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Urban Rank-Size Hierarchy

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James W. Fonseca

URBAN RANK-SIZE HIERARCHY:

A MATHEMATICAL INTERPRETATION

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Introduction

Despite seventy years of intensive research, the mathematical nature of the urban rank-size hierarchy is still subject to much debate (Carroll, 1982). Stewart (1958,245) has argued that urban distributions approximate an "S" shaped logistics curve, others suggest a "J" shaped distribution and that linearity in the data exists only in truncated portions of the distribution (Carroll, 1979). The research in this monograph demonstrates a new interpretation of rank-size regularities: data which conform, in general, to a rank-size pattern, approximate the curve of an equiangular spiral. (Figure 1).

It seems prudent first to demonstrate this relationship empirically and then to explore the implications of this fact. Urbanized area data are used because these data are more scientifically defined than city populations or Standard Metropolitan Area data. Rosen and Resnick (1980,180), for example, recognize that city data affect the distribution under analysis by neglecting suburbanization and thus under-estimating the growth of cities. DeCola (1985,1645) makes the case that urban regions other than those defined by political boundaries should be used. Figure 1 is an illustration of the correspondence between 1970 urbanized area data for the United States (Appendix) and a theoretically ideal equiangular spiral described by the formula

$$r = ae^{\theta \cot \alpha} \qquad (1)$$

where a >, o cot $\alpha \neq o$. (When cot $\alpha > o$, the curve spirals outward as θ increases (as in Figure 2); when cot $\alpha < o$, the curve spirals inward as θ increases).

In this formula, r is the radius, a is a constant which sets the dimension of the spiral, e is the base of the natural logarithm, θ is the

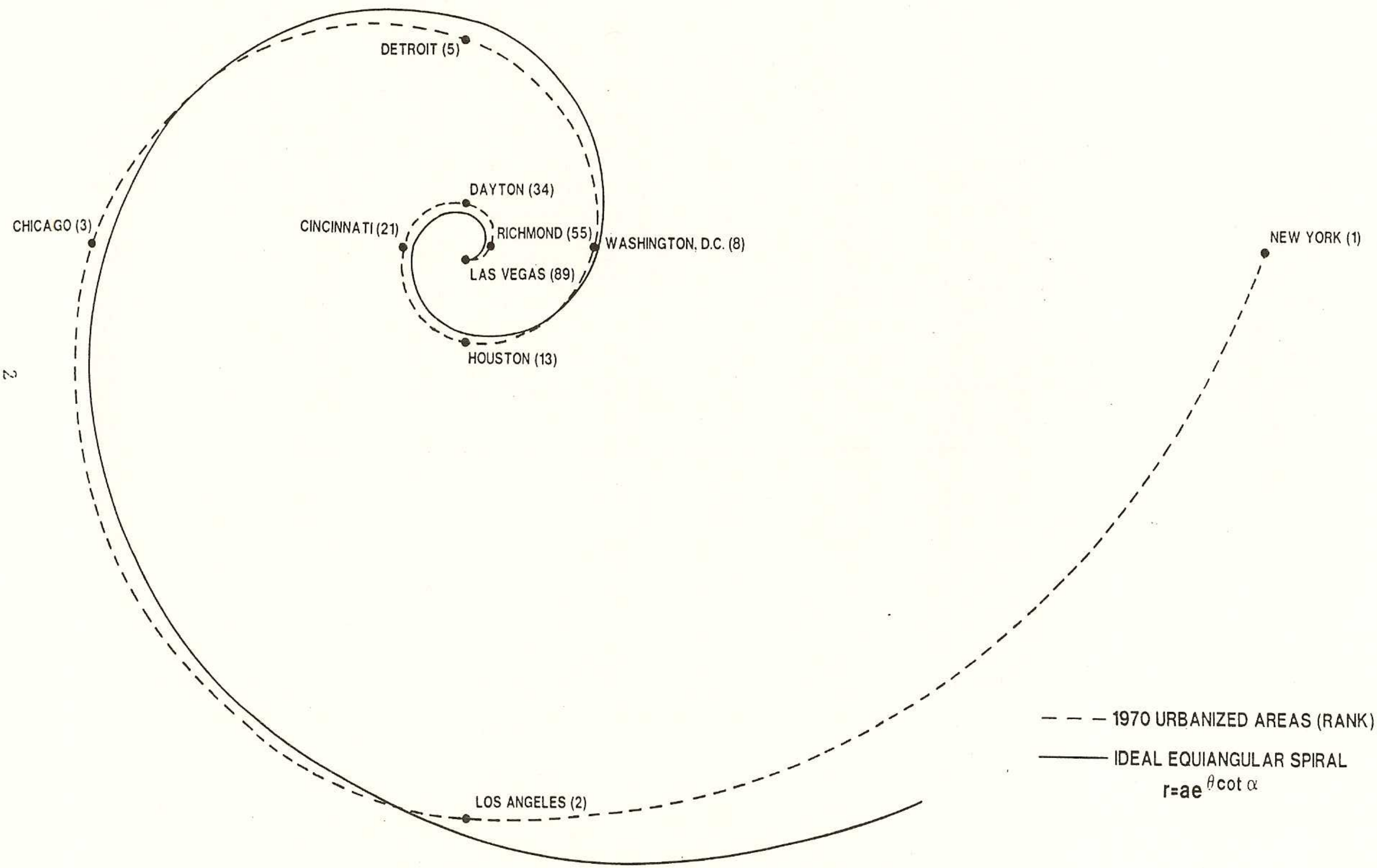

Figure 1: Comparison of 1970 urbanized area data plotted by the spiral constant with an ideal equiangular spiral.

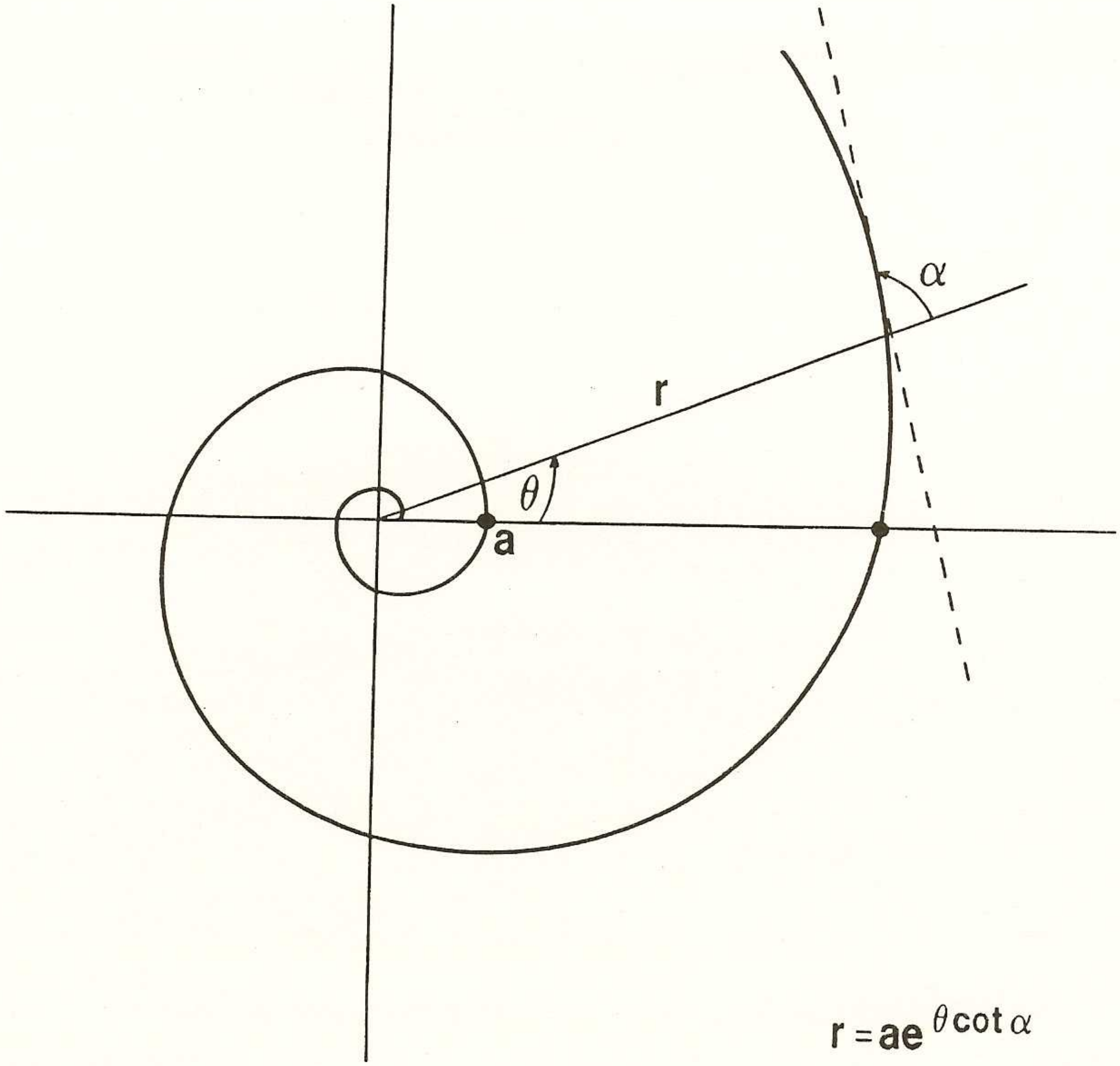

Figure 2. The construction of an equiangular spiral.

angle of the radius from the origin, and α is the constant angle at which a line perpendicular to the radius is tangent to the spiral curve. (Taylor, 1959, 408). These relationships are shown in Figure 2.

To understand the meaning of the data plotted in Figure 1, we must first discuss the idea of Fibonacci numbers. This is because an equiangular spiral is a mathematical plot of a particular Fibonacci sequence. (Huntley 1970, 175). The nth term, U_n, of the Fibonacci sequence is generated by the general recurrence relation (Albertson and Hutchinson, 1988, 198)

$$U_n = U_{n-1} + U_{n-2} \text{ where } U_1 = 1 \text{ and } U_2 = 1$$

This relationship states a simple procedure: the next term in the Fibonacci sequence is the sum of the two preceeding terms. This sequence is displayed below:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . .

A related sequence is the Lucas sequence. Lucas numbers (Rosen, 1988,249), L_i, are defined by $L_1 = 1$ and $L_n = U_{n+1} + U_{n-1}$, $n > 1$, where U_n is the nth Fibonacci number.

1, 1, 3, 4, 7, 11, 18, 29, 47, 123, 199, . . .

Clearly, an infinite number of sequences of this sort can be produced merely by varying the values of the initial terms, U_1 and U_2. Employing only this single recurrence relation leads to an infinite number of sequences whose internal pattern of numbers rests on the choice of initial values of U_1 and U_2. Thus, each sequence can be characterized by an ordered pair of positive integers (U_1, U_2). To each such ordered pair, there corresponds a unique equiangular spiral. The structure of the curves corresponding to these sequences is an equiangular spiral because the length of the radii forming the curve at successive increments of 90° in the angle of the radius are Fibonacci numbers. That is, if the radius of the curve at 0° is 1 unit of length, the radius at 90° is 2 units of length; at 180° , 3 units; at 270°, 5 units, and so on (Figure 3). At 360° the curve begins to form a second layer around the first as the spiral wraps around itself. The term equiangular refers to the constant angle (α in equation 1) maintained between an extension of any radius of the curve and a line tangent to the curve at the point where the two meet.

One method by which urbanized area data arranged by rank-size can be plotted as a spiral is by assigning population as the value of r (the length of the radius). Rank is determined by Fibonacci sequence. Thus the 1970 population of the New York urbanized area, the largest, is the value of r at $\theta = 0°$; the population of the Los Angeles urbanized area, second largest, is the value of r at $\theta = 90°$; the population of Chicago, third largest, is the value of r at $\theta = 180°$. The next Fibonacci rank is 5, so the population of the fifth ranked city, Detroit, is the value of r at $\theta = 270°$; Washington, D.C., the 8th ranked city is the value of r at $\theta = 360°$, etc. These cities are indicated on Figure 3.

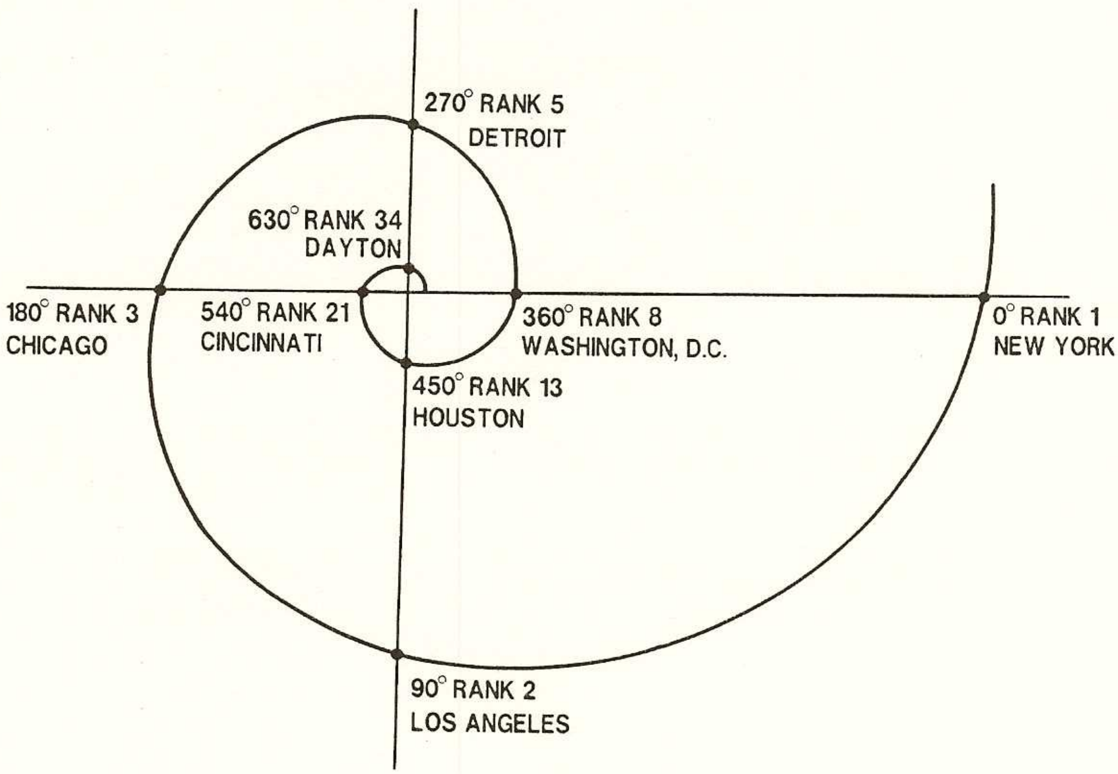

Figure 3. Relationship among urbanized areas, Fibonacci rank and θ on an equiangular spiral.

Once we recognize that an urban system which fits the simple rank-size rule can be conceptualized as a spiral distribution, a further physical analogy offers additional insight. This is the analogy of a conical shell, such as those shown in Figure 4. A conical shell shows how a continuous distribution such as cities in rank-size relation, can simultaneously be considered as occupying distinct levels, as cities in a central place hierarchy. If a conical shell is held in the hand, the levels can be clearly identified and even counted. Yet the shell surface is a continuous distribution that can be traced in screw-like fashion around the outside curves of the shell.

