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Comments on charges and near-horizon data of black rings

Kentaro Hanaki

Department of Physics, University of Michigan,
Ann Arbor, MI 48109-1120, U.S.A.
E-mail: hanaki@umich.edu

Keisuke Ohashi

DAMTP, Centre for Mathematical Sciences, Cambridge University,
Wilberforce Road, Cambridge CB3OWA, U.K.
E-mail: K.Ohashi@damtp.cam.ac.uk

Yuji Tachikawa

School of Natural Sciences, Institute for Advanced Study,
Princeton, New Jersey 08540, U.S.A.
E-mail: yujitach@ias.edu

Abstract: We study how the charges of the black rings measured at the asymptotic infinity are encoded in the near-horizon metric and gauge potentials, independent of the detailed structure of the connecting region. Our analysis clarifies how different sets of four-dimensional charges can be assigned to a single five-dimensional object under the Kaluza-Klein reduction. Possible choices are related by the Witten effect on dyons and by a large gauge transformation in four and five dimensions, respectively.

Keywords: Black Holes, Black Holes in String Theory, Classical Theories of Gravity.

Contents

1. Introduction

One of the achievements of string/M theory is the microscopic explanation for the Bekenstein-Hawking entropy for a class of four-dimensional supersymmetric black holes [1, 2]. The microscopic counting predicts subleading corrections to the entropy, which can also be calculated from the macroscopic point of view, i.e. from stringy modifications to the Einstein-Hilbert Lagrangian [3]. Comparison of the two approaches has proven to be very fruitful, e.g. it has led to the relation to the partition function of topological strings [4]. Beginning in ref. [5], it has been also generalized to non-supersymmetric extremal black holes using the fact that the near-horizon geometry has enhanced symmetry. The analysis has also been extended to rotating black holes [6].

There is a richer set of supersymmetric black objects in five dimensions, including black rings [7], on which we focus. The entropy is still given by the area law macroscopically to leading order, and it can be understood microscopically using a D-brane construction [8, 9].

The understanding of higher-derivative corrections remains more elusive [10 − 12]. One reason for this is that the supersymmetric higher-derivative terms were not known until quite recently [13]. Even with this supersymmetric higher-derivative action, it has been quite difficult to construct the black ring solution embedded in the asymptotically flat spacetime, and it is preferable if we can only study the near horizon geometry. Then the problem is to find the charges carried by the black ring from its data at the near-horizon region.

The usual approach taken in the literature so far is to consider the dimensional reduction along a circle down to four dimensions, and to study the charges there [12, 14 − 16]. Then, the attractor mechanism fixes the scalar vacuum expectation values (vevs) and the metric at the horizon by the electric and magnetic charges [17, 18]. Conversely, the magnetic charge can be measured by the flux, and the electric charge can be found by taking the variation of the Lagrangian by the gauge potential. In this way, the entropy as a function of charges can be obtained from the analysis of the near-horizon region alone [5, 6]. Nevertheless, it has not been clarified how to reconcile the competing proposals [8, 9, 19 − 21] of the mapping between the four- and five-dimensional charges of the black rings embedded in the asymptotically flat spacetime.

Thus we believe it worthwhile to revisit the identification of the charges directly in five dimensions, with local five-dimensional Lorentz symmetry intact. It poses two related problems because of the presence of the Chern-Simons interaction in the Lagrangian. One is that, in the presence of the Chern-Simons interaction, the equation of motion of the gauge field is given by

$$d \star F = F \wedge F, \tag{1.1}$$

which means that the topological density of the gauge field itself becomes the source of electric charge. To put it differently, the attractor mechanism for the black rings [22] determines the scalar vevs at the near-horizon region via the magnetic dipole charges only, and the information about the electric charges seems to be lost. Then the electric charge of a black ring seems to be diffusely distributed throughout the spacetime. Eq.(1.1) can be rewritten in the form

$$d(\star F - A \wedge F) = 0, \tag{1.2}$$

then $\int_\Sigma (*F - A \wedge F)$ is independent of Σ. This integral is called the Page charge. Similar analysis can be done for angular momenta, and Suryanarayana and Wapler [23] obtained a nice formula for them using the Noether charge of Wald.

There is a second problem remaining for black rings, which stems from the fact that A is not a well-defined one-form there because of the presence of the magnetic dipole. It makes $\int_\Sigma (\star F - A \wedge F)$ ill-defined, because in the integral all the forms are to be well-defined. The same can be said for the angular momenta. The aim of this paper is then to show how this second problem can be overcome, and to see how the near-horizon region of a black ring encodes its charges measured at the asymptotic infinity.

In section 2, we use elementary methods to convert the integral at the asymptotic infinity to the one at the horizon. We apply our formalism to the supersymmetric black ring and check that it correctly reproduces known values for the conserved charges. We will show how the gauge non-invariance of $\int A \wedge F$ can be solved by using two coordinate patches and a compensating term along the boundary of the patches. Then in section 3 we will see that our viewpoint helps in identifying the relation of the charges under the Kaluza-Klein reduction along S^1. We will see that the change in the charges under a large gauge transformation in five dimensions maps to the Witten effect on dyons [24] in four dimensions. Proposals in the literature [8, 9, 19–21] will be found equivalent under the transformation. We conclude with a summary in section 4. In appendix A the geometry of the concentric rings is briefly reviewed.

2. Near-horizon data and conserved charges

To emphasize essential physical ideas, we discuss the problem first for the minimal supergravity in five dimensions. Later in this section we will apply the technique to the case with vector multiplets. The bosonic part of the Lagrangian of the minimal supergravity theory is

$$S = \frac{1}{8\pi G} \int \left(\frac{1}{2} \star R - F \wedge \star F - \frac{4}{3\sqrt{3}} A \wedge F \wedge F \right). \tag{2.1}$$

Our metric is mostly plus, and $R_{\mu\nu}$ is defined to be positive for spheres. We define the Hodge star operator for an n-form as

$$\star \left(dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_{n-1}} \right) = \frac{\sqrt{-g}}{(5-n)!} \epsilon^{\mu_0 \cdots \mu_{n-1}}{}_{\mu_n \cdots \mu_4} dx^{\mu_n} \wedge \cdots \wedge dx^{\mu_4} \,. \tag{2.2}$$

with the Levi-Civita symbol $\epsilon_{01234} = +1$ and $\epsilon^{01234} = -1$ defined in local Lorentz coordinates. The equations of motion are

$$R_{\mu\nu} = -\frac{1}{3} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + 2 F_{\mu\rho} F_{\nu}{}^{\rho}, \tag{2.3}$$

$$d \star F = -\frac{2}{\sqrt{3}} F \wedge F. \tag{2.4}$$

2.1 Electric charges

From the equation of motion of the gauge field (2.4), we see that $F \wedge F$ is the electric current for the charge $\int \star F$. Thus, the charge is distributed diffusely in the spacetime as was emphasized e.g. in [25]. However, the equation (2.4) can also be cast in the form

$$d \left(\star F + \frac{2}{\sqrt{3}} A \wedge F \right) = 0. \tag{2.5}$$

At the asymptotic infinity, $A \wedge F$ decays sufficiently rapidly, so that we have

$$\int_\infty \star F = \int_\infty \left(\star F + \frac{2}{\sqrt{3}} A \wedge F \right) = \int_\Sigma \left(\star F + \frac{2}{\sqrt{3}} A \wedge F \right), \tag{2.6}$$

where the subscript ∞ indicates that the integral is taken at S^3 at the asymptotic infinity, and Σ is an arbitrary three-cycle surrounding the black object. Thus we can think of the electric charge as the integral of the quantity inside the bracket, which is called the Page charge.

JHEP12(2007)057

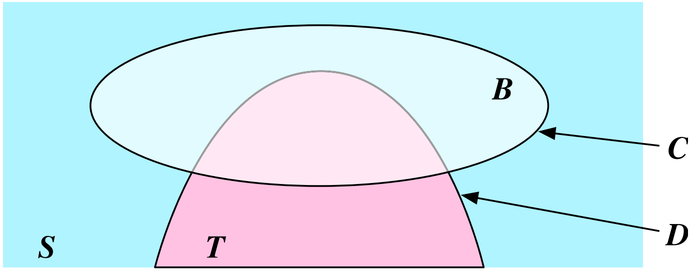

Figure 1: Coordinate patches used to define $\int_C A \wedge F$ consistently without ambiguity.

