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Theoretical Market Areas under Euclidean Distance

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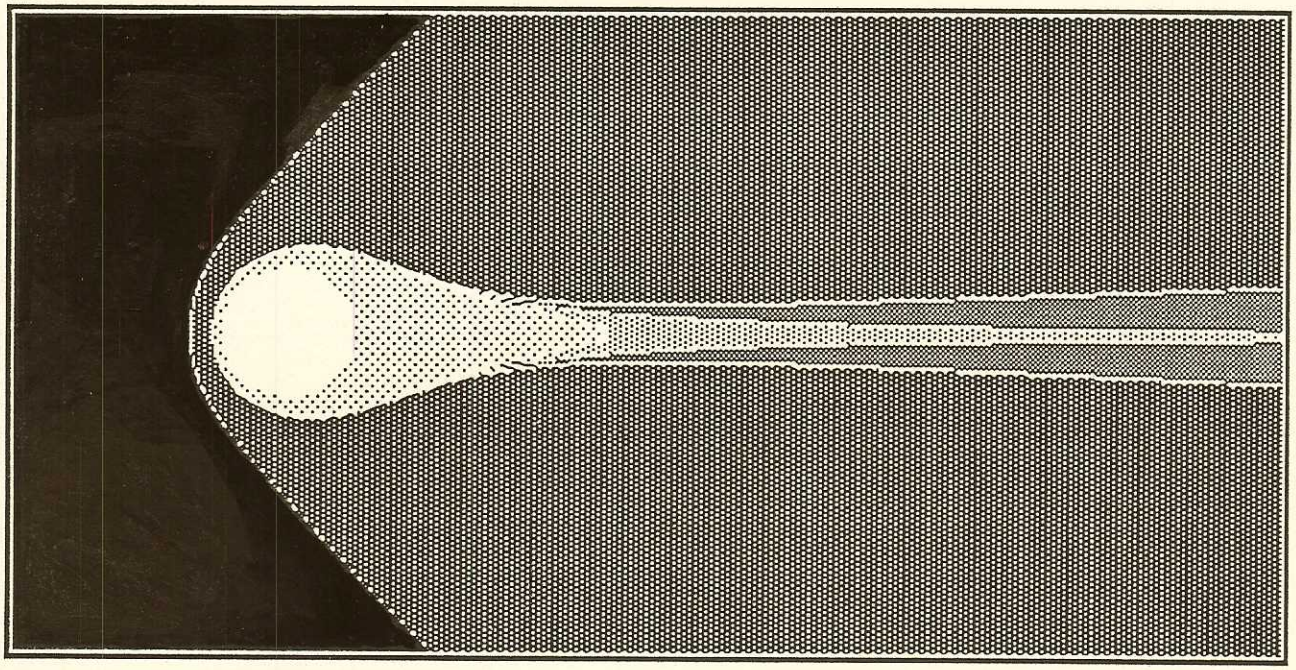

THEORETICAL MARKET AREAS
UNDER EUCLIDEAN DISTANCE

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ISBN: 1-877751-12-X

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Abstract

Though already initiated by Rau in 1841, the economic theory of the shape of two-dimensional market areas has long remained concerned with a representation of transportation costs as linear in distance. In the general gravity model, to which the theory also applies, this corresponds to a decreasing exponential function of distance deterrence. Other transportation cost and distance deterrence functions also appear in the literature, however. They have not always been considered from the viewpoint of the shape of the market areas they generate, and their disparity asks the question whether other types of functions would not be worth being investigated. There is thus a need for a general theory of market areas : the present work aims at filling this gap, in the case of a duopoly competing inside the Euclidean plane endowed with Euclidean distance.

Résumé

Bien qu'ébauchée par Rau dès 1841, la théorie économique de la forme des aires de marché planaires s'est longtemps contentée de l'hypothèse de coûts de transport proportionnels à la distance. Dans le modèle gravitaire généralisé, auquel on peut étendre cette théorie, ceci correspond au choix d'une exponentielle décroissante comme fonction de dissuasion de la distance. D'autres fonctions de coût de transport ou de dissuasion de la distance apparaissent cependant dans la littérature. La forme des aires de marché qu'elles engendrent n'a pas toujours été étudiée ; par ailleurs, leur variété amène à se demander si d'autres fonctions encore ne mériteraient pas d'être examinées. Il paraît donc utile de disposer d'une théorie générale des aires de marché : ce à quoi s'attache ce travail en cas de duopole, dans le cadre du plan euclidien muni d'une distance euclidienne.

Acknowledgments

Our thanks go first to the undaunted secretaries of GEOG, Mrs. Vielvoye and Riguelle, who performed the typing ; then to Pr. J.-F. Thisse (Department of Geography and Centre for Operations Research and Econometrics, Université Catholique de Louvain, Louvain-la-Neuve) for giving us his opinion about this work and drawing our attention to the study of the measure of market areas in relation with equilibrium problems ; to Pr. R. Lavendhomme (Department of Algebra, Geometry, and Mathematical Logics, and

Department of Pure and Applied Mathematics, Université Catholique de Louvain, Louvain-la-Neuve) for a clarifying discussion about mathematical topology and planar curve theory ; to Pr. M. Sonis (Department of Geography, Bar-Ilan University) for his suggestion to examine the constrained gravity model ; and to Prs. F. Aurenhammer, B. Boots, Y. Papageorgiou, W. Tobler, and many others, for the information and encouragement they provided.

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1. *Introduction*

In a central place system, space can generally be partitioned into urban market areas. The market area of a centre is the area where the influence of that centre is greater than (dominant with respect to) that of any other one. The model defining the mode of spatial urban influence is usually also suitable for determining the theoretical market area of every centre in a given system. A number of papers are devoted to market areas in the context of *central place theory* ; see e.g. Alao et al. (1977), Mulligan (1981). A theoretical paper by Webber (1974) relates population density with market areas (extension, town spacing). Geographers like Illeris (1967), Berry and Lamb (1974), Huff and Lutz (1979), have often delineated theoretical market areas by means of a *gravity model* of urban influence ; they commonly represent the gravitational deterrence of distance by a decreasing power function, and the indifference line between two centres is then circular, as shown by Tuominen (1949) and Godlund (1956). But the spatial flows, independently of market areas, are also modelled by means of other deterrence functions. Allen and Sanglier (1981) employ a decreasing power function, not of the distance alone but of the delivered price. And all the *spatial interaction models* use a decreasing exponential function of distance.

The *economic theory of the shape of market areas*, regarding the deterministic problem of a consumer making his choice between two places where a commodity is sold at two different mill prices, has reached a higher development. The transportation costs being proportional to distance, the indifference line is known to be a branch of a hyperbola if the transportation rates are the same for both centres but the f.o.b. prices differ. It is a circle if the f.o.b. prices are equal but the transportation rates differ ; and in the general case, it is a particular quartic curve, the Cartesian oval. This results from a long series of rediscoveries and developments described by Shieh (1985), Robine (1985), and also partially by Ponsard (1983). As background, the main names are those of Rau (1841), Launhardt (1882), Cheysson (1887), Fetter (1924), Palander (1935), Schneider (1935), and Hyson and Hyson (1950). Quadratic transportation costs have been studied too and give rise to a straight indifference line ; see Aurenhammer (1983).

There is clearly room for some research in both fields. The implications on the shape of market areas of the assumptions of Allen and Sanglier (1981) and of the spatial interaction model do not seem to have been examined yet. And there is no clear reason to ignore other possible transportation cost or deterrence functions than those presently used. But why should we bother, after all, about a general theory of market areas ? It seems so easy to draw conclusions from the study of the classical cases mentioned above, at least when the transportation rates are the same for both centres. The market area of the less attractive centre is obviously convex. It is finite when the transportation cost function is strictly concave : we shall see indeed that the case studied by Tuominen and by Godlund is equivalent in terms of the economic theory of market areas to the choice of a logarithmic transportation cost function, which is strictly concave. One property, perhaps, may hurt the common sense : when transportation costs are quadratic in distance, the less attractive centre may not belong to its own market area. Would it be because the relation of costs to distance is strictly convex ? Intuition gives here the right answer. But all the other properties we have suggested are false ... This was precisely the starting point of our research : noticing that some properties of market areas were not necessarily the same with concave as with convex transportation costs. We then tried to detect those properties through simulation, with a power transportation cost function. This immediately showed that the linear case was a threshold, but that the quadratic case was another one. Something else than the concavity or convexity of the transportation cost function was thus at work. What could it be ?

This work is organized as follows. In Section 2 we detail the gravitational and deterministic models of consumer behaviour to which we have alluded above, and see how the market areas they generate fit into a unique model precised in Section 3. Sections 4 to 11 study various properties of those areas, and are extended by a study of Descartes' ovals in Section 13 and by Section 12 which describes the effect of a relaxation of the very slight assumptions made in Section 3. We refer the reader wanting to get a condensed overall view of our main results concerning the shape of market areas to a communication of ours [Hanjoul et al. (1986)]. The quantitative aspects are summarized in Hanjoul and Thill (1986 and 1987), in the context of demand analysis and equilibrium theory. Although we consider two centres only, it is possible to draw some conclusions about any number of centres. Denote indeed by Z_{jk} the area where the influence of centre j is greater than that of k. If C is the set of all centres, the market area Z_j is the intersection of the areas Z_{jk}, for all $k \in C$. The boundary of Z_j is consequently made of pieces of boundaries of market areas defined by couples of centres, and our statements obtain for each of those pieces separately.

Note about the figures

The figures representing market areas are inserted in rectangular frames. The left edge of those frames is a part of the y-axis, and centre j is thus symmetric of centre k with regard to that edge ; unless otherwise mentioned, either implicitly (centre j appearing together with centre k) or explicitly (Fig. 10.3b). The horizontal stroke in the lower left corner of the frame is the distance unit ; if a number is written above it, however, the stroke represents the distance unit multiplied by that number.

2. *Modelling consumers' behaviour*

2.1. *The gravity model*

Consider two urban centres j and k and any place i in the geographical space. A first mathematical model used to represent the interaction between places and which can help to define market areas is the gravity model whose general form may be written [Sheppard (1978)] :

$$t_{ij} = K\, m_i G(A_j)\, F(\delta_{ij}) \qquad (1)$$

where t_{ij} is the amount of interaction between j and i due to the relative influence exerted by j on i; m_i is a function of the attributes of place i; $G(A_j)$ is a function of the attractivity A_j of centre j; $F(\delta_{ij})$ is a decreasing function of the distance δ_{ij} between i and j; K is a proportionality factor. All these functions are assumed to be continuous and strictly positive.

As we are considering two centres only, the market area Z_k of centre k is the set of places where the influence of k is stronger than that of j, i.e., where $t_{ik} \geqslant t_{ij}$. Equation (1) shows that such is the case when

$$F(\delta_{ik}) \;/\; F(\delta_{ij}) \geqslant G(A_j) \;/\; G(A_k). \qquad (2)$$

If we define an increasing function h of distance and an index Q of comparative attractivity as follows :

$$h(\delta) = -\ell n\; F(\delta) \qquad (3)$$
$$Q = \ell n\, [G(A_j)/G(A_k)],$$

then (2) becomes

$$h(\delta_{ij}) - h(\delta_{ik}) \geqslant Q \qquad (4)$$

which describes market area Z_k as the set of places i where the proximity of k wrt. that of j (i.e., k with respect to that of j) surpasses or equals the advantage of j in terms of attractivity.

The introduction of function h simplifies the statements, as will be seen throughout the paper. That function, which enables us to write F (δ) in the more familiar form $e^{-h(\delta)}$, may (but not necessarily) be interpreted as a transportation cost, or as a perceived disutility associated with distance deterrence. The first interpretation is of course wrong in the model of Allen and Sanglier (1981). And when the deterrence function is the more frequent power function $\delta^{-\beta}$, it is impossible to consider h (δ), here equal to $\beta \ln \delta$, as a transportation cost : h (δ) is indeed < 0 when $\delta < 1$. Anyway, the sign of h (δ) does not matter : inequality (4) shows that we may add any constant number to function h without modifying the market areas, so that the positivity of h (δ) cannot bring any useful information. What is really important is the difference between h (δ) and the value of h at some reference distance : e.g. h (0).

Several related interaction issues are reducible to inequality (4). Once submitted to a logarithmic transformation, the linear interaction formula of Isard (1960, p. 512) is not different from that of Sheppard. Also, as noted by Beckmann (1971 and 1972), if we want to determine from Sheppard's model the area where the *sales ratio* t_{ik}/t_{ij} of centre k is at least equal to some value σ, or equivalently where its *market share* $t_{ik}/(t_{ik} + t_{ij})$ is at least equal to $\sigma/(1+\sigma)$, we are back again to inequality (4) : the constant Q must simply be increased by $\ln \sigma$. In this interpretation the demarcation line described by h (δ_{ij}) - h (δ_{ik}) = Q is called an *isoshare line*. The constrained interaction models [see e.g. Wilson (1970)] are similarly amenable to ours, as they are described by equality (1) with K replaced by the product $E_i B_j K$; $E_i = 1$ if the model is not consumption-constrained, and $B_j = 1$ if it is not production-constrained. The constant Q has thus to be increased by $\ln (B_j/B_k)$ if the productions t_j and t_k are known a priori. Two interesting properties emerge in that case. First, a change in the deterrence function or in the set of consumers implies a change in Q as

$$B_c = t_c/K\ G\ (A_c)\ \Sigma_i E_i m_i\ F\ (\delta_{ic})$$

Second, if additional centres appear, and if the model is simultaneously consumption-constrained, Q is also modified because the coefficients B_c are then interrelated through the coefficients E_i, for which a similar equality holds. A fourth issue is that of the area where the interaction difference $t_{ik} - t_{ij}$ is greater than some given value D. That area is defined by

$$G(A_k)\ F\ (\delta_{ik}) - G\ (A_j)\ F\ (\delta_{ij}) \geqslant D/K\ m_i.$$

It will appear from Section 2.2 that this area can be studied by means of our model if m_i is the same for all i. Section 2.2 will also show how to treat problems in which the delivered price, not the distance, is the argument of the deterrence function ; or in which distance is affected by a coefficient depending on the firm considered. In particular, in the model of Allen and Sanglier (1981), the equation of the indifference line becomes in the most complex case

$$(p_j + r_j\ \delta_{ij})\ /\ (p_k + r_k\ \delta_{ik}) = [G\ (A_j)/G\ (A_k)]^{1/\beta}$$

which clearly describes a Cartesian oval, a circle, or one branch of a hyperbola.

2.2. *A deterministic interaction model*

In a 1976 paper, Beckmann assumes that the utility u_{ij}, for a possible consumer located at a place i, of interacting with a centre j is expressed by :

$$u_{ij} = \gamma(A_j) - \eta(\delta_{ij})$$

where A_j is the attractivity of centre j; functions γ and η are of course increasing wrt. their argument. The consumer consequently chooses to interact with centre j rather than with centre k ($\Rightarrow$ $t_{ik} = 0$) when $u_{ij} > u_{ik}$, and vice versa; when $u_{ij} = u_{ik}$, we may assume that it interacts equally with both centres, i.e. $t_{ij} = t_{ik}$. If we limit ourselves to that deterministic model, a place i belongs to the market area Z_k of centre k iff. (if and only if) $u_{ij} \leqslant u_{ik}$, i.e.,

$$\eta(\delta_{ij}) - \eta(\delta_{ik}) \geqslant \gamma(A_j) - \gamma(A_k), \qquad (5)$$

which is again equivalent to (4) if we adapt as follows the definitions of h and Q :

$$h = \eta$$
$$Q = \gamma(A_j) - \gamma(A_k).$$

Assuming all the consumers to be similar, we get the same inequality (5) when Z_k is defined as the area where $u_{ik} - u_{ij} \geqslant \varepsilon$: we just have to increase Q by ε ; when Z_k is the set where $u_{ik}/u_{ij} \geqslant \varepsilon$, the transformation that is studied hereafter allows again to use (5). On the other hand, notice that the model is related to the gravity one : after introducing it, Beckmann adds stochastic terms to u_{ij} and u_{ik} and shows that it is so possible to derive the consumption-constrained gravity model under a few assumptions and approximations.

In particular, when $u_{ij} = - p_j - r\delta_{ij}$ (where p_j is the mill price at j, and where r is a constant transportation rate), the demarcation line is now the set of points where the delivered prices relative to both centres are equal. As such, it has received the name of *isostant* from Schilling (1925). That problem has been studied by Rau (1841) and his successors. They generally also allow, however, the transportation rate r to depend on the centre with which interaction occurs; i.e., they write r_j and r_k instead of r. In other words, the accessibilities to the two centres are different. When $p_j = p_k$, i.e., when the f.o.b. price is uniform, the inequality of market area Z_k becomes

$$\delta_{ij} \,/\, \delta_{ik} \geqslant r_k/r_j,$$

which has been known since the Greek mathematician Apollonius of Perga († 180 BC) to represent either a disk containing k if $r_k > r_j$, or the exterior of an open disk containing j if $r_j > r_k$. As that inequality of Z_k may be written

$$\ell n \; \delta_{ij} - \ell n \; \delta_{ik} \geqslant \ell n \; (r_k/r_j) \, ,$$

we are back again to our own model, although h is not here a transportation cost any more.

When $p_j \neq p_k$ and $r_j \neq r_k$, the common boundary of the two market areas is a quartic curve called Cartesian oval in honour of Descartes (1637) who first discovered the family of such curves as a solution to a problem in optics. This case is also amenable to our model. The inequality of market area Z_k is now indeed

$$r_j \; \delta_{ij} - r_k \; \delta_{ik} \geqslant p_k - p_j \; ;$$

if we define the ratio

$$K = \frac{p_k - p_j}{r_k - r_j},$$

that inequality may be rewritten

$$r_j(K + \delta_{ij}) \geqslant r_k(K + \delta_{ik}).$$

If $K \geqslant 0$, this is equivalent to

$$\ell n(K + \delta_{ij}) - \ell n(K + \delta_{ik}) \geqslant \ell n\ (r_k/r_j),$$

and the properties of market areas can be studied through our model. Interestingly, $K \geqslant 0$ means that the cheapest centre is also the most accessible one, and thus indubitably the most attractive. The complete amenability of that case to our model might be viewed as resulting from the joint effect of the differences in accessibilities and prices.

At the opposite, things become more difficult when $K < 0$, i.e., when the cheapest centre is the least accessible one. To apply our model, we have now to distinguish between four regions of the plane :

1) $\delta_{ij} \geqslant - K \leqslant \delta_{ik}$. Here the inequality of Z_k can be treated in the same way as when $K \geqslant 0$.

2) $\delta_{ik} \leqslant - K \leqslant \delta_{ij}$. That region is obviously a part of Z_k.

3) $\delta_{ij} \leqslant - K \leqslant \delta_{ij}$. That region is a part of Z_j.

4) $\delta_{ij} \leqslant - K \geqslant \delta_{ik}$. The inequality of Z_k can here be given the following form:

$$[- \ell n(- K - \delta_{ij})] - [- \ell n\ (- K - \delta_{ik})] \geqslant \ell n\ (r_j/r_k)$$

and can again be studied by means of our model.

For this fourth situation to happen, it is necessary to have $\delta_{jk} \leqslant - 2\ K$, which means that the middle of the segment [jk] belongs to the market area of the most expensive centre. This never occurs in our model : see Prop. 4.3. It is thus important to understand that although Descartes'ovals can be described by our model, their properties differ, when $K < 0$, from those of the market areas appearing in the genuine model. We shall resume this point in

Section 13. Of course, the more general problem where Z_k is defined by

$$r_j\, c(\delta_{ij}) - r_k\, c(\delta_{ik}) \geqslant p_k - p_j,$$

where c is some increasing function of distance, can be similarly studied. In particular, when $c(\delta) = \delta^2$, the demarcation line is straight if $r_j = r_k$, or circular if $r_j \neq r_k$; see Section 5.5, Prop. 4.7c and 4.7e.

3. *Delimitation of the problem and definitions*

Given some increasing real function h, our problem is to describe the two following market areas :

$$Z_j = \{i;\ h(\delta_{ij}) - h(\delta_{ik}) \leqslant Q\}$$

$$Z_k = \{i;\ h(\delta_{ij}) - h(\delta_{ik}) \geqslant Q\} \qquad (6)$$

where δ_{ij} and δ_{ik} are the *Euclidean* distances between i and j and between i and k, respectively ; Q is the *attractivity index*, and we shall refer to $|Q|$ as to the *absolute* attractivity index in our Conclusion.

The issue, a purely geometrical one, is clearly akin to the once fashionable studies about curves as recollected in the treatises of Salmon (1879)or Gomes Texeira (1908). However,our approach is rather different. In particular, those authors do not seem to have paid much attention to questions considered in our Sections 6 and 8 (see eg. the rash discussion about inflexion points in Salmon, item 204).

From now on we suppose that function h is continuous on $]0, +\infty[$, right-continuous lato sensu at 0, i.e.,

$$h(0) = \lim_{\substack{\delta \to 0 \\ >}} h(\delta),$$

and strictly increasing, to ensure that Z_k and Z_j are closed sets and that $Z_j \cap Z_k$ is 1-dimensional when $\neq \emptyset$ (see Section 12).

We shall refer to function h as to the *transportation cost function,* although it is seen from Section 2 that this is not its only possible interpretation. The index Q is related with the *attractivities* of the two centres; without loss of generality, we assume that the attractivity of centre j is greater than that of centre k, i.e.,

$$Q > 0.$$

As area Z_k is then obviously smaller than Z_j, it seems most of the time convenient to focus on the properties of Z_k without mentioning the complementary ones of Z_j. On the other hand, some properties are easier to enounce in reference to $Z_j \cap Z_k$, which is the common boundary of Z_j and Z_k and whose definition writes

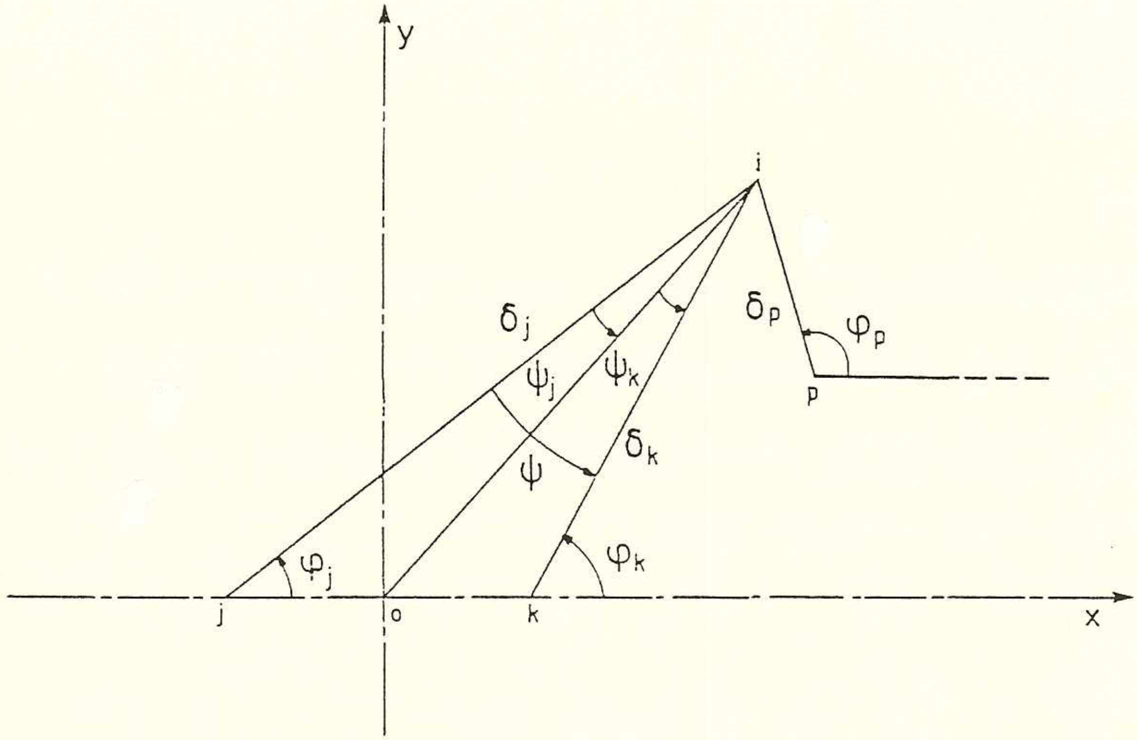


Fig. 3.1. The systems of rectangular and polar coordinates.

$$Z_j \cap Z_k = \{i \; ; \; h(\delta_{ij}) - h(\delta_{ik}) = Q \}. \qquad (7)$$

To make the writing more concise, some compact notation is used. Index i is dropped when this entails no confusion, so we write δ_j, δ_k instead of δ_{ij}, δ_{ik}; $h(\delta_j)$ and $h(\delta_k)$ are simplified into h_j and h_k, and similarly for the derivatives of h. Also, we set

$$\Delta h_i = h(\delta_{ij}) - h(\delta_{ik}),$$

usually shortened into Δh and called the transportation cost difference.

In order to be able to describe accurately the position of any point wrt. the centres, we assimilate the plane to $\mathbb{R}^2$. The point (0,0), called o, is the middle of the segment [jk]. The centres themselves are $j = (-\delta_{ok}, 0)$ and $k = (\delta_{ok}, 0)$, with the exception of Proposition 4.6.a. The straight lines are referred to as vertical lines when parallel to the y-axis, horizontal lines when parallel to the x-axis, and oblique lines otherwise. We also attach to any point p of the plane a system of polar coordinates (δ_p, φ_p), the angle φ_p (abbreviation for φ_{pi}) being defined as in Fig. 3.1.

Last, to avoid any confusion with a negative power, we denote the inverse relation of any function or relation f by $f^{\wedge}$.

4. *Elementary properties of market areas*

A first question, which is not related to the particular space and distance considered, is whether it is possible to deduce the transportation cost function from the knowledge of the market areas. Obviously that identification can usually not be complete, as the range of distances concerned by the equation of $Z_j \cap Z_k$ (from which that identification should be carried on) is generally not $[0, +\infty[$ but

$$I = [\min_{i \in Z_j \cap Z_k} \delta_k, \max_{i \in Z_j \cap Z_k} \delta_j[.$$

(The determination of that interval is made in Corollary 6.2). On the other hand it is obvious from (4) that replacing $h(\delta)$ by $\mu h(\delta) + \lambda$ has no effect either on Δh or on market areas if $\mu > 0$ and Q is replaced by μQ. In particular, constant fixed transportation costs bear no influence on market areas; they do determine, however, which market areas j and k belong to (see Section 12.2).

But function h may be transformed more deeply without altering the market areas. Let us indeed denote by Z_j and Z_k (respectively $\tilde{Z}_j$ and $\tilde{Z}_k$) the market areas associated with a transportation cost function h (resp. $\tilde{h}$) and a value Q (resp. $\tilde{Q}$) of the attractivity constant. The general property is then (we set $\mathbb{N}^* = \mathbb{N} - \{0\}$):

(4.1) (a) *If h and $\tilde{h}$ are related as follows for all $\delta > 0$:*

$$\tilde{h}(\delta) = \frac{\tilde{Q}}{Q} [h(\delta) + f(h(\delta))] \quad (8)$$

where f is a constant or periodic function with period Q/n for some $n \in \mathbb{N}^*$, then $\tilde{Z}_k = Z_k$ and $\tilde{Z}_j = Z_j$;

(b) *If $\tilde{Z}_k = Z_k$ and $\tilde{Z}_j = Z_j$, then h and $\tilde{h}$ are related by (8) for every δ of the interval I and for some function f having the properties mentioned in (a).*

The equality between market areas in item (a) is equivalent to the equality of the boundaries, i.e., $Z_j \cap Z_k = \tilde{Z}_j \cap \tilde{Z}_k$, which itself derives from the fact that the two following sets

$$A = \{(\delta_j, \delta_k); \quad h_j - h_k = Q\}$$

$$\tilde{A} = \{(\delta_j, \delta_k); \quad \tilde{h}_j - \tilde{h}_k = \tilde{Q}\}$$

(where no subscript 'i' is implied this time) are equal; the set of the coordinates (δ_j, δ_k) of points of $Z_j \cap Z_k$ (resp. $\tilde{Z}_j \cap \tilde{Z}_k$) is the subset of A (resp. $\tilde{A}$) satisfying the triangular inequalities

$$\begin{aligned} \delta_j + \delta_k &\geq \delta_{jk} \\ |\delta_j - \delta_k| &\leq \delta_{jk} . \end{aligned} \qquad (9)$$

The inclusion $A \subseteq \tilde{A}$ is obvious : if $(\delta_j, \delta_k) \in A$, then $h_j - h_k = Q$ so that $f(h_j) = f(h_k)$ and $\tilde{h}_j - \tilde{h}_k = \tilde{Q}$. As to the fact that $\tilde{A} \subseteq A$, take now some $(\delta_j, \delta_k) \in \tilde{A}$ and let $\delta_j' = h^{\wedge}(h_k + Q)$; then $(\delta_j', \delta_k) \in A$. As $A \subseteq \tilde{A}$, we have also $(\delta_j', \delta_k) \in \tilde{A}$, which implies that $\delta_j' = \tilde{h}^{\wedge}[\tilde{Q} + \tilde{h}(\delta_k)] = \delta_j$. Hence $(\delta_j, \delta_k) = (\delta_j', \delta_k) \in A$ and $\tilde{A} \subseteq A$: Q.E.D.

Now for item (b). For every point i of $Z_j \cap Z_k$, i.e., of $\tilde{Z}_j \cap \tilde{Z}_k$, we have $h_j - h_k = Q$ and $\tilde{h}_j - \tilde{h}_k = \tilde{Q}$; so that

$$\frac{h_j}{Q} - \frac{\tilde{h}_j}{\tilde{Q}} = \frac{h_k}{Q} - \frac{\tilde{h}_k}{\tilde{Q}} .$$

On the range of distances to j and k concerned with $Z_j \cap Z_k$, i.e., on $[\min \{\delta_k; i \in Z_j \cap Z_k\}, \max \{\delta_j; i \in Z_j \cap Z_k\}[$ (the interval is open on the right because the upper bound may be $+\infty$), we thus see that the function f defined by $f[h(\delta)] = h(\delta)/Q - \tilde{h}(\delta)/\tilde{Q}$ must be either constant or periodical with period Q/n for some $n \in \mathbb{N}^*$. On the range H of $h(\delta)$ corresponding to the above-mentioned range of δ, we have indeed $f(\xi) = f(\xi + Q)$ for any $\xi \in H$. (We could also say, equivalently, that f is periodical with period Q/n, with $n \in \mathbb{N}^* \cup \{+\infty\}$). Hence the statement.

Prop. 4.1. shows that the identification of the transportation cost function (which identification should probably be carried out by means of Fourier's or Lagrange's transforms) does not produce a unique result unless some assumptions are made. What such assumptions should be is a problem we have left out of the field of our investigation, but we may point out the fact that the requirement that function h should be concave would not suffice : the expressions $\ell n(\delta) + \sin[0.5\ \ell n(\delta)]$ and $-(\delta+1)^{-1} - \sin[0.5\ (\delta+1)^{-1}]$, e.g., are concave and increasing wrt. δ.

Let us illustrate Prop. 4.1 by a striking example. The function h defined by $h(\delta) = \delta^2 + \sin \delta^2$ is continuous and strictly increasing; so we may call it a transportation cost function. It is known (see Prop. 4.7c) that when the transportation cost is given by δ^2 the demarcation line $Z_j \cap Z_k$ is vertical. Consequently, function h also yields a vertical demarcation line when $Q = 2\pi n$, for all $n \in \mathbb{N}$; see Fig. 4.1.

The following proposition is also obvious from (4) :

(4.2) *Market areas Z_j and Z_k are symmetric with respect to the x-axis $\mathbb{R} \times \{0\}$.*
The properties of Z_k and Z_j above the axis of x consequently characterize the whole market areas. This point will be used throughout the paper. It is also worth noticing that the distances δ_j and δ_k constitute a system of coordinates of $\mathbb{R} \times \mathbb{R}_+$, in the sense that to every point in $\mathbb{R} \times \mathbb{R}_+$ corresponds exactly one couple (δ_j, δ_k) and that every such couple is associated with one point at most in $\mathbb{R} \times \mathbb{R}_+$. Those coordinates are called *dipolar*; see Jones (1979).

(4.3) *Market area Z_k is strictly included in the part of the plane at the right of the y-axis $\{0\} \times \mathbb{R}$ and has no point on that y - axis* ; i.e., $Z_k \subset \mathbb{R}^*_+ \times \mathbb{R}$.
This derives from (6), where $Q > 0$ entails $\delta_j > \delta_k$ in Z_k. The property implies $\mathbb{R}_- \times \mathbb{R} \subset Z_j$, which says that any point i is more influenced by j than by k when both $A_j > A_k$ (i.e., $Q > 0$) and $\delta_j \leq \delta_k$ (i.e., $i \in \mathbb{R}_- \times \mathbb{R}$), if we refer to the gravity model of Section 2.1.

(4.4) *Let i be a point moving on $Z_j \cap Z_k$. The distances from i to all the points of $[jk]$ increase or decrease simultaneously.*
The proof is as follows. First, δ_j and δ_k increase or decrease simultaneously since $\delta_j = h^{\wedge}[h(\delta_k) + Q]$; h and $h^{\wedge}$ are indeed strictly increasing functions.

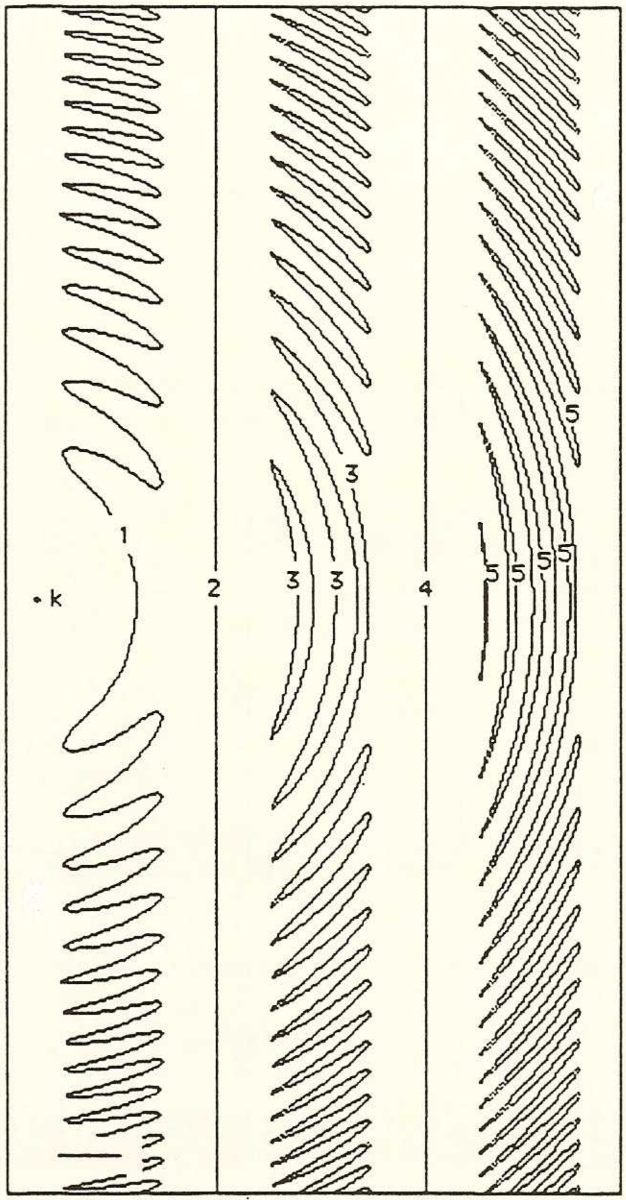


Fig. 4.1. Market areas generated by the t.c.f. $\delta^2 + \sin \delta^2$ when $\delta_{jk} = 0.96$ and $Q = n\pi$, for indicated values of n.

Second, take any point $p \in [jk]$. The following relations hold :

$$\delta_j^2 = \delta_{jp}^2 + \delta_p^2 + 2\delta_{jp}\ \delta_p\ \cos\varphi_p$$

$$\delta_k^2 = \delta_{kp}^2 + \delta_p^2 - 2\delta_{kp}\ \delta_p\ \cos\varphi_p .$$

Eliminating $\cos\varphi_p$ makes δ_p^2 appear as an increasing function of both δ_j^2 and δ_k^2; hence the property.

The next two propositions deal with the influence of two basic data of the problem : the attractivity constant Q and the distance δ_{jk} between the two centres. They show that the market area Z_k of the less attractive centre shrinks when the difference in attractiveness between the centres increases or when the distance between them decreases.

(4.5) *Market area Z_k is a strongly decreasing function of Q.*
The meaning of 'strongly decreasing' is as follows. Suppose that Z_k' and Z_j' are the market areas corresponding to another value Q' of the attractivity constant. The property (see Fig. 4.2.) signifies that

$$Q' < Q \Rightarrow Z_k \subseteq Z_k' - Z_j' .$$

Otherwise said, if parameter Q decreases, any point in Z_k remains in Z_k' (as $\Delta h \geq Q \Rightarrow \Delta h > Q'$), and the points of the previous boundary $Z_j \cap Z_k$ cannot remain on the new boundary (as $\Delta h = Q \Rightarrow \Delta h > Q'$). This is immediately understandable from (7) which shows that the boundaries $Z_j \cap Z_k$ corresponding to various values of the attractivity constant Q are *iso-Δh lines* of the plane.

Proposition 4.5 may be particularized to some interesting points of Z_k. For instance, if we call x_r and x_ℓ the abscissae of the extreme right (if any) and left points r and ℓ of $Z_k \cap (\mathbb{R} \times \{0\})$, *$x_\ell$ is a strictly increasing and x_r a strictly decreasing function of Q.*

(4.6) (a) *If k is fixed and if j is allowed to move on the x-axis at the left of k, then Z_k is a strongly increasing function of δ_{jk};*
(b) *Whatever the way the positions of the centres vary, the measure $|Z_k|$ of area Z_k is a strictly increasing function of δ_{jk} when $|Z_k|$ is neither 0 nor $+\infty$.*

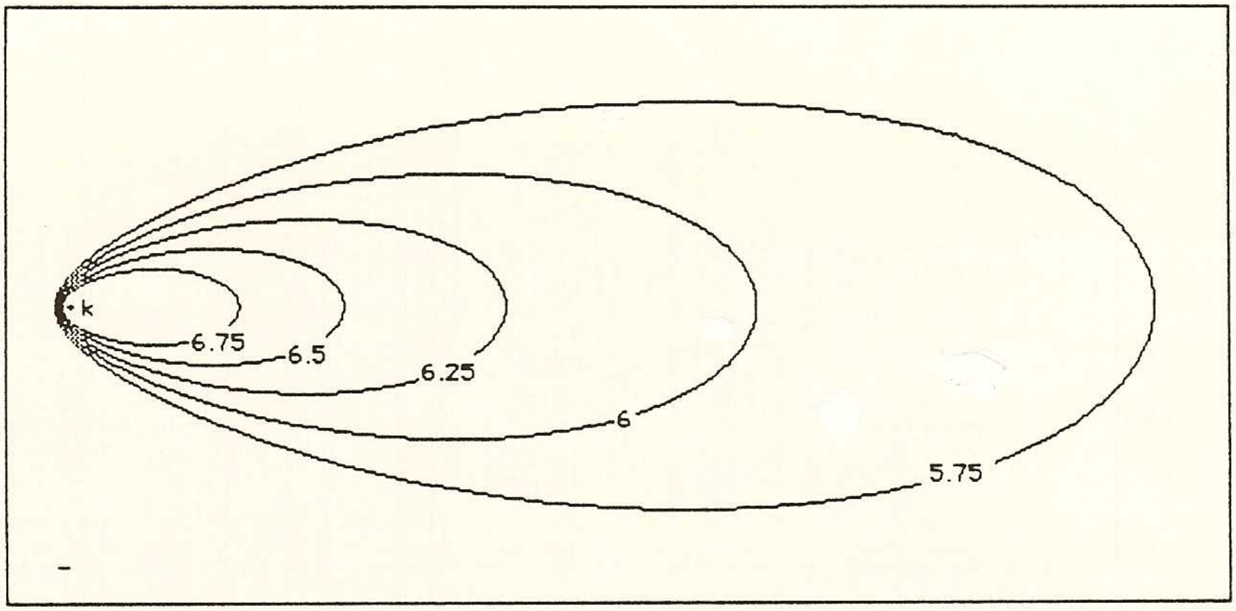


Fig. 4.2. Market areas generated by the power t.c.f. $\delta^{0.9}$ when $\delta_{jk} = 10$, Q being as indicated.

Subproposition(b) is straightforwardly derived from (a), as the change of Z_k for a variation of the location of centres j and k may always be studied through some appropriate variable coordinate system. As to point (a) : consider the market areas for a given value 2 L of the distance between centres. We have thus j = (-L,0) and k = (L,0). Let now j move toward the left, without modifying the system of axes. Point k remains equal to (L,0), whereas j has become (-L',0) with L' > L. Consequently, for all points of the half-plane $\mathbb{R}_+ \times \mathbb{R}$, the distance to j and thus the transportation cost difference Δh have strictly increased. In particular, as $Z_k \subset \mathbb{R}_+ \times \mathbb{R}$ (Prop. 4.3), we have $(\Delta h)' > \Delta h \geqslant Q$ on Z_k, so that $Z_k \subseteq Z'_k - Z'_j$. Notice that Prop. 4.6a may be particularized in the same way as Prop. 4.5 : *x_ℓ is a strictly decreasing and x_r (if r exists) a strictly increasing function of δ_{jk} if, k remaining fixed, j is allowed to move at the left of k on the x-axis.*

The next proposition lists cases where Z_k is known to have a simple shape with well-established properties ; 'coth' stands for hyperbolic cotangent. The deterrence function F (δ), if we refer to the gravity model, then takes one of the forms $\delta^{-\beta}$, exp $(-\beta\delta)$, exp $(-\beta\delta^2)$, $(K + \delta)^{-\beta}$, $(K + \delta^2)^{-\beta}$. As said in our introduction, the first case has already been solved in this context ; the two sequent ones, in the economic theory of market areas. Remember that β may be replaced by 1 (Prop. 4.1).

(4.7) (a) *If $h = \ell n$, Z_k is an Apollonian disk relative to points j and k ;*
(b) *If $h = .$, Z_k is the convex area limited by the right branch of a hyperbola with foci j and k ;*
(c) *If $h = .^2$, Z_k is the half-plane $x \geqslant Q/2\delta_{jk}$;*
(d) *If $h = \ell n\,(K + .)$ for some $K > 0$, Z_k is convex, bounded, more extended in the direction of x than in that of y, and the indifference line is a Cartesian oval ;*
(e) *If $h = \ell n\,(K + .^2)$ for some $K > 0$, Z_k is a disk with centre $[\delta_{ok}\, \coth\,(Q/2), 0]$ and radius $[\delta_{jk}^2\, /\, (4sh^2\,(Q/2)) - K]^{1/2}$.*

All those properties are easily proved. As to (a), the inequality of Z_k may be written $\delta_j/\delta_k \leqslant e^Q$, which Apollônios († 180 BC) has shown to describe a disk the centre of which lies on the x-axis at the right of k. Point (b) is obvious since the indifference line is described by $\delta_j - \delta_k = Q$ which clearly represents the right branch of a hyperbola. As regards point (c), which would have already been examined by Laguerre († 1886), δ_j and δ_k can be expressed as functions of x and y, as follows :

$$\delta_j^2 = (x + \delta_{ok})^2 + y^2$$
$$\delta_k^2 = (x - \delta_{ok})^2 + y^2 \qquad (10)$$

from which appears the well-known Euclidean relation

$$\delta_j^2 - \delta_k^2 = 2\, x\, \delta_{jk} \ . \qquad (11)$$

As the inequality concerning Z_k becomes $\delta_j^2 - \delta_k^2 \geqslant Q$, the proof of this item is complete. It has already been shown in Section 2.2 that the indifference line is a Cartesian oval in point (d). The properties of Z_k will be established in Section 13, where they correspond to the case $r_j < r_k$. The last item (e) is a consequence of Sections 2.2 and 5.5. The inequality concerning Z_k may in that case be written successively as

$$\ell n\ (K + \delta_j^2) - \ell n\ (K + \delta_k^2) \geqslant Q$$
$$K + \delta_j^2 \geqslant e^Q\ (K + \delta_k^2),$$

i.e., because of (10)

$$(e^Q - 1)\ (y^2 + x^2 + \delta_{ok}^2) - (e^Q + 1)\ 2\, x\, \delta_{ok} + (e^Q - 1)\ K \leqslant 0$$

or, if we set $\lambda = (e^Q + 1)/(e^Q - 1) = \coth\ (Q/2)$,

$$y^2 + x^2 - 2\, \lambda\, x\, \delta_{ok} + \delta_{ok}^2 + K \leqslant 0$$
$$y^2 + (x - \lambda\delta_{ok})^2 \leqslant \delta_{ok}^2\ (\lambda^2 - 1) - K = 4\ \delta_{ok}^2\ e^Q/(e^Q - 1)^2 - K.$$

Items (a), (d), and (e) call for two remarks. First : should we specify $K \geqslant 0$ instead of $K > 0$, item (a) would be a particular case of (d) and of (e). The centre - independent of K - and radius mentioned in (e) are also valid for (a) if we substitute Q/2 by Q. Second : in (a) and (e), we know the radii of the disk, thus their superficies too. As far as we know, and apart from the approximate results of Section 11, this is the only case where such an analytic expression of the measure $|Z_k|$ is available.

This proposition provides us with a set of instances of the general properties we shall discover. Let us thus have a closer look at their main apparent features. In items (a), (b), and (d), function h is concave, which is empirically appealing. In item (b) it is linear, i.e., concave and convex at the same time. In those three cases, centre k lies in Z_k. In item (c), function h is strictly convex. It is more complex in (e) : it is here strictly convex on $[0, \sqrt{K}]$ and concave beyond ; the marginal cost or disutility grows with distance in short trips, but decreases when the trip length exceeds $\sqrt{K}$; also, $h'(0) = 0$. The centre of Z_k then lies at the right of k. In both cases (c) and (e), k may belong to its own market area or not. In item (d), and also somehow in (b), Z_k is more extended in the direction defined by j and k than in the perpendicular one (Fig. 13.3b). In all items, Z_k is a convex set.

Item (d) corresponds to the model of Allen and Sanglier (1981) in the particular case of identical mill prices and transportation rates but different attractivities ; see Section 2.1. Item (e) may result from a generalization of that model to quadratic transportation costs. Those two particular transportation costs functions are depicted in Fig. 4.3.

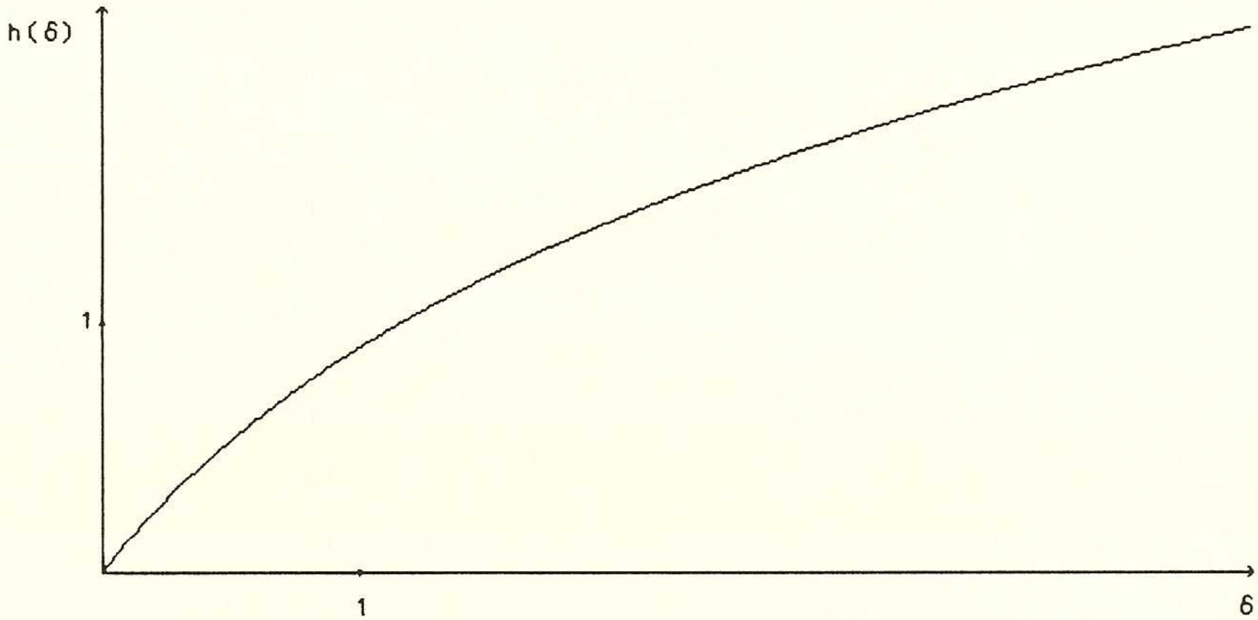


Fig. 4.3a. The t.c.f. 1.3 ln (K + δ), which generates Cartesian ovals as indifference lines. The value of K is here 1.

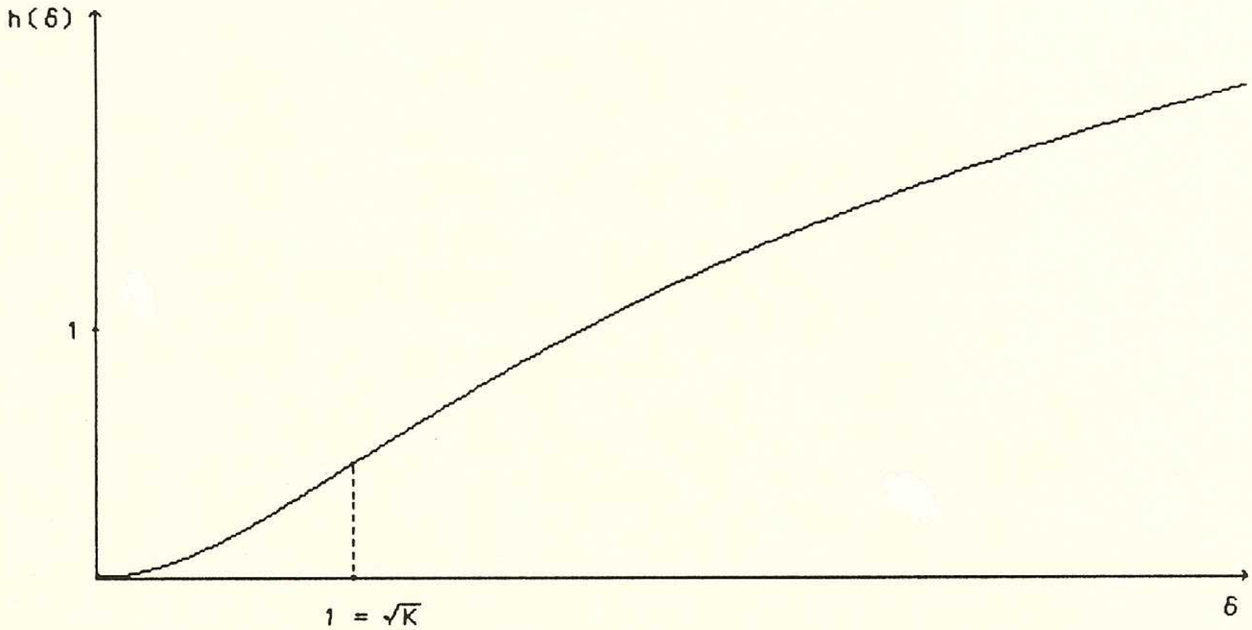


Fig. 4.3b. The t.c.f. 0.65 ln $(K + \delta^2)$, which generates circles as indifference lines. The inflexion point of the function occurs when $\delta = \sqrt{K}$. The number K is here equal to 1.

5. *Variation of the transportation cost difference along specific curves*

In this section we review how the transportation cost difference Δh varies along some curves of the plane . The next sections 6 and 8 make use of those 'microscopic' results to produce more elaborate statements, yet in their simple form they already yield immediate information about Z_j and Z_k in the following way. Suppose that a point i belongs to Z_k. If point i moves (thus creating a curve) so as to increase Δh, it obviously does not leave Z_k since Δh remains larger than Q : intuitively, it moves inward relatively to Z_k (or away from Z_j). When Δh decreases, point i intuitively moves outward relatively to Z_k or toward Z_j. *Mutatis mutandis*, a similar property holds when $i \in Z_j$.

Figure 5.1 illustrates some points of this section by indicating, on curves of the appropriate type, the direction in which Δh increases.

From now on, the statements may implicitly suppose the derivability of function h. They remain valid, however, when h is *derivable almost everywhere*, i.e., derivable except perhaps at a denumerable set $\{\delta_n \; ; \; n \in \mathbb{N}\}$ of values of its argument, with $\delta_n \leqslant \delta_{n+1} \; \forall n \in \mathbb{N}$. The derivative $h'(\delta)$ is then to be replaced by the *subgradient* or *generalized derivative* $\partial h(\delta)$ of h, i.e., the interval between the left and right derivatives of h at δ. The partial derivatives mentioned in the propositions are to be understood similarly as intervals. Fig. 8.3 and its discussion in Section 8.3 give an instance of that enlarged applicability of the statements.

5.1. *Move along circles centred on a point of* $[jk]$ (Fig. 5.1 a)

(5.1) *For any point* $p \in [jk]$, $(\partial \Delta h / \partial \varphi_p)_{\delta_p} < 0$ *on* $(\mathbb{R} \times \mathbb{R}_+) - \{p\}$.

If we move above the x-axis (i.e., inside $\mathbb{R} \times \mathbb{R}_+$) on a circle centred on p (i.e., with constant δ_p) so as to increase φ_p, it is obvious that δ_k increases strictly and δ_j decreases strictly when $p \in \,]jk[$. When p = j (resp. k), δ_j (resp. δ_k) remains constant while δ_k increases (resp. δ_j decreases) strictly. As a consequence, Δh decreases strictly in every case.

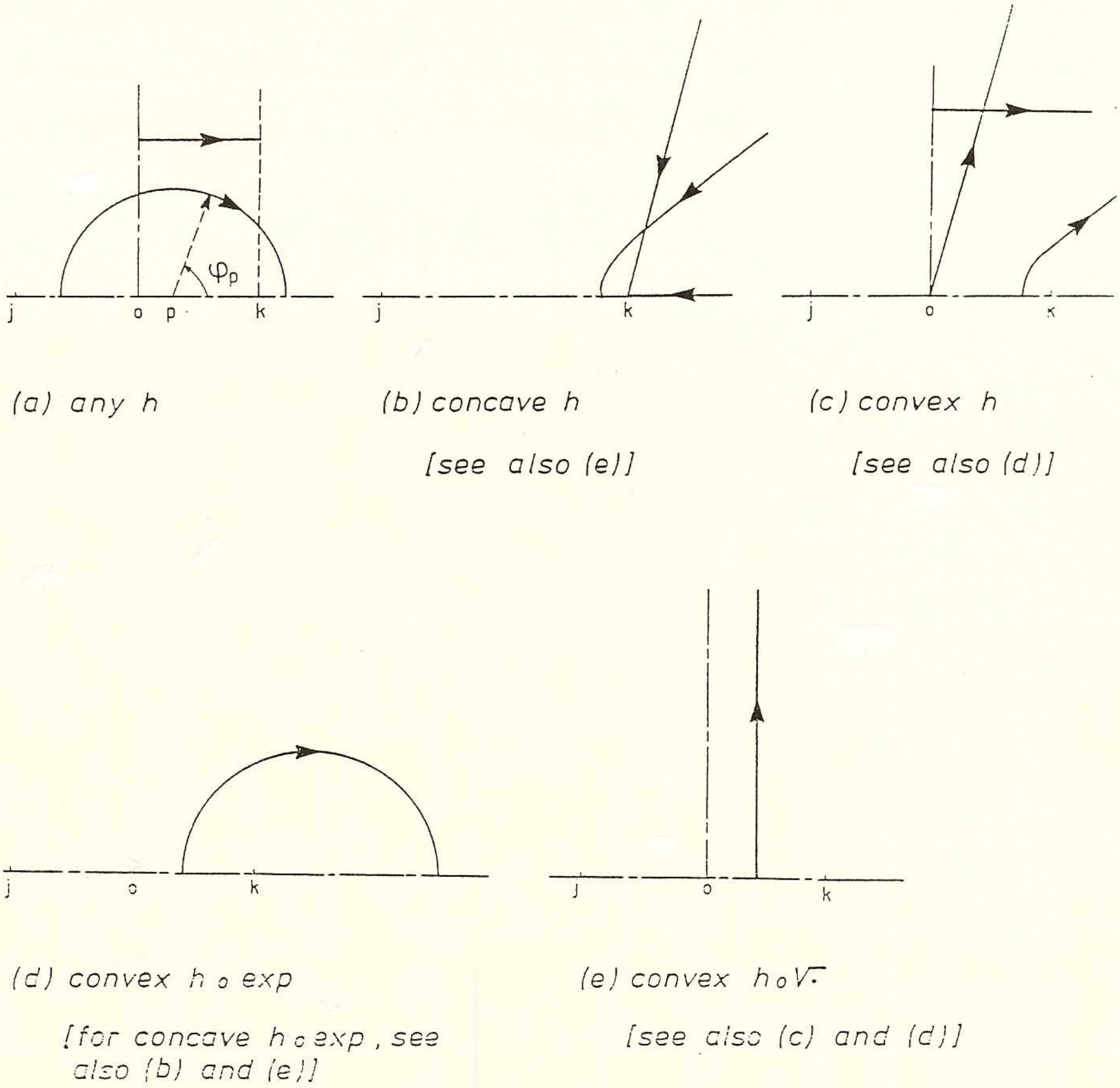


Fig. 5.1. Direction of increase of the transportation cost difference for several typical curves. (In d, the circle is Apollonian wrt. the centres j and k).