The Spiral Constant

Why should the rank-size relationship closely parallel the curves of an equiangular spiral? The key to understanding the linkage lies with the mathematical constant ϕ derived from

$$(1 \pm \sqrt{5})/2 \approx 1.61803...$$

In addition to its relation to the constant angle of a spiral generated by the Fibonacci series (as in equation 1), the spiral constant, also known as the golden mean, can be derived from many sources in mathematics and geometry. To cite just two examples (Huntley, 1970, 24-25, 42):

1. Any two intersecting diagonals of a regular pentagon divide each other into two segments having the ratio ϕ:1;
2. If the edge of a regular decagon is 1, the radius of a circle circumscribed around the decagon is ϕ:1

In addition to, and in part because of, its recurrence in geometric figures, numerous aesthetic qualities have been ascribed to the ratio.

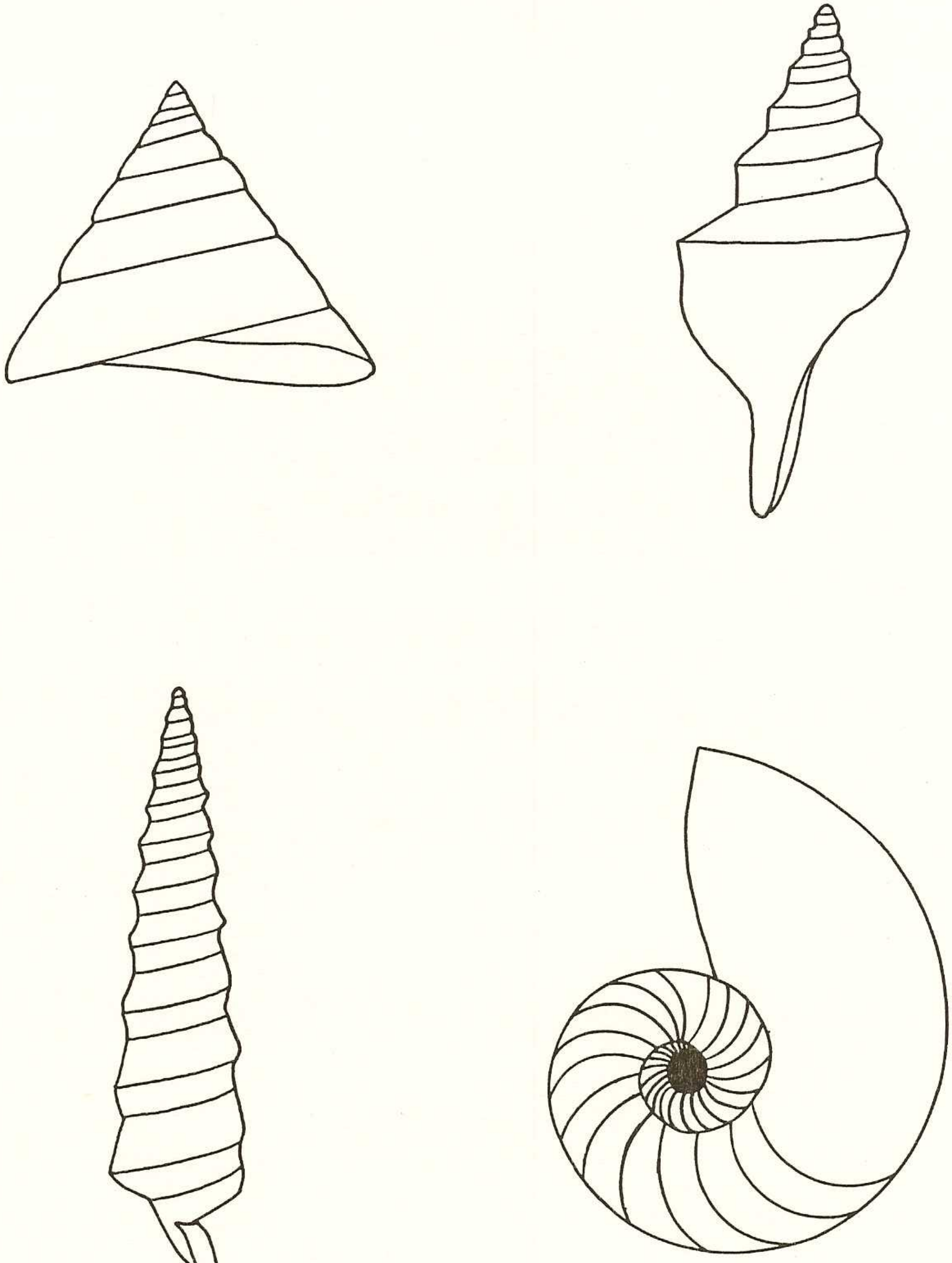

Figure 4. Variety of spiral shells.

Indeed, the symbol ϕ was selected to designate this ratio in honor of Phidias, a Greek sculptor who made use of the ratio in his works. Another example from the Greeks is that the facade of the Parthenon is a rectangle, the sides of which are in the ϕ ratio. In modern architecture, Le Corbusieur is known for his use of spirals and the golden mean in his work. Tufte (1983, 189) refers to this ratio as a venerable but dubious rule of aesthetic proportion. In this paper, the term spiral constant rather than phi will be used to avoid confusion with Euler's formula, but the symbol ϕ will be used.

The aesthetic appeal of rectangles, in particular, those with sides in ϕ proportion has not gone unnoticed by designers and marketers of goods. One will find that a large proportion of everyday goods have a rectangular shape approximately in ϕ proportion: houses, rooms, desks and table tops, books, cigarette packages, newspaper pages, briefcases, television sets, toasters, microwave ovens, and even 3x5 and 5x8 index cards (although no one ever accused these last two items of having much aesthetic appeal).

It has been suggested that the aesthetic appeal of objects in ϕ proportion derives from our subconscious familiarity with numerous objects in the natural world, especially organic objects, which have dimensions in ϕ proportion. Many average size relationships in portions of human anatomy can be approximately described by ϕ: the ratio of one's height to the height of one's navel; the length and width of one's head are two examples (Ghyka, 1977, 97). Needless to say, the frequent occurrence of the spiral constant in the most unexpected places in mathematical formulas as well as in the physical world has led, on occasion, to endowment of the ratio with mystic qualities. The ratio has thus been called the Golden Mean, the Golden Ratio or the Divine Proportion. Figures in the proportion of the spiral constant are sometimes given names such as Golden Rectangles and Golden Triangles.

Mathematicians do not attempt to answer "why" the spiral constant appears repeatedly and in apparently unconnected places in both the mathematical and physical worlds. The spiral constant, like pi, simply occurs.

<u>Other Derivations of the Ratio of the Spiral Constant</u>

If we construct a golden rectangle such that ($\phi = \frac{\ell}{w}$) as in Figure 5, It follows that $\ell = \frac{w}{2} + r$

By the Pythagorean theorem, $r^2 = \frac{w^2}{4} + w^2 = \frac{5\,w^2}{4}$

Combining, $\ell = \frac{w}{2} + \frac{\sqrt{5}(w)}{2} = \frac{(1 + \sqrt{5})w}{2}$

Therefore $\phi = \frac{\ell}{w} = \frac{(1+\sqrt{5})w}{2} = \frac{1 + \sqrt{5}}{2} \approx 1.61803...$

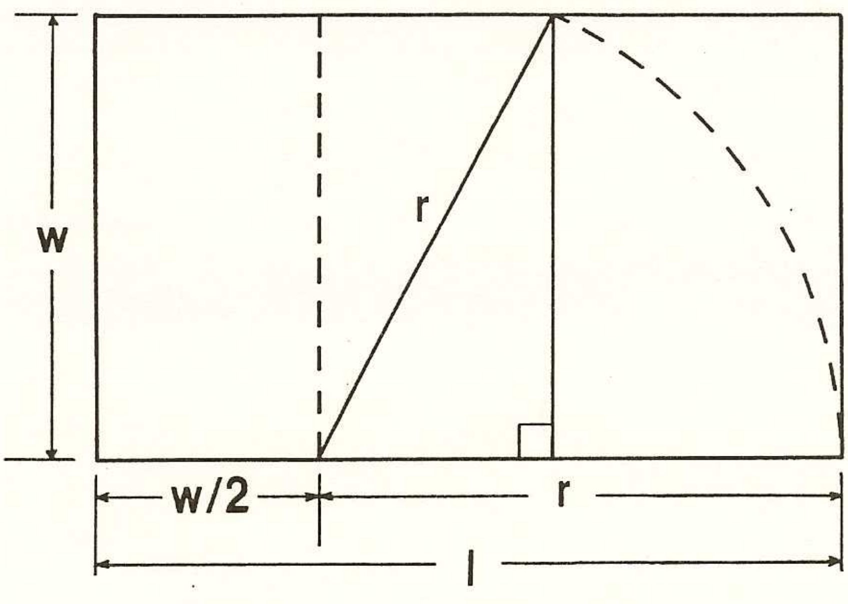

Figure 5. A golden rectangle and the derivation of the spiral constant.

Figure 5 also serves to introduce the idea of the gnomon. A gnomon is a portion of a figure which has been added to another figure so that the whole is of the same shape as the smaller figure (Huntley, 1970, 169). Figure 6 shows a series of gnomons generated from a triangle and a rectangle. Note that both figures can be shown to contain within them the points needed to generate an equiangular spiral. The edges of the gnomons are in ϕ proportion. The concept of the gnomon and the preceeding discussion of proportion in line segments illustrates the fact that the ratio represented by the spiral constant is the only growth ratio by which a new unit can grow in proportion to the old unit and still retain the same shape. This is exactly the process of growth in a spiral shell. As noted by D'Arcy Thompson (1961, 179):

> In the growth of a shell we can conceive no simpler law than this, namely that it shall widen and lengthen in the same unvarying proportions: and this simplest of laws is that which Nature tends to follow. The shell, like the creature within it, grows in size but does not change its shape; and the existence of this constant relativity of growth, or constant similarity of form, is of the essence, and may be made the basis of a definition, of the equiangular spiral.

The ϕ ratio thus is a mathematical representation of a kind of growth in which an initial proportion is retained unaltered, that is, in which size increases but shape remains the same. This is isomorphic as opposed to allometric growth (Woldenberg, 1971; Coffey, 1981). A second way of viewing the phenomenon, as outlined by Huxley (1972, 163) is that the ratio of the spiral constant expresses simple-interest growth. In the case of shells, new material produced by growth is turned into non-living material as soon as it is formed and does not contribute to any further new material, so that growth is an additive, rather than multiplicative process. Of course, as in any interest curve, the amount of growth can be seen as the product of the initial amount and interest, but this is still different from growth as a

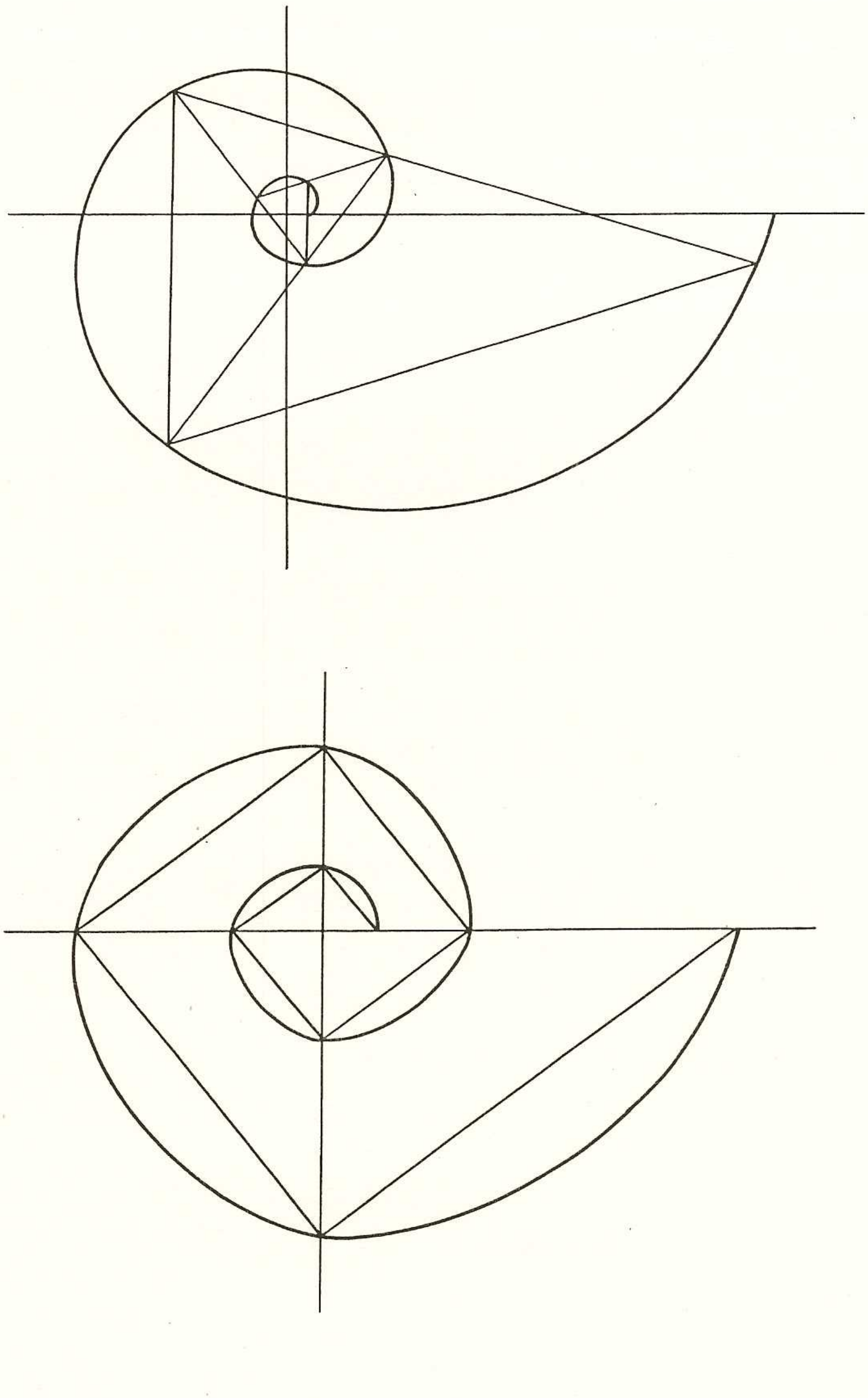

Figure 6. Equiangular and rectangular spirals.

multiplicative process. In the latter process, new material produced by the growth process itself grows and, in turn, produces new material in an exponential manner. The spiral constant appears to reflect some kind of accretionary, isometric growth, a system in which size increases while shape is retained.

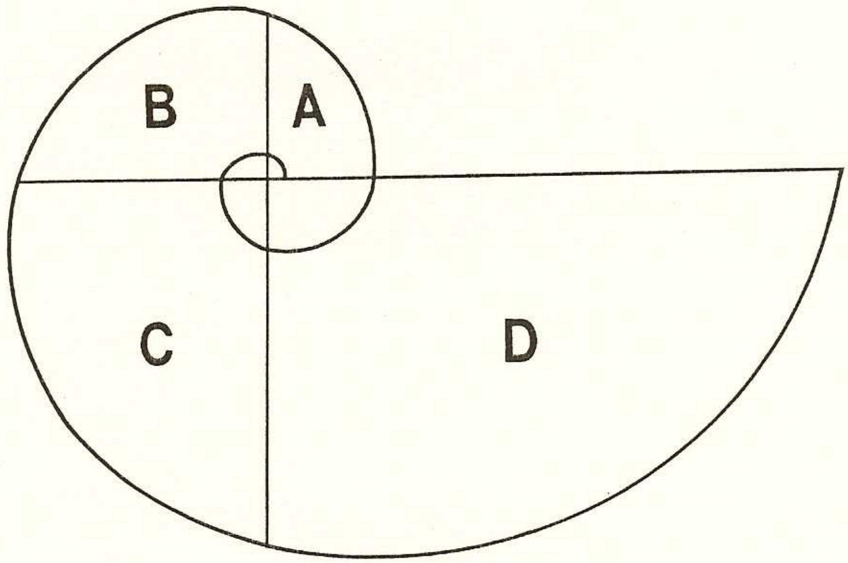

Figure 7. Sections of an equiangular spiral. Although greatly different in size, sections A, B, C and D are identical in shape. (After Davis and Hersh, 1986.)

