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Condensing momentum modes in 2-d 0A string theory with flux

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Abstract: We use a combination of conformal perturbation theory techniques and matrix model results to study the effects of perturbing by momentum modes two dimensional type 0A strings with non-vanishing Ramond-Ramond (RR) flux. In the limit of large RR flux (equivalently, $\mu = 0$) we find an explicit analytic form of the genus zero partition function in terms of the RR flux q and the momentum modes coupling constant α. The analyticity of the partition function enables us to go beyond the perturbative regime and, for $\alpha \gg q$, obtain the partition function in a background corresponding to the momentum modes condensation. For momenta such that $0 < p < 2$ we find no obstruction to condensing the momentum modes in the phase diagram of the partition function.

Keywords: M(atrix) Theories, Conformal Field Models in String Theory.

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1. Introduction

The open/closed string correspondence is one of the fundamental concepts in the modern understanding of string theory. This correspondence provides, in various cases, a non-perturbative definition of string theory.

The AdS/CFT correspondence is perhaps one of the best studied instances of the open/closed string correspondence. Another very important case is string theory in two dimensions where the open string side of the correspondence is described via a matrix model. The main attraction of the open/closed string correspondence in two dimensions resides in the ability to obtain exact results on both sides of the correspondence.

The simplest case of two-dimensional duality is provided by the $c = 1$ model. The open string side is described by an exactly solvable random matrix model with inverted harmonic potential. The closed string side is a Liouville theory which has been solved using the conformal bootstrap.

Recently, two new non-supersymmetric two-dimensional string theories have been formulated and their corresponding matrix models identified [1, 2]. In the spirit of the renormalization group flow, it is natural to study the deformation of the above correspondence, that is to study the relationship between the two descriptions after adding operators or expectation values to these theories.

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In this paper we study the deformation of two-dimensional type 0A string theory by momentum modes. We employ a technique successfully applied to the $c = 1$ model by G. Moore in [3] (see also [4]). This technique uses a combination of conformal perturbation theory and matrix model results. In recent years, the beautiful results of [3, 4] have been reproduced and improved using alternative techniques [5, 6]. In particular, the study of adding momentum and winding perturbations to the $c = 1$ model has explicitly revealed the rich mathematical structure of integrable systems in these models [7 − 10]. Various physical aspects related to phase transitions have been confirmed and reinterpreted. Most remarkably among them are the connection with the Euclidean two-dimensional black hole [6] and to time-dependent backgrounds [10 − 13].

The idea that some two-dimensional black holes admit a matrix model description has a long history. A prominent role has been played by a deformation of the inverted harmonic oscillator matrix model due to Jevicki and Yoneya (JY) [14]. This precise matrix model has resurfaced recently as it describes type 0A string theory in the presence of D-branes [2].

Indeed, there was some evidence that certain aspects of the deformed matrix model match their counterpart in the 0A two-dimensional black hole [15]. A closer examination, however, showed that the thermodynamics of two-dimensional type 0A black holes does not match that of the deformed matrix model [16 − 18]. It was suggested in [16], that the two-dimensional 0A black hole has properties similar to that of a different deformation of the c=1 matrix model considered by Boulatov and Kazakov [5] and applied to the $c = 1$ black hole in [6]. Kazakov and Tseytlin [19] compared the matrix model deformed by vortices with the exact two-dimensional black hole obtained in [20] and found some qualitative agreement. Despite much effort, the existence of a direct correspondence between two-dimensional Lorentzian black holes and matrix models is still under scrutiny [21].

Irrespectively of the ultimate relationship of perturbed two-dimensional string theories with two-dimensional black holes, our work is interesting in its own right as it provides an explicit expression for the partition function of the Jevicki-Yoneya (JY) model in the presence of momentum modes. From the matrix model point of view we are computing the effect of adding momentum modes in a model that provides a non-perturbatively calculable unitary S matrix [22]. Other interesting nonperturbative aspects have been discussed in, for example, [23 − 26].

The paper is organized as follows. In section 2 we review the work of Moore in [3], outlining the strategy that we will follow and introducing most of the notation. Section 3 contains our main result, the partition function of the two-dimensional type 0A string theory perturbed by momentum modes. Using the dual matrix model description in terms of free fermions, the deformed JY model, we find an explicit analytical expression for the genus zero partition function, in the limit of a vanishing Fermi energy. In section 4 we analyze the phase diagram in terms of the three parameters: the momentum p, the RR flux q and the coupling constant of the momentum modes α. We conclude in section 5 with comments on the approximations used in this paper and some open problems. In appendix A we apply the Lagrange Inversion Formula to obtain and analytic expression for the partition function and comment on its analytic continuation.

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2. Review of the gravitational Sine-Gordon model ($c = 1$ perturbed by momentum modes)

In this section we review Moore's analysis [3]. Similar calculations were also performed in [4], and in [6]. This review should provide much of the notation and the logical framework we will use in the next section.

The Sine-Gordon (SG) model coupled to two-dimensional gravity is given by the following action:

$$S(\mu,\lambda) = \int d^2z\sqrt{\hat{g}}\left(\frac{1}{8\pi}(\nabla\phi)^2 + \frac{\mu}{8\pi\gamma^2}e^{\gamma\phi} + \frac{Q}{8\pi}\phi R(\hat{g})\right) + \int d^2z\sqrt{\hat{g}}\left(\frac{1}{8\pi}(\nabla X)^2 + \lambda e^{\xi\phi}\cos(pX)\right), \tag{2.1}$$

where $\hat{g}$ is a background metric, R its curvature, μ the cosmological constant. The theory is conformal for $\gamma = \sqrt{2}$, $Q = \sqrt{8}$, $\xi = \gamma(1 - |p|/2)$ in the convention where $\alpha' = 1$.