One problem about the Page charge is that, even in the case where A is a globally defined one-form, it changes its value under a large gauge transformation. It is completely analogous to the fact that $\int_C A$ for an uncontractible circle C is only defined up to an integral multiple of 2π under a large gauge transformation. Indeed, let us parametrize C by $0 \leq \theta \leq 2\pi$ and we perform a gauge transformation by $g(\theta) \in U(1)$, i.e. we change A to $A + i^{-1}g^{-1}dg$. Such a continuous $g(\theta)$ can be written as $g(\theta) = \exp(i\phi(\theta))$. Then $\int_C A$ changes by $\int_C d\phi(\theta) = \phi(2\pi) - \phi(0)$, which can jump by a multiple of 2π. Thus, $\int_C A$ is invariant under a small gauge transformation $\phi(0) = \phi(2\pi)$ but is not under a large gauge transformation $\phi(0) \neq \phi(2\pi)$. Exactly the same analysis can be done for $\int_\Sigma A \wedge F$, and it changes under a large gauge transformation along C if Σ contains intersecting one-cycle C and two-cycle S and $\int_S F \neq 0$. However, this non-invariance under a large gauge transformation poses no problem if Σ is at the asymptotic infinity of the flat space, because we usually demand that A should decay sufficiently rapidly there, which removes the freedom to do a large gauge transformation.

These facts are well-known, and have been utilized previously e.g. in [14]. It is the manifestation of the fact that there are several notions of electric charges in the presence of Chern-Simons interactions, as clearly discussed by Marolf in ref. [26]. One is the Maxwell charge which is gauge-invariant but not conserved, and another is the charge which is conserved but not gauge-invariant. In our case $\int \star F$ is the Maxwell charge and $\int(\star F + (2/\sqrt{3})A \wedge F)$ is the Page charge. Yet another notion of the charge is the quantity which generates the symmetry in the Hamiltonian framework, which can be constructed using Noether's theorem and its generalization by the work of Wald and collaborators [27 – 30]. The charge thus constructed is called the Noether charge, and in our case it agrees with the Page charge.

Unfortunately, the manipulation above cannot be directly applied to the black rings with dipole charges. It is because A is not a globally well-defined one-form, and the integrals are not even well-defined. The way out is to generalize the definition of $\int_C A \wedge F$ to the case A is a $U(1)$ gauge field defined using two coordinate patches, so that

$$\int_B F \wedge F = \text{“}\int_{\partial B} A \wedge F\text{”} \tag{2.7}$$

holds. Then the manipulation (2.6) makes sense. The essential idea is to introduce a term localized in the boundary of the patches which compensates the gauge variation. Copsey and Horowitz [31] used similar subtlety associated to the gauge transformation between patches to study how the magnetic dipole enters in the first law of the black rings.

Let us assume the whole spacetime is covered by two coordinate patches, S and T, see figure 1. We denote the boundary of two regions by $D = \partial S = -\partial T$. The gauge field A is represented as well-defined one-forms A_S and A_T on the patches S and T, respectively. These two are related by a gauge transformation, $A_S = A_T + \beta$ with $d\beta = 0$ on the boundary D. Suppose the region B has the boundary $C = \partial B$. Then we have

$$\begin{aligned}\int_B F\wedge F &= \int_{B\cap S} F\wedge F + \int_{B\cap T} F\wedge F && (2.8)\\ &= \int_{C\cap S + D\cap B} A_S\wedge F + \int_{C\cap T - D\cap B} A_T\wedge F && (2.9)\\ &= \left(\int_{C\cap S} A_S\wedge F + \int_{C\cap T} A_T\wedge F\right) + \int_{D\cap B}(A_S\wedge F - A_T\wedge F) && (2.10)\\ &= \left(\int_{C\cap S} A_S\wedge F + \int_{C\cap T} A_T\wedge F\right) + \int_{C\cap D} A_S\wedge\beta. && (2.11)\end{aligned}$$

Now we define the symbol $\int_M A\wedge F$ for a three-cycle M to mean

$$\text{“}\int_M A\wedge F\text{”} \equiv \int_{M\cap S} A_S\wedge F + \int_{M\cap T} A_T\wedge F + \int_{D\cap M} A_S\wedge\beta, \tag{2.12}$$

then the relation (2.7) holds as is. The important point here is that we need a term $\int_{D\cap M} A_S\wedge\beta$ which compensates the gauge variation localized at the boundary of the coordinate patches.

One immediate concern might be the gauge invariance of the definition (2.12), but it is guaranteed for $C = \partial B$ from the very fact the relation (2.7) holds. It is because its left hand side is obviously gauge invariant. For illustration, consider the case $\partial B = C_1 - C_2$. The Page charges measured at C_1, C_2 themselves are affected by a large gauge transformation, but their difference is not. When one takes C_1 as the asymptotic infinity, it is conventional to set the gauge potential to be zero there, thus fixing the gauge freedom. Then the Page charge at the cycle C_2 is defined without ambiguity.

In the following, we drop the quotation marks around the generalized integral “$\int A\wedge F$”. We believe it does not cause any confusion.

2.2 Angular momenta

The technique similar to the one we used for electric charges can be applied to the angular momenta, and we can obtain a formula which expresses them by the integral at the horizon. There is a general formalism, developed by Lee and Wald [27], which constructs the appropriate integrand from a given arbitrary generally-covariant Lagrangian, and the expression for the angular momenta was obtained in [23, 32]. Instead, here we will construct a suitable quantity in a more down-to-earth and direct method. We will see that the integrand contains the gauge field A without the exterior derivative, and that it is

JHEP12(2007)057

ill-defined in the presence of magnetic dipole. We will use the technique developed in the last section to make it well-defined.

Firstly, the angular momentum corresponding to the axial Killing vector ξ can be measured at the asymptotic infinity by Komar's formula

$$J_\xi = -\frac{1}{16\pi G}\int_\infty \star\nabla\xi, \tag{2.13}$$

where $\nabla\xi$ is an abbreviation for the two-form $\nabla_\mu\xi_\nu dx^\mu \wedge dx^\nu = d\xi$. Using the Killing identity, the divergence of the integrand is given by

$$d\star\nabla\xi = 2\star R_{\mu\nu}\xi^\mu dx^\nu, \tag{2.14}$$

which vanishes in the pure gravity. Thus, the angular momentum of a black object of the pure gravity theory can be measured by $\int_S \star\nabla\xi$ for any surface S which surrounds the object.

Let us analyze our case, where the equations of motion are given by (2.3) and (2.4). We need to introduce some notations: $\mathcal{L}_\xi$ denotes the Lie derivative along the vector field ξ, $\iota_\xi\omega$ denotes the interior product of a vector ξ to a differential form ω, i.e. the contraction of the index of ξ to the first index of ω. Then $\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d$ when it acts on the forms. For a vector ξ and a one-form A, we abbreviate $\iota_\xi A$ as $(\xi\cdot A)$.

We will take the gauge where gauge potentials are invariant under the axial isometry $\mathcal{L}_\xi A = 0$. It can be achieved by averaging over the orbit of the isometry ξ. We furthermore assume that every chain or cycle we use is invariant under the isometry ξ, then any term of the form $\iota_\xi(\cdots)$ vanishes upon integration on such a chain or cycle.

Under these assumptions, the difference of the integral of $\star\nabla\xi$ at the asymptotic infinity and at C is evaluated with the help of the Einstein equation (2.3) to be

$$\int_\infty \star\nabla\xi - \int_C \star\nabla\xi = 2\int_B \star R_{\mu\nu}\xi^\mu dx^\nu = 4\int_B (\iota_\xi F)\wedge\star F \tag{2.15}$$

where B is a hypersurface connecting the asymptotic infinity and C. We dropped the term $\int \iota_\xi(\star F^2)$ because it vanishes upon integration.

The right hand side can be partially-integrated using the following relations: one is

$$d\left[\star(\xi\cdot A)F\right] = -(\iota_\xi F)\wedge\star F - (\xi\cdot A)\frac{2}{\sqrt{3}}F\wedge F \tag{2.16}$$

and another is

$$d\left[(\xi\cdot A)A\wedge F\right] = (\xi\cdot A)F\wedge F - (\iota_\xi F)\wedge A\wedge F \tag{2.17}$$

$$= \frac{3}{2}(\xi\cdot A)F\wedge F - \frac{1}{2}\iota_\xi(A\wedge F\wedge F) \tag{2.18}$$

of which the last term vanishes upon integration. Thus we have

$$dX_\xi[A] = -(\iota_\xi F)\wedge\star F \tag{2.19}$$

modulo the term of the form $\iota_\xi(\cdots)$, where

$$X_\xi[A] \equiv \star(\xi \cdot A)F + \frac{4}{3\sqrt{3}}(\xi \cdot A)A \wedge F. \tag{2.20}$$

$X_\xi[A]$ is not a globally well-defined form. Thus, to perform the partial integration of the right hand side of (2.19), compensating terms along the boundary of the coordinate patches need to be introduced, just as we did in the previous section in the analysis of the Page charge.

