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Integrable deformations of $\hat{c} = 1$ strings in flux backgrounds

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Abstract: We study $d = 2$ 0A string theory perturbed by tachyon momentum modes in backgrounds with non-trivial tachyon condensate and Ramond-Ramond (RR) flux. In the matrix model description, we uncover a complexified Toda lattice hierarchy constrained by a pair of novel holomorphic string equations. We solve these constraints in the classical limit for general RR flux and tachyon condensate. Due to the non-holomorphic nature of the tachyon perturbations, the transcendental equations which we derive for the string susceptibility are manifestly non-holomorphic. We explore the phase structure and critical behavior of the theory.

Keywords: Integrable Field Theories, Matrix Models.

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Contents

1. Introduction

Two-dimensional string theory has been fruitfully used to explore problems for which we lack the technical prowess and/or conceptual framework to address in more realistic, higher-dimensional string theories. This has been made possible mostly by the matrix models dual to the $d = 2$ bosonic (for a review see [1−3]) and Type 0 strings [4, 5], which provide non-perturbative definitions for those theories. Of interest in the present article are

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Ramond-Ramond flux backgrounds which were recently investigated using the matrix dual descriptions in [6]. In this work we further study non-trivial backgrounds in Type 0A string theory by deforming the theory with an integrable set of momentum modes in the presence of both RR flux (q) and tachyon condensate (μ).

Although the integrable structure of the $d = 2$ bosonic string has been thoroughly explored (for example, in [7−16]), such structures for 0A string theory are somewhat less developed. The integrable properties of the 0A matrix model were first studied using the Toda lattice hierarchy [17] in the early '90s [18], when it was known as the deformed matrix model [19]. More recently this has been discussed by [20, 21]. Alternatively, following [22], perturbative techniques have been utilized to address momentum mode deformations in [23]. A few of the lowest order correlators were computed and a pattern for $\mu = 0$ was discerned; summing the infinite perturbative series lead to an expression for the 0A partition function. As discussed in [11] for the $c = 1$ matrix model, this is equivalent to solving the string equation constraining the integrable Toda lattice hierarchy. However in all of these previous works, the authors studied limits where either $q \to 0$ or $\mu \to 0$ to obtain tractable results regarding the partition function or string susceptibility.

In contrast, we obtain transcendental equations for the genus zero string susceptibility parameterized by non-trivial tachyon condensate, Ramond-Ramond flux and momentum mode perturbations simultaneously. To do so we take advantage of the complex coordinate $\mu + iq$ identified in [6] and introduce a complexified Toda lattice hierarchy. This hierarchy is based on shifts of both the energy and angular momentum quantum numbers of the single-fermion Hilbert space of the dual 0A matrix model description. The utilization of the fermion angular momentum as a dynamical variable is reminiscent of the proposed non-critical M-theory [24, 25] perspective on 0A string theory, although we introduce it for purely mathematical ends. The complex nature of the integrable structure provides an additional real constraint (string equation) on the operator algebra of the theory, relative to previous integrability analyses. This additional constraint allows us to solve for the susceptibility equations in the dispersionless, i.e. classical, limit.

This paper is organized as follows. Section 2 contains a review of the treatment of integrable momentum mode perturbations to the $c = 1$ matrix model. We introduce our notation and philosophy here. In section 3, we address integrable deformations of the 0A matrix model. We show how the system is intractable when there is only one string equation as a constraint. We then introduce the complexified Toda lattice hierarchy which provides two such constraints. We solve this system in the dispersionless limit, obtaining equations for the perturbed susceptibility. In section 4 we analyze the susceptibility equations, exploring the critical behavior and phase diagram of 0A string theory. We close with section 5 wherein we include some preliminary results on perturbations which take advantage of the holomorphic structure of the complexified Toda hierarchy. Finally, a number of technical appendices are included to elucidate points made within the text as well as providing an alternate derivation of the complex string equations.

While discussing the bosonic string and the $c = 1$ matrix model, our units are such that $\alpha' = 1$. When discussing the Type 0 strings, we will use $\alpha' = \frac{1}{2}$ units.

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2. Integrable perturbations of the $c = 1$ matrix model

Before turning to integrable deformations of 0A string theory, we will now provide a short review of such deformations in the $c = 1$ model, first studied in [7]. The purpose of this detour is to introduce our logic and notation in an example where we closely follow previous analyses before moving on to a treatment of 0A where we will depart significantly from prior work by other authors.

2.1 Chiral quantization and the energy representation

The $c = 1$ matrix model is usefully described by a system of non-interacting fermions. The operator algebra of these fermions is given by

$$\begin{aligned} [\hat{x}_+, \hat{x}_-] &= i, \\ [\hat{x}_\pm, \hat{\epsilon}] &= \pm i\hat{x}_\pm, \\ \{\hat{x}_+, \hat{x}_-\} &= 2\hat{\epsilon}. \end{aligned} \tag{2.1}$$

where $\hat{\epsilon}$ is the energy and $\hat{x}_\pm = \frac{\hat{p}\pm\hat{x}}{\sqrt{2}}$ are light-cone coordinates in the single-particle phase space. The commutator of $\hat{x}_\pm$ with $\hat{\epsilon}$ indicates that these operators have simple expressions in the energy basis, $\hat{x}_\pm \sim e^{\pm i\partial_\epsilon}$. To produce the correct commutator and anti-commutator between $\hat{x}_+$ and $\hat{x}_-$ requires the addition of certain dressing phases. Suitable energy representations which reproduce all of the relations (2.1) are

$$[\hat{x}_\pm]_\epsilon = \pm S^{\mp 1/2}(\epsilon)\,\hat{\omega}^{\pm 1} S^{\pm 1/2}(\epsilon) \tag{2.2}$$

where $\hat{\omega}$ is the shift operator

$$\hat{\omega} \equiv e^{i\partial_\epsilon}, \tag{2.3}$$

and $S(\epsilon)$ is

$$S(\epsilon) = e^{-i\pi/4}\sqrt{\frac{\Gamma\left(\frac{1}{2} - i\epsilon\right)}{\Gamma\left(\frac{1}{2} + i\epsilon\right)}} \equiv e^{i\phi_0(\epsilon)}, \tag{2.4}$$

which can be understood as the $\epsilon \to -\infty$ scattering amplitude for the fermions.[1]

Some comments on the representations above are in order. Although we use the term "operator" to describe $\hat{\omega}$, as well as using a hat, it should be made clear that in this text we will use it exclusively in the energy basis as a shorthand for the derivative (2.3). Thus, expressions containing $\hat{\omega}$ will not be treated as operator statements but rather as basis specific statements. One could, of course, adopt the alternative viewpoint that $\hat{\omega}$ is an operator and (2.3) is its energy representation, but we will not do so.

Additionally, we wish to mention that the energy basis for the $\hat{x}_\pm$ operators is actually somewhat subtle and depends on the sign of ϵ. In studying the $c = 1$ matrix model, one considers only fermions with negative energy, which in perturbation theory are localized in one of the quadrants of the x_+x_- plane. The definitions (2.2) have been chosen to describe the quadrant with $\pm x_\pm > 0$. Fortunately this subtlety will not occur for 0A where it is necessary to consider both signs of ϵ.

[1]See appendix A for more details on light-cone quantization and calculation of $S(\epsilon)$.

2.2 Unitary transformations and the lax formalism

Consider some unitary transformation $\hat{U}$ acting on the system of fermions. We choose to employ a passive transformation picture where the state kets remain unchanged and the operators in (2.1) transform as $\hat{\mathcal{O}} \to \hat{U}\hat{\mathcal{O}}\hat{U}^{-1}$. In particular, we will label the transformed operators as

$$\hat{L}_{\pm} = \hat{U}\hat{x}_{\pm}\hat{U}^{-1}, \tag{2.5}$$
$$\hat{M} = \hat{U}\hat{\epsilon}\hat{U}^{-1}. \tag{2.6}$$

As our notation suggests, these transformed operators are precisely the Lax and Orlov-Shulman operators of the Toda lattice hierarchy when $\hat{U}$ is chosen appropriately. It follows trivially that the commutators and anti-commutators (2.1) are preserved by the unitary transformation under consideration. This leads to

$$\hat{L}_{\pm}\hat{L}_{\mp} = \hat{M} \pm \frac{i}{2}. \tag{2.7}$$

In the context of the Toda lattice hierarchy, (2.7) are known as the string equations. Unlike the usual treatment where separate $\hat{M}_{\pm}$ are introduced and then equated as an additional constraint, we posit only one Orlov-Shulman operator from the outset. Thus we are studying the constrained Toda hierarchy, *ab initio.*

An integrable set of momentum mode deformations to the $c = 1$ model are generated by the transformations

$$\hat{U}_{\pm} = e^{i\sum_{n>0}\tilde{b}_{\pm n}(\hat{\epsilon};\{t\})\,(\pm\hat{x}_{\pm})^{-n/R}}e^{\mp i\tilde{\phi}(\hat{\epsilon};\{t\})/2}e^{i\sum_{n>0}t_{\pm n}\,(\pm\hat{x}_{\pm})^{n/R}}. \tag{2.8}$$

The $t_{\pm n}$ are real constants[2], and the operators $\tilde{b}$ and $\tilde{\phi}$ are unspecified functions which vanish when all $t_{\pm n} = 0$. We will constrain the undetermined functions $\tilde{b}$ and $\tilde{\phi}$ such that $\hat{U}_{+} \equiv \hat{U}_{-} \equiv \hat{U}$. We understand $\hat{U}_{+}$ ($\hat{U}_{-}$) to be the natural form of $\hat{U}$ acting on the $\hat{x}_{+}$ ($\hat{x}_{-}$) operator. This is not to be confused with different bases; the expression $\hat{U}_{+} = \hat{U}_{-}$ indicates an equality between operators, not that $\hat{U}_{\pm}$ are different representations of the same operator. Thus they are equal in any given basis.

The operators in (2.8) are of interest since they allow us to represent momentum mode deformations to the worldsheet action in terms of transformations on the single fermion Hilbert space. The operators $(\pm\hat{x}_{\pm})^{n/R}$ are the single-particle representations of the matrix model operators $\mathrm{Tr}(M \pm \dot{M})^{n/R}$ which create states of Euclidean momentum $\frac{n}{R}$ [26]. In the regions $x_{\pm} \to \infty$ the operators $\hat{U}_{\pm}$ behave like

$$\hat{U}_{\pm} \sim e^{\mp i\tilde{\phi}(\hat{\epsilon};\{t\})/2}e^{i\sum_{n>0}t_{\pm n}\,(\pm\hat{x}_{\pm})^{n/R}}, \tag{2.9}$$

demonstrating that asymptotically $\hat{U}_{\pm}$ creates a coherent state of tachyons (plus some zero mode). The factor with negative powers of $\hat{x}_{\pm}$ is present to allow the equality of $\hat{U}_{+}$ and $\hat{U}_{-}$.

[2] Although we will be more general in our analysis, in order to produce real deformations to the worldsheet action the signs of the t's must be such that $t_n = -t_{-n}$ for all n. This will be true for the 0A theory as well.

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Now consider these operators in the energy basis. From (2.2), we have

$$\left[e^{i\sum_{n>0} t_{\pm n}(\pm\hat{x}_\pm)^{n/R}}\right]_\epsilon = S^{\mp 1/2}(\epsilon)\, e^{i\sum_{n>0} t_{\pm n}\hat{\omega}^{\pm n/R}} S^{\pm 1/2}(\epsilon)\,, \tag{2.10}$$

and so find

$$\left[\hat{U}_\pm\right]_\epsilon = e^{i\sum_{n>0} b_{\pm n}(\epsilon;\{t\})\hat{\omega}^{\mp n/R}} e^{\mp i\phi(\epsilon;\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\hat{\omega}^{\pm n/R}} S^{\pm 1/2}(\epsilon)\,, \tag{2.11}$$

where we have introduced

$$\begin{aligned} \phi(\epsilon;\{t\}) &\equiv \tilde{\phi}(\epsilon;\{t\}) + \phi_0(\epsilon)\,, \\ b_{\pm n}(\epsilon;\{t\}) &\equiv \tilde{b}_{\pm n}(\epsilon;\{t\})\, S^{\mp 1/2}(\epsilon)\, S^{\pm 1/2}(\epsilon \mp in/R)\,. \end{aligned} \tag{2.12}$$

Using the energy basis and the equivalent forms of the transformation $\hat{U}$, we can make explicit the connection of the above with the Lax formalism. We write

$$\begin{aligned} \hat{L}_\pm &= \hat{U}_\pm \hat{x}_\pm \hat{U}_\pm^{-1}, \\ \hat{M}_\pm &= \hat{U}_\pm \hat{\epsilon} \hat{U}_\pm^{-1}. \end{aligned} \tag{2.13}$$

It should be emphasized that the above is simply a rewriting of (2.6) and that

$$\hat{M} = \hat{M}_+ = \hat{M}_-, \tag{2.14}$$

which follows from the equality of $\hat{U}$ and both $\hat{U}_\pm$. In the ϵ basis, we have

$$\begin{aligned} \left[\hat{L}_\pm\right]_\epsilon &= \pm \hat{W}_\pm \hat{\omega}^{\pm 1} \hat{W}_\pm^{-1}, \\ \left[\hat{M}_\pm\right]_\epsilon &= \hat{W}_\pm \epsilon \hat{W}_\pm^{-1}, \end{aligned} \tag{2.15}$$

where $\hat{W}_\pm$ are the dressings

$$\hat{W}_\pm = e^{i\sum_{n>0} b_{\pm n}(\epsilon;\{t\})\hat{\omega}^{\mp n/R}} e^{\mp i\phi(\epsilon;\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\hat{\omega}^{\pm n/R}}. \tag{2.16}$$

We wish to emphasize that the $\hat{W}_\pm$, as functions of ϵ and $\hat{\omega}$, are basis specific expressions.