5.2. *Radial move from centre k or from the middle of [jk]* (Fig. 5.1b,5.1c,5.2)

(5.2) (a) *If function h is concave, then* $(\partial\Delta h/\partial\delta_k)_{\varphi_k} \leq 0$ *on* $(\mathbb{R}_+ \times \mathbb{R})-\{k\}$;

(b) *If function h is convex, then* $(\partial\Delta h/\partial\delta_o)_{\varphi_o} \geq 0$ *on* $(\mathbb{R}_+ \times \mathbb{R})-\{o\}$.

As to (a), we have indeed (Fig. 5.2 a)

$$\delta_j^2 = \delta_k^2 + \delta_{jk}^2 + 2\delta_{jk}\,\delta_k \cos\varphi_k ;$$

hence,

$$\left(\frac{\partial\delta_j}{\partial\delta_k}\right)_{\varphi_k} = \frac{\delta_k + \delta_{jk}\cos\varphi_k}{\delta_j} = \cos\psi$$

and consequently

$$\left(\frac{\partial\Delta h}{\partial\delta_k}\right)_{\varphi_k} = h_j' \cos\psi - h_k' = h_j' - h_k' - h_j' (1-\cos\psi)$$

$$\leq 0 \text{ if h is concave and if } \delta_j \geq \delta_k \text{ (ie., } i \in \mathbb{R}_+ \times \mathbb{R})$$

To prove (b) we start from those equalities (Fig.5.2b)

$$\delta_j^2 = \delta_o^2 + \delta_{ok}^2 + \delta_{jk}\,\delta_o \cos\varphi_o$$

$$\delta_k^2 = \delta_o^2 + \delta_{ok}^2 - \delta_{jk}\,\delta_o \cos\varphi_o$$

which entail

$$\left(\frac{\partial\delta_j}{\partial\delta_o}\right)_{\varphi_o} = \frac{\delta_o + \delta_{ok}\cos\varphi_o}{\delta_j} = \cos\psi_j$$

$$\left(\frac{\partial\delta_k}{\partial\delta_o}\right)_{\varphi_o} = \frac{\delta_o - \delta_{ok}\cos\varphi_o}{\delta_k} = \cos\psi_k .$$

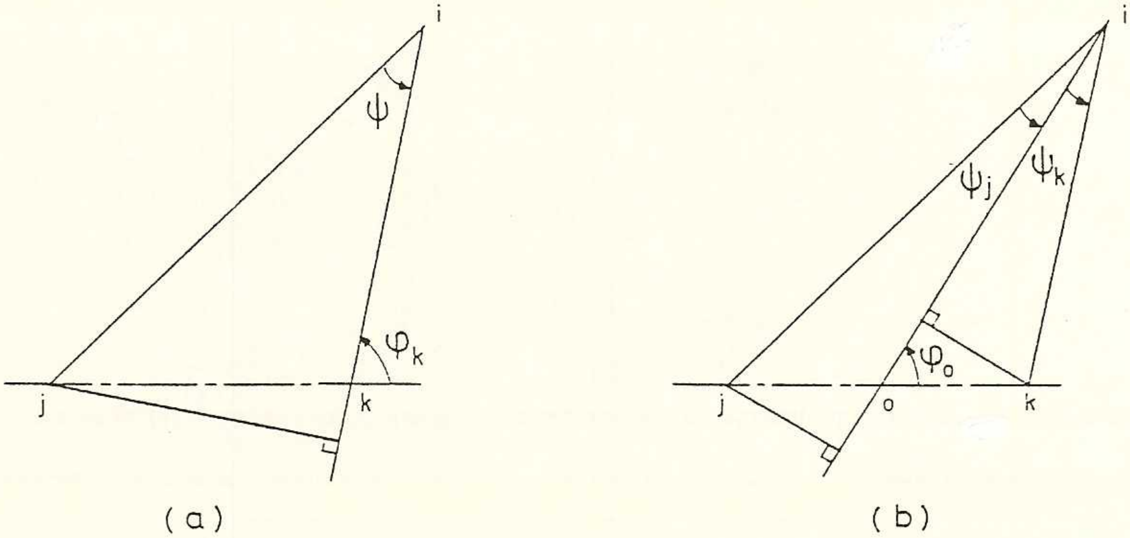


Fig. 5.2. Proof of Prop. 5.2.

Because of symmetry (Prop. 4.2), it is enough considering $i \in \mathbb{R}^2_+ - \{o\}$, so we have $0 \leq \psi_j \leq \frac{\pi}{2}$ and $\psi_j \leq \psi_k \leq \pi$, and we may write

$$\left(\frac{\partial \Delta h}{\partial \delta_o}\right)_{\varphi_o} = h'_j \cos \psi_j - h'_k \cos \psi_k$$

$$= h'_k (\cos \psi_j - \cos \psi_k) + (h'_j - h'_k) \cos \psi_j$$

$$\geq 0 \text{ if } h \text{ is convex .}$$

5.3. *Move along straight lines parallel to* $[jk]$ (Fig. 3.1 ; 5.1 a,b,c)

(5.3) (a) $(\partial \Delta h/\partial x)_y > 0$ *in the strip* $[0, \delta_{ok}] \times \mathbb{R}$;

(b) *If function h is convex, then* $(\partial \Delta h/\partial x)_y > 0$ *on* $\mathbb{R}_+ \times \mathbb{R}^*$;

(c) *If function h is concave [resp. convex], then* $(\partial \Delta h/\partial x)_y \leq 0$ *[resp.* ≥ 0 *] on the x-axis on the right of k, ie. on* $]\delta_{ok}, +\infty[\times \{0\}$.

According to (10), we know that

$$\left(\frac{\partial \delta_j}{\partial x}\right)_y = \frac{x+\delta_{ok}}{\delta_j} = \cos \varphi_j$$

$$\left(\frac{\partial \delta_k}{\partial x}\right)_y = \frac{x-\delta_{ok}}{\delta_k} = \cos \varphi_k$$

and thus

$$\left(\frac{\partial \Delta h}{\partial x}\right)_y = h'_j \cos \varphi_j - h'_k \cos \varphi_k . \tag{13}$$

As we are on $\mathbb{R}_+ \times \mathbb{R}$, we have $\cos \varphi_j > 0$. Subproposition (a) then holds because $\varphi_k \geq \frac{\pi}{2}$ and $\cos \varphi_k \leq 0$ on $[0, \delta_{ok}] \times \mathbb{R}$ (notice that, in this proof and in the ones that follow , $h'(\delta) \neq 0$ $\forall \delta > 0$ when h is globally concave or globally convex, since h is strictly increasing). Subproposition (b) may be proved for instance in this way : on $\mathbb{R}^{*2}_+$, to which we may restrict the study (Prop. 4.2), $\varphi_j < \varphi_k$; so (10) implies

$$\left(\frac{\partial \Delta h}{\partial x}\right)_y = \cos \varphi_j \,(h'_j - h'_k) + h'_k \,(\cos \varphi_j - \cos \varphi_k)$$

$$> 0 \text{ if h is convex, as } \cos \varphi_j > 0.$$

As to (c), $\varphi_j = \varphi_k = 0$ on the set under study, so that $(\partial\Delta h/\partial x)_y$ reduces to $h'_j - h'_k$. When h is convex, $(\partial\Delta h/\partial x)_y$ is consequently ≥ 0 in accordance with (b), and ≤ 0 when h is concave.

5.4. *Crossing between indifference lines*

In this subsection we make a comparison of the market areas induced by two different transportation cost functions, h and $\tilde{h}$. As both are strictly increasing and thus injective functions, it is possible to define a function $T = h \circ \tilde{h}^{\wedge}$, i.e.,

$$T\,[\,\tilde{h}(\delta)\,] = h(\delta), \; \forall \delta \geq 0. \tag{14}$$

Function T translates one type of transportation cost into the other. The variation of T is expressed by its derivative wrt. $\tilde{h}(\delta)$, i.e.,

$$T'[\,\tilde{h}(\delta)\,] = h'(\delta) \;/\; \tilde{h}'(\delta) \tag{15}$$

and its concavity or convexity may be determined for instance from

$$T''\,[\,\tilde{h}(\delta)\,] = \frac{h'(\delta)}{\tilde{h}'^2(\delta)} \left[\frac{h''(\delta)}{h'(\delta)} - \frac{\tilde{h}''(\delta)}{\tilde{h}'(\delta)} \right]. \tag{16}$$

This leads to the following interpretation. An increasing marginal transportation cost (h convex) is represented by an increasing slope h' ; if $\tilde{h}'$ rises less (more) in percentage than h' everywhere, then T' increases (decreases) and T is convex (concave). A decreasing marginal transportation cost (h concave), which is more realistic, is represented by a decreasing slope h' ; if $\tilde{h}'$ goes down faster (slower) in percentage than h' everywhere, then T' increases (decreases) and T is convex (concave). Consequently, when h is concave or convex, the concavity of T can represent two situations : (1) one in which the transportation cost function h is convex with a slope h' increasing slower than $\tilde{h}'$ in percentage ($\tilde{h}$ 'more convex' than h) ; (2) one in which the transportation cost function h is concave with a slope h' decreasing faster than $\tilde{h}'$ in percentage ($\tilde{h}$ 'less concave' than h). Similarly the convexity of T may arise from a transportation cost function h more convex than $\tilde{h}$ or less concave than $\tilde{h}$.

The next general proposition shows a decrease (increase) in Δh when moving away from k along the boundary $\tilde{Z}_j \cap \tilde{Z}_k$ and when T is concave (convex) :

(5.4) *If $h(\delta)$ is a concave [resp. convex] expression of $\tilde{h}(\delta)$, i.e., if $h'/\tilde{h}'$ is a decreasing [resp. an increasing] function, then $(\partial\Delta h/\partial\delta_k)_{\Delta\tilde{h}} \leqslant 0$ [resp. $\geqslant 0$].*

Of course, saying that $h(\delta)$ is a concave or convex expression of $\tilde{h}(\delta)$ is just another way, maybe less abstract, to express that T is concave or convex. To come to this conclusion, we made three simplifications :

1° according to Proposition 4.4, δ_k may be replaced by δ_j in that sentence, or by δ_p, for any p ∈ [jk] ; 2° when T is *strictly* concave or convex, the inequality holds strictly too ; 3° in order to determine the sign of $(\partial\Delta h/\partial\delta_k)_{\Delta\tilde{h}}$ at some point i, it is enough to know that T is concave, or convex, on the interval $]\tilde{h}_k,\tilde{h}_j[$; or, still less than that, to compare $T'(\tilde{h}_j)$ to $T'(\tilde{h}_k)$. Those remarks are also valid for Proposition 5.5.

To prove Prop. 5.4, we consider the infinitesimal variations of Δh and $\Delta\tilde{h}$:

$$d\Delta h = h'_j\, d\delta_j - h'_k\, d\delta_k$$
$$d\Delta\tilde{h} = \tilde{h}'_j\, d\delta_j - \tilde{h}'_k\, d\delta_k .$$

If the infinitesimal move of point i occurs inside $\tilde{Z}_j \cap \tilde{Z}_k$, $\Delta\tilde{h}$ is constant and $d\Delta\tilde{h} = 0$ here above. The differential $d\delta_j$ may thus be eliminated, and we find :

$$\frac{d\Delta h}{d\delta_k} = \left(\frac{\partial \Delta h}{\partial \delta_k}\right)_{\Delta \tilde{h}} = \frac{h_j' \, \tilde{h}_k'}{\tilde{h}_j'} - h_k'$$

$$= \tilde{h}_k' \left(\frac{h_j'}{\tilde{h}_j'} - \frac{h_k'}{\tilde{h}_k'} \right)$$

$$= \tilde{h}_k' \left[T'(\tilde{h}_j) - T'(\tilde{h}_k) \right] \quad \text{because of (15).}$$

The property follows immediately.

Proposition 5.4 may be used to make a comparison between the market areas derived from a family of transportation cost functions. Take the case of $h(\delta) = \delta^a$, for instance. For $\tilde{h}(\delta) = \delta^{\tilde{a}}$, we have $T[\tilde{h}(\delta)] = \delta^a = (\delta^{\tilde{a}})^{a/\tilde{a}}$. Suppose $\tilde{a} > a$; T is then a concave function. If i is a point on $Z_j \cap Z_k$ and we move on the curve $\tilde{Z}_j \cap \tilde{Z}_k$ passing through i, Prop. 5.4 and 4.4 show that we remain in Z_j when we move away from [jk], or in Z_k when we move towards [jk]. Otherwise said, when $Z_j \cap Z_k$ is constrained to pass through some fixed point i, Proposition 5.4 may tell us about the influence on $Z_j \cap Z_k$ of the choice of a transportation cost function. Fig. 5.3 illustrates this for $h(\delta) = \delta^a$.

We may also combine Propositions 4.7 and 5.4 (see Fig. 5.1b,c,d,e) :

(5.5) (a) [*Appolonian circles relative to j and k* : $\tilde{h}(\delta) = \ln \delta$]
When $h \circ \exp$ *is concave* [*resp. convex*], *then* $(\partial \Delta h/\partial \delta_k)_{\delta_j/\delta_k} \leq 0$ [*resp.* ≥ 0];

(b) [*Hyperbolae with foci j and k* : $\tilde{h}(\delta) = \delta$]. *When h is concave* [*resp. convex*] , *then* $(\partial \Delta h/\partial \delta_k)_{\delta_j - \delta_k} \leq 0$ [*resp.* ≥ 0];

(c) [*Vertical lines* : $\tilde{h}(\delta) = \delta^2$]. *When* $h \circ \sqrt{.}$ *is concave* [*resp. convex*], *then* $(\partial \Delta h/\partial y)_x \leq 0$ [*resp.* ≥ 0] *on* $\mathbb{R} \times \mathbb{R}_+$.

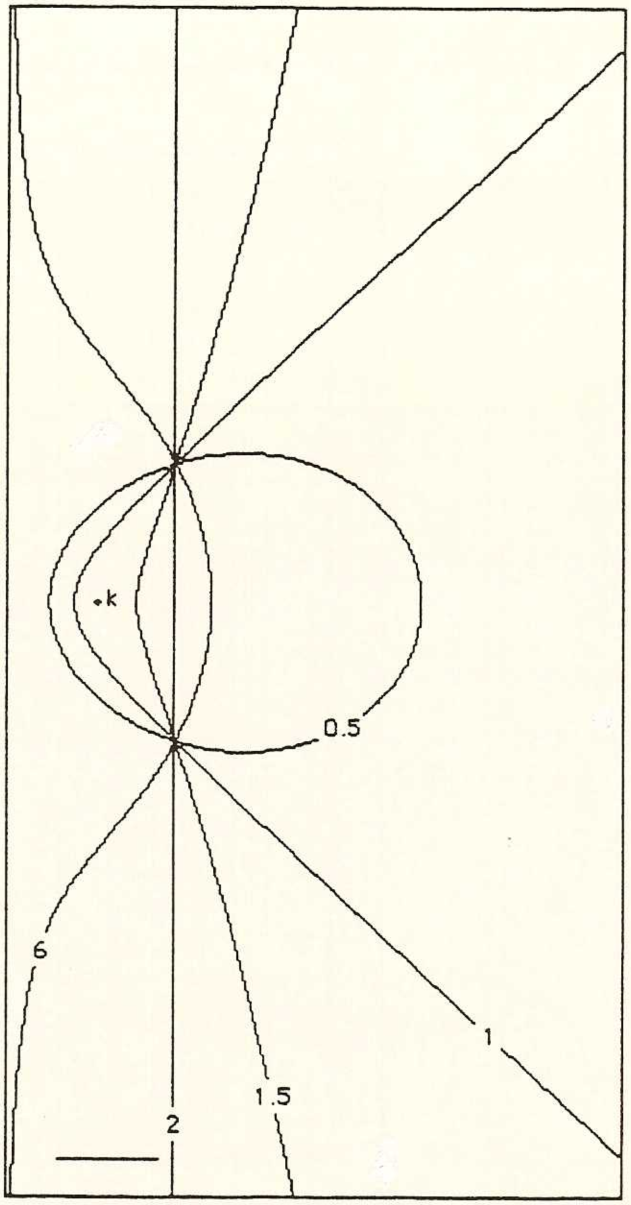


Fig.5.3. Market boundaries generated by the power t.c.f. δ^a for indicated values of a when δ_{jk}= 1.75 , Q being chosen so as to let the indifference line pass through the point (1.625, 1.375).

(See the note that follows Propositions 5.4). This is a straight application of Prop. 5.4, except for one detail : in (c) we have taken equality (11) into account, as well as the fact that y and δ_k increase simultaneously on any vertical line above the x-axis. Notice also that Prop. 5.3c may be viewed as an application of Prop. 5.5b, as $[\delta_{ok}, +\infty[\times \mathbb{R}$ is the right-hand branch of the degenerate hyperbola with equation $\delta_j - \delta_k = \delta_{jk}$.

As to the concavity or convexity of $h \circ \exp$ and $h \circ \sqrt{.}$, they are immediately known in some important cases :

(5.6) (a) *When h is convex, or when* $h(\delta) = \delta^a \ \forall \delta \geq 0$ *with* $a > 0$, *then* $h \circ \exp$ *is convex ;*
(b) *When h is concave, or when* $h(\delta) = \delta^a \ \forall \delta \geq 0$ *with* $0 < a \leq 2$, *then* $h \circ \sqrt{.}$ *is concave ;*
(c) *When* $h(\delta) = \delta^a \ \forall \delta \geq 0$ *with* $2 \leq a$, *then* $h \circ \sqrt{.}$ *is convex.*

Except for $h(\delta) = \delta^a$, the proof is that any increasing convex [resp. concave] function of a convex [resp. concave] function is convex [resp. concave]. Reciprocally, we have too

(5.7) (a) *When* $h \circ \exp$ *is concave, h is concave ;*
(b) *When* $h \circ \sqrt{.}$ *is convex, h is convex.*

5.5. *Tangency between indifference lines and specific curves*

We have already pointed out that (δ_j, δ_k) constitutes a system of coordinates of the half-plane $\mathbb{R} \times \mathbb{R}_+$. More generally, the same is true for $(\tilde{h}_j, \tilde{h}_k)$ if $\tilde{h}$ is any strictly increasing function ; we additionally require $\tilde{h}$ to be continuous. Representing the market areas in a system of axes where $\tilde{h}_j$ and $\tilde{h}_k$ replace x and y will learn us some of their properties ; see the example where $\tilde{h}(\delta) = \delta^2$ in Fig. 5.4. In this respect the present work is related to the general study of map transformations of geographic space made by Tobler (1961).

Not all ordered pairs (δ_j,δ_k) [and hence $(\tilde{h}_j,\tilde{h}_k)$] can exist in the plane used so far (no subscript 'i' is here implicit in writing δ_j, $\tilde{h}_j$, etc.). Otherwise said, the value of y corresponding to (δ_j,δ_k) may be imaginary, whereas x is always real according to (11) ; y is computed from the formula

$$y^2 = \frac{\delta_j^2 + \delta_k^2}{2} - \frac{(\delta_j^2 - \delta_k^2)^2}{4\,\delta_{jk}^2} - \delta_{ok}^2 , \qquad (17)$$

derived from (10) and (11). We are thus led to define the set $R(\tilde{h})$ of 'feasible' pairs $(\tilde{h}_j, \tilde{h}_k)$, i.e., corresponding to points of our Euclidean plane ; $R(\tilde{h})$ is simply the set of such pairs that verify the triangular inequalities (9) :

$$R(\tilde{h}) = \{ (\tilde{h}_j,\tilde{h}_k);\ \delta_j + \delta_k \geqslant \delta_{jk} \text{ and } |\delta_j - \delta_k| \leqslant \delta_{jk} \} .$$

The boundary of $R(\tilde{h})$ is the set of the coordinates $(\tilde{h}_j, \tilde{h}_k)$ of the x-axis $\mathbb{R}\times\{0\}$, as the three equations $\delta_j + \delta_k = \delta_{jk}$, $\delta_j - \delta_k = \delta_{jk}$ and $\delta_k - \delta_j = \delta_{jk}$ respectively describe the interval $[jk]$ and the straight half-lines $[\delta_{ok}, +\infty[\times\{0\}$ and $]-\infty,-\delta_{ok}]\times\{0\}$. We also set

$$Z_k^*(\tilde{h}) = \{(\tilde{h}_j,\tilde{h}_k);\ \tilde{h}_j - \tilde{h}_k \geqslant Q\},$$

and do similary for $Z_j^*(\tilde{h})$. Let us call $\chi_{\tilde{h}}$ the function which associates every pair (x,y) of $\mathbb{R}^2$ with the corresponding couple $(\tilde{h}_j,\tilde{h}_k)$.

We have of course

$$\chi_{\tilde{h}} < Z_k > = Z_k^*(\tilde{h}) \cap R(\tilde{h})$$

$$\chi_{\hat{\tilde{h}}} < Z_k^*(\tilde{h}) > = Z_k$$

and similarly for Z_j.

The set $Z_j^*(\tilde{h}) \cap Z_k^*(\tilde{h})$, the intersection of which with $R(\tilde{h})$ is $\chi_{\tilde{h}}<Z_j \cap Z_k>$, is the graph of the relation

$$\tilde{h}_k = (\tilde{h}\circ h^\wedge)[(h \circ \tilde{h}^\wedge)(\tilde{h}_j) - Q]$$

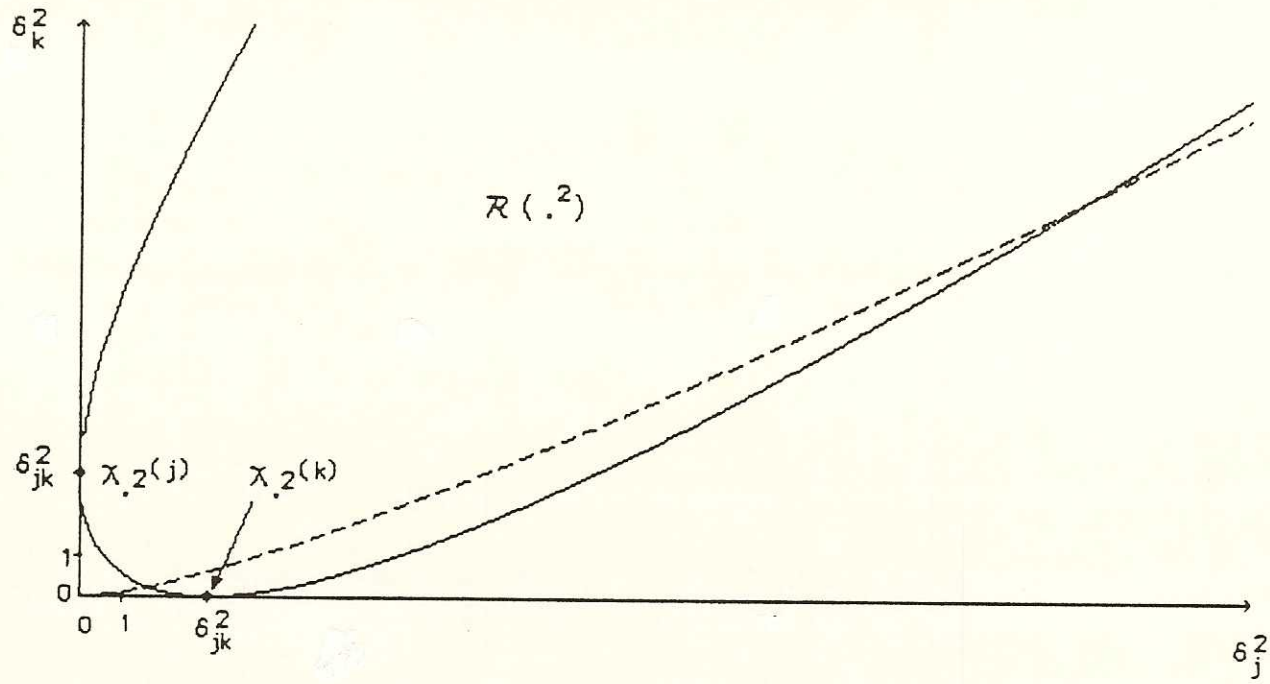


Fig. 5.4a. The set $\mathcal{R}(.^2)$ of couples (δ^2_j, δ^2_k) corresponding to points of the upper half-plane $\mathbb{R} \times \mathbb{R}_+$. The dotted curve is the image of the indifference line corresponding to the value 0.5 of the distance exponent a in Fig. 5.3.

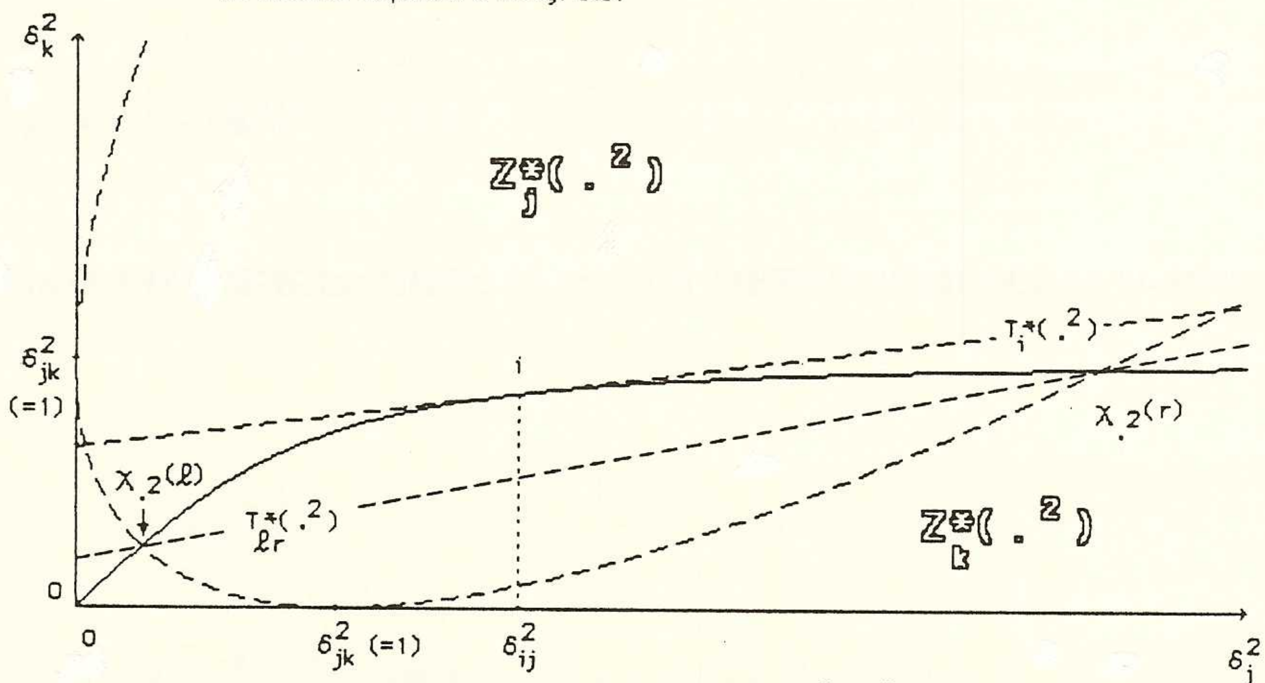


Fig. 5.4b. The market areas as seen through the coordinates (δ^2_j, δ^2_k) when the t.c.f. is δ^{-4}, the parameters Q and δ_{jk} being both equal to 1.

and its slope is strictly positive as h and $\tilde{h}$ are strictly increasing functions. As to $Z_j^*(\tilde{h})$ and $Z_k^*(\tilde{h})$, their situation with regard to their common boundary is easily found ; see the example of Fig.5.4b. Let us indeed start from some pair $(\tilde{h}_j, \tilde{h}_k)$ of $Z_j^*(\tilde{h}) \cap Z_k^*(\tilde{h})$ and diminish $\tilde{h}_j$ (and thus h_j) while keeping $\tilde{h}_k$ (and h_k) constant : Δh decreases, so that we are now inside $Z_j^*(\tilde{h})$. Consequently, $Z_j^*(\tilde{h})$ is on the left of $Z_j^*(\tilde{h}) \cap Z_k^*(\tilde{h})$ and above it, whereas $Z_k^*(\tilde{h})$ is under $Z_j^*(\tilde{h}) \cap Z_k^*(\tilde{h})$ and on its right.

Let us now take some point $i \in Z_j \cap Z_k$ and draw the $\tilde{h}$-tangent $T_i(\tilde{h})$ at i to $Z_j \cap Z_k$, i.e., the curve tangent to $Z_j \cap Z_k$ at i and having an equation of the form

$$\tilde{h}_k = \alpha \tilde{h}_j + \zeta .$$

Its image $\chi_{\tilde{h}} < T_i(\tilde{h}) >$ is the intersection of R with the tangent $T_i^*(\tilde{h})$ to $Z_j^*(\tilde{h}) \cap Z_k^*(\tilde{h})$ at the corresponding pair $(\tilde{h}_j, \tilde{h}_k)$, which tangent has the same equation as $T_i(\tilde{h})$. Hence it is clear that $T_i(h)$ will lie completely inside Z_j if the set $Z_k^*(\tilde{h})$ is convex, or inside Z_k if $Z_j^*(\tilde{h})$ is convex. In order to determine if $Z_k^*(\tilde{h})$ or $Z_j^*(\tilde{h})$ is convex, we have simply to compute the second derivative of h_k wrt. $\tilde{h}_j$ on $Z_j^*(\tilde{h}) \cap Z_k^*(\tilde{h})$. The equation of that set may be written as

$$(h \circ \tilde{h}^{\wedge})(\tilde{h}_j) - (h \circ \tilde{h}^{\wedge})(\tilde{h}_k) = Q \ ;$$

from which, with obvious notation,

$$(h \circ \tilde{h}^{\wedge})'_j \, d\tilde{h}_j - (h \circ \tilde{h}^{\wedge})'_k \, d\tilde{h}_k = 0$$

and

$$\left(\frac{\partial \tilde{h}_k}{\partial \tilde{h}_j}\right)_{\Delta h} = \frac{(h \circ \tilde{h}^{\wedge})'_j}{(h \circ \tilde{h}^{\wedge})'_k} \ .$$

So we get the second derivative :

$$\left(\frac{\partial^2 \tilde{h}_k}{\partial \tilde{h}_j^2}\right)_{\Delta\tilde{h}} = \frac{(h\circ\tilde{h}^\wedge)''_j\,(h\circ\tilde{h}^\wedge)'_k - (h\circ\tilde{h}^\wedge)'_j\,(h\circ\tilde{h}^\wedge)''_k\,.\dfrac{(h\circ\tilde{h}^\wedge)'_j}{(h\circ\tilde{h}^\wedge)'_k}}{(h\circ\tilde{h}^\wedge)'^2_k}$$

$$= \frac{(h\circ\tilde{h}^\wedge)'^2_j}{(h\circ\tilde{h}^\wedge)'_k}\left[\frac{(h\circ\tilde{h}^\wedge)''_j}{(h\circ\tilde{h}^\wedge)'^2_j} - \frac{(h\circ\tilde{h}^\wedge)''_k}{(h\circ\tilde{h}^\wedge)'^2_k}\right]$$

$$= \frac{(h\circ\tilde{h}^\wedge)'^2_j}{(h\circ\tilde{h}^\wedge)'_k}\left[\left(\frac{1}{(h\circ\tilde{h}^\wedge)'}\right)'_k - \left(\frac{1}{(h\circ\tilde{h}^\wedge)'}\right)'_j\right]$$

$$= \frac{(h\circ\tilde{h}^\wedge)'^2_j}{(h\circ\tilde{h}^\wedge)'_k}\left[\left(\frac{\tilde{h}'}{h'}\circ\tilde{h}^\wedge\right)'_k - \left(\frac{\tilde{h}'}{h'}\circ\tilde{h}^\wedge\right)'_j\right].$$

As a consequence, the second derivative $(\partial^2\tilde{h}_k / \partial\tilde{h}_j^2)_{\Delta h}$ is $\geqslant 0$ (resp. $\leqslant 0$) when the function $(\tilde{h}'/h')\circ\tilde{h}^\wedge$ is concave (resp. convex).

For any $i \in Z_j \cap Z_k$ we denote by $D_i^*(\tilde{h})$ the set of pairs $(\tilde{h}_j,\tilde{h}_k)$ defined by the inequality $\tilde{h}_k \leqslant \alpha\,\tilde{h}_j + \zeta$ and the boundary of which is $T_i^*(\tilde{h})$, and we set $D_i(\tilde{h}) = \hat{\chi}_{\tilde{h}} < D_i^*(\tilde{h}) >$. When Z_k is bounded we call $D_{\ell r}^*(\tilde{h})$ the set of pairs $(\tilde{h}_j,\tilde{h}_k)$ described by a similar inequality and the boundary of which is the straight line defined by $\chi_{\tilde{h}}(\ell)$ and $\chi_{\tilde{h}}(r)$, and we set $D_{\ell r}(\tilde{h}) = \hat{\chi}_{\tilde{h}} < D_{\ell r}^*(\tilde{h}) >$. The mathematical development of $(\partial^2\tilde{h}_k/\partial\tilde{h}_j^2)_{\Delta h}$ made here above then yields these general properties.

(5.8) *If $(\tilde{h}'/h')\circ\tilde{h}^\wedge$ is concave [resp. convex], then $D_i(\tilde{h}) \subseteq Z_k$ [resp. $Z_k \subseteq D_i(\tilde{h})$] for any $i \in Z_j \cap Z_k$; and, if moreover Z_k is bounded, $Z_k \subseteq D_{\ell r}(\tilde{h})$ [resp. $D_{\ell r}(\tilde{h}) \subseteq Z_k$].*

Of course, the concavity or convexity of $(\tilde{h}'/h') \circ \tilde{h}^{\wedge}$ means the concavity or convexity of the expression $\tilde{h}'(\delta)/h'(\delta)$ wrt. $\tilde{h}(\delta)$. As to the values of α and ζ for $D_i(\tilde{h})$ and $T_i(\tilde{h})$, they are easily found :

$$\alpha = \left(\frac{\partial \tilde{h}_k}{\partial \tilde{h}_j}\right)_{\Delta h} = \frac{h'_j \ \tilde{h}'_k}{h'_k \ \tilde{h}'_j} \tag{18}$$

$$\zeta = \frac{\tilde{h}'_k}{h'_k}\left(\frac{\tilde{h}_k \ h'_k}{\tilde{h}'_k} - \frac{\tilde{h}_j \ h'_j}{\tilde{h}'_j}\right)$$

Concerning the $\tilde{h}$-tangent $T_i(\tilde{h})$, the property could be expressed in a way more alike to the other propositions of this section : if $(\tilde{h}'/h') \circ \tilde{h}^{\wedge}$ is convex [resp. concave], then Δh is quasi-concave [resp. quasi-convex] on $T_i(\tilde{h})$ for any $i \in Z_j \cap Z_k$ and reaches its maximum [resp. minimum] at i.

In two cases at least, Prop. 5.8 yields interesting results. When $\tilde{h}$ is defined as $\tilde{h}(\delta) = \delta$, i.e., when $\tilde{h}$ is the identity function $1_{\mathbb{R}_+}$, the associated curves $T_i(1_{\mathbb{R}_+})$ are *Cartesian ovals*. If for instance $h(\delta)$ is given by δ^a, i.e., if $h = .^a$, $a > 0$, we have $\tilde{h}'(\delta)/h'(\delta) = \delta^{1-a}/a$, so that the ovals are contained in Z_k when $0 < a \leqslant 1$ and in Z_j when $a \geqslant 1$; but the interest is here mainly theoretical. The second case occurs when $\tilde{h} = .^2$. It is then found that the curves $T_i(.^2)$ are *circles* centred on the axis of x. Their equation may indeed be written successively as follows :

$$\delta_k^2 = \alpha\delta_j^2 + \zeta$$

$$y^2 + (x-\delta_{ok})^2 = \alpha\,[\,y^2 + (x+\delta_{ok})^2\,] + \zeta$$

$$x^2 + y^2 - 2\,x\,\delta_{ok}\,\frac{1+\alpha}{1-\alpha} + \delta_{ok}^2 - \frac{\zeta}{1-\alpha} = 0$$

$$\left(x - \delta_{ok}\,\frac{1+\alpha}{1-\alpha}\right)^2 + y^2 = \delta_{ok}^2\,\frac{4\,\alpha}{(1-\alpha)^2} + \frac{\zeta}{1-\alpha}$$

which shows that $T_i(.^2)$ is the circle of centre $[\delta_{ok}\ (1 + \alpha)\ /\ (1 - \alpha),\ 0\ ]$ and passing through point i. We shall come back to that result in Sections 6.4 and 8.5.

If we compare Prop. 5.8 to the preceding ones of this Section 5, we see that it is closely related to Prop. 5.4, which considers the first derivative of $\tilde{h}'(\delta)\ /\ h'\ (\delta)$ wrt. $\tilde{h}\ (\delta)$, whereas Prop. 5.8 uses its second derivative. This is not surprising as Prop. 5.4 can be easily derived from (18) by comparing the $\tilde{h}$-slopes α of $Z_j \cap Z_k$ and $\tilde{Z}_j \cap \tilde{Z}_k$, the second of which is equal to 1. For both propositions the most interesting case is when $\tilde{h}\ (\delta) = \delta^2$, as will appear in the next sections. The function $\sqrt{.}\ /\ (h' \circ \sqrt{.}\ )$ consequently appears as particularly important in the study of the shape of market areas.

On the other hand, Prop. 5.8 differs from all the preceding ones of this section in that its conditions are not additive. For those statements indeed, if the conditions are verified for a set of transportation cost functions, they remain so for any positive-coefficient linear combination of those functions. However Prop. 5.8 is partially additive. We have indeed the following

LEMMA 5.1. *If f and g are two concave functions, then their parallel association $f \,/\!/\, g$, defined as*

$$f \,/\!/\, g = \frac{1}{\frac{1}{f} + \frac{1}{g}},$$

is concave provided that $f + g > 0$.

We have borrowed the concept from the domain of electricity, where that formula describes the parallel association of two impedances. Computing the second derivative $(f \,/\!/\, g)''$ suffices for the proof :

$$(f \,/\!/\, g)'' = \frac{f^2 g'' + f'' g^2}{(f + g)^2} - 2\,\frac{(f'g - fg')^2}{(f + g)^3} .$$

The result does not depend, of course, on the differentiability of the functions.

As the parallel association of functions $\tilde{h}'/h'_\nu$, with $\nu \in \{1, \dots n\}$, is $\tilde{h}' / \sum_{\nu=1}^{n} h'_\nu$, we have because of the associativity of '//' :

(5.9) *If $(\tilde{h}'/h'_\nu) \circ \tilde{h}^{-}$ is concave for any $\nu \in \{1, \dots n\}$ and if h is a positive-coefficient linear combination of those transportation cost functions h_ν, $\nu \in \{1, \dots n\}$, then $(\tilde{h}'/h') \circ \tilde{h}^{-}$ is also concave.*

A last remark. The discussion made in Section 2 about Descartes' ovals can be extended to the curves T_i $(\tilde{h})$. These are boundaries of market areas corresponding to the transportation cost function ℓn $[\tilde{h}$ (δ) + $\beta/$ $(\alpha -1)]$ and to the value $-\ell n$ α of the constant Q, when $\zeta \leqslant 0$ and $\alpha \in]0,1[$; the case where $\alpha \geqslant 1$ and $\zeta \geqslant 0$ is impossible, since T_i $(\tilde{h})$ would then lie at the left of the y-axis. When that condition is not met, the present theory nevertheless provides a way to study the properties of the curves T_i $(\tilde{h})$; see Section 2.2. The same is true, of course, for the boundaries $T_{\ell r}$ $(\tilde{h})$ of the areas $D_{\ell r}$ $(\tilde{h})$.

6. *Remarkable bounds and asymptotic behaviour : spatial extension*

The market areas that appear for given values of Q and δ_{jk} and a given function h can be studied by simulation. We may draw the curve $Z_j \cap Z_k$ by means of a compass, when function h is easily invertible, using the formula $\delta_j = \hat{h} \; [\; Q + h \; (\delta_k) \;]$ for instance. We may also compute Δh at every point of a grid, and then compare Δh to Q in order to see if the point belongs to Z_k or to Z_j, or both. We have implemented those methods on a computer to check our theoretical results and choose values of the parameters that should produce patterns of market areas in which the features we want to emphasize clearly appear.

Such an approach is insufficient, however, to predict the modification of the market areas induced by a change in function h. A purpose of the present paper is precisely to show that apparently similar transportation cost functions may produce definitely different patterns of market areas. The analytical approach is then necessary to master the properties of our model, through the knowledge of the family of functions associated with each of them.

Another aspect of simulation is its double spatial limitation : in *extension* and *precision*. We cannot study the influence of the centres over an infinite area, neither can we decide if all the points of the area under study are more influenced by the one centre or the other. And the wider that investigation area, the smaller the density of the points considered. One may be satisfied with such a limited approach if it corresponds to the real spatial extension of the problem and to the spatial dispersion of the data. One may also wonder if a change in the scale of the study is likely to bring a change in the apparent properties. With both points of view in mind, we shall use the analytical approach in a way complementary to simulation :i.e., we shall study the properties of the market areas up to the infinitely great and up to the infinitesimal. That distinction *grosso modo* marks the separation between Sections 6 and 8.

The present Section 6 is dedicated to the spatial extension of market areas. Essentially, the problem is of finding simple sets containing Z_k or contained in it, thereby constituting upper and lower bounds on Z_k. We begin by comparing it with the elementary sets ϕ, {k}, and {j}, and by asking the question whether Z_k is bounded, i.e., whether it is possible to encompass it with some circle. That is the matter of subsections 6.1, 6.2, and 6.3.

Thereafter in subsections 6.4, 6.5, and 6.7, we exploit the results of Section 5 in order to find circles, hyperbolas, or vertical lines, delimitating Z_k more or less precisely. The aim is at becoming able to decide whether some characteristic regions are contained in Z_k or in Z_j, or contain Z_k, leaning only on the knowledge of some points or pieces of Z_k, Z_j, or $Z_j \cap Z_k$, and on some hypotheses about the transportation cost functions. Of course the particular shapes of those bounds also yield some properties of the shape of the market areas themselves.

Hyperbolic and vertical bounds are particularly interesting when Z_k is not bounded. The subsections 6.6, 6.8 and 6.9 complete the information about this case. A first characteristic of the market areas is then the pair of symmetric directions taken by the demarcation line when the distance to the centres becomes infinite, and how those directions vary with Q and δ_{jk}. Then a question arises whether there are some points of the plane for which the acute angle formed by those directions contains Z_k or is contained in it ; this leads us to the classical study of the possible straight asymptotes of the curve $Z_j \cap Z_k$, which also learns us if there is some distance above which $Z_j \cap Z_k$ is nearly linear. We have in fact to distinguish between a *limiting* line and an *asymptotic* one. The distance between the former and $Z_j \cap Z_k$ tends towards zero when the distance to the centers tends towards infinity. The latter is a limiting line which is not crossed any more by $Z_j \cap Z_k$ beyond some distance to the centres; an *asymptote* is a straight asymptotic line. This distinction between limiting and asymptotic lines is only used to express our lack of knowledge regarding some cases. Our investigation also allows us to know if the boundaries of the hyperbolic and vertical bounds are asymptotes of $Z_j \cap Z_k$, when those bounds may thus be considered as tight.

6.1: *Emptiness of market area* Z_k

(6.1) (a) $Z_k \neq \phi \Leftrightarrow Z_k \cap [\delta_{ok}, +\infty[\times \{0\} \neq \phi$;

(b) *If h is concave, then :* $Z_k \neq \phi \Leftrightarrow k \in Z_k$;

(c) *If h is convex, then :* $Z_k \neq \phi \Leftrightarrow Z_k$ *is unbounded.*

The proposition still holds if 'Z_k' *is replaced everywhere by* '$Z_k - Z_j$'.

We give the proof concerning Z_k only ; the one regarding $Z_k - Z_j$ is similar. The second implication ($\Leftarrow$) of item (a) is quite plain. The first one ($\Rightarrow$) relies on Prop. 5.1 : if Z_k contains some point i , then it also contains the intersection point i' of the circle passing through i and centred on k with that part of the x-axis which lies at the right of k, i.e., $[\delta_{ok}, +\infty[\times\{0\}$. Accordingly, the study about the emptiness of Z_k may be restricted to that set. As we have there $\delta_j = x + \delta_{ok}$ and $\delta_k = x - \delta_{ok}$, the derivative of Δh wrt.x is :

$$\left(\frac{\partial \Delta h}{\partial x}\right)_{y=0} = h'_j - h'_k$$

which expression is negative when h is concave and positive when h is convex. In the former case, Δh is decreasing wrt.x. Consequently if point i lies in Z_k at the right of k on the x-axis, i.e., if $i \in Z_k \cap [\delta_{ok}, +\infty[\times \{0\}$, then $[ki] \subseteq Z_k$ since $\Delta h \geqslant Q$ defines Z_k : so a fortiori $k \in Z_k$. In the latter case, Δh is increasing wrt.x, so that $i \in Z_k \cap [\delta_{ok}, +\infty[\times \{0\}$ entails $[x, +\infty[\times \{0\} \subseteq Z_k$, and Z_k is unbounded. The implications '$\Leftarrow$' of items (b) and (c) being obvious, the proof is complete.

Whether the conditions '$k \in Z_k$' or 'Z_k is unbounded' are met is easy to check. The first one is equivalent to $\Delta h_k \geqslant Q$. This means in particular that Z_k is never empty when $h(\delta) = \ln(\delta)$, for we have then $\Delta h_k = +\infty$; when $h(\delta) = \delta$, on the contrary, Z_k vanishes when $\delta_{jk} < Q$. Let us now study the second condition.

6.2. *Boundedness of market area* Z_k

A set E is *bounded* iff. its diameter is finite ; in other words, iff. it is contained in some disk. Market area Z_j is of course unbounded as $\mathbb{R}_- \times \mathbb{R} \subset Z_j$. As to the boundedness conditions on Z_k, they are related to the limiting value of Δh when point i moves towards infinity, since '$\Delta h \geqslant Q$' defines Z_k. This is the philosophy of the next Prop. 6.2. That result emphasizes the importance of the ratio δ_{jk}/Q ; this will be confirmed in Section 10.2 by the study of the dicentral approximation of market areas when $h(\delta) = \delta^a$.

From now on we shall denote by $h'(+\infty)$ the limiting value $\lim_{\delta \to +\infty} h'(\delta)$, if any.

(6.2) (a) Z_k *is unbounded* $\Leftrightarrow$ $Z_k \cap [\,\delta_{ok}, +\infty\,[\, \times \{0\}$ *is unbounded* ;

(b) *If* $\delta_{jk}\, h'(+\infty) > Q$, *then* $Z_k - Z_j$ *is unbounded* ;
$< Q$, *then* Z_k *is bounded* ;

(c) *If* $\delta_{jk}\, h'(+\infty) = Q$ *and h is concave* [*resp. strictly concave*] Z_k [*resp.* $Z_k - Z_j$] *is unbounded* ;

(d) *If* $\delta_{jk}\, h'(+\infty) = Q$ *and h is convex, then the next equivalences hold for some* $\delta^* \geqslant 0$:

Z_k *is unbounded*

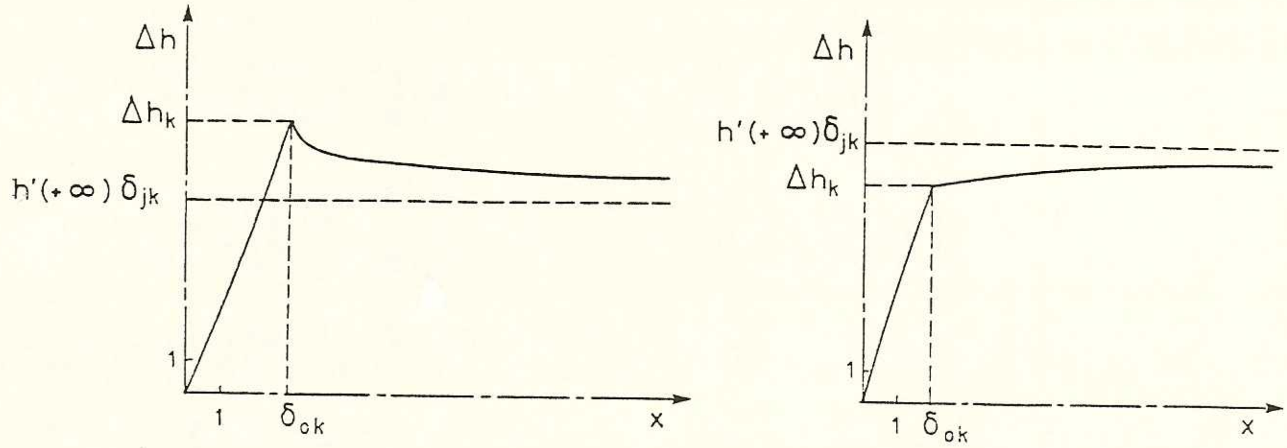
$\Leftrightarrow$ $Z_k \neq \phi$

$\Leftrightarrow$ *h is linear on* $[\,\delta^*, +\infty\,[$ *and nonlinear on* $[\,\delta^{**}, +\infty\,[$ *for any* $\delta^{**} \in\,]\,0, \delta^*\,[$

$\Leftrightarrow$ Z_k *is the straight half-line* $[\,\delta_{ok} + \delta^*, +\infty\,[\, \times \{0\}$;

(e) *If* $\delta_{jk}\, h'(+\infty) = Q$ *and h is strictly convex,* Z_k *is empty and thus bounded.*

The proof of item (a) is similar to that of Prop. 6.1 a. As to item (b), [Fig. 6.1], the limiting value of Δh when point i moves towards infinity along the x-axis in the positive direction is given by $\delta_{jk}\, h'(+\infty)$. This is obvious from the mean-value theorem, which states that for any i on the half-axis $[\,\delta_{ok}, +\infty\,[\, \times \{0\}$



(a) concave h

ex.: $h(\delta)=\delta+\sqrt{\delta}$, $\delta_{ok}=3$

(b) convex h

ex.: $h(\delta)=2\delta-\sqrt{1+\delta}$, $\delta_{ok}=2$

Fig. 6.1. Evolution of the transportation cost difference with x along the x-axis.

we have $\Delta h = h'(\delta)(\delta_j - \delta_k) = h'(\delta)\,\delta_{jk}$ for some $\delta \in]\,\delta_k, \delta_j\,[$. Accordingly, when $\delta_{jk}\, h'(+\infty) > Q$ there is some $\tilde{\delta} \geq 0$ for which we must have $\Delta h = h(\delta_k + \delta_{jk}) - h(\delta_k) > Q$, $\forall\, \delta_k \geq \tilde{\delta}$, as h and Δh are continuous functions. The straight half-line $[\,\delta_{ok} + \tilde{\delta}, +\infty\,[\times \{0\}$ is consequently included in $Z_k - Z_j$ which is thus unbounded. The proof of the boundedness of Z_k when $\delta_{jk}\, h'(+\infty) < Q$ is similar. As to point (c), function h being concave, $h'(\delta) \geq h'(+\infty)$; so, for any $i \in [\,\delta_{ok}, +\infty\,[\times \{0\}$, we have $\Delta h = h'(\delta)\,\delta_{jk} \geq h'(+\infty)\,\delta_{jk} = Q$. The whole straight half-line $[\,\delta_{ok}, +\infty\,[\times \{0\}$ is then a part of Z_k. When function h is strictly concave, we have $\Delta h > Q$ on the set $[\,\delta_{ok}, +\infty\,[\times \{0\}$ which is consequently included in $Z_k - Z_j$. Item (e) directly derives from item (d).

We now come to the chain of equivalences listed in (d). The first one, 'Z_k is unbounded $\Leftrightarrow Z_k \neq \phi$', has already been proved : see Prop. 6.1 c. Here follow the other ones, in which 'Z_k is unbounded' is accordingly replaced by '$Z_k \neq \phi$' :

i) $Z_k \neq \phi \Leftrightarrow$ *h is linear beyond some* δ^*. (Due to the continuity of h, the assertions '$\exists \delta^*$: h is linear on $[\,\delta^*, +\infty\,[$' and '$\exists \delta^*$: h is linear on $[\,\delta^*, +\infty\,[$ and non-linear on $[\,\delta^{**}, +\infty\,[\ \forall \delta^{**} \in]\,0, \delta^*\,[$' are equivalent). ($\Rightarrow$) As $Z_k \neq \phi$, some point i' on $[\,\delta_{ok}, +\infty\,[\times \{0\}$ belongs to Z_k (Prop. 6.1a) ; so $\Delta h_{i'} \geq Q$. As Δh is an increasing function of x on $[\,\delta_{ok}, +\infty\,[\times \{0\}$ (see Prop. 5.3c), we have also $\Delta h_{i'} \leq h'(+\infty)\,\delta_{jk} = Q$. Consequently $\Delta h_{i'} = \Delta h_i = Q$ for every $i \in [\,x_{i'}, +\infty[\times \{0\}$; i.e., $h(\delta_{jk} + \delta) - h(\delta) = Q \ \ \forall\ \delta \geq \delta_{ki'}$. As h is convex, this implies the linearity of h for $\delta \geq \delta_{ki'}$. ($\Leftarrow$) If h is linear for $\delta \geq \delta^*$, Δh is constant on $[\,\delta_{ok} + \delta^*, +\infty\,[\times \{0\}$ and its value there is $\delta_{jk}\, h'(+\infty) = Q$. That straight half-line is then necessarily in Z_k, and $Z_k \neq \phi$.

ii) *h is linear for $\delta \geq \delta^*$ and nonlinear on $[\delta^{**}, +\infty[$ $\forall$ $\delta^{**} \in]0, \delta^*[$ $\Leftrightarrow$ Z_k =* $[\delta_{ok} + \delta^*, +\infty[\times \{0\}$. ($\Rightarrow$) The linearity of h for $\delta \geq \delta^*$ implies the constancy of Δh on $[\delta_{ok} + \delta^*, +\infty[\times \{0\}$, and its value is there δ_{jk} $h'(+\infty) = Q$; so that this set is in Z_k (in $Z_k \cap Z_j$, more precisely). Furthermore, all the points that do not belong to that half-line lie outside Z_k. Consider first a point i of the x-axis between k and $(\delta_{ok} + \delta^*, 0)$, the origin of the half-line ; i.e., $i \in [k, (\delta_{ok} + \delta^*, 0)[$. Then the convexity of h implies

$$\Delta h = h(\delta_{jk} + \delta_k) - h(\delta_k) < h(\delta_{jk} + \delta^*) - h(\delta^*) = h'(+\infty)\ \delta_{jk} = Q,$$

the inequality being strict because of the nonlinearity of h on $[\delta_k, +\infty[$. So $i \notin Z_k$. Second take $i \notin [\delta_{ok}, +\infty[\times \{0\}$. According to the proof of Prop. 6.1 a, $\Delta h < \Delta h_{i'}$ for some $i' \in [\delta_{ok}, +\infty[\times \{0\}$; as we have just seen that $\Delta h_{i'} \leq Q$, we have $\Delta h < Q$, and $i \notin Z_k$. ($\Leftarrow$) . As Δh is continuous, the fact that Z_k is a straight half-line entails that $Z_k = Z_j \cap Z_k$. That is to say, $\Delta h = Q$ on Z_k ; or $h(\delta_{jk} + \delta) - h(\delta) = Q$ $\forall$ $\delta \geq \delta^*$. Being a convex function, h must then be linear on $[\delta^*, +\infty[$. And if it were also linear on $[\delta^{**}, +\infty[$ with $0 < \delta^{**} < \delta^*$, Δh should be constant on $[\delta_{ok} + \delta^{**}, +\infty[$, which should then be a part of Z_k ; whereas it is not so. This completes the proof. Q.E.D.