The accumulative growth discussed by Sahal (1978, 1374) results in the sort of self-similarity noted in so many real-world applications by Mandelbrot (1983). Goodchild and Mark (1987) state that probability distributions governing the sizes of self-similar phenomena result in Pareto distributions and rank-size relations. They offer this definition of self-similarity; a definition which expresses in words the diagram of the shell in Figure 7: "A feature is said to be self-similar if any part of the feature, appropriately enlarged, is indistinguishable from the feature as a whole" (1987, 268). Arlinghaus (1985) discusses self-similarity as a key component of algorithms which generate fractal structures; indeed she shows that the geometry of central place theory is a subset of the theory of fractal geometry. Further, it appears that all three of the interrelated concepts which characterize fractals can be related to rank-size. In addition to self-similarity, these three concepts include response of measure to scale and the recursive subdivision of space. The response of measure to scale, known to many geographers through the problem of determining the length of a shoreline, applies to rank-size in the issue of determining the level at which the urban system ends. Are towns, villages or hamlets to be included in a particular urban hierarchy? If so, the rank-size distribution may take on strange shapes. The successive levels of the urban hierarchy defined by the spiral constant are successive divisions of the same areas into tributary regions, that is, recursive divisions of space. Hofstadter, (1979, 495-548) offers a philosophical survey of the concept of self-similarity. The concepts of central place, self-similarity, rank-size, fractals and the spiral constant are interrelated in more than a superficial way.

This understanding of the spiral constant as a reflection of growth in size without change in shape, offers an opportunity to review several observations which have accumulated in rank-size research and to suggest a possible synthesis of some of these findings. For example the hexagon, the geometric shape of market areas in central place theory, could be the basic clue to further understanding of the rank-size relationship. Some researchers who have already explored relationships between rank-size and central place concepts include Beckman (1958), Beguin (1985), Higgs (1970), Parr (1969 and 1970) and Berry (1971). The development of a hierarchical network of larger and larger hexagons is an obvious example of a system increasing in size while retaining its shape. Indeed, as will be shown, hexagons can provide points for the construction of an equiangular spiral. While it is not immediately apparent how hexagons could combine to produce a ratio of k = 1.618, it is an intriguing concept. As will be discussed, the slope of -1 in a rank-size distribution may be a reflection of the isometry of this hexagonal hierarchy. Gilbrat's law and the Yule distribution, assumptions used to explain how a central place hierarchy can be modified by stochastic processes to result in a rank-size distribution, can also be conceptualized as mechanisms to retain the constancy of shape. Gilbrat's law, which proposes that the growth rate is the same for each size class, means that the urban areas maintain the same proportion to each other as they grow - - it is, in effect, a restriction on the equation that produces isometric growth.

Generating Rank-Size by the Spiral Constant

While it is clear that numbers in Fibonacci sequence are additive by definition, a series of values in ϕ proportion to each other should also be additive which can be shown as follows: The golden section or spiral constant satisfies the following identity:

$$\phi = 1 + \frac{1}{\phi} \qquad (2)$$

This can be proven as follows:

$$1 + \frac{1}{\phi} = 1 + 2/(\sqrt{5}+1) = (\sqrt{5}+3)/(\sqrt{5}+1)$$

Multipling by $(\sqrt{5}-1)/(\sqrt{5}-1)$ yields

$$(5 + 3\sqrt{5} - \sqrt{5} - 3)/(5-1) = (2 + 2\sqrt{5})/(4)$$

or $$(1 + \sqrt{5})/(2) = \phi$$

Returning to the identity (2), if we multiply both sides by ϕ, we see that $\phi^2 = \phi + 1$

and in turn $\phi^3 = \phi^2 + \phi^1$

and in general

$$\phi^n = \phi^{n-1} + \phi^{n-2} \qquad (3)$$

(Cook, 441; Ghyka, 8)

Table 1 shows this relationship.

TABLE 1

Values of successsive powers of ϕ

ϕ^0	=	1.000000
ϕ^1	=	1.618034
ϕ^2	=	2.618034
ϕ^3	=	4.236068
ϕ^4	=	6.854102
ϕ^5	=	11.090170
ϕ^6	=	17.944273
ϕ^7	=	29.034443
ϕ^8	=	46.978716
ϕ^9	=	76.013160
ϕ^{10}	=	122.991880
ϕ^{11}	=	199.005040
ϕ^n	=	$\phi^{n-1} + \phi^{n-2}$

Thus the sequential powers of the spiral constant are, in themselves, a sequence. This relationship is unique to ϕ. There is no other series for which the values of successive powers can be generated by addition of the values of the two preceeding powers. (Cook, 1914, 442)

If U_n and U_{n+1} represent the nth and the (n+1)st Fibonacci numbers, then

$$\phi \rightarrow (U_{n+1})/(U_n) \text{ as } n \rightarrow \infty \qquad (4)$$

(Huntley, 141-44)

As shown in Table 2, quotients with higher Fibonacci numbers approximate ϕ more closely than do lower Fibonacci numbers. This can be seen also by taking the identity $\phi = 1 + \frac{1}{\phi}$ and viewing it as the continued fraction

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$$

TABLE 2

Approximate value of ϕ to five decimal places

8/5	=	1.60000
13/8	=	1.62500
21/13	=	1.61538
34/21	=	1.61905
55/34	=	1.61765
89/55	=	1.61818
144/89	=	1.61798
233/144	=	1.61806
377/233	=	1.61803

If we examine the sequence at each substitution, we get 1+1, 1 + 1/1+1, 1 + 1/1+1/1+1... which yields 2, 3/2, 5/3 or, $(U_{n+1})/U_n$, thus each successive quotient approximates ϕ more and more closely as we saw in Table 2, and gives empirical motivation for the validity of (4).

When applying Fibonacci numbers to an equiangular spiral, the extent to which a set of ranked data can be plotted as an equiangular spiral depends on the extent to which empirical values at Fibonacci intervals in proportion to each other approximate the spiral constant. That is, for the empirically derived curve to correspond to the equiangular spiral, the ratio of population size of the largest urbanized area to that of the second largest urbanized area must be approximately 1.61803..., as will the ratios of the third to second largest, the fifth to third largest, and so on. Expressed another way,the application of ϕ to the ranking of urbanized areas can be demonstrated by examining the population of the urbanized areas in rank intervals which match the Fibonacci series and by comparing the empirical data to values derived from successive division by 1.618 of the population of the largest city (Tables 3 and 4). For example, in 1980 the population of the largest urbanized area (New York - Northeast New Jersey) was 15,590,274. Division of this figure by 1.618 yields 9,635,522 as the expected or predicted value for the second largest urbanized area. The actual value of the second urbanized area is 9,479,436, (Table 4). The symbol "f" will be used to indicate rank position in the Fibonacci sequence (that is standard ranks 5 and 8 are the third and fourth terms in a Fibonacci sequence). In effect, the first column of predicted values is generated from the population of the largest city divided by sequential powers of ϕ.

TABLE 3

Actual and predicted population of U.S. urbanized areas, 1970

Urbanized Area	Census Rank	Fibonacci Rank (f)	Actual Population	Predicted Population $(P_1)/(\phi^f)$	(p_f/ϕ)
New York-Northeast, NJ	1	0	16,206,841	--	--
Los Angeles-Long Beach, CA	2	1	8,351,266	10,016,589	10,016,589
Chicago-Northwest, IN	3	2	6,714,578	6,190,723	5,161,475
Detroit, MI	5	3	3,970,584	3,826,157	4,149,925
Washington, DC	8	4	2,481,459	2,364,745	2,454,007
Houston, TX	13	5	1,677,863	1,461,523	1,533,658
Cincinnati, OH	21	6	1,110,514	903,290	1,036,998
Dayton, OH	34	7	685,942	558,276	686,350
Richmond, VA	55	8	416,563	345,041	423,944
Las Vegas, NV	89	9	236,681	213,251	257,456
New London-Norwich, CT	144	10	139,121	131,799	146,280
Great Falls, MT	233	11	70,905	81,458	85,983

TABLE 4

Actual and predicted population of U.S. urbanized areas, 1980

Urbanized Area	Census Rank	Fibonacci Rank (f)	Actual Population	Predicted Population	
				$(P_1)/(\phi^f)$	(P_f/ϕ)
New York-Northeast, NJ	1	0	15,590,274	--	--
Los Angeles-Long Beach, CA	2	1	9,479,436	9,635,522	9,635,522
Chicago-Northwest IN	3	2	6,779,799	5,955,205	5,858,737
Detroit, MI	5	3	3,809,327	3,680,596	4,190,234
Boston, MA	8	4	2,678,762	2,274,782	2,354,343
Minneapolis-St. Paul, MN	13	5	1,787,564	1,405,922	1,655,601
Denver, CO	21	6	1,352,070	868,926	1,104,799
Sacramento, CA	34	7	796,266	537,037	835,643
Albany-Schen.-Troy, NY	55	8	490,015	331,914	492,130
Colorado Springs, CO	89	9	276,872	205,138	302,852
Lancaster, PA	144	10	157,385	126,785	171,120
Elkart-Goshen, IN	233	11	83,920	78,359	97,271

This method of predicting urban values is based on the structure of the Fibonacci sequence in that each successive term in the sequence can be predicted once the first term is known. The second term is then approximately 1.61803 times the first term, the third term is approximately 1.61803 times that of the second term and so forth.

A second method of using ϕ to generate predicted population values is shown in the column on the far right of Tables 3 and 4. The actual population at each rank is divided by ϕ to predict the population of the city of the following rank in Fibonacci order, that is

$$p = (p_1)/(\phi^f) \tag{5}$$

This method of predicting urban values is not based solely on the first term, but rather on (4) above, which shows that of any successive terms in a Fibonacci sequence, $(U_{n+1})/(U_n) \approx 1.61803$. The larger term is used to predict the following smaller term. A third method of predicting values, not given here, would be to use the smaller value to predict the next larger one in the sequence.

Table 5 illustrates the average of these values for the United States urbanized area data for 1980. The data in Table 5 are derived as follows: the actual populations of cities in Fibonacci rank are calculated as a ratio. For example, in 1980 the ratio of the population of the 5th ranked city (Detroit) to that of the 8th ranked city, (Boston) is calculated. The ratio is 1.422 (3,809,327/2,678,762). (See Table 5.) When the calculations were completed for all the successive pairs of urbanized areas listed in Table 4, an average of all the values was taken. This average for the 1, 2, 3, 5 ...series for 1980 was 1.6175. The process was repeated for 1980 for the 1, 3, 4, 7...Lucas series and then both series were calculated for 1970 data in Table 3. The average of these values closely approximates the value of

TABLE 5

Average ratios between 1980 urbanized areas in Fibonacci rank

Rank	Urbanized Area	Population	Ratio*
1	New York	15,590,274	
2	Los Angeles - Long Beach	9,479,436	1.6446
3	Chicago - NW Indiana	6,779,799	1.3982
5	Detroit	3,809,327	1.7799
8	Boston	2,678,762	1.4220
13	Minneapolis - St. Paul	1,787,564	1.4986
21	Denver	1,352,070	1.3221
34	Sacramento	796,266	1.6980
55	Albany - Schenectady - Troy	490,015	1.6250
89	Colorado Springs	276,872	1.7698
144	Lowell, MA	157,385	1.7592
233	Elkhart - Goshen, IN	83,920	1.8754
	Average		= 1.6175
	Average (excluding Rank 2/Rank 1)		= 1.6148

*Ratio of the urban area in the row to the lower ranked urban area.

ϕ whether the 1, 2, 3, 5 Fibonacci series or the 1, 3, 4, 7 sequence is used. (Table 6)

TABLE 6

Mean ratio of urbanized areas

Sequence	Year	Mean Ratio
1,2,3,5,8...	1970	1.653
1,3,4,7,11...	1970	1.650
1,2,3,5,8...	1980	1.618
1,3,4,7,11..	1980	1.628

There is some evidence from traditional rank-size research that the largest city in many urban hierarchies is larger than predicted by the usual rank-size formula. In an attempt to explore this possibility, a second set of calculations was performed the same way on the same data in Tables 3 and 4 except that the ratio of the largest city to the second largest was excluded. Table 7 shows the result. In 1970 the modified calculations were substantially closer to the expected value of 1.618 but the 1980 calculations were slightly further apart; however, in both the 1979 and 1989 cases the value 1.618 produces sequences of values descriptive of city size at the time, demonstrating the dynamic character of this procedure.

TABLE 7

Mean ratio of urbanized area excluding largest city

Sequence	Year	Mean Ratio
2,3,5,8...	1970	1.624
3,4,7,11...	1970	1.621
2,3,5,8...	1980	1.615
3,4,7,11...	1980	1.626

Another way of demonstrating that the ϕ ratio and, in turn, the equiangular spiral can be used to describe the rank-size structure of the urbanized areas of the United States is by examining the additive nature of the populations of cities of sequential Fibonacci ranks. As was demonstrated in (3), data which are in ϕ proportion are also additive and thus addition can be used to predict the value of the next datum. For example, if the spiral constant is reflected in the rank-size hierarchy of urbanized areas of the United States, the population of the city of rank 233 added to the population of the city of rank 144 should approximately equal the population of the city of rank 89. Similarly, at the higher end of the scale, the population of the largest urbanized area should be approximately the size of the sum of the population of the second and third largest urbanized areas. Table 8 shows this additive nature of the population of urbanized areas by using empirical data from 1980. For the first entry in Table 8 the 1980 population of the New York urbanized area is 15,590,274. The model predicts that it should be equal to the population of Los Angeles (9,479,436) added to that of Chicago (6,779,799). The sum of the populations of the two smaller cities is 16,259,235. In the far right column the ratio of the actual population of New York to the estimated one is 96%. All the estimates on Table 8 fall within 83% to 115% of actual values.

One last way of showing conclusively that the spiral constant and, in turn, Fibonacci sequences, are related to the simple rank-size relationship would be by mathematically demonstrating this fact. Tables 9 and 10 give empirical evidence by comparing values of urban size predicted by the simple rank-size formula, with values predicted by successive division by ϕ for the 1, 2, 3, 5. . . and 1, 3, 4, 7. . . Fibonacci sequences. The first row in Table 9, for example, shows that the city of second rank calculated by the simple rank-size formula is predicted to have a value of .50 of that of the

TABLE 8

Additive nature of the population of urbanized areas

Rank	Actual Population (1980)	Population Estimated by Adding Two Preceeding Lower Ranked Values		(Actual)/ (Estimate) (%)
1	15,590,274	16,259,235	(2 + 3)	.96
2	9,479,436	10,589,126	(3 + 5)	.90
3	6,779,799	6,488,089	(5 + 8)*	1.04
5	3,809,327	4,466,326	(8 + 13)	.85
8	2,678,762	3,139,634	(13 + 21)	.85
13	1,787,564	2,148,336	(21 + 34)	.83
21	1,352,070	1,286,281	(34 + 55)	1.05
34	796,266	766,887	(55 + 89)	1.04
55	490,015	434,257	(89 + 144)	1.13
89	276,872	241,305	(144 + 233)	1.15
144	157,385	--		
233	83,920	--		

*Example: The population of the 3rd largest city (6,779,799) can be approximated by adding the populations of the 5th largest city (3,809,327) and the 8th largest city (2,678,762). (3,809,327 + 2,678,762) = 6,488,089.

largest city. By the ϕ method of calculating rank-size, the second ranked city would have a population of .618 of that of the largest city. The value in the right hand column expresses the former value as a proportion of the latter (.5/.618 = .81). There appears to be a fairly direct relationship between the two sets of predicted values with a systematic bias such that the value generated by the simple rank-size rule is approximately 85% of the value generated by the ϕ method of calculation.