There are various motivations for considering the above problem. Coupling the SG model with two-dimensional gravity helps understand aspects of the SG model as a quantum field theory. For example, properties like the existence of certain RG trajectories are expected to be insensitive to coupling to gravity.

The interpretation that is more along the lines of our interest here is as follows. Consider the free action of two uncompactified real fields ϕ and X:

$$S_{\text{Liouville}} + S_{\text{Gaussian}} = \int d^2z\sqrt{g}\left(\frac{1}{8\pi}(\nabla\phi)^2 + \frac{Q}{8\pi}\phi R(g)\right) + \int d^2z\sqrt{g}\frac{1}{8\pi}(\nabla X)^2. \tag{2.2}$$

It is natural to consider, in the spirit of RG, perturbing this action by an operator of the form

$$\sum_i e^{\xi_i\phi}\mathcal{O}_i, \tag{2.3}$$

where $\mathcal{O}_i$ are operators of the $c = 1$ Gaussian model, the values of ξ_i are selected such that the dressed operator has conformal dimension one. For the choice of $\mathcal{O}_i$ made in (2.1), the problem formulated above is that of *perturbing the* $c = 1$ *model by momentum modes.*

The central object is the partition function defined as

$$Z = \left\langle e^{-S(\mu,\lambda)}\right\rangle. \tag{2.4}$$

In the limit were λ is very small, the techniques of conformal perturbation theory become available to us. Namely, we can think of Z as a series expansion of the form:

$$Z = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\langle\left(\cos pXe^{\xi\phi}\right)^n\rangle_{\lambda=0} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2}\lambda^2)^n}{(n!)^2}\langle(e^{ipX+\xi\phi})^n(e^{-ipX+\xi\phi})^n\rangle_{\lambda=0}. \tag{2.5}$$

As can be seen from the above expression, we will be interested in correlators of the form

$$\left\langle\prod V_{q_i}e^{\frac{1}{2}\lambda(V_p+V_{-p})}\right\rangle \equiv \sum_{n_1,n_2\geq 0}\frac{\lambda^{n_1+n_2}}{2^{n_1+n_2}n_1!n_2!}\left\langle\prod V_{q_i}(V_p)^{n_1}(V_{-p})^{n_2}\right\rangle. \tag{2.6}$$

where

$$V_p = \int d^2z\sqrt{\hat{g}}e^{\xi\phi}e^{ipX}. \tag{2.7}$$

A remarkable aspect of the duality between two-dimensional string theories and matrix models is that one can actually compute all the correlators in (2.6) using matrix models [27, 3].

In the matrix model framework it is convenient to work with rescaled operators and coupling:

$$\mathcal{J}_{\pm p} = \frac{\Gamma(p)}{\Gamma(-p)}V_{\pm p} = \frac{\Gamma(p)}{\Gamma(-p)}\int d^2z\sqrt{g}e^{\xi\phi}e^{\pm ipX}, \quad \alpha = \frac{\Gamma(-p)}{\Gamma(p)}\lambda. \tag{2.8}$$

Note that with the above notation the partition function takes the simple form of:

$$Z = \left\langle e^{\alpha\mathcal{J}_p + \alpha\mathcal{J}_{-p}}\right\rangle. \tag{2.9}$$

In principle, one could proceed to evaluate $\left\langle \mathcal{J}_p^n\mathcal{J}_{-p}^n\right\rangle$ using the prescription of [27]. However, a simpler way to evaluate it is by inserting a zero momentum operator inside the correlators. The simplification comes about due to a couple of observations made in [27]. First, note that introducing $\mathcal{J}_0$ into a correlator is equivalent to differentiating the correlator with respect to μ. Second, since μ always enters as $p + i\mu$ we see that differentiation with respect to μ is equivalent to differentiation with respect to p. Inspecting the general form of the amplitudes in [27] we see that differentiating with respect to momentum turns the θ-functions into δ-functions making integration a simple task. In other words, inserting $\mathcal{J}_0$ into the amplitudes has the advantage of turning a complicated integration into a manageable combinatorial formula presented in appendix A of [3]:

$$\begin{aligned} A_n(\mu,p) &\equiv \mu^{-np}\left\langle \mathcal{J}_0\mathcal{J}_p^n\mathcal{J}_{-p}^n\right\rangle \\ &= i(-1)^n(n!)^2\sum_{k=1}^{n}\frac{(-1)^k}{k}\sum_{a_i,b_i}\left(\prod_{i=1}^{k}b_i^2 - \prod_{i=1}^{k}a_i^2\right)\mathcal{C}(a_1,\ldots,b_k)\prod_{i=1}^{k}\frac{R_{a_ip}R^*_{b_ip}}{(a_i)!^2(b_i)!^2}, \end{aligned} \tag{2.10}$$

where R_p is the bounce factor of $c = 1$ for momentum p. The sum is over all partitions of $n = a_1 + b_1 + \ldots + a_k + b_k$ with $a_i, b_i \geq 0$ such that

$$\mathcal{C}^{-1}(a_1,\ldots,b_k) \equiv (a_1+b_1)(b_1+a_2)(a_2+b_2)\ldots(a_k+b_k)(b_k+a_1), \tag{2.11}$$

is nonzero. The last step in recovering the amplitudes that enter in the partition function involves integrating over μ.