Let us see how those statements apply when $h(\delta) = \delta^a$ $\forall \delta \geq 0$, with $a > 0$ (Fig.5.3). If $0 < a < 1$, we have $h'(+\infty) = 0$; according to Prop. 6.2 b, Z_k is then bounded. If $a > 1$, $h'(+\infty) = +\infty$: Z_k is unbounded (notice however that the shape of Z_k is not the same for $1 < a < 2$ as for $a > 2$). Last, when a = 1, $h'(+\infty) = 1$. In this case, if $\delta_{jk} > Q$, Z_k is unbounded ; while $Z_k = \phi$ if $\delta_{jk} < Q$ (see Prop. 6.1c). If $\delta_{jk} = Q$, we may refer to Prop. 6.2 c as well as to Prop. 6.2d, as h is now linear. Both indicate that Z_k is unbounded, but Prop. 6.2d tells us moreover that Z_k degenerates into the straight half-line $[\delta_{ok}, +\infty[\times \{0\}$.

An example of a more complex case is given by the transportation cost function $h(\delta) = \delta + \ell n\, \delta$ (Fig. 6.2). Here market area Z_k is bounded (and nonempty) when $Q > \delta_{jk}$ and unbounded when $Q \leq \delta_{jk}$.

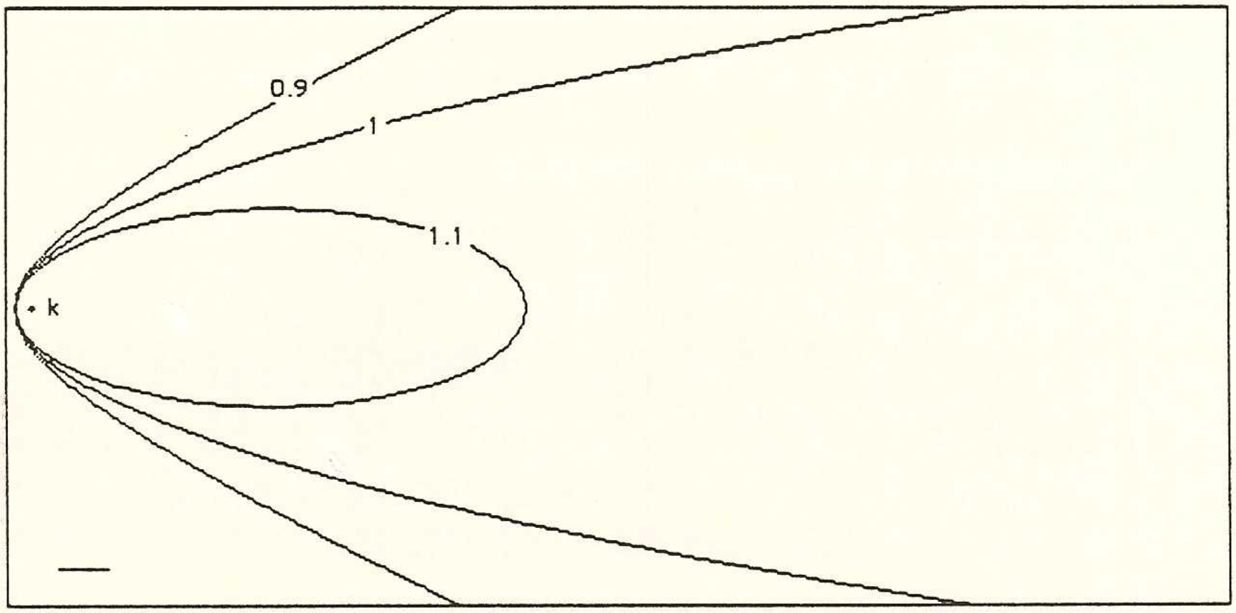


Fig. 6.2. Market areas generated by the t.c.f. $\delta + \ln \delta$ when $\delta_{jk} = 1$, for indicated values of Q. When Q = 1, the indifference line has no straight limiting line; when Q < 1, it has two oblique ones crossing at the point $(-Q / \sqrt{1 - Q^2}, 0)$.

6.3. *Position of centres relatively to market areas*

This section is related with the two preceding ones 6.1 and 6.2, as well as with Sections 8.1 and 8.2, in that the field of study can be reduced to the x-axis ; and more precisely, as appears from the next proposition, to the half-line $[\delta_{ok}, +\infty[\times \{0\}$. As to centre j, we already know from Prop. 4.3 that it belongs to its own market area ; this is not necessarily true for centre k : see Fig. 5.3.

(6.3) (a) *Centre* $k \in Z_k$ *iff.* $\ell \in]ok]$;

(b) *If h is concave and* $Z_k \neq \phi$, *then* $k \in Z_k$;

(c) *When h is convex and nonlinear, then* $]h'(\delta_{jk}) - h(0), h'(+\infty)\delta_{jk}[$ *is not empty and, if Q belongs to that interval,* $k \notin Z_k \neq \phi$.

According to Prop. 5.3a, we have indeed $\Delta h < \Delta h_k$ for any $i \in]ok[$. Consequently if $k \notin Z_k$, i.e., if $\Delta h_k < Q$, then $\Delta h < Q$ and $]ok] \cap Z_k = \phi$: so $\ell \notin]ok]$. If $k \in Z_k$, $]ok] \cap Z_k \neq \phi$; and as $\ell \in \mathbb{R}_+^* \times \{0\}$ (Prop. 4.3), we see that $\ell \in]ok]$. Item (a) is thus proved. Item (b) repeats Prop. 6.1b. As to item (c), it is obvious that $k \notin Z_k$ iff. $\Delta h_k < Q$, and that $Z_k \neq \phi$ if $Q < \delta_{jk} h'(+\infty)$ (Prop. 6.1c and 6.2b). The only problem is to prove that $\Delta h_k < \delta_{jk} h'(+\infty)$. As Δh is increasing wrt. x on $[\delta_{ok}, +\infty[\times \{0\}$ because h is convex (Prop. 5.3b), and as we have seen that $\delta_{jk} h'(+\infty)$ is the limiting value of Δh on the right of the x-axis, we should have $\Delta h_k = \delta_{jk} h'(+\infty)$ only if Δh were constant on $[\delta_{ok}, +\infty[\times \{0\}$. But this would mean that $h(\delta + \delta_{jk}) - h(\delta) = Q$ $\forall \delta \geqslant 0$; as h is convex, this would entail the linearity of that same function. Item (c) is so verified ab absurdo.

6.4. *Circular bounds*

The main result is here the one that follows from Prop. 5.8. For any $i \in Z_j \cap Z_k$, we call C_i the disk the centre and radius of which have been determined according to Prop. 5.8 (if $\alpha \neq 1$) when $h = \cdot^2$. When i is not a point of the x-axis, C_i is the disk tangent to $Z_j \cap Z_k$ at i and the centre of which lies on the x-axis. On the other hand we define $C(i_1, i_2)$ as the disk centred on i_1 and the circle of which passes through i_2. In particular when Z_k is bounded C [(ℓ + r)/2, ℓ], which is equal to C [(ℓ + r)/2,r], is the disk of which [ℓr] is a diameter. In the next proposition, for simplicity's sake, we have not repeated that the properties of the function $\sqrt{.}$ / (h'∘$\sqrt{.}$) are those of $\delta/h'(\delta)$ wrt. to δ^2.

(6.4) (a) *When $\sqrt{.}$ / $(h'\circ\sqrt{.})$ is concave,*

(i) *if Z_k is bounded, $Z_k \subseteq C[(\ell + r)/2, r]$;*

(ii) *if $\sqrt{.}$ / $(h'\circ\sqrt{.})$ is strictly increasing, $C_i \subseteq Z_k$, $\forall i \in Z_j \cap Z_k$, and $Z_k = \bigcup_{i \in Z_j \cap Z_k} C_i$;*

(iii) *$\sqrt{.}$ / $(h'\circ\sqrt{.})$ is not strictly decreasing ;*

(b) *When $\sqrt{.}$ / $(h'\circ\sqrt{.})$ is convex ,*

(i) *if Z_k is bounded, $C[(\ell + r)/2, r] \subseteq Z_k$;*

(ii) *if $\sqrt{.}/(h'\circ\sqrt{.})$ is strictly increasing, $Z_k \subseteq C_i$, $\forall i \in Z_j \cap Z_k$, and $Z_k = \bigcap_{i \in Z_j \cap Z_k} C_i$;*

(iii) *if $\sqrt{.}/(h'\circ\sqrt{.})$ is strictly decreasing, $C_i \subseteq Z_j$, $\forall i \in Z_j \cap Z_k$, and $Z_j = \bigcup_{i \in Z_j \cap Z_k} C_i$.*

Prop. 6.4 is a mere application of Prop. 5.8 . The only problem is to know whether $D_i(.^2)$ and $D_{\ell r}(.^2)$ correspond to C_i and C [(ℓ + r)/2,r] or to the respective closures of their exteriors ('exterior' is in this case equivalent to 'complementary area'). As to $D_{\ell r}(.^2)$, we have $D_{\ell r}(.^2)$ = C [(ℓ + r) / 2, r] in any case, as $D^*_{\ell r}(.^2) \cap R$ is obviously finite (see Fig. 5.4b) and must thus be equal to $\chi_{.^2}$< C [(ℓ + r)/2,r] >.

We have now to see whether $D_i(.^2)$ is equal to C_i or to the closure of its exterior. As $\delta_k^2 \leqslant \alpha\ \delta_j^2 + \zeta$ is equivalent (when $\alpha \neq 1$) to

$$(\alpha - 1) \left[\left(x - \delta_{ok} \frac{1 + \alpha}{1 - \alpha}\right)^2 + y^2 \right] \geqslant \delta_{ok}^2 \frac{4\alpha}{\alpha - 1} - \zeta ,$$

the problem boils down to knowing whether $\alpha < 1$ or $\alpha > 1$. Equality (18) shows that $\alpha < 1$ when $\tilde{h}'(\delta) / h'(\delta)$ is strictly increasing wrt. δ^2 (or δ), and that $\alpha > 1$ when it is strictly decreasing. The proof of Prop. 6.4 is then immediate.

In particular, the case where $\delta/h'(\delta)$ would be concave and strictly decreasing wrt. δ^2 is impossible, for it would lead to $Z_j \subseteq C_i$. However, $\delta/h'(\delta)$ may have that property when δ is in a vicinity of $[\delta_k, \delta_j]$. The inclusion then makes sense in a vicinity of point i : there is some vicinity V of i(i.e., a set including some open set containing i) for which $Z_j \cap V \subseteq C_i \cap V$. This is what happens in Fig. 8.1b if $\delta_k \leqslant 3\,\delta^*/5$. Similar remarks can be made for all the items of Prop. 6.4. Also, relatively to any particular point i of $Z_j \cap Z_k$, the condition that $\delta/h'(\delta)$ should be strictly increasing or decreasing may be replaced by the local one that $\alpha < 1$ or $\alpha > 1$. A final note : when $\alpha = 1$, the area $\delta_k^2 \leqslant \alpha\delta_j^2 + \zeta$ degenerates into the half-plane $[x, +\infty[\times \{0\}$; it should not be difficult for the reader to foresee what ensues when $\delta/h'(\delta)$ is concave or convex wrt. δ^2.

Prop. 6.4 has the interesting consequence that we are able to get information about the smallest disk circumscribed to Z_k and about a largest disk inscribed in Z_k. Let us denote by s (a *summit*) a point maximizing the ordinate y on Z_k, if any ; and by s' its orthogonal projection on the x-axis.

(6.5) (a) *If $\sqrt{.} / (h' \circ \sqrt{.})$ is concave,*

(i) *if Z_k is bounded, $C[(\ell + r)/2, r]$ is the smallest disk containing Z_k ;*

(ii) *if a summit s exists and $\sqrt{.} / (h' \circ \sqrt{.})$ is strictly increasing, $C(s', s)$ is a largest disk contained in Z_k ;*

(b) *If $\sqrt{.} / (h' \circ \sqrt{.})$ is convex,*

(i) *if Z_k is bounded, $C[(\ell + r)/2, r]$ is a largest disk contained in Z_k ;*

(ii) *if* $\sqrt{.}\ /\ (h' \circ \sqrt{.})$ *is strictly increasing, a unique summit* s *exists and* $C\ (s',s)$ *is the smallest disk containing* Z_k.

The proof, being obvious, is omitted.

The bounds recorded in Prop. 6.4 are *tight* in the sense that, for Prop. 6.4a eg., C_i is the largest disk contained in Z_k and the circle of which passes through i ; etc. This is guaranteed by the symmetry of Z_k wrt. the x-axis. Let us detail the case of item (a) for instance. If point i is not on the x-axis, a disk contained in Z_k, tangent to $Z_j \cap Z_k$ at i but larger than C_i would have its centre lying in Z_k on the normal to $Z_j \cap Z_k$ at i (ie., on the straight line joining i to the centre of C_i), but farther from i than the centre of C_i ; and so it would encompass the symmetric of i wrt. to the x-axis, and contain points of $Z_j - Z_k$: a contradiction.

For continuity reasons, this remains true when i is on the x-axis. More generally, anyway, it is obvious that

(6.6) *If point* i *lies on the x-axis,* C_i *is the osculating disk to* $Z_j \cap Z_k$ *at* i,

i.e., the disk the circle of which is locally most alike to $Z_j \cap Z_k$ at i ; the radius of C_i is then the *radius of curvature* of $Z_j \cap Z_k$ at i. This may be useful when drawing $Z_j \cap Z_k$.

We now give two other circular bounds on Z_k. They are necessarily less precise than the above-mentioned ones, but the conditions for their application are different from those of Prop. 6.4. We denote by A_i the Apollonian area relative to points j and k and the circle of which passes through a point i ; the area $A_{i'}$ is described by the inequality $\delta_j/\delta_k \geqslant \delta_{i'j}/\delta_{i'k}$. When $i \in \mathbb{R}^*_+ \times \mathbb{R}$, A_i is a disk ; it is the closure of the exterior of a disk when $i \in \mathbb{R}^*_- \times \mathbb{R}$. The abcissa of the centre of the disk is

$$x_c = \delta_{ok} \frac{\delta_j^2 + \delta_k^2}{\delta_j^2 - \delta_k^2} = \frac{\delta_o^2 + \delta_{ok}^2}{2\,x}. \qquad (19)$$

When Z_k is empty, we set $C\ (k,r) = C\ (k,\ell) = A_r = A_\ell = \phi$.

(6.7) (a) *If* Z_k *is bounded,* $Z_k \subseteq C\ (k,r)$;

(b) *If* $k \in Z_k$, $C\ (k,\ell) \subseteq Z_k$;

(c) $C\ (j,\ell) \subset Z_j$.

(6.8) (a) *If* $h \circ \exp$ *is convex, ie. if* $h\ (\delta)$ *is a convex expression of* $\ln \delta$, *then*

(i) *if* Z_k *is bounded,* $Z_k \subseteq A_r$;

(ii) *if* $k \in Z_k$, $A_\ell \subseteq Z_k$;

(b) *If* $h \circ \exp$ *is concave, ie. if* $h\ (\delta)$ *is a concave expression of* $\ln \delta$, *then* $Z_k \subseteq A_\ell$ *and, if* Z_k *is bounded,* $A_r \subseteq Z_k$.

Those properties obviously derive from Prop. 5.1 and 5.5a, and from the fact that when Z_k is not empty centre k either belongs to Z_k or to $]o\ell[$ (Prop. 6.3a). Also, regarding Prop. 6.8b, we know from Prop. 5.7a that the concavity of $h \circ \exp$ entails the concavity of h itself ; which in turn implies that $k \in Z_k$ if $Z_k \neq \phi$ (Prop. 6.3b). Here is for instance a proof of Prop. 6.7c. For every $p \in [j\ell]$, the circle centred on j and passing through ℓ is in Z_j according to Prop. 5.1. As $C\ (j,\ell)$ is the union of all such circles, we have finally $C\ (j,\ell) \subset Z_j$.

Prop. 6.4 and 6.5 show that the maximal distance between any two points of Z_k, i.e., its diameter Φ_k, may be equal to the length L_x of the projection of Z_k upon the x-axis or to the length L_y of its projection upon the y-axis. From another point of view, the ratio L_y/L_x may be <1 or >1, which means that Z_k is somehow directed as the segment [jk] or perpendicular to it. When $k \in Z_k$, Prop. 6.7b and 6.8 may be used to produce additional information about the ratios L_y/L_x and Φ_k/L_x. Notice that L_x and L_y are $\leqslant \Phi_k$, so that $L_y/L_x \leqslant \Phi_k/L_x \geqslant 1$.

COROLLARY 6.1 *When $L_x \neq 0$,*

(a) *if $\sqrt{.}\ /\ (h' \circ \sqrt{.}\ )$ is concave and strictly increasing, then $L_x = \Phi_k$, and $L_y/L_x \leqslant 1$ when L_y is finite ;*

(b) *if $\sqrt{.}\ /\ (h' \circ \sqrt{.}\ )$ is convex and strictly increasing, $L_y = \Phi_k$ and $1 \leqslant L_y/L_x$;*

(c) *if $k \in Z_k$, $L_y/L_x \leqslant \Phi_k/L_x < 2$; if moreover $h \circ \exp$ is convex, $L_y/L_x \leqslant \Phi_k\ /L_x \leqslant 1 + \delta_{ok}/(\delta_{ok} + L_x)$;*

(d) *if $h \circ \exp$ is concave and $L_x \geqslant \delta_{ok}$, $1 - \delta^2_{ok}\ /\ L^2_x \leqslant L_y\ /\ L_x \leqslant \Phi_k\ /\ L_x < 2$.*

Items (a) and (b) derive from Prop. 6.5 . As to item (c), if $k \in Z_k$, we have $\delta_{kr} \leqslant L_x$;and $L_x \in \mathbb{R}^*_+$ implies that Z_k is bounded and that $Z_k \subset C\ (k,r)$, so that $\Phi_k < 2\ \delta_{kr}$ (the inclusion is strict as $k \neq r$: see Prop. 5.1 ; and k is $\neq$ r because $k = r$ would imply $Z_k = C\ (k,k) = \{k\}$ and $L_x = 0$). When $h \circ \exp$ is convex, $Z_k \subseteq A_r$, so that Φ_k is not larger than the diameter of A_r, which is $x_r - \delta^2_{ok}/x_r$. As that expression is increasing wrt. x_r and as $k \in Z_k$ implies that $x_r \leqslant \delta_{ok} + L_x$, the proof of (c) is completed. When $h \circ \exp$ is concave, $A_r \subseteq Z_k$ and $L_y \geqslant x_r - \delta^2_{ok}\ /\ x_r$. The hypothesis $L_x \geqslant \delta_{ok}$ implies that the smallest possible value of x_r is L_x. Hence the lower bound on L_y/L_x. On the other hand h is now concave (Prop. 5.7a) and Z_k is not empty since $L_x > 0$; so item (c) yields the upper bound on Φ_k/L_x (the specification $L_x < +\infty$ is useless because it follows from the concavity of $h \circ \exp$ that $h'(+\infty) = 0$, from which we know that Z_k is bounded : see Prop. 6.2b).

A second consequence of Prop. 6.7 concerns the points ℓ and r (see also Prop. 4.4) :

COROLLARY 6.2 *Point ℓ is the point of $Z_j \cap Z_k$ closest to [jk]; if Z_k is bounded, point r is the point of $Z_j \cap Z_k$ farthest from [jk].*

Let us now have a look at two examples of application of all those properties. Take first the monomial transportation cost function : δ^a, with $a > 0$. Here $\delta/h'(\delta)$ is equal to δ^{2-a}/a, or $(\delta^2)^{1-a/2}/a$, so that, for every $i \in Z_j \cap Z_k$,

$C_i \subseteq Z_k$ if $a \in]\,0,2\,]$ and $C_i \subset Z_j$ if $a > 2$. The abscissa of the centre $\bar{c}_i$ of C_i is

$$x_{c_i} = \delta_{ok} \frac{\delta_j^{2-a} + \delta_k^{2-a}}{\delta_j^{2-a} - \delta_k^{2-a}} . \qquad (20)$$

The similarity between (19) and (20) is not casual, as $\ell n\ (\delta)$ is a kind of limit of δ^a when $a \gtrless 0$; see Section 10. On the other hand , $h(\delta) = \delta^a = \exp\,(a\ \ell n\ \delta)$ is convex wrt. $\ell n\ \delta$; if $a < 1$, in particular, we thus have $A_\ell \subseteq Z_k \subseteq A_r$.
The second example is $h\ (\delta) = -\ \delta^{-a}$, with $a > 0$. In opposition with the previous one, $\delta/h'(\delta)$ is now $(\delta^2)^{1+a/2}/a$ so that for every $i \in Z_j \cap Z_k$ we have $Z_k \subseteq C_i$. The ratio L_y/L_x is consequently now in the range $]\,1,2\,[$, instead of $]\,0,1\,[$. As to $h \circ \exp$, it is now concave, as $h\ (\delta) = -\ \exp\ (-a\ \ell n\ \delta)$; so that $A_r \subseteq Z_k \subseteq A_\ell$.

6.5. *Hyperbolic bounds*

We define the two following hyperbolic areas :

$$H_\ell = \{i\ ;\ \delta_j - \delta_k \geqslant \delta_{j\ell} - \delta_{k\ell} \text{ and } \delta_j \geqslant \delta_{j\ell}\ \}$$

$$H_\infty = \{i\ ;\ \delta_j - \delta_k \geqslant Q/h'(+\infty)\}.$$

From now on we implicitly assume, when necessary, that $h'\ (+\infty) \in \mathbb{R}_+ \cup \{+\infty\}$.
To reach a full understanding of the proposition that follows, it is necessary to notice that those areas may in some cases degenerate into obvious bounds, making the proposition useless.

When $\ell \in]\ ok\ ]$, the boundary of H_ℓ is a branch of a hyperbola (degenerating into $[\,\delta_{ok}, +\infty\,[\times \{0\}$ if $k = \ell$) with foci j and k ; the specification $\delta_j \geqslant \delta_{j\ell}$ in the definition of H_ℓ is superfluous because ℓ is then the closest point of the branch to centre j (Corollary 6.2). But it may happen (Prop.6.3a and c) that $k \in]\,o\ell\,[$; in that case H_ℓ degenerates into $[\,x_\ell, +\infty\,[\times \{0\}$, and the specification $\delta_j \geqslant \delta_{j\ell}$ is now necessary. The information yielded by Prop. 6.9b about H_ℓ is nothing but a particular case of Prop. 8.3b when those cases of degeneracy occur.

The definition of H_∞ shows that it degenerates into $\mathbb{R}_+ \times \mathbb{R}$ if $h'(+\infty) = +\infty$. It obviously also degenerates into the straight half-line $[\delta_{ok}, +\infty[\times \{0\}$ when $Q = h'(+\infty)\,\delta_{jk}$, and becomes empty when $Q > h'(+\infty)\delta_{jk}$.

(6.9) (a) *If h is concave,* $H_\infty \subseteq Z_k \subseteq H_\ell$;

(b) *If h is convex,* $H_\ell \subseteq Z_k \subseteq H_\infty$.

The proof concerning H_ℓ directly derives from Prop. 5.5b. As to H_∞, the mean-value theorem tells us that $h_j - h_k = h'(\delta)(\delta_j - \delta_k)$ for some $\delta \in]\delta_k, \delta_j[$. If h is concave for instance we have $h'(\delta) \geqslant h'(+\infty)$ so that $h_j - h_k \geqslant h'(+\infty)(\delta_j - \delta_k)$: hence $h'(+\infty)(\delta_j - \delta_k) \geqslant Q$ implies $h_j - h_k \geqslant Q$, i.e., $H_\infty \subseteq Z_k$. The argument is similar when h is convex.

Let us study how Prop. 6.9 can be used for the monomial transportation cost function. The bound H_∞ is here completely useless : when $a \in]0,1[$, it degenerates into ϕ ; when $a = 1$, it is equal to Z_k ; and when $a > 1$, it degenerates into $\mathbb{R}_+ \times \mathbb{R}$. The bound H_ℓ is effective when $a \in]0,1[$. When $a \geqslant 2$, although H_ℓ is not necessarily degenerate, it is less tight than the bound V_ℓ studied in the next section. It is only when $a \in]1,2[$ that the inclusion of H_ℓ in Z_k provides some original information if $\ell \in]ok[$.

The bounds H_ℓ and H_∞ are of special interest if the transportation cost function $h(\delta)$ is a positive-coefficient linear combination of δ with expressions like δ^a (with $a \in]0,1[$) ,or $\ell n\,\delta$,or $-\delta^{-a}$(with $a > 0$).The transportation cost function is then concave and $H_\infty \subseteq Z_k \subseteq H_\ell$, and neither of the bounds is a priori degenerate. On the other hand, an illustration of Prop. 6.9b is given by $h(\delta) = \delta - \sqrt{\delta + K}$ with $K \geqslant 1/4$: then $H_\ell \subseteq Z_k \subseteq H_\infty$, with no a priori degeneracy of the bounds.

A question which we have not raised so far is whether the boundary of H_∞ is a limiting curve of $Z_j \cap Z_k$. As that boundary of H_∞ has itself two straight asymptotes, the problem boils down to the determination of whether those straight asymptotes are limiting curves of $Z_j \cap Z_k$; we shall come back to this issue in Sections 6.8 and 6.10.

6.6. *Asymptotic direction*

Suppose that Z_k is not bounded and let a point i move towards infinity on $Z_j \cap Z_k$ above the x-axis. If the angle $\varphi_{i'}$ under which i is seen from some fixed point i' of the plane tends towards some limit that we denote by φ_∞, the angle $\varphi_{i''}$ must tend towards the same limit for any point i" of the plane. We call that angle φ_∞ the *asymptotic direction* of $Z_j \cap Z_k$. When φ_∞ exists it is easily seen (Fig. 6.3) that

$$\lim_{\substack{\delta_k \to +\infty \\ i \in Z_j \cap Z_k}} (\delta_j - \delta_k) = \delta_{jk} \cos \varphi_\infty \ .$$

As that limit is also equal to $Q / h'(+\infty)$ (see the proof of Prop. 6.6), we finally get

$$\cos \varphi_\infty = \frac{Q}{h'(+\infty)\ \delta_{jk}} \ . \tag{21}$$

Conversely, when Z_k is unbounded and $h'(+\infty)$ exists, $Z_j \cap Z_k$ has the asymptotic direction given by (21).

It can be seen in particular that the boundary of H_∞ (Section 6.3) has the same asymptotic direction as $Z_j \cap Z_k$. Another interesting point is that the property $0 \leqslant \cos \varphi_\infty \leqslant 1$ (as $\varphi_\infty \in [0, \pi/2]$) is perfectly coherent with the conditions of unboundedness of Z_k ; see Prop. 6.2. Formula (21) is also in accordance with Prop. 4.5 : as Z_k is a decreasing function of the attractivity constant Q, so must be the angle φ_∞.

6.7. *Vertical bounds and vertical limiting lines*

Assuming that $h''(+\infty)$ exists lato sensu, we define the following vertical half-planes :

$$V_\ell = \{i;\ x \geqslant x_\ell\}$$

$$V_\infty = \{\ i;\ x \geqslant \frac{Q}{\delta_{jk}\ h''(+\infty)}\ \} \ .$$

Of those two sets, the only one that may degenerate is V_∞. This happens when $h''(+\infty) = 0$ and when $h''(+\infty) = +\infty$; in the first case, we have $V_\infty = \phi$; in the second one, $V_\infty = \mathbb{R}_+ \times \mathbb{R}$. Notice that $h''(+\infty) \geqslant 0$, as will appear from the proof of Prop. 6.10.

(6.10) (a) *If $h''(+\infty) \neq 0$, the boundary of V_∞ is a limiting line of $Z_j \cap Z_k$;*

(b) *If $h''(+\infty) = 0$, $Z_j \cap Z_k$ has no vertical limiting line ;*

(c) *If $h \circ \sqrt{.}$ is concave, $V_\infty \subseteq Z_k \subseteq V_\ell$;*

(d) *If $h \circ \sqrt{.}$ is convex, $V_\ell \subseteq Z_k \subseteq V_\infty$;*

(e) *If $h \circ \sqrt{.}$ is concave or convex and $h''(+\infty) \neq 0$, or if $h''(+\infty) = +\infty$, the boundary of V_∞ is asymptotic to $Z_j \cap Z_k$.*

The proof relies again on the mean-value theorem, now applied to the function $h \circ \sqrt{.}$. From this and from equality (8) we deduce that

$$\Delta h = h(\sqrt{\delta_j^2}) - h(\sqrt{\delta_k^2}) = (h \circ \sqrt{.})'(\delta^2)(\delta_j^2 - \delta_k^2) = \frac{h'(\delta)}{\delta} x \delta_{jk} \qquad (22)$$

for some $\delta \in]\,\delta_k, \delta_j\,[$. As $h''(+\infty)$ is supposed to exist lato sensu, we have according to de l'Hospital's rule, even if $h'(+\infty) \neq +\infty$,

$$\lim_{\delta \to +\infty} \frac{h'(\delta)}{\delta} = h''(+\infty).$$

Consequently, $h''(+\infty) \geqslant 0$ as announced, and (22) entails that, if $h''(+\infty) > 0$,

$$x_\infty = \lim_{\substack{\delta_k \to +\infty \\ i \in Z_j \cap Z_k}} x = \frac{Q}{h''(+\infty)\,\delta_{jk}} . \qquad (23)$$

As $h''(+\infty) > 0$ implies that $h'(+\infty) = +\infty$, Z_k is unbounded and the limit (23) makes sense. Hence the first item. Conversely if there is a vertical limiting line, Z_k is unbounded and we may use (23), which shows that $h''(+\infty) \neq 0$. This proves item (b) by contraposition.

As to the third item : the inclusion of Z_k in V_ℓ is a mere application of Prop. 5.5c ; and as $h \circ \sqrt{.}$ is concave, we have

$$(h \circ \sqrt{.})' \; (\delta^2) \; = \frac{h'(\delta)}{2\,\delta} \geqslant (h \circ \sqrt{.}\,)' \; (+\infty) = \frac{h''(+\infty)}{2} ,$$

so that if $i \in V_\infty$, then $\Delta h \geqslant h''\,(+\infty) \times \delta_{jk} \geqslant Q$ because of (22), and $i \in Z_k$. The argument is similar for item (d). And item (e) obviously results from items (c) and (d), from the proof of item (a), and from Prop. 4.3.

In the case of the monomial transportation cost function δ^a, Prop. 6.10 shows that $Z_j \cap Z_k$ has no vertical limiting curve when $a \in [\,1, 2\,[$, and that the axis of y is the straight asymptote of $Z_j \cap Z_k$ when $a > 2$ (see Fig. 5.3). When the transportation cost is for instance a positive-coefficient linear combination of δ^2 with expressions like $\ell n\ \delta$ or δ^a with $a \in\,]\,0,2\,[$, we face a more complex situation where the vertical asymptote is not the y-axis.

6.8. *Oblique limiting lines*

Suppose now that $Q \in\,]\,\delta_{jk}\,h'\,(+\infty),\ +\infty\,[$, so that Z_k is unbounded and $Z_j \cap Z_k$ has a non-vertical and non-horizontal asymptotic direction : $\varphi_\infty \notin \{0,\ \frac{\pi}{2}\}$. We construct a new system of axes by rotating the present system clockwise with an angle $\pi/2 - \varphi_\infty$. The new abscissa x' is related thus to the old coordinates :

$$x' = x \sin \varphi_\infty - y \cos \varphi_\infty \quad . \qquad (24)$$

As the x'-axis is perpendicular to the asymptotic direction, the existence of a limiting straight line of $Z_j \cap Z_k$ is equivalent to that of the limit of x' when point i moves towards infinity along $Z_j \cap Z_k$. In the system of polar coordinates (δ_o, φ_o) associated with point o, we have $x = \delta_o \cos \varphi_o$ and $y = \delta_o \sin \varphi_o$; hence x' may be rewritten

$$\begin{aligned} x' &= \delta_o \;(\cos \varphi_o \sin \varphi_\infty - \sin \varphi_o \cos \varphi_\infty) \\ &= \delta_o \; \frac{\cos^2 \varphi_o \sin^2 \varphi_\infty - \sin^2 \varphi_o \cos^2 \varphi_\infty}{\cos \varphi_o \sin \varphi_\infty + \sin \varphi_o \cos \varphi_\infty} = \delta_o^2 \; \frac{\cos^2 \varphi_o - \cos^2 \varphi_\infty}{\delta_o \sin\,(\varphi_o + \varphi_\infty)} \\ &= \frac{x^2 - \delta_o^2 \cos^2 \varphi_\infty}{\delta_o \;\sin(\varphi_o + \varphi_\infty)} \quad . \end{aligned}$$

We now express x and δ_o^2 as functions of δ_k and the difference $\delta_j - \delta_k$, in short $\Delta\delta$. As $x = (\delta_j^2 - \delta_k^2)/\,2\delta_{jk}$ and $\delta_o^2 = (\delta_j^2 + \delta_k^2)/2 - \delta_{ok}^2$, this yields

$$x = \frac{\Delta\delta}{2\delta_{jk}}\,(2\delta_k + \Delta\delta)$$

$$\delta_o^2 = \delta_k^2 + \delta_k\,\Delta\delta + \frac{(\Delta\delta)^2}{2} - \delta_{ok}^2$$

so that

$$x' = \frac{\delta_k\,(\delta_k + \Delta\delta)\,[\dfrac{(\Delta\delta)^2}{\delta_{jk}{}^2} - \cos^2\varphi_\infty\,] + C}{\delta_o\,\sin\,(\varphi_o + \varphi_\infty)}$$

where

$$C = \frac{(\Delta\delta)^4}{4\delta_{jk}^2} - [\,\frac{(\Delta\delta)^2}{2} - \delta_{ok}^2]\,\cos^2\varphi_\infty \;.$$

As $\Delta\delta \in\,]\,0,\ \delta_{jk}\,[$, C is bounded above and C/δ_o tends towards 0 when $\delta_k \to +\infty$. Let us now consider the difference $(\Delta\delta/\delta_{jk} - \cos\varphi_\infty)$. According to the mean-value theorem, $\Delta h = h'(\delta)\,\Delta\delta = Q$ for some $\delta \in\,]\,\delta_k,\ \delta_j\,[$; taking also (21) into account, we have :

$$\frac{\Delta\delta}{\delta_{jk}} - \cos\varphi_\infty = \frac{Q}{\delta_{jk}}\,[\frac{1}{h'(\delta)} - \frac{1}{h'(+\infty)}\,]\;.$$

Let us now set $f\,(1/\delta) = 1/h'(\delta)$, so that, using again the mean-value theorem, we find for some $\tilde{\delta} > \delta$:

$$\frac{\Delta\delta}{\delta_{jk}} - \cos\varphi_\infty = \frac{Q}{\delta_{jk}}\,[\,f\,(\frac{1}{\delta}) - f\,(0)\,] = \frac{Q}{\delta_{jk}}\,.\,\frac{1}{\delta}\,.\,f'(1/\tilde{\delta})$$

$$= \frac{Q}{\delta_{jk}}\,.\,\frac{\tilde{\delta}^2\,h''\,(\tilde{\delta})}{\delta\,h'^2\,(\tilde{\delta})}\;. \qquad (26)$$

The abscissa x' may consequently be rewritten as

$$x' = \frac{\delta_k\,(\delta_k + \Delta\delta)\,(\Delta\delta\,/\,\delta_{jk} + \cos\varphi_\infty)\,Q\,\tilde{\delta}^2\,h''\,(\tilde{\delta})/\,\delta_{jk}\;\;\delta\,h'^2\,(\tilde{\delta}) + C}{\delta_o\,\sin\,(\varphi_o + \varphi_\infty)}\;,$$

so that we get, noticing that $\Delta\delta/\delta_{jk} \to \cos\varphi_\infty$ when $\delta_k \to +\infty$:

$$x'_\infty = \lim_{\substack{\delta_k \to +\infty \\ i \in Z_j \cap Z_k}} x' = \frac{2Q \cos\varphi_\infty}{h'^2(+\infty)\,\delta_{jk}\,\sin(2\varphi_\infty)} \lim_{\substack{\delta_k \to +\infty \\ i \in Z_j \cap Z_k}} \frac{\delta_k^2\,\tilde{\delta}^2\,h''(\tilde{\delta})}{\delta\,\delta_o} .$$

As δ_k/δ and δ_k/δ_o tend towards 1, we finally have, given (21) :

$$x'_\infty = \frac{\operatorname{cotg}\varphi_\infty}{h'(+\infty)} \lim_{\delta \to +\infty} [\delta^2 h''(\delta)] . \qquad (27)$$

That formula can be used for instance when the transportation cost h (δ) is a positive-coefficient linear combination of δ with $\ell n\,\delta$, or with functions of the type $-\delta^{-a}$ with $a > 0$. In such a case $Z_j \cap Z_k$ has two oblique limiting lines, which pass through point o iff. $\ell n\,\delta$ is not one of the functions composing the linear combination ; see Fig. 6.2 and 6.4.

We may also combine what we know about hyperbolic bounds with (27) ; this yields

(6.11) *When* $h'(+\infty) \in\,]\, Q/\delta_{jk}, +\infty\, [$, *the half-hyperbola* $\delta_j - \delta_k = Q/h'(+\infty)$ *is a limiting curve of* $Z_j \cap Z_k$ *iff.* $\lim_{\delta \to +\infty} \delta^2 h''(\delta) = 0$. *If moreover*

(a) *h is concave or convex, that half-hyperbola is asymptotic to* $Z_j \cap Z_k$;

(b) *h is convex, the oblique lines* $|y| = x\,tg\,\varphi_\infty$ *are asymptotes of* $Z_j \cap Z_k$.

When h (δ) = $\delta - \delta^{-4}$, for instance, $Z_j \cap Z_k$ has two oblique asymptotes $y = x\,tg\,\varphi_\infty$ and $y = -x\,tg\,\varphi_\infty$; see Fig.6.4. Prop. 6.11 also trivially applies, of course, when h (δ) = δ.

6.9. *Horizontal limiting lines*

Replacing δ_j by $\delta_k + \Delta\delta$ in the expression (17) of y^2, we get

$$y^2 = \delta_k^2 \left(1 + \frac{\Delta\delta}{\delta_k}\right) \left[1 - \frac{(\Delta\delta)^2}{\delta_{jk}^2}\right] - \delta_{ok}^2 \left[1 - \frac{(\Delta\delta)^2}{\delta_{jk}^2} \right]^2. \qquad (28)$$

If $Z_j \cap Z_k$ has a horizontal asymptotic direction , ie. if $\varphi_\infty = 0$, (21) implies that $Q = \delta_{jk}\, h'(+\infty)$; setting $f(1/\delta^2) = 1/h'(\delta)$, we then derive similarly to (26) that, for some $\delta \in]\, \delta_k, \delta_j\, [$ and some $\tilde{\delta} > \delta$, we have when point $i \in Z_j \cap Z_k$:

$$1 - \frac{\Delta\delta}{\delta_{jk}} = - \frac{\tilde{\delta}^3 \; h''(\tilde{\delta}) \; h'(+\infty)}{2\, \delta^2 \, h'^2(\tilde{\delta})} \quad . \tag{29}$$

If we let δ_k tend toward infinity, $h'(\delta)$ tends toward $h'(+\infty)$, $\Delta\delta \to \delta_{jk}$ (as $\varphi_\infty = 0$), and $\delta_k / \delta \to 1$. The limiting value of y^2 is consequently

$$y^2_\infty = \lim_{\substack{\delta_k \to +\infty \\ i \in Z_j \cap Z_k}} y^2 = - \lim_{\delta \to +\infty} [\, \delta^3 \, h''(\delta)\,] \, / \, h'(+\infty) . \tag{30}$$

According to that formula, when the limit of $\delta^3 h''(\delta)$ is > 0, y^2_∞ should be strictly negative : the conclusion is that Z_k must be bounded in such a case. This is again coherent with Prop. 6 .2 d, as the strict positivity of that limit entails the strict convexity of function h beyond some value $\underline{\delta}$ of its argument.

When $y_\infty = 0$, the horizontal limiting line, here reduced to the x-axis, is of course an symptote of $Z_j \cap Z_k$:

(6.12) *When $Q = h'(+\infty)\, \delta_{jk}$ and h is concave, the x-axis is asymptotic to $Z_j \cap Z_k$ iff. $\lim_{\delta \to +\infty} \delta^3 h''(\delta) = 0$.*

Applying those results for instance to transportation costs $h(\delta)$ that are positive-coefficient linear combinations of δ with functions of the type $-\delta^{-a}$ with $a \geqslant 1$ shows that the indifference line $Z_j \cap Z_k$ is then endowed, at the threshold value $\delta_{jk}\, h'(+\infty)$ of the attractivity constant Q, with a horizontal limiting line which is the x-axis unless one of those functions is $-1/\delta$; see the examples of Fig.6.2 and 6.5. When $Q < \delta_{jk}\, h'(+\infty)$, we have seen that there are two oblique limiting lines with such transportation cost functions. This is an illustration of the following property, which is obvious if one compares (27) and (30) :

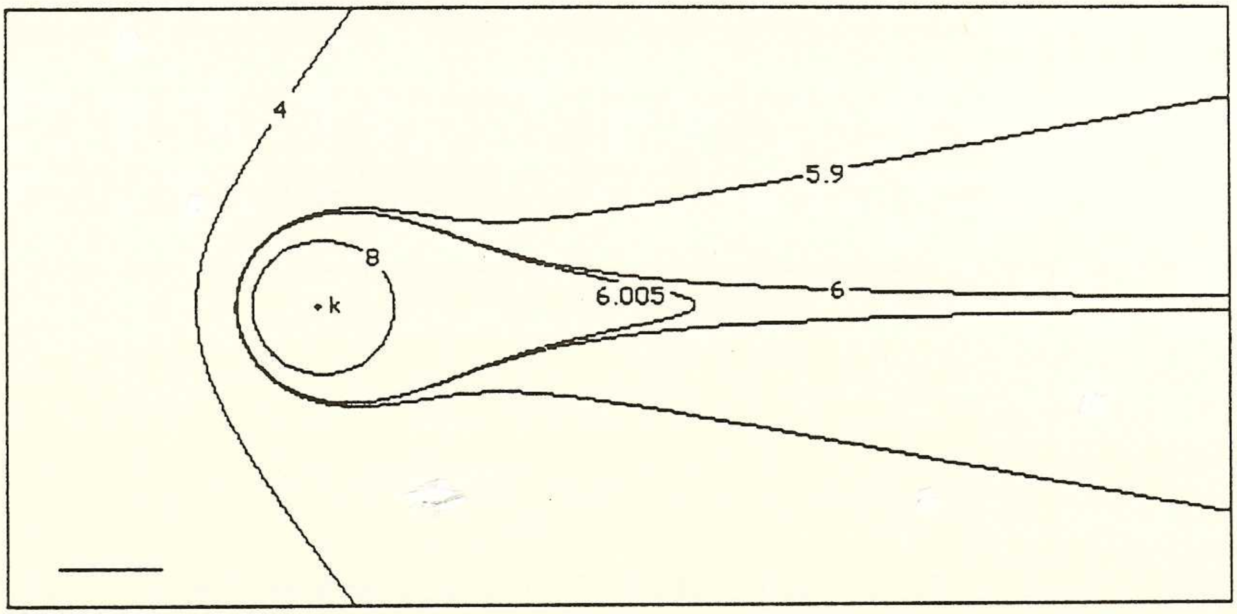


Fig. 6.4. Market areas generated by the t.c.f. $\delta - (\delta + 0.1)^{-4}$ when $\delta_{jk} = 6$, for indicated values of Q. The x-axis is the horizontal asymptote of the indifference line when Q = 6 ; the superficies of Z_k is then finite, however. When Q < 6, the half-hyperbola $\delta_j - \delta_k = Q$ is asymptotic to the indifference line and contained in Z_k.

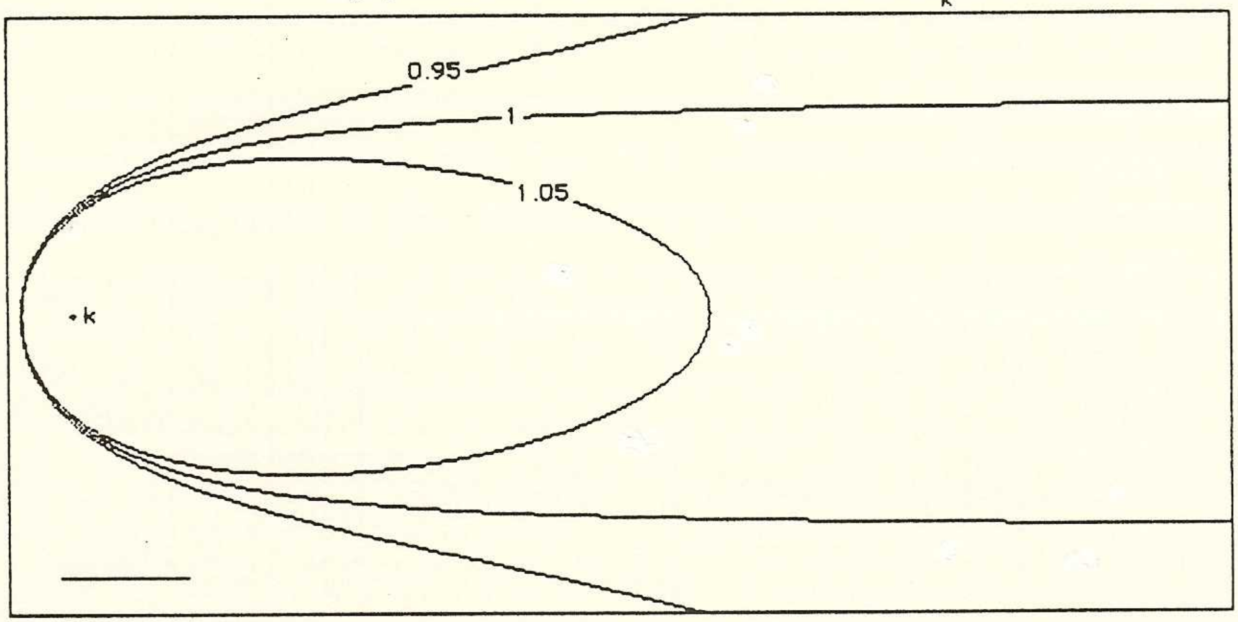


Fig. 6.5 . Market areas when the t. c. f. is $\delta - 1.5/\delta$ and $\delta_{jk} = 1$, for indicated values of Q. When Q = 1, the indifference line has two horizontal limiting lines; when Q < 1, it is asymptotic to the half-hyperbola $\delta_j - \delta_k = Q$, which is contained in Z_k.

(6.13) *If $Z_j \cap Z_k$ has one horizontal limiting line or two when $Q = \delta_{jk}\, h'\,(+\infty)$, then it has when $Q < \delta_{jk}\, h'\,(+\infty)$ two oblique limiting lines passing through point* o .

On the contrary, when $h\,(\delta) = \delta + \ell n\,(\delta)$ for instance, there is no horizontal limiting line if $Q = \delta_{jk}\, h'\,(+\infty)$, but there are two oblique ones if $Q < \delta_{jk}\, h'\,(+\infty)$. Those oblique lines do not cross at o but at some point of the x-axis on the left of o ; h is indeed concave, so that $x' < 0$: see formula (27) and Fig. 6.2.

If we now include in the linear combinations considered above functions of the type δ^a with $0 < a < 1$, there is neither oblique nor horizontal limiting line any more.

Notice also that formulae (27) and (30) and Prop. 6.11, 6.12, and 6.13 trivially apply when $h\,(\delta) = \delta$, as Z_k then shrinks to the straight half-line $[\,\delta_{ok}, +\infty\,[\, \times \{0\}$ when $Q = \delta_{jk}$, and has two oblique asymptotes passing through o when $Q < \delta_{jk}$.

The knowledge thus acquired of the shape of market areas at the threshold value $\delta_{jk}\, h'\,(+\infty)$ of Q, which marks the limit under which Z_k is unbounded and above which it is bounded, allows because of Prop. 4.5 to predict some features of the market areas when Q is close to $\delta_{jk}\, h'\,(+\infty)$. This is particularly perceptible in the case of Fig. 6.4.

7. *Variation of y with x along the indifference line*

This section provides information about the *slope* of the boundary line $Z_j \cap Z_k$. It establishes the formulae of the first and second derivatives of y wrt. x on the demarcation line $Z_j \cap Z_k$. The second derivative will be used in the proof of Prop. 8.6 to show the convexity of market area Z_k in some cases. The first derivative leads of course to the second one, and also to some properties which we shall mention although they may be seen as corollaries of subsequent ones of Section 8.

A simple way to arrive at the expression of the first derivative is to consider $\partial y^2/\partial x$, as $\partial y^2/\partial x = 2y\, \partial y/\partial x$. The first equality of (10) yields

$$\frac{\partial y^2}{\partial x} = \frac{\partial \delta_k^2}{\partial x} - 2\,(x - \delta_{ok}). \qquad (31)$$

On $Z_j \cap Z_k$, we have

$$(h \circ \sqrt{.})_j' \; d\delta_j^2 \; - \; (h \circ \sqrt{.})_k' \; d\delta_k^2 = 0;$$

given (11), this may be rewritten

$$(h \circ \sqrt{.})_j' \; (d\delta_k^2 + 2\,\delta_{jk}\; dx) - (h \circ \sqrt{.})_k' \; d\delta_k^2 = 0,$$

from which

$$\left(\frac{\partial \delta_k^2}{\partial x}\right)_{\Delta h} = \frac{2\delta_{jk}\,(h \circ \sqrt{.})_j'}{(h \circ \sqrt{.})_k' - (h \circ \sqrt{.})_j'} \; . \qquad (32)$$

So we deduce from (31) and (32) that

$$\left(\frac{\partial y^2}{\partial x}\right)_{\Delta h} = 2\left(\frac{(h \circ \sqrt{.})_k' + (h \circ \sqrt{.})_j'}{(h \circ \sqrt{.})_k' - (h \circ \sqrt{.})_j'}\,\delta_{ok} - x\right). \qquad (33)$$

The first consequence is that we are now able to compute the first of the derivatives we are looking for :

$$\left(\frac{\partial y}{\partial x}\right)_{\Delta h} = \frac{\dfrac{(h\circ\sqrt{.})'_k + (h\circ\sqrt{.})'_j}{(h\circ\sqrt{.})'_k - (h\circ\sqrt{.})'_j}\,\delta_{ok} - x}{y} \quad . \tag{34}$$

That formula immediately leads to the following properties :

(7.1) *On* $\mathbb{R}^2_+$,

(a) *When* $h\circ\sqrt{.}$ *is convex,* $(\partial y/\partial x)_{\Delta h} \leqslant 0$;

(b) *When* $h\circ\sqrt{.}$ *is concave but* h *is convex,* $(\partial y/\partial x)_{\Delta h} \geqslant 0$;

(c) *When* $h\circ\sqrt{.}$ *is concave,* $(\partial y/\partial x)_{\Delta h} \geqslant 0$ *if* $x \leqslant \delta_{ok}$.

Prop. 6.10 now appears as a consequence of Prop. 7.1, as far as the vertical bound V_ℓ is concerned. Prop. 7.1 itself may be derived from Prop. 8.3. But here its proof relies directly on (34), from which item (a) is obvious; when $h\circ\sqrt{.}$ is concave, the sign of $(\partial y/\partial x)_{\Delta h}$ is the sign of $(h\circ\sqrt{.})'_k\ (\delta_{ok} - x) + (h\circ\sqrt{.})'_j\ (\delta_{ok} + x)$, which expression yields item (c) and may be written [for $(h\circ\sqrt{.})'\ (\delta^2) = h'\ (\delta)\ /\ 2\delta$] as $(h'_j\ \cos\varphi_j - h'_k\ \cos\varphi_k)\ /2$, yielding item (b) (see the proof of Prop. 5.3).

Let us now tackle the second derivative. Considering again y^2 before y, we see that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{y}\left[\frac{1}{2}\ \frac{\partial^2 y^2}{\partial x^2} - \left(\frac{\partial y}{\partial x}\right)^2\right] . \tag{35}$$

All we have yet to do is to find an expression of $(\partial^2 y^2/\partial x^2)_{\Delta h}$.
To simplify, we set $\hbar = h\circ\sqrt{.}$ and $\Sigma\hbar' = \hbar'_j + \hbar'_k$. We already know $(\partial\delta^2_k\ /\ \partial x)_{\Delta h}$ from (32); a similar argument shows that $(\partial\delta^2_j\ /\ \partial x)_{\Delta h} = -2\delta_{jk}\ \hbar'_k / \Delta\hbar'$. We then compute from (33) :

$$\left(\frac{\partial^2 y^2}{\partial x^2}\right)_{\Delta h} = 2\left[-1 + \frac{\delta_{ok}}{(\Delta \hbar')^2}\left[\Delta\hbar'\left(\hbar''_j \frac{2\delta_{jk}\hbar'_k}{\Delta\hbar'} + \hbar''_k \frac{2\delta_{jk}\hbar'_j}{\Delta\hbar'}\right) + \Sigma\hbar'\left(\hbar''_k \frac{2\delta_{jk}\hbar'_j}{\Delta\hbar'} - \hbar''_j \frac{2\delta_{jk}\hbar'_k}{\Delta\hbar'}\right)\right]\right]$$

$$= 2\left[-1 + \frac{\delta^2_{jk}}{(\Delta\hbar')^3}\left[\hbar''_j\,\hbar'_k\,(\Delta\hbar' - \Sigma\hbar') + \hbar''_k\,\hbar'_j\,(\Delta\hbar' + \Sigma\hbar')\right]\right]$$

$$= 2\left[-1 + \frac{2\delta^2_{jk}}{(\Delta\hbar')^3}\left(\hbar''_k\,\hbar'^2_j - \hbar''_j\,\hbar'^2_k\right)\right]$$

$$= 2\left[-1 + \frac{2\delta^2_{jk}\,\hbar'^2_k\,\hbar'^2_j}{(\Delta\hbar')^3}\left(\frac{\hbar''_k}{\hbar'^2_k} - \frac{\hbar''_j}{\hbar'^2_j}\right)\right].$$

Noticing that $\hbar''/\hbar'^2 = (-1/\hbar')' = -2\,[\sqrt{.}/(h'\circ\sqrt{.})]'$, and defining $g = \sqrt{.}/(h'\circ\sqrt{.})$, we finally deduce from (34) that

$$\left(\frac{\partial^2 y}{\partial x^2}\right)_{\Delta h} = \frac{1}{y}\left(\frac{4\,\delta^2_{jk}\,\hbar'^2_k\,\hbar'^2_j\,\Delta g'}{(\Delta\hbar')^3} - 1\right) - \frac{1}{y^3}\left(\frac{\Sigma\hbar'}{\Delta\hbar'}\,\delta_{ok} + x\right)^2. \quad (36)$$

8. *The standpoint of mathematical topology : spatial structure*

The properties listed in this section are usually introduced in the mathematical field of *general topology*. Spinedness, however, seems a concept of our own, but we include it here because of its resemblance to starshapedness. Also, we have thought it interesting to mention visibility as another point of view on starshapedness.

Mark that only two of the properties deserve the name of *topological* ones : these are arcwise-connectedness and simple connectedness. A property is topological iff. it is maintained through any homeomorphism ; see eg. Lipschutz (1965).

8.1. *Arcwise-connectedness*

A set E is *arcwise connected*, i.e., *unipartite*, iff. any two points of E can be joined by a *path*(i.e., a continuous application of $[0,1]$ into the plane) contained in E. Otherwise said, E is all of a piece (but can have holes : see Section 8.2). A maximal arcwise connected subset of a set E (be E unipartite or not) is an *arcwise connected component* of E. The arcwise connected components of E constitute a partition of E.

(8.1)(a) *Z_j is arcwise connected ;*

(b) *The unipartite components of $Z_k \cap [\delta_{ok}, +\infty[\times \{0\}$ are the intersections of the unipartite components of Z_k with $[\delta_{ok}, +\infty[\times \{0\}$, or equivalently the circular projections of the unipartite components of Z_k around k on $[\delta_{ok}, +\infty[\times \{0\}$;*

(c) *Z_k has no more than one unbounded arcwise connected component ;*

(d) *If h is concave or convex, Z_k is arcwise connected.*

The items (b), (c), and (d) remain true if '$Z_k - Z_j$' replaces 'Z_k'.

Item (d) applies for instance to the monomial transportation cost function δ^a : Z_k is then arcwise connected anyway. Conversely, Fig. 8.1a displays a case where h is first concave, then convex. It can be shown, using Prop. 8.1b, that Z_k may then be bipartite as occurs here.