Let S and T be two coordinate patches, $D = \partial S = -\partial T$ be their common boundary, and $A_S = A_T + \beta$ as before. Let us call the correction term $Y_\xi[\beta, A_S]$ and we define

$$\int_M X_\xi[A] \equiv \int_{M\cap S} X_\xi[A_S] + \int_{M\cap T} X_\xi[A_T] + \int_{M\cap D} Y_\xi[\beta, A_T]. \tag{2.21}$$

We demand that it satisfies

$$\int_{\partial B} X_\xi[A] = \int_B (\iota_\xi F) \wedge \star F. \tag{2.22}$$

Then $Y[\beta, A]$ should solve

$$dY_\xi[\beta, A_T] = X_\xi[A_S] - X_\xi[A_T]. \tag{2.23}$$

The right hand side is automatically closed since $dX_\xi[A]$ is gauge invariant. Thus the equation above should have a solution if there is no cohomological obstruction. Indeed, substituting (2.20) in the above equation, we get

$$Y_\xi[\beta, A_T] = (\xi \cdot \beta)Z - \frac{2}{3\sqrt{3}}\left[2(\xi \cdot \beta)\beta \wedge A_T + (\xi \cdot A_T)\beta \wedge A_T\right] \tag{2.24}$$

modulo $\iota_\xi(\cdots)$, where dZ should satisfy

$$dZ = \star F + \frac{2}{\sqrt{3}} A_T \wedge F, \tag{2.25}$$

the right hand side of which is closed using the equation of motion (2.4). Unfortunately there seems to be no general way to write Z as a functional of A and β. We need to choose Z by hand for each on-shell configuration. With these preparation, we can finally integrate the right hand side of (2.15) partially and conclude that

$$\int_C \left(\star\nabla\xi + 4X_\xi[A]\right). \tag{2.26}$$

is independent under continuous deformation of C.

Taking C to be the 3-sphere at the asymptotic infinity, the terms $X[A]$ vanish too fast to contribute to the integral. Then, the integral above is proportional to the Komar integral at the asymptotic infinity. Thus we arrive at the formula

$$J_\xi = -\frac{1}{16\pi G}\int_\Sigma \left(\star\nabla\xi + 4\star(\xi \cdot A)F + \frac{16}{3\sqrt{3}}(\xi \cdot A)A \wedge F\right), \tag{2.27}$$

where Σ is any surface enclosing the black object. The right hand side is precisely the Noether charge of Wald as constructed in [23, 32].

The contribution $\int \star\nabla\xi$ to the angular momentum is gauge invariant but is not conserved. It is expected, since the matter energy-momentum tensor carries the angular momentum. The rest of the terms in (2.27) was obtained by the partial integral of the contribution from the matter energy-momentum tensor, and can also be obtained by constructing the Noether charge. The price we paid is that it is now not invariant under a gauge transformation.

2.3 Example 1: the black ring

Let us check our formulae against known examples. First we consider the celebrated supersymmetric black ring in five dimensions [7].

2.3.1 Geometry

It has been known [33] that any supersymmetric solution of the minimal supergravity in the asymptotically flat $\mathbb{R}^{1,4}$ can be written in the form

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}ds^2(\mathbb{R}^4) \tag{2.28}$$

where f and ω is a function and a one-form on $\mathbb{R}^4$, respectively. For the supersymmetric black ring [7], we use a coordinate system adopted for a ring of radius R in the $\mathbb{R}^4$ given by

$$ds^2(\mathbb{R}^4) = \frac{R^2}{(x-y)^2}\left[\frac{dx^2}{1-x^2} + (1-x^2)d\phi_1^2 + \frac{dy^2}{y^2-1} + (y^2-1)d\phi_2^2\right] \tag{2.29}$$

with the ranges $-1 \le x \le 1$, $-\infty < y \le -1$ and $0 \le \phi_{1,2} < 2\pi$.[1] ϕ_1, ϕ_2 were denoted by ϕ, ψ in ref. [7].

The solution for the single black ring is parametrized by the radius R in the base $\mathbb{R}^4$ above, and two extra parameter q and Q. More details can be found in appendix A. q controls the magnetic dipole through S^2 surrounding the ring,

$$\frac{1}{2\pi}\int_{S^2} F = \frac{\sqrt{3}}{2}q. \tag{2.30}$$

Conserved charges measured at the asymptotic infinity are as follows:

$$\mathbf{Q} = \frac{1}{4\pi G}\int_\infty \star F = \frac{\sqrt{3}\pi}{2G}Q, \tag{2.31}$$

$$J_1 = -\frac{1}{16\pi G}\int_\infty \star\nabla\xi_1 = \frac{\pi}{8G}q(3Q-q^2), \tag{2.32}$$

$$J_2 = -\frac{1}{16\pi G}\int_\infty \star\nabla\xi_2 = \frac{\pi}{8G}q(6R^2+3Q-q^2) \tag{2.33}$$

where ξ_1, ξ_2 are the vector fields ∂_{ϕ_1}, ∂_{ϕ_2} respectively.

[1] We fix the orientations so that $\int_\Sigma dx\wedge d\phi_1\wedge d\phi_2 > 0$ and $\int_{S^2} dx\wedge d\phi_1 < 0$ for S^2 surrounding the ring.

JHEP12(2007)057

There is a magnetic flux through S^2 surrounding the ring, so we need to introduce two patches S, T. We choose S to cover the region $x < 1 - \epsilon$ and T to cover $1 - \epsilon < x < 1$, with infinitesimal ϵ. The boundary D is at $x = \epsilon$ and parametrized by $0 \le \phi_1, \phi_2 < 2\pi$. We choose the gauge transformation between the two patches to be

$$A_T = A_S + \frac{\sqrt{3}}{2} q d\phi_1 \tag{2.34}$$

which is chosen to make A_T smooth at the origin of $\mathbb{R}^4$.

The horizon is located at $y \to -\infty$ and has the topology $S^1 \times S^2$. The gauge potential near the horizon is

$$A_S = -\frac{\sqrt{3}}{4}\left(q + \frac{Q}{q}\right) d\psi - \frac{\sqrt{3}}{4} q(x+1) d\chi, \tag{2.35}$$

while the geometry near the horizon is given as

$$ds^2 = 2dvdr + \frac{4\ell}{q} r dv d\psi + \ell^2 d\psi^2 + \frac{q^2}{4}(d\theta^2 + \sin^2\theta d\chi^2) \tag{2.36}$$

where $r = r(y)$ is chosen so that $r \to 0$ corresponds to $y \to -\infty$, v is a combination of t and y, $x = \cos\theta$, $\psi = \phi_2 + C_1/r + C_0$ for suitably chosen $C_{0,1}$, $\chi = \phi_1 - \phi_2$, and

$$\ell^2 = 3\left(\frac{(Q - q^2)^2}{4q^2} - R^2\right). \tag{2.37}$$

It is a direct product of an extremal Bañados-Teitelboim-Zanelli (BTZ) black hole with horizon length $2\pi\ell$ and curvature radius q and of a round two-sphere with radius $q/2$.

ℓ is a more physical quantity characterizing the ring than R is, so it is preferable to express J_2, (2.33), using ℓ in the form

$$J_2 = \frac{\pi}{8G} q \left[-2\ell^2 + \frac{3Q^2}{2q^2} - \frac{q^2}{2}\right]. \tag{2.38}$$

Our objective is to reproduce the conserved charges, (2.31), (2.32) and (2.38), purely from the near-horizon data, (2.35) and (2.36).

2.3.2 Electric charge

We use the formula (2.6) to get the electric charge. Using the form of the gauge field near the horizon (2.34) and (2.35), we obtain

$$\begin{aligned}
\mathbf{Q} &= \frac{1}{4\pi G}\frac{2}{\sqrt{3}}\int_\Sigma A \wedge F \\
&= \frac{1}{4\pi G}\frac{2}{\sqrt{3}}\left(\int_{S\cap\Sigma} A_S \wedge F + \int_{D\cap\Sigma} A_S \wedge \beta\right) \\
&= \frac{\sqrt{3}\pi}{2G}\left(\frac{Q+q^2}{2} + \frac{Q-q^2}{2}\right) = \frac{\sqrt{3}\pi}{2G} Q,
\end{aligned} \tag{2.39}$$

which correctly reproduces the charge measured at the asymptotic infinity. Vanishing of $\int_\Sigma \star F$ at the horizon means that all the Maxwell charge of the system is carried outside of the horizon in the form of $\int F \wedge F$, while all of the Page charge is still inside the horizon.

One important fact behind the gauge invariance of the calculation above is that the integral $\int A_S$ along the ψ' direction is not just defined mod integer, but is well-defined as a real number. It is because the circle along ψ, which is not contractible in the near-horizon region, becomes contractible in the full geometry.