The relationship is even more striking in Table 10. After the seventh-ranked city, the values generated by the simple rank-size rule and the values predicted by division by ϕ from the Lucas series 1, 3, 4, 7, 11... approach unity. That is, the two methods (rank-size and the spiral constant) generate essentially the same set of data. Clearly, here is another distribution different from the lognormal, the Pareto and the Yule which can generate the rank-size relationship. Yet the rank-size and the spiral constant methods of predicting urban hierarchies are not identical. Is one method perhaps a "shadow" or "phantom" index of the other? Here is a startling phenomenon: examination of Table 10 and Figure 8, which plots both measures on log-log paper, suggests that the simple rank-size rule may "work" simply because division by 2, for example, generates a value close to that generated by division by 1.618; division by 3 is close to division by 2.618; division by 4 is close to division by 4.236; division by 7 is approximately equal to division by 6.854; division by 11 is close to division by 11.090, etc. At the higher ranks the pairs of values become essentially identical. The advantage of this formulation of the rank-size rule is that one can clearly see the workings of the mechanism that generates rank-size. A mathematical model transforming a probability distribution is not needed. In addition what makes this formulation unique is not just its relative simplicity, but the fact that

TABLE 9

Comparison of predicted population calculated by rank-size and the spiral constant (Fibonacci Series 1, 2, 3, 5. . .)

Rank	Value of ϕ	Predicted by Rank-Size	Predicted by ϕ	(Rank-Size Population)/ (ϕ Prediction)
1				
2	1.618	.5000	.6180	.81
3	2.618	.3333	.3820	.87
5	4.236	.2000	.2361	.85
8	6.854	.1250	.1459	.86
13	11.090	.0769	.0902	.85
21	17.944	.0476	.0557	.85
34	29.034	.0294	.0344	.85
55	46.979	.0182	.0213	.85
89	76.013	.0112	.0132	.85
144	122.992	.0069	.0081	.85
233	199.005	.0043	.0050	.86

TABLE 10

Comparison of predicted population calculated by rank-size and the spiral constant
(Fibonacci Series 1, 2, 3, 4, 7 . . .)

Rank	Value of ϕ	Predicted by Rank-Size	Predicted by ϕ	(Rank-Size Population)/ (ϕ Prediction)
1				
2	1.618	.5000	.6180	.81
3	2.618	.3333	.3820	.87
4	4.236	.2500	.2361	1.06
7	6.854	.1429	.1459	.98
11	11.090	.0909	.0902	1.01
18	17.944	.0556	.0557	1.00
29	29.034	.0345	.0344	1.00
47	46.979	.0213	.0213	1.00
76	76.013	.0132	.0132	1.00
123	122.992	.0081	.0081	1.00
199	199.005	.0050	.0050	1.00

some properties of the spiral constant indicate that it may be a more meaningful measure of the urban hierarchy than the rank-size distribution.

When two hypotheses offer equal explanatory power, we assume that the more valid of the two is generally accepted to be that which is more sound, theoretically (based on mathematical logic). A frequently offered, although by no means totally accepted, explanation for the rank-size relationship is based on a central place hierarchy subjected to random disturbance. As mentioned earlier, this explanation depends on the additional assumptions expressed by Gilbrat's Law and the Yule Theorem. As we have seen empirically demonstrated, however, the rank-size relation and the ϕ relation are closely related, thus any evidence supporting the simple rank size relationship can also be interpreted as supporting the ϕ relation as well. Which then is more valid or need they be viewed as competing strategies?

The Slope of the Rank-Size Distribution

Some insight into the problem of which index is more valid can be gained by examination of the factor of slope in the linear log-log graph. The simple rank-size rule is usually regarded as a special case of the more general rank-size rule

$$r_i(p_i^q) = K \qquad (6)$$

where q is an exponent with a value of 1 in the simple rank-size rule. In rank-size research the exponent of 1 generates a slope of -1 on a log-log graph. Usually this slope of unity is seen as coincidental, an accompanying but not necessarily theoretically important accessory characteristic of the rank-size relationship, despite the frequency with which an exponent near unity occurs when empirical data are examined. A slope near unity is commonly

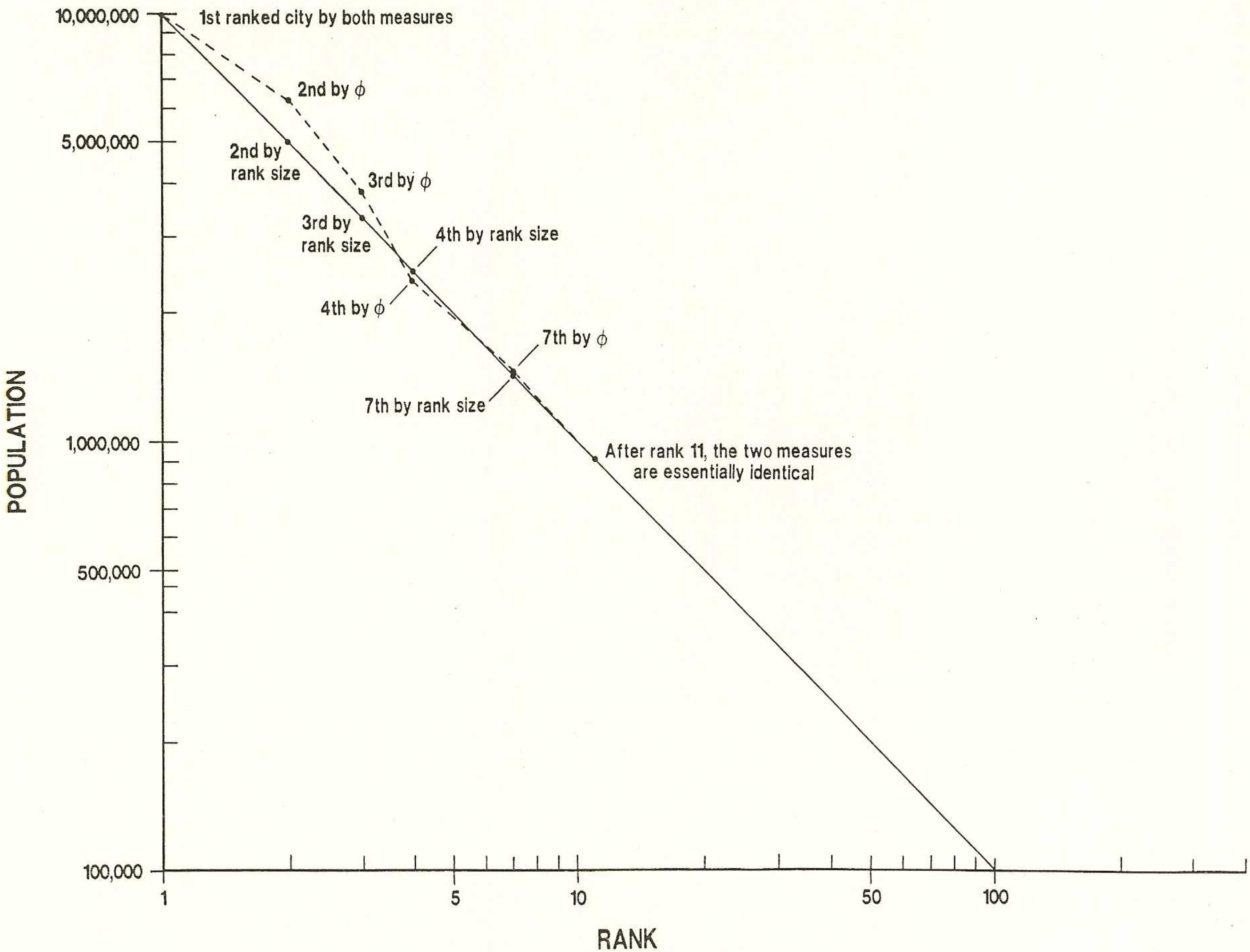

Figure 8. Comparison of urban hierarchy plotted by ϕ with one plotted by rank-size. (P_1 = 10,000,000. Rank 1-100. For Fibonacci series 1, 2, 3, 4, 7, 11 ...)

found in studies of the urban system of the United States. The research in this paper suggests that the slope of -1 (an exponent of 1.0 and an angle of 45° in the slope of the function on log-log graph paper) is not coincidental, but rather derives from the fact that Fibonacci sequences and the ϕ function when plotted on a log-log graph, both approach a slope of -1 (Figure 8). In fact the difference in the higher values between the two indices may be responsible for the frequently noted "concavity" in the empirical data. This issue of slope is thus worth exploring at some length.

Frequently the line which best approximates the slope of graphed rank-size data is close to unity and the slope is simply left out of the equation by omission of the exponent q. In fact the phrase "simple rank-size rule" means the rank size formula is used but the exponent, q, is ignored. Inclusion of the exponent allows the formula to be adapted to rank-size distributions which are log-linear but which do not exhibit a slope of -1. Because the rank-size rule can be adapted in this fashion, it is tempting, but incorrect, to assume that the simple rank-size rule is a special or odd case of a more general, more useful formula. Beguin, for example, suggests that "The case where the general slope is 1 is not particularly significant. . ." and that it "appears as some empiric average observation" (Beguin 1984, 754). Rosen and Resnick (1980,166) state that the rank-size rule is a special case of the Pareto distribution where the exponent equals 1. Instead, this research argues that the simple rank-size rule _is_ the general formula and other distributions should be explained as variations of the general principle. The situation is complicated by the fact that the ranked empirical data never exactly equal a slope of 1.000. It does not necessarily follow that this slope, thus, is coincidental; indeed it would be startling, given the myriad forces impacting upon urban systems and the inaccuracies inherent in the collection of data, if the slope of the empirical data of any rank-size hierarchy exactly equaled unity, although it does come close for aggregate United States data (Rosen and Resnick 1980,171).

Almost every rank-size researcher has considered, in some fashion, the issue of the slope of unity. Z. K. Zipf, (1949, 131) in his classic work, Human Behavior and the Principle of Least Effort noted that "it is only natural to ask why the value, p = 1 (the slope) should be of critical significance..." He interpreted the slope as a result of the equation of the generalized harmonic series and tried to explain the slope in a partially developed argument concerning competing forces of unity (clustering and homogenizing forces) and disunity (scattering and differentiating forces) (Zipf 1949, 130-31). This is not unlike Coffey's definition (1981, 227) that in general,..." a hierarchical spatial system is a dynamic equilibrium state or, if you prefer, a compromise between the opposing tendencies of organization and disorganization, between complete agglomeration and complete disagglomeration" or Sahal's (1978, 1374-75) description of accumulative growth and differential growth.

Although mainly dealing with business firms rather than urban areas, Steindl (1965) sees the slope of unity in a rank-size distribution as analagous to a center of gravity about a mean. He argues that given an initial distribution of different sizes of firms, concentration takes place which results in the elimination of large numbers of smaller firms and the creation of a few larger ones. Similar principles obviously apply to urban areas. Some firms continue to grow while numerous smaller firms attempt to get started. In theory, a variety of mechanisms operate to maintain equilibrium. For example, continuous development of larger firms implies a successful industry and thus more small firms enter. If the growth rate of firms declines, mortality of firms is increased and the process is reversed.

The significance of the slope of unity is that, in theory, at this value, growth of firms can continue indefinitely. Steindl states that the

> tendency to concentrate is to some extent endemic, which explains why the Pareto coefficient hovers round a level of 1.1 or so in most cases. The growth of industry, even while equilibrium - in terms of the model - lasts and the mean size of the firm is stable, leads to tension owing to the fact that some firms become very large in relation to the rest of the industry. This change is connected with a statistical measure of "concentration" in a different sense - the share of the largest one, two, or three, etc., firms; it is obvious that concentration in this sense of the term proceeds continuously in a growing industry, while the equilibrium of our model is undisturbed and the mean size of the firm remains stable.

Woldenberg (1971) and Coffey (1981) also attach significance to the slope of -1. They see it as <u>the</u> differentiating factor between allometric and isometric growth. Both allometric and isometric growth can be expressed as a straight line when plotted on log-log graph paper. When two factors reflecting allometric growth are plotted (body weight to surface area, to use an organic example), one grows faster than the other and this is reflected in an exponent and slope other than -1.0. Isometry is a special case of growth which, by definition, is different from allometric growth because both factors grow at the same rate. Visually, this can be seen on the log-log graph by the maintenance of a 45° angle, or equidistance, between the line marking growth and the two axes reflecting the growth. Woldenberg (1971, 4-5) explains that strong allometric growth, a strong influence in one direction rather than another, cannot continue for long because of size limitations. This correlates with Steindl's observation that growth at a slope of -1, isometric growth, can theoretically continue indefinitely. Coffey (1981, 194) summarizes Bertalanffy's concern with the exponent (slope) as follows:

> Briefly, the allometric relationship may be viewed as an expression of competition within a given system, with each system component taking its share of the available resources of the total system according to its capacity, as expressed by the exponent. The exponent may thus be regarded as a "growth partition coefficient" that expresses the capacity of a component to seize its share

> of the resources. An exponent indicating positive allometry, having a value greater than one when the equation is dimensionally balanced, signifies that the component in question captures a proportionately larger share of the resources than either the total system or a second component. Conversely, an exponent less than one indicates that a component captures a share proportionately less than the system or a second component.

It follows that an exponent of exactly one means the components are securing resources exactly in proportion to their respective sizes (Gilbrat's Law, or the law of proportionate effect, again).

It should also be noted that the slope of unity has significance in physics. In the logarithmic plot of functions which illustrate exponential decay with time, Shire and Weber (1982,27) note that special significance is attached to the time at which the exponent of the function equals unity. (The exponent changes as the process goes through time.) At that instant, approximately 37% of the decay has occurred and this instant is called the one-over-e time (e for the natural logarithm), the relaxation time, or the time constant of the process. This measure of time, which is system-dependent, may be related to Sahal's statement (1979,1375) related to the evolution of a system: "According to the framework provided by the Pareto distribution, the appropriate concept is then one of system-specific time; that is, the passage of time measured by successive events which are specific to the system under consideration."

Despite the general acknowledgement of some unexplained significance to the slope of unity, rank-size researchers usually believe the slope of -1.0 derived from the simple rank-size rule to be a special case of a more general rank-size process. Carroll (1982, 1) calls the simple rank-size rule the "restrictive rank-size rule" and in a discussion of the "misleading rank-size versus primate continuum", he states "In particular, the rank-size rule with exponent of unity is too restrictive to be the limiting case against which primacy is contrasted" (1982, 3). Berry (1971, 148) is aware of the slope of unity but interprets it as a measure of balance between cities in the system and cities entering the system, that is, a reflection of the Yule distribution:

> . . . unity is therefore the value of the exponent to be expected if one is dealing with a fixed number of cities (lognormal case) as opposed to the case in which the number of cities is growing. If (the part of growth attributed to the entry of new cities reaching a threshold size) is substantial, as in the case of a system of cities rapidly expanding in numbers above threshold, q (the exponent) will exceed unity. If, on the other hand, the number of centers in the system is shrinking . . . q will be less than 1.0. In general, a value of q exceeding unity implies proportionately more small places, whereas q's of less than one arise with increasing concentration of the urban population in larger cities.