In order to solve the string equation (2.7), we first must have the expansion of the Orlov-Shulman operator in powers of the Lax operators. There are two equivalent expansions obtained by resumming the expansions of $\hat{M}_\pm$ in powers of $\hat{\omega}$ in the energy representation in terms of the Lax operators

$$\hat{M}_\pm = \hat{\epsilon} \mp \sum_{n>0} \frac{n t_{\pm n}}{R} \left(\pm \hat{L}_\pm\right)^{n/R} + \sum_{n>0} v_{\pm n}(\hat{\epsilon};\{t\}) \left(\pm \hat{L}_\pm\right)^{-n/R}. \tag{2.17}$$

The $v_{\pm n}(\hat{\epsilon};\{t\})$ are undetermined functions which are in principle calculable. By introducing the functions $v_{\pm k}$ we have not introduced any more unknown functions, but rather simply reorganized the unknown $b_{\pm k}$ into the $v_{\pm k}$.

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2.3 Dispersionless limit and solving the string equation

The above formalism is a compact way to record an infinite hierarchy of finite-difference equations. These are obtained by expanding both sides of the string equation (2.7) in powers of $\hat{\omega}$, with all factors of $\hat{\omega}$ moved to the right. Matching the coefficients to each term in the series provides an infinite set of finite difference equations for the various undetermined functions in the operators $\hat{U}$. The most interesting of these functions is the zero-mode $\phi(\epsilon;\{t\})$ which provides the density of states via $\rho(\epsilon) = \frac{\partial_\epsilon \phi(\epsilon)}{2\pi}$.

While the system of finite difference equations is in principle soluble, the study of these equations is not technically practical. Firstly, there are an infinite number of such equations and, secondly, even a finite set of finite difference equations is generally difficult to solve. Instead we will take advantage of the simplification occurring in the "dispersionless limit", when the lattice spacing goes to zero [14, 15, 12, 13]. In the fermion language, this is the classical limit, $\hbar \to 0$. Since we use $\hbar = 1$ units this is accomplished by considering the regime $|\epsilon| \gg 1$, which is the genus zero limit of the dual string theory. Since the Orlov-Shulman operator scales as ϵ and the Lax operators scale as $\sqrt{\epsilon}$, the string equation in this regime is

$$M = L_- L_+. \tag{2.18}$$

In this limit, the Lax operators are given by

$$L_\pm = \pm e^{-\partial_\epsilon \phi/2} \omega^{\pm 1} \left(1 + \sum_{k>0} a_{\pm k}\left(\epsilon;\{t\}\right) \omega^{\mp k/R} \right) + \cdots , \tag{2.19}$$

where the dots represent subleading terms in ϵ and the hat on ω as been dropped to indicate it is now a classical variable. The functions $a_{\pm k}(\epsilon;\{t\})$ are calculable from the functions in the transformation $\hat{U}$ but we will shortly solve for them using algebraic constraints, bypassing the need to ever know the precise form of the undetermined functions in $\hat{U}$.

We now take advantage of the expansion of M in terms of the Lax variables (2.17). To simplify matters, we consider the case where all the couplings vanish except $t_{\pm n}$ for some particular n

$$M_\pm = \epsilon \mp \frac{n t_{\pm n}}{R} \left(\pm L_\pm\right)^{n/R} + \sum_{k>0} v_{\pm k}\left(\epsilon\right) \left(\pm L_\pm\right)^{-k/R}. \tag{2.20}$$

These two expansions will effectively provide two string equations, $M_\pm = L_- L_+$, constraining the $a_{\pm k}$ coefficients strongly. Substituting (2.19) into the above, we find

$$\begin{aligned} M_\pm &= \epsilon \mp \frac{n t_{\pm n}}{R} e^{-n\partial_\epsilon \phi/2R} \omega^{\pm n/R} \left(1 + \sum_{k>0} a_{\pm k}\left(\epsilon\right) \omega^{\mp k/R} \right)^{n/R} \\ &\quad + \sum_{k>0} v_{\pm k}\left(\epsilon\right) e^{\frac{k}{R}\partial_\epsilon \phi} \omega^{\mp k/R} \left(1 + \sum_{\ell>0} a_{\pm \ell}\left(\epsilon\right) \omega^{\mp \ell/R} \right)^{-k/R}. \end{aligned} \tag{2.21}$$

Now compare (2.21) with

$$L_+ L_- = -e^{-\partial_\epsilon \phi} \left(1 + \sum_{k>0} a_{+k}\left(\epsilon\right) \omega^{-k/R} \right) \left(1 + \sum_{k>0} a_{-k}\left(\epsilon\right) \omega^{k/R} \right). \tag{2.22}$$

Since the highest (lowest) power of ω in $M_\pm$ is $\omega^{\pm n/R}$, power matching in the string equation implies $a_{\pm k} = 0$ if $k > n$. Furthermore, upon examining other coefficients in the string equation one can see all the constraints are satisfied for $a_{\pm k} = 0$ for $k \neq n$.

Bearing this in mind, we match the coefficients for the $\omega^{\pm n/R}$ terms in $M_\pm = L_+ L_-$ and obtain

$$a_{\mp n}(\epsilon) = \pm \frac{n t_{\pm n}}{R} e^{(1-n/2R)\partial_\epsilon \phi}, \tag{2.23}$$

which when substituted into the equation obtained from the ω^0 terms results in[3]

$$|\epsilon| - e^{-\partial_\epsilon \phi} = \frac{n^2}{R^2} t_{+n} t_{-n} e^{(1-n/R)\partial_\epsilon \phi} \left(\frac{n}{R} - 1 \right). \tag{2.24}$$

We wish to emphasize the importance of (2.24). This equation provides a transcendental equation for the density of states, and hence the free energy, in the presence of momentum mode perturbations. Our goal in the sections ahead is to obtain equations such as (2.24) for the density of states in 0A string theory perturbed by momentum modes.

As a quick consistency check of our method, we see that (2.24) produces the correct density of states for the $c = 1$ theory when there is no perturbation,

$$\rho(\mu) = \frac{1}{2\pi} \partial_\epsilon \phi \bigg|_{\epsilon = -\mu} = -\frac{1}{2\pi} \log \mu. \tag{2.25}$$

For more details see, for example, [2]. A more non-trivial check is that (2.24), with $n = 1$ and a suitable redefinition of couplings, is precisely T-dual to the genus zero susceptibility equation obtained in [11] for the $c = 1$ theory perturbed by winding modes.

The beauty of the procedure above is that there was never a need to know the undetermined functions $v_{\pm k}$. We have only assumed that they are such that it is possible to match the coefficients for the powers of ω which we do not examine. The fact that knowledge of the functions $v_{\pm k}$ is irrelevant to obtaining the equation for the density of states is not limited to the simple example above. Suppose that some finite number of couplings are turned on, but that $t_{\pm k} = 0$ for $k > N$. Once again, a general argument indicates $a_{\pm k}$ vanish if $k > N$. Matching the coefficients for $\omega^{k/R}$ with $0 \leq k \leq N$ in $M_+ = L_+ L_-$, we obtain $N + 1$ constraints, none of which contain $v_{\pm k}$. Similarly, examining the $\omega^{-k/R}$ coefficients with $0 < k \leq N$ in $M_- = L_+ L_-$ leads to another N equations which do not involve the $v_{\pm k}$. Together, these $2N + 1$ constraints are sufficient to solve for $\partial_\epsilon \phi$ and the non-vanishing $a_{\pm k \leq N}$. We will find in the next section that the most straightforward application of this method to 0A string theory will not be so tractable.

3. Integrable deformations of the 0A matrix model

We would like to perform deformations of the 0A matrix model analogous to those of the previous section. This has been previously attempted [18, 20, 21, 23], but these authors were only able to obtain solutions by setting μ or q to zero. We will first demonstrate the intractability of the most straightforward organization of the integrable structure of the

[3]Although this result has been derived for $\epsilon < 0$, it can be derived formally for positive energy as well.

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0A matrix model. Then we consider an integrable structure based on the $2+1$ dimensional perspective for the 0A matrix model eigenvalues. Through this viewpoint we are able to find additional string equations which can be solved in the dispersionless limit with both μ and q non-zero.

3.1 The 0A matrix model

It is well-known [4] that type 0A string theory in $d = 2$ can be described by a gauged matrix model similar to that of the $d = 2$ bosonic string. As is common with matrix models, the singlet sector can be reduced to the dynamics of a many-body problem of non-interacting fermions in $1+1$ dimensions. The single-particle Hamiltonian is

$$\hat{\epsilon}_{0A} = \frac{1}{2}\left(\hat{p}^2 - \hat{x}^2 + \frac{q^2 - 1/4}{\hat{x}^2}\right), \tag{3.1}$$

where we consider q to simply be some parameter. Proceeding as in the $c = 1$ model, we put the operator algebra into the light-cone form [26]

$$\begin{aligned} \left[\hat{B}_+, \hat{B}_-\right] &= 4i\hat{\epsilon}_{0A}, \\ \left[\hat{B}_\pm, \hat{\epsilon}_{0A}\right] &= \pm 2i\hat{B}_\pm, \\ \left\{\hat{B}_+, \hat{B}_-\right\} &= 2\left(\hat{\epsilon}_{0A}^2 + q^2 - 1\right), \end{aligned} \tag{3.2}$$

where

$$\hat{B}_\pm = \hat{x}_\pm^2 + \left(q^2 - 1/4\right)\hat{x}^{-2}. \tag{3.3}$$

We can easily obtain from (3.2) the following identity

$$\hat{B}_\pm \hat{B}_\mp = (\hat{\epsilon}_{0A} \pm i)^2 + q^2. \tag{3.4}$$

We will hereafter drop the subscript "0A" on the energy.

From the algebra (3.2) we can find the energy basis representations

$$\left[\hat{B}_\pm\right]_\epsilon = S^{\mp 1/2}(\epsilon, q)\hat{\omega}^{\pm 2} S^{\pm 1/2}(\epsilon, q), \tag{3.5}$$

where $\hat{\omega}$ is defined in (2.3) and $S(\epsilon, q)$ is the scattering phase[4]

$$S(\epsilon, q) = 2^{-i\epsilon} i^{|q|} \frac{\Gamma\left(1/2\left(1 + |q| - i\epsilon\right)\right)}{\Gamma\left(1/2\left(1 + |q| + i\epsilon\right)\right)} \equiv e^{i\phi_0(\epsilon, q)}. \tag{3.6}$$

3.2 Intractability of the 0A lax system

Besides simplifying the 0A operator algebra the $\hat{B}_\pm$ play the role of momentum modes in the free fermion representation of 0A string theory [26]. Thus, following the analysis for the $c = 1$ model, we wish to consider unitary transformations generated by powers of $\hat{B}_\pm$ to implement momentum mode perturbations. However, we will see that a straightforward application of this method will not lead to a solvable string equation in this instance.

[4]The attentive reader may notice that this differs slightly from the result in [6]. This is due to our somewhat different normalization of the fermion states. Note that the exclusion of the $i^{|q|}$ factor in [6] does not affect the calculation of the density of states, $\rho(\epsilon) \sim \phi'(\epsilon)$. See appendix B for more details.