2.3.3 Angular momenta

The integral of the right hand side of (2.25) can be made arbitrarily small by choosing very small ϵ, so that we can forget the complication coming from the choice of Z. Then for $\xi_1 = \partial_{\phi_1} = \partial_\chi$, we have

$$\begin{aligned} J_1 &= \frac{1}{16\pi G}\left[-\int_{-1<x<1-\epsilon} \frac{16}{3\sqrt{3}}(\xi \cdot A_S)A_S \wedge F + \int_{x=1-\epsilon} \frac{16}{3\sqrt{3}}(\xi \cdot \beta)\beta \wedge A_T\right] \\ &= \frac{1}{16\pi G}\frac{16}{3\sqrt{3}}(2\pi)^2\left(\frac{\sqrt{3}}{2}\right)^3\left[\frac{1}{4}(q^3+qQ)+\frac{1}{2}(-q^3+qQ)\right] \\ &= \frac{\pi}{8G}q(3Q-q^2), \end{aligned} \tag{2.40}$$

reproducing (2.32).

For $\xi_\psi = \partial_\psi = \partial_{\phi_1} + \partial_{\phi_2}$, we have a contribution from $\int \star\nabla\xi_\psi = 4\pi^2 q\ell^2$. Adding contribution from $X[A]$, we obtain

$$J_\psi = \frac{\pi}{8G}\left(-2q\ell^2 - \frac{q^3}{2} + 3qQ + \frac{3Q^2}{2q}\right) \tag{2.41}$$

which matches with $J_1 + J_2$, see (2.32) and (2.38).

The second and the third terms in the expression above are obtained by the partial integration of the contribution from the angular part of the energy-momentum tensor of the gauge field. In this sense, a part of the angular momentum is carried outside of the horizon and the part proportional to ℓ^2 is carried inside the horizon. However, the Noether charge of the black ring resides purely inside of the horizon.

2.4 Example 2: concentric black rings

The concentric black-ring solution constructed in ref. [34] is a superposition of the single black ring we discussed in the last subsection. We focus on the case where all the rings lie on a plane in the base $\mathbb{R}^4$. For the superposition of N rings, the full geometry is parametrized by $3N$ parameters q_i, Q_i and R_i, $(i = 1, \dots, N)$. q_i is the dipole charge and R_i is the radius in the base $\mathbb{R}^4$ of the i-th ring. For more details, see appendix A. We order the rings so that $R_1 < R_2 < \cdots < R_N$. The conserved charges measured at infinity are known to be

$$\mathbf{Q} = \frac{\sqrt{3}\pi}{2G}\left[\sum_{i=1}^N (Q_i - q_i^2) + s^2\right], \tag{2.42}$$

$$J_1 = \frac{\pi}{8G}\left[2s^3 + 3s\sum_{j=1}^{N}(Q_j - q_j^2)\right], \tag{2.43}$$

$$J_2 = \frac{\pi}{8G}\left[2s^3 + 3s\sum_{j=1}^{N}(Q_j - q_j^2) + 6\sum_{i=1}^{N} q_i R_i^2\right] \tag{2.44}$$

where s is an abbreviation for the sum of the magnetic charges, i.e. $s = \sum_{i=1}^{N} q_i$. Our aim is to reproduce these results from the near-horizon data.

The near-horizon metric of i-th ring has the form (2.36) with q, Q, R replaced with q_i, Q_i and R_i, respectively. The horizon radius ℓ_i is given by

$$\ell_i^2 = 3\left(\frac{(Q_i - q_i^2)^2}{4q_i} - R_i^2\right). \tag{2.45}$$

Since each ring has a magnetic dipole charge, we introduce coordinate patches S and T_i so that the gauge field is non-singular in each patch. Let T_i be the patch covering the region between $(i-1)$-th and i-th ring and S be a patch covering the outer region. More precisely, we introduce the ring coordinate (2.29) for each of the ring, and choose S to cover $-1+\epsilon < x_i < 1-\epsilon$ for each ring while T_i to cover $1-\epsilon < x_i < 1$ for the i-th ring and $-1 < x_{i-1} < -1+\epsilon$ for the $(i-1)$-th ring. Then, near the i-th horizon the gauge field on S is given by

$$A_S = -\frac{\sqrt{3}}{4}\left[\left(\frac{Q_i}{q_i} - q_i + 2s\right)d\psi + \left(q_i(1+x) + 2\sum_{j=i+1}^{N} q_j\right)d\chi\right]. \tag{2.46}$$

Its ψ component is determined in appendix A, while the coefficient for $d\chi$ is determined so that the field strength is reproduced, the gauge field is non-singular except for $x = \pm 1$ for the 1st to $(N-1)$-th rings and non-singular except for $x = -1$ for the N-th ring. The gauge field on T_i is given by

$$A_{T_i} = A_S + \frac{\sqrt{3}}{2}\sum_{j=i}^{N} q_j d\phi_1. \tag{2.47}$$

The electric charge is given by using (2.6) and $\beta_i = A_S - A_{T_i} = -\frac{\sqrt{3}}{2}\sum_{j=i}^{N} q_j d\phi_1$ as

$$\begin{aligned}
\mathbf{Q} &= \frac{1}{4\pi G}\frac{2}{\sqrt{3}}\sum_{i=1}^{N}\int_{\Sigma_i\cap S} A_S\wedge F + \frac{1}{4\pi G}\frac{2}{\sqrt{3}}\sum_{i=1}^{N}\left(\int_{\Sigma_i\cap\partial S} A_S\wedge\beta_i + \int_{\Sigma_{i-1}\cap\partial S} A_S\wedge\beta_i\right)\\
&= \frac{\sqrt{3}\pi}{4G}\sum_{i=1}^{N}\left[(Q_i - q_i^2) + 2sq_i\right] + \frac{\sqrt{3}\pi}{4G}\sum_{i=1}^{N}(Q_i - q_i^2)\\
&= \frac{\sqrt{3}\pi}{2G}\left[\sum_{i=1}^{N}(Q_i - q_i^2) + s^2\right].
\end{aligned} \tag{2.48}$$

This correctly reproduces the known result (2.42).

JHEP12(2007)057

Let us move onto the evaluation of the angular momenta. Note that for certain configurations of charges, the concentric black rings develop singularities on the rotation axes. While the condition for the absence of singularities has not been known fully, it was pointed out in ref. [34] that there is no singularity on the rotation axes if all

$$\Lambda_i = \frac{Q_i - q_i^2}{q_i} \tag{2.49}$$

are equal. We will show that we can obtain the correct angular momenta if this condition is satisfied.

The angular momentum associated with $\xi_1 = \partial_{\phi_1} = \partial_\chi$ is given by

$$\begin{aligned} J_1 =& -\frac{1}{16\pi G}\frac{16}{3\sqrt{3}}\sum_{i=1}^{N}\int_{\Sigma_i\cap S}(\xi_1\cdot A_S)A_S\wedge F \\ &-\frac{1}{16\pi G}\frac{16}{3\sqrt{3}}\sum_{i=1}^{N}\left(\int_{\Sigma_i\cap\partial T_i}+\int_{\Sigma_{i-1}\cap\partial T_i}\right)(\xi_1\cdot\beta_i)\beta_i\wedge A_{T_i}. \end{aligned} \tag{2.50}$$

After summing up terms, we have

$$J_1 = \frac{\pi}{8G}\left[2s^3 + 6\sum_{i=1}^{N}(Q_i - q_i^2)\sum_{j=i+1}^{N} q_j + 3\sum_{i=1}^{N}(q_i(Q_i - q_i^2))\right]. \tag{2.51}$$

If the condition (2.49) is satisfied, J_1 computed above matches (2.43) and we have

$$J_1 \to \frac{\pi}{8G}\left[2s^3 + 3\Lambda_i s^2\right]. \tag{2.52}$$

Finally, let us consider the angular momentum associated with $\xi_\psi = \partial_\psi = \partial_{\phi_1} + \partial_{\phi_2}$. In addition to (2.50) with ξ_1 being replaced by ξ_ψ, here we have to consider two more contributions. Namely,

$$-\frac{1}{16\pi G}\sum_{i=1}^{N}\int_{\Sigma_i}\star\nabla\xi_\psi - \frac{1}{16\pi G}\frac{8}{3\sqrt{3}}\sum_{i=1}^{N}\left(\int_{\Sigma_i\cap\partial T_i}+\int_{\Sigma_{i-1}\cap\partial T_i}\right)(\xi_\psi\cdot A_{T_i})\beta_i\wedge A_{T_i}. \tag{2.53}$$

It is easy to check that the sum of each term is given by

$$J_\psi = \frac{\pi}{8G}\left[6\sum_{i=1}^{N} q_i R_i^2 + 4s^3 + 6s\sum_{i=1}^{N}(Q_i - q_i^2)\right]. \tag{2.54}$$

When evaluated under the condition (2.49), this gives

$$J_\psi \to \frac{\pi}{8G}\left[6\sum_{i=1}^{N} q_i R_i^2 + 4s^3 + 6\Lambda_i s^2\right] \tag{2.55}$$

and agrees with J_ψ given as the sum of (2.43) and (2.44).