Since we are interested in the genus zero partition function, it is worth considering the asymptotic form of the bounce factor for the $c = 1$ model

$$R_p^{\mu\to\infty} = \exp\left[p\psi + \sum_{n\geq 1}\frac{i^np^{n+1}}{(n+1)!}\left(\frac{d}{d\mu}\right)^n(\log\mu + \psi)\right] = 1 + \frac{ip^2}{2\mu} + \ldots, \tag{2.12}$$

where,

$$\psi \equiv \sum_{k\geq 1}\frac{(-1)^kB_{2k}}{2k}(1 - 2^{-2k+1})\frac{1}{\mu^{2k}}, \tag{2.13}$$

and B_{2k} are the Bernoulli numbers. With this expression for the bounce factor Moore finds that the genus zero amplitude is given by

$$A_n^{h=0}(\mu,p) \equiv \mu^{-np}\langle \mathcal{J}_0 \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle = n!\mu^{-2n+1}\frac{\Gamma(n(1-p)+n-1)}{\Gamma(n(1-p)+1)}(1-p)^n p^{2n}\,. \tag{2.14}$$

Integrating with respect to μ we find that the needed correlators are:

$$\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle = -\mu^{np-2n+2} n! p^{2n}(1-p)^n \frac{\Gamma(n(1-p)+n-2)}{\Gamma(n(1-p)+1)}. \tag{2.15}$$

An insightful way of assembling the answer was presented in [6].[1] Using that

$$\sum_{n\geq 0} \frac{\Gamma(na+b-1)}{n!\Gamma(n(a-1)+b)}(-z)^n = \frac{(1-s)^{b-1}}{b-1}, \quad \text{where} \frac{s}{(1-s)^a} \equiv z, \tag{2.16}$$

one obtains an expression for the susceptibility $\chi = \partial_\mu^2 Z$ of the form:

$$\chi = -\ln\mu + \ln(1-s), \tag{2.17}$$

where in this case

$$z = \alpha^2 p^2 (p-1)\mu^{p-2}, \qquad a = 2-p, \qquad b = 1. \tag{2.18}$$

This expression for the susceptibility can be rewritten as

$$\mu\, e^{\chi} + \alpha^2\, p^2\, (p-1)e^{(2-p)\chi} = 1. \tag{2.19}$$

The main advantage of the above expression is that it allows finding the large α behavior in the region where μ can be turned off. As a bonus, the KPZ scaling in the new coupling [3, 6] can be verified automatically. Namely, we find that in this limit

$$\chi_{\mu=0} = -\frac{1}{2-p}\ln\alpha^2 p^2 (p-1). \tag{2.20}$$

3. Type 0A perturbed by momentum modes

In this section we discuss perturbing two-dimensional type 0A string theory by momentum modes following the techniques of [3, 4].

The matrix model description of type 0A with q unit of fluxes[2] was recently established [2] to be the Jevicki-Yoneya (JY) matrix model [14]. Essentially, this is a matrix model with the following potential:

$$V(x) = -\frac{x^2}{2} + \frac{q^2 - 1/4}{2x^2}. \tag{3.1}$$

[1] In appendix A we included a derivation of this formula as an application of the Lagrange Inversion Formula.

[2] Recently, a clarification of the meaning of q has been given [28] as the sum of the two distinct fluxes and was denoted by $\hat{q}$. A similar interpretation was put forward previously in [18] based on a thermodynamical analysis of the low energy supergravity action.

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This model was solved a decade ago, its non-perturbative S-matrix and the explicit form for some of the amplitudes were discussed in [22, 29].

Our goal is to compute the partition function of the two-dimensional type 0A string theory, *with non-vanishing RR flux*, and in the presence of momentum perturbations. Formally, we would like to compute:

$$Z = \langle \exp(\lambda \cos(pX) e^{\xi\phi}) \rangle_{0A} \,. \tag{3.2}$$

where $\xi = (1-p)$, and λ is the coupling of the momentum perturbation and we now work in the convention where $\alpha' = 1/2$. In practice, however, we lack a worldsheet action analogous to (2.1). Nevertheless, via the string/matrix model correspondence we take the momentum correlators in two-dimensional type 0A to be those of momentum operators in the JY matrix model. Then, we interpret (3.2) as defined with the momentum correlators computed using the matrix models. Within conformal perturbation theory, the partition function is composed of building blocks similar to the $c = 1$ case, that is, the partition function is obtained from correlators of the form $\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$.

3.1 Bounce factor

Our starting point is the bounce factor of the JY matrix model:

$$R(p) = \left(\frac{4}{q^2 + \mu^2 - 1/4} \right)^{p/2} \frac{\Gamma(\frac{1}{2}(1+q+p-i\mu))}{\Gamma(\frac{1}{2}(1+q-p+i\mu))}. \tag{3.3}$$

As in the $c = 1$ model, we build the genus zero partition function in the presence of momentum modes perturbatively in the coupling constant λ, with $\lambda \ll \mu$. This amounts to expanding the bounce factor as a series in inverse powers of μ. However, as a result of having $\mu \to \infty$ in this expansion, the dependence on the RR flux will be washed out. To avoid this we choose an alternative limit in which the RR flux scales with μ:

$$q = \mu f \,. \tag{3.4}$$

Introducing the notation

$$\mu_1 = \frac{\mu}{2}(if + a), \qquad \mu_2 = \frac{\mu}{2}(if - a) \,, \tag{3.5}$$

up to an overall p-independent phase we can rewrite the bounce factor (3.3) as

$$\begin{aligned} R(p) = (-\mu_1\mu_2)^{p/2} \left(\frac{-16}{16\mu_1\mu_2 + 1} \right)^{p/2} \exp \Bigg(& \frac{p}{2}\psi(\mu_1) + \frac{p}{2}\psi(\mu_2) \\ & + \sum_{n \geq 1} \frac{p^{n+1}}{2^{n+1}(n+1)!} (\partial_{\mu_1}^n \ln \mu_1 + (-)^n \partial_{\mu_2}^n \ln \mu_2 + \partial_{\mu_1}^n \psi(\mu_1) + (-)^n \partial_{\mu_2}^n \psi(\mu_2)) \Bigg), \end{aligned} \tag{3.6}$$

where ψ is defined as in (2.13). In (3.5), a is a marker introduced for later purposes, with $a = 1$ its canonical value. Introducing a allows, among other things, to turn off μ (by setting $a = 0$) without turning off the RR flux at the same time. Also, the limit $f = 0$ is expected to bring us back to the $c = 1$ bosonic string.