For the 0A matrix model an integrable set of momentum mode deformations is generated by the unitary transformations

$$\hat{U}_{\pm} = e^{i\sum_{n>0}\tilde{b}_{\pm n}(\hat{\epsilon};q,\{t\})\hat{B}_{\pm}^{-n/R}} e^{\mp i\tilde{\phi}(\hat{\epsilon};q,\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\hat{B}_{\pm}^{n/R}}. \tag{3.7}$$

Once again, $\tilde{\phi}$ and the $\tilde{b}$ are some undetermined functions which vanish when all of the couplings $\{t\}$, are turned off. As in the treatment of the $c = 1$ theory, the operators (3.7) are constrained by $\hat{U}_+ = \hat{U}_-$. In the energy basis these operators are given by

$$\left[\hat{U}_{\pm}\right]_{\epsilon} = e^{i\sum_{n>0} b_{\pm n}(\epsilon;q,\{t\})\hat{\omega}^{\mp 2n/R}} e^{\mp i\phi(\epsilon;q,\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\hat{\omega}^{\pm 2n/R}} S^{\pm 1/2}(\epsilon, q), \tag{3.8}$$

where we have introduced

$$\begin{aligned}\phi(\epsilon;q,\{t\}) &\equiv \tilde{\phi}(\epsilon;q,\{t\}) + \phi_0(\epsilon,q), \\ b_{\pm n}(\epsilon;q,\{t\}) &\equiv \tilde{b}_{\pm n}(\epsilon;q,\{t\})\, S^{\mp 1/2}(\epsilon,q)\, S^{\pm 1/2}(\epsilon \mp 2in/R, q).\end{aligned} \tag{3.9}$$

We now define Lax and Orlov-Shulman operators as in the $c = 1$ theory through unitary transformations on the 0A operators

$$\begin{aligned}\hat{L}_{\pm} &= \hat{U}_{\pm}\hat{B}_{\pm}\hat{U}_{\pm}^{-1}, \\ \hat{M}_{\pm} &= \hat{U}_{\pm}\hat{\epsilon}\hat{U}_{\pm}^{-1}.\end{aligned} \tag{3.10}$$

where once again we have $\hat{M}_+ = \hat{M}_-$ due to the equivalence of the unitary transformations $\hat{U}_{\pm}$. In the energy basis these operators are represented as

$$\begin{aligned}\left[\hat{L}_{\pm}\right]_{\epsilon} &= \hat{W}_{\pm}\hat{\omega}^{\pm 2}\hat{W}_{\pm}^{-1}, \\ \left[M_{\pm}\right]_{\epsilon} &= \hat{W}_{\pm}\epsilon\hat{W}_{\pm}^{-1},\end{aligned} \tag{3.11}$$

with the dressings given by

$$\hat{W}_{\pm} = e^{i\sum_{n>0} b_{\pm n}(\epsilon;q,\{t\})\hat{\omega}^{\mp 2n/R}} e^{\mp i\phi(\epsilon;q,\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\hat{\omega}^{\pm 2n/R}}. \tag{3.12}$$

Since this is simply a unitary transformation, the operator algebra remains intact so we immediately get the string equation

$$\hat{L}_{\pm}\hat{L}_{\mp} = \left(\hat{M} \pm i\right)^2 + q^2, \tag{3.13}$$

which was also obtained in [18, 20]. This equation is perfectly correct but its use in reaching the density of states is limited. The problem lies in the appearance of a quadratic power of the Orlov-Shulman operator as we will see below.

In the dispersionless limit we take $\epsilon, q \gg 1$. The above string equation becomes

$$L_+ L_- = M^2 + q^2, \tag{3.14}$$

and the Lax operators can be expanded as

$$L_{\pm} = e^{-\partial_\epsilon \phi}\omega^{\pm 2}\left(1 + \sum_{k>0} a_{\pm k}(\epsilon;\{t\})\,\omega^{\mp 2k/R}\right). \tag{3.15}$$

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Once again, we examine a simple case where the only non-zero couplings are $t_{\pm n}$ for some n. The expansion of M in terms of the Lax operators is similar to (2.20)

$$M_{\pm} = \epsilon \mp \frac{2nt_{\pm n}}{R} L_{\pm}^{n/R} + \sum_{k>0} v_{\pm k}(\epsilon)\, L_{\pm}^{-k/R}. \tag{3.16}$$

We examine the non-negative powers of $L_{\pm}$ that appear in $M_{\pm}^2$ (denoted $M_{\pm,\geq}^2$), since only these will contribute to the coefficients in which we are interested

$$M_{\pm,\geq}^2 = \epsilon^2 + \left(\frac{4nt_{\pm n}}{R}\right)^2 L_{\pm}^{2n/R} \mp \frac{2nt_{\pm n}}{R}\left(\epsilon L_{\pm}^{n/R} + \sum_{k=1}^{n} L_{\pm}^{(n-k)/R} v_{\pm k}(\epsilon)\right). \tag{3.17}$$

Compare this with

$$L_+L_- = e^{-\partial_\epsilon \phi}\left(1 + \sum_{k>0} a_{+k}(\epsilon)\,\omega^{-2k/R}\right)\left(1 + \sum_{k>0} a_{-k}(\epsilon)\,\omega^{2k/R}\right). \tag{3.18}$$

Once again, matching of coefficients implies the $a_{\pm k}$ truncate. In this case we have $4n+1$ non-vanishing functions; $\partial_\epsilon \phi$ and the $a_{\pm k}$ for $k \leq 2n$. However, one cannot obtain a closed set of equations for these variables. The $v_{\pm k}$ appear in the $2n+1$ equations one obtains from examining the coefficients to $\omega^{2k/R}$ with $0 \leq k \leq 2n$ in $L_+L_- = M_+^2 + q^2$. This is also true of the $2n$ coefficients of $\omega^{-2k/R}$ with $0 < k \leq 2n$ obtained from the M_- string equation. Examining more coefficients will simply increase the number of $v_{\pm k}$ involved. One must then solve the whole infinite set of equations.

One can see that the origin of the above intractability is the appearance of the unknown functions $v_{\pm k}$ multiplying non-negative powers of $L_{\pm}$ in the string equation once (3.17) is substituted into (3.14). This would be alleviated were the string equation linear in the Orlov-Shulman operator. We now turn to such a way of organizing the 0A system.

3.3 Flux and the complex basis

We have found that the Toda Lattice based on shifts of the single-particle energy do not lead to solvable string equations. In fact, it appears as if we are missing information. To remedy this, we recall that the one-dimensional Hamiltonian (3.1) can be understood as the effective radial Hamiltonian for fermions moving in a two-dimensional harmonic oscillator [4]. In this description, the RR-flux is the angular momentum operator, $\hat{q}$. In terms of the Cartesian coordinates and momenta in the plane, the energy and angular momentum are given by

$$\hat{\epsilon} = \frac{1}{2}\left(\hat{p}_1^2 + \hat{p}_2^2 - \hat{x}_1^2 - \hat{x}_2^2\right), \tag{3.19}$$
$$\hat{q} = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1\,. \tag{3.20}$$

It is convenient to complexify these operators

$$\hat{z}_{\pm} \equiv \left(\frac{\hat{p}_1 \pm \hat{x}_1}{\sqrt{2}}\right) + i\left(\frac{\hat{p}_2 \pm \hat{x}_2}{\sqrt{2}}\right), \tag{3.21}$$
$$\hat{y} \equiv \hat{\epsilon} + i\hat{q}\,, \tag{3.22}$$

which obey the algebra

$$\begin{aligned}
\left[\hat{\bar{z}}_+, \hat{z}_-\right] &= 2i\,, \\
\left[\hat{z}_-, \hat{y}\right] &= -2i\hat{z}_-\,, \\
\left[\hat{\bar{z}}_+, \hat{y}\right] &= 2i\hat{\bar{z}}_+\,,
\end{aligned} \tag{3.23}$$

where the bar above an operator denotes its Hermitian conjugate. Other commutators are trivial or obtained by Hermitian conjugation of (3.23). For convenience we also record the relation

$$\hat{z}_-\hat{\bar{z}}_+ = \hat{y} - i. \tag{3.24}$$

We can obtain the previous algebra (3.2) by noting that

$$\hat{B}_\pm = \hat{z}_\pm\hat{\bar{z}}_\pm\,. \tag{3.25}$$

Recalling that $\hat{B}_\pm$ are the vertex operators for momentum modes, it is evident that in the complex coordinates we should consider unitary transformations generated by powers of $\hat{z}_\pm\hat{\bar{z}}_\pm$ to implement momentum mode deformations. This is done in the following section. But first let us obtain the (ϵ, q)-basis for the operators above.

Define the shift derivative $\hat{\eta} = e^{2i\partial}$ with Hermitian conjugate $\hat{\bar{\eta}} = e^{2i\bar{\partial}}$ where

$$\begin{aligned}
\partial \equiv \partial_y &= \frac{1}{2}\left(\partial_\epsilon - i\partial_q\right), \\
\bar{\partial} \equiv \partial_{\bar{y}} &= \frac{1}{2}\left(\partial_\epsilon + i\partial_q\right).
\end{aligned} \tag{3.26}$$

Examining the algebra (3.23), we see that $\hat{\bar{z}}_+ \sim \hat{\eta}$ and $\hat{z}_- \sim \hat{\eta}^{-1}$ will reproduce the commutators with $\hat{y}$. To reproduce the final commutator and (3.24), we dress the appropriate power of $\hat{\eta}$ with the scattering phase[5]

$$\begin{aligned}
\left[\hat{z}_-\right]_{y,\bar{y}} &= S^{1/2}\left(y,\bar{y}\right)\hat{\eta}^{-1}S^{-1/2}\left(y,\bar{y}\right), \\
\left[\hat{\bar{z}}_+\right]_{y,\bar{y}} &= S^{-1/2}\left(y,\bar{y}\right)\hat{\eta}\,S^{1/2}\left(y,\bar{y}\right),
\end{aligned} \tag{3.27}$$

with other operators defined by Hermitian conjugation and $S\left(y,\bar{y}\right)$ given by

$$S\left(y,\bar{y}\right) = 2^{-i\,Re(y)}i^{Im(y)}\frac{\Gamma\left(\frac{1}{2}\left(1-iy\right)\right)}{\Gamma\left(\frac{1}{2}\left(1+i\bar{y}\right)\right)}. \tag{3.28}$$

3.4 Complexified lax formalism

Now we act with some unitary operator, $\hat{U}$, obtaining the complex Lax and Orlov-Shulman operators[6]

$$\hat{Z}_\pm \equiv \hat{U}\hat{z}_\pm\hat{U}^{-1}, \tag{3.29}$$

$$\hat{Y} \equiv \hat{U}\hat{y}\hat{U}^{-1}, \tag{3.30}$$

[5]Without loss of generality we have assumed that $q > 0$. For $q < 0$, it is important that $|q|$ appears in the scattering amplitude dressing $\hat{\eta}$. However, this eventually leads to the same susceptibility equation (3.44).

[6]Note that the algebra of all operators quadratic in the z's is $\mathfrak{sp}_4$. It may be possible to construct generalized Toda lattices from other Lie algebras, building in and out Lax operators from the positive and negative roots, and Orlov-Shulman operators from the elements of the Cartan subalgebra. Such a construction is a subject for future research.

and similarly for their Hermitian conjugates. The algebra (3.23) is preserved by the unitary transformation as is, most importantly, the string equation

$$\hat{Z}_{-}\hat{\bar{Z}}_{+} = \hat{Y} - i. \tag{3.31}$$

This is essentially a complexified version of the string equation for the $c = 1$ theory. To implement momentum mode perturbations we use the unitary transformations (3.7) generated by $B_{\pm}$. The y basis representations are easily obtained from (3.8) by noting that $\hat{\omega}^2 = \hat{\eta}\hat{\bar{\eta}}$

$$\left[\hat{U}_{\pm}\right]_{y,\bar{y}} = e^{i\sum_{n>0} b_{\pm n}(y,\bar{y};\{t\})\left(\hat{\eta}\hat{\bar{\eta}}\right)^{\mp n/R}} e^{\mp i\phi(y,\bar{y};\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\left(\hat{\eta}\hat{\bar{\eta}}\right)^{\pm n/R}} S^{\pm 1/2}\left(y,\bar{y}\right). \tag{3.32}$$

As with the previous examples we impose $\hat{U}_{+} = \hat{U}_{-}$ and define the transformed operators

$$\hat{Z}_{\pm} \equiv \hat{U}_{\pm}\hat{z}_{\pm}\hat{U}_{\pm}^{-1}, \tag{3.33}$$

$$\hat{Y}_{\pm} \equiv \hat{U}_{\pm}\hat{y}\,\hat{U}_{\pm}^{-1}, \tag{3.34}$$

where $\hat{Y}_{+} = \hat{Y}_{-} = \hat{Y}$. The y basis expressions are given by

$$\left[\hat{Z}_{-}\right]_{y,\bar{y}} = \hat{W}_{-}\hat{\eta}^{-1}\,\hat{W}_{-}^{-1}, \tag{3.35}$$

$$\left[\hat{\bar{Z}}_{+}\right]_{y,\bar{y}} = \hat{W}_{+}\hat{\eta}\,\hat{W}_{+}^{-1}, \tag{3.36}$$

$$\left[\hat{Y}_{\pm}\right]_{y,\bar{y}} = \hat{W}_{\pm}\hat{y}\,\hat{W}_{\pm}^{-1}, \tag{3.37}$$

where the dressings are

$$\hat{W}_{\pm} = e^{i\sum_{n>0} b_{\pm n}(y,\bar{y};\{t\})\left(\hat{\eta}\hat{\bar{\eta}}\right)^{\mp n/R}} e^{\mp i\phi(y,\bar{y};\{t\})/2} e^{i\sum_{n>0} t_{\pm n}\left(\hat{\eta}\hat{\bar{\eta}}\right)^{\pm n/R}}. \tag{3.38}$$

By reorganizing the $\hat{\eta}$ expansions of the two equivalent forms of $\hat{Y}$ we find the expansions of the Orlov-Shulman operator in terms of the Lax operators

$$\hat{Y}_{\pm} = \hat{y} \mp \sum_{k>0} \frac{2k}{R} t_{\pm k}\hat{Z}_{\pm}^{k/R}\hat{\bar{Z}}_{\pm}^{k/R} + \sum_{k>0} v_{\pm k}\left(\hat{y},\hat{\bar{y}}\right)\hat{Z}_{\pm}^{-k/R}\hat{\bar{Z}}_{\pm}^{-k/R}. \tag{3.39}$$

In the dispersionless limit, we simply drop the hats in the above. The string equation becomes

$$Y_{\pm} = Z_{-}\bar{Z}_{+}, \tag{3.40}$$

and the Lax variables have expansions

$$Z_{-} = e^{-\partial\phi}\eta^{-1}\left(1 + \sum_{k>0} a_{-k}(y,\bar{y})\left(\eta\bar{\eta}\right)^{k/R}\right), \tag{3.41}$$

$$\bar{Z}_{+} = e^{-\partial\phi}\eta\left(1 + \sum_{k>0} a_{+k}(y,\bar{y})\left(\eta\bar{\eta}\right)^{-k/R}\right), \tag{3.42}$$

and similarly for their conjugates. Conveniently and unsurprisingly, we obtain only powers of $\eta\bar{\eta} = \omega^2$ in the expansion of (3.40).