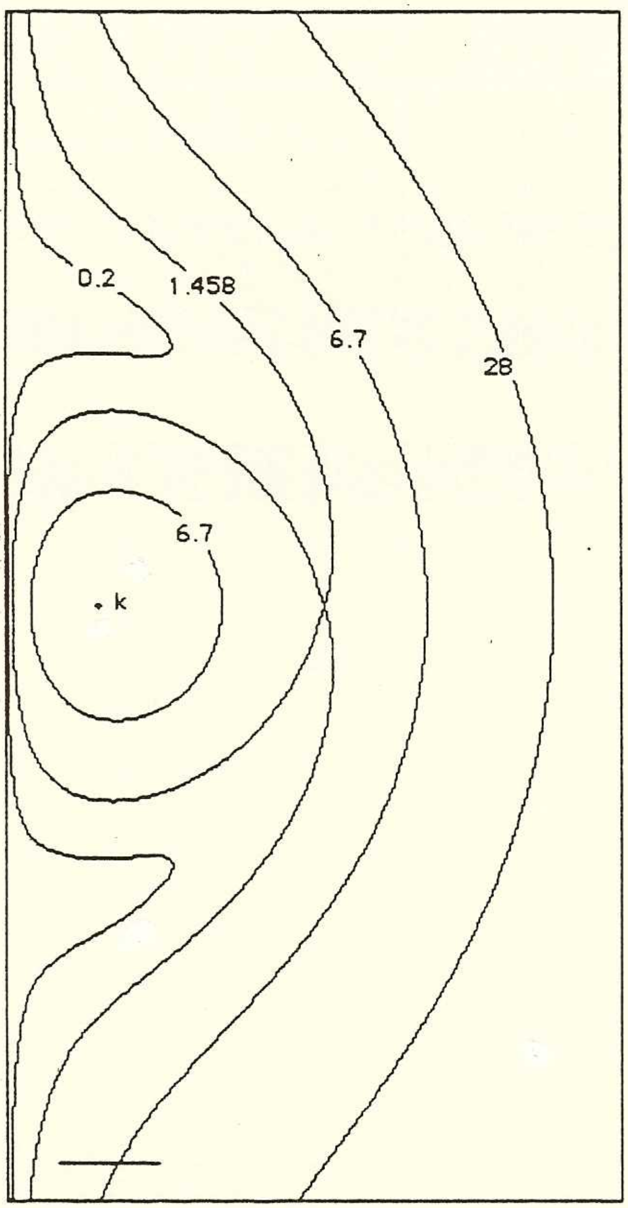


Fig. 8.1a. Market areas produced by a t.c.f. first concave, then convex : $h(\delta) = (\delta - 3.1)^3$, $\delta_{jk} = 1.8$, indicated values of Q.

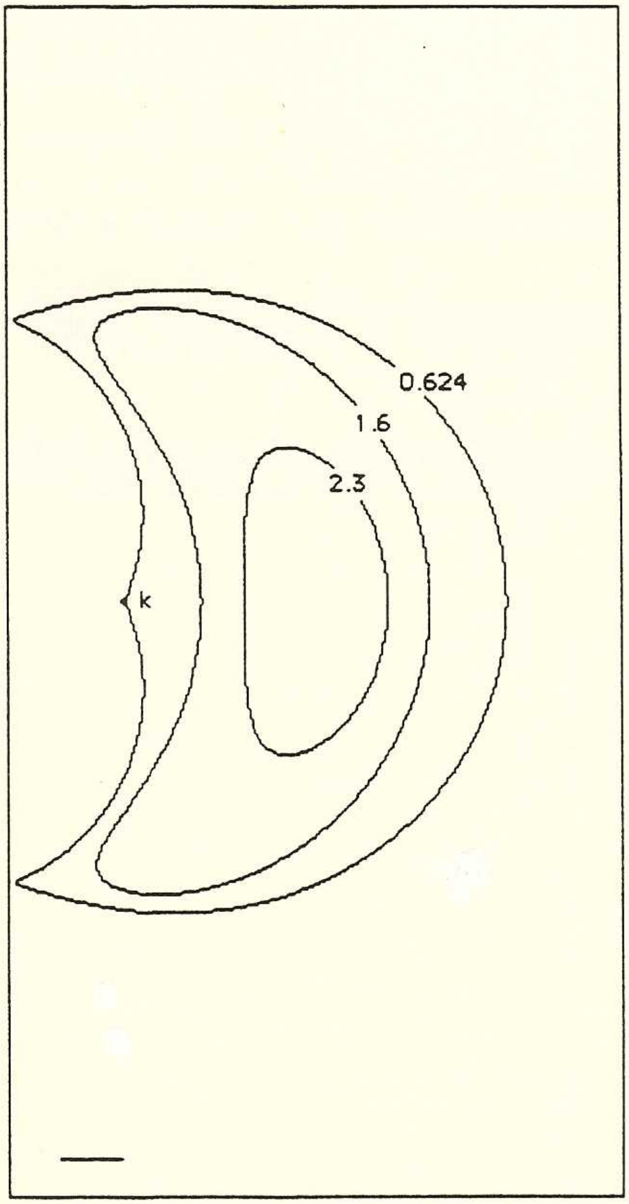


Fig. 8.1b. Market areas produced by a t.c.f. first convex, then concave : $h(\delta) = (\delta - 4.8)^{1/3}$, $\delta_{jk} = 3.6$, indicated values of Q.

The proofs of Prop. 8.1a and b rely upon Prop. 5.1. This one entails that any point of Z_j either is in the half-plane $\mathbb{R}_- \times \mathbb{R}$ or can be joined to it by an arc Γ of a circle centred on any point $p \in [jo]$, with $\Gamma \subset Z_j$. Any two points of Z_j may consequently be joined together by a path contained in Z_j, and item (a) is proved ; see Fig. 8.2a.

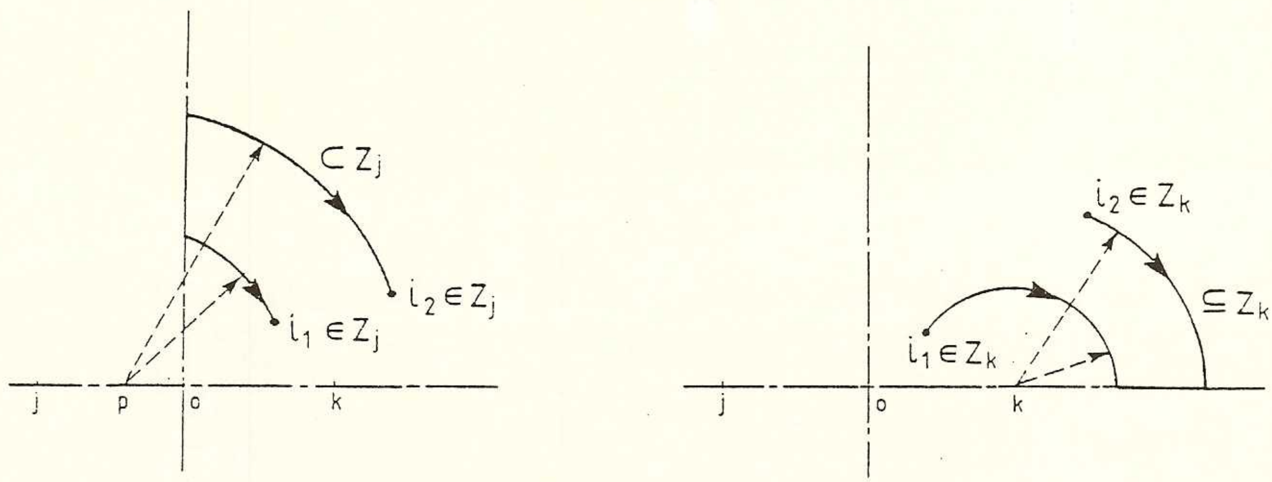
Let us now come to the other items, for which we give the proof concerning Z_k only. As to item (b) : of course, we mean by circular projection of a point i around k the intersection point of the circle centred on k and passing through i with the half-axis $[\delta_{ok}, +\infty[\times \{0\}$. Let us denote that point by $\gamma(i)$. The first step is to prove that, C being a unipartite component of Z_k, we have $\gamma <C> = C \cap [\delta_{ok}, +\infty[\times \{0\}$. The argument is straight as $\gamma <C> \subseteq C \cap [\delta_{ok}, +\infty[\times \{0\}$ (from Prop. 5.1) and obviously

$$C \cap [\delta_{ok}, +\infty[\times \{0\} = \gamma < C \cap [\delta_{ok}, +\infty[\times \{0\} > \subseteq \gamma <C> .$$

(Of course, we have also $Z_k \cap [\delta_{ok}, +\infty[\times \{0\} = \gamma <Z_k>$, and this will ease our notation).

The second step is to show that when a set C is a unipartite component of Z_k, then $\gamma <C>$ is a unipartite component of $\gamma <Z_k>$, ie. a maximal unipartite subset of $\gamma <Z_k>$. First, let us examine why $\gamma <C>$ is unipartite. Consider indeed two points i_1 and i_2 of $\gamma <C>$. According to Prop. 5.1, $\gamma <C> \subseteq C$; so i_1 and i_2 may be linked by a path $P \subseteq C$. But Prop. 5.1 again shows that $\gamma <P>$, which is obviously a path between i_1 and i_2, is contained in $\gamma <C>$. So $\gamma <C>$ is unipartite. Were it not maximal, $\gamma <C>$ would be strictly included in a broader unipartite subset of $[\delta_{ok}, +\infty[\times \{0\}$. We could thus find a point $i' \in \gamma <Z_k>$, not contained in $\gamma <C>$ but linkable to it by some path P' included in $\gamma <Z_k>$. Being outside $\gamma <C>$, i' would also be outside C, as $\gamma <C> = C \cap [\delta_{ok}, +\infty[\times \{0\}$; P'being included in Z_k, C would not be a maximal unipartite component of Z_k : a contradiction. (To be exhaustive we ought also to prove that if C' is a unipartite component of $\gamma < Z_k >$, then $C' = \gamma < C >$ for some unipartite component C of Z_k ; but this is rather obvious).

Item (c) directly derives from item (b) : as a subset of a straight half-line, $\gamma <Z_k>$ must have no more than one unipartite component. As we know from Prop. 6.2a the equivalence 'C bounded $\Leftrightarrow$ $\gamma <C>$ bounded', the proof is complete.



(a) Z_j *is arcwise connected*

(b) Z_k *is arcwise connected when h is concave or convex*

Fig. 8.2. Arcwise-connectedness.

Item (d) also is a mere application of item (b). We have seen (cf. the proof of Prop. 6.1 b and c) that Δh is a monotonic function of x on $[\delta_{ok}, +\infty[\times \{0\}$ when h is concave or convex. Consequently, if i_1 and i_2 belong to $\gamma <Z_k>$, we have for any $i \in [i_1\ i_2]$:

$$\Delta h \geqslant \min \{ \Delta h_{i_1}, \Delta h_{i_2} \} \geqslant Q$$

so that $[i_1\ i_2] \subseteq \gamma <Z_k>$. So $\gamma <Z_k>$ and Z_k are unipartite; see Fig. 8.2b.

8.2. *Simple connectedness*

A set E is *simply connected* iff. every closed path of E is homotopic to the constant path, i.e., is contractable to a point. In $\mathbb{R}^2$, that means the impossibility of drawing in E a path that would surround a point of $\mathbb{R}^2 - E$. Otherwise said, E has no hole, it has no Emmenthal-like structure. Some authors additionally require that E should be arcwise connected, but when the above definition is used E may independently be arcwise or simply connected. The two properties are related, however :

LEMMA 8.1 *If $\mathbb{R}^2 - E$ is arcwise connected and unbounded, E is simply connected.*

Indeed, were E not simply connected, it would be possible to draw in E a path around some $i \in \mathbb{R}^2 - E$. If $\mathbb{R}^2 - E$ is arcwise connected, it should consequently be bounded. Concerning market areas, we get the following properties :

(8.2) (a) *Z_k is simply connected ;*

(b) *Z_j [resp. $Z_j - Z_k$] is simply connected iff. $Z_k - Z_j$ [resp. Z_k] is unbounded and arcwise connected ;*

(c) *If h is convex, or if $Q < h'(+\infty)\,\delta_{jk}$ and h is concave, Z_j is simply connected.*

We give the proofs about the simple connectedness of Z_k and Z_j only ; that concerning $Z_j - Z_k$ is perfectly similar. Item (a) is a consequence of Lemma 8.1, as $Z_j - Z_k$ is arcwise connected (same argument as with Prop. 8.1a) and of course unbounded. The implication ($\Leftarrow$) of item (b) is identical to Lemma 8.1. As to ($\Rightarrow$) : suppose that $Z_k - Z_j$ is bounded or not unipartite. In the first case, it is possible to surround $Z_k - Z_j$ with a path included in Z_j , and Z_j is not simply connected. In the second one, $(Z_k - Z_j) \cap \mathbb{R}_+ \times \{0\}$ is not unipartite (Prop. 8.1b). So we can find on $\mathbb{R}_+ \times \{0\}$ two points i_1 and i_2 of $Z_k - Z_j$ separated by a point $i' \in Z_j$, with $x_{i_1} < x_{i_2}$. According to Prop. 5.1, the circle passing through i' and centred on any point p of [jo] is then contained in Z_j ; as it surrounds $i_1 \in Z_k - Z_j$, market area Z_j is not simply connected. Finally, item (c) is derived from Prop. 8.2b, 8.1d, and 6.2b.

Simple connectivity thus appears as another viewpoint on boundedness and arcwise-connectedness. It is easily seen, in particular, that Z_j is not simply connected when $h = .^a$ with $0 < a < 1$, provided that $Z_k - Z_j \neq \phi$, but that Z_j is simply connected when $a \geqslant 1$.

As to the distinction between Z_k and $Z_k - Z_j$ or between Z_j and $Z_j - Z_k$, notice that if a set E is simply connected, then its interior is also simply connected, as the suppression of the points of the boundary of E cannot make possible to draw new paths inside E. That is why we did not mention in item (a) that $Z_k - Z_j$ is simply connected. But $Z_j - Z_k$ may be simply connected without Z_j possessing the same property. If we lower the attractivity constant Q in the example of Fig.8.1a, the two unipartite components Z_k^1 and Z_k^2 will grow. When Q = 1.458, Z_k^1 and Z_k^2 will share a unique point lying on the x-axis ; so that there will be in fact one unipartite component $Z_k^1 \cup Z_k^2$ for Z_k, instead of two, $Z_k^1 - Z_j$ and $Z_k^2 - Z_j$, for $Z_k - Z_j$. In that case $Z_j - Z_k$ is simply connected while Z_j is not. But the distinction is of course more mathematical than geographical ...

8.3. *Spinedness*

Let us set this definition : a set E is *spined* by a straight line D iff. for every $i \in E$, the segment $[ii']$ is a subset of E, i' being the orthogonal projection of i on D. It then follows from Prop. 5.2 and 5.3 that :

(8.3) (a) *The part of Z_k lying at the left of k, ie. $Z_k \cap [0, \delta_{ok}] \times \mathbb{R}$, is spined by the straight line perpendicular to $[jk]$ and passing through k, ie. $\{\delta_{ok}\} \times \mathbb{R}$;*

(b) *If h is convex, Z_j is spined by the y-axis ;*

(c) *If $h \circ \sqrt{.}$ is concave, Z_k is spined by the x-axis ;*

(d) *If $h \circ \sqrt{.}$ is convex, Z_j is spined by the x-axis.*

The property of spinedness is not independent of the ones studied so far. In particular, it is easily seen that if a set $E \subseteq \mathbb{R}^2$ is spined by a straight line D, then E is simply connected and, if $\mathbb{R}^2 - E$ is moreover contained in one of the two half-planes delimited by D, $\mathbb{R}^2 - E$ is also simply connected.

According to that it is possible to give, for instance, a new proof of the simple connectedness of Z_j and Z_k when function h is convex. Noticing that Prop. 8.3 may be extended to the interior of the concerned areas, and using Prop. 8.3b, we can indeed build those two logical chains :

$$\text{h convex} \Rightarrow Z_j \text{ spined} \Rightarrow Z_j \text{ simply connected ;}$$

$$\left.\begin{array}{l} \text{h convex} \Rightarrow Z_j - Z_k \text{ spined by the y-axis} \\ \qquad Z_k \subseteq \mathbb{R}_+ \times \mathbb{R} \end{array}\right\} \Rightarrow Z_k \text{ simply connected.}$$

The property is also related to arcwise-connectedness : when h is convex and $h \circ \sqrt{.}$ is convex or concave, (as is the case if $h = .^a$ with $a \geqslant 1$) Prop. 8.3b, c, and d entail that any two points i_1 and i_2 of Z_k (resp. Z_j) may be linked by a Manhattan-like path contained in Z_k (resp. Z_j).

Last, Prop. 8.3b also gives a straightforward proof that when h is convex, $Z_k \neq \phi$ iff. Z_k is unbounded (Prop. 6.1c).

In the case of the monomial transportation cost function $.^a$, Prop. 8.3 entails that Z_k is spined by the x-axis when $a \in]\ 0,\ 2\ ]$; and that Z_j is spined by the y-axis when $a \geqslant 1$, and by the x-axis when $a \geqslant 2$. On the contrary, in the two examples of Fig. 8.1 neither Z_k nor Z_j are spined by the axes of x or y ; item (a) is the only one that can be used (if we do not consider separately the properties of the unipartite components of Z_k, as far as Fig. 8.1a is concerned).

Fig.8.3 shows possible shapes of market areas when function h is made of linear pieces . In Fig. 8.3a, h is concave ; in Fig. 8.3b, it is convex. Though surprising, those shapes are coherent with Prop. 8.3. In the first case h is concave but not convex : no wonder if Z_k is not spined by the y-axis. In the second case, neither Z_j nor Z_k are spined by the x-axis ; and indeed $h \circ \sqrt{.}$ is neither convex nor concave.

8.4. *Starshapedness and visibility*

A set E is *starshaped* wrt. a point p iff., for any $i \in E$, the segment [ip] is contained in E. It is starshaped wrt. to a set S iff. it is starshaped wrt. to every $p \in S$.

(8.4) (a) *If function h is concave, Z_k is starshaped wrt. k ;*

(b) *If function h is convex, Z_j is starshaped wrt. $\mathbb{R}_- \times \{0\}$, the nonpositive part of the x-axis.*

The proof is straightforward. Item (a) is just another formulation of Prop.5.2a. Similarly, Prop. 5.2b yields that Z_j is starshaped wrt. o when h is convex, and item b is obtained by combining this with Prop. 8.3b.

As was the case for spinedness, starshapedness pours a new light on preceding properties. It is immediately seen that if a set E is starshaped wrt.

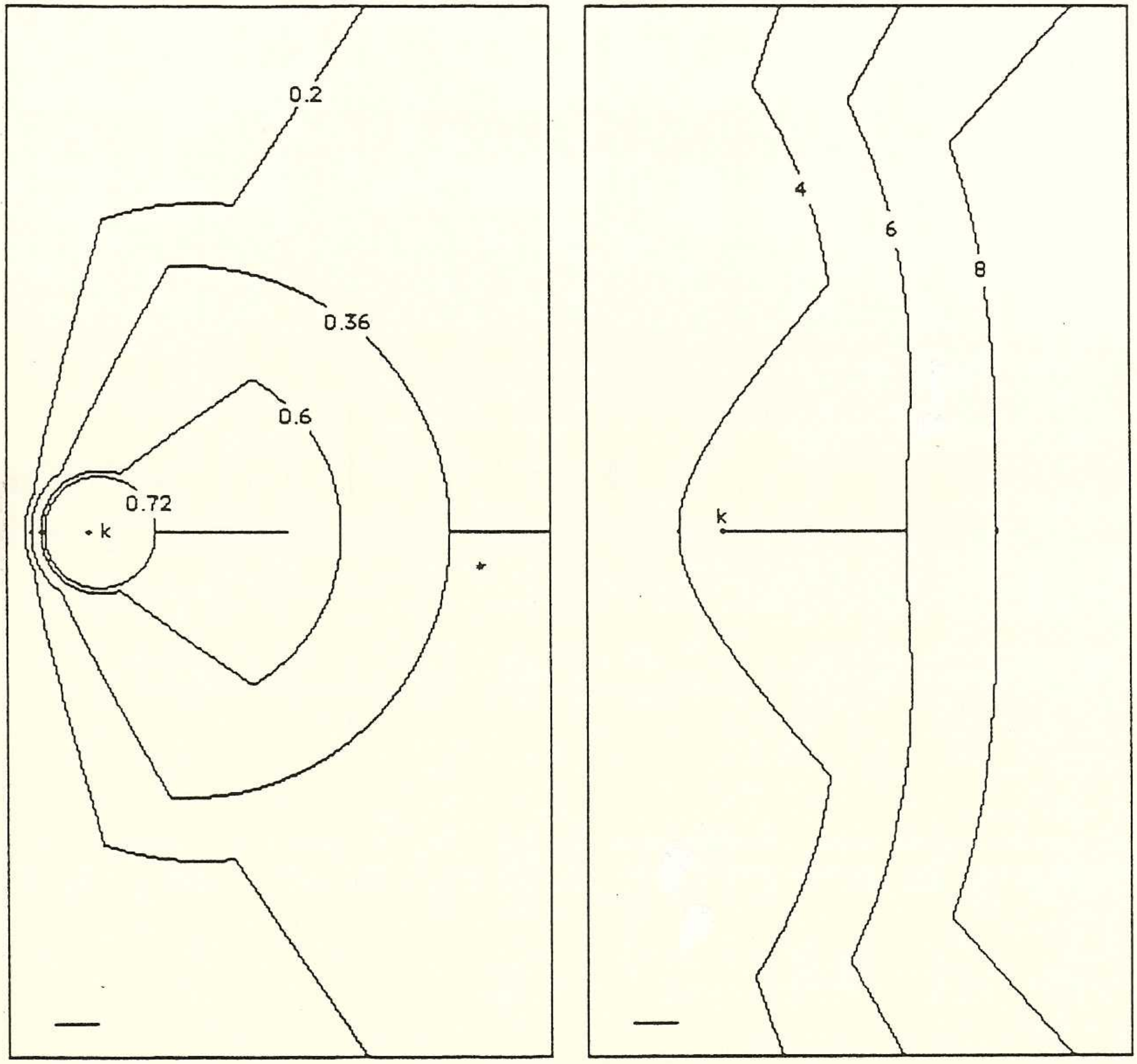


Fig. 8.3a. $h(\delta) = \min\{\delta, 0.2\,\delta + 1.2, 0.1\,\delta + 2\}$, $\delta_{jk} = 3.6$, indicated values of Q. Notice the infinite 1-dimensional tail of Z_k when $Q = 0.36$ and the finite one when $Q = 0.72$.

Fig. 8.3b. $h(\delta) = \max\{2\,\delta, \delta + 10\}$, $\delta_{jk} = 6$, indicated values of Q. Notice that when $Q = 6$, Z_k has a one-dimensional tail ending at k.

a point p, E is arcwise connected, as $\{i_1, i_2\} \subseteq E \Rightarrow [i_1\, p] \cup [p\, i_2] \subseteq E$. The set E is also simply connected : otherwise some point $i \in \mathbb{R}^2 - E$ would be surrounded by a closed path $\Gamma \subseteq E$; calling i' an intersection point of Γ with the straight half-line originating in i and the prolongation of which passes through p, we should have $[pi'] \subseteq E$ and hence $i \in E$, a contradiction. If moreover E is unbounded, $\mathbb{R}^2 - E$ is simply connected because of Lemma 8.1. The propositions 8.1d (when function h is concave) and 8.2c may consequently be considered as deriving from Prop. 8.4. The Propositions 6.1b and 6.1c about the possible emptiness of market area Z_k are also corollaries of Prop. 8.4.

Prop. 8.4 can be illustrated by most of the examples given so far. See in particular Fig. 5.3, 8.1 and 8.3.

A concept very close to starshapedness is that of visibility, introduced by Goldman (1963) in the field of convex programming and by Hurter et al. (1975) in location theory. Adapting slightly their definitions, we say that a set E is *entirely visible* from a point p iff., for all $i \in E$, no other point of E is on the segment $]ip]$, i.e., $]ip] \cap E = \phi$. It is entirely visible from a set S iff. it is entirely visible from every $p \in S$. So we have concerning the border $Z_j \cap Z_k$ a proposition parallel to Prop. 8.4 :

8.5 (a) *If h is strictly concave [resp. strictly convex] , $Z_j \cap Z_k$ is entirely visible from k [resp. $\mathbb{R}_- \times \{0\}$] ;*

(b) *If h is concave [resp. convex] , $(Z_j \cap Z_k) -]\delta_{ok}, +\infty[\times \{0\}$ is entirely visible from k [resp. $\mathbb{R}_- \times \{0\}$].*

The restriction expressed in item (b) corresponds to the possibility for $Z_j \cap Z_k$ to have a nondenumerable intersection with the straight half-line $]\delta_{ok}, +\infty[\times \{0\}$ (see eg. Prop. 6.2d).

A remarkable case occurs when the t.c.f. is linear : $h(\delta) = \delta$. That function being concave and convex, both items of Prop. 8.4 (or 8.5) can be applied. As a consequence, *when $h(\delta) = \delta$, each market area is starshaped wrt. its centre.* That property obtains whatever the number of centres ; see Fig. C1A.

8.5. *Convexity*

This mathematical concept corresponds to the intuitive idea of a round shape. A set E is *convex* iff., for any two points i_1 and i_2 of E, the segment $[i_1 i_2]$ is in E. The set E is *strictly convex* iff. $\forall i_1, i_2 \in E$, the open segment $]i_1 i_2[$ is in the interior of E, i.e., E minus its boundary. In other words, E is convex iff. it is starshaped wrt. itself. A convex set E is thus arcwise and simply connected ; it is also spined by any straight line containing a diameter of E, a diameter being understood here as a segment $[i'i'']$ for which i' and i" belong to E and maximize the distance $\delta_{i'i''}$ on E. Convexity is thus a strong property. It is then perhaps not surprising that our results in this matter lack the polarization observed in the preceding statements between the concavity and convexity of h or of some derived function ; the items (c) and (d) of Prop. 8.6 even concern only particular classes of transportation cost functions.

8.6 (a) *Except when* $Z_j = \mathbb{R}^2$, Z_j *is convex iff.* $Z_j \cap Z_k$ *is a straight line parallel to the y-axis ;* Z_j *is never strictly convex when* $Z_j \neq \mathbb{R}^2$;

(b) *If the function* $\sqrt{.}/(h' \circ \sqrt{.})$ *is convex and strictly increasing,* Z_k *is strictly convex ;*

(c) *When* $h = .^a$, *(i) if* $0 < a \leq 1$, Z_k *is strictly convex ;*

(ii) if $a = 2$, Z_k *is convex, but not strictly ;*

(iii) if $1 < a < 2$ *or* $a > 2$, Z_k *is not convex.*

(d) *When* $h = \ln(K + .^a)$ *with* $a = 1$ *or* 2, Z_k *is strictly convex.*

Item (a) is obvious as $\mathbb{R}_- \times \mathbb{R} \subset Z_j$. Item (b) is a direct consequence of Prop. 6.4b; or of formula (36) as we have here $\Delta g' \geq 0$ and $\Delta h' < 0$. Item (c) is proved by using (36) to show that when $h = .^a$ with $0 < a < 1$, $(\partial^2 y/\partial x^2)_{\Delta h}$ is < 0 on $\mathbb{R}_+^{*2}$ and finite unless $y = 0$. As those computations are heavy, the reader is referred to Appendix 1. When $a = 1$, $Z_j \cap Z_k$ is a branch of a hyperbola, and Z_k is consequently strictly convex. When $a > 1$, we know from (21) that $\varphi_\infty = \pi/2$: exactly as in item (a) regarding Z_j, this implies, if Z_k is convex and not empty, that $Z_j \cap Z_k$ is vertical. As this is true only if $a = 2$, and as $Z_k \neq \phi$ if $a > 1$ (because $h'(+\infty) = +\infty$: see

Prop. 6.1c and 6.2b), the proof of this item is complete. Item (d) derives from item (b) when a = 2, and from Prop. 13.7a (see Section 2.2).

Still about items (c,i) and (d), two facts are worth noticing. The first one is that the function $\sqrt{.}$ / $(h' \circ \sqrt{.})$ is here concave and strictly increasing : item (b) cannot be used to prove item (c); and indeed, as shown in Fig. 5.3 and 13.3b, Z_k is characterized by $L_y/L_x \leqslant 1$ in accordance with Corollary 6.1a. The second fact is that the convexity of Z_k does not here derive from the possible concavity of Δh seen as a function of (x,y) : Prop. 8.6c entails that Δh is quasi-concave wrt. (x,y); it is not concave, however. To show this it is enough considering the evolution of Δh along the positive part of the x-axis, for instance : Δh is convex and increasing wrt. x on [ok], then convex and decreasing on $[\, \delta_{ok}, +\infty\, [\times \{0\}$.

We already know that item (a) is illustrated by $h = .^2$; and we have seen how Prop. 4.1 can then be used to find all the transportation cost functions for which $Z_j \cap Z_k$ is vertical. Item (b) applies for instance to the unbounded unipartite component Z_k^1 of market area Z_k in Fig. 8.1a, as it can be computed that the required condition is verified inside the range of distances to j and k that concerns Z_k^1; we have thus also here $L_y/L_x > 1$. Item (b) also applies when $h = -.^{-a}$ with $a > 0$, and when $h = \ell n\,(K + \delta^2)$ with $K > 0$.

Notice, conversely, that we have met situations where function h is concave but where Z_k is obviously not convex : see Fig. 8.3a and 6.4. In the first case, the nonconvexity of Z_k is clearly related with the nonderivability of h at some given distance. In the second, however, function h is perfectly smooth. On the other hand, when h is partially convex and concave, Z_k may be convex, as indicated by Prop. 6.2d (Fig. 4.3b), or not ; see Fig. 4.1 and 8.1. All this shows that the convexity of Z_k is absolutely independent of the concavity of function h.

9. *Properties of the measures of Z_k and of the extra territory of centre j*

We deal here with *quantitative properties* of market areas. We have already seen in Section 4 that the measure $|Z_k|$ is a decreasing function of the attractivity constant Q. The mathematical developments of Section 5.2 make it easy to refine our knowledge, and we so settle that $|Z_k|$ is most of the time *convex* wrt. Q. The question was asked to us in relation with a paper of Jaskold-Gabscewicz and Thisse (1985) about the theory of the equilibrium. in location and price of non-cooperative firms. It has indeed an economic significance when Q is simply the difference $p_k - p_j$ between the f.o.b. prices proposed at the two centres, if a continuum of identical and inelastic-demand customers is spread over the plane; $|Z_k|$ is then proportional to the total demand addressed to centre k.

When the transportation cost function h is convex, $|Z_k|$ is very simply related to Q. If $Q \geq \delta_{jk}\ h'(+\infty)$, $|Z_k| = 0$ as Z_k is then either empty or reduced to a straight half-line (Prop. 6.1c, 6.2b, 6.2d). If $Q < \delta_{jk}\ h'(+\infty)$, the angle φ_∞ is > 0 according to (21), so that $|Z_k| = +\infty$. But the method used here above to study $|Z_k|$ may now also be applied to $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$, which can be finite when the axis of y is the vertical asymptote of the demarcation line $Z_j \cap Z_k$. The area $Z_j \cap \mathbb{R}_+ \times \mathbb{R}$ has an obvious interpretation : it is the *extra territory* gained by centre j on centre k due to the difference between their attractivities. We find here a result remarkably symmetrical of the preceding one : we establish a sufficient and seemingly not very restrictive condition for $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$, i.e., $|\mathbb{R}_+ \times \mathbb{R} - Z_k|$, to be a *concave* function of Q.

We have said that $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$ may be finite : the proof and conditions are given in section 9.1. We also treat there the similar problem that exists for $|Z_k|$ when h is concave and Q takes the threshold value $\delta_{jk}\ h'(+\infty)$. If the x-axis is an asymptote of $Z_j \cap Z_k$ as may then happen, we shall see -surprisingly enough- that $|Z_k|$ may still be finite despite its endless tail, before suddenly jumping to infinity when Q becomes less than $\delta_{jk}\ h'(+\infty)$; see Fig. 9.1.

It could also seem interesting to possess more information than Prop. 4.5 about the dependence of those areas on the distance δ_{jk} between the centres. When $h = \ell n$ or $\ell n\,(.^2 + K)$ with $K > 0$, $|Z_k|$ is obviously convex in δ_{jk} ; see Prop. 4.7e. But what about other cases ?
An approximate answer will be given in Section 11 when the transportation cost function is $.^a$.

9.1. *Finiteness*

Let us begin with $Z_j \cap \mathbb{R}_+ \times \mathbb{R}$. We have already mentioned that the mean-value theorem allows to write, if $i \in Z_j \cap Z_k$, that $\exists \delta \in]\delta_k, \delta_j[$ for which $\hbar'(\delta^2)(\delta_j^2 - \delta_k^2) = Q$; i.e., given (11),

$$\hbar'(\delta^2)\ 2 \times \delta_{jk} = Q \,. \qquad (37)$$

Consequently, if $\hbar'$ is invertible ,i.e., if $\hbar$ is strictly concave or strictly convex, we deduce from (10) the inequalities

$$\hbar'^{\wedge}(Q/2x\delta_{jk}) - (x+\delta_{ok})^2 < y^2 < \hbar'^{\wedge}(Q/2\,x\delta_{jk}) - (x-\delta_{ok})^2 \,. \qquad (38)$$

In particular, when $\hbar$ is strictly convex, we know from Prop. 7.1 that y is a decreasing function of x above the x-axis; $Z_j \cap Z_k$ must then have a vertical asymptote $x = x_\infty$. If we also require that $h''(+\infty) = +\infty$, then $x_\infty = 0$ (see Section 6.7) and the measure of $Z_j \cap \mathbb{R}_+ \times \mathbb{R}$, the part of Z_j at the right of the y-axis, may be computed, if $(x,y) \in Z_j \cap Z_k \cap \mathbb{R}^2_+$, as

$$|Z_j \cap \mathbb{R}_+ \times \mathbb{R}| = 2\int_0^{x_\ell} y\, dx \,. \qquad (39)$$

As the inequalities (38) imply that

$$\sqrt{\hbar'^{\wedge}(Q/2x\ \delta_{jk})} - x - \delta_{ok} \leqslant y \leqslant \sqrt{\hbar'^{\wedge}(Q/2x\delta_{jk})},$$

$|Z_j \cap \mathbb{R}_+ \times \mathbb{R}| < +\infty$ iff. $[\hbar'^{\wedge}(Q/2x\delta_{jk})]^{1/2}$ is integrable between 0 and x_ℓ. To deal with that last issue we use Cauchy's criterion. Obviously that integrability obtains if one has for some $K > 0$ and some $\varepsilon' > 0$, for every x between 0 and some $\bar{x}$:

$$\sqrt{\hbar'^{\wedge}(Q/2x\delta_{jk})} \leqslant K\ x^{-1+\varepsilon'}.$$

Setting $\delta = Kx^{-1+\varepsilon'}$ and $\varepsilon = \varepsilon'/(1-\varepsilon')$, and remembering that $\hbar'(\delta^2) = h'(\delta)/2\delta$, one easily finds that the inequality is equivalent to

$$\frac{\hbar'(\delta)}{\delta^{2+\varepsilon}} \geqslant \frac{Q}{\delta_{jk}\, K^{1+\varepsilon}} \,.$$

As the values of K, ε', and $\bar{x}$ are left free, this is the same as requiring that, for some $\varepsilon > 0$,

$$\lim_{\delta \to +\infty} \frac{h'(\delta)}{\delta^{2+\varepsilon}} > 0.$$

Similarly, a sufficient condition for the integral to be infinite is that $[h' \hat{} (Q/2x\delta_{jk})]^{1/2} \geqslant K\, x^{-1}$. Developing that inequality and applying de l'Hospital's rule to both results, and noticing that we only need in fact the integrability of y near the y-axis so that Prop. 6.10e dispenses us with requiring the convexity of $h \circ \sqrt{.}$, we finally get

(9.1) *When* $h''(+\infty) = +\infty$,

(a) *if* $\lim_{\delta \to +\infty} [h(\delta)/\delta^{3+\varepsilon}] > 0$ *for some* $\varepsilon > 0$, $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}| < +\infty$;

(b) *if* $\lim_{\delta \to +\infty} [h(\delta)/\delta^3] < +\infty$, $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}| = +\infty$.

In the particular case of the transportation cost function $.^a$, that proposition indicates that $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}| < +\infty$ iff. $a > 3$ (see Fig. 5.3).

Let us come now to the measure $|Z_k|$, when the y-axis is the asymptote of $Z_j \cap Z_k$. We assume that h is concave; $h \circ \sqrt{.}$ is thus strictly concave (as $h' > 0$), so that y is a function of x if $i \in Z_j \cap Z_k \cap \mathbb{R}_+^2$ (see Prop. 5.5c) and we may write

$$|Z_k| = 2 \int_{x_\ell}^{+\infty} y \, dx \ .$$

It then appears from the expression (28) of y^2 that $|Z_k| < +\infty$ iff. $\delta_k (1 - \Delta\delta/\delta_{jk})^{1/2}$ may be integrated on $[x_\ell, +\infty[$, as $\Delta\delta$ tends in the present case towards δ_{jk} when $\delta_k \to +\infty$. Using again Cauchy's criterion, we see that $|Z_k| < +\infty$ if

$$\delta_k \sqrt{1 - \frac{\Delta\delta}{\delta_{jk}}} \leqslant K\, x^{-1-\varepsilon'}$$

beyond some value $\bar{x}$ of x and for some strictly positive K and ε'. If we now set $f(1/\delta^{4+2\varepsilon'}) = 1/h'(\delta)$, we derive, similarly to (26) and (29), that for some $\delta \in]\delta_k, \delta_j[$ and some $\tilde{\delta} > \delta$

$$1 - \frac{\Delta\delta}{\delta_{jk}} = - \frac{\tilde{\delta}^{5+2\varepsilon'}\, h''(\tilde{\delta})\, h'(+\infty)}{(4+2\varepsilon')\ \delta^{4+2\varepsilon'} h'^2(\tilde{\delta})} .$$

Cauchy's condition may consequently be written as

$$\frac{\delta_k^2\, \tilde{\delta}^{5+2\varepsilon'} x^{2+2\varepsilon'} h''(\tilde{\delta})}{\delta^{4+2\varepsilon'}} \geqslant - \frac{K^2\ (4+2\varepsilon')\, h'^2(\tilde{\delta})}{h'(+\infty)} .$$

When $\delta_k \to +\infty$, $\delta_k/\delta \to 1$; moreover here, as $y \to 0$, $x/\delta \to 1$. As the values of K and $\bar{x}$ are left free, the condition is reduced as follows; Cauchy's non-integrability condition is similarly treated.

(9.2) *When* $Q = \delta_{jk}\, h'(+\infty)$,

(a) *if* $\lim\limits_{\delta \to +\infty} [\delta^{5+\varepsilon} h''(\delta)] > -\infty$ *for some* $\varepsilon > 0$, $|Z_k| < +\infty$;

(b) *if* $\lim\limits_{\delta \to +\infty} [\delta^5 h''(\delta)] < 0$, $|Z_k| = +\infty$.

In item (a), we have taken into account the fact that the mentioned condition entails that when $\delta \to +\infty$, $\delta^3 h''(\delta) \to 0$ [see (30) and Prop. 6.12].

If we apply this to transportation costs of the form $\delta - b\delta^{-a}$ $(a, b > 0)$ when $Q = \delta_{jk}$, we find that $|Z_k|$ is finite iff. $a > 3$. The exponent 3 plays thus here the same threshold role as in Prop. 9.1. As $|Z_k|$ is a continuous function of Q as long as that measure remains finite (because the hypothesis of the strict monotonicity of function h implies that h is not constant on any disk with strictly positive radius; see Section 12.3), it appears that when $a > 3$, nothing in the behaviour of $|Z_k|$, if Q decreases towards δ_{jk}, lets us foresee the sudden explosion of $|Z_k|$ to infinity when Q goes past δ_{jk} ; see Fig. 6.4. and 9.1.

9.2 *Dependence on attractivity*

We already know from Prop. 4.5 that the measure $|Z_k|$ must be a strictly decreasing function of the attractivity constant Q. But we can go further. According to Prop. 8.4a, when h is concave, Z_k is starshaped wrt. centre k; the measure $|Z_k|$ may thus be computed as

$$|Z_k| = \int_0^{\pi} \bar{\delta}_k^{\,2}(\varphi_k, Q) \cdot d\varphi_k$$

where $\bar{\delta}_k$ is the distance between k and the point of Z_k most remote from k in the direction φ_k; this point belongs to $Z_j \cap Z_k$.

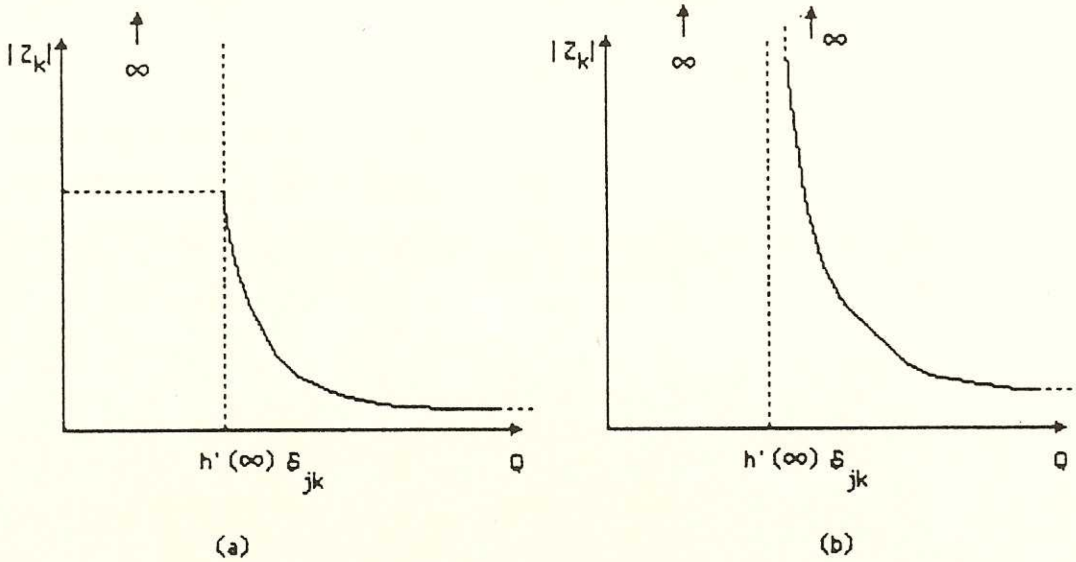


Fig. 9.1. The superficies $|Z_k|$ is decreasing and often convex wrt. Q;
it may be finite or not when $Q = h'(\infty)\,\delta_{jk}$.

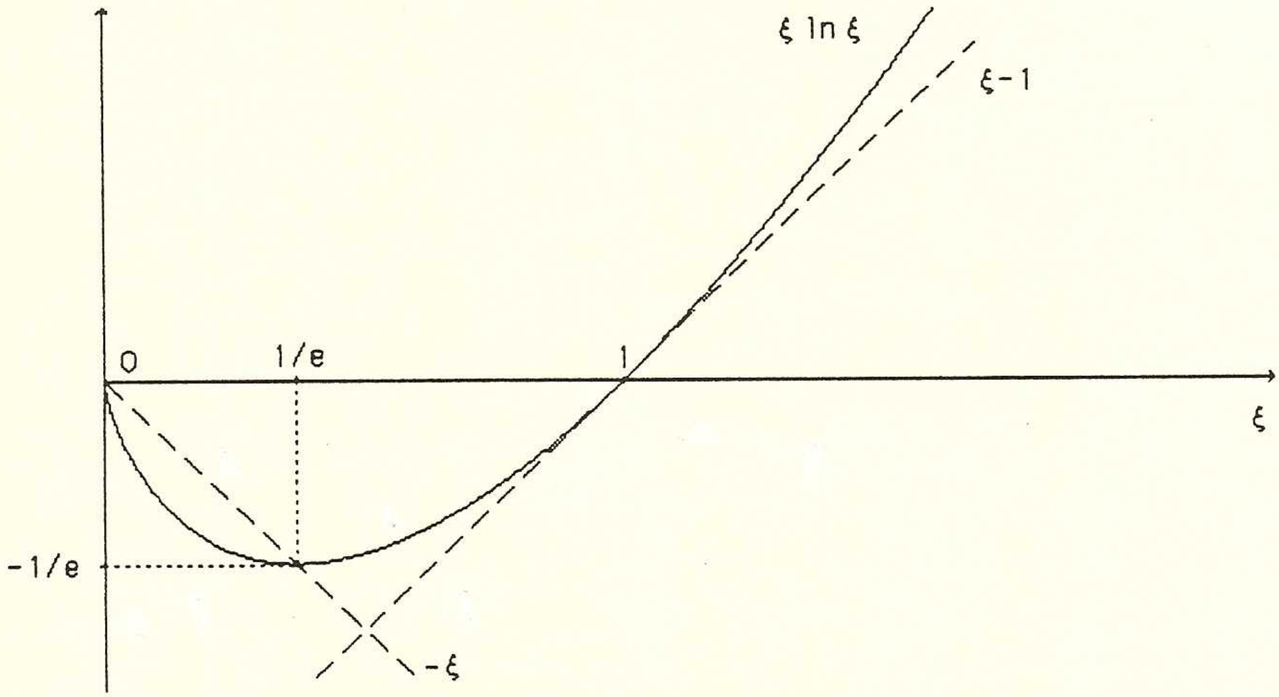


Fig. 10.1. The function $\xi \ln \xi$ and the other functions used to establish the relation between Σh and the dependence of Δh on the distance exponent a when $h = \delta^a$.

The distance $\bar{\delta}_k$ is of course strictly decreasing wrt. Q : this is shown by Prop. 4.5. Consequently, when the angle φ_k is fixed, $\bar{\delta}_k$ is convex wrt. Q iff. Δh is convex wrt. δ_k. The concavity of $\bar{\delta}_k$ wrt. Q can be studied in the same way. The formulas of $(\partial\Delta h/\partial\delta_k)_{\varphi_k}$ and $(\partial\delta_j/\partial\delta_k)_{\varphi_k}$ established in Section 5.2 yield :

$$\left(\frac{\partial^2\Delta h}{\partial\delta_k^2}\right)_{\varphi_k} = h_j''\left(\frac{\partial\delta_j}{\partial\delta_k}\right)^2_{\varphi_k} + h_j'\left(\frac{\partial^2\delta_j}{\partial\delta_k^2}\right)_{\varphi_k} - h_k''$$

$$\left(\frac{\partial^2\delta_j}{\partial\delta_k^2}\right)_{\varphi_k} = \frac{\delta_j - (\delta_k + \delta_{jk}\cos\varphi_k)\cos\psi}{\delta_j^2} = \frac{\sin^2\psi}{\delta_j} .$$

Hence,

$$\left(\frac{\partial^2\Delta h}{\partial\delta_k^2}\right)_{\varphi_k} = h_j''\cos^2\psi - h_k'' + \frac{h'_j}{\delta_j}\sin^2\psi$$

$$\geqslant h_j'' - h_k'' + \frac{h'_j}{\delta_j}\sin^2\psi \qquad \text{as h is concave.}$$

It is then easy to find a condition ensuring the convexity wrt. Q of $\bar{\delta}_k$, and consequently the strict convexity wrt. Q of $\bar{\delta}_k^{\,2}$ and $|Z_k|$:

(9.3) *If h is concave and h' is convex , $|Z_k|$ is strictly decreasing and strictly convex wrt. Q when $|Z_k|$ is not 0 or $+\infty$.*

The two conditions are verified for many functions among which δ^a and $-\delta^{-a}$ (with $a > 0$), $\ln\delta$, atg δ, etc. It should be noticed that h' cannot be concave when h is concave and increasing : which explains why h' is then often convex. It is also convex on $[0,\delta^*]$ if $h(\delta) = \text{sign}(\delta-\delta^*)\,|\delta-\delta^*|^a$ with $a>1$, as in Fig. 8.1a, so that the property even holds for the bounded component of Z_k in that example. On the contrary, Prop. 9.3 does not apply if function h is concave and piecewise linear, as in the example of Fig. 8.3a.

When function h is convex, Z_k is now starshaped wrt. O, and we have

$$|Z_j \cap \mathbb{R}_+ \times \mathbb{R}| = \int_0^{\pi/2} \bar{\delta}_o^2(\varphi_o, Q)\, d\varphi_o$$

where $\bar{\delta}_o$ is defined as the maximal distance between o and a point of Z_j in the direction φ_o. According to Prop. 4.5 , that distance is strictly increasing wrt. Q; so that when the angle φ_o is fixed, $\bar{\delta}_o^2$ is concave (resp. convex) wrt. Q iff. Δh is convex (resp. concave) wrt. δ_o^2. Computing the first two derivatives of Δh wrt. δ_o^2 by means of the formulae of Section 5.2, we find :

$$\left(\frac{\partial \Delta h}{\partial \delta_o^2}\right)_{\varphi_o} = \hbar_j' \left(1 + \frac{\delta_{ok} \cos \varphi_o}{\delta_o}\right) - \hbar_k' \left(1 - \frac{\delta_{ok} \cos \varphi_o}{\delta_o}\right)$$

$$\left(\frac{\partial^2 \Delta h}{\partial (\delta_o^2)^2}\right)_{\varphi_o} = \hbar_j'' \left(1 + \frac{\delta_{ok} \cos \varphi_o}{\delta_o}\right)^2 - \hbar_k'' \left(1 - \frac{\delta_{ok} \cos \varphi_o}{\delta_o}\right)^2$$

$$- \frac{1}{2} \hbar_j' \frac{\delta_{ok} \cos \varphi_o}{\delta_o^3} - \frac{1}{2} \hbar_k' \frac{\delta_{ok} \cos \varphi_o}{\delta_o^3}$$

$$= (\hbar_j'' - \hbar_k'') \left(1 + \frac{\delta_{ok}^2 \cos^2 \varphi_o}{\delta_o^2}\right)$$

$$+ \frac{\delta_{ok} \cos \varphi_o}{2\delta_o^3} (4 \cdot \hbar_j'' \delta_o^2 - \hbar_j' + 4 \hbar_k'' \delta_o^2 - \hbar_k').$$

Let us now consider the disk C (j,ℓ). On the circle of that disk, the ratio δ_o / δ_j is minimal at the point which minimizes δ_o ; i.e., at point ℓ. The Apollonian circles (relative to the points j and o) corresponding to values of δ_o/δ_j lower than $\delta_{o\ell}/\delta_{j\ell}$ are thus contained in $C(j,\ell)$. Since $C(j,\ell) \subset Z_j$ (Prop. 6.7c), this implies that, on $Z_j \cap Z_k$, the ratio δ_o/δ_j is also minimal at ℓ. Hence, if we denote by ε the ratio $\delta_{o\ell}/\delta_{ok}$ and if we assume that $\hbar$ is convex [so that $\hbar''(\delta) \geq 0$],

$$4 \hbar''_j \delta_o^2 - \hbar'_j \geq 4 \hbar''_j \delta_j^2 \, \varepsilon^2/(\varepsilon + 1)^2 - \hbar'_j.$$

As $\delta_k < \delta_j$ on $Z_j \cap Z_k$, we have there $\delta_o/\delta_k > \delta_o/\delta_j \geq \varepsilon/(\varepsilon + 1)$, so that the inequality here above holds too if we replace 'j' by 'k'.

Noticing now that

$$(h \circ \sqrt[a]{\cdot}\,)''(\delta^a) = [\, \hbar \circ (.)^{2/a}]''(\delta^a) = \frac{\hbar''(\delta^2)\delta^2 - (a/2-1)\hbar'(\delta^2)}{a^2 \, \delta^{2a-2}} ,$$

we see that $4\,\hbar''_j\,\delta_o^{\,2} - \hbar'_j$ and $4\,\hbar''_k\,\delta_o^{\,2} - \hbar'_k$ are $\geqslant 0$ if the function $h \circ {}^a\sqrt{.}$ is convex, where the index a is given by the equality $a/2 - 1 = (\varepsilon + 1)^2/\,4\varepsilon^2$. Interestingly, that convexity also implies the convexity of $\hbar$ if $a \geqslant 2$. It is then easy to state what follows :

(9.4) *If the transportation cost function h is twice differentiable and if, for some $a > 2.5$, the functions $(h \circ \sqrt{.})'$ and $h \circ {}^a\sqrt{.}$ are convex on $\mathbb{R}_+$ then the measure $|\,Z_j \cap \mathbb{R}_+ \times \mathbb{R}\,|$ of the extra territory of centre j, when $\neq \infty$, is strictly concave wrt. Q on the range*

$$[h(\delta_{ok}\ (1 + \varepsilon)) - h\ (\delta_{ok}|1 - \varepsilon|)),\ \infty\ [,$$

where

$$\varepsilon = \frac{1}{\sqrt{2(a-2)}-1}\ .$$

One point does not immediately appear from the discussion above : the reason why the concavity is *strict*. Suppose indeed that it is not ; i.e., that $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}\,|$ is linear wrt. Q on some range of Q. Under the conditions of Prop. 9.4, the derivative $[\partial^2\Delta h/\partial(\delta_o^{\,2})^2]_{\varphi_o}$ is $\geqslant 0$; on that range of Q, it must thus be equivalent to zero. The proof of Prop. 9.4 shows that $\hbar''$ must then be equal to zero (instead of being only $\geqslant 0$) for all δ^2 on the corresponding range of $\delta_k^{\,2}$; otherwise we should have at least the strict inequality $4\,\hbar''_k\delta_o^{\,2} - \hbar'_k > 0$ and the above - mentioned derivative would be > 0. But it is easily seen that $\hbar''$ is equivalent to zero if and only if $h\,(\delta) = \delta^2$ (remember that constants and coefficients may be dropped in the study of function h). That case being forbidden by Prop. 9.1b, the proof is complete.

As could be expected, ε, as given by Prop. 9.4, is a decreasing function of a : the higher the value of a, the wider the range of Q on which we are sure that the measure of the extra territory of j is concave wrt.Q. It is also worth noticing that, under the conditions of Prop. 9.4, $\delta_o^{\,2}$ and consequently δ_o are strictly concave functions of Q when the angle φ_o is fixed.

When $h = .^{a'}$, the conditions on $(h \circ \sqrt{.}\,)'$ and $h \circ {}^a\sqrt{.}$ boil down to $a \leqslant a' \geqslant 4$; this value 4 of the exponent of the t.c.f. is remarkably close to the threshold value 3 indicated by Prop. 9.1. The first of the two remarks just made showing that the index a is best chosen equal to a', Prop. 9.4 thus sounds in this case : *When $h = .^a$ with $a \geqslant 4$, the measure of the extra territory of centre j is strictly concave wrt. Q on the range ... etc.* On the other hand, when $a = 4 \leqslant a'$, Prop. 9.4 yields $\varepsilon = 1$; this lets appear, as a corollary, that, *when $h = .^a$ with $a \geqslant 4$, the measure of the extra territory of j is strictly concave wrt. Q if Q is large enough for k to belong to Z_j.*

Not surprisingly, more accurate statements may still be found, at least in some particular cases. When $h(\delta) = \delta^4$, it appears that

$$\left(\frac{\partial^2 \Delta h}{\partial (\delta_o{}^2)^2}\right)_{\varphi_o} = \frac{2(3\delta_o{}^2 - \delta_{ok}{}^2)\,\delta_{ok} \cos \varphi_o}{\delta_o{}^3} .$$

Consequently, $1/\sqrt{3}$ is here a better value of ε than the value 1 proposed by Prop. 9.4.

The present section 9.2 is extended in the sections 11.1 and particularly 11.2 through some approximate properties.

10. *Dependence of the market areas on the exponent a when* $h(\delta) = \delta^a$

This issue is particularly interesting because it makes us feel how delicate the problems involving distance units can be. We shall meet again such a difficulty in section 11.1, where it will appear that the limit of the area Z_k when $h(\delta) = \delta^a$ and the ratio Q/a is kept constant while $a \underset{>}{\to} 0$, depends on the distance unit. We shall now see that the evolution of the market areas when a varies and Q remains unchanged also differs if $\delta_{jk} > 1$ and if $\delta_{jk} < 1$, and more generally depends on the ratio between δ_{jk} and the distance unit.

We have already discussed in sections 5.4 the evolution of market areas when the exponent a varies, if one point of the plane is supposed to belong anyway to the demarcation line. In the present section, as in Prop. 11.2, the spatial parameter that is kept invariable is the transportation cost associated with the distance unit, as $h(1) = 1 \ \forall a$.