In Berry's explanation, as in many of the others, a balance of competing forces is seen as the mechanism generating a slope of unity. His explanation also dovetails with that of Woldenberg, above, in that rapid growth of cities or very slow growth produces allometric rather than isometric growth because growth progresses faster along one axis rather than another.

The slope of empirical data which are linear but which do not approximate a slope of -1, has not been a major concern of this research. As ϕ and the simple rank-size formula are essentially the same measure after rank seven, the ϕ formula could be adapted mathematically to fit other slopes just as the simple rank-size formula can be adapted. Near the end of this paper, the application of a systematic bias to ϕ will show how this can be accomplished.

The Spiral Constant and Concavity of Rank-Size Distributions

The slope of -1, and the exponent of 1 in the simple rank-size equation, are interrelated phenomena that have occupied the attention of many rank-size researchers. Both the simple rank-size rule and the ϕ method of deriving rank-size produce a slope of -1 on a log-log graph; the former equation does so exactly; the latter approximates that slope after rank seven. Figure 8 shows how the two derivations compare. Given the preceeding discussion of the significance of the slope of -1 in the rank-size relationship, is the ϕ method of deriving the rank-size distribution the less accurate of the two methods? The answer appears to be "no" when empirical data are taken into account. Note that the line representing the graph of the values derived from ϕ , in a sense, bulges, or is deflected upward from the line derived from the simple rank-size rule. This phenomenon is referred to as being concave to the origin of the graph, and far from being an unexpected aberration when compared to empirical data, rank-size distributions which are concave to the origin at the higher ranks are a frequently occurring and largely unexplained phenomenon.

Ijiri and Simon, in discussing Pareto and Yule distributions applied to sizes of business firms state quite clearly: "When the fit between theory and data is examined, systematic differences can be seen between the theoretical and empirical curves. The most important of these is that the empirical data, when plotted on log-log graph paper, almost always exhibits a noticeable concavity toward the origin" (Ijiri and Simon 1977, 12). In their work they propose various adjustments in the rank-size formula to produce such concavity. In general terms they see the concavity as a result of growth processes of firms in which (1) the rate at which new firms develop is not constant but decreases gradually over time or (2) growth is largely static yet sizes of some firms continue to increase or decrease in a fashion which follows Gilbrat's law. Beguin (1982, 230-31) summarizes work by Parr, Vining, Malecki and others, and notes that "This concavity is well known in the literature and it is frequently observed in empirical data (although not always located in the same sections of the curve..." He later states that "...the concavity is essentially observed in the highest level of the hierarchy..." (Beguin 1982, 230-31). He suggests that one consider the concavity as a systematic upward deviation from the rank-size formula. Beguin discusses diseconomies of scale in urban growth in conjunction with rank-size as the possible producers of this distribution. Sahal also cites this phenomenon: "It should be noted that the empirical distributions of the Pareto Law invariably exhibit the observed anomalies. Typically they tend to be concave upward and are linear or nearly so only over the lower range of data" (Sahal, 1981, 292-93). He attributes this, again, to the general phenomenon of the inability of simple laws to accurately describe values across whole ranges of data. Thus, although the hypothesis is not empirically tested in this research, the ϕ formulation clearly offers the possibility that it more closely approximates urban empirical data than does the rank-size rule because ϕ predicts concavity to the origin in the upper tail of the distribution. Although it is not an explanation for the phenomenon, it is easily seen that the concave deviation from the straight line of the rank-size distribution is the result of the numerical differences between ϕ and rank-size in ranks up to rank 11 which were shown on Figure 8 and discussed in reference to Tables 9 and 10. That is, the deviation results from the graphical plot of 2 instead of 1.618; 3 instead of 2.618; 4 instead of 4.236, etc.

The Spiral Constant and the Primate City

At first glance, the spiral constant may seem unable to offer any insight into the phenomenon of the primate city. A look again at Table 9 will illustrate this. The simple rank-size formulation suggests that the second largest city should be one-half the size (.50) of the largest city and the third largest should be one-third, (.33) of that size. The ϕ method of deriving rank-size suggests the second largest city should be .618 the size of the largest city and the third largest city should be .381 the size of the largest city. Thus the ϕ formulation projects larger expected second and third ranked cities than does the simple rank-size formula, so a primate distribution lands even farther from the mark in the ϕ formulation than in the simple rank-size formulation. Expressed another way, if primate cities seem "larger than they should be" by rank-size, they are larger still by the ϕ formula. On the other hand, if one argues that primacy reflects a situation where the largest city usurps much of the population and many of the functions of the second, or second and third largest cities, then the ϕ formula offers a more plausible measure of primacy because the second and third largest cities, as calculated by Φ, offer more to usurp. That is, (.618 + .381 = .999) compared to (.5 + 333 = .833). Berry (1971) and Carroll (1982) remind us that rank-size and primate distributions are not, in some way, opposite kinds of distributions. Vapharsky (1969, 584) stated "It can be observed that primacy and rank-size rule are not mutually exclusive models. Rather, a perfect fit to the rank-size rule of all cities in an area except the largest is compatible with a high level of primacy." Stewart (1958) also studied the interplay of primacy and rank-size and concluded that

> The results diverge considerably from the rank-size rule. For most countries not only is the second city much less than half the size of the largest but other ratios also differ somewhat from the rule. The size difference between second and third cities, and between third and fourth cities, is larger than the rule postulates. The fourth and fifth cities are close together in size.

He goes on to conclude that "Well-structured areas of urban dominance tend to have an S-shaped, rather than a linear logarithmic, distribution of towns by size." Table 11 shows the data Stewart saw as diverging from the rank-size rule:

Based on the additive properties of the the spiral constant derivation of rank-size (the sum of the populations of the fifth and the third largest cities should approximate the population of the second largest for example),

TABLE 11

Median size of five largest cities as a fraction of the largest city

Area	Largest	Second	Third	Fourth	Fifth
72 Countries	1	.315	.200	.140	.120
Australia (states)	1	.076	.0405	.024	.019
Brazil (states)	1	.210	.135	.105	.0785
Canada (provinces)	1	.340	.220	.140	.078
India (states)	1	.440	.365	.280	. . .
United States (states)	1	.435	.310	.200	.165
U.S.S.R. (republics)	1	.375	. . .	. . .	. . .
Rank-Size Rule	1	.500	.333	.250	.200

Source: Stewart, 1958.

Table 12 shows the expected values for the second largest cities for all areas Stewart analyzed in Table 11 for which the fifth largest city was given. Rather than the chaos that Stewart found, a good deal of empirical regularity can be demonstrated by the spiral constant using the very same data Stewart used.

The spiral constant offers new insight into primacy because unlike the traditional rank-size formulation and unlike central place theory, the

Φ formulation of the urban hierarchy suggests that in the series 1, 2, 3, 5, 8, 13. . . urban areas of ranks 1, 2 and 3 are separate, distinct levels. As

TABLE 12

Expected size of Stewart's second largest cities based on additive values

Area	Stewart's Second Largest City	Ranks (5+3)
72 Countries	.315	.320
Australia (states)	.076	.060
Brazil (states)	.210	.214
Canada (provinces)	.340	.298
Unites States (states)	.435	.475
Rank-Size Rule	.500	.533

shown in Table 13, it is not until the fourth ranked city that urban areas share a level; that is 1, 2, 3 and 4-5 are four levels of the hierarchy. A possible interpretation of this structure derived from the spiral constant is that the two or three largest cities of an urban system all serve a national market. Primate cities, as some have already suggested, may represent a situation in which the largest city has usurped the functions of the second and/or third largest cities. Beguin (1984, 754) makes a similar case when, in a discussion of concave distributions, he argues that the concept of primacy does not have to be restricted to an arbitrary number of large cities. Vapharsky (1969, 589) also wrote of "group primacy" in regions of Argentina. Expressed another way, the spiral constant offers theoretical grounds to argue that an urban system can be expected to have its three cities share a national

TABLE 13

Levels within an urban hierarchy defined by the spiral constant

	NUMBER OF CITIES AT EACH LEVEL
1	1
2	1
3	1
4 5	2
6 7 8	3
9 10 11 12 13	5
14 15 16 17 18 19 20 21	8

Although cities of ranks 2 and 3 are clearly smaller than and subordinate to city 1, it is not until cities of rank 4 and 5 that levels of function are duplicated. That is, city 3 is smaller than and different in function from city 2, whereas city 5 is smaller than city 4, but similar in function; cities 6, 7 and 8 are similar in function, etc.

market with some overlap of functions (New York, Los Angeles and Chicago come to mind). As has been noted in the research, (Lensky, 1965), nations small in area may be more likely to have these functions concentrated in a single city. Another advantage offered by the spiral constant approach is that the 1,3,4,7 hierarchy may fit some nations better than the 1,2,3,5 hierarchy; that is, there are multiple patterns of distribution generated by the spiral constant. This fits in well with research which argues that primate cities should be analyzed from a perspective other than that of an aberration from expected distributions (Rose, 1966, for example).

affords wide latitude in plotting data. Even when the plotted data exhibit a "good fit", it is striking how far off the mark are predictions of the ranks of specific urban areas. Anyone who has had the experience of plotting a rank-size curve knows how slight a difference even several thousands of population can make in the placement of a dot on the graph in the high and middle ranges of many sets of urban data. Rapoport (1978,847) reminds us that some monotonically decreasing curve will describe just about any group of objects arranged according to size. Secondly, the existence of a straight line on a graph also depends, to an extent, on the portion of the urban distribution selected for analysis -- the lowest levels of the urban hierarchy frequently deviate markedly from the projected straight line (and the mathematical formula) and thus are frequently omitted from analysis because attention is given to the largest urban areas. The largest cities, as well, usually deviate significantly from the values predicted by the formula or the straight line on a graph. This is the phenomenon known as the primate city. Attempts to accommodate the lower portion of the curve often displace the upper portion and vice-versa. Sahal (1981,294) has discussed this as a problem of results being sensitive to the origin of the independent variable. Applied to the simple rank-size rule this means, for example, that instead of attempting to predict the size of the 100th largest city by dividing the population of the largest city by 100, we could, with equal logic, but with different results, predict the size of the largest city by multiplying the population of the city of rank 100 by 100. This follows, of course, from equation (7) above in which the product of these two factors is a constant. Rosing (1966) relates how Zipf accomplished a similar end by determining the population of the largest city (New York), not by census data, but by the computation of the y intercept of a regression line through the ranking of the 100 largest cities on double log paper.

There is evidence that a full urban distribution may be " S " shaped, reflecting a growth or logistics curve (Stewart 1958, 245), or " J " shaped, and that the linear orientation of the data exists only in portions of the distribution. Parr and Jones (1983, 284-85) for example, describe the rank-size distribution as a lognormal distribution if truncated at a sufficiently high level. Carroll (1979) offers the greatest clarity on these issues. He points out that the rank-size distribution is more properly classified as a relation between two variables than as a probability distribution. The rank-size distribution can be derived from three kinds of the class of skew probability distributions. All of these probability distributions -- the lognormal, the Pareto and the Yule, are J-shaped and highly skewed in the upper tail. Each of the distributions is unique although the upper tails are quite similar, therefore, if the examination of a set of urban data excludes the smaller urban centers, any or all three distributions might apply. Carroll states

> We have seen that the law of proportionate effect results in the lognormal distribution. This law with a lower threshold results in the Pareto. And, this law with a lower threshold at which new units enter in a constant rate gives the Yule distribution.

Nader (1984) suggests that a non-logarithmic approach to rank-size data yields results superior to that of a logarithmic model. Perhaps such non-linear distributions are polymodal and reflect mixtures of two or more rank-size distributions, as has been reported for some distributions in geology (Krombein and Graybill 1965,108,126). Another perspective on this issue is that of Sahal (1981) who states quite simply that it is a general characteristic of simple laws that they do not hold over the entire range of variables. Clearly the rank-size relationship is still open to interpretation, and it is was intent of this research paper to suggest even another method which can generate a distribution similar to the rank-size rule.

Rank-Size and Central Place Theory

In addition to this uncertainty about the accurate mathematical portrayal of the structure of urban size distributions, geographers face a theoretical dilemma, which has not been satisfactorily explained, between two of the most intensively researched problems in the field of geography; rank-size research and central place research. Expressed very simply, the dilemma is that presented when the theoretical forms of the two distributions are contrasted. Central place theory suggests the existence of step-like plateaus or clusters of cities of similar sizes at intervals along the urban hierarchy. Rank-size research suggests a linear, sloping function with a continuous gradient rather than steps. Recall our earlier discussion of the spiral shell and how distinct levels as well as a continuous distribution can be visualized. It has been mathematically demonstrated that with certain assumptions a hypothetical central place hierarchy can be randomly disturbed to generate a rank-size hierarchy (Berry 1971, 144-46; Parr 1970; Beguin, 1985).

Briefly, these two necessary assumptions are those known as Gilbrat's Law and the Yule Theorem. In Berry's words, (1971) Gilbrat's Law of proportionate effect is that ". . . in very small time intervals city growth is small relative to city size, and that absolute growth is proportional to city size (i.e. the growth rate is the same for each size class of cities; . . .)". Carroll (1979) notes that Gilbrat's law is simply the basic underlying assumption of the lognormal distribution. The Yule distribution, is, in effect, a way of dealing with the "tail" of a distribution which usually deviates from the straight line of the plot of a rank-size relationship. The Yule distribution is a modification of the rank-size formulation (as Parr and Jones note) which results in a truncated probability distribution in which the tail falls below a threshold size for the objects under consideration, in this case urban areas. The number or quantity of new cities entering the system over a given time is a kind of birthrate into the rank-size hierarchy. (Berry 1971, 147-48). Ijiri and Simon (1977) and Aitchison and Brown (1963) describe these two modifications of the rank-size relationship in greater detail but with application to size of business firms rather than urban centers.

In summary, the traditional view of rank-size is that it is a situation of entropy in which numerous stochastic processes acting on a central place hierarchy disturb the distribution and produce a rank-size hierarchy. (Carroll, 1982, 5-8). Coffey's, (1981, 215) summary of Simon's work (1955) expresses this view well: " . . . the rank-size regularity is the manifestation of a set of stochastic relationships that offset the development of any phenomenon." More specifically, Simon believes the rank-size distribution to represent the equilibrium slope of any general growth process in which each element initially has a random size and thereafter grows in an exponential manner proportional to its size. Moreover, the stability of the rank-size relationship in various systems over both time and space suggests that it may be the manifestation of a steady-state condition in which the distribution is affected by a myriad of small random forces. From a statistical perspective, these explanations make sense. As we know from Carroll, what is being accomplished is that the rank-size portion of a particular skew distribution results from modification of certain probability distributions by incorporation of assumptions which result in the use of Gilbrat's law and Yule's theorem. What is less satisfying are the highly unrealistic assumptions necessary to accomplish these manipulations; namely the assumption that all classes of urban areas grow at the same rate in the case of Gilbrat's law, and the assumption of a constant and steady rate of new entrants into the urban system in the case of Yule's theorem. In any case, since Gilbrat's law and Yule's theorem are needed to generate rank-size, clearly, as Thomas (1985) concludes, the regularity of rank-size measurements will depend on the number of cities included in the urban system and the minimum population threshold that defines or limits the urban system.

Directions for Further Research

It appears from the preceeding material that further research on the spiral constant as the organizing concept for the rank-size relationship is profitable. Because the concept seems to lead in so many directions, one researcher cannot hope to explore all leads in a short amount of time. Therefore, this author will suggest a variety of paths of exploration, citing where possible specific, testable hypotheses. By their very nature these hypotheses are speculative.