To be explicit, assume all $t_{\pm k} = 0$ except $t_{\pm n}$. The matching of coefficients once again implies only $a_{\pm n}$ are non-vanishing. Since the $v_{\pm k}$ do not multiply positive powers of the Lax variables in the string equation, we have few enough variables to solve the equations imposed by matching of coefficients. We obtain

$$a_{\pm n} = \pm \frac{2n}{R} t_{\mp n} e^{2\partial\phi} e^{-\frac{n}{R}(\partial\phi + \bar{\partial}\phi)}, \tag{3.43}$$

by matching the $(\eta\bar{\eta})^{\pm n/R}$ coefficients. The order $(\eta\bar{\eta})^0$ coefficient matching then gives the complex susceptibility equation

$$e^{-2\partial\phi} - y = \frac{4n^2}{R^2} t_{+n} t_{-n} e^{-\frac{2n}{R}(\partial\phi + \bar{\partial}\phi)} \left(\left(1 - \frac{n}{R}\right) e^{2\partial\phi} - \frac{n}{R} e^{2\bar{\partial}\phi} \right). \tag{3.44}$$

The above equation is perhaps more transparent when split into real and imaginary parts

$$\begin{aligned} \epsilon - e^{-\partial_\epsilon \phi} \cos(\partial_q \phi) &= \frac{4n^2 t_{-n} t_{+n}}{R^2} e^{(1-2n/R)\partial_\epsilon \phi} \left(\frac{2n}{R} - 1 \right) \cos(\partial_q \phi), \\ q - e^{-\partial_\epsilon \phi} \sin(\partial_q \phi) &= \frac{4n^2 t_{-n} t_{+n}}{R^2} e^{(1-2n/R)\partial_\epsilon \phi} \sin(\partial_q \phi). \end{aligned} \tag{3.45}$$

The equation (3.44) (or alternatively the real and imaginary parts (3.45)) constitutes the main result of our paper, as it contains the information necessary to extract the genus zero susceptibility for 0A string theory in the presence of momentum mode perturbations. A few comments are in order. First, we would like to emphasize that (3.44) has been derived in full generality, with both non-vanishing cosmological constant μ and RR flux q. To the best of our knowledge, this is the first instance in which a set of string equations have been solved in a flux background. Second, we have arrived at these results by exploiting the integrability of the 0A matrix model. We have found that perturbations to the matrix model can be cast as a complexified Toda system, and have derived the string equations constraining it. In the dispersionless limit, these string equations yield (3.44).

4. Analysis of the 0A susceptibility equations

We wish to analyze the complex susceptibility equation (3.44), with the goal of obtaining the phase diagram of the system. Given the transcendental nature of the equation, this is a difficult task. We begin by specializing to the case of no Ramond-Ramond flux and perform an analysis T-dual to that of [11]. We find critical behavior which obstructs smoothly moving through parameter space to the momentum mode condensate phase for $R > 2n$. We then proceed to the case of non-trivial flux and explore the critical behavior present there.

Before diving into the intricacies of these transcendental equations, let us digress briefly to make some general comments. Firstly, the perturbation couplings $t_{\pm n}$ must be such that the worldsheet Lagrangian is deformed by a real operator. For the 0A theory, the unitary transformations generating the momentum perturbations have the asymptotic forms $\hat{U}_\pm \sim e^{it_{\pm n}\hat{B}_\pm^{n/R}}$, which act on the single-fermion Hamiltonian as $\hat{H} \to \hat{H} \mp \frac{2nt_{\pm n}}{R}\hat{B}_\pm^{n/R}$. This translates, in the absence of flux, to deforming the worldsheet Lagrangian as

$$\mathcal{L} \to \mathcal{L} - t_n \mathcal{V}_{2n/R} + t_{-n} \mathcal{V}_{-2n/R}. \tag{4.1}$$

Given the form of the vertex operators $\mathcal{V}_p \sim e^{ipX}$, we see that a real deformation requires that $t_n = -t_{-n} \equiv t$ for some real t. This is not expected to change in the presence of RR flux. We will thus make the substitution $t^2 = -t_n t_{-n}$ hereafter.

Secondly, we will pursue a thermodynamic analysis of 0A string theory in what follows. This can be accomplished with our susceptibility equations since $\partial_\epsilon \phi(\epsilon) = 2\pi\rho(\epsilon)$, where $\rho(\epsilon)$ is the density of states. The grand canonical partition function, with fixed q, is then

$$\ln \mathcal{Z} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\epsilon \, \partial_\epsilon \phi(\epsilon) \ln(1 + e^{2\pi R(\mu - \epsilon)}). \tag{4.2}$$

It follows that

$$\begin{aligned} \partial_\mu \ln \mathcal{Z} &= R \int_{-\infty}^{\infty} d\epsilon \, \frac{\partial_\epsilon \phi(\epsilon)}{1 + e^{2\pi R(\epsilon - \mu)}} \\ &\approx R \int_{-\infty}^{\mu} d\epsilon \, \partial_\epsilon \phi(\epsilon) \, , \end{aligned} \tag{4.3}$$

where in the second line, we have kept only the leading term as $|\mu| \to \infty$. We thus arrive at the genus zero expression for the susceptibility

$$\chi(\mu) \equiv \frac{1}{R} \partial_\mu^2 \ln \mathcal{Z} = \partial_\mu \phi(\mu). \tag{4.4}$$

In keeping with this notation we will make the replacement $\epsilon = \mu$ henceforth.

4.1 Relevant perturbations at $q = 0$

We are studying in this paper deformations of the string worldsheet theory by tachyon vertex operators of definite Euclidean momentum. In order to have an effect on the long wavelength dynamics of the theory, and hence the semi-classical spacetime geometry, this must be a relevant deformation in the sense of worldsheet renormalization group flow. This consideration puts a bound on the radius of compactification that we consider.

Let us consider the bosonic string for illustrative purposes. Before gauge fixing the worldsheet metric, the vertex operators we are considering are simply

$$\mathcal{V}_p = e^{ipX}, \tag{4.5}$$

and have conformal dimension $\Delta = \frac{p^2}{4}$ in $\alpha' = 1$ units. If $\mathcal{V}_p = e^{ipX}$ is to be relevant then $|p| < 2$. This same bound applies to the canonical ghost picture[7] Type 0 vertex operators at zero Ramond-Ramond flux in $\alpha' = \frac{1}{2}$ units. To obtain the susceptibility equation (3.44) for the 0A matrix model, we have considered perturbations with momenta $p = \pm\frac{2n}{R}$. Thus, for a given n, we have a bound on the Euclidean time radius, $R > n$. Although this has been argued for the case of vanishing flux we will shortly see indications that this bound should be extended to $q \neq 0$.

[7] That is, picture $(-1, -1)$ for NS-NS operators and picture $(-\frac{1}{2}, -\frac{1}{2})$ for R-R operators.

4.2 Scaling behavior of $c = 1$

We begin by considering the scaling behavior in μ of the perturbed 0A theory with $q = 0$. One might expect perturbed 0A with no flux to be identical to the perturbed $c = 1$ matrix model, and in this section we will highlight the similarities of these two theories. In section 4.5 we will see that 0A contains novel behavior resulting from divergences in the q-derivatives of the free energy, even at strictly vanishing flux.

To explore the similarities between $c = 1$ and 0A with $q = 0$, it is most useful to use the component form of the susceptibility equation. For vanishing flux, the second equation in (3.45) is satisfied with $\sin(\partial_q\phi) = 0$. When all of the $t_{\pm n}$ vanish, $\cos(\partial_q\phi) = \mathrm{sign}(\mu)$ at large $|\mu|$ and $q = 0$; we will assume this is so when the perturbations are non-trivial. Thus the first equation of (3.45) reduces to

$$|\mu| = e^{-\partial_\mu\phi} + \frac{4n^2t^2}{R^2}\left(1 - \frac{2n}{R}\right)e^{(1-2n/R)\partial_\mu\phi}. \tag{4.6}$$

This is identical to the result (2.24) for the $c = 1$ theory perturbed by modes with momentum $p = \pm\frac{2n}{R}$. This factor of 2 is accounted for by the fact that only perturbations by $\hat{B}_\pm \sim (\hat{x}_\pm)^2$ have been used in our treatment of the 0A theory. Our result (4.6) is thus T-dual to that obtained through studies of the $c = 1$ theory perturbed by even winding modes [11, 12].

We find that (4.6) can be expressed more simply by using the positive-definite dimensionless parameter[8]

$$\Lambda = \frac{4n^2}{R^2}t^2|\mu|^{\frac{2n}{R}-2}. \tag{4.7}$$

This variable replaces the perturbation coupling t^2 with a coupling that runs with the worldsheet cosmological constant. Recalling that $R > n$ for any relevant perturbation, we see that the unperturbed theory is located at $\Lambda = 0$ by sending $t^2 \to 0$ and $|\mu| \to \infty$. Similarly, condensation of the momentum mode perturbation occurs as $\Lambda \to \infty$ via the opposite limits, $t^2 \to \infty$ and $|\mu| \to 0$. For $R < n$ these limits break down, and the variable Λ no longer has a clear limit in which the theory is unperturbed. Although there will not be a bound from worldsheet relevance when we study $q \neq 0$, we will see a similar behavior in the corresponding dimensionless variable.

To take advantage of the dimensionless variable we introduce the reduced susceptibility $\tilde{\chi}$ through

$$\chi \equiv \partial_\mu\phi = \tilde{\chi} - \frac{1}{2(1-\frac{n}{R})}\log\left(\frac{4n^2}{R^2}t^2\right). \tag{4.8}$$

Then (4.6) reduces to

$$\Lambda^{\frac{1}{2(n/R-1)}} = e^{-\tilde{\chi}} + \left(1 - \frac{2n}{R}\right)e^{(1-2n/R)\tilde{\chi}}, \tag{4.9}$$

indicating that $\tilde{\chi}$ is function only of Λ and R.

[8] The parameter Λ is a natural variable since, being linear in the coupling t^2, it organizes the perturbative expansion around the 0A string without flux. Another natural variable, $\tilde{\Lambda} = \Lambda^{-\frac{1}{2(1-n/R)}}$, is linear in $|\mu|$ and associated with the expansion near the theory with momentum mode condensate.