The influence of the distance unit immediately appears if we try to know for which values of the exponent a of the transportation cost function market area Z_k is empty. A detailed study of all the possible rankings of the three numbers δ_{jk}, Q, and 1 -the distance unit- yields the results that follow. From now on we denote by $Z_k[a]$ and $Z_j[a]$ the market areas corresponding to the value a of the distance exponent. Together with the threshold values of a are indicated the corresponding areas $Z_k[a]$. Of course, we know that $Z_k \neq \phi$ when $a > 1$; and that when $a \leq 1$, $Z_k \neq \phi$ iff. $\delta_{jk}^a \geq Q$ (Prop. 6.1 b).

(10.1) *When* $h = .^a$ *with* $a > 0$,

(a) *if* $Q \leq \min\{1, \delta_{jk}\}$, $Z_k \neq \phi$; $\lim_{a \underset{>}{\to} 0} Z_k[a] = \{k\}$;

(b) *if* $1 < Q < \delta_{jk}$, $Z_k \neq \phi$ *iff.* $a \geq a^*$, *where* $a^* = \ln Q / \ln \delta_{jk} \in]0,1[$; $Z_k[a^*] = \{k\}$;

(c) *if* $1 < Q = \delta_{jk}$, $Z_k \neq \phi$ *iff.* $a \geq 1$; $Z_k[1] = [\delta_{ok}, +\infty[\times \{0\}$;

(d) *if* $1 \leq \delta_{jk} < Q$, $Z_k \neq \phi$ *iff.* $a > 1$; $\lim_{a \underset{>}{\to} 1} Z_k[a] = \phi$;

(e) *if* $\delta_{jk} < Q < 1$, $Z_k \neq \phi$ *iff.* $a > 1$ *or* $0 < a \leq a^*$, *where* $a^* = \ln Q / \ln \delta_{jk} \in]0,1[$; $\lim_{a \underset{>}{\to} 0} Z_k[a] = Z_k[a^*] = \{k\}$; $\lim_{a \underset{>}{\to} 1} Z_k[a] = \phi$.

In the first four cases, Z_k becomes nonempty above a threshold value of the distance exponent. In item d, Z_k is unbounded as soon as $a > 1$, but becomes infinitely remote if $a \overset{\rightarrow}{>} 1$; that is why $Z_k[1] = \lim_{a \overset{\rightarrow}{>} 1} Z_k[a] = \phi$. The fifth item is completely different : Z_k first grows, starting from the limiting area {k}; it shrinks again to {k} when $a = a^*$, vanishes when $a^* < a \leq 1$ and reappears when $a > 1$; see Fig. 10.3b. Item b is illustrated by Fig. 10.3a.

That last case shows that the evolution of Z_k wrt. a is not necessarily obvious. To study it in general, we introduce a definition. Let us consider that $Z_k[a] = Z_j[a] = \phi$ if $a \leq 0$. We set that Z_k is *increasing* (resp. *strongly increasing*) wrt. a at $\tilde{a}$ iff. $\exists \eta > 0$ for which, $\forall \varepsilon \in]0,\eta]$, $Z_k[\tilde{a}-\varepsilon] \subseteq Z_k[\tilde{a}]$ (resp. $\subseteq Z_k[\tilde{a}] - Z_j[\tilde{a}]$) and $Z_k[\tilde{a}] \subseteq Z_k[\tilde{a}+\varepsilon]$ (resp. $\subseteq Z_k[\tilde{a}+\varepsilon] - Z_j[\tilde{a}+\varepsilon]$). We define in the same manner the (strong) decreasingness of Z_k wrt. a at $\tilde{a}$. It is easy to verify that when such a property holds at all $a \in \mathbb{R}$, we find the global notion of (strong) increasingness or decreasingness defined in Section 4.

It can be shown, concerning Prop. 10.1, that Z_k is strongly increasing wrt. the exponent a at its threshold values in items b, c, and d. In item a also, unless $Q = \delta_{jk} = 1$: in that case Z_k is increasing wrt. a at 0 but not strongly; Z_k cannot indeed be strongly monotonic wrt. a at any value of a, for centre k then belongs to $Z_j \cap Z_k$ $\forall a > 0$. In item e, Z_k is strongly increasing wrt. a at 0 and 1, but strongly decreasing at a^*.

How can we now determine, given any value of the distance exponent, whether Z_k is increasing, or decreasing, wrt. a ? A first important property has already been found as an illustration of Prop. 5.4. We have seen there that if $h = .^a$ and $\tilde{h} = .^{\tilde{a}}$, $(\partial \Delta\tilde{h} / \partial \delta_k)_{\Delta h} > 0$ iff. $\tilde{a} > a$. Consequently, if a point i of $Z_j[a] \cap Z_k[a]$ belongs to $Z_k[a+\varepsilon]$ $\forall \varepsilon \in [0,\eta]$, then all the points of $Z_j[a] \cap Z_k[a]$ that are farther from [jk] than point i are in $Z_k[a+\varepsilon] - Z_j[a+\varepsilon]$, $\forall \varepsilon \in]0,\eta]$. Conversely, if point $i \in Z_j[a+\varepsilon]$ $\forall \varepsilon \in [0,\eta]$, we have $Z_j[a] \cap Z_k[a] \cap \overset{\circ}{C}(k,i) \subset Z_j[a+\varepsilon] - Z_k[a+\varepsilon]$, $\forall \varepsilon \in]0,\eta]$.

Thus we have to determine, at least at some chosen points of $Z_j \cap Z_k$, if Δh is locally increasing or decreasing wrt.a. The natural way is then to examine the derivative $\partial \Delta h / \partial a$:

$$\frac{\partial \Delta h}{\partial a} = \delta_j^a \, \ell n \, \delta_j - \delta_k^a \, \ell n \, \delta_k .$$

In particular, we have :

(10.2) *When $h = .^a$ and $a > 0$,*

(a) if $\delta_{j\ell}{}^a \ln \delta_{j\ell} \geq \delta_{k\ell}{}^a \ln \delta_{k\ell}$, Z_k is increasing wrt. a ;

(b) if $a < 1$ and $\delta_{jr}{}^a \ln \delta_{jr} \leq \delta_{kr}{}^a \ln \delta_{kr}$, Z_k is decreasing wrt. a.

If the inequalities hold strictly, the properties of Z_k are strong.

We can get a better view of the issue if we get deeper into the analysis of $\partial\Delta h/\partial a$. Let us set $f(\xi) = \xi \ln \xi$; $\partial\Delta h/\partial a$ may be written $[f(\delta_j^a) - f(\delta_k^a)]/a$. That function f is ≤ 0 on $[0,1]$, ≥ 0 on $[1, +\infty[$, strictly decreasing on $[0,1/e]$, strictly increasing on $[1/e, +\infty[$, and convex. From that short description we may already state that (see Fig. 10.1)

$$\delta_k \geq e^{-1/a} \text{ or } \delta_j \geq 1 \Rightarrow \partial\Delta h/\partial a > 0 ;$$
$$\delta_j \leq e^{-1/a} \Rightarrow \partial\Delta h/\partial a < 0 .$$

But it is easy to go further. We have indeed :

LEMMA 11.1 $\forall\ \xi_1, \xi_2$ *such that* $0 < \xi_1 < \xi_2$:

(a) $\xi_1 + \xi_2 \geq 1 \Rightarrow \xi_2 \ln \xi_2 - \xi_1 \ln \xi_1 > 0$;

(b) $\xi_1 + \xi_2 \leq 2/e \Rightarrow \xi_2 \ln \xi_2 - \xi_1 \ln \xi_1 < 0$.

If $\xi_1 \geq 1/e$, item a is obvious as f is strictly increasing on $[1/e, +\infty[$. If now $\xi_1 < 1/e$, we have $\xi_1 \ln \xi_1 < -\xi_1$ as f is convex and as $\lim_{\xi \to 0,\ \xi > 0} \xi \ln \xi = 0$ and $(1/e)\ln(1/e) = -1/e$. On the other hand, we have $\xi \ln \xi \geq \xi - 1$ for any ξ (and so for ξ_2) because f is convex and because $f'(1) = 1$. The summation of those two inequalities yields $\xi_2 \ln \xi_2 - \xi_1 \ln \xi_1 > \xi_2 + \xi_1 - 1$; hence the result. As to item (b) : developing f around its minimizer 1/e yields

$$\xi \ln \xi = -1 + (\xi - 1/e)^2 / 2\tilde{\xi}$$

where $\tilde{\xi}$ is between 1/e and ξ. For any two values of ξ symmetric wrt. 1/e, the one at the left of 1/e consequently corresponds to a higher value of $\xi \ln \xi$, because $1/\tilde{\xi}$ is higher. As f is decreasing on $[0,1/e]$ and increasing on $[1/e, +\infty[$, it is then clear that whatever ξ_1, ξ_2, if $0 < \xi_1 < \xi_2$ and

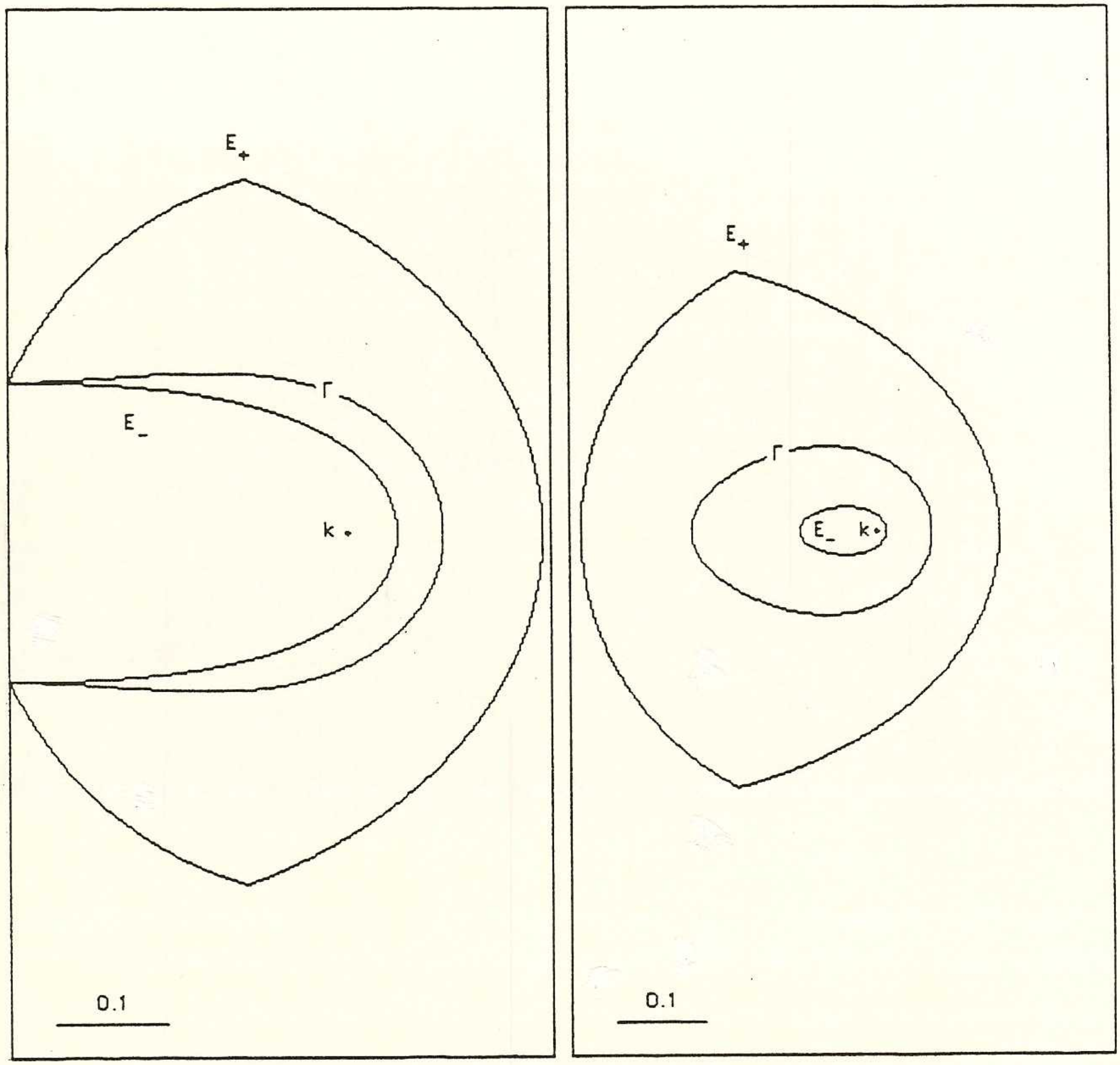
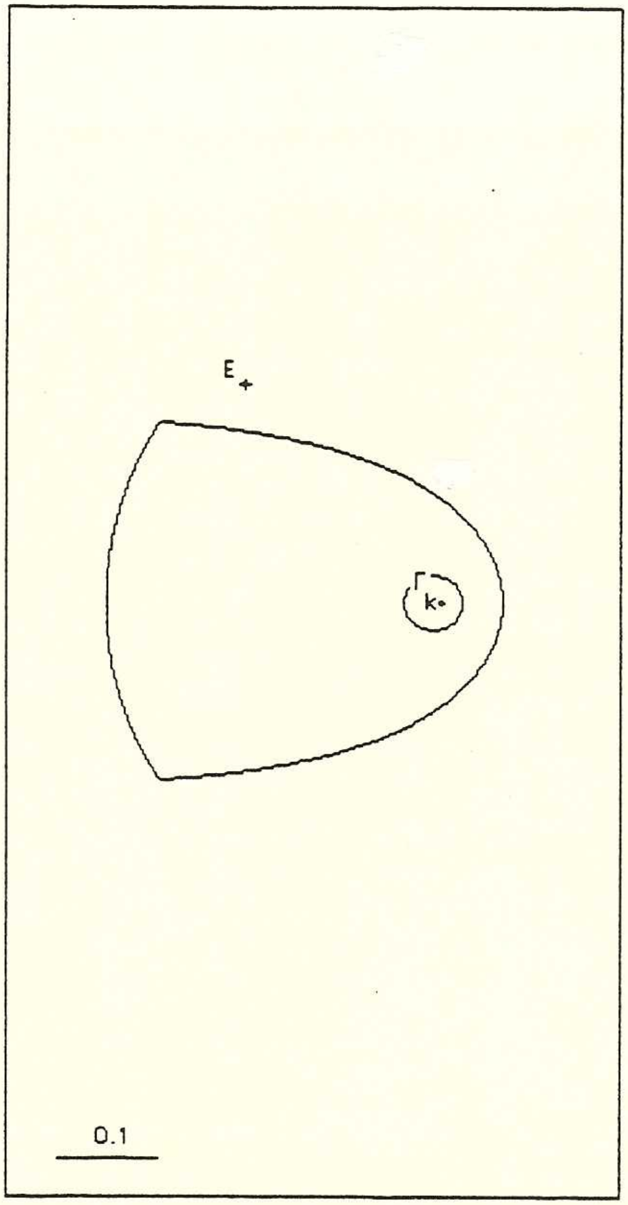


Fig.10.2a. $k \in E_-$ and $\Gamma \cap \{0\} \times \mathbb{R} \neq \emptyset$. Those properties occur iff. (i) $a \leq 1$ and $\delta_{jk} \leq 2e^{-1/a}$ or (ii) $a \geq 1$ and $\delta_{jk} \leq (2/e)^{1/a}$. Here $a = 0.9$ and $\delta_{jk} = 0.6$.

Fig.10.2b. $k \in E_-$ and $\Gamma \cap \{0\} \times \mathbb{R} = \emptyset$. Those properties occur iff. $a < 1$ and $2e^{-1/a} < \delta_{jk} \leq (2/e)^{1/a}$. Here $a = 0.9$ and $\delta_{jk} = 0.68$.



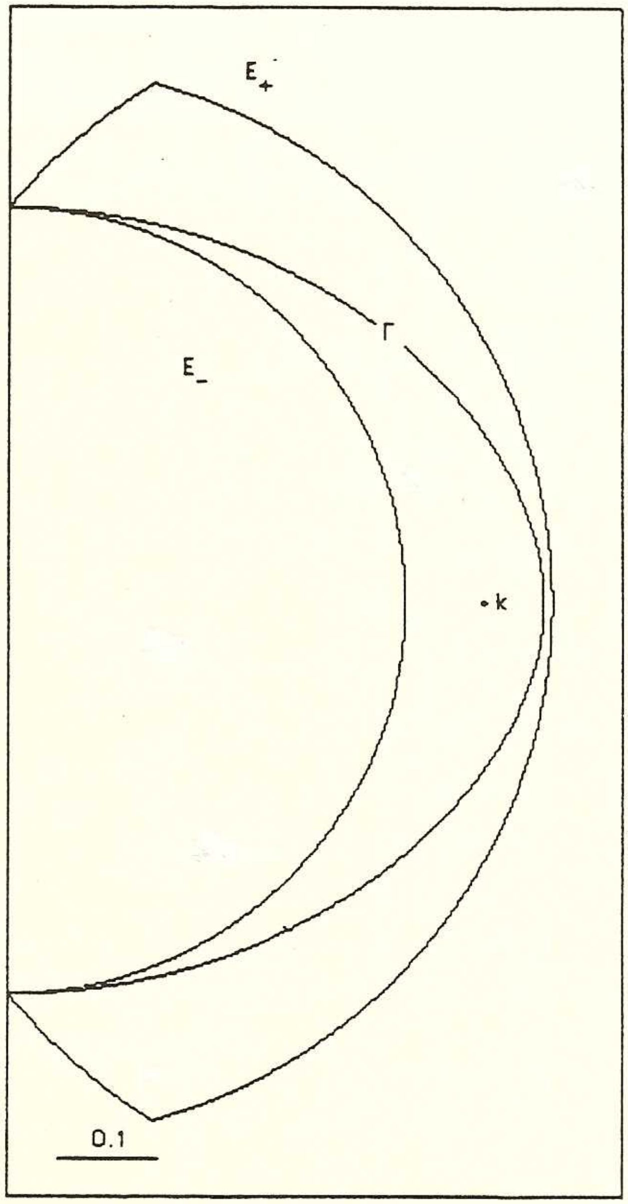


Fig. 10.2c. $E_- = \emptyset$ and $\partial \Delta h/\partial a \leq 0$ at k. Those properties occur iff. (i) $a \leq 1$ and $(2/e)^{1/a} < \delta_{jk} \leq 1$ or (ii) $1 \leq a < 1/\ln 2$ and $2\,e^{-1/a} < \delta_{jk} \leq 1$. Here $a = 0.9$ and $\delta_{jk} = 0.85$.

Fig. 10.2d. $k \notin E_-$, $\Gamma \cap \{0\} \times \mathbb{R} \neq \emptyset$, and $\partial \Delta h/\partial a \leq 0$ at k. Those properties occur iff. (i) $1 < a \leq 1/\ln 2$ and $(2/e)^{1/a} < \delta_{jk} \leq 2\,e^{-1/a}$ or (ii) $a > 1/\ln 2$ and $(2/e)^{1/a} < \delta_{jk} \leq 1$. Here $a = 2$ and $\delta_{jk} = 0.93$.

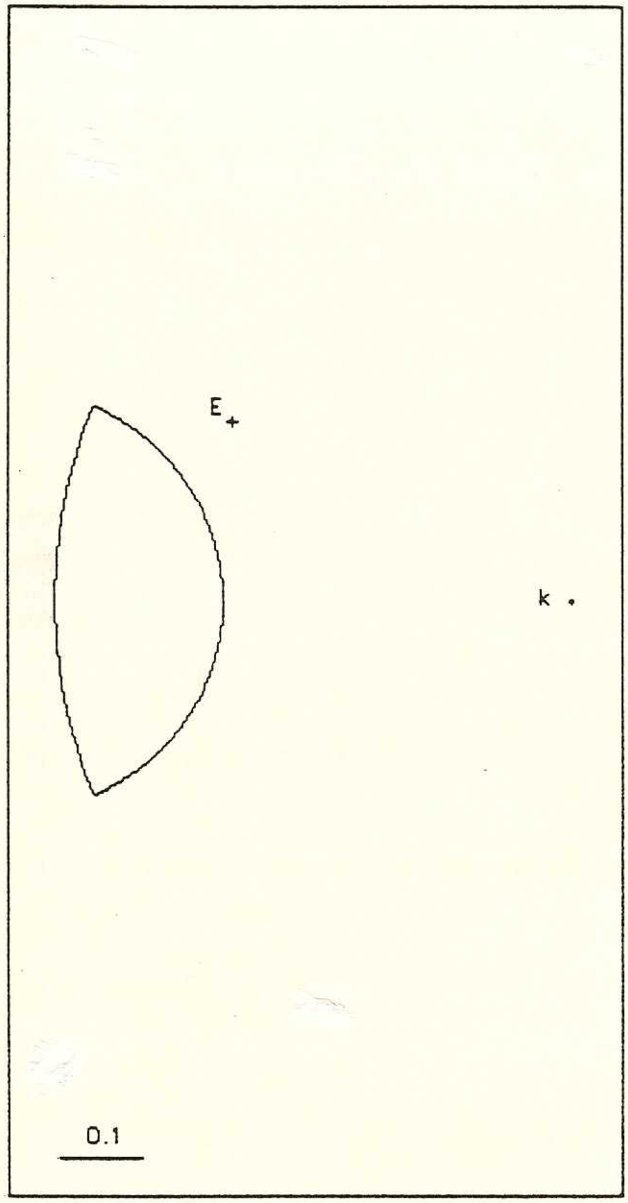


Fig. 10.2e. $k \in E_+$ and $E_- = \Gamma = \emptyset$. Those properties may occur only if (i) $1 < a < 1/\ln 2$ and $1 \leq \delta_{jk} \leq (1 - 1/e)^{1/a} + e^{-1/a}$ or (ii) $a \geq 1/\ln 2$ and $2e^{-1/a} < \delta_{jk} < (1 - 1/e)^{1/a} + e^{-1/a}$.

Here $a = 2$ and $\delta_{jk} = 1.32$.

Another disposition, with $E_- = \emptyset \neq \Gamma$ and $k \in E_+$, seems theoretically predictible under those conditions, but we could not detect any instance of it.

$1/e - \xi_1 \geq \xi_2 - 1/e$, we shall have $f(\xi_2) - f(\xi_1) < 0$. So item (b) is proved.

Summarizing all those results, we have

(10.3) *When* $h = .^a$ *with* $a > 0$ *and* $i \in \mathbb{R}_+^* \times \mathbb{R}$,

(a) *if* $\delta_k \geq e^{-1/a}$ *or* $\delta_j^a + \delta_k^a \geq 1$, $\partial\Delta h/\partial a > 0$;

(b) *if* $\delta_j^a + \delta_k^a \leq 2/e$, $\partial\Delta h/\partial a < 0$.

So the change of Δh wrt. a appears related with the value of Σh. Let us denote by E_+ and E_-, respectively, the sets of points of $\mathbb{R}_+ \times \mathbb{R}$ satisfying the conditions of items a and b. It is then clear that $E_- \subseteq \{ i \in \mathbb{R}_+ \times \mathbb{R} ; \partial\Delta h/\partial a \leq 0 \} \subset \mathbb{R}_+ \times \mathbb{R} - E_+$. The threshold relations between δ_{jk} and a at which E_- or $\mathbb{R}_+ \times \mathbb{R} - E_+$ vanishes will be studied in Prop. 10.6.
Another important though obvious property is that
(10.4) *The areas* E_-, $\mathbb{R}_+ \times \mathbb{R} - E_+$, *and* $\{i \in \mathbb{R}_+ \times \mathbb{R} ; \partial\Delta h/\partial a \leq 0 \}$ *are bounded.*
The inclusion of Z_k into the area $\partial\Delta h/\partial a \leq 0$ is thus possible only if $a \leq 1$.
Some minor and easily proved properties help us to understand the examples of Fig. 10.2. When $a \leq 1$, E_- and $\mathbb{R}_+ \times \mathbb{R} - E_+$ are starshaped wrt. k. When $a \geq 1$, E_- and $\mathbb{R}_+ \times \mathbb{R} - E_+$ are convex, and E_- is starshaped wrt. o . The set $\partial\Delta h/\partial a = 0$ is composed of the y-axis and of a path Γ that either has 2 points of the y-axis as extremities ; or meets it at o or nowhere and is a loop ; or is empty. The curve Γ meets the y-axis where E_- and E_+ meet each other.

When the attractivity constant Q and the distance exponent a are given, Prop. 10.3 allows us to determine ranges of values of the intercentral distance δ_{jk} for which Z_k is strongly increasing or decreasing wrt. a. We know from Prop. 10.2 that if $(\partial\Delta h/\partial a)_{i=\ell} > 0$, then Z_k is strongly increasing wrt. a. Suppose that ℓ is at the right of centre k, the exponent a being > 1. The conditions of item a in Prop. 10.3 may be expressed as : $\delta_{k\ell} \geq e^{-1/a}$ or $\delta_{k\ell} \geq [(1-Q)/2]^{1/a}$;i.e., equivalently : $(\delta_{ok} + e^{-1/a}, 0) \in Z_j$ or $(\delta_{ok} + [(1-Q)/2]^{1/a}, 0) \in Z_j$. If we use the last form of the conditions, we have to specify that $a > 1$: if a were < 1, their meaning would be reversed. We have also to notice that if $Q \geq 1$, the conditions are satisfied in their initial form. Finally, they may be rewritten :

$$Q \geq [\delta_{jk} + e^{-1/a}]^a - 1/e$$

or

$$Q \geq [\delta_{jk} + [(1- Q)/2]^{1/a}]^a - (1-Q)/2 ;$$

and we get so the second inequality of the first item of the next proposition.

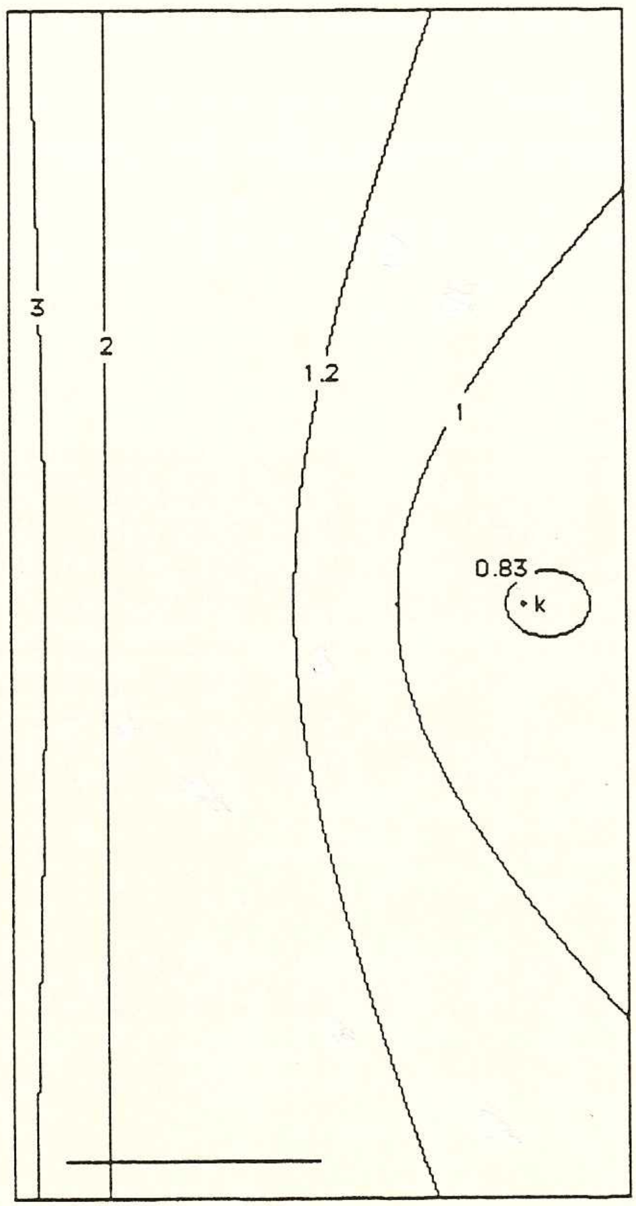


Fig. 10.3a. Evolution of Z_k wrt. a when h = .^{a}: δ_{jk} = 4 , Q = 3, indicated values of a.

According to Prop. 10.6a, Z_k is strongly increasing in a for all values of a, as $\delta_{jk} \geq 2$.

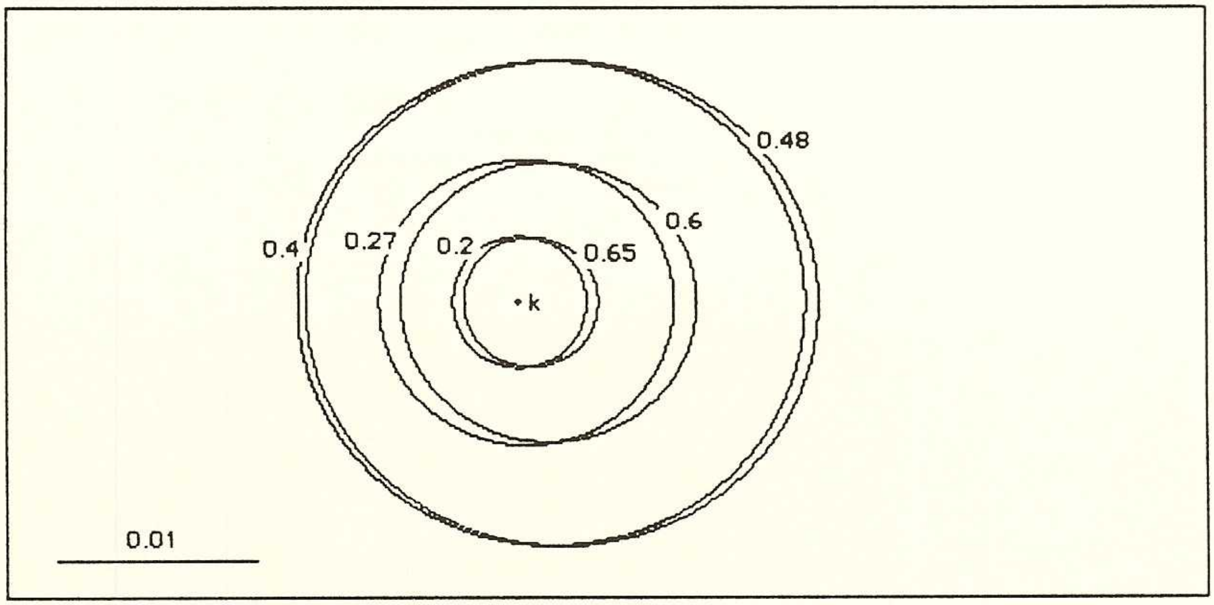


Fig. 10.3b. Evolution of Z_k wrt. a when $h = .^a$ and $\delta_{jk} < Q \leq 2/e$, as an illustration of Prop. 10.1e and Prop. 10.5 : $\delta_{jk} = 0.37$, $Q = 0.5$, indicated values of a. Market area Z_k is nonemtpty iff. a either is > 1 or belongs to an interval which is approximately]0, 0.6971[. If it can be proved that each of the sets of values of a (0 < a < 1) where Z_k is respectively increasing and decreasing wrt. a is an interval, a recursive calculation based on Prop. 10.2 and on its proof shows that Z_k should be increasing wrt. a iff. a is inside an interval $I \approx]0, 0.4251]$, and decreasing wrt. a iff. $a \geq a'$ where $a' \approx 0.4562$. Less accurate (thus smaller) intervals can be more easily computed on basis of Prop. 10.5. Notice that Z_k is very small compared to [jk] and that the left edge of the figure is not the y-axis.

The other items are obtained similarly. In item b, we have added the condition $\delta_{jk}^{a} \geq Q$ to ensure that Z_k be nonempty.

Notice also that explicit conditions on Q instead of δ_{jk} can be derived by considering only the conditions $\delta_{k\ell} \geq 1/e$ or $\delta_{jr} \leq 1/e$ - so the first of the two inequalities here above - or by approximating more complex conditions like the second of those two inequations. We have not mentioned such conditions on Q because they are necessarily less tight than those of Prop. 10.5, and because a number of ways could be explored to find such conditions.

(10.5) *When* $h = .^{a}$ *and* $a > 0$,

(a) *if* $Q \geq 1$, *or* *if* (*i*) $\delta_{jk} \geq \min\{(\frac{1+Q}{2})^{1/a} + (\frac{1-Q}{2})^{1/a},\ (Q+\frac{1}{e})^{1/a} + e^{-1/a}\}$,

or (*ii*) $a > 1$ *and* $\delta_{jk} \leq \max\{(\frac{1+Q}{2})^{1/a} - (\frac{1-Q}{2})^{1/a},$

$(Q+\frac{1}{e})^{1/a} - e^{-1/a}\}$,

then Z_k *is strongly increasing wrt.* a [*and nonempty in* (*i*) *and* (*ii*)];

(b) *if* $Q < 2/e$, $a \in\ ]0,1[$, *and* $\delta_{jk} \in [Q^{1/a},\ (\frac{1}{e} + \frac{Q}{2})^{1/a} + (\frac{1}{e} - \frac{Q}{2})^{1/a}]$

(*which interval is not empty*),

then Z_k *is strongly decreasing wrt.* a *and nonempty.*

It appears from item a that for any values of Q and a, there are values of δ_{jk} for which Z_k is strongly increasing wrt.a and nonempty; and from item b, that there are values of δ_{jk} for which Z_k is strongly decreasing wrt. a and nonempty, provided that $Q < 2/e$ and $0 < a < 1$. Similarly, we may try to eliminate Q from the conditions and see if there are values of δ_{jk} and a for which Z_k is strongly increasing wrt.a and nonempty whatever Q, or for some values of Q, and what the values of δ_{jk} are for which Z_k is strongly decreasing wrt. a and nonempty for some values of Q (certainly not for all : see Prop. 10.3a). Rather than to elaborate on Prop. 10.5, it is easier to use Prop. 10.3 directly. Here are the results :

(10.6) *When* $h = .^{a}$ *and* $a > 0$,

(*a*) *if* $a < 1$ *and* $\delta_{jk} \geq 1$, *or if* $a > 1$ *and* $\delta_{jk} \geq (1-1/e)^{1/a} + e^{-1/a}$,

then Z_k *is strongly increasing wrt.* a ;

(b) *if $a < 1$ and $\delta_{jk} > 2e^{-1/a}$, or if $a > 1$, then for some values of the attractivity constant Q market area Z_k is nonempty and strongly increasing wrt. a ;*

(c) *if $a < 1$ and $\delta_{jk} < (2/e)^{1/a}$, then for some values of the attractivity constant Q market area Z_k is nonempty and strongly decreasing wrt. a ;*

(d) *if $a > 1$, then Z_k is not decreasing wrt. a..*

Let us now see the proof. The minimum of Σh is necessarily to be found on the x-axis. Take indeed any point i of the plane ; if we denote by i' its orthogonal projection on the x-axis, we have $\delta_{ji'} \leqslant \delta_j$ and $\delta_{ki'} \leqslant \delta_k$, so that $\Sigma h_{i'} \leqslant \Sigma h$. Moreover, if point i is on the x-axis at the right of centre k, we have $\delta_{kk} = 0 \leqslant \delta_k$ and $\delta_{jk} \leqslant \delta_j$: so that $\Sigma h_k \leqslant \Sigma h$. Every minimizer of Σh on $\mathbb{R}_+ \times \mathbb{R}$ must thus belong to the segment [ok] . On that set, we have

$$\left(\frac{\partial \Sigma h}{\partial x} \right)_{a;y=0} = a \, (\delta_j^{a-1} - \delta_k^{a-1}).$$

When $a < 1$, we so have $(\partial \Sigma h/\partial x)_{a,y=0} < 0$; the minimum of Σh is found at centre k and is δ_{jk}^a . The items a and c of Prop. 10.6 are then simply derived by comparison of that value with the numbers 1 and 2/e quoted in Prop. 10.3. More precisely, regarding item c, the condition ensures that $\partial \Delta h/\partial a < 0$ in a vicinity of centre k, and in particular at points of the x-axis on the right of k. Those points are the points r for some values of Q, and Prop. 10.2b may be applied to them. As to item b : the existence of points of]ok] for which $\partial \Delta h/\partial a > 0$ (and thus allowing to use Prop. 10.2a) is guaranteed by Prop. 10.3 if $\delta_k \geqslant e^{-1/a}$ for some i ∈]ok], or if $\Sigma h \geqslant 1$ for some i ∈ [ok]. As the maximum of Σh on [ok] is now at o, those conditions reduce to : $\delta_{ok} > e^{-1/a}$ or $2\delta_{ok} \geqslant 1$; the second one, being stronger, is eliminated.

When $a > 1$, $(\partial \Sigma h/\partial x)_{a,y=0} > 0$. The minimum of Σh now occurs at o and is $2\,\delta_{ok}^a$; comparing it with 2/e yields item b. But the proof of item a is different. We have to assign δ_{jk} to such a range as to ensure that the sets $\mathring{C}_k\,(e^{-1/a})$, i.e., $\{i;\ \delta_{ik} < e^{-1/a}\}$, and $\{i;\ \Sigma h < 1\}$ have an empty intersection. This is the case if the second set itself is empty, ie. if $2\delta_{ok}^a \geqslant 1$, or $\delta_{jk} \geqslant 2 \times 2^{-1/a}$. Suppose now that the set is not empty. The intersection is still empty iff. the point p or $(\delta_{ok} - e^{-1/a}, 0)$, where the half-line $]-\infty, \delta_{ok}] \times \{0\}$ crosses the boundary of $C_k(e^{-1/a})$, verifies the inequality $\Sigma h_p \geqslant 1$ and is on the right of point o. It is clear indeed that if $\Sigma h_p < 1$, the intersection sought for contains points at the right of p and is not empty ; conversely, if $\Sigma h_p \geqslant 1$, then $\Sigma h \geqslant 1$ on the circle centred on j and passing through p. As that circle encompasses o which belongs to $\{ i; \Sigma h < 1 \}$ if that set is not empty (as Σh is now minimum at o), and as $\{i ; \Sigma h < 1 \}$ is convex (as Σh is now a convex function of i), we have $\{i; \Sigma h < 1 \} \subset \mathring{C}(j, p)$. As $\mathring{C}(j,p) \cap \mathring{C}(k, p) = \phi$, the proof is nearly complete —we have yet to express that p is on the right of o and that $\Sigma h_p \geqslant 1$. This yields :

$$e^{-1/a} \leqslant \delta_{ok}$$

$$(\delta_{jk} - e^{-1/a})^a + 1/e \geqslant 1$$

or equivalently

$$\delta_{jk} \geqslant 2e^{-1/a}$$

$$\delta_{jk} \geqslant (1-1/e)^{1/a} + e^{-1/a} \quad .$$

The first inequality is implied by the second one and may thus be cancelled. Moreover as $(.)^{1/a}$ is here a concave function, we have

$$\frac{(1-\frac{1}{e})^{1/a} + (\frac{1}{e})^{1/a}}{2} \leqslant \left(\frac{1 - \frac{1}{e} + \frac{1}{e}}{2}\right)^{1/a} = 2^{-1/a} ,$$

so that the other sufficient condition for $C(k,p) \cap \{ i; \Sigma h < 1 \}$ to be empty, i.e., $\delta_{jk} \geqslant 2\times 2^{-1/a}$, also disappears. The proof of item a is thus complete.

Last, item d derives from the inexistence of point r when $a > 1$ and from the boundedness of the area where $\partial \Delta h/\partial a < 0$ (Prop. 10.4).

The results of this section will be completed in the next one by approximate ways to compute $|Z_k|$ and $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$, and in particular by an approximate formula of $|Z_k|$ when $a \in]0,1[$ (see Prop. 11.8).

11. *Some limiting properties of market areas*

In this section we have grouped together two types of limiting results. The first one concerns parametrized families of transportation cost functions. The problem is here to find the limits the market areas tend toward when the parameter tends toward values at which the function looses some of the properties defining a transportation cost function. The job is not done, of course, for any family of transportation cost functions ; we have chosen to examine what happens when $h(\delta) = \delta^a$ (with $a > 0$), if the distance exponent a tends towards 0 or $+\infty$. However, the method could be applied to other functions, particularly to $-\delta^{-a}$ $(a > 0)$.

The second type of results is obtained by letting the intercentral distance δ_{jk} tend toward 0. A similar way is successfully followed in electricity to assess the resulting field of two spatially close charges q and -q : this constitutes the theory of dipoles. Our study shows that the quantity δ_{jk}/Q, already pointed out in section 6.2, is determining and comparable with the dipolar moment $q\ \delta_{jk}$ of electricity. This is why we call δ_{jk}/Q a *dicentral moment*. The approach is valid for any transportation cost function ; and such are some of the statements obtained. But here again the main results are found when we restrict the study to the function $h\ (\delta) = \delta^a$.

The section is thus focused on the monomial function δ^a. The reason for this lies not only in its nice mathematical properties. We are also naturally led through this whole paper to deepen our knowledge of that family of transportation cost functions because two of the three 'classical' functions (see Prop. 4.7) namely δ and δ^2, belong to that family. Moreover, section 11.1 shows that the third classical function, $\ln \delta$, is somehow another member of the family.

The interest taken in the study of limiting properties is based, of course, on the hope that they will still hold, more or less, in situations close to the limiting ones. In both subsections 11.1 and 11.2, the appreciation of the suitability of the limits of Z_k as approximations of Z_k brings us back to the issue of the *distance unit*, already met in section 10.

11.1 *Extreme shapes of market areas when* $h = .^a$

Call Z_k (i') the market area of centre k corresponding to the value of the attractivity constant Q for which a point $i' \in Z_j \cap Z_k$; i.e., $Q = \Delta h_{i'}$. Denote by $\mathring{C}$ (i", i') the interior of the disk C (i", i') ; i.e., $\mathring{C}(i'', i') = \{i \; ; \delta_{i''} < \delta_{i'' i'}\}$. Then we have

(11.1) *If the transportation costs are represented by* $h = .^a$ *with* $a > 0$, *then* :

$$\lim_{a \gtrless 0} Z_k(i') = A_{i'}$$

$$\lim_{a \to +\infty} Z_k(i') = \mathbb{R}_+ \times \mathbb{R} - \mathring{C}(j,i') .$$

To simplify the statements we have not considered what happens at the boundary $Z_j \cap Z_k$. With a stricter conception, the limit of Z_k when $a \to +\infty$ for instance would be the area

$$\mathbb{R}_+^* \times \mathbb{R} - \{i; \delta_j < \delta_{ji'} \text{ or } (\delta_j = \delta_{ji'} \text{ and } \delta_k > \delta_{ki'})\}.$$

Similar remarks hold for Prop. 11.2 and 11.4.

Let us see the proof. The set Z_k (i') corresponds to the inequality

$$\delta_j^a - \delta_k^a \geqslant \delta_{ji'}^a - \delta_{ki'}^a \quad . \tag{40}$$

If we divide both members by the exponent a, we see, using de l'Hospital's rule, that when $a \gtrless 0$, Z_k (i') tends to be defined by

$$\ell n\,(\delta_j/\delta_k) \geqslant \ell n\,(\delta_{ji'}/\delta_{ki'}).$$

The proof of the first item is then obvious.

For the second item, we first consider a point i of $\mathbb{R}_+^* \times \mathbb{R} - C(j,i')$. The largest of the four distances δ_j, δ_k, $\delta_{ji'}$, and $\delta_{ki'}$ is then δ_j. Dividing (40) by δ_j yields

$$1 - (\delta_k/\delta_j)^a \geqslant (\delta_{ji'}/\delta_j)^a - (\delta_{ki'}/\delta_j)^a.$$

All those three ratios being < 1, it is clear that point i must belong to Z_k when a becomes large enough. Dividing (40) by $\delta_{ji'}$ similarly shows that every point of $\mathring{C}(j,i')$ must belong to Z_j when a becomes large enough.

The limiting shapes obtained in Prop. 11.1 can be derived in another perspective. Denote by A (ρ) the area $\delta_j/\delta_k \geqslant \rho$, and by $\mathring{C}_j$ (L) the interior of the disk of radius L, centred on j. A proof similar to that of Prop. 11.1 shows the following :

(11.2) *If the transportation cost function is $.^a$ with $a > 0$, then, λ and L being strictly positive constants,*

$$\lim_{\substack{a \gtrless 0 \\ Q/a = \lambda}} Z_k = A(e^{\lambda}) \;,$$

$$\lim_{\substack{a \to +\infty \\ Q = L^a}} Z_k = \mathbb{R}_+ \times \mathbb{R} - \mathring{C}_j \;(L) \;.$$

Interestingly, as we shall have the opportunity to recall in section 11.2, the first limit cannot be used as an approximation of Z_k when the exponent a becomes close to zero. One might think indeed in that case, in view of Prop. 11.2, that Z_k should be well approximated by the Apollonian disk $A(e^{Q/a})$. The problem is that this disk is modified if we change the distance unit, as :

(11.3) *The disks $A(e^{Q/a})$ and $C_j\,(Q^{1/a})$ are respectively instable and stable under a change in the distance unit.*

Such a change means indeed that the new distances are $\upsilon\delta$, where δ represents the previous distances and υ is > 0. In order to keep the market areas unaltered, we have then to replace Q by υ^a Q : hence the inequality $(\upsilon\delta_j)^a - (\upsilon\delta_k)^a \geqslant \upsilon^a Q$ is equivalent to $\delta_j^a - \delta_k^a \geqslant Q$ and still describes Z_k. More exactly, Q remains at its value but the transportation cost function becomes $\upsilon^{-a}(.)^a$ instead of $.^a$; however, as we have pointed out in Prop. 4.1, this is mathematically equivalent to keeping the old transportation cost function $.^a$ and modifying Q into $\upsilon^a Q$.

Unfortunately the inequality $\delta_j/\delta_k \geqslant e^{Q/a}$ is then transformed into $\upsilon\delta_j/\upsilon\delta_k \geqslant e^{\upsilon^a Q/a}$: instead of $A(e^{Q/a})$, we find the disk $A(e^{\upsilon^a Q/a})$. Using $A(e^{Q/a})$ as an approximation of Z_k would thus make little sense, unless the distance unit were optimized according to some criterion. Anyway, Prop. 6.8 already contains much information about the relations between market areas and Apollonian disks.

The difficulty disappears for the area $\mathring{C}_j\ (Q^{1/a})$: its inequality $\delta_j \geqslant Q^{1/a}$ becomes $\upsilon\delta_j \geqslant (\upsilon^a\ Q)^{1/a}$, and remains thus unchanged. In this respect we may use $\mathbb{R}_+ \times \mathbb{R} - \mathring{C}_j\ (Q^{1/a})$ as an approximation of Z_k for large values of the distance exponent. More precisely, we have here a lower bound on Z_k, as $\delta_j^a - \delta_k^a \geqslant Q \Rightarrow \delta_j^a \geqslant Q$. This is certainly not the best bound of the type : the largest disk centred on j and contained in Z_j is $C\,(j,\ell)$, as shown by Prop. 6.7, and is itself outperformed by the osculating disk C_ℓ when $a \geqslant 2$ (Prop. 6.6 and 7.1). Nevertheless, the measure $|\ C_j\ (Q^{1/a}) \cap \mathbb{R}_+ \times \mathbb{R}\ |$ provides us with a quickly computed lower bound on $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}\ |$:

$$|C_j(Q^{1/a}) \cap \mathbb{R}_+ \times \mathbb{R}\ | = \max\ \{0,\ Q^{2/a}\ \text{acos}\ (\delta_{ok}/Q^{1/a}) - \delta_{ok}\sqrt{Q^{2/a} - \delta_{ok}^2}\}.$$

This allows us also to give a tentative answer to the question asked in Section 9 about the dependence of $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}\ \ |$ on the intercentral distance δ_{jk}. When $C_j(Q^{1/a}) \cap \mathbb{R}_+^* \times \mathbb{R} \neq \phi$, ie. when $\delta_{ok} < Q^{1/a}$, we have :

$$\left(\frac{\partial^2 |C_j\ (Q^{1/a})\ \cap \mathbb{R}_+ \times\ \mathbb{R}\ |}{\partial \delta_{jk}{}^2} \right)_Q = \frac{\delta_{jk}}{\sqrt{Q^{2/a} - \delta_{ok}^2}} > 0.$$

So $|\ C_j\ (Q^{1/a}) \cap\ \mathbb{R}_+ \times \mathbb{R}\ |$ is convex wrt. δ_{jk}, which will also be the case for the dicentral approximation used in section 11.2. As to the dependence on Q, we find when $Q > \delta_{ok}^a$:

$$\left(\frac{\partial^2\ |C_j\ (Q^{1/a}) \cap \mathbb{R}_+ \times \mathbb{R}\ |}{\partial Q^2} \right)_{\delta_{ok}} = \frac{2Q}{a^2}[(2-a)\ \text{acos}\ (\delta_{ok}/Q^{1/a}) + \text{cotg acos}\ (\delta_{ok}/\ Q^{1/a})\ ],$$

which resembles Prop. 9.4 : there is some threshold value of Q, say $Q_t(a)$, such that $Q_t\ (a) > \delta_{ok}^a$ and that $|C_j(Q^{1/a}) \cap \mathbb{R}_+\ \times\ \mathbb{R}\ \ |$ is convex wrt. Q if $Q \leqslant Q_t\ (a)$. But it is also strictly concave wrt. Q if $Q > Q_t\ (a)$, and $Q_t\ (a)$ is increasing wrt. a ; those two properties cannot be deduced from Prop. 9.4 regarding Z_k. Regarding the dependence of $|\ Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$ on a (see Section 10), it is clear that $|\ C_j(Q^{1/a} \cap \mathbb{R}_+\ \times\ \mathbb{R}\ |$ is increasing or decreasing in a according as Q is < 1 or > 1.

We have left aside the problem of the accuracy of those bounds or approximations. They must certainly not be used, however, when $a \leqslant 2$, as the shape of Z_k is then opposite to that of $\mathbb{R}_+ \times \mathbb{R} - \overset{\circ}{C}_j\ (Q^{1/a})$; and the measure $|C_j(Q^{1/a}) \cap \mathbb{R}_+ \times \mathbb{R}|$ has nothing to do with $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$ when $a \leqslant 3$, as this one is infinite according to Prop. 9.1. Remember also that formula (38) can be used to produce bounds on $|Z_k|$ and $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$.

In view of Prop. 11.1, the properties discovered for the transportation cost function $.^a$ become very clear as linking in a continuous manner the four pivot cases $a \gtrsim 0$, $a = 1$, $a = 2$, and $a \to +\infty$. Market area Z_k is convex and $L_y/L_x \leqslant 1$, when $a \gtrsim 0$ as well as when $a \lesssim 1$; when $a \gtrsim 1$, Z_k clearly opens itself towards the right, rejecting point r to infinity. The demarcation line $Z_j \cap Z_k$ has a vertical asymptote parallel to the y-axis when $a = 2$: it is $Z_j \cap Z_k$ itself ; but it has also such an asymptote, rejected to infinity, when $a \gtrsim 1$. The line $Z_j \cap Z_k$ turns its convexity at point ℓ to the left when $a = 1$; this is also true when $a = 2$ -in which case it is also turned to the right, like $\lim_{a \to +\infty} (Z_j \cap Z_k)$. In projective geometry, parallel lines meet at infinity ; so that we may consider that the y-axis is a kind of an asymptote of $Z_j \cap Z_k$ when $a = 2$; and it is also an asymptote of $\lim_{a \to +\infty} (Z_j \cap Z_k)$. And the properties of the disk C_i defined in section 6.4 are obvious : C_i is equal to Z_k when $a \gtrsim 0$; when $a = 1$, the orientation of $Z_j \cap Z_k$ shows that C_i is included in Z_k, not in Z_j ; when $a \lesssim 2$, C_i is equal to Z_k ; when $a \gtrsim 2$, to Z_j ; and finally, when $a \to +\infty$, C_i is either $\mathbb{R}_- \times \mathbb{R}$ or $C\ (j,i)$, the union of which two sets constitutes Z_j, precisely.

It may seem interesting to relate the present study to Section 10, in order to have more information about the shape of Z_k when $a \gtrsim 0$ and Q is maintained at some fixed value. One could hope that Z_k would tend towards the Apollonian disk $A(e^{Q/a})$ if $a \gtrsim 0$, when $Z_k \to \{k\}$ if $a \gtrsim 0$, i.e., when $Q \leqslant 1$ (see Prop. 10.1). In some way this is true, of course, as the limit of $A(e^{Q/a})$ is then $\{k\}$ too. But if we ask for instance whether the ratio $|Z_k \,\Delta\, A(e^{Q/a})| / |A(e^{Q/a})|$, which is a measure of the *relative* difference between Z_k and $A(e^{Q/a})$, tends towards 0 in any case, the answer is : no. If it were true, the center c of $A(e^{Q/a})$ would indeed belong to Z_k if the distance exponent a were small enough. Now, the abscissa of c is

$$x_c = \delta_{ok} \frac{e^{2Q/a} + 1}{e^{2Q/a} - 1} ;$$

so that it belongs to $Z_k - Z_j$ iff.

$$(e^{2Q} - 1)/ Q > [(e^{2Q/a} - 1) / \delta_{jk}]^a ;$$

if the inequality is reversed, $c \in Z_j - Z_k$. If we let a tend toward 0, the right-hand member tends toward e^{2Q}. Consequently, if $(e^{2Q} - 1)/Q > e^{2Q}$, $c \in Z_k - Z_j$ for all positive values of a smaller than some $\bar{a} > 0$; if $(e^{2Q} - 1) / Q < e^{2Q}$,

$c \in Z_j - Z_k \quad \forall a \in]0,\bar{a}]$. Those conditions are respectively equivalent to $Q < Q^*$ and to $Q > Q^*$, where $Q^* \simeq 0.796812$. It is thus not true that Z_k would tend toward $A(e^{Q/a})$ whatever Q when $a \overset{\to}{>} 0$ and $Q \leqslant 1$. Of course this does not mean that more positive statements cannot be found, but we have not thought it useful to carry on with the investigation.

11.2 *Dicentral approximation*

We have already used the mean-value theorem to show that $\Delta h = 2 \times \delta_{jk} \hbar' (\tilde{\delta}^2)$ for some $\tilde{\delta} \in]\delta_k, \delta_j[$. As the distance δ_o is also between δ_k and δ_j, and as $| \delta_j - \delta_k | \leqslant \delta_{jk}$, it seems reasonable to try δ_o as an approximation of $\tilde{\delta}$. This amounts to approximating Z_k by the area

$$\mathcal{Z}_k = \{i;\ i \neq o \text{ and } 2 \times \delta_{jk} \hbar' (\delta_o^2) \geqslant Q\} = \{i\ ;\ i \neq o \text{ and } h' (\delta_o) \cos\varphi_o \geqslant Q/\delta_{jk}\},$$

and similarly for $\mathcal{Z}_j$, point o belonging now to $\mathcal{Z}_j$; see Fig. 11.1. The approximate market area $\mathcal{Z}_k$ is thus an increasing function of the ratio δ_{jk}/Q.

Another way to introduce $\mathcal{Z}_k$ is to consider some strictly positive number μ and to let Q and δ_{jk} tend toward zero simultaneously while keeping the ratio δ_{jk}/Q equal to μ. The distances δ_j and δ_k then tend toward δ_o, and so does $\tilde{\delta}$. Calling $\mathcal{Z}_k(\mu)$ the set

$$\mathcal{Z}_k(\mu) = \{i;\ i \neq o \text{ and } 2 \times \hbar'(\delta_o^2) \geqslant 1/\mu\} = \{i;\ i \neq o \text{ and } h'(\delta_o)\cos\varphi_o \geqslant 1/\mu\},$$

we consequently have :

(11.4) *If μ is some strictly positive constant ,* $\displaystyle\lim_{\substack{\delta_{jk} \overset{\to}{>} 0 \\ \delta_{jk} = Q\mu}} Z_k = \mathcal{Z}_k(\mu)$.

