat{A}t}\hat{B}\,\mathrm{d}t, \quad C = \hat{C}$$

$$V_1 = \int_0^h e^{\hat{A}t}\hat{V}_1 e^{\hat{A}^T t}\,\mathrm{d}t, \quad V_2 = \hat{V}_2$$

$$R_1 = \hat{R}_1, \quad R_{12} = 0, \quad R_2 = \hat{R}_2$$

The design equations (40-43) for the control problem can be solved using exactly the same techniques as in the previous example for the estimation problem. For the strictly proper case ($D_c = 0$), a series of controllers was designed with $n_c = 1, 2, \ldots, 10$, where the $n_c = 10$ result is the LQG solution. The gains for the case $n_c = 4$, for example, are given by

$$A_c = \begin{bmatrix} 0.9317 & 0.1572 & -0.2130 & -0.005038 \\ 0.0137 & 0.6879 & 0.2519 & 0.4085 \\ 0.3330 & -0.0580 & 0.7713 & -0.2602 \\ 0.05980 & -0.3297 & 0.3918 & 0.3005 \end{bmatrix}$$

$$B_c = \begin{bmatrix} -0.4920 & -0.2166 \\ 0.6179 & -0.5959 \\ 0.2253 & -0.02572 \\ 0.07221 & -0.4863 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 0.05864 & -0.3094 & -0.01815 & 0.2409 \\ -0.1301 & 0.1463 & -0.1945 & -0.07192 \end{bmatrix}$$

Figure 1 summarizes the results for each order, where the rms controller performance is given by

$$J_x^{\frac{1}{2}} = \boldsymbol{E}\left[\lim_{k \to \infty} x(k)^T R_1 x(k)\right]^{\frac{1}{2}}$$

These results provide a tradeoff study of performance versus controller order that can be used to assess processor requirements.

IV. Conclusion

Optimality conditions have been obtained for the problems of least-squares, reduced-order (i.e., fixed-order), discrete-time modeling, estimation, and control. These conditions comprise systems of two, three, and four matrix equations, respectively, coupled by an oblique projection which determines the optimal system gains. When the order of the estimator or controller is equal to the order of the plant, the oblique projection becomes the identity matrix and the estimation and control results specialize to the standard discrete-time Kalman filter and linear-quadratic-Gaussian results. The design results are applied to two illustrative examples. For a third-order stirred-tank problem, filters of first and second order are obtained, and, for a simply supported Euler beam example with five flexible modes (i.e., 10 states), a series of reduced-order controllers with 1, 2, ..., 9 poles is obtained. The latter results illustrate the tradeoff between control-system performance and controller order.

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