1) The spiral constant may be able to predict urban distributions more accurately than rank-size. To test this hypothesis a large number of national urban systems, perhaps twenty or thirty in number, should be tested. Both the spiral constant and rank-size can be used to predict the urban distributions and the closer correlation in the majority of cases will be the more accurate predictor. Some mathematical difficulties present themselves in that experimentation with the formula for generating an equiangular spiral will be necessary to determine how such a spiral can be best adjusted to provide a best fit for the data. This is a process analogous to adjusting the exponent of the rank-size formula.

2) The spiral constant may help explain the concavity in rank-size distributions. Most rank-size research views the concavity in the empirical data as a deviation from a theoretically expected straight line. The spiral constant formulation suggests that the concavity in the data is quite

predictable based on theoretically expected values. This hypothesis, and in general, the ability of the formula using the spiral constant to predict urban distributions can be tested statistically. Statistical tests should correlate empirical urban data and values predicted by the formula for an equiangular spiral as described above. These correlations should be compared to those derived by the usual rank-size formula. In addition, for the same sets of data, separate correlations should be run for just the ten largest cities, those where the spiral constant may be able to predict the concavity in the rank-size distribution. This hypothesis is a subset of the first, more general, hypothesis above.

3) An understanding of the spiral constant can help in analyzing the primate city phenomenon. The structure of the spiral constant formula provides a variety of systematic ways to experiment with primate city data. For example, recall that the population of the eighth and the fifth largest cities combined should equal, approximately, the size of the third largest city. Thus the spiral constant gives us a method of predicting an expected size of the third largest city independent of the large primate city. Similarly, once the expected size of the third largest city is known, the size of the second largest city can be predicted, and, in turn, the largest city. If the second and third largest cities are substantially less than the predicted figure, this may be an indication that the primate city has usurped some functions of the other cities. In any case the population of the primate city can be hypothetically re-allocated to second and third ranked cities to see if a "normal" spiral curve can be generated. The examination of many primate distributions should provide insight into this process. As a general comment, it should be noted that what are often seen as three separate issues may in fact be related: the fit of the expected distribution to the straight-line portion of the rank-size distribution, primacy and concavity. The spiral

constant may help explain concavity or primacy even if it does not fit all categories of urban data better than does the rank size rule, or similarly, it may help explain concavity but not primacy. Hopefully it may shed some light on all three issues.

4) The spiral constant offers a new way to conceptualize levels of urban functions in urban hierarchies. If there is validity to the concepts of a rank-size relationship based on the spiral constant, this validity may derive from functionally different structural levels at Fibonacci intervals along the rank-size distribution. In a Christaller hierarchy (k=3), functionally different levels would be found at ranks 1, 3, 9, 27, 81. . . In the hierarchy derived by the spiral constant these functionally different levels could be at 1, 2, 3, 5, 8, 13. . ., or 1, 3, 4, 7, 11, 18. . ., or 1, 2, 5, 7, 12, 19. . ., etc. Considered as a hypothesis, the numbers of central place functions could be examined to see if there are decreasing levels of urban functions with decreasing rank. In this research, of course, we have used urbanized area data as a substitute for functional level. Detailed investigation of functional level of urban areas must look at the actual number of functions in each urban area. A rank-size relationship may be evident in the functional data. If not, the average number of functions at each Fibonacci level based on urbanized area data can be used. The fact that one set of data provides several Fibonacci sequences for examination allows a number of tests to be applied to the same set of data. In other words, one set of urban data allows the three Fibonacci sequences listed above to be examined. In addition, the finer differentiation of an urban hierarchy expected from the phi formulation allows more vigorous testing than that possible from simple rank-size. Indeed there is no theoretical basis in the traditional rank-size formulation to predict varying levels of functions. These levels are either assumed from central place theory or they are inferred

from breaks on the graph of the empirical data. The spiral constant, on the other hand, offers some mathematical justification for stating a priori, that urban centers ranked 4 and 5 for example, should have a different number of functions than urban centers ranked 6, 7 and 8, and these in turn should be at a different functional level from those ranked 9 through 13, etc.

To test this hypothesis the number and kinds of functions in the U.S. urban hierarchy can be examined. Differences in functions within levels should be less than differences between levels, that is, the number of different functions available in urban centers ranked 4th and 5th should be closer than the numbers of functions in cities ranked 6th, 7th and 8th. In short, differences in numbers of functions within levels should be less than differences between levels. There may also be a Fibonacci hierarchy in the number of functions as well. This can be easily tested when the data are collected for the first test. Expressed simply, when the numbers of functions of urban centers are ranked, do they approximate a Fibonacci sequence?

While it may be poor scientific practice to base an expectation on intuitive reasoning, there is something inherently unsatisfactory in the gross differentiation of the urban hierarchy provided by traditional Christaller hierarchies. Even the finest Christaller hierarchy, k=3, provides no differentiation in level between cities of ranks 4 and 8, or between ranks 10 and 26, 28 and 80, or 81 and 243. Expressed in 1980 U.S. population data, the k=3 hierarchy predicts no difference in function between those aforementioned pairs of cities with populations of 4,113,000 and 2,679,000; 2,413,000 and 1,078,000; 1,009,000 and 312,000; 306,000 and 79,000. The spiral constant formulation, in contrast, predicts twelve levels between and including ranks 1 and 243, namely 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 and 233.

5) The spiral constant may offer new perspectives on integration of rank-size research and central place theory. Hypothesizing twelve levels of

urban function rather than six in a given range of an urban hierarchy may be intuitively pleasing, but in addition to empirical testing, sound theory is needed as to why this might occur. Attempting to match a hexagonally based central place hierarchy with a rank-size distribution derived from the spiral constant requires a k=1.618 hierarchy. This is evident from the fact that, if functionally different levels are found at ranks 3, 5, 8, 13, 21, etc., on average the ratio of number of centers of one level to numbers at the preceeding level is 1.618:1. The column entitled "number of cities at each level" in Table 13 illustrates this. (Note that the number of cities at each level is, itself, an element of a Fibonacci sequence.)

In order to speculate how the equivalent of a k=1.618 hierarchy can be generated from a regular central place hierarchy, it seems wise to begin with the standard central place pattern which is closest in area, that of k=3. One method of generating a k=1.618 hierarchy from a k=3 hierarchy is by introducing a systematic bias that increases the number of centers which evolve into higher order centers. The ratio of 3/1.618 which approximates 1.85, will accomplish the task. Assume that the ideal conditions of an isotropic plain do not exist over most of the territory of a national urban system, and that instead, barriers are found due to terrain, uneven distribution of population, incomplete accessibility of each point to every other point, and so forth. Assume further that a k=3 hexagonal network was evolving but the above factors introduced a systematic bias such that as each level of the hierarchy evolved, 1.85 higher level centers resulted for each one predicted by central place theory. For example, if a Christaller-type hierarchy of 1, 3, 9, 27, 81, 243, 729 was "expected," 729 lowest order centers would produce not 243 higher level centers, but rather, 450 centers (243 x 1.85). From these 450 centers, combination of hexagons in the k=3 pattern would tend to produce 150 of the next level center, but instead, 277.

centers (150 x 1.85) would be produced. Table 14 shows these figures. Note that the numbers of the resulting centers are another Fibonacci series (with some variation due to rounding). As such the average ratio of numbers of centers at successive levels is 1.618. The resulting figures could be plotted on a log-log graph with an approximate slope of -1, or as an equiangular spiral.

This demonstration is offered as illustrative of one possible way in which a traditional central place hierarchy can be manipulated in a somewhat plausible manner to result in a rank-size distribution. Such manipulation might seem less contrived when it is recalled that the generally accepted demonstration of how a central place hierarchy results in a rank-size distribution is based on stochastic processes applied to the central place hierarchy. The main difference between the two approaches appears to be that one is based on random factors and the other on random factors with a systematic bias. There is some research which argues for a random factor which disturbs the regular central place hierarchy and results in the rank-size distribution. In fact, this is way the generation of rank size from central place is usually explained. Beckmann (1958), Berry (1971) and Beguin (1985) all explain this random factor as deviation about the midpoint of each hierarchical level. Beguin (1985, 440) describes it best: "A large random diversity of empirical local conditions describing the heterogeniety of geographical space might be represented by some random factor. Such a factor influences the sizes of local urban centers, and thus generates differences between observed urban populations and theoretical populations derived from

TABLE 14

k=3 hierarchy modified by a systematic bias of 1.85

Arrangement of centers from k=3	k=3 centers modified by 1.85
729	729
243	451
81	278
27	172
9	106
3	66
1	41
	25
	16
	10
	6
	4
	2
	1

Note that the number of centers in the right column is, approximately, a Fibonacci sequence.

central place theory." Further, he states (1985,437) "the random factor should thus generate as many positive as negative differences between the empirical values p; and the population P of the median center of any level m." Clearly, a systematic bias from a beginning point or end point for each level can result in the same distribution as positive and negative deviations around a midpoint. It is difficult to think of a way to test this hypothesis which is different from the test suggested earlier: test for levels in the urban hierarchy based on number of different functions. If the levels on average are separated by ratios approximating the spiral constant, it is evidence that the urban hierarchy in that case is based on what in a Christaller model would be called k=1.618. Such a network could possibly be modeled by fractal generation of a central place hierarchy (Arlinghaus, 1985).

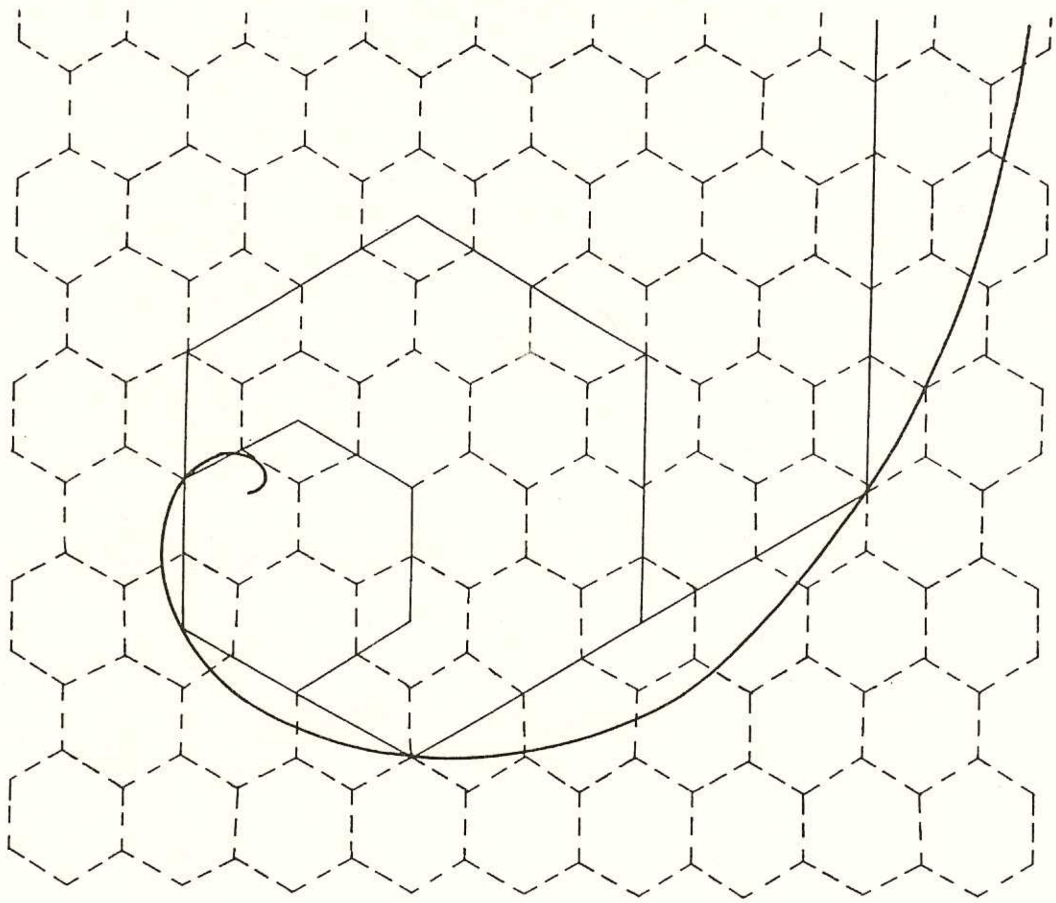

Figure 9. An equiangular spiral generated from a network of hexagons. (After Thompson, 1961.)

Although the k=3 hierarchy seems the most logical hierarchy to be converted into one which might reflect the influence of growth by the spiral constant intervals, the k=4 hierarchy also offers possibilities. As was mentioned earlier in a discussion of gnonomic growth, almost any simple regular geometric figure can be used to demonstrate how a set increment of growth can produce an equiangular spiral. Squares, rectangles and triangles are used to illustrate this in many works dealing with gnomonic growth. Interestingly, an equiangular spiral derived from corresponding points in a system of hexagons (Figure 9) is based on hexagons which are four times as large as the preceeding one. Note in the figure that the hexagons are not related to each other in the usual manner of a k=4 hierarchy. This occurrence can be seen as an invitation to explore new ways of combining hexagons into logical and economically rational hierarchies.

6) The spiral constant may help explain why different urban systems have different slopes. Due to the importance attached to urban distributions with a slope of -1, little has been written in this paper about rank-size distributions which deviate significantly from this slope. The argument has been made that the slope of -1 is a slope of theoretical significance, not of incidental significance. It could be argued that urban distributions substantially different from those with slopes of -1 are incomplete, still evolving, or deviant in some fashion and it could be argued that some corrective factor should be applied to the formula. In the case of the formula for the equiangular spiral, a variable could be added or the angle of tangency adjusted. The formula would no longer be that of an equiangular spiral and it would no longer have a slope of -1 on a log-log graph, but this was the desired result. This adjustment is equivalent to that accomplished by manipulation of the exponent in the traditional rank-size formula. When the rank-size exponent is manipulated, theoretical reasons why a particular

exponent should be assigned are not offered; it is simply the exponent needed to achieve a line of best fit.

If, as we have considered, a slope of -1 is a reflection of a systematic bias of 1.85 (related to the ratio of 8:5) at work in the central place hierarchy, could other slopes reflect other systematic biases? Suppose the systematic bias due to non-isotropic conditions resulted in ratios of 9:5 or 8:4, what slopes would result? Statistical testing of this hypothesis would require examination of many urban systems in an attempt to see if there are consistently recurring values of slope; that is, if values of slopes are not randomly distributed around -1.0, but rather cluster around set values. If such recurring values are found, detailed comparative investigation of the structure of the urban system of each slope class is warranted. Standardized methods of determining slope must be employed to avoid non-comparative data which, as Carroll states, has hindered rank-size research in the past. It may be possible to examine slope data from already-completed research such as that of Rosen and Resnick (1980) who looked at 44 countries.

The spiral constant may help generate new conceptualizations of the urban hierarchy. There is still debate over the validity of central place theory and the extent to which approximations of hexagonal market areas exist. Suppose market areas combine with nearest neighbors in some way to produce dendritic market regions analogous to river drainage patterns? Consider three examples of branching patterns cited by Huntley (1970). He shows how stems and flowers of certain plants can produce Fibonacci sequences as does Coxeter (1961, 169-72) with the facets on the face of a pineapple. The same kind of pattern can be found in the branching pattern illustrating the genealogical table of a bee or the possible histories of an atomic electron (in terms of gain and loss of charge). Dendritic patterns are frequently found in spatial systems. Certain types of such patterns generate fibonacci sequences.