JHEP12(2007)057

2.5 Generalization

It is straightforward to generalize the techniques we developed so far to the supergravity theory with n of $U(1)$ vector fields A^I, $(I = 1, \ldots, n)$. There are $(n-1)$ vector multiplets because the gravity multiplet also contains the graviphoton field which is a vector field. The scalars in the vector multiplet are denoted by M^I, which are constrained by the condition

$$\mathcal{N} \equiv c_{IJK} M^I M^J M^K = 1. \tag{2.56}$$

c_{IJK} is a set of constants. The action for the boson fields is given by

$$S = \frac{1}{16\pi G} \int \left[\star R - a_{IJ} dM^I \wedge \star dM^J - a_{IJ} F^I \wedge \star F^J - c_{IJK} A^I \wedge F^J \wedge F^K \right] \tag{2.57}$$

where R is the Ricci scalar, and

$$a_{IJ} = -\frac{1}{2} \left(\mathcal{N}_{IJ} - \mathcal{N}_I \mathcal{N}_J \right). \tag{2.58}$$

In the last expression, $\mathcal{N}_I = \partial \mathcal{N} / \partial M^I$ and $\mathcal{N}_{IJ} = \partial^2 \mathcal{N} / \partial M^I \partial M^J$. This is the low-energy action of M-theory compactified on a Calabi-Yau manifold M with $n = h_{1,1}(M)$, and

$$6 c_{IJK} = \int \omega_I \wedge \omega_J \wedge \omega_K \tag{2.59}$$

is the triple intersection of integrally-quantized two-forms ω_I on M. The action for the minimal supergavity (2.1) is obtained by setting $n = 1$, $c_{111} = (2/\sqrt{3})^3$, and $a_{11} = 2$.

As for the calculation of the electric charges, one only needs to put the indices I, J, K to the vector fields and the result is

$$\mathbf{Q}_I = \frac{1}{8\pi G} \int \left[\star a_{IJ} F^J + \frac{3}{2} c_{IJK} A^J \wedge F^K \right]. \tag{2.60}$$

As for the angular momenta, there is extra terms coming from the energy-momentum tensor of the scalar fields in the right hand side of (2.15). Its contribution to the angular momenta vanishes upon integration, so that the result is

$$J_\xi = -\frac{1}{16\pi G} \int \left[\star \nabla \xi + 2 \star a_{IJ} (\xi \cdot A^I) F^J + 2 c_{IJK} (\xi \cdot A^I) A^J \wedge F^K \right]. \tag{2.61}$$

For a more complicated Lagrangian, e.g. with charged hypermultiplets and/or with higher-derivative corrections, it is easier to utilize the general framework set up by Wald, than to find the partial integral in (2.15) by inspection. The charge constructed by this technique has an important property [27] that it acts as the Hamiltonian for the corresponding local symmetry in the Hamiltonian formulation of the theory, and it reproduces the Page charge and the angular momenta (2.61). Consequently, the charge as the generator of the symmetry is not the gauge-invariant Maxwell charge, but the Page charge which depends on a large gauge transformation.

The integrands in the expressions above are not well-defined as differential forms when there are magnetic fluxes, thus it needs to be defined appropriately as we did in the previous sections. Generically, we would like to rewrite the integral of a gauge invariant form ω on a region B to the integral of $\omega_{(1)}$ satisfying

$$d\omega_{(1)} = \omega \tag{2.62}$$

on its boundary ∂B. The problem is that $\omega_{(1)}$ may depend on the gauge. On two patches S and T, it is represented by differential forms $\omega^S_{(1)}$ and $\omega^T_{(1)}$ respectively. Since ω is gauge-invariant, we have $d\omega^S_{(1)} = d\omega^T_{(1)}$. Thus, if we take a sufficiently small coordinate patch, we can choose $\omega^{(S,T)}_{(2)}$ such that

$$d\omega^{(S,T)}_{(2)} = \omega^S_{(1)} - \omega^T_{(1)}. \tag{2.63}$$

Then one defines the integral of $\omega_{(1)}$ on $C = \partial B$ via

$$\int_C \omega_{(1)} \equiv \int_{C\cap S} \omega^S_{(1)} + \int_{C\cap T} \omega^T_{(1)} + \int_{C\cap D} \omega^{S,T}_{(2)}, \tag{2.64}$$

where $D = \partial S = -\partial T$. The equations (2.62), (2.63) are the so-called descent relation which is important in the understanding of the anomaly. It will be interesting to generalize our analysis to the case where there are more than two patches and multiple overlaps among them. Presumably we need to include higher descendants $\omega^{(S_1,\ldots,S_n)}_{(n)}$ as the correction term at the boundary of n patches $S_1, \ldots, S_n$ in the definition of the integral (2.64).

3. Relation to four-dimensional charges

We have seen how the near-horizon data of the black rings encode the charges measured at the asymptotic infinity. We can also consider rings in the Taub-NUT space [19 − 21] instead in the five-dimensional Minkowski space. Then the theory can also be thought of as a theory in four dimensions, via the Kaluza-Klein reduction along S^1 of the Taub-NUT space. It has been established [35] that supersymmetric solutions for five dimensional supergravity nicely reduces to supersymmetric solutions for the corresponding four dimensional theory.

In four dimensions, there are no problems in defining the charges, because the equations of motion and Bianchi identities yield the relations

$$dF^I = 0, \qquad dG_I = d(\star(g^{-2}_{IJ})F^J + \theta_{IJ}F^J) = 0 \tag{3.1}$$

where $(g^{-2})_{IJ}$ are the inverse coupling constants and θ_{IJ} are the theta angles. The electric and magnetic charges can be readily obtained by integrating G_I and F^I over the horizon. Then it is natural to expect that our formulae for the charges will yield the four-dimensional ones after the Kaluza-Klein reduction. One apparent problem is that the Page charges changes under a large gauge transformation, whereas the four-dimensional charges are seemingly well-defined as is. We will see that a large gauge transformation corresponds to the Witten effect on dyons in four-dimensions.

3.1 Mapping of the fields

First let us recall the well-known mapping of the fields in four and five dimensions. The details can be found e.g. in [11, 12, 15, 16]. When we reduce a five-dimensional $\mathcal{N}=2$ supergravity with n vector fields along S^1, it results in a four-dimensional $\mathcal{N}=2$ supergravity with $(n+1)$ vector fields. The metrics in respective dimensions are related by

$$ds^2_{5d} = e^{2\rho}(d\psi - A^0)^2 + e^{-\rho} ds^2_{4d}, \tag{3.2}$$

where we take the periodicity of ψ to be 2π so that e^ρ is the five-dimensional radius of the Kaluza-Klein circle. The factor in front of the four-dimensional metric is so chosen that the four-dimensional Einstein-Hilbert term is canonical.

The gauge fields in four and five dimensions are related by

$$A^I_{5d} = a^I(d\psi - A^0) + A^I_{4d} \tag{3.3}$$

where $I = 1, \ldots, n$. It is chosen so that a gauge transformation of A^0 do not affect A^I_{4d}. We need to introduce coordinate patches when there is a flux for A^I_{5d}. We demand that gauge transformations used between patches should not depend on ψ so that a^I are globally well-defined scalar fields.

Then, by the reduction of the five-dimensional action (2.57), the action of four-dimensional gauge fields is determined to be[2]

$$\begin{aligned}\mathcal{L} = &-\left[\frac{1}{2}e^{3\rho} + e^\rho a_{IJ} a^I a^J\right] F^0 \wedge \star F^0 - c_{IJK} a^I a^J a^K F^0 \wedge F^0 \\ &+2e^\rho a_{IJ} a^I F^0 \wedge \star F^J + 3c_{IJK} a^I a^J F^0 \wedge F^K \\ &-e^\rho a_{IJ} F^I \wedge \star F^J - 3c_{IJK} a^I F^J \wedge F^K. \end{aligned}\tag{3.4}$$

Partial integrations are necessary to bring the naive Kaluza-Klein reduction to the form above. The resulting Lagrangian above follows from the prepotential

$$F(X) = \frac{c_{IJK} X^I X^J X^K}{X^0}, \tag{3.5}$$

if one defines special coordinates $z^I = X^I/X^0$ by

$$z^I = a^I + ie^\rho M^I. \tag{3.6}$$

This relation can be checked without the detailed Kaluza-Klein reduction. Indeed, the ratio of a^I and M^I in (3.6) can be fixed by inspecting the mass squared of a hypermultiplet, and the fact a^I should enter in z^I linearly with unit coefficient is fixed by the monodromy.

[2] We take the following conventions in four dimensions: The orientations in four and five dimensions are related such that $\int_{5d} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\psi = 2\pi \int_{4d} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The Levi-Civita symbol in four dimensions is defined by $\epsilon_{0123} = +1$ and $\epsilon^{0123} = -1$ in local Lorentz coordinates.