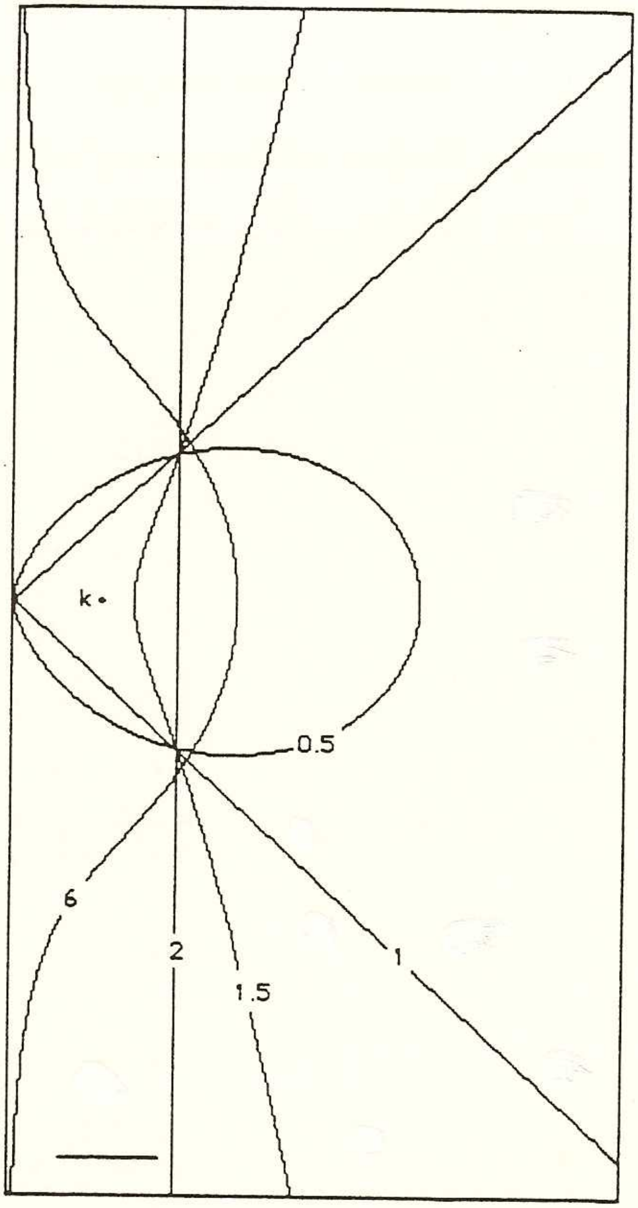


Fig. 11.1a. Dicentral approximation of Fig. 5.3 with the same values of Q, δ_{jk}, and a.

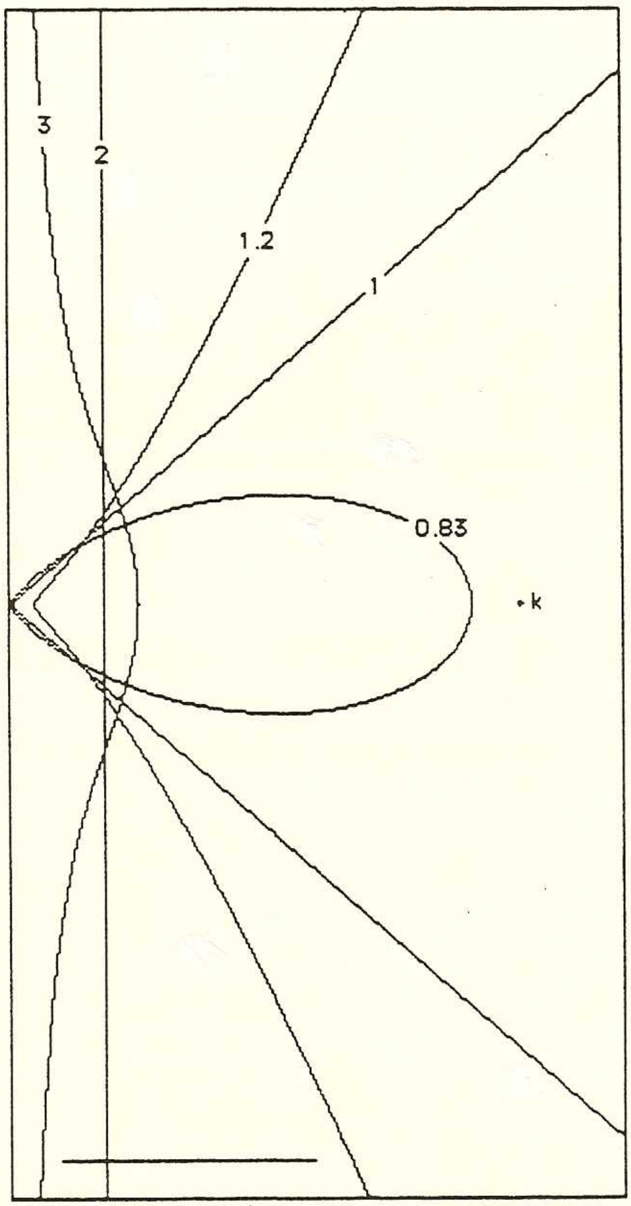


Fig. 11.1b. Dicentral approximation of Fig. 10.3a ; i.e., when h = $\dot{a}$, $\mu = 4/3$, and for indicated values of a. The result is here a bit disappointing : but, as the dicentral approximation is obtained by letting δ_{jk} tend toward zero, it is clear that the sufficient conditions of Prop. 10.6a cannot be met by the approximation.

In particular when $h = .^a$ the inequality of $\mathcal{Z}_k(\mu)$ is $a\, x\, \delta_o^{a-2} \geq 1/\mu$ or $a\, \delta_o^{a-1} \cos \varphi_o \geq 1/\mu$. A change in the distance unit (cf. section 11.1) transforms the inequality of $\mathcal{Z}_k\ (\delta_{jk}/Q)$, i.e., $\mathcal{Z}_k$, into $a(\upsilon\delta_o)^{a-1} \geq \upsilon^a Q/\upsilon\delta_{jk}$, which defines the same area although the dicentral moment itself has become $\upsilon^{1-a}\ \delta_{jk}/Q$. More generally :

(11.5) *The area $\mathcal{Z}_k$ is stable under a change in the distance unit if the transportation cost function h is $.^a$ or $-(.)^{-a}$ with $a > 0$, or $\ln$ (.).*

From the viewpoint which has led us in section 11.1 to eliminate the disk $A(e^{Q/a})$ and to accept $\mathbb{R}_+ \times \mathbb{R} - \mathring{C}_j\ (Q^{1/a})$, that statement justifies the use of $\mathcal{Z}_k$ as an approximation of Z_k in the mentioned cases. The precision of the approximation will not be examined here, but could be approached by means of inequality (38).

We speak here about an approximation, not a lower or upper bound. When $h(\delta) = \ln \delta$, for instance, Z_k is an Apollonian disk ; we then have $x_\ell = \delta_{ok}$ th $(Q/2)$ and $x_r = \delta_{ok}/$ th $(Q/2)$, where 'th' is the hyperbolic tangent. On the other hand, $\mathcal{Z}_k \cup \{o\}$ is a disk centred on the x-axis, and its circle passes through o and through $r' = (\delta_{jk}/Q, 0)$. As $x_{r'} < x_r$, none of the two disks can contain the other : $Z_k\ \Delta\ (\mathcal{Z}_k \cup \{o\}) \neq \phi$. When Q is high enough, we even have $x_{r'} < x_\ell$, and $Z_k \cap \mathcal{Z}_k = \phi$.

The interest of replacing Z_k by $\mathcal{Z}_k$ is that the properties of $\mathcal{Z}_k$ are easier to study than those of Z_k, whereas they should not differ much in view of Prop. 11.4. As $\mathcal{Z}_k\ (\mu)$ is obtained by letting j and k tend towards o, the propositions concerning centre k are to be modified accordingly. So we have in particular : If h is concave, then : $\mathcal{Z}_k\ (\mu) \neq \phi \Leftrightarrow$ o belongs to the boundary of $\mathcal{Z}_k\ (\mu)$. Another difference, here from Prop. 8.6c, is that when $h(\delta) = \delta$, $\mathcal{Z}_k\ (\mu)$ is convex but not strictly, as $\mathcal{Z}_j\ (\mu) \cap \mathcal{Z}_k\ (\mu)$ degenerates in two straight open half-lines originating in o.

Two quantities at least are easy to compute with the dicentral approximation. Here is the first one :

(11.6) *If $h \circ \sqrt{.}$ is strictly concave or strictly convex and $i \in \mathcal{Z}_j\ (\mu) \cap \mathcal{Z}_k\ (\mu) \cap \mathbb{R}_+^2$, then*

$$y = \sqrt{(h \circ \sqrt{.})'^{\wedge}\ (1/2x\mu) - x^2}.$$

The second quantity is $\overline{\delta}_o$ (φ_o, Q) (see section 9.2), here the distance δ_o from o to the point of $\mathcal{Z}_j$ (μ) ∩ $\mathcal{Z}_k$ (μ) in the direction given by the angle φ_o. We obviously have $\overline{\delta}_o(\varphi_o, Q) = h'^{\wedge}(1/\mu \cos \varphi_o)$, if h' is invertible. Hence we also have a possible way to approximate the measures $|Z_k|$ or $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$:

(11.7) *If h is strictly concave,* $|\mathcal{Z}_k(\mu)|$ *is given by* $\int_0^{\pi/2}[h'^{\wedge}(1/\mu \cos\varphi_o)]^2 d\varphi_o$; *if h is strictly convex, that integral is* $|\mathcal{Z}_j(\mu) \cap \mathbb{R}_+ \times \mathbb{R}|$.

When $h(\delta) = \ell n\ \delta$, that formula leads to $|\mathcal{Z}_k| = \pi\delta_{jk}^2 / 4Q^2$, which we may compare with the measure of the area of the Apollonian disk Z_k: $|Z_k| = \pi\delta_{jk}^2/4\mathrm{sh}^2\, Q$, where 'sh' is the hyperbolic sinus. In this case $|\mathcal{Z}_k|$ clearly works as an approximation of $|Z_k|$. If we now apply Prop. 11.7 when $h(\delta) = \delta^a$, we find that the integral of Prop. 11.7 is

$$\int_0^{\pi/2} (a\,\mu \cos \varphi_o)^{2/(1-a)} d\varphi_o .$$

When the exponent 2/(1-a) of the integrand is integer, the integral can be computed analytically ; a similar result also holds when $h(\delta) = -\delta^{-a}$, and we find :

(11.8) (a) *When* $h(\delta) = \delta^a$ *with* $a > 0$, *if for some* $p \in \mathbb{N}$

(i) $a = (p-1)/p$, *then* $|\mathcal{Z}_k(\mu)| = (a\mu)^{2p} \dfrac{\pi\ (2p)!}{2^{2p+1}\ p!^2}$;

(*ii*) $a = (2p-1)/(2p+1)$, *then* $|Z_k(\mu)| = (a\mu)^{2p+1} \dfrac{2^{2p} p!^2}{(2p+1)!}$.

(b) *When* $h(\delta) = -1/\delta$, $|\mathcal{Z}_k(\mu)| = \mu$.

So we ware in possession of an approximation of $|Z_k|$ when h = $.^a$ if a = 1/2, 2/3, 3/4 ... or if a = 1/3, 3/5, 5/7 ... (Fig. 11.2). The formulas are proved in Appendix 2.
It would certainly be interesting to deepen the study of the behaviour of $|\mathcal{Z}_k(\mu)|$ wrt. a and μ on the basis of Prop. 11.7 and 11.8 and to compare it with the results of Section 10.

Another remarkable property :

(11.9) *When $h'^{\wedge}$ is a continuous function or $h=1_{\mathbb{R}}$: the areas $\mathcal{Z}_k(\mu)$ and $\mathcal{Z}_k(\mu')$ are homothetic wrt. o for any $\mu, \mu' \in \mathbb{R}_+^*$ iff. $h(\delta)$ has the form δ^a or $-\delta^{-a}$ with $a>0$, or $\ln \delta$.*

The proof is in Appendix 3. In those cases, the *shapes* of $\mathcal{Z}_k(\mu)$ and $\mathcal{Z}_j(\mu)$ are thus independent of the value of μ. When point i is a summit of $\mathcal{Z}_k(\mu)$ or an inflexion point of $\mathcal{Z}_j(\mu) \cap \mathcal{Z}_k(\mu)$, the angle φ_o is consequently independent of the dicentral moment μ. This allows us to give a simple characterization of such points, whereas it would be difficult, if not impossible, in the case of the market areas themselves. Either by direct calculation or by taking the limit of formulas met in section 7, we find when $h(\delta) = \delta^a$:

$$\left(\frac{\partial y}{\partial x}\right)_{\Delta h} = \frac{a-1 + tg^2 \varphi_o}{(2-a)\ tg\ \varphi_o}$$

$$\left(\frac{\partial^2 y}{\partial x^2}\right)_{\Delta h} = \frac{(a-1)\ (tg^2 \varphi_o - a+1)}{(2-a)^2\ \delta_o \sin^3 \varphi_o} .$$

The proof is in Appendix 4. Hence we easily deduce :

(11.10) *When $h(\delta) = \delta^a$:*

(i) *if $0 < a < 1$: $\mathcal{Z}_k(\mu)$ has a unique summit s, and $tg\ \varphi_{os} = \sqrt{1-a}$;*

(ii) *if $1 < a \neq 2$: $\mathcal{Z}_j(\mu) \cap \mathcal{Z}_k(\mu) \cap \mathbb{R}_+^2$ has a unique inflexion point s, and $tg\ \varphi_{os} = \sqrt{a-1}$.*

The unicity of the inflexion point is particularly interesting for we have failed so far, regarding $Z_j \cap Z_k$, to prove it as well as to find any example with more than one inflexion point.

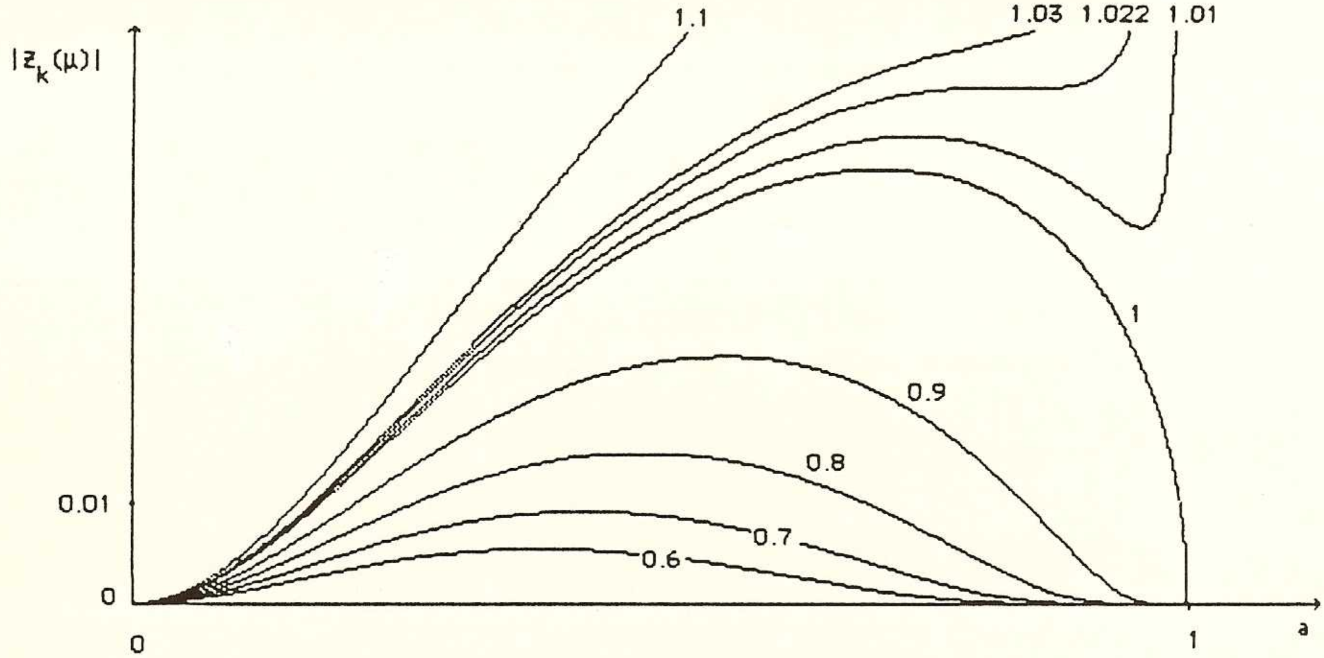


Fig. 11.2a : for values of the dicentral moment smaller than 1 or close to 1 ;

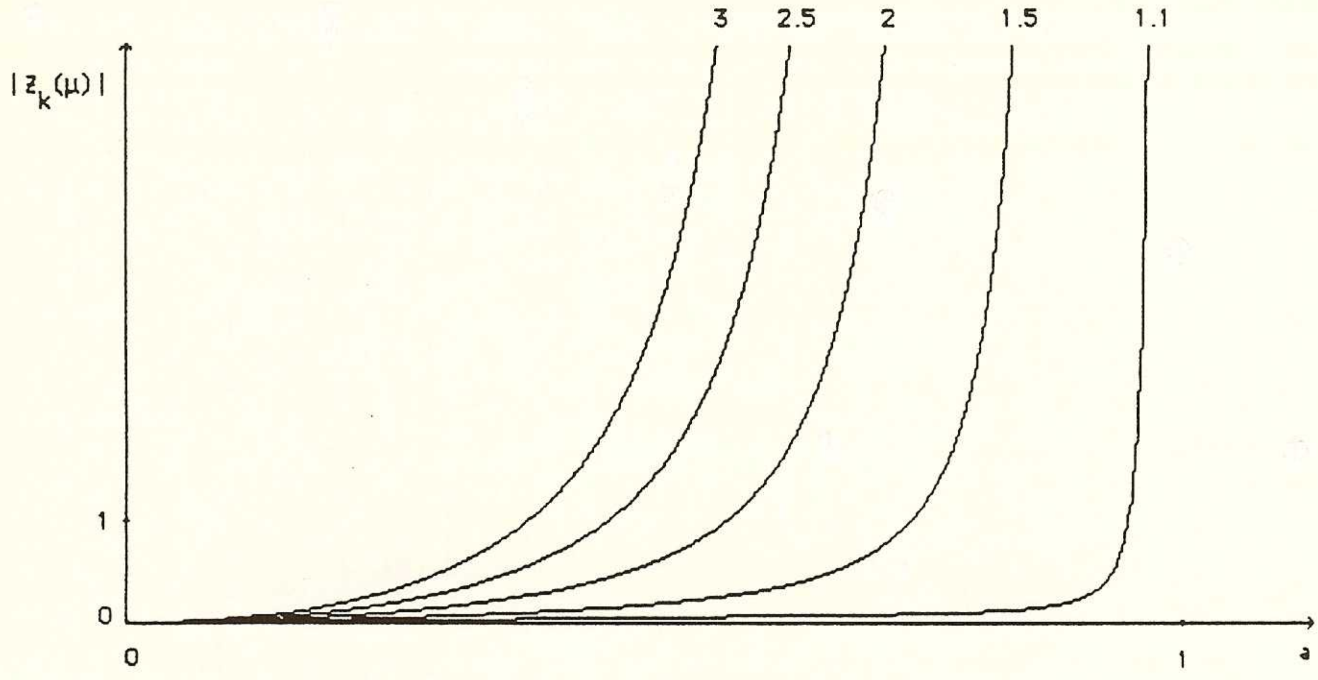


Fig. 11.2b : for values of the dicentral moment larger than 1.

Fig. 11.2. Dependence of the superficies of the dicentral approximation of Z_k on the distance exponent a when the t.c.f. is δ^a. The approximate curves here above respect Prop. 11.8 and the facts -which can be deduced from Prop. 11.7- that $|z_k(0)| = 0$, that $\partial |z_k(\mu)| / \partial a \to 0$ when $a \underset{>}{\to} 0$, and that $|z_k(\mu)| = \mu^{2/(1-a)} |z_k(1)|$. As confirmed by Prop. 11.7, when $a \underset{<}{\to} 1$, $|z_k(\mu)| \to 0$ if $\mu \leq 1$ and $\to \infty$ if $\mu > 1$.

To end this section, let us reexamine the dependence of the superficies of Z_k on Q and δ_{jk}. The convexity or concavity of the integral of Prop. 11.7 wrt. Q is now directly related to that of the function $h'^{\frown 2}$. As the first two derivatives of $h'^{\frown 2}$ are

$$(h'^{\frown 2})'[h'(\delta)] = \frac{2\delta}{h''(\delta)}$$

and thus

$$(h'^{\frown 2})''[h'(\delta)] = \frac{2[h''(\delta) - \delta h'''(\delta)]}{h''^3(\delta)} = -\frac{8\delta^3 (h' \circ \sqrt{.})''(\delta^2)}{h''^3(\delta)},$$

it is clear that when h is concave, $h'^{\frown 2}$ is convex iff $h' \circ \sqrt{.}$ is convex. The conditions that h should be concave and h' convex which appear in Prop. 9.3 are thus sufficient for $h'^{\frown 2}$ to be a convex function. The relation with Prop. 9.4 is more difficult to appreciate. If h is the power function $.^a$ and if we let δ_{jk} tend toward zero in Prop. 9.4, we reach the conclusion that $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$ is concave in Q on the range $[0, \infty[$ for all $a \geqslant 4$. The direct application of Prop. 11.7 shows, however, that it is true for all $a > 3$, as the function $h' \circ \sqrt{.}$ is then convex iff. $0 < a \leqslant 1$ or $a \geqslant 3$ (but the value 3 is excluded by Prop. 9.1b). This is probably an indication that a better statement than Prop. 9.4 could be found.

As to the dependence of the integral of Prop. 11.7 on δ_{jk}, we similarly have

$$[h'^{\frown 2}(1/.)]'[1/h'(\delta)] = -\frac{2\delta h'^2(\delta)}{h''(\delta)}$$

and thus

$$[h'^{\frown 2}(1/.)]''[1/h'(\delta)] = \frac{2h'^3(\delta)[h'(\delta)h''(\delta) + 2\delta h''^2(\delta) - \delta h'(\delta)h'''(\delta)]}{h''^3(\delta)}$$

It seems difficult to derive any simple law from the latter expression, but it is good to write it down. Who knows ? Nevertheless we may, instead of looking for general statements, examine a particular transportation cost function. When h is the power function δ^a, we have $h'^{\frown 2}(1/\xi) = (a\xi)^{2/(1-a)}$, which expression is convex in ξ for all $a > 0$, except when $a = 1$. The measure $|Z_k|$ is thus convex in δ_{jk} if $0 < a < 1$, and $|Z_j \cap \mathbb{R}_+ \times \mathbb{R}|$ is also convex in δ_{jk} if $a > 3$ (see Prop. 9.1b). The complementarity observed between Prop. 9.3 and 9.4 thus disappears in the case of δ_{jk}.

12. *Relaxation of the assumptions about the transportation cost function h*

Three assumptions have been made about h since the beginning of this paper : that h should be increasing, without any plateau (i.e., any maximal open interval on which h is constant), and continuous. From a mathematical standpoint those limitations are a bit frustrating. This is also true for the geographer. The possibility of finding more pleasant to make a short trip than to stay at home, for instance, would deny the first of those properties. The comparison with the Manhattan distance case, on the other hand, rises this question : is it impossible for the demarcation line to degenerate into a demarcation area ? The answer is that it is indeed possible under some conditions when function h possesses plateaux. This seems a reason good enough to put that hypothesis under study. Finally, it will be seen that the release of the continuity assumption is complementary of that latter point, and so finds here its place. Of course, the present section does not aim at building a detailed theory, but rather at delineating the issues through some examples. Only the degeneracy of the demarcation line will be studied formally.

12.1 *Increasingness*

As we now assume that h is not increasing, let us first consider the opposite case, i.e., when h is strictly decreasing. As the inequality of Z_k may be written

$$(-h_k) - (-h_j) \geqslant Q ,$$

the market areas are symmetric wrt. the y-axis of what they would be if the so-called transportation cost function were -h instead of h. If h (δ) is equal to -δ for instance, the market area Z_k is the part of the plane at the left of the left branch of a hyperbola having j and k as foci. Generally speaking, as the function -h is strictly increasing, all the preceding sections can be used *mutatis mutandis* to describe the case of a continuous and strictly decreasing function h.

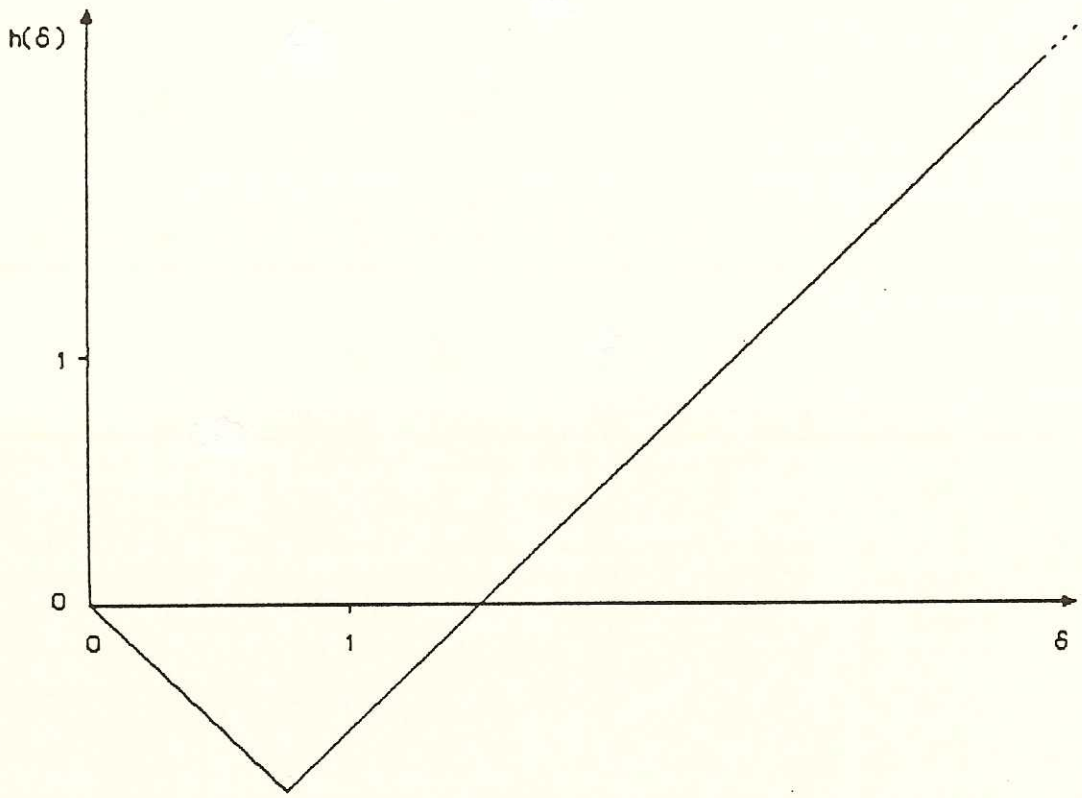


Fig. 12.1a. A nonmonotonic t.c.f. : max { - δ , δ - 1.5}.

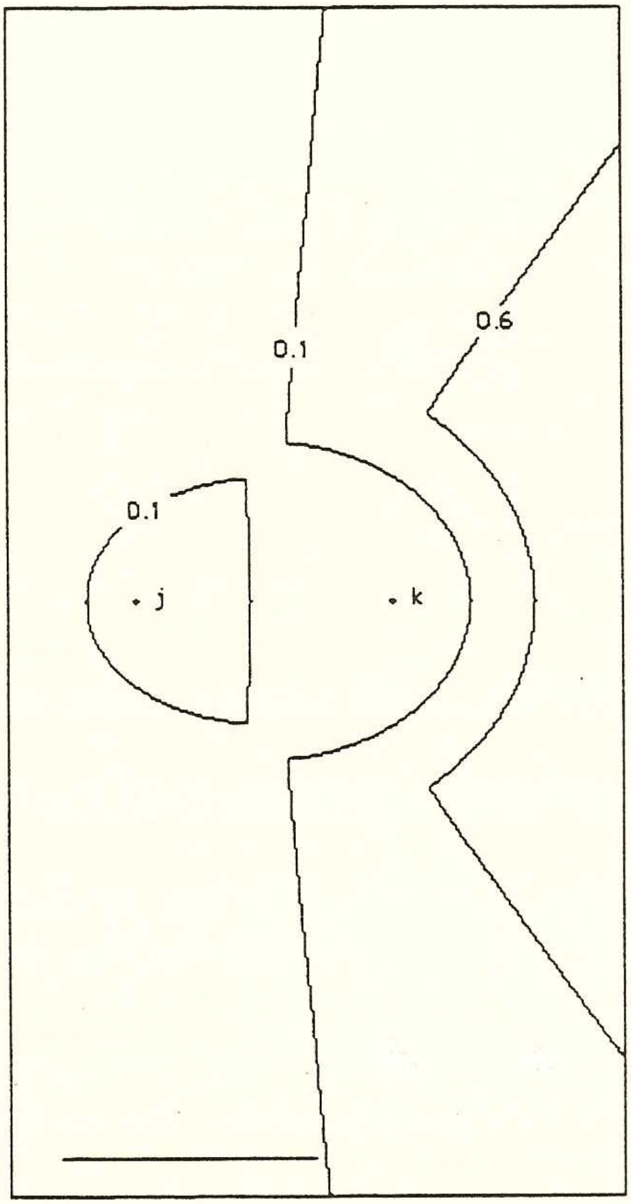


Fig. 12.1b. Market areas generated by the t.c.f. max { - δ, δ - 1.5} when $\delta_{jk} = 1$, for indicated values of Q.

When function h is increasing on some intervals and decreasing on others, we may expect the pattern of market areas to be intermediate between the two genuine cases studied so far. Let us illustrate this by a little example. Suppose that the consumers find it more pleasant to make a short trip, up to a distance δ^*, than to stay at home if they can. More precisely, let us assume that, $\forall \delta$,

$$h(\delta) = \max \{- \delta, \delta - \delta^* \}.$$

Consequently, $h(0) = 0$; then the perceived disutility $h(\delta)$ decreases and is minimal when $\delta = \delta^*/2$; thereafter it increases again and, when δ becomes $> \delta^*$, displacement is considered less useful than not moving. The following results are not difficult to derive under that hypothesis; see Fig. 12.1. If $Q > \delta_{jk}$, everybody goes to centre j : $Z_k = \phi$. If $\delta^* < \delta_{jk}$ and $Q < \delta_{jk} - \delta^*$, Z_k is not empty but is just as it would be if $h(\delta)$ were simply defined as δ. If $\delta^* \leqslant 2\delta_{jk}$ and $|\delta_{jk} - \delta^*| < Q \leqslant \delta_{jk}$, Z_k is the hyperbolic area $\delta_j - \delta_k \geqslant Q$, minus the ellipse $\delta_j + \delta_k < Q + \delta^*$. Now if $Q \leqslant \min \{\delta_{jk}, \delta^* - \delta_{jk}\}$ -which implies that $\delta^* > \delta_{jk}$-, market area Z_k is bipartite, its two unipartite subsets being separated by the y-axis. The subset Z_k^1 of Z_k at the right of that axis is as Z_k in the preceding case, whereas the subset Z_k^2 of Z_k at the left of that axis is the intersection of the hyperbolic area $\delta_k - \delta_j \geqslant Q$ (which is symmetric of the previous one) and the ellipse $\delta_j + \delta_k \leqslant \delta^* - Q$.

Particularly interesting are the facts that we have then both $k \in Z_j$ and $j \in Z_k$, and that Z_k is not arcwise connected. This is an illustration of a general property that only relies on the assumption that h is a function, i.e., that to every distance δ corresponds exactly one value $h(\delta)$:

(12.1) *If the transportation cost relation h is a function and if a point $(x, y) \in Z_k$, then $(-x,y) \notin Z_k$.*

The proof is immediate : $\Delta h_{(x,y)} = -\Delta h_{(-x,y)}$, so that

$$(x,y) \in Z_k \Leftrightarrow \Delta h_{(x,y)} \geqslant Q \Leftrightarrow \Delta h_{(-x,y)} \leqslant - Q \Rightarrow (-x,y) \notin Z_k.$$

In particular, Prop. 12.1 implies that the y-axis is contained in the symmetric difference $Z_j \,\Delta\, Z_k$, and that the pair $\{j,k\}$ is not contained in Z_k.

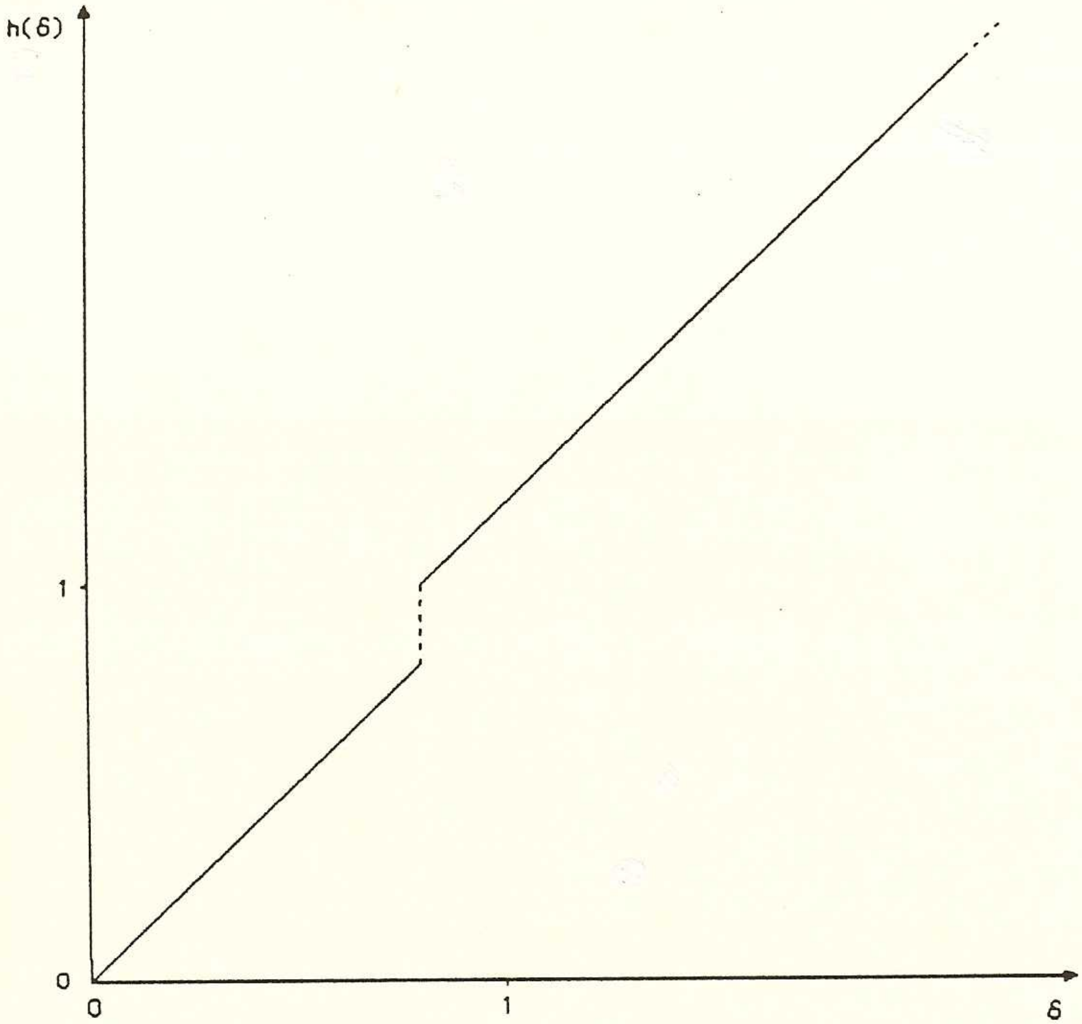


Fig. 12.2a. A noncontinuous t.c.f. : h(δ) is δ when δ < 0.8, and δ + 0.2 when δ ≥ 0.8.

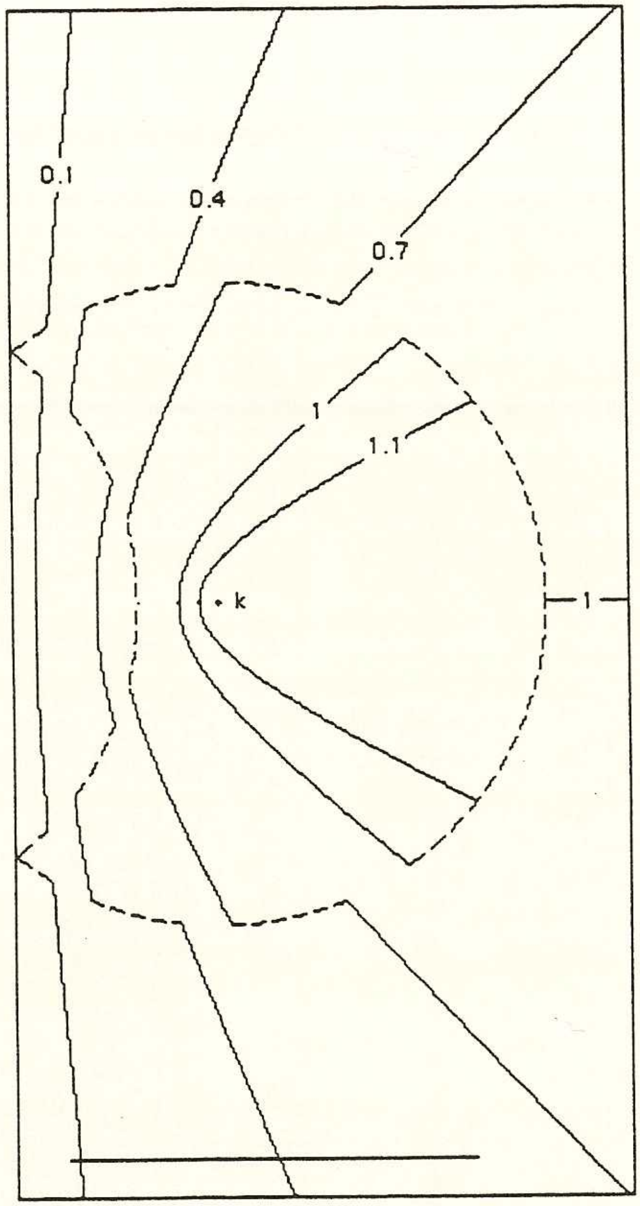


Fig. 12.2b. Market areas generated by a discontinuous t.c.f.; $\delta_{jk} = 1$, indicated values of Q. The t.c.f. is δ when $\delta < 0.8$, and $\delta + 0.2$ when $\delta \geq 0.8$. That part of the common boundary of the market areas which is not a subset of the indifference line has been dotted.

12.2 *Continuity*

The simplest discontinuity happens when h is not continuous at 0 lato sensu (see Section 3). The only points where this may have some effect are the centres j and k, as we have there $\delta_j = 0$ and $\delta_k = 0$, respectively. Two situations may be found, according to whether h (0) is strictly larger than $\lim_{\delta \gtrless 0} h(\delta)$ or strictly smaller. The first one could model for instance a situation similar to that used as an example in the preceding section 12.1 ; the second one may express the presence of indivisibilities in transportation. (In both cases the discontinuity of h when δ=0 may possibly result from the fact that the centres are described as dimensionless points for mathematical convenience, so that h (0) might represent the average transportation cost inside a centre).

We may compare the market areas with what they become when function h is replaced by h*, defined as $h^*(\delta) = h(\delta)$ if $\delta > 0$ and $h^*(0) = \lim_{\delta \gtrless 0} h(\delta)$, if that limit exists. When $h(0) > h^*(0)$, as $\Delta h_k < \Delta h^*_k$ (ie. $\Delta h_j > \Delta h^*_j$) we can see that $k \in Z_k \Rightarrow k \in Z^*_k$ and that $j \in Z_j \Rightarrow j \in Z^*_j$, the converses being not necessarily true. Even if h* is concave and strictly increasing it may so happen that $k \in Z_j$ and $j \in Z_k$. When $h(0) < h^*(0)$, similarly, $k \in Z^*_k \Rightarrow k \in Z_k$ and $j \in Z^*_j \Rightarrow j \in Z_j$: indivisibilities in transportation reinforce the belonging of the centres to their own market area. If h* is strictly increasing, we consequently have that $j \in Z_j$; if moreover h* is concave, $k \in Z_k$ -but Prop. 6.3b applies here directly to function h, because h is concave on $\mathbb{R}_+$ and because the proof of Prop. 6.3b does not require the continuity of h (nor its strict monotonicity).

When function h has a discontinuity elsewhere than at 0, the shape of the market areas may be modified. Such a discontinuity seems likely to happen eg. if some technical constraint obliges to shift from a transportation mode to a more expensive one beyond some threshold distance. Let us study the following example : $h(\delta) = \delta$ if $\delta < \delta^*$, and $h(\delta) = \delta + b$ if $\delta \geqslant \delta^*$ (Fig.12.2). Instead of describing only the indifference line $Z_j \cap Z_k$, we shall consider the common boundary of Z_j and Z_k, which includes $Z_j \cap Z_k$. It can easily be seen that this boundary is the indifference line $\tilde{Z}_j \cap \tilde{Z}_k$, where function h is replaced by the following binary relation $\tilde{h}$:

$$\forall\ \delta \neq \delta^*: \delta\ \tilde{h}\ \xi \Leftrightarrow \xi = h(\delta) \ ,$$

$$\delta^*\ \tilde{h}\ \xi \Leftrightarrow \xi \in [\delta^*, \delta^* + b] .$$

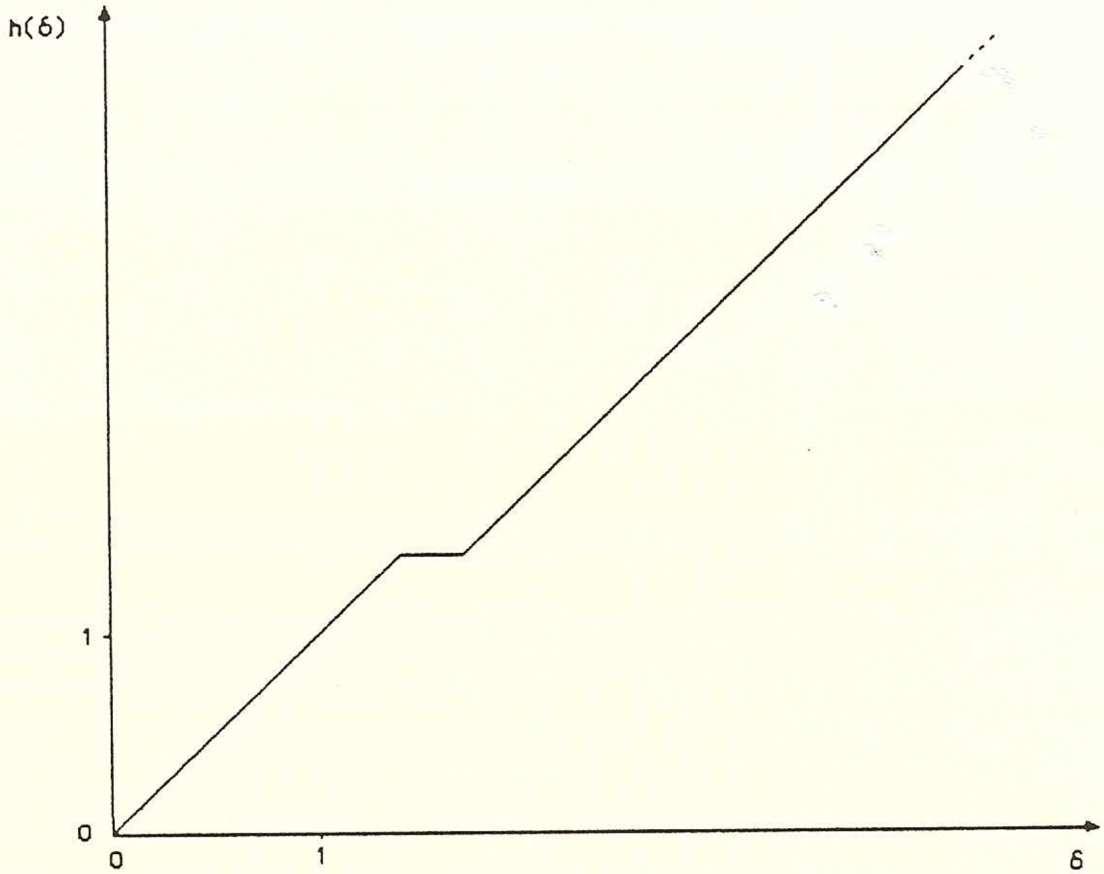


Fig. 12.3a. A t.c.f. with one plateau : min {δ , max{1.4, δ - 0.3}} .

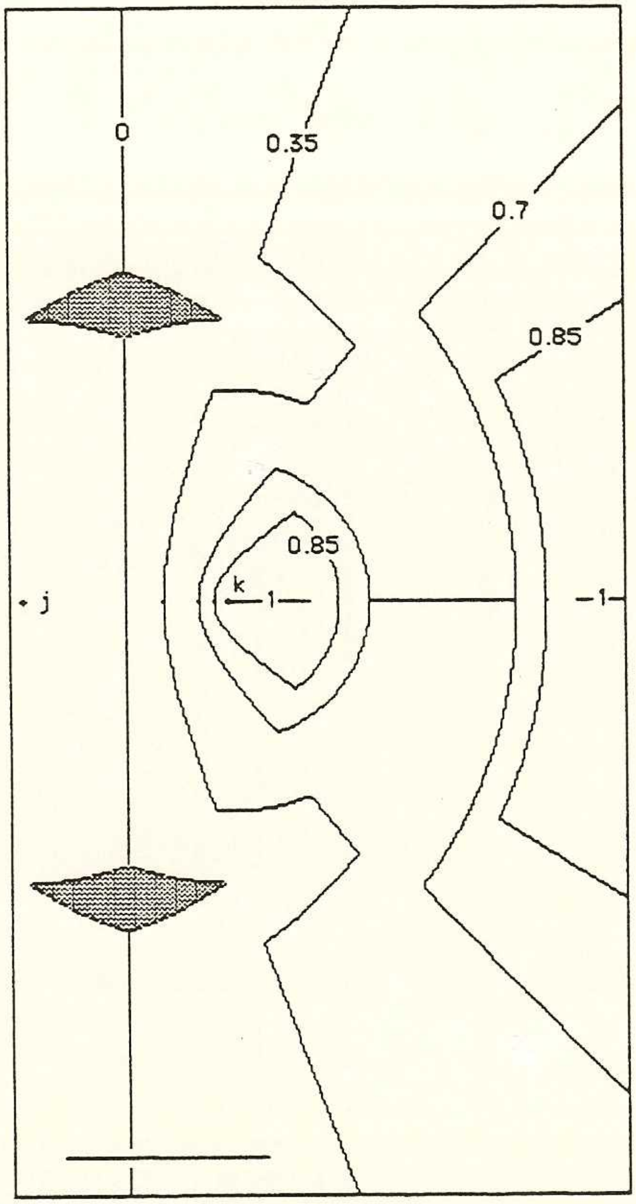


Fig. 12.3b. Market areas generated by the t.c.f min {δ, max{1.4, δ − 0.3}} when $\delta_{jk} = 1$, for indicated values of Q. Two-dimensional subsets appear in the indifference zone if Q = 0. The market area of centre k is partly one-dimensional if Q = 0.7, and completely if Q = 1.

Suppose that point ℓ, the point of $Z_j \cap Z_k$ closest to j and k, is such that $\delta_{j\ell} < \delta^*$. Starting from that point we move inside $\tilde{Z}_j \cap \tilde{Z}_k$ above the x-axis so as to increase δ_j and δ_k. At the beginning, $\tilde{Z}_j \cap \tilde{Z}_k$ coincides with the hyperbola $\delta_j - \delta_k = Q$. When δ_j reaches δ^*, δ_j stops increasing, contrarily to δ_k. This means that we are now progressing anticlockwise on a circle of radius δ^* and centred on j. Thereafter we have again to increase δ_j ; if δ_k has not yet reached δ^*, as in Fig. 12.2b (Q = 0.4), δ_k increases too and we are on the hyperbola $\delta_j - \delta_k = Q - b$. Then δ_k reaches δ^* and stops, whereas δ_j still increases : we are on a circle centred on k, with radius δ^*. Finally δ_j and δ_k increase simultaneously again and we are back to the first hyperbola.

Although this has not been discussed, as being of minor interest, Fig.12.2b makes a distinction between $\tilde{Z}_j \cap \tilde{Z}_k$ and $Z_j \cap Z_k$. Of course nothing special would have happened if $\delta_{k\ell}$ had been $\geq \delta^*$: $Z_j \cap Z_k$ would have been equal to the hyperbola $\delta_j - \delta_k = Q$. It can be deduced from the triangular inequations (9) that the discontinuity affects the market areas iff. $b < \delta_{jk} < 2\delta^* + b$. This is perhaps the first point to retain from this example : discontinuities do not necessarily modify the shape of market areas. The second one is that their possible effect, if function h is increasing, is to produce excrescences on market area Z_k.

12.3 *Absence of plateaux*

Let us first consider an example where function h has a single plateau (Fig.12.3) :

$$
\begin{aligned}
h(\delta) &= \delta && \text{if } \delta \leq \delta_{(1)} \\
&= \delta_{(1)} && \text{if } \delta_{(1)} \leq \delta \leq \delta_{(2)} \\
&= \delta_{(1)} + \delta - \delta_{(2)} && \text{if } \delta \geq \delta_{(2)}
\end{aligned}
\quad .
$$

A reasoning similar to that hold in Section 12.2 shows that the presence of a plateau has somewhat the effect of carving market area Z_k if function h is increasing. The cleft may be deep enough to cut off a piece of Z_k. Also, the plateau does not necessarily bear an effect on Z_k. In this particular problem it could be shown that the phenomenon appears iff. $Q \leq \min \{\delta_{jk}, \delta_{(1)}, \delta_{(1)} + \delta_{(2)} - \delta_{jk}\}$ or $\delta_{jk} - 2\delta_{(2)} \leq Q \leq \delta_{jk}$.

We now come to the issue of the possible total or partial degeneracy of $Z_j \cap Z_k$ into an area. Such an areal degeneracy means that some vicinity of some point i' of $Z_j \cap Z_k$ is contained in $Z_j \cap Z_k$. Consequently it must be possible to let δ_j vary in a vicinity of $\delta_{ji'}$, while keeping δ_k constant, without

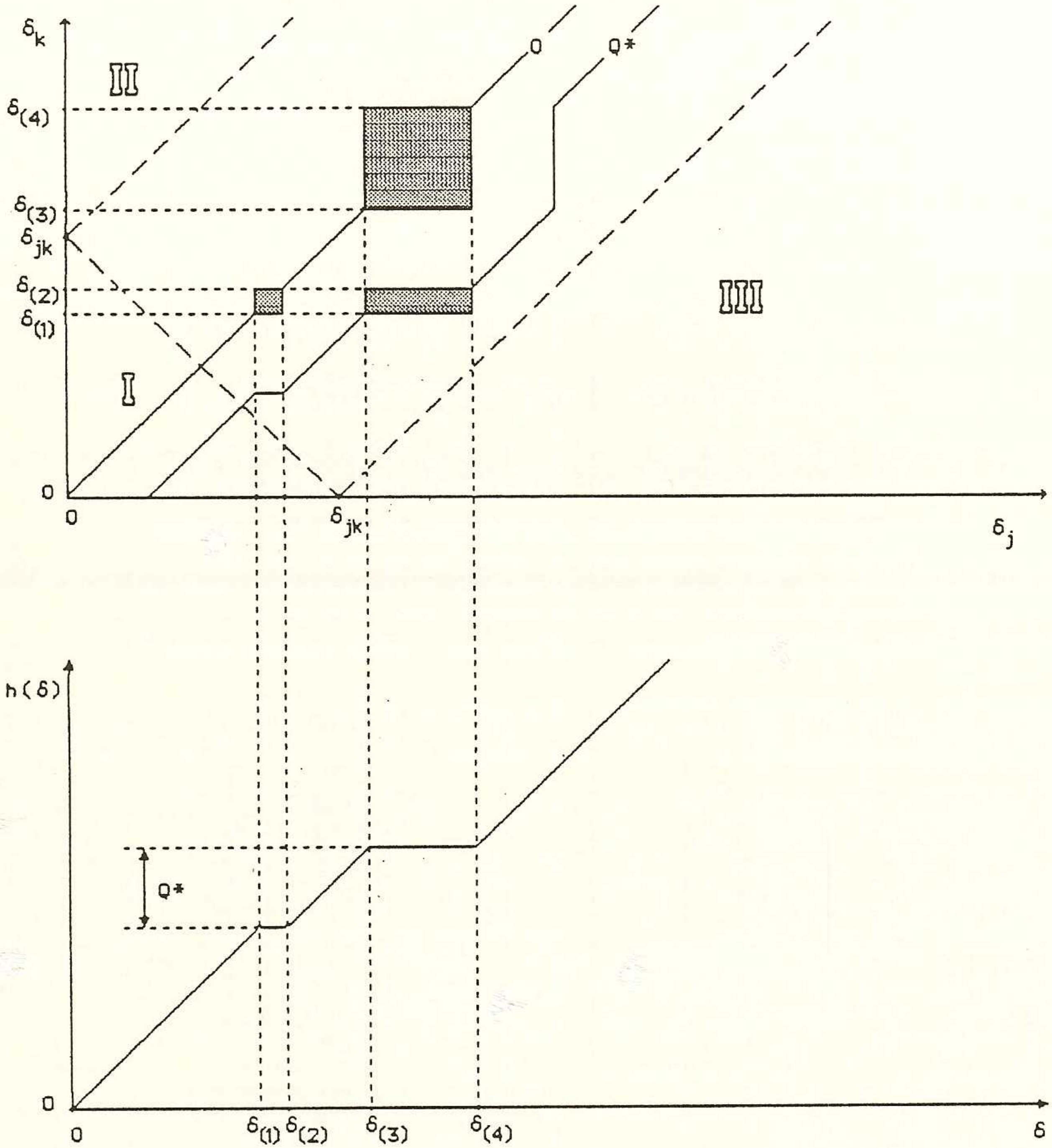


Fig. 12.4 a and b. If the t.c.f. presents two plateaux, the indifference zone may be partly two-dimensional when $Q = 0$ and when Q is equal the difference $Q*$ between the values taken by the t.c.f. on the plateaux. That property is studied here in the coordinates (δ_j , δ_k) for the case depicted in Fig. 12.4c.

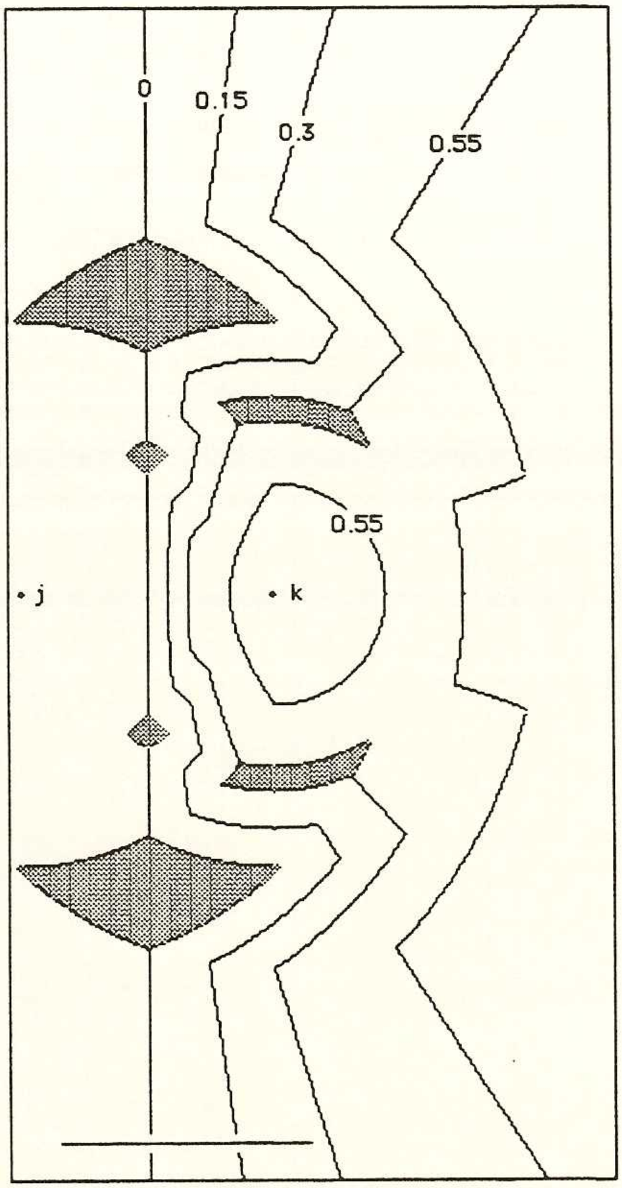


Fig. 12.4c. Market areas generated by the t.c.f. min {δ, max {0.7, δ - 0.1}, max {1, δ - 0.5}}, which has two plateaux; δ_{jk} = 1, indicated values of Q. The indifference zone is partly two-dimensional when Q = 0 and when Q = 0.3.

modifying Δh. This means that h is constant in that vicinity of $\delta_{ji'}$. Similarly h must be constant in a vicinity of $\delta_{ki'}$. Function h thus has at least two plateaux if $Q \neq 0$, or one if $Q = 0$.