Goodchild and Mark (1987) cite Batty (1985) who connected recursive algorithms to the branching process used in simulation of networks and trees. Kapteyn, a turn-of-the-century mathematician, designed a mechanical analog to show how trickling sand could generate a lognormal distribution similar to a rank-size hierarchy (Aitchison and Brown, 1963, 22-24). The machine accomplished this by subdividing the flow of sand in a branching pattern. Could a branching pattern rather than hexagonal pattern of service areas better account for the rank-size regularity in urban systems?

APPENDIX A

RANK OF URBANIZED AREAS BY POPULATION 1970

RANK	URBANIZED AREA	POPULATION
1.	New York, NY-Northeastern NJ	16,206,841
2.	Los Angeles-Long Beach, CA	8,351,266
3.	Chicago, IL-Northwestern IN	6,714,578
4.	Philadelphia, PA-NJ	4,021,066
5.	Detroit, MI	3,970,584
6.	San Francisco-Oakland, CA	2,987,850
7.	Boston, MA	2,652,575
8.	Washington, DC-MD-VA	2,481,489
9.	Cleveland, OH	1,959,880
10.	St. Louis, MO-IL	1,882,944
11.	Pittsburg, PA	1,846,042
12.	Minneapolis-St. Paul, MN	1,704,423
13.	Houston, TX	1,677,863
14.	Baltimore, MD	1,579,781
15.	Dallas, TX	1,338,684
16.	Milwaukee, WI	1,252,457
17.	Seattle-Everett, WA	1,238,107
18.	Miami, FL	1,219,661
19.	San Diego, CA	1,198,323
20.	Atlanta, GA	1,172,778
21.	Cincinnati, OH-KY	1,110,514
22.	Kansas City, MO-KS	1,101,787

RANK	URBANIZED AREA	POPULATION
23.	Buffalo, NY	1,086,594
24.	Denver, CO	1,047,311
25.	San Jose, CA	1,025,273
26.	New Orleans, LA	961,728
27.	Phoenix, AZ	863,357
28.	Portland, OR-WA	824,926
29.	Indianapolis, IN	820,259
30.	Providence-Pawtucket-Warwick, RI-MA	795,311
31.	Columbus, OH	790,019
32.	San Antonio, TX	772,513
33.	Louisville, KY-IN	739,396
34.	Dayton, OH	685,942
35.	Fort Worth, TX	676,944
36.	Norfolk-Portsmouth, VA	668,259
37.	Memphis, TN-MS	663,976
38.	Sacramento, CA	633,732
39.	Fort Lauderdale-Hollywood, FL	613,797
40.	Rochester, NY	601,361
41.	San Bernardino-Riverside, CA	583,597
42.	Oklahoma City, OK	579,788
43.	Birmingham, AL	558,099
44.	Akron, OH	542,775
45.	Jacksonville, FL	529,585
46.	Springfield-Chicopee-Holyoke, MA-CT	514,308
47.	St. Petersburg, FL	495,159
48.	Omaha, NE-IA	491,776

RANK	URBANIZED AREA	POPULATION
75.	Trenton, NJ-PA	274,148
76.	Newport News-Hampton, VA	268,263
77.	Davenport-Rock Is.-Moline, IA-IL	266,119
78.	Austin, TX	264,499
79.	Fresno, CA	262,908
80.	Mobile, AL	257,816
81.	Des Moines, IA	255,824
82.	Baton Rouge, LA	249,463
83.	Worcester, MA	247,416
84.	Peoria, IL	247,121
85.	Oxnard-Ventura-Thousand Oaks, CA	244,653
86.	Canton, OH	244,279
87.	Columbia, SC	241,781
88.	Harrisonburg, PA	240,751
89.	Las Vegas, NV	236,681
90.	Shreveport, LA	234,564
91.	Aurora-Elgin, IL	232,917
92.	Spokane, WA	229,620
93.	Lansing, MI	229,518
94.	Charlestown, SC	228,399
95.	Fort Wayne, IN	225,184
96.	Chattanooga, TN-GA	223,580
97.	Wilkes-Barre, PA	222,830
98.	Little Rock-N. Little Rock, AK	222,616
99.	Corpus Christi, TX	212,820
100.	Columbus, GA-AL	208,616

RANK	URBANIZED AREA	POPULATION
101.	Rockford, IL	206,084
102.	Madison, WI	205,457
103.	Colorado Springs, CO	204,766
104.	Scranton, PA	204,205
105.	Lawrence-Haverhill, MA-NH	200,280
106.	Lorain-Elyria, OH	192,265
107.	Knoxville, TN	190,502
108.	Jackson, MS	190,060
109.	Stamford, CT	184,898
110.	Lowell, MA	182,731
111.	Utica-Rome, NY	180,355
112.	Ann Arbor, MI	178,605
113.	Bakersfield, CA	176,155
114.	Erie, PA	175,263
115.	Reading, PA	167,932
116.	Huntington-Ashland, WV-KY-OH	167,583
117.	Binghamton, NY	167,224
118.	Pensacola, FL	166,619
119.	Savannah, GA	163,753
120.	Fayetteville, NC	161,370
121.	Stockton, CA	160,373
122.	Lexington, KY	159,538
123.	Charleston, WV	157,662
124.	Greenville, SC	157,073
125.	Waterbury, CT	156,986
126.	Roanoke, VA	156,621

RANK	URBANIZED AREA	POPULATION
127.	Joliet, IL	155,500
128.	Lincoln, NE	153,443
129.	Raleigh, NC	152,289
130.	Greensboro, NC	152,252
131.	Kalamazoo, MI	152,083
132.	Lubbock, TX	150,135
133.	Ogden, UT	149,727
134.	Augusta, GA-SC	148,953
135.	Brockton, MA	148,844
136.	Saginaw, MI	147,552
137.	Huntsville, AL	146,565
138.	Winston-Salem, NC	142,584
139.	Evansville, IN	142,476
140.	Fall River, MA-RI	139,392
141.	Eugene, OR	139,255
142.	Montgomery, AL	138,983
143.	Duluth-Superior, MN-WI	138,352
144.	Atlantic City, NJ	134,016
145.	New Bedford, MA	133,667
146.	Topeka, KS	132,108
147.	Cedar Rapids, IA	132,008
148.	New Britain, CT	131,349
149.	Santa Barbara, CA	129,774
150.	Appleton, WI	129,532
151.	Green Bay, WI	129,105
152.	Macon, GA	128,065

RANK	URBANIZED AREA	POPULATION
153.	Amarillo, TX	127,010
154.	York, PA	123,106
155.	Biloxi-Gulfport, MS	121,601
156.	Springfield, MO	121,340
157.	Springfield, IL	120,794
158.	Waco, TX	118,843
159.	Racine, WI	117,408
160.	Lancaster, PA	117,097
161.	Port Arthur, TX	116,474
162.	Beaumont, TX	116,350
163.	Waterloo, IA	112,881
164.	Norwalk, CT	106,707
165.	Portland, ME	106,599
166.	Modesto, CA	106,107
167.	Muskegon-Muskegon Hgts., MI	105,716
168.	Provo-Orem, UT	104,110
169.	Pueblo, CO	103,300
170.	Durham, NC	100,764
171.	Petersburg-Colonial Hgts., VA	100,617
172.	Champaign-Urbana, IL	100,417
173.	Decatur, IL	99,693
174.	Reno, NV	99,687
175.	Meriden, CT	98,454
176.	Wichita Falls, TX	97,564
177.	Johnstown, PA	96,146
178.	Sioux City, IA-NE-SD	95,937

RANK	URBANIZED AREA	POPULATION
179.	Lawton, OK	95,687
180.	Manchester, NH	95,140
181.	Springfield, OH	93,653
182.	High Point, NC	93,547
183.	Seaside-Monterey, CA	93,284
184.	Salem, OR	93,041
185.	Wheeling, WV-OH	92,944
186.	McAllen-Pharr-Edinburg, TX	91,141
187.	Hamilton, OH	90,912
188.	Abilene, TX	90,571
189.	Monroe, LA	90,567
190.	Muncie, IN	90,427
191.	Lake Charles, LA	88,260
192.	Tuscaloosa, AL	85,875
193.	Steubenville-Weirton, OH-WV	85,492
194.	Fargo-Moorhead, ND-MN	85,446
195.	Boise City, ID	85,187
196.	Kenosha, WI	84,262
197.	Texas City-La Marque, TX	84,054
198.	Altoona, PA	81,795
199.	Odessa, TX	81,645
200.	Terre Haute, IN	80,908
201.	Anderson, IN	80,704
202.	Lafayette-West Lafayette, IN	79,117
203.	Jackson, MI	78,572
204.	Lafayette, LA	78,544

RANK	URBANIZED AREA	POPULATION
205.	Bay City, MI	78,097
206.	Fitchburg-Leominster, MA	78,053
207.	Tallahassee, FL	77,851
208.	Mansfield, OH	77,599
209.	St. Joseph, MO-KS	77,223
210.	Albany, GA	76,512
211.	Fort Smith, AR-OK	75,517
212.	Sioux Falls, SD	75,146
213.	Santa Rosa, CA	75,083
214.	Vineland-Millville, NJ	73,579
215.	Ashville, NC	72,451
216.	Bristol, CT	71,732
217.	Billings, MT	71,197
218.	Great Falls, MT	70,905
219.	Lynchburg, VA	70,842
220.	Lima, OH	70,295
221.	Laredo, TX	70,197
222.	Bloomington-Normal, IL	69,392
223.	Gainesville, FL	69,329
224.	Boulder, CO	68,634
225.	Gadsden, AL	67,706
226.	Dansbury, CT	66,651
227.	Dubuque, IA-IL	65,550
228.	Lewiston-Auburn, ME	65,212
229.	San Angelo, TX	63,884
230.	LaCrosse, WI-MN	63,373

RANK URBANIZED AREA	POPULATION
231. Pittsfield, MA	62,872
232. Salinas, CA	62,456
233. Galveston, TX	61,809
234. Nashua, NH	60,961
235. Pine Bluff,AK	60,907
236. Midland, TX	60,371
237. Tyler, TX	59,781
238. Columbia, MO	59,231
239. Texarkana, TX-AR	58,570
240. Wilmington, NC	57,645
241. Simi Valley, CA	56,936
242. Rochester, MN	56,604
243. Oshkosh, WI	55,480
244. Sherman-Denison, TX	55,343
245. Owensboro, KY	53,133
246. Brownsville, TX	52,627
247. Bryan-College Station, TX	51,395
248. Harlingen-San Benito, TX	50,469

APPENDIX B

RANK OF URBANIZED AREAS BY POPULATION 1980

RANK	URBANIZED AREA	POPULATION
1.	New York, NY-Northeastern NJ	15,590,274
2.	Los Angeles-Long Beach, CA	9,479,436
3.	Chicago, IL-Northwestern IN	6,779,799
4.	Philadelphia,PA-NJ	4,112,933
5.	Detroit, MI	3,809,327
6.	San Francisco-Oakland, CA	3,190,698
7.	Washington, DC-MD-VA	2,763,105
8.	Boston, MA	2,678,762
9.	Dallas-Fort Worth, TX	2,451,390
10.	Houston, TX	2,412,664
11.	St. Louis, MO-IL	1,848,590
12.	Pittsburgh, PA	1,810,038
13.	Minneapolis-St. Paul, MN	1,787,564
14.	Baltimore, MD	1,755,477
15.	Cleveland, OH	1,752,424
16.	San Diego, CA	1,704,352
17.	Atlanta, GA	1,613,357
18.	Miami, FL	1,608,159
19.	Phoenix, AZ	1,409,279
20.	Seattle-Everett, WA	1,391,535
21.	Denver, CO	1,352,070
22.	San Jose, CA	1,243,952

RANK	URBANIZED AREA	POPULATION
23.	Milwaukee, WI	1,207,008
24.	Cincinnati, OH-KY	1,123,412
25.	Kansas City, MO-KS	1,097,793
26.	New Orleans, LA	1,078,299
27.	Portland, OR-WA	1,026,144
28.	Fort Lauderdale-Hollywood, FL	1,008,526
29.	Buffalo, NY	1,002,285
30.	San Antonio, TX	944,893
31.	Indianapolis, IN	836,472
32.	Columbus, OH	833,648
33.	St. Petersburg, FL	833,337
34.	Sacramento, CA	796,266
35.	Providence-Pawtucket-Warwick,RI-MA	796,250
36.	Memphis, TN-AK-MS	774,551
37.	Norfolk-Portsmouth, VA	770,784
38.	Louisville, KY-IN	761,002
39.	San Bernardino-Riverside, CA	705,175
40.	Oklahoma City, OK	674,322
41.	Salt Lake City, UT	674,201
42.	Birmingham, AL	606,085
43.	Rochester, NY	606,070
44.	Jacksonville, FL	598,015
45.	Dayton, OH	595,059
46.	Honolulu, HI	582,463
47.	Orlando, FL	577,235
48.	Tampa, FL	520,912

RANK	URBANIZED AREA	POPULATION
49.	Nashville-Davidson, TN	518,325
50.	Akron, OH	515,720
51.	Omaha, NE-IA	512,438
52.	Hartford, CT	510,034
53.	Springfd.-Chicopee-Holyoke,MA-CT	505,822
54.	Richmond, VA	491,627
55.	Albany-Schenectady-Troy,NY	490,015
56.	West Palm Beach, FL	487,044
57.	Toledo, OH-MI	485,440
58.	El Paso, TX	454,159
59.	Tucson, AZ	450,059
60.	Tulsa, OK	443,350
61.	Las Vegas, NV	432,874
62.	Albuquerque, NM	418,206
63.	Bridgeport, CT	410,998
64.	Scranton-Wilkes-Barre,PA	406,517
65.	Wilmington, DE-NJ-MD	406,112
66.	Tacoma, WA	402,077
67.	Youngstown-Warren, OH	383,398
68.	Allentown-Bethlehem-Easton,PA-NJ	381,734
69.	Austin, TX	379,560
70.	Syracuse, NY	379,284
71.	Oxnard-Ventura-Thousand Oaks, CA	377,695
72.	Grand Rapids, MI	374,744
73.	New Haven, CT	368,061
74.	Charlotte, NC	350,715

RANK	URBANIZED AREA	POPULATION
75.	Baton Rouge, IL	350,657
76.	Flint, MI	331,931
77.	Fresno, CA	331,551
78.	Newport News-Hampton, VA	328,576
79.	Charleston, SC	328,572
80.	Columbia, SC	311,561
81.	Wichita, KS	305,752
82.	Sarasota-Bradenton, FL	305,431
83.	Chattanooga, TN-GA	301,515
84.	Mobile, AL	295,493
85.	Little Rock-N.Little Rock, AK	295,133
86.	Davenport-Rock Is.-Moline, IA-IL	285,024
87.	Knoxville, TN	284,708
88.	Harrisburg, PA	278,296
89.	Colorado Springs, CO	276,872
90.	Worcester, MA	276,022
91.	Des Moines, IA	267,192
92.	Spokane, WA	266,709
93.	Jackson, MS	265,051
94.	Shreveport, LA	263,827
95.	Peoria, Il	261,418
96.	Trenton, NJ-PA	260,751
97.	Lansing, MI	254,704
98.	Augusta, GA-SC	251,250
99.	Corpus Christi, TX	245,854
100.	Canton, OH	244,888