JHEP12(2007)057

3.2 Mapping of the charges

In many references including ref. [12, 16, 23], the charge of the black object in five dimensions is defined to be the charges in four dimensions after the dimensional reduction determined from the Lagrangian (3.4). It was motivated partly because the analysis of the charge in five dimensions was subtle due to the presence of the Chern-Simons interaction, whereas we studied how we can obtain the formula for the charges which has five-dimensional general covariance in section 2. Now let us compare the charges thus defined in four- and five- dimensions.

Firstly, the magnetic charge

$$q^0 = \frac{1}{2\pi}\int_C F^0 \tag{3.7}$$

in four dimensions counts the number of the Kaluza-Klein monopole inside C. It is also called the nut charge. The other magnetic charges in four dimensions

$$q^I = \frac{1}{2\pi}\int_C F^I \tag{3.8}$$

come directly from the dipole charges in five dimensions, as long as the surface C does not enclose the nut. When C does contain a nut, the Kaluza-Klein circle is non-trivially fibered over C. Thus, the surface C cannot be lifted to five dimensions. We will come back to this problem in section 3.5.

The formulae for the electric charges follow from the Lagrangian:

$$Q_I = \frac{1}{2\pi}\int \left[\star 2e^\rho a_{IJ}(F^J - a^J F^0) + 6c_{IJK}a^J F^K - 3c_{IJK}a^J a^K F^0\right], \tag{3.9}$$

$$Q_0 = \frac{1}{2\pi}\int \left[\star e^{3\rho}F^0 - \star 2e^\rho a_{IJ}a^I(F^J - a^J F^0) + 2c_{IJK}a^I a^J a^K F^0 - 3c_{IJK}a^I a^J F^K\right]. \tag{3.10}$$

It is easy to verify that the five-dimensional Page charges (2.60) and the Noether charge J_ψ (2.61) for the isometry ∂_ψ along the Kaluza-Klein circle are related to the four-dimensional electric charges via

$$Q_I = -\frac{4G}{\pi}\mathbf{Q}_I, \qquad Q_0 = -\frac{4G}{\pi}J_\psi. \tag{3.11}$$

An important point in the calculation is that the compensating term on the boundary of the coordinate patches vanishes, since a^I and F^J_{4d} are globally well-defined.

Thus we see that the four-dimensional charges are not the reduction of the gauge-invariant Maxwell charges $\int \star F$ or that of the gauge-invariant "Maxwell-like" part of the angular momentum, $\int \star\nabla\xi$. They are rather the reduction of the Page or the Noether charges, which change under a large gauge transformation.

3.3 Reduction and the attractor

In the literature, the attractor equation is often analyzed after the reduction to four dimensions [12, 15, 16], while the five-dimensional attractor mechanism for the black rings in [22] only determines the scalar vacuum expectation values via the magnetic dipoles. As we saw in the previous sections, the electric charges at the asymptotic infinity are encoded by the Wilson lines along the horizon. We show that how these five-dimensional consideration reproduces the known attractor solution [36, 37] in four-dimensions.

The five-dimensional metric is characterized by the magnetic charges q^I through the horizon, and the physical radius of the horizon $\ell = e^\rho$ there. From the attractor mechanism for the black rings [22], the near-horizon geometry is of the form $AdS_3 \times S^2$, and the curvature radii are q and $q/2$ in each factor, where $q^3 = c_{IJK} q^I q^J q^K$. The scalar vevs are fixed to be proportional to the magnetic dipoles, i.e. $M^I = q^I/q$.

For the calculation of electric charges the Wilson lines a^I along the horizon are also important. Then we can evaluate the Page charges and angular momenta on the horizon to obtain

$$Q_I = 6c_{IJK} a^J q^K, \qquad Q_0 = q\ell^2 - 3c_{IJK} a^I a^J q^K. \tag{3.12}$$

We can solve the equations above for ℓ and a^I so that we have the formula for the four-dimensional special coordinates z^I in terms of the charges. The result is

$$z^I = a^I + ie^\rho M^I = \frac{1}{6} D^{IJ} Q_I + i\sqrt{\frac{\hat{Q}_0}{D}} q^I \tag{3.13}$$

where

$$D_{IJ} = c_{IJK} q^K, \quad D^{IJ} D_{JK} = \delta^I_K \tag{3.14}$$

and

$$D = q^3 = c_{IJK} q^I q^J q^K, \quad \hat{Q}_0 = q\ell^2 = Q_0 + \frac{1}{12} D^{IJ} Q_I Q_J. \tag{3.15}$$

It is the well-known solution of the attractor equation in four-dimensions with $q^0 = 0$ [36, 37].

Thus, the combination of the attractor mechanism in five dimensions and the technique of Page charges yield the attractor mechanism in four dimensions. The point is that the Wilson lines a^I along the horizon of the black string carry the information of its electric charges. Conversely, the Wilson line at the horizon is determined by the electric charge. The horizon length is also determined by the angular momentum. In this sense, the attractor mechanism for the black rings also fixes all the relevant near-horizon data by means of the charges, angular momenta and dipoles.

3.4 Gauge dependence and monodromy

Let us now come back to the question of the variation of the Page charges under large gauge transformations. The problem is that the integral $\int_C A \wedge F$ depends on the shift $A \to A + \beta$ for $d\beta = 0$ if C has a non-contractible loop ℓ and $\int_\ell \beta \neq 0$. In the spacetime which asymptotes to $\mathbb{R}^{4,1}$, the large gauge transformation can be fixed by demanding that the gauge potential vanishes at the asymptotic infinity.

In the present case of reduction to four dimensions, however, the gauge potential along the Kaluza-Klein circle is one of the moduli and is not a thing to be fixed. More precisely, if the ψ direction is non-contractible, a large gauge transformation associated to the Kaluza-Klein circle corresponds to a shift $a^I \to a^I + t^I$ where t^I are integers. In four-dimensional language it is the shift

$$z^I \to z^I + t^I, \tag{3.16}$$

and the gauge variation of the Page charge translates to the variation of the electric charge under the transformation (3.16). It is precisely the Witten effect on dyons [24] if one recalls the fact that the dynamical theta angles of the theory depends on z^I. In the terminology of $\mathcal{N} = 2$ supergravity and special geometry, it is called the monodromy transformation associated to the shift (3.16), which acts symplectically on the charges (q^I, Q_I) and on the projective special coordinates (X^I, F_I)

For the M-theory compactification on the product of S^1 and a Calabi-Yau, electric charges Q_I and q^I correspond to the number of M2-branes and M5-branes wrapping two-cycles Π^I and four-cycles Σ_I, respectively. The relation (2.59) translates to $6c_{IJK} = \#(\Sigma_I \cap \Sigma_J \cap \Sigma_K)$ in this language. The gauge fields A_I arise from the Kaluza-Klein reduction of the M-theory three-form C on Π^I. Thus, the results above imply that the M2-brane charges transform non-trivially in the presence of M5-branes under large gauge transformations of the C-field.

It might sound novel, but it can be clearly seen from the point of view of Type IIA string theory on the Calabi-Yau. Consider a soliton without D6-brane charge. There, the D2-brane charge Q_I of the soliton is induced by the world-volume gauge field F on the D4 brane wrapped on a four-cycle $\Sigma = q^I \Sigma_I$ through the Chern-Simons coupling

$$\int_\Sigma (F + B) \wedge C \tag{3.17}$$

where B is the NSNS two-form and C is the RR three-form. In this description, a^I is given by $\int_{\Pi^I} B$. The induced brane charge in the presence of the non-zero B-field is an intricate problem in itself, but the end result is that the large gauge transformation $B \to B + \omega$ with $\int_{\Pi^I} \omega = t^I$ changes the D2-brane charge of the system by $6c_{IJK} q^I t^J$. It will be interesting to derive the same effect from the worldvolume Lagrangian [38] of the M5 brane, which is subtle because the worldvolume tensor field is self-dual. The change in the M2-brane charge induce a change in the Kaluza-Klein momentum carried by the zero-mode on the black strings wrapped on S^1, so that Q_0 also changes [2]. The point is that the momentum carried by non-zero modes, $\hat{Q}_0$ defined in (3.15), is a monodromy-invariant quantity.

Before leaving this section, it is worth noticing that if an M2-brane has the worldvolume V, it enters in the equation of motion for $G = dC$ in the following way:

$$d \star G + G \wedge G = \delta_V \tag{3.18}$$

where δ_V is the delta function supported on V. Thus, the quantized M2-brane charge is not the source of the Maxwell charge. It is rather the source of the Page charge. Essentially the same argument in five dimensions, using the specific decomposition (2.28), was made in ref. [39].

3.5 Monodromy and Taub-NUT

If we use the Taub-NUT space in the dimensional reduction, in other words if there is a Kaluza-Klein monopole in the system, the Kaluza-Klein circle shrinks at the nut of the monopole. As the circle is now contractible, one might think that one can no longer do a large gauge transformation and that it is natural to choose $a^I = 0$ at the nut. Nevertheless, from a four-dimensional standpoint the monodromy transformation should be always possible. How can these two points of view be reconciled?