Expanding in the first few orders in inverse powers of μ we get:

$$\begin{aligned}R(p) = &\ 1 + \frac{i}{2}\frac{a}{\mu(f^2+a^2)}p^2 \\ &- \frac{p}{24\mu^2(f^2+a^2)^2}\left(-7f^2 + a^2 + 4p^2f^2 - 4p^2a^2 + 3p^3a^2\right) \\ &- \frac{i}{48}\frac{a}{\mu^3(f^2+a^2)^3}p^2\left(-8a^2 + pa^2 + 4p^2a^2 - 4p^3a^2 + p^4a^2\right. \\ &\left. + 24f^2 - 7pf^2 - 12p^2f^2 + 4p^3f^2\right) + \mathcal{O}(\mu^{-4})\,. \end{aligned} \tag{3.7}$$

Some consistency checks are in order. First, note that for $f = 0$ we basically return to the $c = 1$ model (setting $a = 1$).[3] In this case, the first line in the above expression coincides with the appropriate result quoted in (2.12). A less trivial consistency check can be obtained as follows. Setting $a = 0$ brings us to the case discussed in [22], where the coupling was identified as $M = (\mu f)^2 - 1/4$. The first interesting observation is that in the expansion of the bounce factor all odd powers of μ^{-1} are proportional to a and therefore vanish in the limit $a \to 0$, in perfect agreement with [22]. Taking a to zero in (3.7) we obtain (including a term not written above)

$$\begin{aligned}R(p) = &\ 1 + \left(\frac{7}{24}p - \frac{1}{6}p^3\right)M^{-1} \\ &+ \frac{p(p-2)}{5760}\left(80p^4 - 128p^3 + 536p^2 + 128p + 510)\right)M^{-2} + \mathcal{O}(M^{-3})\,.\end{aligned} \tag{3.8}$$

This reproduces the expansion of the bounce factor of Demeterfi, Klebanov and Rodrigues [22] (equations (12) and (13)). Notice that in the limit when the free fermion Fermi energy μ is vanishing, and one expands the bounce factor in M, the strength of the deformation in (3.1), the infinite series expansion in (3.8) truncates for even integer values of the momentum.

3.2 Correlators

As mentioned before, the building blocks of the partition function are correlators of the form $\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$. They can be computed directly with the methods developed by [27], but the calculations can soon become rather tedious. Instead, as in [3], it is technically simpler to compute correlators with an extra insertion of a zero-momentum vertex operator which we will denote by A_n. This section is dedicated to their evaluation after which we can proceed with the derivation of the partition function.

3.2.1 $\langle \mathcal{J}_0 \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$ correlators

The A_n amplitudes are related to the correlators we need for the partition function by the following equation:

$$A_n = (\sqrt{\mu^2+M})^{-np}\frac{1}{\mu}\partial_a((\sqrt{\mu^2+M})^{np}\mathcal{R}_{n\to n}) = (\sqrt{\mu^2+M})^{-np}\langle \mathcal{J}_0 \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle\,. \tag{3.9}$$

[3]Strictly speaking the $c = 1$ limit is obtained by setting the deformation in (3.1) to zero, which amounts to setting $f = 1/(2\mu)$ in (3.7) and next re-expanding in $\mu \to \infty$. This will precisely reproduce the bounce factor of the $c = 1$ matrix model. In particular, upon making this substitution, the *a priori* infinite series in (3.7) will truncate to order $1/\mu^n$ for an integer value of the momentum $p = n$.

where $\mathcal{R}_{n\to n} = \sqrt{\mu^2+M}^{-np}\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n\rangle$ is the S-matrix element corresponding to the scattering of n tachyons of equal momenta into n tachyons of equal momenta.

This stems form the observation that, as in the case of the $c = 1$ model, in the bounce factor the dependence on the momentum p arises in the combination $p - i\mu a$, with the exception of the prefactor $\sqrt{\mu^2+M}^{-p}$. Hence we can trade again the differentiation with respect to a for a differentiation with respect to the momentum. The latter, after partial integration, when acting on the Heaviside functions of the integrand, yields delta-functions and so, the net effect is to reduce the evaluation of (3.9) to the same simple algebraic computation according to (2.10). Recall that in (2.10) the insertion of the cosmological constant $\mathcal{J}_0$ inside the correlator was done with the same means of differentiating with respect to μ, and to the same end. Thus, (3.9) can be evaluated as in the $c = 1$ case, using the definition of (2.10), where we insert the asymptotic expansion of the bounce factors (3.6). At the expense of being too explicit but with the hope of exemplifying the simplification achieved by the combinatorial formula quoted in (2.10) and obtained in [3] we list the first few terms:

$$\begin{aligned}
-iA_1 &= R_0R_p^* - R_pR_0^*,\\
-iA_2 &= R_{2p}R_0^* - R_0R_{2p}^* - 2R_p^2R^{*2}_0 + 2R_0^2R^{*2}_p,\\
-iA_3 &= R_0R_{3p}^* - R_{3p}R_0^* + 3R_pR_{2p}^* - 3R_{2p}R_p^* + 9R_pR_{2p}R_0^{*2} - 9R_p^*R_{2p}^*R_0^2\\
&\quad +9R_p^2R_p^*R_0^* - 9R_pR_p^{*2}R_0 - 12R_p^3R_0^{*3} + 12R_0^3R_p^{*3},
\end{aligned} \tag{3.10}$$

where $*$ represents complex conjugation. All we need to do at this point is to substitute the asymptotic expansion for the bounce factor (3.6). We present only the first few genus zero correlators:

$$\begin{aligned}
A_1 &= -\frac{ap^2}{\mu(f^2+a^2)},\\
A_2 &= -2!\frac{ap^4[(p-1)^2a^2+(2p-3)f^2]}{\mu^3(a^2+f^2)^3},\\
A_3 &= -3!\frac{ap^6[(p-1)^3(3p-4)a^4+(13p^3-54p^2+78p-40)a^2f^2+(3p-4)(3p-5)f^4]}{\mu^5(a^2+f^2)^5},\\
&\dots
\end{aligned} \tag{3.11}$$

The higher genera correlators correspond to subleading order terms in $1/\mu^{2n-1+h}$.