Suppose now conversely that h possesses at least two plateaux $]\delta_{(1)}, \delta_{(2)}[$ and $]\delta_{(3)}, \delta_{(4)}[$, h being higher on the second plateau than on the first one (we allow here h to be discontinuous, non-increasing or even not to be a function). This will entail an areal degeneracy of $Z_j \cap Z_k$ iff. two conditions are verified : 1° that Q be precisely equal to the difference in h between the two plateaux ; and 2° that there be some δ_k in $]\delta_{(1)}, \delta_{(2)}[$ and some δ_j in $]\delta_{(3)}, \delta_{(4)}[$ verifying the strict triangular inequalities corresponding to (9). Eliminating δ_k and δ_j, or considering the issue in the coordinates (δ_j, δ_k) and expressing that the rectangle $]\delta_{(3)}, \delta_{(4)}[\times]\delta_{(1)}, \delta_{(2)}[$ is not completely contained in one of the three regions I, II and III (see Fig.12.4a), we find what follows :

(12.2) *The interior $(Z_j \cap Z_k)^\circ$ of the indifference zone is not empty iff. the transportation cost relation h possesses at least two plateaux $]\delta_{(1)}, \delta_{(2)}[$ and $]\delta_{(3)}, \delta_{(4)}[$ and these inequalities are satisfied :*

$$\delta_{(1)} - \delta_{(4)} < \delta_{jk}$$
$$\delta_{(3)} - \delta_{(2)} < \delta_{jk}$$
$$\delta_{(2)} + \delta_{(4)} > \delta_{jk} \quad .$$

The zone $Z_j \cap Z_k$ then contains the 2-dimensional set of points

$$[C_j^\circ(\delta_{(4)}) - C_j(\delta_{(3)})] \cap [C_k^\circ(\delta_{(2)}) - C_k(\delta_{(1)})].$$

Of course, such a degeneracy essentially means a *discontinuity* in the evolution of Z_j and Z_k wrt. Q, as Z_k is decreasing wrt. Q whatever the properties of h. Fig.12.4 displays a case of partial areal degeneracy of $Z_j \cap Z_k$, when h is the identity function with insertion of two plateaux.

Even with such a simple function the set $Z_j \cap Z_k$ may become wholly 2-dimensional in three ways (Fig.12.5); the reason for this immediately appears if we look at the problem through the coordinates (δ_j, δ_k) (Fig.12.6). This result can be generalized as follows :

Fig. 12.5. If the t.c.f. is endowed with two plateaux, the indifference zone may have in three different ways a two-dimensional arcwise connected component when Q is equal to the difference between the values taken by the t.c.f. on the two plateaux. The indifference zone may also have two-dimensional subsets when Q = 0, but is then necessarily arcwise connected.

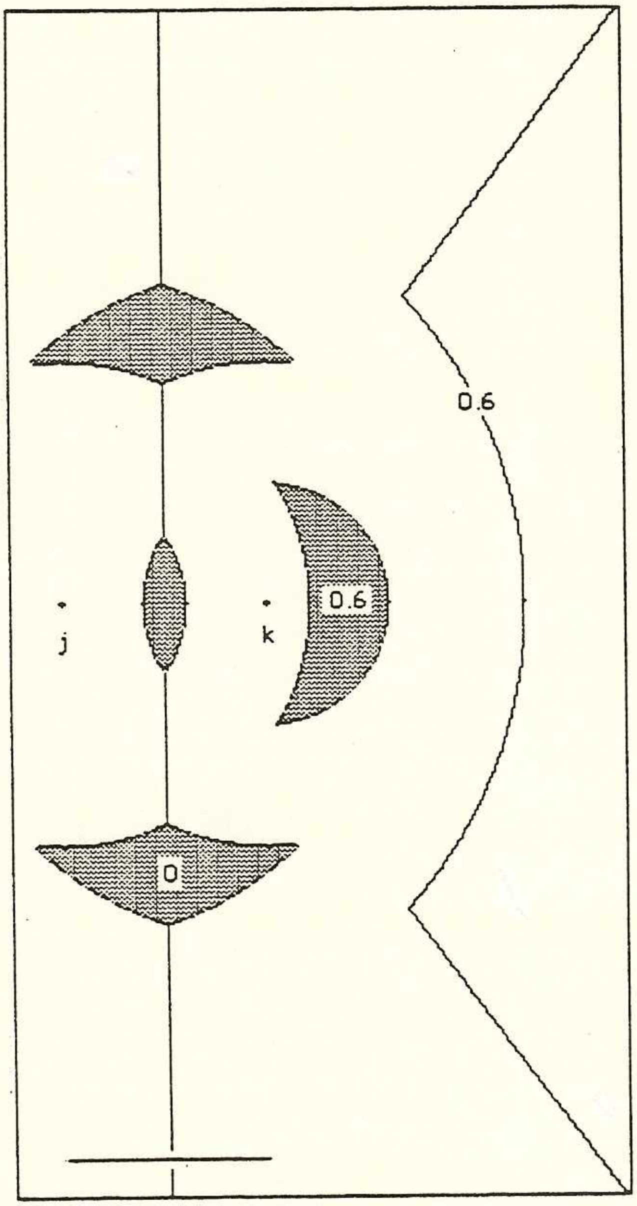


Fig. 12.5a. Market areas generated by the t.c.f. max { min {δ, 0.1}, min {δ - 0.5, 0.7}, δ - 0.95} if δ_{jk}=1, for indicated values of Q. When Q = 0.6, the indifference zone has a crescent-shaped arcwise connected component.

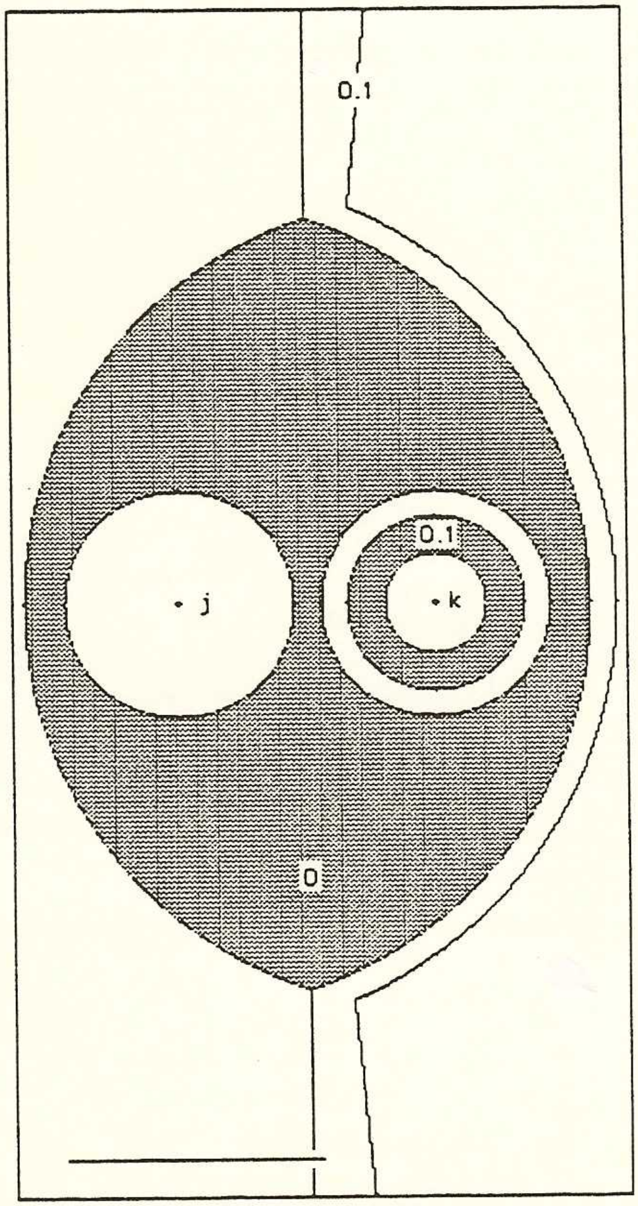


Fig. 12.5b. Market areas generated by the t.c.f. max { min {δ, 0.2}, min {δ - 0.15, 0.3 }, δ - 1.3} if δ_{jk}=1, for indicated values of Q. When Q = 0.1, the indifference zone has a crown-shaped arcwise connected component.

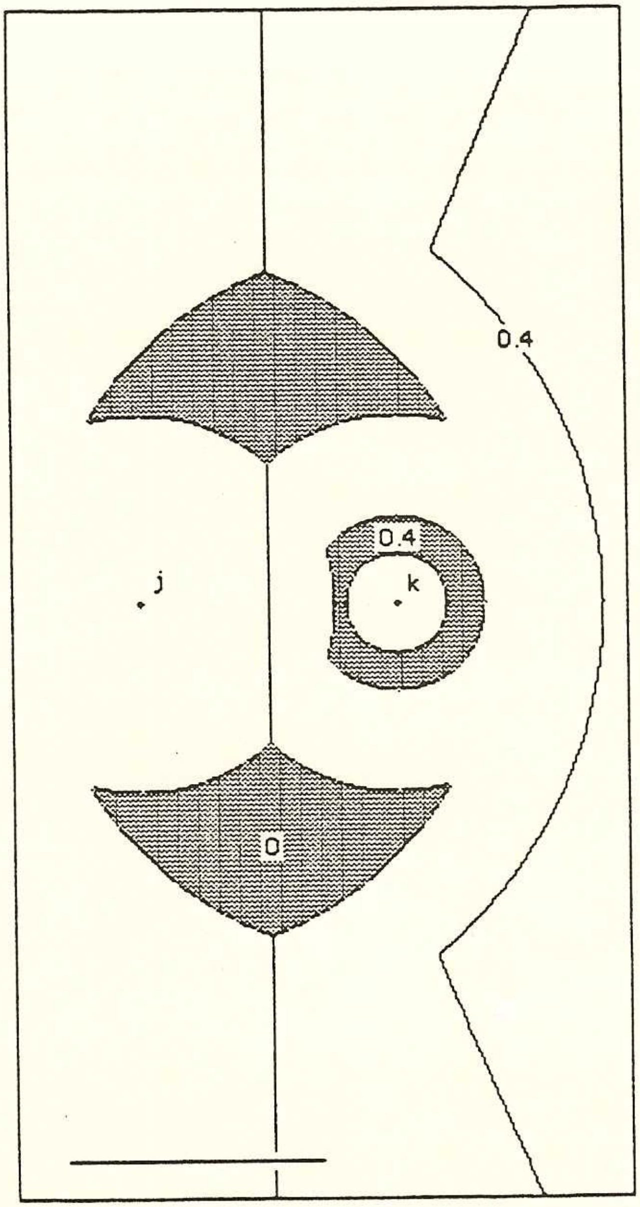


Fig. 12.5c. Market areas generated by the t.c.f. max { min {δ, 0.2}, min {δ - 0.15, 0.6}, δ - 0.8 } if $\delta_{jk}=1$, for indicated values of Q. When Q = 0.4, the indifference zone has an arcwise connected component which is a crescent with a hole.

Fig. 12.6. The indifference zones of Fig. 12.5 as they appear in coordinates (δ_j, δ_k).

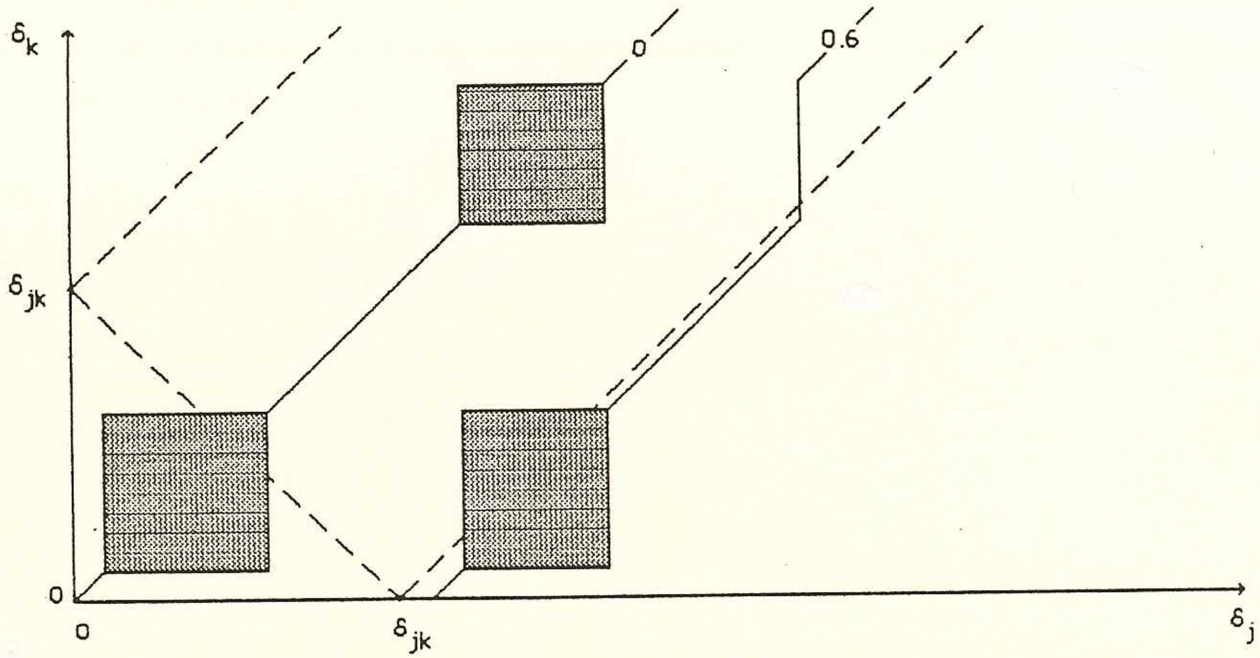


Fig. 12.6a. The indifference zones of Fig. 12.5a as seen through coordinates (δ_j, δ_k).

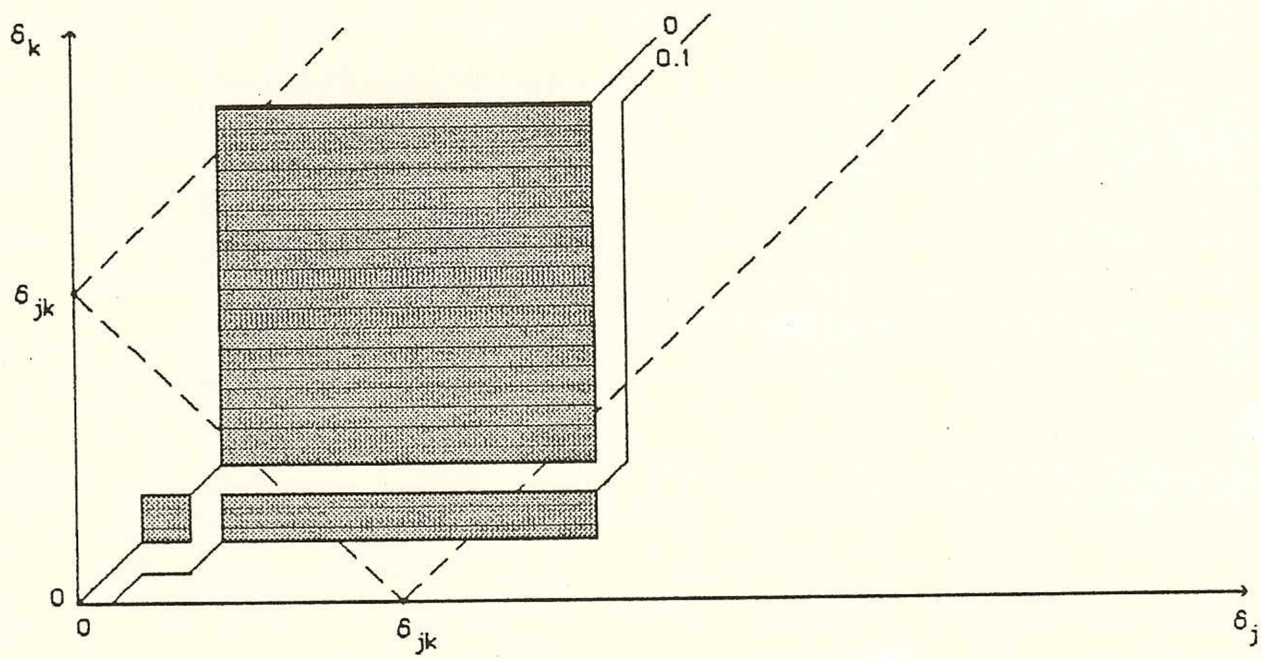


Fig. 12.6b. The indifference zones of Fig. 12.5b as seen through coordinates (δ_j, δ_k).

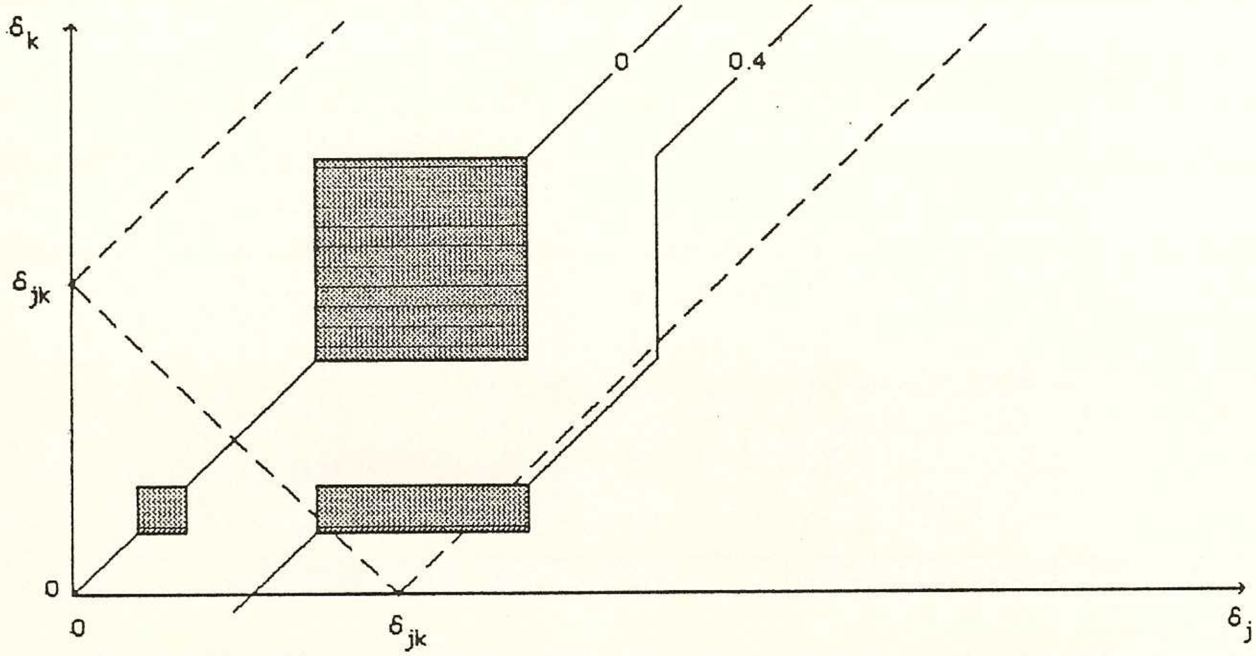


Fig. 12.6c. The indifference zones of Fig. 12.5c as seen through coordinates (δ_j, δ_k).

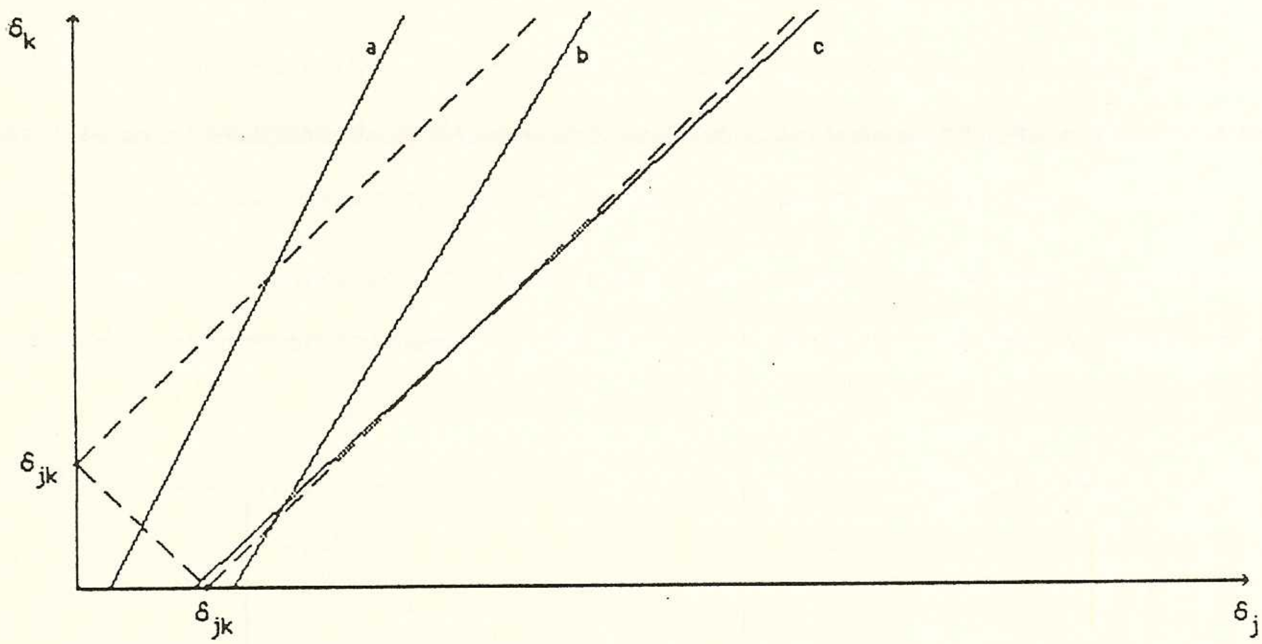


Fig. 13.1. Three Cartesian ovals seen through coordinates (δ_j, δ_k). The transportation rate r_j is equal to 2, and $\delta_{jk} = 1$.

(a) $r_k = 1$, $Q = 0.5$; (b) $r_k = 1.2$, $Q = 2.4$; (c) $r_k = 2.1$, $Q = 1.8$.

(12.3) *When the transportation cost relation h is an increasing function having at least two plateaux* $[\delta_{(1)}, \delta_{(2)}]$ *and* $[\delta_{(3)}, \delta_{(4)}]$ *with* $\delta_{(2)} < \delta_{(3)}$, *the area*

$$[C_j(\delta_{(4)}) - C_j^{c}(\delta_{(3)})] \cap [C_k(\delta_{(2)}) - C_k^{\circ}(\delta_{(1)})]$$

is a 2-dimensional arcwise connected component of the demarcation area $Z_j \cap Z_k$ *if* $\delta_{(4)} - \delta_{(2)} > \delta_{jk}$ *and one of those conditions is satisfied :*

(i) $\delta_{(3)} - \delta_{(1)} > \delta_{jk}$ and $\delta_{(3)} - \delta_{(2)} < \delta_{jk}$;

or (ii) $\delta_{(3)} + \delta_{(1)} < \delta_{jk}$.

If (i) holds, that component is a crescent ;

if (ii) holds and $\delta_{(1)} > 0$, *it encompasses an arcwise connected circular component of* Z_k, *and is either a crescent with a hole or a ring according as* $\delta_{(3)} + \delta_{(2)}$ *is* $> \delta_{jk}$ *or* $\leq \delta_{jk}$.

13. *Descartes'ovals*

As we have shown in Section 2.2, the study of Descartes'ovals has its place in the present work. We shall not use, however, the transformations that allow us to fit them into our model. We shall rather examine them directly and briefly from various points of view adopted in Sections 5, 6 and 8. The arguments are given in a condensed style, as they more or less repeat developments already met in those sections. We also seize this opportunity to come back to the proofs of some properties in a different but equally valid perspective.

13.1 *Dipolar and squared-dipolar coordinates*

Let us look at Fig.13.1 which displays in the dipolar coordinates (δ_j, δ_k) some instances of Cartesian ovals. Each of those ovals is the limit between an area Z_k, defined by the inequality

$$\Delta h = r_j\ \delta_j - r_k\ \delta_k \geqslant Q$$

(where $r_j \neq r_k$), and the corresponding area Z_j. Reasoning as we have done in Sections 5.5. and 6.4, we see these four properties appear :

(13.1) *Z_j and Z_k are arcwise connected and, when bounded, simply connected.*

(13.2) *Z_j is bounded $\Leftrightarrow$ Z_k is not bounded $\Leftrightarrow$ $r_j > r_k$; and $Z_k \subset A\ (r_k/r_j)$.*

(13.3) (a) *Z_j is not empty ;*

(b) *Z_k is empty iff. $Q > r_j\ \delta_{jk}$ (ie. $k \notin Z_k$) and $r_j < r_k$.*

(13.4) (a) *$j \in Z_j$;*

(b) *$k \in Z_k$ iff. $Q \leqslant r_j\ \ \delta_{jk}$; $k \notin Z_k \neq \phi$ iff. $Q > r_j\ \delta_{jk}$ and $r_j > r_k$;*

(c) *$o \in Z_k$ $\Rightarrow$ $Q < r_j\ \delta_{ok}$ and $r_j > r_k$.*

Those properties are obvious once it has been noticed that the slope of the $1_{\mathbb{R}}$ - straight line $Z_j \cap Z_k$ is the ratio r_j/r_k in the coordinates (δ_j, δ_k) ; except for the second part of Prop. 13.2, which is simply deduced from the definition of Z_k and from the fact that $Q > 0$.

This dipolar representation allows a first classification of Descartes' ovals into four types :

(i) $Q < r_j \delta_{jk}$ and $r_j > r_k$: Z_j is bounded, $k \in Z_k - Z_j$; and $o \in Z_j$ iff. $Q \geq (r_j - r_k)\, \delta_{ok}$;

(ii) $Q < r_j \delta_{jk}$ and $r_j < r_k$: Z_k is bounded, $k \in Z_k - Z_j$, $o \in Z_j$;

(iii) $Q \geq r_j \delta_{jk}$ and $r_j > r_k$: Z_j is bounded, $[jk] \subset Z_j$;

(iv) $Q \geq r_j \delta_{jk}$ and $r_j < r_k$: $Z_k = \phi$.

Consider now the coordinates (δ_j^2, δ_k^2) (Fig. 13.2). As the equation of $Z_j \cap Z_k$ may be written

$$\delta_k^2 = (r_j^2 \, \delta_j^2 + Q^2 - 2Qr_j \sqrt{\delta_j^2})/r_k^2,$$

we see that δ_k^2 is a strictly convex function of δ_j^2 on that curve, the slope of which is, wrt. (δ_j^2, δ_k^2) :

$$\left(\frac{\partial \delta_k^2}{\partial \delta_j^2} \right)_{\Delta h} = \frac{r_j \delta_k}{r_k \delta_j} \quad .$$

As we have mentioned in Section 6.4, the disk C_i at any point $i \in Z_j \cap Z_k$ then contains Z_j when that ratio is > 1, or is contained in Z_k when it is < 1. Market area Z_j is bounded, of course, in the first case ; conversely Z_k is bounded iff. the slope $(\partial \delta_k^2 / \partial \delta_j^2)_{\Delta h}$ is < 1 for any i of $Z_j \cap Z_k$. Also, if that slope is > 1 at the point of $Z_j \cap Z_k$ closest to centres j and k and that we shall call ℓ, then it is > 1 too at any other point of $Z_j \cap Z_k$. Consequently, the situation where $Z_j \subset C_i$ $\forall i \in Z_j \cap Z_k$ and that where $C_i \subset Z_k$ $\forall i \in Z_j \cap Z_k$ are easily characterized :

(13.5) (a) *If* $r_j < r_k$, *then* $C_i \subset Z_k$, $\forall i \in Z_j \cap Z_k$;

(b) *If* $r_j > r_k$ *and* $Q > (r_j + r_k)\, \delta_{jk}$ [*ie.* $A(r_j/r_k) \cap Z_k = \phi$, *or if* $r_j > r_k$ *and* $Q < (r_j - r_k)\, \delta_{jk}$ [*ie.* $(0, \delta_{ok}\sqrt{3}) \notin Z_j$, *which point is the summit of the equilateral triangle having* $[jk]$ *as basis ;i.e.,* $A(r_j/r_k) \cap Z_j = \phi]$,

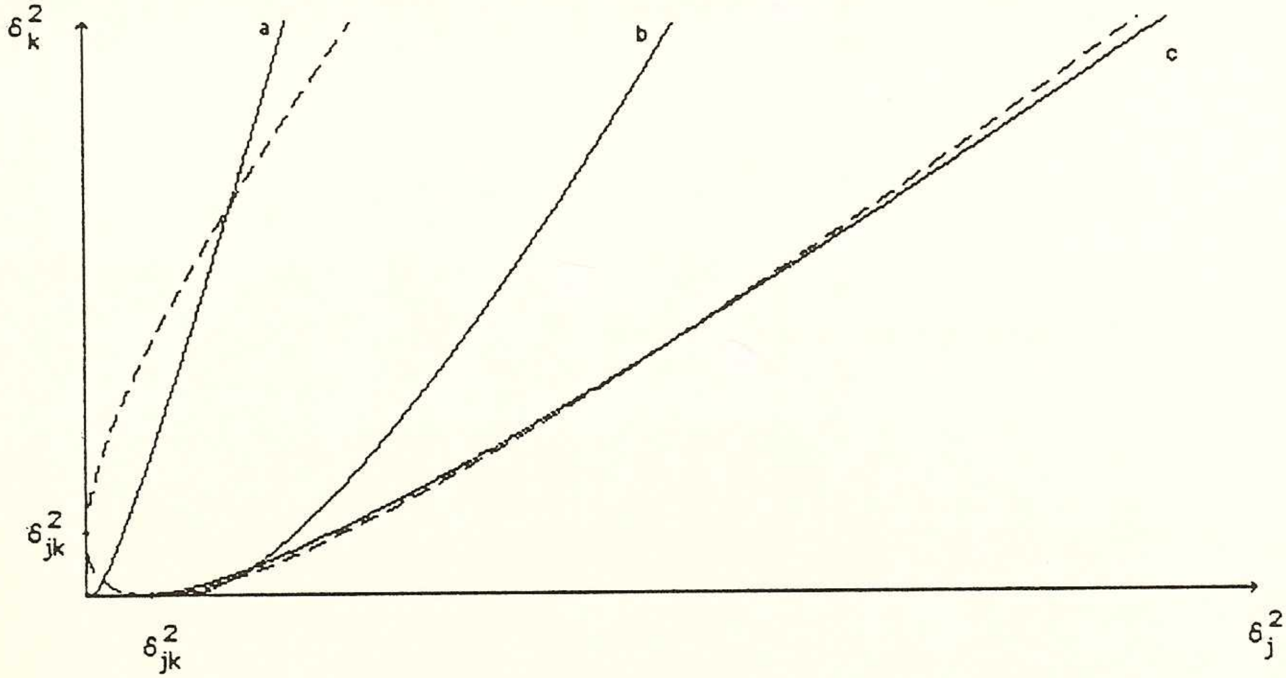


Fig. 13.2a. The Cartesian ovals of Fig. 13.1, now seen through coordinates (δ_j^2, δ_k^2).

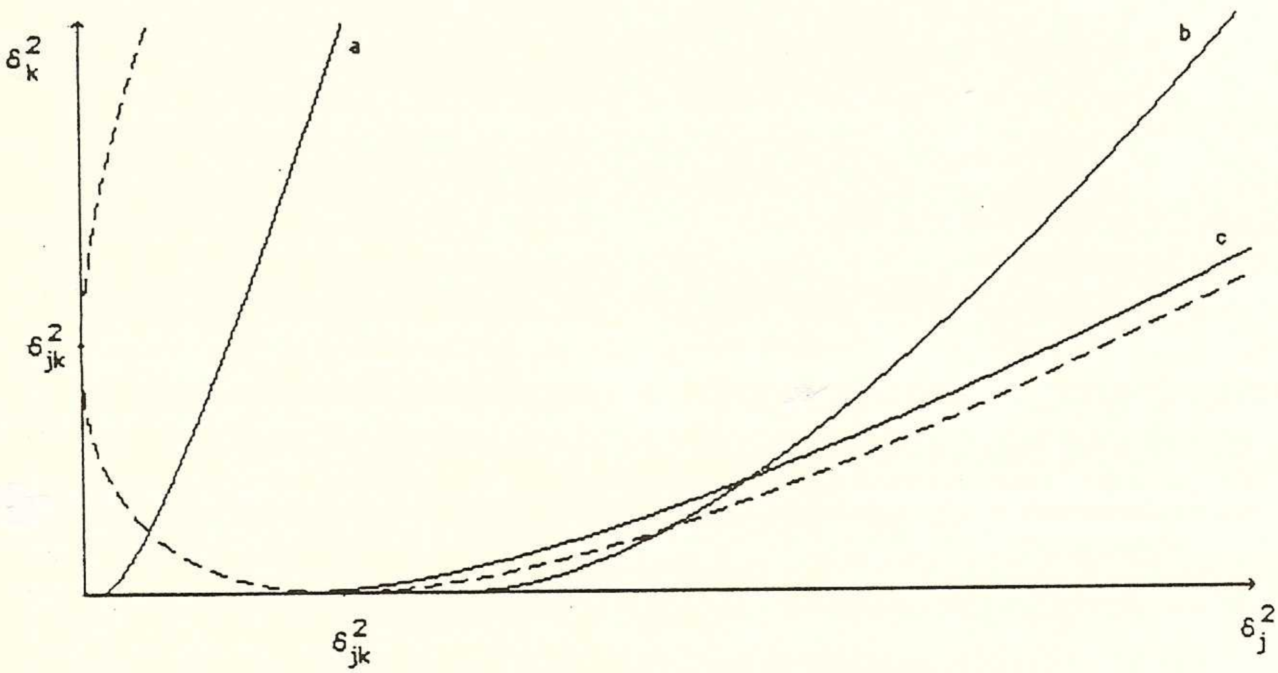


Fig. 13.2b. The same figure with another scale. The cause of the nonconvexity of Z_j in (b) can be clearly seen : the slope of the oval in those coordinates is < 1 at its intersection with the (real) x-axis, and becomes > 1 farther.

then $Z_j \subset C_i$, $\forall\, i \in Z_j \cap Z_k$.

Item (a) is obvious, as Prop. 13.2 tells us that Z_k is bounded iff. $r_j < r_k$. As to item (b), we have to make a distinction between two cases : $Q > r_j\ \delta_{jk}$ (or $k \notin Z_k$) and $Q < r_j\ \delta_{jk}$ ($k \notin Z_j$). Point ℓ can indeed be determined by the equation of $Z_j \cap Z_k$ and by the equality $\delta_{j\ell} - \delta_{k\ell} = \delta_{jk}$ in the first case or $\delta_{j\ell} + \delta_{k\ell} = \delta_{jk}$ in the second. The condition $r_j\ \delta_{k\ell} > r_k\ \delta_{j\ell}$ so becomes $Q > \delta_{jk}\ (r_j + r_k)$ or $Q < \delta_{jk}\ (r_j - r_k)$, respectively ; hence the result. The conditions $Q < (r_j - r_k)\ \delta_{jk}$ and $A\ (r_j/r_k) \cap Z_j = \phi$ are equivalent because the value of Δh at the point p of the boundary of that Apollonian disk which also belongs to [jk] is $(r_j - r_k)\ \delta_{jk}$. Consequently, if $(r_j - r_k)\ \delta_{jk} \leqslant Q$, then $p \in A\ (r_j/r_k) \cap Z_j \neq \phi$. If $(r_j - r_k)\ \delta_{jk} > Q$, then $p \notin Z_j$; as p is the point of $A\ (r_j/r_k)$ closest to j, the circle of $A\ (r_j/r_k)$ -on which $\Delta h = \delta_j (r_j^2 - r_k^2)/r_j$- does not intersect Z_j ; and as $j \in Z_j - A\ (r_j/r_k)$ [Prop.13.4] and $k \in A\ (r_j/r_k)$, none of the sets Z_j and $A\ (r_j/r_k)$ contains the other : we have thus $Z_j \cap A\ (r_j/r_k) = \phi$. The equivalence between '$Q > \delta_{jk}(r_j + r_k)$' and '$A(r_j/r_k) \cap Z_k = \phi$' is similar.

As we have seen in Section 6.4, Prop. 13.5 has remarkable consequences. In particular : if $r_j < r_k$, we have $L_x > L_y$; if $r_j > r_k$ and if $Q \geqslant (r_j+r_k)\delta_{jk}$ or $\leqslant (r_j - r_k)\ \delta_{jk}$, Z_j is convex and its projections L_{xj} and L_{yj} on the axes of x and y are such that $L_{xj} < L_{yj}$ (see Fig. 13.3a).

When we face the situation where $r_j > r_k$ and $(r_j - r_k)\ \delta_{jk} < Q < (r_j+r_k)\delta_{jk}$, it is easy to prove that Z_j is not spined by the x-axis by considering the intersection of Z_j with vertical lines, i.e., straight lines with slope 1 in the coordinates (δ_j^2, δ_k^2). Although Z_j is thus not convex, something remains of Prop. 13.5. If we define the *x-hull* of any set E as

$$x\text{-hull}\ (E) = \{i\ ;\ \exists\ i' \in E\ :\ i \in [\,i'\pi_x(i')\,]\},$$

where $\pi_x(i')$ is the orthogonal projection of i' on the x-axis, the coordinates (δ_j^2, δ_k^2) show that, whatever the value of Q,

(13.6) *For any* i *being at the same time on the boundaries of* Z_j *and of its x-hull, we have, if* C_i *exists at* i *and* $r_j > r_k$, *x-hull* $(Z_j) \subset C_i$.

It is then clear that, when $r_j > r_k$, the x-hull of Z_j is convex even if Z_j is not,

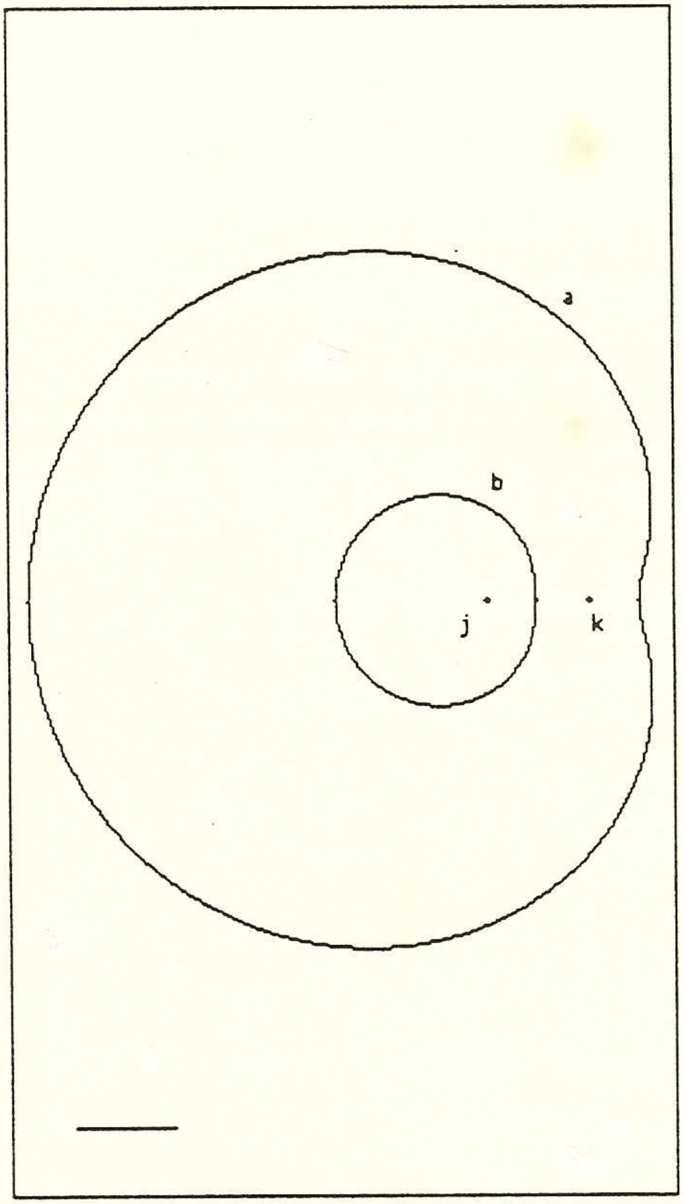


Fig. 13.3a. The Cartesian ovals (a) and (b) of Fig. 13.1 and 13.2. Both enclose the corresponding market areas Z_j, which are more extended in the y- than in the x-direction.

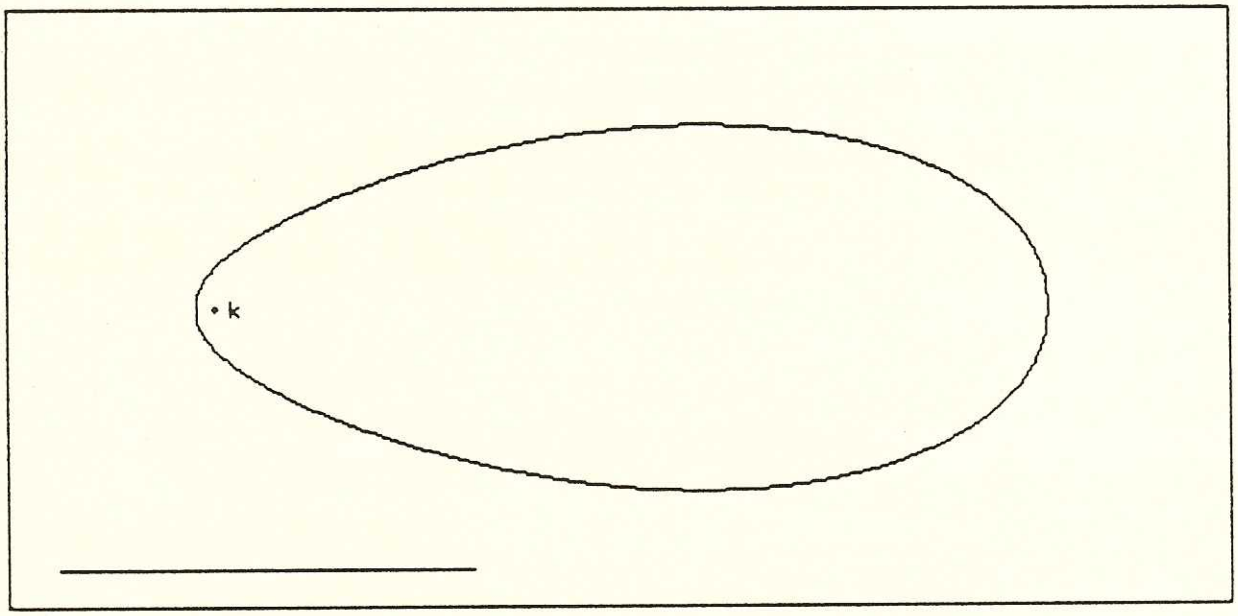


Fig. 13.3b. The Cartesian oval (c) of Fig. 13.1 and 13.2. It encloses market area Z_k, which is more extended in the x- than in the y-direction.
The same market areas appear with the t.c.f. ln(δ +18) if Q = ln 1.05.

and that the ratio L_{yj}/L_{xj}, which is the same for Z_j and its x-hull, remains > 1.

13.2 *Convexity and starshapedness*

We are already able to deduce from Prop. 8.6b and from the discussion about Prop. 13.5 the conditions under which Z_j is convex when $r_j > r_k$. We now venture into the study of the convexity of Z_k when $r_k > r_j$ (Fig. 13.4). It is perfectly possible, in the case of Descartes' ovals, to express $|y|$ as a function of x. But that expression and its first and second derivatives are so intricate that it seems preferable to follow another and certainly more instructive way.

Considering Z_k in the coordinates (δ_j^2, δ_k^2) shows that, r_k being $> r_j$, Z_k is spined by the x-axis. As it is also symmetric wrt. that axis, its convexity is thus equivalent to that of $Z_k \cap \mathbb{R}_+ \times \mathbb{R}$, above the x-axis. That last property might result from the concavity of Δh wrt. (x,y) above the x-axis. That concavity obtains if, for any point v of the x-axis and on any half-line originating in v, we have $\partial^2\Delta h / \partial\delta_v^2 \leqslant 0$. Because of its continuity, Δh is then also concave wrt. x on all horizontal lines. Generalizing the method used for the study of radial variations of Δh centred on the points k and o in Sections 5.2 and 9, we define for any point v of the x-axis and any point i of the plane the two angles

$$\psi_{jv} = | \varphi_v - \varphi_j | = | j \hat{\imath} v |$$
$$\psi_{kv} = | \varphi_v - \varphi_k | = | k \hat{\imath} v |.$$

Whatever the position of v, this implies that $(\partial\delta_j/\partial\delta_v)_{\varphi_v} = \cos\psi_{jv}$ and that $(\partial\delta_k/\partial\delta_v)_{\varphi_v} = \cos\psi_{kv}$. Hence

$$\left(\frac{\partial\Delta h}{\partial\delta_v} \right)_{\varphi_v} = r_j \cdot \cos\psi_{jv} - r_k \cos\psi_{kv} . \qquad (41)$$

Like in Section 9.2, we have $(\partial^2\delta_j/\partial\delta^2_v)_{\varphi_v} = \sin^2\psi_{jv}/\delta_j$ and $(\partial^2\delta_k/\partial\delta_v^2)_{\varphi_v} = \sin^2\psi_{kv}/\delta_k$. On the other hand, the following trigonometric relations hold :

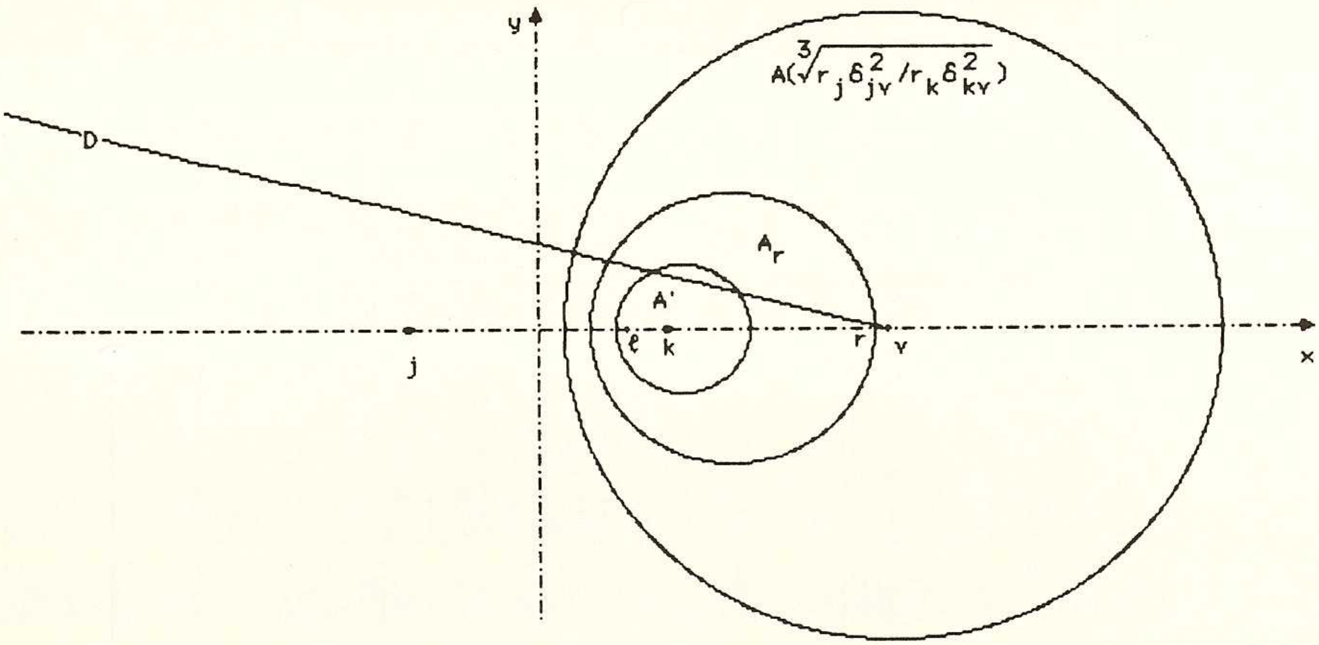


Fig. 13.4a. A point v for which Δh is concave over $D \cap Z_k$, for any straight half-line D originating in v. The area A' is the disk $A(\sqrt{r_k \delta_{jr}^3 / r_j \delta_{kr}^3})$.

$r_j = 1$, $r_k = 1.5$, $\delta_{jk} = 2$, $x_v = 2.7$, $Q = 1.2$.

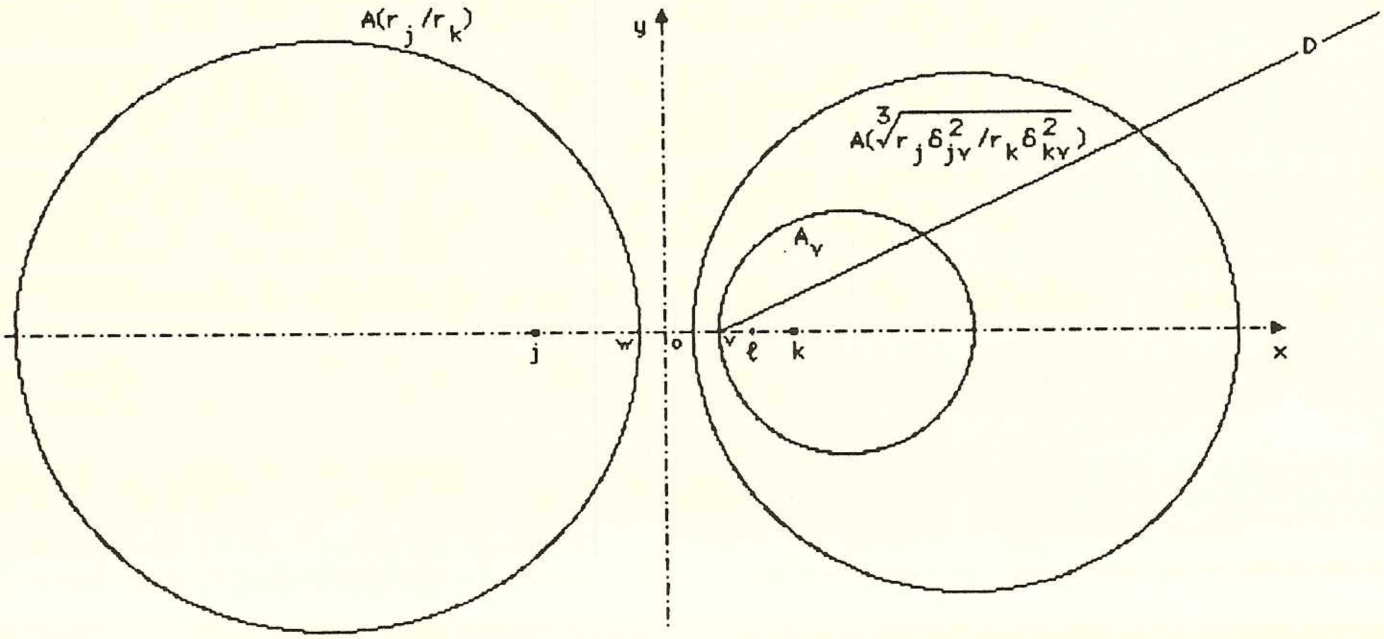


Fig. 13.4b. A point v for which Δh is quasi-concave on any straight half-line D originating in v. Here $x_v = 0.42$; the other data are as in Fig. 13.4a.

Fig. 13.4. Proof of Prop. 13.7a : Z_k is convex when $r_j < r_k$.

$$\frac{\sin\ \psi_{jv}}{\delta_{jv}} = \frac{\sin\varphi_v}{\delta_j} \quad \text{and} \quad \frac{\sin\psi_{kv}}{\delta_{kv}} = \frac{\sin\varphi_v}{\delta_k} \ . \qquad (42)$$

We consequently obtain that

$$\left(\frac{\partial^2\Delta h}{\partial\delta_v^2}\right)_{\varphi_v} = \frac{r_j\ \sin^2\psi_{jv}}{\delta_j} - \frac{r_k\ \sin^2\psi_{kv}}{\delta_k}$$

$$= \left(\frac{r_j\ \delta_{jv}^2}{\delta_j^3} - \frac{r_k\ \delta_{kv}^2}{\delta_k^3}\right) \sin^2\varphi_v \ . \qquad (43)$$

So that $(\partial^2\Delta h/\partial\delta_v^2)_{\varphi_v}$ is $\leqslant 0$ inside $A(\sqrt[3]{r_j\ \delta_{jv}^2/r_k\delta_{kv}^2})$, and $\geqslant 0$ outside. In particular, v itself lies in that area if it belongs to A (r_j/r_k), which contains $\mathbb{R}_+ \times \mathbb{R}$ as $r_j < r_k$.

It is thus clear that we cannot require Δh to be concave everywhere above the x-axis. But is it possible that Δh be concave over Z_k at least ? It appears from the coordinates (δ_j, δ_k) that the smallest Apollonian disk containing Z_k is A_r, ie. $A(\delta_{jr}/\delta_{kr})$, r being the rightmost element of Z_k on the x-axis. So we just have to manage to include A_r into $A(\sqrt[3]{r_j\delta_{jv}^2/r_k\delta_{kv}^2})$; this is the case iff. $(\delta_{jr}/\delta_{kr})^3 \geqslant r_j\delta_{jv}^2/r_k\delta_{kv}^2$, ie. iff. v is outside A $(\sqrt{r_k\delta_{jr}^3/r_j\delta_{kr}^3})$, which is an Apollonian disk , as $r_k > r_j$ and $\delta_{jr} > \delta_{kr}$. So that we cannot have Δh concave over Z_k : when v is inside A $(\sqrt{r_k\ \delta_{jr}^3/r_j\ \delta_{kr}^3})$, we must find something else. Before coming to this, notice an important point : r itself obviously lies outside the disk , so that Z_k is starshaped wrt. r.

We have so far considered only the second derivative of Δh wrt. δ_k; let us return to the first one. According to (41), a sufficient condition for it to be $\leqslant 0$ is that $\cos\psi_{jv} \leqslant \cos\psi_{kv}$ and $\cos\psi_{kv} \geqslant 0$, ie. $\psi_{kv} \leqslant \psi_{jv}$ and $\psi_{kv} \leqslant \pi/2$. The second condition is satisfied iff. i is outside the disk C [(v + k)/2,k] ('outside'meaning of course: 'in $\mathbb{R}^2$ minus the interior of'). When v belongs to [jk], the first condition is equivalent to $\sin\psi_{kv} \leqslant \sin\psi_{jv}$, for we have here $\psi_{jv} + \psi_{kv} \leqslant \pi$; (42) shows that it is also equivalent to $\delta_j/\delta_k \leqslant \delta_{jv}/\delta_{kv}$,i.e., i is outside A_v. As C [(v + k)/2,k] is $\subset A_v$ because $v \in [jk]$, we are left with this : $(\partial\Delta h/\partial\delta_v)_{\varphi_v}$ is < 0 outside A_v. Let us now restrict the study to $[jk] \cap A(r_j/r_k)$, which includes [ok] .We have just seen that v is outside $A(r_j/r_k)$ iff. $v \in A(\sqrt[3]{r_j\delta_{jv}^2/r_k\delta_{kv}^2})$, where $(\partial^2\Delta h/\partial\delta_v^2)_{\varphi_v} \leqslant 0$. If that last area contains itself A_v, outside which $(\partial\Delta h/\partial\delta_v)_{\varphi_v} < 0$, it is clear that Δh will be quasi-concave wrt. δ_v on any direction φ_v ; if we start from v, Δh will indeed be first concave, then decreasing. This will thus be the case if $\delta_{jv}/\delta_{kv} \geqslant \sqrt[3]{r_j\delta_{jv}^2/r_k\delta_{kv}^2}$; which boils down to $\delta_{jv}/\delta_{kv} \geqslant r_j/r_k$, ie. again to : $v \in A(r_j/r_k)$. So we see that Δh is quasi-concave wrt. δ_v on all the half-lines originating at v, if $v \in [jk] \cap A(r_j/r_k)$. In particular, as [ok] is a part of that segment, Z_k is starshaped wrt. k and wrt. ℓ.

The starshapedness of Z_k wrt. ℓ and r entails that it is starshaped wrt.[ℓr]. Suppose indeed that i belongs to Z_k; then $[\ell i] \subset Z_k$; and, due to the starshapedness wrt. r, the whole triangle formed by i, ℓ, and r, is a part of Z_k. This fills the last gap in our demonstration. Let us summarize the essence of our information. Calling w the extreme left point of $A(\sqrt[3]{r_k\delta_{jr}^3/r_j\delta_{kr}^3})$, we have seen that, v being a point of the x-axis and D being any half-line above the x-axis,originating at v :

- if v is at the left of w or at the right of r, Δh is concave wrt. δ_v on the range $D \cap A(\sqrt[3]{r_j\delta_{jv}^2/r_k\delta_{kv}^2})$, that includes $D \cap Z_k$;

- if $v \in [w\ell]$, Δh is quasi-concave wrt. δ_v on D ;

- if $v \in [\ell r]$, Z_k is starshaped wrt. δ_v.