Firstly, the fact that the five-dimensional spacetime is smooth at the nut only requires that the gauge field strength is zero there and that the integral of the gauge potential is an integer. There should be a patch around the nut in the five-dimensional spacetime in which A^I should be smooth, but it is not the patch connected to the asymptotic region of the Taub-NUT space where a^I is defined.

A similar problem was studied in ref. [40]. There, it was shown how the winding number can still be conserved in the background with the nut, where the circle on which strings are wound degenerates. A crucial role is played by the normalizable self-dual two-form ω localized at the nut, which gives the worldvolume gauge field A of the D6-brane realized as the M-theory Kaluza-Klein monopole via $C = A \wedge \omega$. It should enter in the worldvolume Lagrangian in the combination $dA + B$, and the large gauge transformation affects the contribution from B.

Indeed, the Kaluza-Klein ansatz of the gauge fields (3.3), one can make the combined shift

$$a^I \to a^I + t^I, \qquad A^I_{4d} \to A^I_{4d} + t^I A^0 \tag{3.19}$$

without changing the five-dimensional gauge field strengths. Therefore, the magnetic charge also transforms as

$$q^I \to q^I + t^I q^0. \tag{3.20}$$

The action of the transformation (3.16) on the electric charges then becomes

$$Q_I \to Q_I + 6c_{IJK}t^J q^K + 3c_{IJK}t^J t^K q^0, \tag{3.21}$$
$$Q_0 \to Q_0 - Q_I t^I - 3c_{IJK}t^I t^J q^K - c_{IJK}t^I t^J t^K q^0, \tag{3.22}$$

which is exactly how the projective coordinates

$$X^0, \quad X^I, \quad F_I = 3c_{IJK}X^J X^K/X^0, \quad F_0 = -c_{IJK}X^I X^J X^K/(X^0)^2. \tag{3.23}$$

get transformed by the monodromy $a^I \to a^I + t^I$. It was already noted in ref. [21] that the same symmetry acts on the functions which characterize the supersymmetric solution on the Taub-NUT, (V, K^I, L_I, M) in their notation. The point is that it modifies the five-dimensional Page charges, and hence the four-dimensional charges.

If we neglect quantum corrections coming from instantons wrapping the Kaluza-Klein circle, it is allowed to do the monodromy transformation $z^I \to z^I + t^I$ even with continuous parameters t^I. It maps a solution of the equations of motion to another, and the electric charges in four-dimensions depends continuously on the vevs for the moduli a^I at the asymptotic infinity. The issue concerning the stability of the solitons can be safely ignored.

JHEP12(2007)057

In the analyses in refs. [19 − 21], their proposals for the identification of four-dimensional electric charges Q_I and of five-dimensional ones $\mathbf{Q}_I$ were different from one another. The source of the discrepancy in the identification is now clear after our long discussion. It can be readily checked that the differing proposals for the identification can be connected by the monodromy transformation with $t^I = \frac{1}{2}q^I$. Namely, the charges in the five-dimensional language are transformed as

$$\frac{4G}{\pi}\mathbf{Q}_I \to \frac{4G}{\pi}\mathbf{Q}_I - 3c_{IJK}q^J q^K, \quad J_\psi \to J_\psi - J_\phi \tag{3.24}$$

for $Q_0 \gg q^3$ limit.[3] Thus they are equivalent under a large gauge transformation.

The analysis above also answers the question raised in section 3.2 how the dipole charges in five dimensions are related in the magnetic charges in four dimensions in the presence of the nut. It is instructive to consider the case of a black ring in the Taub-NUT space. From a five-dimensional viewpoint, the dipole charge is not a conserved quantity measurable at the asymptotic infinity. Correspondingly, the surface of the Dirac string necessary to define the gauge potential can be chosen to fill the disc inside the black ring only, and not to extend to the asymptotic infinity. It was what we did in section 2.3.1 in defining the coordinate patches. However, the gauge transformation required to achieve it necessarily depends on the ψ coordinate, which is the direction along the Kaluza-Klein circle. Hence it is not allowed if one carries out the reduction to four dimensions. In this case, the Dirac string emanating from the black ring necessarily extends all the way to the spatial infinity, thus making the magnetic charge measurable at the asymptotic infinity. A related point is that dipole charges enter in the first law of black objects because of the existence of two patches [31].[4] It is easier to understand it after the reduction because now it is a conserved quantity measurable at the asymptotic infinity.

As a final example to illustrate the subtlety in the identification of the four- and five-dimensional charges, let us consider a two-centered Taub-NUT space with centers $\mathbf{x}_1$ and $\mathbf{x}_2$. There is an S^2 between two centers, and one can introduce a self-dual magnetic fluxes q^I through it. Although the Chern-Simons interactions put some constraint on the allowed q^I, there is a supersymmetric solution of this form [44]. In this configuration, the Wilson lines a^I at $\mathbf{x}_1$ and $\mathbf{x}_2$ necessarily differ by the amount proportional to the flux, and one cannot simultaneously make them zero. An important consequence is that the magnetic charges F^I_{4d} of the nuts at $\mathbf{x}_2$ and $\mathbf{x}_2$ necessarily differ, in spite of the fact that the geometry and the gauge fields in five dimensions are completely symmetric under the exchange of $\mathbf{x}_1$ and $\mathbf{x}_2$.

4. Summary

In this paper, we have first clarified how the near-horizon data of black objects encode the

[3] We noticed that a small discrepancy proportional to $c_{IJK}q^I q^J q^K$ remains, which is related to the zero-point energy of the conformal field theory of the black string. Its effect on the entropy is subleading in the large Q_0 limit.

[4] The authors of [31] used the approach to the first law developed in [41]. There is another understanding of appearance of the dipole charges in the first law [42] if one follows the approach in [43].

JHEP12(2007)057

conserved charges measured at asymptotic infinity. Namely, the existence of the Chern-Simons coupling means that $F \wedge F$ is a source of electric charges, thus it was necessary to perform the partial integration once to rewrite the asymptotic electric charge by the integral of $A \wedge F$ over the horizon. Since F has magnetic flux through the horizon, $A \wedge F$ cannot be naively defined, and we showed how to treat it consistently. Likewise, we obtained the formula for the angular momenta using the near-horizon data.

Then, we saw how our formula for the charges in five dimensions is related to the four-dimensional formula under Kaluza-Klein reduction. We studied how the ambiguity coming from large gauge transformations in five dimensions corresponds to the Witten effect and the associated monodromy transformation in four dimensions.

It is now straightforward to obtain the correction to the entropy of the black rings, since we now have the supersymmetric higher-derivative action [13], the near-horizon geometry [45 – 47], and also the formulation developed in this paper to obtain conserved charges from the near-horizon data alone. It would be interesting to see if it matches with the microscopic calculation.

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A. Geometry of concentric black rings

Any supersymmetric solution in the asymptotically flat $\mathbb{R}^{1,4}$ is known to be of the form [33]

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}ds^2(\mathbb{R}^4) \tag{A.1}$$

where f and ω is a function and a one-form on $\mathbb{R}^4$, respectively. We parametrize the base $\mathbb{R}^4$ in the Gibbons-Hawking coordinate system

$$ds^2(\mathbb{R}^4) = H[dr^2 + r^2(d\theta^2 + \sin^2\theta d\chi^2)] + H^{-1}(2d\psi + \cos\theta d\chi)^2 \tag{A.2}$$

where (r, θ, ϕ) parametrize a flat $\mathbb{R}^3$, the periodicity of ψ is 2π and $H = 1/r$. Our notation mostly follows the one in ref. [34], with the change $\psi_{\text{there}} = 2\psi_{\text{here}}$. The quantities f, ω and the gauge field $F = dA$ are determined by three functions K, L and M on the flat $\mathbb{R}^3$. The relations we need are

$$f^{-1} = H^{-1}K^2 + L, \qquad \iota_{\partial_\psi}\omega = 2H^{-2}K^3 + 3H^{-1}KL + 2M, \tag{A.3}$$

$$F = \frac{\sqrt{3}}{2}d[f(dt+\omega)] - \frac{1}{\sqrt{3}}G^+, \qquad \iota_{\partial_\psi}G^+ = -3d(H^{-1}K) \tag{A.4}$$

where $G^+ = f(d\omega + \star d\omega)/2$ is a self-dual two-form on $\mathbb{R}^4$.