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4.3 Critical behavior of $c = 1$

The critical behavior of $c = 1$ string theory can be inferred from (4.9). We differentiate this relation with respect to Λ and obtain

$$\partial_\Lambda \tilde{\chi} = \frac{\Lambda^{\frac{1}{2(n/R-1)}-1} e^{\tilde{\chi}}}{2(n/R-1)} \left[\left(1-\frac{2n}{R}\right)^2 e^{2(1-n/R)\tilde{\chi}} - 1\right]^{-1} \tag{4.10}$$

$$= \frac{\Lambda^{\frac{1}{2(1-n/R)}-1} e^{\tilde{\chi}}}{2(n/R-1)} \left[\left(1-\frac{2n}{R}\right)\left(2-\frac{2n}{R}\right) e^{(1-2n/R)\tilde{\chi}} - \Lambda^{\frac{1}{2(n/R-1)}}\right]^{-1}. \tag{4.11}$$

In the second line we have substituted the susceptibility equation into (4.10) to obtain a relation for later comparison with the case $q \neq 0$. As is clear from (4.10), $\partial_\Lambda \tilde{\chi}$ has a pole at a finite value of $\tilde{\chi}$

$$\tilde{\chi}_c = \frac{1}{(n/R-1)} \log\left|1-\frac{2n}{R}\right|. \tag{4.12}$$

To confirm that this is critical behavior, we must check that the equation of state is satisfied. Substituting (4.12) into (4.9), we obtain the critical surface in the $\Lambda - R$ plane

$$\Lambda_c^{\frac{1}{2(n/R-1)}} = \operatorname{sign}\left(1-\frac{2n}{R}\right)\left(2-\frac{2n}{R}\right)\left|1-\frac{2n}{R}\right|^{\frac{n}{R}\left(1-\frac{n}{R}\right)^{-1}}. \tag{4.13}$$

Since Λ is positive definite, we see that there is no critical behavior for $n < R < 2n$; this can also be discerned from (4.11). We discard the solution for $R < n$ as the perturbation is irrelevant in that region and solve (4.13) for Λ_c

$$\Lambda_c(R) = \left(1-\frac{2n}{R}\right)^{-2n/R}\left[2\left(1-\frac{n}{R}\right)\right]^{-2(1-n/R)}, \qquad R > 2n\,. \tag{4.14}$$

This cleanly divides the parameter space of the theory as can be seen in figure (1). Thus, this critical behavior provides an obstruction to smoothly perturbing from the unperturbed 0A phase to the momentum mode condensate phase with $R > 2n$. Beyond this, for $\Lambda > \Lambda_c(R)$, there do not exist real solutions to the equation of state (4.9).

Alternatively, let us plot the right-hand-side of (4.9) (call it $f(\tilde{\chi})$) in figure (2). The behavior described above is associated with a local minimum at $\tilde{\chi} = \tilde{\chi}_c$, where $f(\tilde{\chi}_c) = \Lambda_c^{\frac{1}{2(n/R-1)}}$. Again, we find that there exist no solutions for $\Lambda > \Lambda_c(R)$. More generally, all local extrema of $f(\tilde{\chi})$ will be associated with critical points since $\partial_\Lambda \tilde{\chi} \sim 1/(\partial_{\tilde{\chi}} f)$.

The above critical behavior can be understood thermodynamically by recalling from (4.4) that $\chi = \frac{1}{R}\partial_\mu^2 \ln \mathcal{Z}$. Then, using (4.8), we see that $\Lambda_c(R)$ is the surface where there is a divergence in

$$\partial_\mu^3 \ln \mathcal{Z}\,\Big|_{t,R} \sim \partial_\Lambda \tilde{\chi}\,|_R\,. \tag{4.15}$$

In the vicinity of this critical surface one has the behavior

$$\partial_\mu^3 \ln Z \sim (\Lambda - \Lambda_c)^{-1/2}, \tag{4.16}$$

typical of pure $d = 2$ gravity, known sometimes as a $c = 0$ model [27]. The physical picture of this phase transition is that the worldsheet field, X, associated with Euclidean time freezes out by settling into one of the minima of the cosine potential in (4.1).

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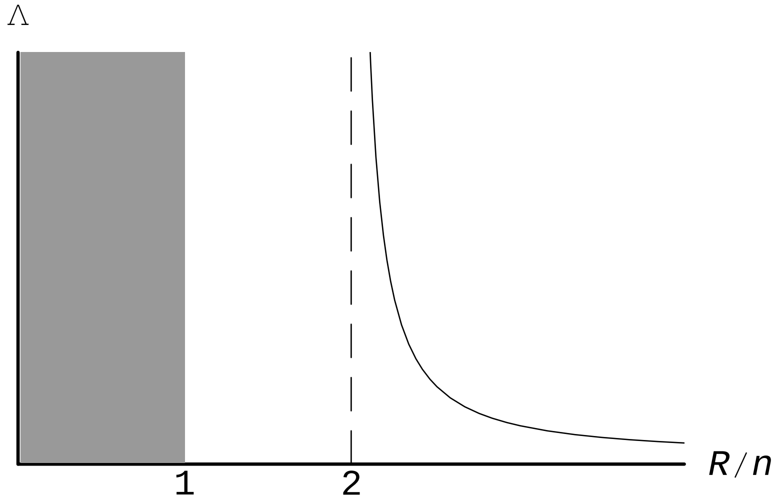

Figure 1: The phase diagram for $c = 1$ theory perturbed by momentum modes. For $\Lambda = 0$, the system is in the unperturbed 0A phase, and for $\Lambda \to \infty$, in the momentum mode condensate phase. For $n < R < 2n$ one can transition freely between these two phases, whereas for $R > 2n$, there will be an obstruction given by the curve $\Lambda_c(R)$, associated with the phase transition to $c = 0$. For $\Lambda > \Lambda_c(R)$, there are no solutions to the reduced susceptibility equation (4.9). The shaded region for $R < n$ is forbidden due to the irrelevance of the operators $\mathcal{V}_{\pm n/R}$.

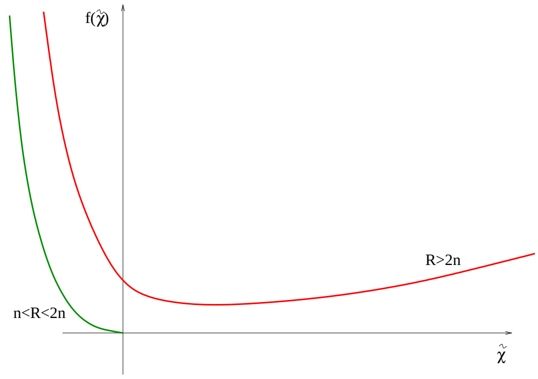

Figure 2: The right-hand-side of (4.9) is plotted as a function of $\tilde{\chi}$ for two values of R, as indicated.

4.4 Perturbative analysis with non-trivial flux

Having analyzed the susceptibility equations at $q = 0$, we would now like to study them in the presence of RR flux. To gain some confidence in the veracity of (3.44), we perform two consistency checks.

We first examine the limit of vanishing momentum mode perturbations; this is the case studied in [6]. Solving (3.44) with $t^2 = 0$ we find

$$\partial_\mu \phi(\mu) = (\partial + \bar{\partial})\phi = -\frac{1}{2}\ln(y\bar{y}) = -\frac{1}{2}\ln\left(\mu^2 + q^2\right), \tag{4.17}$$

which does indeed agree with the known result to leading order in μ^2+q^2. We shall use this expression for the unperturbed partition function as a boundary condition for the non-linear PDE system (3.45).

We next utilize perturbation theory to expand about the solution (4.17). Through the 0A string/matrix model duality we expect that the worldsheet theory perturbed as in (4.1) should be related to the perturbed matrix model free energy through

$$\ln \mathcal{Z} = \left\langle e^{t_{+n}\mathcal{T}_{2n/R} - t_{-n}\mathcal{T}_{-2n/R}} \right\rangle . \tag{4.18}$$

In perturbation theory, the right-hand side is evaluated as a sum of tachyon correlators in the unperturbed worldsheet theory. Through (4.4), we can then interpret $\partial_\mu \phi$ as a generating functional for the tachyon correlators on the sphere

$$\partial_\mu \phi = R^{-1}\partial_\mu^2 \ln \mathcal{Z} = -\frac{1}{2}\ln\left(\mu^2+q^2\right) + R^{-1}\sum_{m>0}\frac{t^{2m}}{(m!)^2}\partial_\mu^2 \left\langle \mathcal{T}^m_{2n/R}\mathcal{T}^m_{-2n/R}\right\rangle . \tag{4.19}$$

In order to compute these correlators, we first eliminate $\partial_q\phi$ from (3.45),

$$e^{-2\partial_\mu\phi} = \frac{\mu^2}{\left[1-(2n/R-1)\left(4n^2t^2/R^2\right)e^{-2(n/R-1)\partial_\mu\phi}\right]^2} + \frac{q^2}{\left[1-\left(4n^2t^2/R^2\right)e^{-2(n/R-1)\partial_\mu\phi}\right]^2}, \tag{4.20}$$

then expand $\partial_\mu\phi$ in powers of t^2 around (4.17). In appendix D we compute these tachyon correlators to $\mathcal{O}\left(t^6\right)$ and find that they agree with those calculated in [23]. This gives us great confidence in our result for general flux and momentum mode perturbation.

4.5 Critical behavior in flux vacua

We are now in position to construct the $d=2$ 0A phase diagram. We are going to follow the same strategy as outlined in the previous section, but in a flux background. Turning back to the complex equation (3.44), we begin by rewriting it in terms of a *complex* dimensionless variable

$$\Lambda = \frac{4n^2t^2}{R^2}y^{-2(1-n/R)} = \frac{4n^2t^2}{R^2}y^{-2(1-n/R)} . \tag{4.21}$$

This gives

$$\Lambda^{-\frac{1}{2(1-n/R)}} = e^{-2\tilde{\chi}} + e^{-\frac{2n}{R}(\tilde{\chi}+\bar{\tilde{\chi}})}\left[\left(1-\frac{n}{R}\right)e^{2\tilde{\chi}} - \frac{n}{R}e^{2\bar{\tilde{\chi}}}\right] \tag{4.22}$$

where $\tilde{\chi} = \partial\phi + \ln(\frac{4n^2t^2}{R^2})$ is the complex reduced susceptibility. Of course, we will also need the complex conjugate of equation (4.22).

Let us pause to make a brief point alluded to in section 4.1. Just as in the $q=0$ case, the relationship between the dimensionless parameter (4.21) and the unperturbed theory becomes ill-defined for $R<n$. In the absence of flux, it is understood that we must have $R>n$ so that the perturbation of the worldsheet Lagrangian by $\mathcal{V}_{2n/R}$ is relevant. We have no such worldsheet understanding with $q\neq 0$, but we still have a breakdown in our variables. This suggests that the relevance bound $R>n$ may be extended for the case with flux. Following this suggestion, we will only consider $R>n$ in the following.

To make the notation more concise, we find it useful to introduce the complex function

$$A = e^{2(1-\frac{n}{R})\tilde{\chi}} e^{-\frac{2n}{R}\bar{\tilde{\chi}}}. \tag{4.23}$$

The complex equation (4.22) can be written in terms of $A, \bar{A}$ as the system

$$\begin{aligned} \frac{A}{\bar{A}} &= \frac{\bar{\Lambda}^{-\frac{1}{2(1-n/R)}} - (1-\frac{n}{R})\bar{A} + \frac{n}{R}A}{\Lambda^{-\frac{1}{2(1-n/R)}} - (1-\frac{n}{R})A + \frac{n}{R}\bar{A}}, \\ \left(A\bar{A}\right)^{-\frac{1}{1-2n/R}} &= \left|\Lambda^{-\frac{1}{2(1-n/R)}} - (1-\frac{n}{R})A + \frac{n}{R}\bar{A}\right|^2. \end{aligned} \tag{4.24}$$

As in the $q=0$ case, we identify the critical points from the divergences of the third order derivatives of the partition function, or more precisely of the derivatives $\partial_\Lambda \tilde{\chi}$ and $\partial_\Lambda \bar{\tilde{\chi}}$. Therefore one begins by taking a Λ-derivative of (4.22) and its complex conjugate

$$\begin{aligned} \frac{\Lambda^{-\frac{1}{2(1-n/R)}-1}}{4(1-\frac{n}{R})^2} &= \left[\frac{\Lambda^{-\frac{1}{2(1-n/R)}}}{1-\frac{n}{R}} - (2-\frac{n}{R})A + \frac{n}{R}\bar{A}\right]\partial_\Lambda \tilde{\chi} + \frac{n}{R}\left(A+\bar{A}\right)\partial_\Lambda \bar{\tilde{\chi}}, \\ 0 &= \frac{n}{R}\left(A+\bar{A}\right)\partial_\Lambda \tilde{\chi} + \left[\frac{\bar{\Lambda}^{-\frac{1}{2(1-n/R)}}}{1-\frac{n}{R}} - (2-\frac{n}{R})\bar{A} + \frac{n}{R}A\right]\partial_\Lambda \bar{\tilde{\chi}}. \end{aligned} \tag{4.25}$$

This is a linear algebraic system for the two derivatives $\partial_\Lambda \partial\tilde{\phi}$ and $\partial_\Lambda \bar{\partial}\tilde{\phi}$. Therefore any divergent (critical) behavior will be associated with the zeros of the discriminant

$$\left|\frac{\Lambda_c^{-\frac{1}{2(1-n/R)}}}{1-\frac{n}{R}} - (2-\frac{n}{R})A_c + \frac{n}{R}\bar{A}_c\right|^2 - (\frac{n}{R})^2(\bar{A}_c + A_c)^2 = 0\,, \tag{4.26}$$

where the subscripts are meant to imply that (4.26) only holds on the critical surface.