We would like to comment on the main difference between the $c = 1$ correlators (which at genus zero are obtained by setting $f = 0$ in (3.11)), and the generic case with both a, f non-vanishing. Namely, in the $c = 1$ case all correlators A_n with $n \geq 2$ vanish for a special value of the momentum, $p = 1$. The reason why this is happening is that for $p = 1$, the $c = 1$ bounce factor is simply $R(p = 1) = 1 + \frac{i}{2\mu}$, and for integer momenta $R(p = n)$ is a degree n polynomial in $1/\mu$. Substituting this into (3.10) one finds that the highest power of $1/\mu$ for a given n is $1/\mu^n$. However, according to KPZ scaling, these correlators should scale with $1/\mu^{2n-1}$. Thus, for $p = 1$, all A_n with $n \geq 2$ must vanish. A short proof by induction shows that $p = 1$ is a zero of order n for the amplitude A_n.

On the other hand, in the 0A case the bounce factor is an infinite series in $1/\mu$ (see (3.7)), and the previous argument does not apply anymore. Indeed, the correlators (3.11) have the right KPZ scaling, and are non-vanishing for $p = 1$ as long as the RR flux is not turned off ($f \neq 0$). It is also worth mentioning that even though there is a similar truncation of the bounce factor that takes place for the 0A bounce factor for even integer values of the momentum, this truncation happens only for $\mu = 0$. In fact, the correlators A_n are odd $2n+1$-point functions which vanish when μ (read a) is zero.

3.2.2 The $\mu \to 0$ limit and $\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$ correlators

As it has been already discussed in the previous section, we obtain the building blocks of the partition function in the presence of momentum modes, $\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$, by performing the integration with respect to a in (3.9).

First, we notice that by setting the RR flux to zero ($f = 0$) we reproduce the correlators of the $c = 1$ model (2.15), as expected. As shown in [3] and reviewed in section 2, at zero RR flux the correlators acquire an expression that can be generalized for all n.

For general values of the Liouville coupling μ and RR flux we have been unable to find a universal expression for all n correlators. Interestingly, there is another limit[4] where such an universal expression can be found. The limit sends the cosmological constant to zero, $\mu \to 0$ (or equivalently $f \gg 1$). In this limit, the correlators which enter in the genus zero 0A partition function can be written as:

$$\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle = -n!(-1)^n q^{np-2n+2}(1-p)p^{2n} \frac{\Gamma(n(2-p)-2)}{\Gamma(n(1-p)+1)}\,, \tag{3.12}$$

where, after taking the limit $f \gg 1$, we reverted to the original notation $\mu f = q$, with q the RR background flux of the two-dimensional type 0A string.[5] We would like to stress that the limit $f \gg 1$ should not be taken prematurely. Even though the correlators (3.11) organize themselves in such a way that in the numerator, which is a polynomial in f, the highest and lowest order term in f can be written as ratios of Euler Γ-functions while the rest of the terms have no apparent structure, as we perform the integral over a all the terms in the numerator are equally contributing to the final result (3.12).

Our formula (3.12) reproduces known results in the literature. Namely, for the 2-point function we obtain:

$$\langle \mathcal{J}_p \mathcal{J}_{-p} \rangle = \frac{1}{2} q^p p\,, \tag{3.13}$$

which coincides with the results of [14, 22, 29]: see for instance eqn (15) in [22]. Recall that the correlators and n-point functions are related by multiplication with leg factors: $< \mathcal{J}_p^n \mathcal{J}_{-p}^n >= \mathcal{R}_{n\to n}(p, \dots p; -p, \cdots -p) q^{2np/2}$. Similarly, we find agreement for the 4-point

[4] This limit was discussed recently by A. Kapustin [24].

[5] Amusingly, the next-to-leading order term in a/f, or equivalently μ/q, has also a universal expression:

$$\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle = -n!(-1)^n q^{np-2n+2} p^{2n} \left((1-p)\frac{\Gamma(n(2-p)-2)}{\Gamma(n(1-p)+1)} + \frac{\mu^2}{2q^2} \frac{\Gamma(n(2-p))}{\Gamma(n(1-p)+1)} \right) \cdots$$

We were unable to organize the other subleading terms in a similar manner.

function (eq. (17) in [22])

$$\langle \mathcal{J}_p^2 \mathcal{J}_{-p}^2 \rangle = q^{2p-2} p^4 \,. \tag{3.14}$$

For comparison the $c = 1$ 2-point function and 4-point function, as given by eqs. (4.17) and (4.40) in [30], are: $\langle \mathcal{J}_p \mathcal{J}_{-p} \rangle = p\mu^p$, and $\langle \mathcal{J}_p^2 \mathcal{J}_{-p}^2 \rangle = p^4(p-1)\mu^{2p-2}$ respectively. The difference between the 0A 4-point function [14, 22, 29] (3.14) and the $c = 1$ model result is reflected in the different dependence on $(1-p)$ encoded in the 0A generic formula (3.12) vs. (2.14).