To construct the concentric black ring solutions, we take N points $\mathbf{x}_i$, $(i = 1, \ldots, N)$ at $r = R_i^2/4, \theta = \pi$ on $\mathbb{R}^3$. The orbit of $\mathbf{x}_i$ along the coordinate ψ is a ring of radius R_i embedded in $\mathbb{R}^4$. We choose functions K, L and M by

$$K = -\frac{1}{2}\sum_{i=1}^{N} q_i h_i, \quad L = 1 + \frac{1}{4}\sum_{i=1}^{N}(Q_i - q_i^2)h_i, \quad M = \frac{3}{4}\sum_{i=1}^{N} q_i(1 - |\mathbf{x}_i|h_i) \tag{A.5}$$

where $h_i(\mathbf{x}) = 1/|\mathbf{x} - \mathbf{x}_i|$ are harmonic functions on $\mathbb{R}^3$. For the case with a single ring, conversion to the ring coordinate used in (2.29) can be achieved via

$$\phi_1 = \psi + \chi/2, \qquad \phi_2 = \psi - \chi/2 \tag{A.6}$$

and

$$\frac{R\sqrt{y^2-1}}{x-y} = 2\sqrt{r}\sin\frac{\theta}{2}, \qquad \frac{R\sqrt{1-x^2}}{x-y} = 2\sqrt{r}\cos\frac{\theta}{2}. \tag{A.7}$$

The behavior of ω and F at the asymptotic infinity, and the near-horizon metric (2.36) are well-known and are not repeated here. The reader is referred to the original article ref. [34]. The gauge potential near the horizon can be obtained by the combination of (A.3) and (A.4). First we have

$$\iota_{\partial_\psi} F = \frac{\sqrt{3}}{2}(-d\iota_{\partial_\psi})[f(dt + \omega)] + \sqrt{3}d(KH^{-1}). \tag{A.8}$$

which can be integrated by inspection. Hence the ψ component of the gauge field is given by

$$\iota_{\partial_\psi} A = \sqrt{3}\left[\frac{H^{-1}KL/2 + M}{H^{-1}K^2 + L} + c\right] \tag{A.9}$$

for some constant c. By demanding $\iota_\psi A \to 0$ as $r \to \infty$, we obtain

$$c = -\frac{1}{2}\sum_{i=1}^{N} q_i. \tag{A.10}$$

Thus, we have

$$\iota_{\partial_\psi} A = -\frac{\sqrt{3}}{4}\left(\frac{Q_i - q_i^2}{q_i} + 2\sum_{i=1}^{N} q_i\right). \tag{A.11}$$

near the i-th horizon. The χ component of the gauge field is fixed by the magnetic dipole through the horizon.

References

[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[2] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [hep-th/9711053].

JHEP12(2007)057

[3] T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [hep-th/0007195].

[4] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[5] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177].

[6] D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058 [hep-th/0606244].

[7] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, *A supersymmetric black ring*, *Phys. Rev. Lett.* **93** (2004) 211302 [hep-th/0407065].

[8] I. Bena and P. Kraus, *Microscopic description of black rings in AdS/CFT*, *JHEP* **12** (2004) 070 [hep-th/0408186].

[9] M. Cyrier, M. Guica, D. Mateos and A. Strominger, *Microscopic entropy of the black ring*, *Phys. Rev. Lett.* **94** (2005) 191601 [hep-th/0411187].

[10] M. Guica, L. Huang, W.W. Li and A. Strominger, *R^2 corrections for 5D black holes and rings*, *JHEP* **10** (2006) 036 [hep-th/0505188].

[11] I. Bena and P. Kraus, *R^2 corrections to black ring entropy*, hep-th/0506015.

[12] R.-G. Cai and D.-W. Pang, *On entropy function for supersymmetric black rings*, *JHEP* **04** (2007) 027 [hep-th/0702040].

[13] K. Hanaki, K. Ohashi and Y. Tachikawa, *Supersymmetric completion of an R^2 term in five-dimensional supergravity*, *Prog. Theor. Phys.* **117** (2007) 533 [hep-th/0611329].

[14] J.F. Morales and H. Samtleben, *Entropy function and attractors for AdS black holes*, *JHEP* **10** (2006) 074 [hep-th/0608044].

[15] G.L. Cardoso, J.M. Oberreuter and J. Perz, *Entropy function for rotating extremal black holes in very special geometry*, *JHEP* **05** (2007) 025 [hep-th/0701176].

[16] K. Goldstein and R.P. Jena, *One entropy function to rule them all*, hep-th/0701221.

[17] A. Strominger, *Macroscopic entropy of $N = 2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[18] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136].

[19] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, *Supersymmetric 4D rotating black holes from 5D black rings*, *JHEP* **08** (2005) 042 [hep-th/0504125].

[20] D. Gaiotto, A. Strominger and X. Yin, *5D black rings and 4D black holes*, *JHEP* **02** (2006) 023 [hep-th/0504126].

[21] I. Bena, P. Kraus and N.P. Warner, *Black rings in Taub-NUT*, *Phys. Rev.* **D 72** (2005) 084019 [hep-th/0504142].

[22] P. Kraus and F. Larsen, *Attractors and black rings*, *Phys. Rev.* **D 72** (2005) 024010 [hep-th/0503219].

[23] N.V. Suryanarayana and M.C. Wapler, *Charges from attractors*, *Class. and Quant. Grav.* **24** (2007) 5047 [arXiv:0704.0955].

JHEP12(2007)057

[24] E. Witten, *Dyons of charge $e\theta/2\pi$*, *Phys. Lett.* **B 86** (1979) 283.

[25] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Black holes of $D = 5$ supergravity*, *Class. and Quant. Grav.* **16** (1999) 1 [hep-th/9810204].

[26] D. Marolf, *Chern-Simons terms and the three notions of charge*, hep-th/0006117.

[27] J. Lee and R.M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725.

[28] R.M. Wald, *On identically closed forms locally constructed from a field*, *J. Math. Phys.* **31** (1990) 2378.

[29] R.M. Wald, *Black hole entropy in the Noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038].

[30] V. Iyer and R.M. Wald, *A comparison of Noether charge and euclidean methods for computing the entropy of stationary black holes*, *Phys. Rev.* **D 52** (1995) 4430 [gr-qc/9503052].

[31] K. Copsey and G.T. Horowitz, *The role of dipole charges in black hole thermodynamics*, *Phys. Rev.* **D 73** (2006) 024015 [hep-th/0505278].

[32] M. Rogatko, *First law of black rings thermodynamics in higher dimensional Chern-Simons gravity*, *Phys. Rev.* **D 75** (2007) 024008 [hep-th/0611260].

[33] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. and Quant. Grav.* **20** (2003) 4587 [hep-th/0209114].

[34] J.P. Gauntlett and J.B. Gutowski, *Concentric black rings*, *Phys. Rev.* **D 71** (2005) 025013 [hep-th/0408010].

[35] K. Behrndt, G. Lopes Cardoso and S. Mahapatra, *Exploring the relation between 4D and 5D BPS solutions*, *Nucl. Phys.* **B 732** (2006) 200 [hep-th/0506251].

[36] K. Behrndt et al., *Classical and quantum $N = 2$ supersymmetric black holes*, *Nucl. Phys.* **B 488** (1997) 236 [hep-th/9610105].

[37] M. Shmakova, *Calabi-Yau black holes*, *Phys. Rev.* **D 56** (1997) 540 [hep-th/9612076].

[38] P. Pasti, D.P. Sorokin and M. Tonin, *Covariant action for a $D = 11$ five-brane with the chiral field*, *Phys. Lett.* **B 398** (1997) 41 [hep-th/9701037].

[39] G.T. Horowitz and H.S. Reall, *How hairy can a black ring be?*, *Class. and Quant. Grav.* **22** (2005) 1289 [hep-th/0411268].

[40] R. Gregory, J.A. Harvey and G.W. Moore, *Unwinding strings and T-duality of Kaluza-Klein and H-monopoles*, *Adv. Theor. Math. Phys.* **1** (1997) 283 [hep-th/9708086].

[41] D. Sudarsky and R.M. Wald, *Extrema of mass, stationarity and staticity and solutions to the Einstein Yang-Mills equations*, *Phys. Rev.* **D 46** (1992) 1453.

[42] D. Astefanesei and E. Radu, *Quasilocal formalism and black ring thermodynamics*, *Phys. Rev.* **D 73** (2006) 044014 [hep-th/0509144].

[43] J.D. Brown and J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev.* **D 47** (1993) 1407.

[44] G.W. Gibbons, D. Kastor, L.A.J. London, P.K. Townsend and J.H. Traschen, *Supersymmetric selfgravitating solitons*, *Nucl. Phys.* **B 416** (1994) 850 [hep-th/9310118].

JHEP12(2007)057

[45] A. Castro, J.L. Davis, P. Kraus and F. Larsen, $5D$ *attractors with higher derivatives*, *JHEP* **04** (2007) 091 [hep-th/0702072].

[46] A. Castro, J.L. Davis, P. Kraus and F. Larsen, $5D$ *black holes and strings with higher derivatives*, *JHEP* **06** (2007) 007 [hep-th/0703087].

[47] M. Alishahiha, *On* R^2 *corrections for* $5D$ *black holes*, *JHEP* **08** (2007) 094 [hep-th/0703099].