To reiterate, the 0A critical surface is constrained by (4.26), where $A, \bar{A}$ are defined through (4.24). To obtain the critical surface we must first eliminate $A_c, \bar{A}_c$ in terms of Λ_c and R. We proceed by combining (4.26) and the first equation in (4.24), and find that we can solve for the critical $A_c, \bar{A}_c$ from a quartic equation

$$(2x-1)z_c^4 + 4L_{1,c}(3x-2)z_c^3 + 24L_{1,c}^2(x-1)z_c^2 + 16L_{1,c}^3(x-2)z_c - 16L_{1,c}^2(L_{1,c}^2 + L_{2,c}^2) = 0\,, \tag{4.27}$$

where

$$z = 4(x-1)Re(A)\,, \qquad L_1 = Re(\Lambda^{-\frac{1}{2(1-n/R)}})\,, \qquad L_2 = Im(\Lambda^{-\frac{1}{2(1-n/R)}})\,, \qquad x = \frac{n}{R} \tag{4.28}$$

The powers appearing in the definitions of $L_{1,2}$ are such that $L_1 \sim \mu$ and $L_2 \sim q$ (with proportionality factors depending on R, t).

Before studying the entire critical surface, let us consider the zero flux limit, $L_2 = 0$, where we expect to recover the results of section 4.3. In this case we find that the solutions to the quartic equation are given by $Re(A_c) = \left(\frac{2}{2x-1}, -2, -2, -2\right)\frac{L_{1,c}}{4(x-1)}$. The degeneracy observed among the roots in this special case is only accidental. The first (non-degenerate) root corresponds to the $c=1$ critical surface (4.14), as can be verified

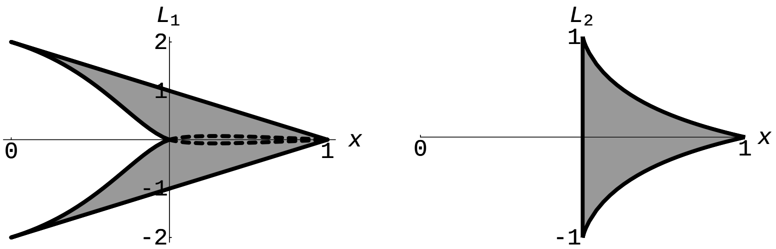

Figure 3: The left plot depicts a slice through the 0A critical surface with $L_2 \sim q = 0$. The inner curve $L_1 = L_1(x)$ (closer to the x-axis) is the $c = 1$ critical curve discussed in section 4.3, where $\partial_\mu^2 \phi$ diverges, while the outer line is a new feature of the 0A critical surface, where $\partial_q^2 \phi$ diverges. The dotted lines indicate solutions of (3.45) where $\partial_\mu^2 \phi$ diverges, but which are probably not accessible from the unperturbed 0A theory since they are completely enclosed by other critical surfaces. The right plot takes $L_1 = 0$ and shows the dependence on the flux parameter L_2.

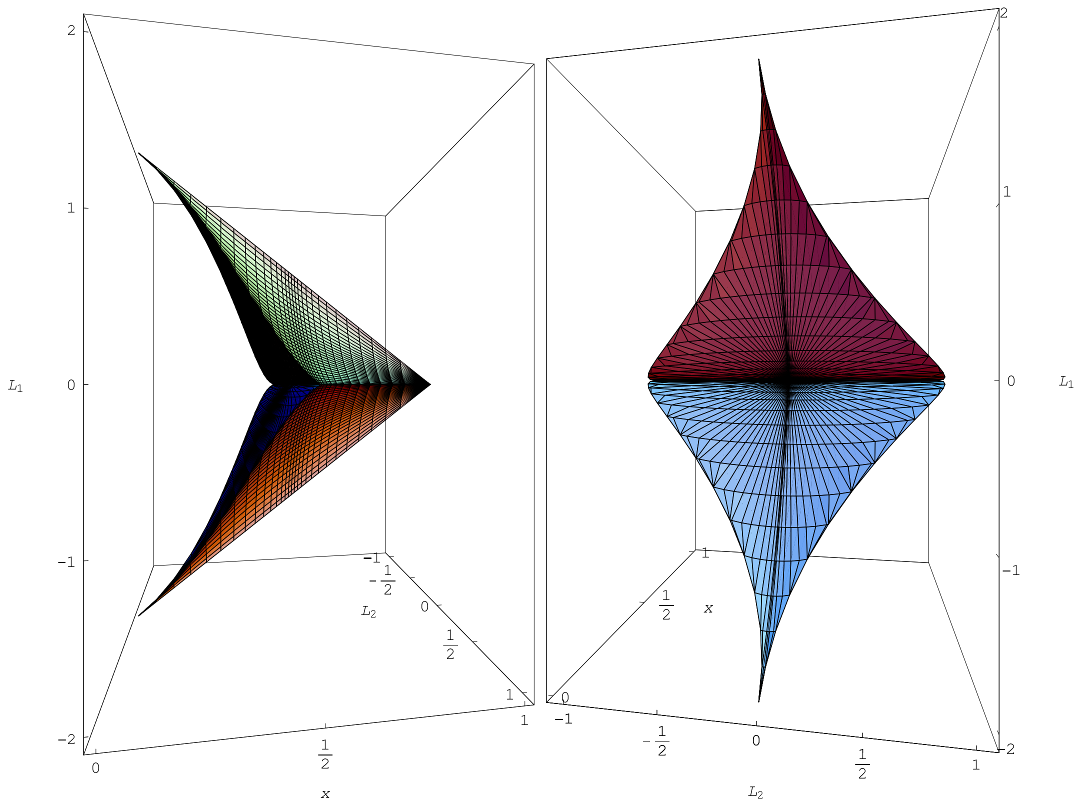

Figure 4: The left plot depicts the 0A critical surface, with emphasis on the dependence of L_1 on x. The right plot depicts the same surface, but from "below", which emphasizes the extent in the $L_{1,2}$ plane. To get oriented, compare the left 3d surface with the left cross-section in figure (3). The "tentacles" of the left surface continue all the way to $x = 0$, where they collapse to the points $L_1 = \pm 2$, $L_2 = 0$. At $x = 1$, the surface collapses to a point at $L_1 = L_2 = 0$.

by comparison with (4.11). By expanding (4.24) around this non-degenerate root, while keeping the RR flux zero, we can extract the divergent behaviour of $\partial_\mu^2 \phi \sim (L_1 - L_{1,c})^{-1/2} \sim (Re(\Lambda) - Re(\Lambda_c))^{-1/2}$. This was expected, and it is in accord with (4.16).

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The second (degenerate) root is novel, and corresponds to a straight line, $|L_1| = 2\,(1-x)$, where $\partial_q^2\phi$ diverges. The critical behaviour at zero RR flux is given by $\partial_q^2\phi \sim (L_1 - L_{1,c})^{-1} \sim (Re(\Lambda) - Re(\Lambda_c))^{-1}$. This clearly distinguishes $c = 1$ perturbed by momentum modes from 0A with $q = 0$, and the same perturbation. In figure (3) we include plots of the intersection of the 0A critical surface with the $L_2 = 0$ and $L_1 = 0$ planes.

The full 0A critical surface is found by substituting the solutions of the quartic into the remaining (second) equation of (4.24). This constraint defines a surface in the 3d space parameterized by L_1, L_2 and x, which were introduced in (4.28). We used Mathematica to generate the plots of the 0A critical surface shown in figure (4). We have plotted the 0A surface in terms of the finite range variable $x \in (0, 1)$ (i.e. $R > n$) and as a function of the real and imaginary parts of the dual "Sine-Liouville" dimensionless parameter, $\Lambda^{-\frac{1}{2(1-n/R)}}$. Note that $L_1^2 + L_2^2 \to \infty$ is the regime identified with the unperturbed 0A string theory. The entire 0A critical surface is symmetric with respect to the $L_2 = 0$ plane as well as the $L_1 = 0$ plane. It is also closed.

The line $L_1 = L_2 = 0$ of figures (3) and (4) is the momentum mode condensate (Sine-Liouville) phase. Fixing $L_2 = 0$ (as in the left plot of figure (3)) it would seem that the critical surface obstructs a smooth connection to the unperturbed 0A regime $|L_1| \gg 1$. For $x \in (0, \frac{1}{2})$ the obstruction includes the familiar $c = 1$ critical curve (4.14). The novel (linear) component of the critical curve obstructs the entire range of $x \in (0, 1)$. In figure (4) we see that, for the range $x \in (0, \frac{1}{2})$ the obstruction can be avoided by taking advantage of non-trivial flux L_2, while for $x \in (\frac{1}{2}, 1)$ the Sine-Liouville regime remains covered by the critical surface. Note that this is precisely the opposite of the case in $c = 1$, where $x \in (0, \frac{1}{2})$ is obstructed and $x \in (\frac{1}{2}, 1)$ is not [11].

5. Outlook & holomorphic perturbations

There are a number of obvious extensions of the line of development presented in this work. We have studied the perturbation of 0A string theory by momentum mode operators in the presence of Ramond-Ramond flux. It would be of interest to address also the effect of perturbing by winding operators when flux is present; some prior work in this direction includes [20, 21]. Although of more technical difficulty (since the matrix model is no longer in its singlet sector), such perturbations may shed light on the conjectured 0A black hole [28]. Also, the study of the integrable structure of 0B string theory with non-trivial flux is of obvious interest. The analysis of winding and momentum mode perturbations in that case would provide another probe of T-duality for Type 0 strings. All of these directions would provide a more detailed picture of the vacuum structure of two-dimensional string theory.

There is a more novel direction which is suggested by the complexified Toda lattice that we have introduced. We have not fully explored this structure for we have only examined deformations by the momentum modes, which are given by $\hat{B}_\pm \sim e^{2i\partial_\epsilon}$. Formally, it is natural to further exploit the complex nature of the system by studying perturbations which preserve the holomorphicity existing at vanishing perturbation [6]. Such perturbations involve the flux in a non-trivial way.

To make this more concrete, we consider the transformations

$$\hat{U}_{\pm} = e^{i\sum_{n>0}\left(\tilde{b}_{\pm n}(\hat{y};\{t\})\hat{z}_{\pm}^{-n/R}+\text{h.c.}\right)} e^{\mp i\left(\tilde{\phi}(\hat{y};\{t\})/2+\text{h.c.}\right)} e^{i\sum_{n>0}\left(t_{\pm n}\hat{z}_{\pm}^{n/R}+\text{h.c.}\right)}, \tag{5.1}$$

for some complex parameters $\{t\}$ and undetermined functions $\tilde{b}$ and $\tilde{\phi}$ and where "h.c." denotes Hermitian conjugate. We will call these "holomorphic perturbations" since (5.1) factorizes into an operator dependent only on $\hat{y}$ and $\hat{\eta}^{\pm 1}$ and one dependent only on $\hat{\bar{y}}$ and $\hat{\bar{\eta}}^{\pm 1}$. Thus, this is much like two decoupled copies of the transformation (2.8) for the $c = 1$ matrix model.

The string equation (3.31) is the same under this transformation as when perturbed by momentum modes. The expansions of the complex Orlov-Shulman operator in terms of the Lax operators are now

$$\hat{Y}_{+} = \hat{y} - \frac{2n}{R}\bar{t}_{+n}\hat{\bar{Z}}_{+}^{n/R} + \sum_{k>0} \bar{v}_{+k}\left(\hat{y}\right)\hat{\bar{Z}}_{+}^{-k/R}, \tag{5.2}$$

$$\hat{Y}_{-} = \hat{y} + \frac{2n}{R}t_{-n}\hat{Z}_{-}^{n/R} + \sum_{k>0} v_{-k}\left(\hat{y}\right)\hat{Z}_{-}^{-k/R}, \tag{5.3}$$

for some undetermined functions $v_{\pm k}$. Combining the above expansions with the string equation we obtain the holomorphic susceptibility equation

$$y - e^{-2\partial\phi} = \left(\frac{2n}{R}\right)^2 t_{-n}\bar{t}_{+n}e^{2(1-n/R)\partial\phi}\left(\frac{n}{R}-1\right). \tag{5.4}$$

Unlike (3.44), the above result does not mix $\partial\phi$ and $\bar{\partial}\phi$ and so preserves the holomorphic factorization of the unperturbed solution.

It is important to keep in mind that the physical interpretation of the holomorphic perturbations is quite different from that of the momentum mode perturbations studied throughout this work. In the absence of flux, perturbing by the matrix model operators $\hat{B}_{\pm}$ has the clear interpretation of adding tachyon vertex operators to the string worldsheet action; this can be generalized to a flux background since $\hat{B}_{\pm}$ still generate small deformations in the Fermi sea at finite q. The holomorphic perturbations above are less clearly interpreted, and perhaps should be understood as generating coherent states of D-branes or discrete states. Alternatively, the interpretation may lie outside of 0A string theory itself and require ideas from non-critical M-theory [24, 25]. It would be of great interest to understand the full physical interpretation of these perturbations and the complex Toda structure introduced herein.

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A. Chiral quantization of $c = 1$ matrix model

Here we will briefly review the chiral quantization of the fermion system dual to $d = 2$ bosonic string theory. Since these fermions are non-interacting we will make do with single-particle quantization.