Note that the role of the genus expansion which was originally played by μ is now played by q in precise agreement with the KPZ scaling. An interesting point to address is that of the order of limits. In the original works of [22, 29] the strategy was to set the cosmological constant to zero at the beginning of the calculations. This was also suggested in works by Jevicki and Yoneya [14]. Here, and in the approximation considered by Kapustin, we have started with a nonzero cosmological constant (nonzero a) and obtained a formula in the limit of large flux which is basically $f/a \gg 1$. In the end, we have found that the expression derived for the 2n-point functions $\langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$ is not sensitive to the order of limits. This independence of the order of limits hints to the existence of a deeper relation between the couplings μ and q beyond the extreme limits when either of them is effectively zero.

3.3 Partition function

The sum

$$\sum_n \frac{(\alpha^2)^n}{2^n n!^2} \langle \mathcal{J}_p^n \mathcal{J}_{-p}^n \rangle$$

can be performed after first differentiating twice with respect to q and using (2.16). Thus, upon taking the limit $\mu \to 0$, the 0A string partition function admits an analytic expression

$$\partial_q^2 Z = \partial_q^2 Z_{n=0} + (1-p) \sum_{n \geq 1} \frac{1}{n!} (-q^{p-2} p^2 \alpha^2)^n \frac{\Gamma(n(2-p))}{\Gamma(n(1-p)+1))} = -\ln\, q + (1-p) \ln(1-s) \,, \tag{3.15}$$

where, for us,

$$\frac{s}{(1-s)^{2-p}} = q^{p-2} p^2 \alpha^2 \equiv z \,. \tag{3.16}$$

Let us contrast the current situation with the $c = 1$ model [3, 6]. While z in the $c = 1$ model could have been positive for $p > 1$, or negative for $p < 1$, in our case we see that z is always positive. Moreover, now z varies monotonically with s for all $0 < p < 2$. Recall that in the $c = 1$ string one had to distinguish between a monotonic z behavior with s for $1 < p < 2$, and a non-monotonic one for $0 < p < 1$. In the latter case, z was bounded by a critical value Z_c reached for $(dz/ds)|_{Z_c} = 0$ for $p < 1$. In the vicinity of the extremum, one finds the susceptibility $\chi = \partial_\mu^2 Z$ being proportional to $(z - Z_c)^2$, behavior that is characteristic to a $c = 0$ system. The physical interpretation is that the $c = 1$ field X decouples by settling into the minima of the cosine potential corresponding to the turning on of the momentum modes. Thus $p = 1$ is a critical point associated with the phase transition from the $c = 1$ string to a $c = 0$ model coupled to gravity. We will soon see that this decoupling is absent in our case.

Returning to the two-dimensional 0A string, and similarly defining $\chi = \partial_q^2 Z$, we find that χ obeys:

$$q^{\frac{1}{1-p}}\, e^{\frac{1}{1-p}\chi} + \alpha^2 p^2 q^{\frac{p(2-p)}{1-p}}\, e^{\frac{2-p}{1-p}\chi} = 1\,. \tag{3.17}$$

Sending $\alpha \to 0$ in the above expression brings us back to the expected answer

$$\chi_{\alpha=0} = -\ln q\,. \tag{3.18}$$

Alternatively, we can directly explore the limit $q \to \infty$ (instead of $\alpha \to 0$), by redefining $\chi = -\ln q + \hat{\chi}$, with $\hat{\chi}$ finite for large flux and constrained by

$$1 = e^{\frac{\hat{\chi}}{1-p}} + \alpha^2 p^2 q^{p-2} e^{\frac{2-p}{1-p}\hat{\chi}}\,. \tag{3.19}$$

However, the limit that we are interested in is $\alpha \gg q$, or equivalently $q \to 0$. This regime can be probed by exploiting the analyticity of the equation (3.17) which allows us to re-expand the partition function around a background provided by the momentum modes. It is clear from (3.17) that sending $q \to 0$ cannot be done without assuming that χ blows up at the same time. More precisely we need $\chi = -p\ln q + \tilde{\chi}$, with $\tilde{\chi}$ defined by

$$1 = q e^{\frac{\tilde{\chi}}{1-p}} + \alpha^2 p^2 e^{\frac{2-p}{1-p}\tilde{\chi}}\,. \tag{3.20}$$

We can accomplish the re-expansion of the partition function in a regime where $\alpha \gg q$ by simply observing that the small expansion parameter z in (3.15) corresponds to $s \approx 0$, while a large z corresponds to $s \approx 1$. Therefore, to expand around large z, all that is needed is to replace the term $\ln(1-s)$ in (3.15) by $\ln(s) \equiv \ln(1-t)$. Solving for t yields

$$t/(1-t)^{1/(2-p)} = z^{-1/(2-p)} \equiv y\,. \tag{3.21}$$

Using that $\tilde{\chi}(2-p)/(1-p) = \ln(1-t)$, the relation between the function $\tilde{\chi}$ and the new variable y is given by

$$y = e^{-\frac{1}{1-p}\tilde{\chi}(y)} - e^{\tilde{\chi}(y)}\,. \tag{3.22}$$

Furthermore, from

$$F = \left(p^{\frac{2}{2-p}}\alpha^{\frac{2}{2-p}}\right)^2 \left[-\frac{py^2}{2}\ln(y\alpha^{\frac{2}{2-p}}p^{\frac{2}{2-p}}) + y^2\frac{p-1}{2-p}\ln(\alpha p) + f(y)\right], \tag{3.23}$$

where $\partial_y^2 f = \tilde{\chi}(y)$, we finally arrive at the sought-after expression of the genus zero partition function of the two-dimensional type 0A string theory, in a momentum mode background:

$$\begin{aligned} F = {} & q^2\left(\frac{p}{2}\ln q + \frac{p-1}{2-p}\ln(\alpha p)\right) \\ & - \frac{\left(p^{\frac{2}{2-p}}\alpha^{\frac{2}{2-p}}\right)^2}{4}\left[1+(-4p^2+4p\tilde{\chi}+4p-4)e^{\frac{p\tilde{\chi}}{p-1}}+(3p^2-3p-2p\tilde{\chi})e^{\frac{2\tilde{\chi}}{p-1}}+(3p-2\tilde{\chi}p)e^{2\tilde{\chi}}\right]. \end{aligned} \tag{3.24}$$