So that if i_1 and i_2 belong to $Z_k \cap \mathbb{R}_+^2$, the intersection point v of the x-axis with the straight line (i_1i_2) has anyway some property that shows that $[i_1i_2] \subseteq Z_k$ -unless (i_1i_2) is horizontal, but that inclusion remains true for continuity reasons. The area $Z_k \cap \mathbb{R}_+^2$ is thus convex ; and so is Z_k itself, as it is symmetric wrt. the x-axis and spined by it. The transportation cost difference Δh is consequently quasi-concave on the area where it is $\geqslant 0$,i.e., on the Apollonian disk $A(r_k/r_j)$. Furthermore, it results from our study of $(\partial^2\Delta h/\partial\delta_v^2)_{\varphi_v}$ that Δh cannot remain constant on any straight segment, except perhaps when $\sin\varphi_v = 0$. But it is easily deduced from the formula

$$\left(\frac{\partial\Delta h}{\partial x}\right)_y = r_j\cos\varphi_j - r_k\cos\varphi_k$$

that Δh is not constant on any segment of the x-axis, nor of any horizontal line. Market area Z_k is thus strictly convex [Fig. 13.3b]. Here is now the proposition :

(13.7) (a) *Market area* Z_k *is convex iff.* $r_j < r_k$;

(b) *Market area* Z_j *is convex iff.* $r_j > r_k$ *and* $Q \geq (r_j + r_k)\,\delta_{jk}$ [*ie.* $A(r_j/r_k) \subset Z_j$], *or* $r_j > r_k$ *and* $Q \leq (r_j - r_k)\ \delta_{jk}$ [*ie.* $(C, \delta_{ok}\sqrt{3}) \in Z_k$, *ie.* $A\ (r_j/r_k) \subset Z_k$] ,

(c) *Market area* Z_k *(resp.* Z_j*) is convex iff. strictly convex.*

As we have seen with Prop. 13.6, some interesting properties of Z_j remain valid even when Z_j is not convex. In the next proposition, we recall the implications of Prop. 13.6 concerning convexity, and we add to this a property of starshapedness which nicely completes Prop. 13.7b. Remember that the *convex hull* of a set E is the smallest convex set containing E. We denote by r' the extreme left point of Z_j on the x-axis.

(13.8) *When* $r_j > r_k$,

(a) *the x-hull of market area* Z_j *is its convex hull ;*

(b) *market area* Z_j *is starshaped wrt.* $[r'\ell] - \overset{\circ}{A}\,(r_j/r_k)$.

The first item is obvious, and we give the proof of the second only. If ℓ is at the right of $A(r_j/r_k)$, the property trivially holds because this entails (Prop. 13.7b) that Z_j is convex, i.e., starshaped wrt. any of its points. Let us now assume that ℓ is at the left of $A(r_j/r_k)$, and denote by w' the extreme left point of $A(r_j/r_k)$; it is clear that $w' \in]\,ok[\, \cap Z_j$. It can be shown, by a proof similar to the part of that of Prop. 13.7a that concerns [jk], that Δh is here quasi-convex wrt. δ_v on any straight half-line originating at any $v \in [jw']$. In particular, Z_j is thus starshaped wrt. the points of $[j\ell] \cap [jw']$, ie. of $[j\ell] - \overset{\circ}{A}\,(r_j/r_k)$.

We still have to examine the left extremity of $[r'\ell] - \overset{\circ}{A}\,(r_j/r_k)$, ie. r'. As r' is at the left of j, we have $\psi_{jr'} \leq \psi_{kr'}$ for any point i of the plane ; according to (41), $(\partial\Delta h/\partial\delta_v)_{\varphi_v}$ is thus ≥ 0 provided that $\psi_{jr'} \leq \pi/2$, ie. that

$i \notin \overset{\circ}{C}[(r' + j)/2, j]$. If a point i belongs to Z_j, this entails that $[ir'] - \overset{\circ}{C}[(r' + j)/2, j] \subset Z_j$. But it clearly appears in the coordinates (δ_j, δ_k) that the Apollonian disk determined by r' is a part of Z_j. As it contains $C[(r' + j)/2, j]$, the whole segment $[ir']$ is in Z_j, which is thus starshaped wrt. r'. Z_j is consequently starshaped wrt. $[r'w'] \cap [r'\ell]$, and the proof is complete.

13.3 *Dependence of the measures of market areas on attractivity*

The preceding results make it easy to extend Section 9 to Descartes' ovals. First, equality (43) shows indeed that $(\partial^2 \Delta h / \partial \delta_k^2)_{\varphi_k} > 0$ (or = 0 on the x-axis) and that $(\partial^2 \Delta h / \partial \delta_j^2)_{\varphi_j} < 0$ (= 0 on the x-axis). Second, Z_k is starshaped wrt. k when $r_j < r_k$ (Prop. 13.7a) and Z_j is starshaped wrt. j when $r_j < r_k$ (Prop. 13.8b) : as Z_k is obviously strongly decreasing wrt. Q, this entails that, with the definitions introduced in Section 9, $\overline{\delta}_k$ is strictly decreasing wrt. Q when $r_j < r_k$, and $\overline{\delta}_j$ is strictly increasing wrt. Q when $r_j > r_k$. The combination of the two results implies that $\overline{\delta}_k$ is convex wrt. Q when $r_j < r_k$, as is $\overline{\delta}_j$ when $r_j > r_k$. Hence,

(13.9) (a) *When $r_j < r_k$, $|Z_k|$, if $\neq 0$, is a strictly decreasing and strictly convex function of Q ;*

(b) *When $r_j > r_k$, $|Z_j|$ is a strictly increasing and strictly convex function of Q.*

13.4 *Negative values of the attractivity index*

Contrary to all the previous sections, Δh, when defined as in Section 13.1, is not symmetric with regard to the y-axis. In particular when Q = 0, the indifference line is not the y-axis but the circle of the set $A(r_k/r_j)$.
It thus becomes interesting to view the implications of our statements for all values of the attractivity index Q, be they positive or negative. Clearly, the properties of Z_k when $r_j < r_k$ and $Q > 0$ are those of Z_j when $r_j > r_k$ and $Q < 0$. It is thus possible to express all statements in terms of the situation where $r_j > r_k$.

In this perspective, illustrated in Fig. 13.5, Prop. 13.2 means that the market area of the less accessible centre (here j) is always bounded. Prop. 13.7 indicates that the same market area is nonconvex if and only if Q is in the open interval $](r_j - r_k)\ \delta_{jk}, (r_j + r_k)\ \delta_{jk}[$, and that it is strictly convex when convex (the empty set, to which Z_j is equal when $Q < -r_k\ \delta_{jk}$, is strictly convex). According to Prop. 13.9, its superficies is an increasing and convex function of Q, and this property is strict when $Q \geqslant -r_k\ \delta_{jk}$, i.e., when Z_j is not empty. As to the market area Z_k of the other centre, it is always unbounded, nonconvex (unless $Q < - r_k\ \delta_{jk}$, when Z_k is the whole plane) and its superficies is infinite. Prop. 13.5a and 13.6 also imply that the Cartesian oval is more extended in the x- than in the y-direction when $Q < 0$, and conversely when $Q > 0$;when $Q=0$, it is equally extended in both directions, as being the Apollonian circle $\delta_j/\delta_k = r_k/r_j$.

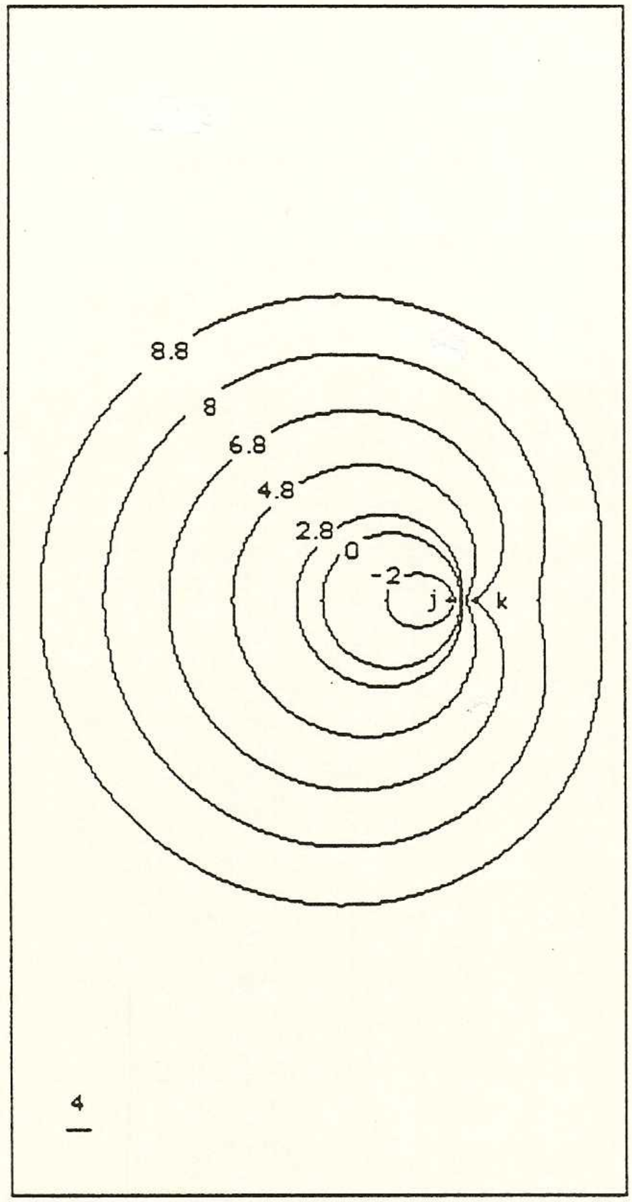


Fig. 13.5a. When $r_j > r_k$, the Cartesian oval encloses Z_j whatever the value of Q. Here $r_j = 1$, $r_k = 1.2$, and $\delta_{jk} = 4$. The area Z_j is convex iff. $Q \notin]2.8, 8.8[$. When Q = 6.8, the oval passes through k, where it has an angulous point.

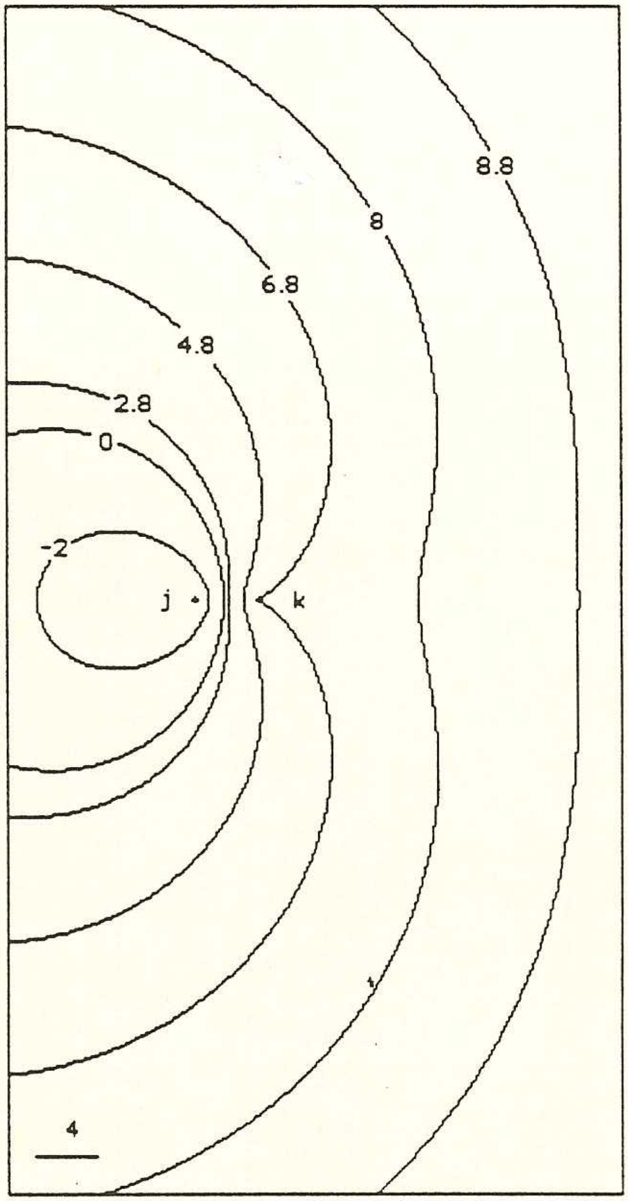


Fig. 13.5b. The same as Fig.13.5a, with another scale.

Conclusion

This work has gathered a harvest of positive results. Two new cases where the indifference line is a well-known curve have been added to the three ones found so far in the literature. Properties that could be expected on the basis of those three classical cases have been proved to hold. Among others : when the so-called transportation cost function is concave, the market area of the less attractive centre is starshaped with regard to that centre and spined by the straight line joining the two centres. When the transportation cost function is convex, on the contrary, that market area may not contain its own centre, but the area of the more attractive centre is starshaped with respect to the middle of the segment joining the centres, and spined by the mediatrix of that segment. Whatever the transportation cost function, the measure of the smaller market area is a decreasing function of the absolute attractivity index $|Q|$ and an increasing one of the distance between centres. That area is convex when a concave power transportation cost function is considered (whether δ^a, with $0 < a \leqslant 1$, or $- \delta^{-a}$, with $a > 0$). The indifference zone is one-dimensional when the transportation cost function is strictly increasing. Etc.

Our study also shows the limits of the classical cases. Some strictly concave transportation cost functions may for instance generate two infinite market areas. Moreover, it happens that the measure of one of those areas then remains finite. The convexity of the market area of the less attractive centre is exceptional when the transportation cost function is convex, and is far from being a rule when the transportation cost function is concave. A purely concave or purely convex transportation cost function yields market areas that are all of a piece, but this does not necessarily remain true in mixed cases. Besides concavity or convexity, other hidden features of the transportation cost function also appear in fact to determine the qualitative and quantitative properties of market areas. The limit of the derivative of the transportation cost function when distance grows toward infinity, as well as the signs of the first and second derivatives of the function $\sqrt{.}/h' \circ \sqrt{.}$ (where h is the transportation cost function), e.g., are particularly significant. Those features of the transportation cost function are usually not immediately detectable, and their interpretation is not obvious. It would be vain, however, to try to explain them in terms of consumer behaviour only. If the Manhattan distance (cf. infra) was considered together with the same models of consumer behaviour, the market areas would be affected, and some determining properties of the transportation cost function could be different. Those properties thus mainly concern the nature of space.

But this work should not be seen as a chamber of horrors.. Although we now better know the limits of the validity of old and new properties, some general trends appear through our exact and approximate statements. It is intuitively satisfying to observe that market areas tend to be functions of the ratio of the attractivity index to the distance between centres. The smaller market area is usually more extended in the direction defined by the two centres than in the perpendicular one. When the transportation cost function is concave, the measure of the smaller market area seems generally convex in the absolute attractivity index and in the distance between centres. When the transportation cost function is convex, the measure of the extra territory appears to be conversely concave in the absolute attractivity index, but convex in the distance between centres.

Market areas -or Voronoi diagrams, Thiessen polygons, Dirichlet domains - thus appear fertile in properties. The present research might be extended in many ways. Extending such works as those of Boots (1980) and Aurenhammer (1983), one might reinvestigate the shape of market areas when there are more than two centres (Fig. C.1) ; in addition to what we have said in our introduction, some particular results are likely to appear. On the other hand, the use of a Euclidean distance is rather restrictive. Other ones could be tried, like the well-known ℓ_p- and block norms [see e.g. Thisse et al. (1984)].Market areas under Manhattan distance, in particular, clearly exhibit some characteristics that are not reducible to the present exposition : we have already mentioned that the indifference zone may then be partially two-dimensional when the transportation cost function is linear, as shown by Lee (1980). It could also be interesting, perhaps, to consider a kind of Euclidean distance varying continuously from point to point ; i.e., a transfer cost function as used by Puu (1979). Other models of consumer behaviour could also be assumed.

It is not necessary, however, to get entangled in too general problems for meeting unsolved issues. Some of the questions we have raised have only received partial or approximate answers, and might be studied further. Nevertheless, we hope that the precise mathematical tools our work provides will clarify the theory of market areas and ease the way for future researchers.

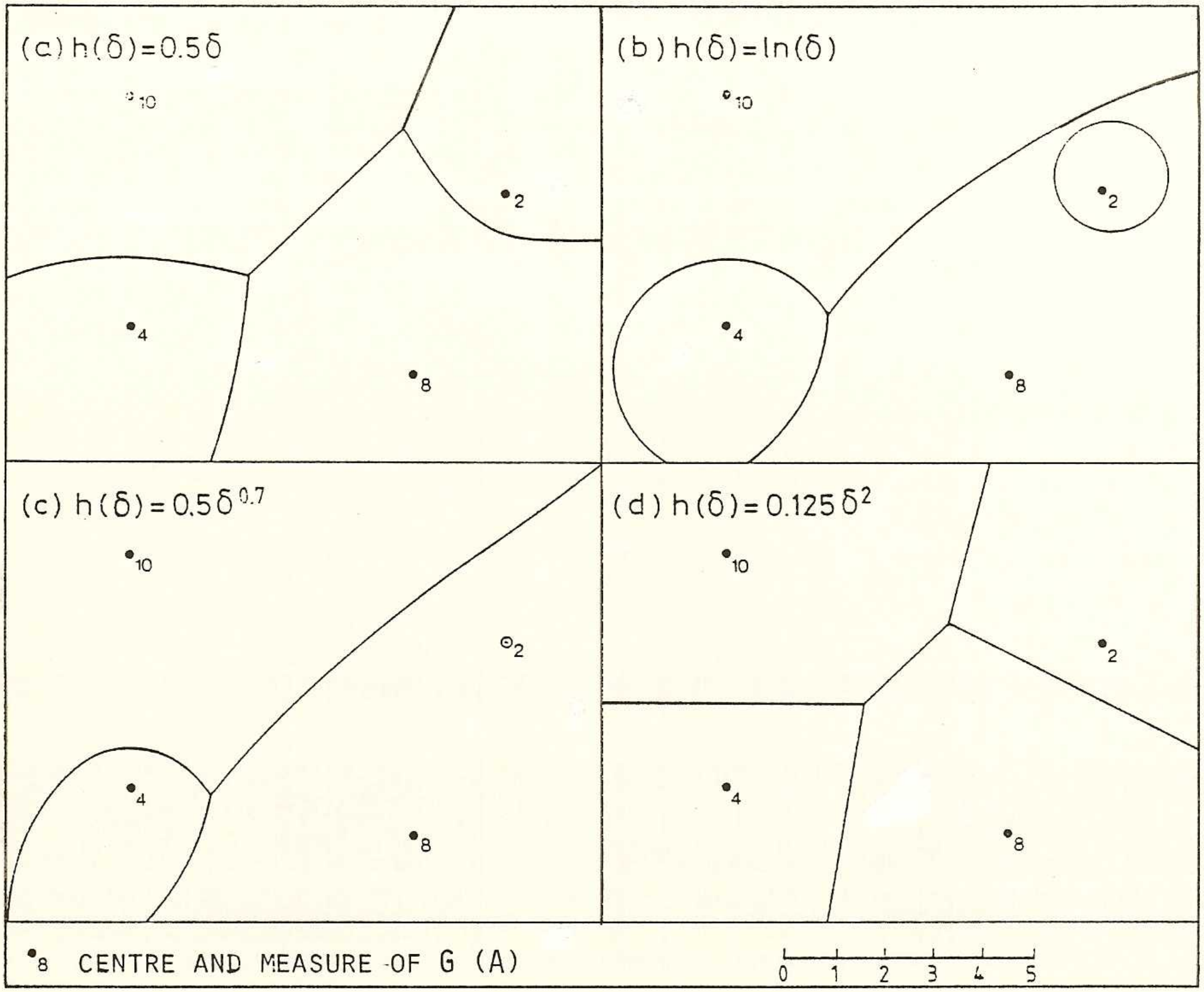


Fig. C.1. Market areas when there are more than two centres. As a consequence of Prop. 8.6, the areas in a, b, and c are convex iff. they are surrounded by market areas of more attractive centres. In d, market areas must always be convex. In a, each market area must always be starshaped wrt. its centre. In d, a market area does not necessarily contain its centre, but it certainly does if surrounded by market areas of less attractive centres.

References

Alao N., M. Dacey, O. Davies, K. Denike, J. Huff, J. Parr, and M. Webber (1977). *Christaller Central Place Structures : An Introductory Statement.* Evanston : Northwestern University.

Allen P.M. and M. Sanglier (1981). 'A Dynamic Model of a Central Place System - II', *Geographical Analysis,* 13, 149-164.

Apollonius Pergæus (†180 BC). *Treatise on Conic Sections.*Edited in modern notation, with introductions, including an essay on the earlier history of the subject, by T.L. Heath (1896). Cambridge : University Press.

Aurenhammer F. (1983) *Power Diagrams : Properties, Algorithms, and Applications.* Report F 120, Institut für Informationsverarbeitung, Technische Universität, Graz, Steiermark, Austria.

Beckmann M. (1971). 'Market Share, Distances, and Potential', *Regional and Urban Economics*, 1, 1-18.

Beckmann M.(1972).'Equilibrium versus Optimum : Spacing of Firms, and Patterns of Market Areas', in R. Funck (ed), *Recent Developments in Regional Science,* London : Pion.

Beckmann M. (1976). 'A Discourse on Distance', *Annals of Regional Science*,2, 1-8.

Berry B. and R. Lamb (1974). 'The Delineation of Urban Spheres of Influence : Evaluation of an Interaction Model', *Regional Studies*, 8, 185-190.

Boots B.,(1980).'Weighting Thiessen Polygons', *Economic Geography*, 56, 248-259.

Cheysson E. (1887). 'La Statistique Géométrique : ses Applications Industrielles et Commerciales', *Le Génie Civil*,X, n. 13 and 14, 206-210 and 224-228.

Descartes R. (1637). *Discours de la Méthode et c., plus la Dioptrique, les Météores et la Géométrie et c.* Facsimile of original edition including an English translation, *The Geometry of René Descartes.* New York : Dover, 1954, Ch. 2.

Fetter F.A. (1924). ' The Economic Law of Market Areas', *Quarterly Journal of Economics*, 39, 520-529.

Godlund S. (1956). 'The Function and Growth of Bus Traffic within the Sphere of Urban Influence', *Lund Studies in Geography B*, 18, 1-80.

Goldman A.P. (1963). *A Theorem on Convex Programming*, paper presented at the MAA Meeting, Indianapolis.

Gomes Teixeira F. (ed) (1915). *Traité des Courbes Spéciales Remarquables Planes et Gauches*. Coimbra : University Press.

Hanjoul P. , H. Beguin, and J.-C. Thill (1986).'Shapes of Market Areas under Euclidean Distance' *Cahiers de Géographie de Besançon*, 28, 107-110.

Hanjoul P. and J.-C. Thill (1986) *Demand Analysis in the Plane : An Attempt at Generalization*. Working Paper SPUR 8601, unité GEOG, Université Catholique de Louvain-la-Neuve, Belgium.

Hanjoul P. and J.-C. Thill (1987). 'Elements of Planar Analysis in Spatial Competition.' *Regional Science and Urban Economics*, 17, 423-439.

Huff D. and J. Lutz (1979). 'Ireland's Urban System', *Economic Geography*, 55, 196-211.

Hurter A.P., M.K. Schaefer, and R.E. Wendell (1975). 'Solutions of Constrained Location Problems', *Management Science*, 22, 51-56.

Hyson C.D. and W.P. Hyson (1950). 'The Economic Law of Market Areas', *Quarterly Journal of Economics*, 64, 319-324.

Illeris S. (1967). 'Funktionelle Regioner i Danmark omkring 1960', *Geografisk Tidsskrift*, 66, 225-251.

Isard W. (1960). *Methods of Regional Analysis : An Introduction to Regional Science*. Cambridge, Mass. : MIT Technology Press.

Jaskold-Gabszewicz J. and J.-F. Thisse (1986). 'Spatial Competition and the Location of Firms', *Fundamentals of Pure and Applied Economics*,5, 1-71.

Jones P.R. (1979). 'The Economic Law of Market Areas', paper presented at the Annual Meeting of the Canadian Association of Geographers.

Launhardt W. (1882). 'Die Bestimmung des zweckmäßigsten Standortes einer gewerblichen Anlage', *Zeitschrift des Vereins Deutscher Ingenieure*,XXVI.

Lee D.T. (1980). 'Two-Dimensional Voronoi Diagrams in the L_p-Metric', *Journal of the Association for Computing Machinery*,27, 604-618.

Lipschutz (1965). *Schaum's Outline of Theory and Problems of General Topology*. New York : Schaum.

Mulligan G. (1981). 'Lösch's Single-Good Equilibrium', *Annals of the Association of American Geographers*, 71, 84-94.

Palander T. (1935). *Beiträge zur Standortstheorie*. Uppsala, Sweden : Almquist Wiksells Boktryckeri.

Ponsard C. (1983). *History of Spatial Economic Theory*. Berlin : Springer-Verlag.

Puu T. (1979). *The Allocation of Road Capital in Two-Dimensional Space : A Continuous Approach*. Amsterdam : North-Holland.

Rau K.H. (1841). 'Report to the Bulletin of the Belgian Royal Society', in W.J. Baumol and S.M. Goldfeld (eds), *Precursors in Mathematical Economics : An Anthology*. London School of Economics and Political Science, 1968, 181-182.

Robine M. (1985). 'Karl Heinrich Rau (1792-1870) : Pionnier de la Théorie des Aires de Marché', *Région et Aménagement du Territoire : Mélanges Offerts au Doyen Joseph Lajugie*, Bordeaux : Bière.

Salmon G. (1879). *Higher Plane Curves*. New York : Chelsea.

Schilling A. (1925). *Die Lehre von Wirtschaften, Wesen und Zusammenhänge*. Berlin : VDI-Verlag.

Schneider E. (1935). 'Bemerkung zu einer Theorie der Raumwirtschaft', *Econometrica*, 3, 79-101.

Sheppard E. (1978). 'Theoretical Underpinnings of the Gravity Hypothesis', *Geographical Analysis*, 10, 386-402.

Shieh Y. (1985).'K. H. Rau and the Economic Law of Market Areas', *Journal of Regional Science*, 25, 191-199.

Thisse J.-F., J.E. Ward , and R.E. Wendell (1984).' Some Properties of Location Problems with Block and Round Norms', *Operations Research*, 32, 1309-1327.

Tobler W. (1961). *Map Transformations of Geographic Space*. Ph. D. Dissertation, University of Washington, Seattle.

Tuominen O. (1949). 'Das Einflußgebiet der Stadt Turku im System der Einflußgebiete SW-Finlands, *Fennia*, 71-5, 1-138.

Webber M. (1974). 'Association between Population Density and the Market Areas of Towns', *Geographical Analysis*, 6, 109-134.

Wilson A.G. (1970). *Entropy in Urban and Regional Modelling*. London : Pion.

Appendix 1 : Z_k is convex when $h = .^a$ and $a \in]0, 1[$

Let $\xi = - \Sigma\, \hbar'/\Delta\hbar'$, $\rho = \delta_j/\delta_k$, and $\lambda = \delta_{jk}^2/\delta_k^2$. Then (34) rewrites :

$$\left(\frac{\partial y}{\partial x}\right)_{\Delta h} = \frac{\xi\delta_{jk}/2 - (\delta_j^2 - \delta_k^2)/2\delta_{jk}}{y} = \frac{\delta_{jk}[\xi\lambda-(\rho^2-1)]}{2y\lambda} \quad . \quad (44)$$

On the other hand, from (10), we have :

$$\begin{aligned} y &= [2(\delta_{jk}^2\ \delta_j^2 + \delta_{jk}^2\ \delta_k^2 + \delta_j^2\ \delta_k^2) - \delta_j^4 - \delta_k^4 - \delta_{jk}^4]\ /4\ \delta_{jk}^2 \\ &= \delta_k^4\ [2\ (\lambda\rho^2 + \lambda + \rho^2) - \rho^4 - 1 - \lambda^2]\ /\ 4\delta_{jk}^2 \\ &= \delta_{jk}^2\ [-\lambda^2 + 2\lambda\ (\rho^2 + 1) - (\rho^2 - 1)^2]\ /\ 4\lambda^2\ . \end{aligned} \quad (45)$$

Combining both results we obtain :

$$\begin{aligned} 1 + \left(\frac{\partial y}{\partial x}\right)_{\Delta h}^2 &= 1 + \frac{[\xi\lambda - (\rho^2-1)]^2}{-\lambda^2+2\lambda(\rho^2+1)-(\rho^2-1)^2} \\ &= \lambda\ \frac{(\xi^2 - 1)\lambda + 2\ [\rho^2(1-\xi) + 1 + \xi]}{-\lambda^2 + 2\lambda(\rho^2 + 1) - (\rho^2 - 1)^2} \quad . \end{aligned}$$

Section 7 and particularly formula (36) have shown that $y\ (\partial^2 y / \partial x^2)_{\Delta h}$ is the difference between $(\partial^2 y^2/\partial x^2)_{\Delta h}/2 + 1$ and that quantity ; and that

$$\frac{1}{2}\left(\frac{\partial^2 y^2}{\partial x^2}\right)_{\Delta h} + 1 = \lambda\zeta \quad \text{where } \zeta = - \frac{4\delta_{jk}^2\ \hbar_k'^2\ \hbar_j'^2\ \Delta g'}{(\Delta\hbar')^3} \quad . \quad (46)$$

Hence ,

$$y\ \left(\frac{\partial^2 y}{\partial x^2}\right)_{\Delta h} = -\lambda\ \frac{\lambda^2\zeta - \lambda\ [2\zeta\ \ (\rho^2+1) - \xi^2+1] + [\zeta(\rho^2-1)^2+2\rho^2(1-\xi)+2+2\xi]}{-\lambda^2 + 2\lambda\ (\rho^2 + 1) - (\rho^2 - 1)^2} \quad .$$

When $h = .^a$, we have :

$$\xi = \frac{\rho^{2-a} + 1}{\rho^{2-a} - 1} \quad \text{and } \zeta = (2-a)\ \frac{\rho^{2-a}\ (\rho^a - 1)}{(\rho^{2-a} - 1)^3} \quad .$$

Introducing this in the expression of $y\ (\partial^2 y/\partial x^2)_{\Delta h}$ here above, we find that

$$y \left(\frac{\partial^2 y}{\partial x^2}\right)_{\Delta h} = \frac{\lambda \rho^{2-2a} \; T(\lambda, \rho)}{(\rho^{2-a}-1)^3 \left[-\lambda^2 + 2\lambda(\rho^2+1)-(\rho^2-1)^2\right]} \qquad (47)$$

where

$$T(\lambda,\rho) = -\lambda^2(2-a)(\rho^a-1) + 2\lambda\left[(4-a)\rho^a+(2-a)\rho^{2+a}-(4-a)\rho^2-(2-a)\right] + C$$

and where C is an expression of ρ (and of a, but a is here a constant number), independent of λ : $(\partial C / \partial\lambda)_\rho = 0$. From the fact that $\delta_j > \delta_k$ on $\mathbb{R}_+^* \times \mathbb{R}$ and from (45), it is clear that $(\partial^2 y/\partial x^2)_{\Delta h}$ and $T(\lambda,\rho)$ have the same sign at every point i of $\mathbb{R}_+^{*2}$. When ρ is fixed, point i is on an Apollonian circle. The minimum of λ on that curve occurs when δ_k is maximum, ie. when $\lambda = (\delta_j - \delta_k)^2/\delta_k^2 = (\rho-1)^2$; conversely, λ is maximum when $\lambda = (\delta_j+\delta_k)^2/\delta_k^2 = (\rho+1)^2$. We have thus to study T on the range $[(\rho-1)^2, (\rho+1)^2]$ of λ. Now, at the beginning of that interval we compute that :

$$\left(\frac{\partial T(\lambda,\rho)}{\partial\lambda}\right)_\rho \left[(\rho-1)^2,\rho\right] = 4\rho^a \, E(\rho),$$

where

$$E(\rho) = -\rho^{2-a} - (2-a)\rho^{1-a} + (2-a)\rho + 1.$$

Let us compute the derivative E' :

$$E'(\rho) = (2-a)\left[1 - \rho^{1-a} - (1-a)\rho^{-a}\right],$$

which is < 0 $\forall\rho > 1$, as $a \in \,]0,1[$. As $E(1) = 0$, we thus have $E(\rho) < 0$, $\forall\rho > 1$: which means that $(\partial T/\partial\lambda)_\rho$ is < 0 when $\lambda = (\rho-1)^2$. But $T(\lambda,\rho)$ itself is concave wrt. λ; $(\partial T/\partial\lambda)_\rho$ is thus < 0 for any $\lambda \geqslant (\rho-1)^2$. Now, when $\lambda \underset{>}{\to} (\rho-1)^2$, the point i on the Apollonian circle determined by the still fixed value of ρ becomes close to the x-axis ; (44) then shows that $|(\partial y/\partial x)_{\Delta h}| \to +\infty$ [it is clear that $\xi > 1$, for $\hbar$ is here strictly concave ; so that $\xi\lambda - (\rho^2-1) \neq 0$ in (41) when $\lambda = \rho^2 - 1$], whereas we see from (46) that $1 + (\partial^2 y^2/\partial x^2)_{\Delta h}/2$ remains finite. Consequently, $y(\partial^2 y/\partial x^2)_{\Delta h}$ and a fortiori $(\partial^2 y/\partial x^2)_{\Delta h}$ tend towards $-\infty$: (45) and (47) then entail that $T[(\rho-1)^2,\rho] < 0$. As we have proved above that $(\partial T/\partial\lambda)_\rho < 0$ when $\lambda \geqslant (\rho-1)^2$, we thus conclude that $T(\lambda,\rho) < 0$, and that $(\partial^2 y/\partial x^2)_{\Delta h} < 0$. Which in turn implies the convexity of Z_k.

Appendix 2. Formulae of the measure of the dicentral approximation when $h = \overset{a}{.}$

The only problem that remains is to integrate $\cos^{2/(1-a)} \varphi$ wrt. φ over $[0, \pi/2]$. We have the following equality :

$$\forall n \in \mathbb{R} - \{1\} : \frac{d}{d\varphi} (\sin \varphi \cos^{n-1} \varphi) = \cos^n \varphi - (n-1) \cos^{n-2} \varphi . \sin^2 \varphi$$
$$= n \cos^n \varphi - (n-1) \cos^{n-2} \varphi.$$

Let us denote by $I(n)$ the integral $\int_0^{\pi/2} \cos^n \varphi \, d\varphi$. It is then clear that, if $n > 1$:

$$I(n) = \frac{1}{n} [\sin \varphi \cos^{n-1} \varphi]_0^{\pi/2} + \frac{n-1}{n} I(n-2)$$
$$= \frac{n-1}{n} I(n-2) \quad . \qquad (48)$$

Consequently, we find these two formulae :

$$\forall p \in \mathbb{N} : \quad I(2p) = \frac{\pi \, (2p)!}{2^{2p+1} \, p!^2} \quad \text{and } I(2p+1) = \frac{2^{2p} \, p!^2}{(2p+1)!} .$$

Those formulae are verified indeed when $p = 0$; and if we substitute p by $p+1$, we find, according to (48),

$$I(2p+2) = \frac{2p+1}{2p+2} I(2p) = \frac{(2p+2)(2p+1)}{4\,(p+1)^2} I(2p) = \frac{\pi (2p+2)!}{2^{2p+3} \, (p+1)!^2}$$

and

$$I(2p+3) = \frac{2p+2}{2p+3} I(2p+1) = \frac{4(p+1)^2}{(2p+3)(2p+2)} I(2p+1) = \frac{2^{2p+2} \, (p+1)!^2}{(2p+3)!} .$$

The formulae thus hold for any $p \in \mathbb{N}$. The remainder of the proof of Prop. 11.9 is obvious.

Appendix 3. Proof of Prop. 11.10 : Homotheticity of dicentral approximations

If we exclude the case of point o, which always belongs to the closure of $Z_j(\mu) \cap Z_k(\mu)$, the homotheticity between $Z_j(\mu) \cap Z_k(\mu)$ and $Z_j(\mu') \cap Z_k(\mu')$ for any $\mu, \mu' > 0$ means that, if $\overline{\delta}_o$ can be determined, ie. if h' is invertible :

$$\forall \mu, \mu' > 0 : \forall \varphi_o \in [0, \pi/2[\; : \frac{\overline{\delta}_o(\varphi_o, \mu)}{\overline{\delta}_o(0, \mu)} = \frac{\overline{\delta}_o(\varphi_o, \mu')}{\overline{\delta}_o(0, \mu')} . \qquad (49)$$

i.e.,

$$\forall\ \mu,\mu' > 0 : \forall\ \varphi \in [0,\pi/2[: h'^{\wedge}(1/\mu\cos\varphi)h'^{\wedge}(1/\mu') = h'^{\wedge}(1/\mu)h'^{\wedge}(1/\mu'\cos\varphi).$$

That condition is satisfied when $h = \ell n$, or $h = .^{a}$ with $1 \neq a > 0$, or $h = -.^{a}$ with $a > 0$. The problem is now to show that when h is derivable on $\mathbb{R}_{+}^{*}$ and $h'^{\wedge}$ is a continuous function, *only* those kinds of functions satisfy (49). Now, one may easily verify that (49) is equivalent to :

$$\forall\ \xi_1,\xi_2 > 0 : h'^{\wedge}(\xi_1)\ h'^{\wedge}(\xi_2) = h'^{\wedge}(\xi_1\xi_2)\ h'^{\wedge}(1).$$

Consequently, for any $\alpha \in \mathbb{N}$,

$$\frac{h'^{\wedge}(\xi^{\alpha})}{h'^{\wedge}(1)} = \left[\frac{h'^{\wedge}(\xi)}{h'^{\wedge}(1)}\right]^{\alpha}; \qquad (50)$$

it is then possible to extend that result, successively, to the situations where : α is a negative integer ($\alpha \in \mathbb{Z}_{-}$), α is rational ($\alpha \in \mathbb{Q}$), or α is real ($\alpha \in \mathbb{R}$) (this last point is by constructing a sequence of rational numbers converging toward α).

As (50) can also be written

$$\ell n\left[\frac{h'^{\wedge} \circ \exp(\alpha\ \ell n\ \xi)}{h'^{\wedge}(1)}\right] = \alpha\ \ell n\left[\frac{h'^{\wedge} \circ \exp(\ell n\ \xi)}{h'^{\wedge}(1)}\right],$$

we see that the function $\ell n\,[h'^{\wedge} \circ \exp / h'^{\wedge}(1)]$ is linear (on $\mathbb{R}_{+}^{*}$), so that, for some $K \in \mathbb{R}^{*}$,

$$\forall\ \xi > 0 : \quad \ell n\left[\frac{h'^{\wedge} \circ \exp(\ell n\,\xi)}{h'^{\wedge}(1)}\right] = K\ \ell n\ \xi \quad .$$

Hence we derive :

$$\forall\ \xi > 0 : h'^{\wedge}(\xi) = h'^{\wedge}(1)\ \xi^{K}$$

$$\forall\ \delta > 0 : \delta = h'^{\wedge}(1)[\,h'(\delta)\,]^{K}$$

$$h'(\delta) = [\,\delta / h'^{\wedge}(1)\,]^{1/K}$$

$$h(\delta) = \frac{K}{K+1}\,[h'^{\wedge}(1)]^{-1/K}\,\delta^{(K+1)/K} + \beta \text{ if } K \neq -1,$$

$$h(\delta) = h'^{\wedge}(1)\ \ell n\ \delta + \beta \qquad \text{if } K = -1.$$

As h'^(1) represents a distance and is thus > 0, h (δ) either assumes the form $b\ \ell n\ \delta + \beta$, where b > 0, or $b\ \delta^a + \beta$ where a and b bear the same sign . The proposition also trivially holding when h (δ) = δ , the proof is complete.

Appendix 4. Variation of y with x along $Z_j(\mu) \cap Z_k(\mu)$ when h = .^{a}

As we have in this case $a\delta_o^{a-1}\cos\varphi_o = 1/\mu$, and as $x = \delta_o \cos\varphi_o$ and $y = \delta_o \sin\varphi_o$, we easily compute the following derivatives, successively :

$$\frac{\partial\delta_o}{\partial\varphi_o} = \frac{-\delta_o \operatorname{tg}\varphi_o}{1-a}$$

$$\frac{\partial x}{\partial\varphi_o} = \frac{\partial\delta_o}{\partial\varphi_o}\cos\varphi_o - \delta_o\sin\varphi_o = \frac{-\delta_o\sin\varphi_o\,(2-a)}{1-a}$$

$$\frac{\partial y}{\partial\varphi_o} = \frac{\partial\delta_o}{\partial\varphi_o}\sin\varphi_o + \delta_o\cos\varphi_o = \frac{\delta_o\cos\varphi_o[(1-a)-\operatorname{tg}^2\varphi_o]}{1-a}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y/\partial\varphi_o}{\partial x/\partial\varphi_o} = \frac{\operatorname{tg}^2\varphi_o-(1-a)}{(2-a)\operatorname{tg}\varphi_o}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\partial x/\partial\varphi_o}\cdot\frac{d\operatorname{tg}\varphi_o}{d\varphi_o}\cdot\frac{\partial^2 y}{\partial\operatorname{tg}\varphi_o\,\partial x}$$

$$= \frac{1-a}{-\delta_o\sin\varphi_o\,(2-a)}\cdot\frac{1}{\cos^2\varphi_o}\cdot\frac{\operatorname{tg}^2\varphi_o-a+1}{(2-a)\operatorname{tg}^2\varphi_o}$$

$$= \frac{(1-a)\,(a-1-\operatorname{tg}^2\varphi_o)}{(2-a)^2\,\delta_o\sin^3\varphi_o}$$

List of main symbols

		First occurence
$\Delta.$	Δf stands for $f(\delta_j) - f(\delta_k)$	p. 16
$.\Delta.$	symmetric difference : $E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1)$	108
$\Sigma.$	Σf stands for $f(\delta_j) + f(\delta_k)$	69
$.^\circ$	E° : interior of area E	93
$\lvert . \rvert$	$\lvert E \rvert$: superficies of area E	21
$.^*(f)$	$E^*(f)$: set of coordinates $[f(\delta_j), f(\delta_k)]$ of area E	37
$.^\frown$	$f^\frown$: inverse relation of function or relation f	16
$.\langle . \rangle$	$f \langle E \rangle = \{ f(i) : i \in E \}$	37
A_p	attractivity of p, if p is a centre *or* the Apollonian area $\delta_j/\delta_k \geqslant \delta_{pj}/\delta_{pk}$	8
$A(\rho)$	Apollonian area $\delta_j/\delta_k \geqslant \rho$	55
C_i	disk whose circle is $T_i(.^2)$	52
$C(i,p)$	disk of centre i and whose circle passes through point p	52
$C_i(R)$	disk of centre i and radius R	102
coth	hyperbolic cotangent, i.e., $[\exp(.) + \exp(-.)] / [\exp(.) - \exp(-.)]$	23
$D_i(f)$	area which appears to be under $T_i(f)$ in the coordinates $[f(\delta_j), f(\delta_k)]$	40
$D_{\ell r}(f)$	area which appears to be under $T_{\ell r}(f)$ in the coordinates $[f(\delta_j), f(\delta_k)]$	40
E_-	area where $\delta_k \geqslant e^{-1/a}$ or $\Sigma \delta^a \geqslant 1$ (when $h = .^a$)	98
E_+	area where $\Sigma \delta^a \leqslant 2/e$ (when $h = .^a$)	98
F	deterrence function of distance	8
G	function expressing the relation of the 'mass' of a centre to its attractivity	8
g	various functions *or* the function $\sqrt{.} / (h' \circ \sqrt{.})$	70
h	(so-called) transportation cost function	8
$\hbar$	function expressing the relation of $h(\delta)$ to δ^2 ; i.e., $h \circ \sqrt{.}$	69
ℓ	leftmost point of Z_k on the x-axis	21
ℓn	Neperian logarithm	8
L_x	length of the projection of the indifference line on the x-axis	56
m_i	'mass' of site i	8

$\mathbb{N}$	set of all nonnegative integers	17
Q	index of comparative attractivity	8
r	rightmost point of Z_k on the x-axis	21
$\mathbb{R}$	set of all real numbers	16
R (f)	set of coordinates [$f(\delta_j)$, $f(\delta_k)$] of the points above the x-axis	37
r_k	rate of transportation to centre k	11
sh	hyperbolic sinus, i.e., the function [exp (.) - exp(-.)]/2	23
th	hyperbolic tangent, i.e., [exp(.) - exp(-.)] / [exp(.) + exp(-.)]	112
t_{ik}	amount of interaction between sites i and k	8
t_k	total interaction of all sites with k	9
$T_i(f)$	curve appearing in the coordinates [$f(\delta_j)$, $f(\delta_k)$] as the tangent to the indifference line passing throught point i	39
$T_{\ell r}(f)$	curve appearing in the coordinates [$f(\delta_j)$,$f(\delta_k)$] as the straight line joining points ℓ and r	43
Z_k	market area of centre k	6
$\mathcal{Z}_k$	dicentral approximation of the market area of centre k	109
μ	dicentral moment ; i.e., the ratio δ_{jk}/Q	109
φ_∞	asymptotic direction of the indifference line	60
Φ_k	diameter of market area Z_k	56
χ_f	function relating the points of the plane to their coordinates [$f(\delta_j)$, $f(\delta_k)$]	37
1_E	identity function defined on the set E	41

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This monograph contains Nystuen's calculations, actually used by Barr to position his abstract tetrahedral sculpture within the earth. Placement of the sculpture vertices in Easter Island, South Africa, Greenland, and Indonesia was chronicled in film by The Archives of American Art for The Smithsonian Institution. In addition to the archival material, this monograph also contains Arlinghaus's solutions to broader theoretical questions–was Barr's choice of a tetrahedron unique within his initial constraints, and, within the set of Platonic solids?

2. Sandra L. Arlinghaus. *Down the Mail Tubes: the Pressured Postal Era, 1853-1984*, 1986.

The history of the pneumatic post, in Europe and in the United States, is examined for the lessons it might offer to the technological scenes of the late twentieth century. As Sylvia L. Thrupp, Alice Freeman Palmer Professor Emeritus of History, The University of Michigan, commented in her review of this work "Such brief comment does far less than justice to the intelligence and the stimulating quality of the author's writing, or to the breadth of her reading. The detail of her accounts of the interest of American private enterprise, in New York and other large cities on this continent, in pushing for construction of large tubes in systems to be leased to the government, brings out contrast between American and European views of how the new technology should be managed. This and many other sections of the monograph will set readers on new tracks of thought."

3. Sandra L. Arlinghaus. *Essays on Mathematical Geography*, 1986.

A collection of essays intended to show the range of power in applying pure mathematics to human systems. There are two types of essay: those which employ traditional mathematical proof, and those which do not. As mathematical proof may itself be regarded as art, the former style of essay might represent "traditional" art, and the latter, "surrealist" art. Essay titles are: "The well-tempered map projection," "Antipodal graphs," "Analogue clocks," "Steiner transformations," "Concavity and urban settlement patterns," "Measuring the vertical city," "Fad and permanence in human systems," "Topological exploration in geography," "A space for thought," and "Chaos in human systems–the Heine-Borel Theorem."

4. Robert F. Austin, *A Historical Gazetteer of Southeast Asia*, 1986.

Dr. Austin's Gazetteer draws geographic coordinates of Southeast Asian place-names together with references to these place-names as they have appeared in historical and literary documents. This book is of obvious use to historians and to historical geographers specializing in Southeast Asia. At a deeper level, it might serve as a valuable source in establishing place-name linkages which have remained previously unnoticed, in documents describing trade or other communications connections, because of variation in place-name nomenclature.

5. Sandra L. Arlinghaus, *Essays on Mathematical Geography–II*, 1987.

Written in the same format as IMaGe Monograph #3, that seeks to use "pure" mathematics in real-world settings, this volume contains the following material: "Frontispiece–the Atlantic Drainage Tree," "Getting a Handel on Water-Graphs," "Terror in Transit: A Graph Theoretic Approach to the Passive Defense of Urban Networks," "Terrae Antipodum," "Urban Inversion," "Fractals: Constructions, Speculations, and Concepts," "Solar Woks," "A Pneumatic Postal Plan: The Chambered Interchange and ZIPPR Code," "Endpiece."

6. Pierre Hanjoul, Hubert Beguin, and Jean-Claude Thill, *Theoretical Market Areas Under Euclidean Distance*, 1988. (English language text; Abstracts written in French and in English.)

Though already initiated by Rau in 1841, the economic theory of the shape of two-dimensional market areas has long remained concerned with a representation of transportation costs as linear in distance. In the general gravity model, to which the theory also applies, this corresponds to a decreasing exponential function of distance deterrence. Other transportation cost and distance deterrence functions also appear in the literature, however. They have not always been considered from the viewpoint of the shape of the market areas they generate, and their disparity asks the question whether other types of functions would not be worth being investigated. There is thus a need for a general theory of market areas: the present work aims at filling this gap, in the case of a duopoly competing inside the Euclidean plane endowed with Euclidean distance.

(Bien qu'ébauchée par Rau dès 1841, la théorie économique de la forme des aires de marché planaires s'est longtemps contentée de l'hypothèse de coûts de transport proportionnels à la distance. Dans le modèle gravitaire généralisé, auquel on peut étendre cette théorie, ceci correspond au choix d'une exponentielle décroissante comme fonction de dissuasion de la distance. D'autres fonctions de coût de transport ou de dissuasion de la distance apparaissent cependant dans la littérature. La forme des aires de marché qu'elles engendrent n'a pas toujours été étudiée ; par ailleurs, leur variété amène à se demander si d'autres fonctions encore ne mériteraient pas d'être examinées. Il paraît donc utile de disposer d'une théorie générale des aires de marché : ce à quoi s'attache ce travail en cas de duopole, dans le cadre du plan euclidien muni d'une distance euclidienne.)

7. Keith J. Tinkler, Editor, *Nystuen—Dacey Nodal Analysis*, 1988.

Professor Tinkler's volume displays the use of this graph theoretical tool in geography, from the original Nystuen—Dacey article, to a bibliography of uses, to original uses by Tinkler. Some reprinted material is included, but by far the larger part is of previously unpublished material. (Unless otherwise noted, all items listed below are previously unpublished.) Contents: " 'Foreward' " by Nystuen, 1988; "Preface" by Tinkler, 1988; "Statistics for Nystuen—Dacey Nodal Analysis," by Tinkler, 1979; Review of Nodal Analysis literature by Tinkler (pre–1979, reprinted with permission; post—1979, new as of 1988); FORTRAN program listing for Nodal Analysis by Tinkler; "A graph theory interpretation of nodal regions" by John D. Nystuen and Michael F. Dacey, reprinted with permission, 1961; Nystuen—Dacey data concerning telephone flows in Washington and Missouri, 1958, 1959 with comment by Nystuen, 1988; "The expected distribution of nodality in random (p, q) graphs and multigraphs," by Tinkler, 1976.

8. James W. Fonseca, *The Urban Rank-size Hierarchy: A Mathematical Interpretation*, 1989.

The urban rank–size hierarchy can be characterized as an equiangular spiral of the form $r = ae^{\theta \cot \alpha}$. An equiangular spiral can also be constructed from a Fibonacci sequence. The urban rank–size hierarchy is thus shown to mirror the properties derived from Fibonacci characteristics such as rank–additive properties. A new method of structuring the urban rank–size hierarchy is explored which essentially parallels that of the traditional rank–size hierarchy below rank 11. Above rank 11 this method may help explain the frequently noted concavity of the rank–size distribution at the upper levels. The research suggests that the simple rank–size rule with the exponent equal to 1 is not merely a special case, but rather a theoretically justified norm against which deviant cases may be measured. The spiral distribution model allows conceptualization of a new view of the urban rank–size hierarchy in which the three largest cities share functions in a Fibonacci hierarchy.

9. Sandra L. Arlinghaus, *An Atlas of Steiner Networks*, 1989.

A Steiner network is a tree of minimum total length joining a prescribed, finite, number of locations; often new locations are introduced into the prescribed set to determine the minimum tree. This Atlas explains the mathematical detail behind the Steiner construction for prescribed sets of n locations and displays the steps, visually, in a series of Figures. The proof of the Steiner construction is by mathematical induction, and enough steps in the early part of the induction are displayed completely that the reader who is well–trained in Euclidean geometry, and familiar with concepts from graph theory and elementary number theory, should be able to replicate the constructions for full as well as for degenerate Steiner trees.

10. Daniel A. Griffith, *Simulating $K = 3$ Christaller Central Place Structures: An Algorithm Using A Constant Elasticity of Substitution Consumption Function*, 1989.

An algorithm is presented that uses BASICA or GWBASIC on IBM compatible machines. This algorithm simulates Christaller $K = 3$ central place structures, for a four–level hierarchy. It is based upon earlier published work by the author. A description of the spatial theory, mathematics, and sample output runs appears in the monograph. A digital version is available from the author, free of charge, upon request; this request must be accompanied by a 5.5–inch formatted diskette. This algorithm has been developed for use in Social Science classroom laboratory situations, and is designed to (a) cultivate a deeper understanding of central place theory, (b) allow parameters of a central place system to be altered and then graphic and tabular results attributable to these changes viewed, without experiencing the tedium of massive calculations, and (c) help promote a better comprehension of the complex role distance plays in the space–economy. The algorithm also should facilitate intensive numerical research on central place structures; it is expected that even the sample simulation results will reveal interesting insights into abstract central place theory.

The background spatial theory concerns demand and competition in the space–economy; both linear and non–linear spatial demand functions are discussed. The mathematics is concerned with (a) integration of non–linear spatial demand cones on a continuous demand surface, using a constant elasticity of substitution consumption function, (b) solving for roots of polynomials, (c) numerical approximations to integration and root extraction, and (d) multinomial discriminant function classification of commodities into central place hierarchy levels. Sample output is presented for contrived data sets, constructed from artificial and empirical information, with the wide range of all possible central place structures being generated. These examples should facilitate implementation testing. Students are able to vary single or multiple parameters of the problem, permitting a study of how certain changes manifest themselves within the context of a theoretical central place structure. Hierarchical classification criteria may be changed, demand elasticities may or may not vary and can take on a wide range of non–negative values, the uniform transport cost may be set at any positive level, assorted fixed costs and variable costs may be introduced, again within a rich range of non–negative possibilities, and the number of commodities can be altered. Directions for algorithm execution are summarized. An ASCII version of the algorithm, written directly from GWBASIC, is included in an appendix; hence, it is free of typing errors.

11. Sandra L. Arlinghaus and John D. Nystuen, *Environmental Effects on Bus Durability*, 1990.

This monograph draws on the authors' previous publications on "Climatic" and "Terrain" effects on bus durability. Material on these two topics is selected, and reprinted, from three published papers that appeared in the *Transportation Research Record* and in the *Geographical Review*. New material concerning "congestion" effects is examined at the national level, to determine "dense," "intermediate," and "sparse" classes of congestion, and at the local level of congestion in Ann Arbor (as suggestive of how one might use local data). This material is drawn together in a single volume, along with a summary of the consequences of all three effects simultaneously, in order to suggest direction for more highly automated studies that should follow naturally with the release of the 1990 U. S. Census data.

12. Daniel A. Griffith, Editor. *Spatial Statistics: Past, Present, and Future*, 1990.

Proceedings of a Symposium of the same name held at Syracuse University in Summer, 1989. Content includes a Preface by Griffith and the following papers:

Brian Ripley, "Gibbsian interaction models";
J. Keith Ord, "Statistical methods for point pattern data";
Luc Anselin, "What is special about spatial data";
Robert P. Haining, "Models in human geography:
problems in specifying, estimating, and validating models for spatial data";
R. J. Martin, "The role of spatial statistics in geographic modelling";
Daniel Wartenberg, "Exploratory spatial analyses: outliers, leverage points, and influence functions";
J. H. P. Paelinck, "Some new estimators in spatial econometrics";
Daniel A. Griffith, "A numerical simplification for estimating parameters of spatial autoregressive models";
Kanti V. Mardia "Maximum likelihood estimation for spatial models";
Ashish Sen, "Distribution of spatial correlation statistics";

Sylvia Richardson, "Some remarks on the testing of association between spatial processes";
Graham J. G. Upton, "Information from regional data";
Patrick Doreian, "Network autocorrelation models: problems and prospects."

Each chapter is preceded by an "Editor's Preface" and followed by a Discussion and, in some cases, by an author's Rejoinder to the Discussion.

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