A.1 Operator algebra

Recall that the single-particle energy for the fermions in the $c = 1$ matrix model is given by

$$\hat{\epsilon} = \frac{1}{2}\left(\hat{p}^2 - \hat{x}^2\right). \tag{A.1}$$

The algebraic structure becomes clearer when we introduce the light-cone coordinates

$$\hat{x}_\pm \equiv \frac{1}{\sqrt{2}}\left(\hat{p} \pm \hat{x}\right), \tag{A.2}$$

which obey the following closed algebra (note $\hbar = 1$)

$$\begin{aligned} [\hat{x}_+, \hat{x}_-] &= i, \\ [\hat{x}_\pm, \hat{\epsilon}] &= \pm i \hat{x}_\pm, \\ \{\hat{x}_+, \hat{x}_-\} &= 2\hat{\epsilon}. \end{aligned} \tag{A.3}$$

The first commutator indicates that $\hat{x}_\pm$ are canonically conjugate, and the second that $\hat{x}_\pm$ are "shift operators" of $\hat{\epsilon}$. The third relation indicates that the Schrodinger equation is first order in both the $x_\pm$ bases, which is a sizable advantage over the second order wave equation in the position basis.

Using the above relations the Schrodinger equation can be expressed in either the $x_\pm$ bases

$$\mp i\left(x_\pm \partial_{x_\pm} + \frac{1}{2}\right)\Psi(x_\pm, t) = i\partial_t \Psi(x_\pm, t). \tag{A.4}$$

Or for energy eigenfunctions, where $\Psi_\epsilon(x_\pm, t) = e^{-i\epsilon t}\psi_\epsilon(x_\pm)$,

$$\left(x_\pm \partial_{x_\pm} + \frac{1}{2}\right)\psi_\epsilon(x_\pm) = \pm i\epsilon\, \psi_\epsilon(x_\pm). \tag{A.5}$$

A.2 States and eigenfunctions

For a given energy, ϵ, there are four states of interest which in ket notation are $|\,\epsilon$, *in*/*out*, $L/R\rangle$. The notation is such that *in* refers to fermions moving toward the origin and *out* refers to those moving away. The label L/R refers to which side of the potential the fermion is localized on. The *in* (*out*) eigenfunctions are most conveniently expressed in the x_- (x_+)

bases. They are as follows

$$\begin{aligned}
\langle x_-|\,\epsilon, in, R\rangle &= \theta(x_-)\frac{x_-^{-i\epsilon-\frac{1}{2}}}{\sqrt{2\pi}}, \\
\langle x_-|\,\epsilon, in, L\rangle &= \theta(-x_-)\frac{(-x_-)^{-i\epsilon-\frac{1}{2}}}{\sqrt{2\pi}}, \\
\langle x_+|\,\epsilon, out, R\rangle &= \theta(x_+)\frac{x_+^{i\epsilon-\frac{1}{2}}}{\sqrt{2\pi}}, \\
\langle x_+|\,\epsilon, out, L\rangle &= \theta(-x_+)\frac{(-x_+)^{i\epsilon-\frac{1}{2}}}{\sqrt{2\pi}}.
\end{aligned} \tag{A.6}$$

The *in* and *out* states are not independent, but rather are connected by the change of basis, $x_- \leftrightarrow x_+$.

We can see that the *in* eigenfunctions are delta-function normalized by inserting a complete set of x_- eigenstates.

$$\begin{aligned}
\langle\,\epsilon', in, a|\,\epsilon, in, b\rangle &= \int_{-\infty}^{\infty} dx_- \langle\,\epsilon', in, a|x_-\rangle\langle x_-|\,\epsilon, in, b\rangle \\
&= \delta(\epsilon'-\epsilon)\delta_{a,b},
\end{aligned} \tag{A.7}$$

where the Roman letters label L/R. Similarly, by inserting a complete set of x_+ states, we can show

$$\langle\,\epsilon', out, a|\,\epsilon, out, b\rangle = \delta(\epsilon'-\epsilon)\delta_{a,b}. \tag{A.8}$$

It is also useful to consider parity eigenstates. Define

$$\begin{aligned}
|\,\epsilon, in, \pm\rangle &= \frac{1}{\sqrt{2}}\left(|\,\epsilon, in, R\rangle \pm |\,\epsilon, in, L\rangle\right), \\
|\,\epsilon, out, \pm\rangle &= \frac{1}{\sqrt{2}}\left(|\,\epsilon, out, R\rangle \pm |\,\epsilon, out, L\rangle\right),
\end{aligned} \tag{A.9}$$

where the third quantum number labels the parity eigenvalue.

A.3 Unperturbed scattering

Scattering in the unperturbed problem involves only a simple change of basis. We want to compute $\langle\,\epsilon', out, a|\,\epsilon, in, b\rangle$. This is easily done by inserting complete sets of states of both $x_\pm$. Note that in performing the calculations, one must use $\langle x_+|x_-\rangle = \frac{1}{\sqrt{2\pi}}e^{ix_+x_-}$ which follows from the canonical commutator $[x_+, x_-] = i$. We will also need use of the integrals [29]

$$\begin{aligned}
\Gamma(z) &= \frac{b^z}{\sin\frac{\pi z}{2}}\int_0^{\infty} dt\,\sin(bt)\,t^{z-1} \\
\Gamma(z) &= \frac{b^z}{\cos\frac{\pi z}{2}}\int_0^{\infty} dt\,\cos(bt)\,t^{z-1}.
\end{aligned} \tag{A.10}$$

The scattering is completely diagonal in the parity basis

$$
\begin{aligned}
\langle\, \epsilon', out, \pm |\, \epsilon, in, \mp\rangle &= 0, \\
\langle\, \epsilon', out, \pm |\, \epsilon, in, \pm\rangle &= \frac{1}{\sqrt{2\pi}} \left(e^{\frac{i\pi}{4}+\frac{\epsilon\pi}{2}} \pm e^{-\frac{i\pi}{4}-\frac{\epsilon\pi}{2}} \right) \Gamma\left(\frac{1}{2} - i\epsilon\right) \delta(\epsilon' - \epsilon) \\
&= e^{i\pi/4} \sqrt{\frac{1 \mp ie^{-\pi\epsilon}}{1 \pm ie^{-\pi\epsilon}}} \sqrt{\frac{\Gamma\left(\frac{1}{2} - i\epsilon\right)}{\Gamma\left(\frac{1}{2} + i\epsilon\right)}} \delta(\epsilon' - \epsilon). \qquad \text{(A.11)}
\end{aligned}
$$

To discuss the $c = 1$ matrix model we consider only the parity odd states. Then we can write the scattering as

$$
|\, \epsilon, in, -\rangle = S(\epsilon) |\, \epsilon, out, -\rangle, \qquad \text{(A.12)}
$$

where the bounce factor is defined as

$$
\begin{aligned}
S(\epsilon) &= e^{i\pi/4} \sqrt{\frac{1 + ie^{-\pi\epsilon}}{1 - ie^{-\pi\epsilon}}} \sqrt{\frac{\Gamma\left(\frac{1}{2} - i\epsilon\right)}{\Gamma\left(\frac{1}{2} + i\epsilon\right)}} \\
&\approx e^{-i\pi/4} \sqrt{\frac{\Gamma\left(\frac{1}{2} - i\epsilon\right)}{\Gamma\left(\frac{1}{2} + i\epsilon\right)}}. \qquad \text{(A.13)}
\end{aligned}
$$

In the second line we have neglected terms of order $e^{\pi\epsilon}$, *i.e.* we consider large negative energy with negligible tunneling. This is appropriate and indeed required in the $c = 1$ model which is defined only by perturbation theory in $\frac{1}{|\epsilon|}$.

The expression (A.12) indicates the *in* and *out* states are not independent. Indeed, (A.12) appears to be the conventional relationship between scattered states, but this is incorrect. Both $|\, \epsilon, in/out, -\rangle$ are solutions to the time-independent Schrodinger equation and so (A.12) is an equality valid at *all finite times*. That is, there is no implicit insertion of the time translation operator evolving time from $t = -\infty$ to $t = +\infty$. To eliminate this redundancy, define

$$
|\, \epsilon\rangle = S^{-1/2}(\epsilon) |\, \epsilon, in, -\rangle = S^{1/2}(\epsilon) |\, \epsilon, out, -\rangle. \qquad \text{(A.14)}
$$

Then we have two equivalent expressions for the same state, given in different bases

$$
\psi_\epsilon^{\pm}(x_\pm) = \langle\, x_\pm |\, \epsilon\rangle = \frac{S^{\pm 1/2}(\epsilon)}{\sqrt{4\pi}} sign(x_\pm) |x_\pm|^{\pm i\epsilon - \frac{1}{2}}. \qquad \text{(A.15)}
$$

Even though we consider the $c = 1$ theory where the fermions are localized on one side of the barrier, to calculate probabilities one should still integrate over all $x_\pm$, i.e. the wavefunctions (A.15) are delta-function normalized on the whole interval $x_\pm \in \mathbb{R}$.

B. Chiral quantization of 0A matrix model

We reproduce here results from the appendix of [6] for our normalization differs in its q-dependence. Consider now particles moving in a plane with a radial harmonic oscillator potential which in Cartesian coordinates has the Hamiltonian

$$
\hat{\epsilon} = \frac{1}{2} \left(\hat{p}_1^2 - \hat{x}_1^2 + \hat{p}_2^2 - \hat{x}_2^2 \right) . \qquad \text{(B.1)}
$$

It is useful to apply chiral quantization as in $c = 1$ to both degrees of freedom,

$$\begin{aligned} \hat{x}_{\pm,i} &\equiv \frac{\hat{p}_i \pm \hat{x}_i}{\sqrt{2}}, \\ [\hat{x}_{\pm,i}, \hat{x}_{\mp,j}] &= \pm i\delta_{ij}\,. \end{aligned} \tag{B.2}$$

For future use, note that the kernel of this transformation is

$$\langle x_i |\, x_{\pm,i}\rangle = \frac{2^{1/4}e^{\pm i\pi/8}}{\sqrt{2\pi}} \exp\left[\mp i\left(\frac{x_i^2}{2} \mp \sqrt{2}x_i x_{\pm,i} + \frac{x_{\pm,i}^2}{2}\right)\right], \tag{B.3}$$

where we have chosen the phase in (B.3) so as to have a simple form for the inner-product

$$\langle x_{+,i} |\, x_{-,i}\rangle = \frac{1}{\sqrt{2\pi}} \exp\left(i x_{+,i} x_{-,i}\right)\,. \tag{B.4}$$

It is not as straightforward to compute the relevant inner products and wave-functions as in the $c = 1$ theory so we must go through a series of changes of variables. First, introduce the conventional polar coordinates

$$\hat{x}_1 = \hat{r}\cos\hat{\theta}\,, \qquad \hat{x}_2 = \hat{r}\sin\hat{\theta}\,. \tag{B.5}$$

So that $|x_1, x_2\rangle$ are delta-function normalized, the kernel of this change of basis is

$$\langle x_1, x_2 |\, r, \theta\rangle = \sqrt{x}\delta\left(x_1 - r\cos\theta\right)\delta\left(x_2 - r\sin\theta\right)\,.. \tag{B.6}$$

The momentum conjugate to θ is the conserved charge $\hat{q} = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1$. It is also of use to introduce polar coordinates in phase space via

$$\hat{x}_{\pm,1} = \hat{r}_\pm \cos\hat{\theta}_\pm\,, \qquad \hat{x}_{\pm,2} = \hat{r}_\pm \sin\hat{\theta}_\pm\,. \tag{B.7}$$

which have an integral kernel analogous to (B.6). The relation between conventional polar coordinates and the phase space polar coordinates is easily derived to be

$$\begin{aligned} \langle r_\pm, \theta_\pm |\, r, \theta\rangle &= \int dx_1 dx_2 dx_{\pm,1} dx_{\pm,2}\, \langle r_\pm, \theta_\pm |\, x_{\pm,1}, x_{\pm,2}\rangle \\ &\qquad \times \langle x_{\pm,1}, x_{\pm,2} |\, x_1, x_2\rangle\, \langle x_1, x_2 |\, r, \theta\rangle \\ &= \frac{e^{\mp i\pi/4}}{2\pi}\sqrt{2rr_\pm}\exp\left[\pm i\left(\frac{r^2}{2} \mp \sqrt{2}rr_\pm\cos\left(\theta - \theta_\pm\right) + \frac{r_\pm^2}{2}\right)\right]. \end{aligned} \tag{B.8}$$