RANK	URBANIZED AREA	POPULATION
101.	Fort Wayne, IN	36,479
102.	Greenville, SC	229,303
103.	South Bend, IN-MI	226,331
104.	Lorain-Elyria, OH	225,331
105.	Bakersfield, CA	222,236
106.	Pensacola, FL	215,995
107.	Fayetteville, NC	215,839
108.	Columbia, GA-AL	214,591
109.	Madison, WI	213,675
110.	Melbourne-Cocoa, FL	212,917
111.	Lawrence-Haverhill, MA-NH	211,428
112.	Ann Arbor, MI	208,782
113.	Raleigh, NC	206,597
114.	Ogden, UT	205,744
115.	Rockford, IL	204,304
116.	Stockton, CA	197,052
117.	Montgomery, AL	196,947
118.	Lexington-Fayett, KY	194,093
119.	Savannah, GA	186,546
120.	Stamford, CT	182,978
121.	Eugene, OR	182,495
122.	Evansville, IN-KY	180,089
123.	Huntington-Ashland, WV-KY-OH	179,840
124.	Biloxi-Gulfport, MS	179,280
125.	Erie, PA	178,338
126.	Brockton, MA	177,784

RANK	URBANIZED AREA	POPULATION
127.	Roanoke, VA	177,475
128.	Lubbock, TX	175,479
129.	Lincoln, NE	173,550
130.	Reading, PA	173,450
131.	Winston-Salem, NC	171,530
132.	Daytona Beach, FL	170,749
133.	Greensboro, NC	170,457
134.	Anchorage, AL	170,247
135.	Provo-Orem, UT	169,699
136.	Joliet, IL	167,475
137.	Reno, NV	162,286
138.	Binghamton, NY	161,312
139.	Waterbury, CT	160,249
140.	Modesto, CA	159,538
141.	Aurora, IL	158,911
142.	McAllen-Pharr-Edinburg, TX	157,423
143.	Lowell, MA-NH	157,412
144.	Lancaster, PA	157,385
145.	Durham, NC	157,287
146.	Utica-Rome, NY	155,238
147.	Kalamazoo, MI	154,990
148.	Huntsville, AL	153,841
149.	Charleston, WV	153,618
150.	Santa Barbara, CA	150,173
151.	Armarillo, TX	149,230
152.	New London-Norwich, CT	148,829

RANK	URBANIZED AREA	POPULATION
153.	Saginaw, MI	146,769
154.	Atlantic City, NJ	146,034
155.	Green Bay, WI	142,747
156.	Appleton, WI	142,151
157.	Fall River, MA-RI	141,510
158.	Fort Myers, FL	140,958
159.	Springfield, MA	139,030
160.	Santa Rosa, CA	137,019
161.	Poughkeepsie, NY	136,571
162.	New Britain, CT	135,817
163.	Cedar Rapids, IA	135,798
164.	Salem, OR	135,747
165.	Boise City, ID	134,848
166.	Waco, TX	134,491
167.	New Bedford, MA	133,274
168.	Duluth-Superior, MN-WI	132,585
169.	Macon, GA	130,871
170.	York, PA	129,336
171.	Topeka, KS	125,936
172.	Lake Charles, LA	123,820
173.	Beaumont, TX	123,729
174.	Santa Cruz, CA	123,226
175.	Springfield, IL	122,806
176.	Waterloo, IA	120,290
177.	Tallahassee, FL	119,341
178.	Racine, WI	118,987

RANK	URBANIZED AREA	POPULATION
179.	Port Arthur, TX	118,562
180.	Seaside-Monterey, CA	115,418
181.	Lakeland, FL	114,360
182.	Lafayette, LA	113,999
183.	Monroe, LA	112,537
184.	Richland-Kennewick, WA	112,171
185.	Pueblo, CO	109,444
186.	Champaign-Urbana, IL	109,278
187.	Texas City-La Marque, TX	109,193
188.	Decatur, IL	107,864
189.	Norwalk, CT	107,550
190.	Portland, MA	107,099
191.	Gastonia, NC	106,884
192.	Elgin, NC	106,593
193.	Petersburg-Colonial Hgts., VA	106,582
194.	Kailua-Kaneohe, HI	105,712
195.	Muskegon-Muskegon Hgts., MI	105,634
196.	Hamilton, OH	105,026
197.	Fargo-Moorhead, ND-MN	104,643
198.	Gainesville, FL	103,768
199.	Portsmouth-Dover-Rochester, NH-ME	103,722
200.	Manchester, NH	102,844
201.	Ashville, NC	102,400
202.	Odessa, TX	101,518
203.	Wheeling, WV-OH	101,049
204.	Spartanburg, SC	100,706

RANK	URBANIZED AREA	POPULATION
205.	High Point, NC	100,089
206.	Abilene, TX	99,763
207.	Tuscaloosa, AL	99,554
208.	Sioux City, IA-NE-SD	96,746
209.	Lawton, OH	96,134
210.	Danbury, CT-NY	95,371
211.	Laredo, TX	94,961
212.	Wichita Falls, TX	94,716
213.	Lynchburg, VA	93,921
214.	Alexandria, LA	92,742
215.	Middletown, OH	91,730
216.	Brownsville, TX	91,611
217.	Muncie, IN	91,479
218.	Lafayette-W. Lafayette, IN	91,380
219.	Johnstown, PA	90,254
220.	Fort Smith, AK-OK	90,021
221.	Kingsport, TN-VA	89,760
222.	Alton, IL	88,994
223.	Vineland-Millville, NJ	88,822
224.	Wilmington, NC	88,763
225.	Albany, GA	88,716
226.	Killeen, TX	88,145
227.	Springfield, OH	86,742
228.	Antioch-Pittburg, CA	86,435
229.	Sioux Falls, SD	85,834
230.	Kenosha, WI	85,742

RANK	URBANIZED AREA	POPULATION
231.	Fort Walton Beach, FL	85,318
232.	Billings, MT	84,328
233.	Elkhart-Goshen, IN	83,920
234.	Bristol, CT	83,601
235.	Bryan-College Station, TX	83,036
236.	Salinas, CA	82,600
237.	Bloomington-Normal, IL	82,397
238.	Boulder, CO	81,239
239.	Jackson, MI	81,178
240.	Yakima, WA	81,085
241.	St. Joseph, MO-KS	79,936
242.	Simi Valley, CA	79,921
243.	Mansfield, OH	78,948
244.	Panama City, FL	78,886
245.	Altoona, PA	78,802
246.	Anderson, IN	78,581
247.	Johnson City, TN	78,473
248.	Fort Collins, CO	78,287
249.	Battle Creek, MI	77,789
250.	Bay City, MI	77,678
251.	Steubenville-Weirton, OH-WV-PA	77,651
252.	Clarksville, TN-KY	77,535
253.	Fitchburg-Leominster, MA	76,652
254.	Burlington, VT	76,528
255.	Anniston, AL	75,614
256.	Nashua, NH	75,299

RANK	URBANIZED AREA	POPULATION
257.	Terre Haute, IN	74,736
258.	Gadsden, AL	74,730
259.	San Angelo, TX	73,994
260.	Tyler, TX	72,927
261.	Jacksonville, NC	72,891
262.	Florence, AL	72,669
263.	Winter Haven, FL	72,560
264.	Eau Claire, WI	72,317
265.	Concord, NC	71,994
266.	Midland, TX	71,606
267.	Fort Pierce, FL	70,450
268.	Lewiston-Auburn, ME	70,108
269.	Lima, OH	70,104
270.	Longview, TX	69,757
271.	Fairfield, CA	69,255
272.	Olympia, WA	68,616
273.	Elmira, NY	68,227
274.	Dubuque, IA-IL	68,149
275.	La Crosse, WI-MN	67,966
276.	Harlingen-San Benito, TX	66,702
277.	Burlington, NC	66,580
278.	Palm Springs, CA	66,431
279.	Hagerstown, MD-PA	66,277
280.	Great Falls, MT	66,256
281.	Monessen, PA	65,884
282.	Houma, LA	65,780

RANK	URBANIZED AREA	POPULATION
283.	Newburgh, NY	65,711
284.	Round Lake Beach, IL	65,676
285.	Columbia, MO	65,380
286.	Pascagoula-Moss Point, MS	65,174
287.	Bremerton, WA	64,536
288.	Annapolis, MD	64,447
289.	Bloomington, IN	63,513
290.	Texarkana, TX-AK	63,474
291.	Parkersburg, WV-OH	63,181
292.	Athens, GA	62,896
293.	Pine Bluff, AK	62,817
294.	Fayetteville-Springdale, AK	62,703
295.	Hickory, NC	62,329
296.	Greeley, CO	62,297
297.	Kankakee, IL	61,451
298.	Galveston, TX	61,382
299.	Kokomo, IN	61,224
300.	Yuba City, CA	61,107
301.	Bismark-Mandan, ND	61,105
302.	Benton Harbor, MI	60,639
303.	Rochester, MN	60,473
304.	Bangor, ME	60,003
305.	Port Huron, MI	59,647
306.	Charlottesville, VA	59,422
307.	Cumberland, MD-WV	59,331
308.	Iowa City, IA	59,295

RANK	URBANIZED AREA	POPULATION
309.	Casper, WY	59,287
310.	Napa, CA	59,277
311.	Visalia, CA	58,957
312.	Williamsport, PA	58,650
313.	Sheboygan, WI	58,531
314.	Cheyenne, WY	58,429
315.	St. Cloud, MN	58,375
316.	Missoula, MT	58,035
317.	Goldsboro, NC	57,670
318.	Joplin, MO	57,658
319.	Pittsfield, MA	57,554
320.	Owensboro, KY	57,549
321.	Santa Maria, CA	57,237
322.	Meriden, CT	57,118
323.	Grand Junction, CO	56,854
324.	Hattiesburg, MS	56,542
325.	Sherman-Denison, TX	56,441
326.	Lancaster, CA	56,328
327.	Florence, SC	56,240
328.	Hemet, CA	55,377
329.	Longview, WA-OR	55,076
330.	Las Cruces, NM	55,072
331.	Warner Robins, GA	54,923
332.	Danville, VA	54,815
333.	Decatur, AL	54,710
334.	Yuma, AZ-CA	54,657

RANK	URBANIZED AREA	POPULATION
335.	Naples, FL	53,675
336.	Bristol, TN-Bristol, VA	53,537
337.	Pocatello, ID	53,401
338.	Temple, TX	53,191
339.	Wausau, WI	52,990
340.	Oshkosh, WI	52,958
341.	Redding, CA	52,867
342.	Lawrence, KS	52,810
343.	Medford, OR	52,469
344.	Taunton, MA	52,334
345.	Grand Forks, ND-MN	52,310
346.	Danville, IL	52,243
347.	Santa Fe, NM	52,042
348.	Dothan, AL	51,976
349.	Chico, CA	51,914
350.	Auburn-Opelika, AL	51,823
351.	Janesville, WI	51,643
352.	Glens Falls, NY	51,382
353.	Newport, RI	51,381
354.	State College, PA	51,298
355.	Rome, GA	51,082
356.	Bellingham, WA	51,025
357.	Anderson, SC	51,014
358.	Sharon, PA-OH	50,933
359.	Rapid City, SD	50,882
360.	Ocala, FL	50,860

RANK	URBANIZED AREA	POPULATION
361.	Rock Hill, SC	50,846
362.	Newark, OH	50,839
363.	Beloit, WI-IL	50,834
364.	Victoria, TX	50,725
365.	Enid, OH	50,601
366.	Jackson, TN	50,338

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6. Pierre Hanjoul, Hubert Beguin, and Jean-Claude Thill, *Theoretical Market Areas Under Euclidean Distance*, 1988. (English language text; Abstracts written in French and in English.)

Though already initiated by Rau in 1841, the economic theory of the shape of two-dimensional market areas has long remained concerned with a representation of transportation costs as linear in distance. In the general gravity model, to which the theory also applies, this corresponds to a decreasing exponential function of distance deterrence. Other transportation cost and distance deterrence functions also appear in the literature, however. They have not always been considered from the viewpoint of the shape of the market areas they generate, and their disparity asks the question whether other types of functions would not be worth being investigated. There is thus a need for a general theory of market areas: the present work aims at filling this gap, in the case of a duopoly competing inside the Euclidean plane endowed with Euclidean distance.

(Bien qu'ébauchée par Rau dès 1841, la théorie économique de la forme des aires de marché planaires s'est longtemps contentée de l'hypothèse de coûts de transport proportionnels à la distance. Dans le modèle gravitaire généralisé, auquel on peut étendre cette théorie, ceci correspond au choix d'une exponentielle décroissante comme fonction de dissuasion de la distance. D'autres fonctions de coût de transport ou de dissuasion de la distance apparaissent cependant dans la littérature. La forme des aires de marché qu'elles engendrent n'a pas toujours été étudiée ; par ailleurs, leur variété amène à se demander si d'autres fonctions encore ne mériteraient pas d'être examinées. Il paraît donc utile de disposer d'une théorie générale des aires de marché : ce à quoi s'attache ce travail en cas de duopole, dans le cadre du plan euclidien muni d'une distance euclidienne.)

7. Keith J. Tinkler, Editor, *Nystuen—Dacey Nodal Analysis*, 1988.

Professor Tinkler's volume displays the use of this graph theoretical tool in geography, from the original Nystuen—Dacey article, to a bibliography of uses, to original uses by Tinkler. Some reprinted material is included, but by far the larger part is of previously unpublished material. (Unless otherwise noted, all items listed below are previously unpublished.) Contents: " 'Foreward' " by Nystuen, 1988; "Preface" by Tinkler, 1988; "Statistics for Nystuen—Dacey Nodal Analysis," by Tinkler, 1979; Review of Nodal Analysis literature by Tinkler (pre–1979, reprinted with permission; post—1979, new as of 1988); FORTRAN program listing for Nodal Analysis by Tinkler; "A graph theory interpretation of nodal regions" by John D. Nystuen and Michael F. Dacey, reprinted with permission, 1961; Nystuen—Dacey data concerning telephone flows in Washington and Missouri, 1958, 1959 with comment by Nystuen, 1988; "The expected distribution of nodality in random (p, q) graphs and multigraphs," by Tinkler, 1976.

8. James W. Fonseca, *The Urban Rank-size Hierarchy: A Mathematical Interpretation*, 1989.

The urban rank–size hierarchy can be characterized as an equiangular spiral of the form $r = ae^{\theta \cot \alpha}$. An equiangular spiral can also be constructed from a Fibonacci sequence. The urban rank–size hierarchy is thus shown to mirror the properties derived from Fibonacci characteristics such as rank–additive properties. A new method of structuring the urban rank–size hierarchy is explored which essentially parallels that of the traditional rank–size hierarchy below rank 11. Above rank 11 this method may help explain the frequently noted concavity of the rank–size distribution at the upper levels. The research suggests that the simple rank–size rule with the exponent equal to 1 is not merely a special case, but rather a theoretically justified norm against which deviant cases may be measured. The spiral distribution model allows conceptualization of a new view of the urban rank–size hierarchy in which the three largest cities share functions in a Fibonacci hierarchy.

9. Sandra L. Arlinghaus, *An Atlas of Steiner Networks*, 1989.

A Steiner network is a tree of minimum total length joining a prescribed, finite, number of locations; often new locations are introduced into the prescribed set to determine the minimum tree. This Atlas explains the mathematical detail behind the Steiner construction for prescribed sets of n locations and displays the steps, visually, in a series of Figures. The proof of the Steiner construction is by mathematical induction, and enough steps in the early part of the induction are displayed completely that the reader who is well–trained in Euclidean geometry, and familiar with concepts from graph theory and elementary number theory, should be able to replicate the constructions for full as well as for degenerate Steiner trees.

Sylvia Richardson, "Some remarks on the testing of association between spatial processes";
Graham J. G. Upton, "Information from regional data";
Patrick Doreian, "Network autocorrelation models: problems and prospects."

Each chapter is preceded by an "Editor's Preface" and followed by a Discussion and, in some cases, by an author's Rejoinder to the Discussion.