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From its definition (3.22), one finds that $\tilde{\chi} \to 0$ as $y \to 0$. Thus, for $q \to 0$, the partition function behaves as

$$F = \frac{(p-2)^2}{4} p^{\frac{2+p}{2-p}} \alpha^{\frac{4}{2-p}} + \mathcal{O}(\mu)\,. \tag{3.25}$$

Let us compare our result with a somehow similar situation: the momentum perturbation of the $c = 1$ model. In [3, 4] and [6] this perturbation was studied and the same KPZ scaling of $\alpha^{\frac{2}{2-p}}$, and $\alpha \gg 1$ was obtained. The result is known as the sine-Liouville theory. Our situation generalizes the previous one in that we start with $\hat{c} = 1$ with RR flux q and zero cosmological constant μ. In the regime where the momentum perturbations become relevant, the original $\hat{c} = 1$ with $(q \neq 0, \mu = 0)$ theory is driven into a new phase which can, at best, be described as a generalization of the sine-Liouville theory. This is inferred from equation (3.25), where wee see that in the regime where the strength of the perturbation, α, sets the scale, we uncover the sine-Liouville KPZ scaling.

It is worth mentioning that using the underlying integrable structure of the 0A matrix model, the derivation of the momentum perturbed partition function is not restricted by perturbation methods to considering one of the cases: either $\mu = 0$ (zero cosmological constant) or $q = 0$ (zero RR flux). This leads to a more detailed picture of the phase diagram of type 0A at arbitrary values of μ and q [31]. The phase transition to a sine-Liouville type theory is confirmed in that picture.

4. Phase diagram

In this section we consider the phase diagram in the (α, p) plane. A natural set of variables for addressing this question are p and z. Basically, $z = 0$ corresponds to the absence of momentum perturbation, that is, to $\alpha = 0$. We are interested in the behavior of the partition function as $z \to \infty$ and in particular will look for singularities as we cover the range of couplings.

Given that

$$z = \alpha^2 \, p^2 \, q^{p-2}, \tag{4.1}$$

we are limited to the region of positive z for all values of p.

Varying z from zero to infinity can be achieved by varying s. Note that the relation between z and s is monotonous. Indeed, using (3.16) we conclude that

$$\partial_s z = \frac{1 + s(1-p)}{(1-s)^{3-p}}. \tag{4.2}$$

Monotonicity breaks when the above expression becomes zero, which happens for

$$s_c = 1/(p-1). \tag{4.3}$$

For $p < 1$ we have $s_c < 0$ and negative s_c implies negative z through (3.16) but this is outside the range of z, which we consider to be positive. For $1 < p < 2$, we have that $s_c > 1$ which is also outside allowed domain for s and z.

Thus, we verify that there is a monotonous relation between z and s and that it is possible to vary z without obstruction in the full range $0 \leq z < \infty$ by taking $0 \leq s < 1$.

In the language of the coupling α, this means that we can vary it in the range $0 \le \alpha < \infty$ with no obstruction, as long as $0 < p < 2$. The expansion for small α, that is, small z, is given by formula (3.15), whereas the expansion for large α is given by,

$$\chi = -p \ln q - \frac{1-p}{2-p} \ln(p^2\alpha^2) + \frac{1-p}{2-p} \sum_{n\ge1} \frac{1}{n!} \frac{\Gamma(\frac{n}{2-p})}{\Gamma(\frac{n}{2-p} - n + 1))} \left(- \frac{q}{p^{\frac{2}{2-p}} \alpha^{\frac{2}{2-p}}} \right)^n , \tag{4.4}$$

where we have introduced the appropriate small parameter (3.21). In appendix A we complement this analysis with a more explicit discussion.

To conclude, let us present an alternative analysis of the phase structure of the partition function. Here we will follow some of the standard techniques for studying series convergence which where applied to the $c = 1$ case in [3]. The main object is the function

$$H(p; z) \equiv \sum_{n=1}^{\infty} \frac{\Gamma(n(2-p))}{n!\Gamma(n(1-p)+1)} z^n . \tag{4.5}$$

The radius of convergence is

$$|z| < R_c = \exp\left((p-2)\ln|p-2| - (p-1)\ln|p-1|\right) . \tag{4.6}$$

There are basically four regions, recall that in our case $z = \alpha^2\, q^{p-2}\, p^2 \ge 0$:

I. $0 < p < 2, \quad 0 \le \alpha^2 q^{p-2} < R_c/p^2$

II. $2 < p < \infty, \quad 0 \le \alpha^2 q^{p-2} < R_c/p^2$

III. $0 < p < 2, \quad \alpha^2 q^{p-2} > R_c/p^2$

IV. $2 < p < \infty, \quad \alpha^2 q^{p-2} > R_c/p^2$

In contrast with the phase diagram of the $c = 1$ model perturbed by momentum modes (Sine-Liouville), two of the phase space regions, distinguished by $0 < p < 1$ and $1 < p < 2$ have coalesced (recall that in our case z stays always positive). As a consequence, the phase transition of the $c = 1$ string in a momentum modes background to the $c = 0$ model coupled to gravity, which took place at $p = 1$, has disappeared from the phase diagram the 0A string.

We have included regions II and IV for completeness. Region II has a singularity but it is expected since it corresponds to non-normalizable α perturbation, that is, an irrelevant perturbation which in the string theory diverges as $\phi \to \infty$ rather than dying off. The partition function in region III is to be computed using eq. (4.4). Remarkably similar formulas were obtained in [3] for regions II and IV.