It is convenient to exchange $\theta_\pm$ for the conjugate momenta $q_\pm$

$$\begin{aligned} \langle r, \theta\, | r_\pm, q_\pm\rangle &= \frac{e^{\pm i\pi/4}}{(2\pi)^{3/2}}\sqrt{2rr_\pm}e^{\mp i(\frac{r^2}{2}+\frac{r_\pm^2}{2})}\int_0^{2\pi} d\theta e^{i\left(q_\pm\theta_\pm + \sqrt{2}rr_\pm\cos(\theta-\theta_\pm)\right)} \\ &= \frac{e^{\pm i\pi/4} i^{|q_\pm|}}{\sqrt{2\pi}}\sqrt{2rr_\pm}e^{\mp i(\frac{r^2}{2}+\frac{r_\pm^2}{2}\mp q_\pm\theta)} J_{|q_\pm|}\left(\sqrt{2}rr_\pm\right), \end{aligned} \tag{B.9}$$

and then calculate

$$\begin{aligned}\langle r_+, q_+ | r_-, q_- \rangle &= \frac{e^{-i\pi/2} i^{|q_-|-|q_+|}}{2\pi} 2\sqrt{r_+ r_-} e^{i\left(r_+^2 + r_-^2\right)/2} \int_0^{2\pi} d\theta e^{i\theta(q_- - q_+)} \\ &\quad \times \int_0^\infty r dr e^{ir^2} J_{|q_+|}\left(\sqrt{2} r r_+\right) J_{|q_-|}\left(\sqrt{2} r r_-\right) \\ &= \delta_{q_+ q_-} i^{|q_+|} \sqrt{r_+ r_-} J_{|q_+|}\left(r_+ r_-\right) . \end{aligned} \tag{B.10}$$

We almost have all of the ingredients for the scattering phase. First we note that the momenta conjugate to $\hat{r}_\pm$ must classically satisfy the Poisson brackets $\{r_\pm, p_\pm\} = 1$. Using $x_{+,i}$ as the generalized coordinates and $x_{-,i}$ as their momenta, we easily obtain $p_\pm = \pm\epsilon/r_\pm$. When promoted to operators this is written

$$\hat{\epsilon} = \pm\frac{1}{2}\left(\hat{r}_\pm \hat{p}_\pm + \hat{p}_\pm \hat{r}_\pm\right) , \tag{B.11}$$

which yields the time-independent wavefunctions

$$\langle r_\mp | \epsilon, in/out \rangle = \frac{1}{\sqrt{2\pi}} r_\mp^{\mp i\epsilon - 1/2}. \tag{B.12}$$

Finally, we compute the scattering phase by combining the results of this appendix,

$$\begin{aligned}\left\langle \epsilon', q', out \right| \epsilon, q, in \rangle &= \int_0^\infty dr_- dr_+ \sum_{q_+, q_-} \left\langle \epsilon', q', out \right| r_+, q_+ \rangle \langle r_+, q_+ | r_-, q_- \rangle \langle r_- q_- | \epsilon, q, in \rangle \\ &= \delta_{qq'} \delta\left(\epsilon - \epsilon'\right) 2^{-i\epsilon} i^{|q|} \frac{\Gamma\left(\frac{1}{2}\left(1 + |q| - i\epsilon\right)\right)}{\Gamma\left(\frac{1}{2}\left(1 + |q| + i\epsilon\right)\right)} . \end{aligned} \tag{B.13}$$

Thus the scattering phase is given by

$$S(\epsilon, q) = 2^{-i\epsilon} i^{|q|} \frac{\Gamma\left(\frac{1}{2}\left(1 + |q| - i\epsilon\right)\right)}{\Gamma\left(\frac{1}{2}\left(1 + |q| + i\epsilon\right)\right)} . \tag{B.14}$$

C. An alternate derivation of the 0A string equations

Consider the compactified, unperturbed, all-genus 0A partition function [6]

$$\ln \mathcal{Z} = -\frac{1}{2} Re \int_0^\infty \frac{dt}{t} \frac{e^{itR(\mu + iq)/2}}{\sinh(\frac{t}{4R}) \sinh(t/2)}, \tag{C.1}$$

where we have neglected terms analytic in μ. Using the notation introduced previously, $y = \mu + iq$, $\partial = \frac{\partial}{\partial y}$, we arrive at the functional equation

$$-4 \sin(\partial/(2R)) \sin(\partial) \ln \mathcal{Z} = \ln(y) \tag{C.2}$$

where we have implicitly regularized the rhs of the above equation. Similarly, we find

$$-4 \sin(\bar{\partial}/(2R)) \sin(\bar{\partial}) \ln \mathcal{Z} = \ln(\bar{y}) . \tag{C.3}$$

Following Kostov [12], we will map these functional equations for the unperturbed partition function into the complex string equation derived in section 3. First, we shall identify the partition function with the $\tau(y,\bar{y})$-function of the complexified Toda hierarchy: $\ln\mathcal{Z} = \ln(\tau)$.

Next, we recall that in the absence of perturbations, the Lax operators are simply dressed complex shift operators

$$Z_- = W_-\eta^{-1}W_-^{-1}, \qquad \bar{Z}_+ = W_+\eta W_+^{-1}, \qquad \text{plus the c.c.,} \tag{C.4}$$

where the dressing functions, with the perturbations turned off are $W_\pm = e^{\mp i\phi_0/2}$. Moreover, given that the scattering phase and the τ-function are related through

$$\phi(\mu;q) = i\ln\left(\frac{\tau(\mu-\frac{i}{2R})}{\tau(\mu+\frac{i}{2R})}\right), \tag{C.5}$$

we derive

$$\frac{W_-(\mu)}{W_+(\mu)} = \frac{\tau(\mu+\frac{i}{2R})}{\tau(\mu-\frac{i}{2R})} \tag{C.6}$$

which in terms of the complex variable y reads

$$\frac{W_-(y,\bar{y})}{W_+(y,\bar{y})} = \frac{\tau(y+\frac{i}{2R},\bar{y}+\frac{i}{2R})}{\tau(y-\frac{i}{2R},\bar{y}-\frac{i}{2R})} \tag{C.7}$$

At this moment it is useful to also recall that the unperturbed partition function, and therefore τ, enjoys a holomorphic factorization, already transparent from (C.1). This fact is a manifestation of the expression for the scattering phase, which factorizes holomorphically in a manifest way. Clearly, this property carries through to the unperturbed dressing operators. We conclude then that the previous equation can be decomposed (up to normalization constants) in

$$\frac{W_-(y)}{W_+(y)} = \frac{\tau(y+\frac{i}{2R})}{\tau(y-\frac{i}{2R})} \tag{C.8}$$

and its complex conjugate.

We have now gathered all the information we need to proceed with the derivation of the string equations. To this end we now turn back to the functional-differential equation (C.2), which when integrated yields

$$y = \frac{\tau(y+\frac{i}{2R}+i,\bar{y})}{\tau(y-\frac{i}{2R}+i,\bar{y})}\frac{\tau(y-\frac{i}{2R}-i,\bar{y})}{\tau(y+\frac{i}{2R}-i,\bar{y})} \tag{C.9}$$

The latter expression, according to the previous discussion on the holomorphic factorization properties of the unperturbed τ-function, reduces to

$$y = \frac{\tau(y+\frac{i}{2R}+i)}{\tau(y-\frac{i}{2R}+i)}\frac{\tau(y-\frac{i}{2R}-i)}{\tau(y+\frac{i}{2R}-i)}\,. \tag{C.10}$$

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In the next step we substitute (C.8), to arrive at

$$y = \frac{W_-(y+i)}{W_+(y+i)} \frac{W_+(y-i)}{W_-(y-i)} \,. \tag{C.11}$$

Equivalently, this equation can be written as

$$\begin{aligned} \Big[W_+(y)^{-1}W_-(y)\Big]\,\eta^{-1}\,\Big[W_-(y)^{-1}W_+(y)\Big]\,\eta &= y - i \\ \eta\Big[W_+(y)^{-1}W_-(y)\Big]\,\eta^{-1}\,\Big[W_-(y)^{-1}W_+(y)\Big] &= y + i \,. \end{aligned} \tag{C.12}$$

Also, since $(W_-)^{-1}W_+ = e^{i\phi_0}$ it is a trivial statement that $(W_-)^{-1}W_+ y[(W_-)^{-1}W_+]^{-1} = y$. Furthermore, based on the Toda hierarchy flow equations, it can be shown that the operator $(W_-)^{-1}W_+$ is independent of the couplings t_n, t_{-n}. This means that (C.12) hold in general. In fact, these equations constitute the constraint among the Lax and Orlov-Shulman operators that we called the *string equations*

$$\bar{Z}_+ Z_- = Y + i \,, \qquad Z_- \bar{Z}_+ = Y - i \,. \tag{C.13}$$

As highlighted by this derivation, the complex nature of the 0A string equations arises naturally from the holomorphicity of the unperturbed partition function.

D. Comparison with low-order correlators

We here confirm the consistency of the complex susceptibility equation (3.45) with calculations of low-order tachyon correlators as stated in section 4.4. To facilitate comparison with [23] we first introduce

$$p \equiv \frac{2n}{R}, \tag{D.1}$$

the momentum of our perturbations, so that (4.20) becomes

$$e^{-2\partial_\mu \phi} = \frac{\mu^2}{\left[1-(p-1)\,p^2 t^2 e^{-(p-2)\partial_\mu \phi}\right]^2} + \frac{q^2}{\left[1 - p^2 t^2 e^{-(p-2)\partial_\mu\phi}\right]^2}. \tag{D.2}$$

We then expand $\partial_\mu \phi$ as

$$-\partial_\mu \phi = \frac{1}{2}\log\left(\mu^2+q^2\right) + \sum_{m>0} \frac{t^{2m}}{(m!)^2} a_m, \tag{D.3}$$

and solve iteratively for the a_m. Recalling (4.19)

$$\partial_\mu \phi = R^{-1}\partial_\mu^2 \log \mathcal{Z} = R^{-1}\left[\partial_\mu^2 \log \mathcal{Z}_0 + \sum_{m>0}\frac{t^{2m}}{(m!)^2}\partial_\mu^2 \left\langle \mathcal{T}^m_{2n/R}\mathcal{T}^m_{-2n/R}\right\rangle\right], \tag{D.4}$$

we can identify the coefficients in (D.3) with the tachyon correlators

$$a_m = -R\partial_\mu^2 \left\langle \mathcal{T}^m_p \mathcal{T}^m_{-p}\right\rangle . \tag{D.5}$$

We reproduce the first few correlators here, as predicted by (D.3):

$$\partial_\mu^2 \langle \mathcal{T}_p \mathcal{T}_{-p} \rangle = -R^{-1} p^2 \left(\mu^2 + q^2\right)^{(p-4)/2} \left[(p-1)\,\mu^2 + q^2\right], \tag{D.6}$$

$$\begin{aligned} \partial_\mu^2 \left\langle \mathcal{T}_p^2 \mathcal{T}_{-p}^2 \right\rangle &= -2R^{-1} \left[p^2 \left(\mu^2 + q^2\right)^{(p-4)/2}\right]^2 \Big[(p-1)^2 (2p-3)\,\mu^4 \\ &\quad + \left(18 - 22p + 7p^2\right) \mu^2 q^2 + (2p-3)\, q^4\Big], \end{aligned} \tag{D.7}$$

$$\begin{aligned} \partial_\mu^2 \left\langle \mathcal{T}_p^3 \mathcal{T}_{-p}^3 \right\rangle &= -6R^{-1} \left[p^2 \left(\mu^2 + q^2\right)^{(p-4)/2}\right]^3 \Big[(3p-5)(3p-4)(p-1)^3 \mu^6 \\ &\quad +3\left(100 - 247p + 239p^2 - 106p^3 + 18p^4\right) \mu^4 q^2 \\ &\quad + 3\left(-100 + 179p - 108p^2 + 22p^3\right) \mu^2 q^4 + (3p-5)(3p-4)\, q^6\Big]. \end{aligned} \tag{D.8}$$

We now wish to compare with the results of [23], which quotes several A_m (see their (3.11)[9]) defined through

$$\left\langle \mathcal{T}_0 \mathcal{T}_p^m \mathcal{T}_{-p}^m \right\rangle = \left(\mu^2 + q^2\right)^{mp/2} A_m. \tag{D.9}$$

These correlators $\left\langle \mathcal{T}_0 \mathcal{T}_p^m \mathcal{T}_{-p}^m \right\rangle$, are related to $\left\langle \mathcal{T}_p^m \mathcal{T}_{-p}^m \right\rangle$, in both $c = 1$ and 0A, via

$$\partial_\mu \left\langle \mathcal{T}_p^m \mathcal{T}_{-p}^m \right\rangle = \left\langle \mathcal{T}_0 \mathcal{T}_p^m \mathcal{T}_{-p}^m \right\rangle. \tag{D.10}$$

This was first recorded in [30] and exploited in [22]. We thus obtain

$$\partial_\mu^2 \left\langle \mathcal{T}_p^m \mathcal{T}_{-p}^m \right\rangle = \partial_\mu \left[\left(\mu^2 + q^2\right)^{mp/2} A_m\right]. \tag{D.11}$$

Using this expression and the A_m in [23] we see that our correlators match, up to an overall normalization factor R.

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