5. Conclusions

Let us comment on some aspects of our calculations and some interesting open problems.

There are several approximations which we had to make in order to arrive at an analytic answer. One particular point that one would like to improve on is relaxing the condition

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of large flux. Note that in this sense we differ from previous results in the literature where the vanishing flux limit was taken [32, 33]. We used perturbative techniques to arrive at an expression for the genus zero two-dimensional 0A partition function perturbed by momentum modes, in the limit of vanishing cosmological constant μ. Exploiting the analyticity of our result we were able to probe regions characterized by arbitrary values of the RR flux q and momentum modes coupling constant α. We explicitly check the existence of a perturbative expansion around large values of α, corresponding to a condensation of momentum modes. The phase diagram analysis shows that for momentum values below 2, such that the momentum mode vertex operator remains relevant, the phase transition to a $c = 0$ system coupled to gravity is absent and there is no obstruction to turning on an arbitrarily large value of α. It would be interesting to study the problem for generic values of μ, q. One would hope that the analysis at intermediate values of q/μ would perhaps uncover a richer phase structure.

We would like to point out the benefits of keeping the Fermi level μ non-vanishing in the intermediate stages of our calculation, even though ultimately we had to assume the limit $\mu \ll 1$. Sending μ to zero prematurely would have left us with only one means of evaluating the two-dimensional 0A correlators $\langle \mathcal{J}^n_p \mathcal{J}^n_{-p} \rangle$, namely integrating the loop momentum following [27]. Instead, keeping μ non-vanishing allows differentiating the correlators with respect to μ, and subsequently turning a tedious integral into a simple algebraic expression, as in [3].

In a sense our calculation can be viewed as part of a more general conjecture mirroring that of Fateev, Zamolodchikov and Zamolodchikov [34]. The FZZ conjecture states (as presented in [6]) that the SL(2)/ U(1) coset CFT, which contains the 2-d black hole, is equivalent to the Sine-Liouville model, $c = 1$ CFT coupled to a Liouville field, with the cosmological constant tuned to zero and the scale set by the winding mode of the $c = 1$ field. We are lead to discuss this relation in the presence of RR flux on both sides of the correspondence. It was conjectured in [35] that in the presence of RR flux perhaps the coset CFT is replaced by a version of gauged WZW models. The Sine-Liouville action now clearly contains the RR flux. Our computation pertains to the limit where the RR flux is large compared to the cosmological constant. It would be interesting to investigate the precise formulation of the conjecture in the presence of fluxes q.

We hope that our results will shed light into the integrable structure of type 0A. In fact, we have partially studied the perturbation by momentum in the framework of the string equation and will report on our findings in an upcoming work [36].

Recently [28] have discussed the finite temperature partition functions for 0A and 0B establishing T duality explicitly. It would be interesting to consider the extension of our work to the Euclidean case when the X field lives in a circle as well as its 0B counterpart. We hope to return to some of the fascinating issues in perturbing two-dimensional string theories with momentum and winding operators.

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A. The Lagrange Inversion Formula applied to the partition function

In this appendix we showed that the main formula used in body of the paper repeatedly ((2.16) and (3.15)) follows as a direct application of a theorem due to Lagrange [37]. Our discussion follows [38, 39].

Theorem. *(The Lagrange Inversion Formula) Let $f(z)$ and $\phi(z)$ be functions of z analytic on and inside a contour $\mathcal{C}$ surrounding a point a, and let t be such that the inequality*

$$|t\phi(z)| < |z-\alpha|, \tag{A.1}$$

is satisfied at all points z on the perimeter of $\mathcal{C}$. Then the equation

$$\xi = \alpha + t\phi(\xi), \tag{A.2}$$

as an equation in ξ has one root in the interior of $\mathcal{C}$; and further any function of ξ analytic on and inside $\mathcal{C}$ can be expanded as a power series in t by the formula

$$f(\xi) = f(\alpha) + \sum_{n=1}^{\infty} \frac{t^n}{n!}\,\frac{d^{n-1}}{dx^{n-1}}\Big[f'(x)(\phi(x))^n\Big]_{x=\alpha}. \tag{A.3}$$

The case we are interested is basically

$$y = 1 - z\,y^a, \tag{A.4}$$

In the formula (A.3) we simply have $f(y) = \ln(y)$ and $\phi(y) = y^a$ and obtain

$$\ln y = -z + \frac{2a-1}{2}z^2 - \frac{(3a-1)(3a-2)}{6}z^3 + \frac{(4a-1)(4a-2)(4a-3)}{24}t^4 \ldots \tag{A.5}$$

which can be rewritten as

$$\ln y = \sum_{n=1}^{\infty} \frac{\Gamma(na)}{n!\Gamma(n(a-1)+1)}(-z)^n, \qquad \text{with} \quad y = 1 - zy^a. \tag{A.6}$$

This is the formula used in the main body of the paper (2.16) and (3.15) with the minor substitution of $y = 1-s$ and for the case of $a = 2-p$.

Let us now discuss the regime of validity of the above expression and its possible continuation. The above expansion (A.6) is valid for

$$|z| < |(a-1)^{a-1}a^{-a}|, \tag{A.7}$$

which coincides with the radius of convergence given in section 4 by equaiton (4.6). Having identified the series in z with $\ln y$, one has a perfect analytic expression near $y = 1$ for the partition function. Now we can analytically continue the natural logarithm. The only problem is with the branch cut $(-\infty, 0]$. However, as explained in the main body, we are interested in $z \in [0,\infty)$ which corresponds to $y \in (0,1]$. Recall that the singularity in the $c = 1$ case reviewed in section 2 appears because z takes negative values for $p < 1$.

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