Institute of Mathematical Geography

MONOGRAPHS

Simulating K=3 Christaller Central Place Structures

Monograph #10

Daniel A. Griffith

IMaGe

SIMULATING K=3 CHRISTALLER CENTRAL PLACE STRUCTURES: AN ALGORITHM USING A CONSTANT ELASTICITY OF SUBSTITUTION CONSUMPTION FUNCTION

by

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The author has donated all royalties for this work to the Mathematical Models and Quantitative Methods Specialty Group of the Association of American Geographers to use as it deems fit to develop the interests of that group.

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THIS MONOGRAPH IS DEDICATED

TO THOSE UNIVERSITY EDUCATORS

WHOSE TUTELAGE

HAS MADE ITS PREPARATION POSSIBLE;

TO DR. LEE R. BEAUMONT,

FORMERLY OF THE SCHOOL OF BUSINESS,

INDIANA UNIVERSITY OF PENNSYLVANIA,

WHO CULTIVATED IN ME

THE NECESSARY SKILLS FOR

PHYSICAL MANUSCRIPT PRODUCTION,

TO DR. LEONARD TEPPER,

DEPARTMENT OF GEOGRAPHY,

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WHO HELPED INSTILL IN ME

THE NECESSARY QUANTITATIVE GEOGRAPHY PERSPECTIVE,

AND TO MR. DOYLE MCBRIDE,

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WHO INSPIRED ME TO STUDY GRADUATE LEVEL MATHEMATICS.

I AM IN DEBT TO THESE FORMER PROFESSORS

FOR THE PROFOUND POSITIVE IMPACT THEY HAVE HAD

ON THE COURSE OF MY PROFESSIONAL CAREER.

PREFACE

This discussion paper presents a summary of the results of research that has been of interest to me for a considerable portion of my academic career. When I embarked on my graduate geography education in 1970, at Indiana University of Pennsylvania, I was impressed by two items that were topical at that time, namely Dacey's "Geometry of central place theory" (1965) and Marble and Anderson's Von Thuenen simulation computer program (in 1970 the computer code was available through the Geography Program Exchange housed at Michigan State University; a subsequent publication describing this work was released by the Association of American Geographers in 1972). One of the first goals I set for my career in those days was to create a computer version of central place theory that was comparable in fundamental ways with what Marble and Anderson had produced for agricultural land use theory; in part this present publication fulfills that aspiration. The contents of this document are a natural extension of my Economic Geography (1986) article.

Throughout my graduate education I had been keenly interested in interfaces between geography and mathematics, in part because my Bachelor of Science degree is in mathematics. Dr. Robert Thomas, now of Michigan State University, and Dr. John Stephens, now of the University of Miami, sparked my attraction to this specialized field of inquiry during the last two years of my undergraduate education. Dr. Leslie Curry further cultivated this curiosity of mine while I undertook my doctoral studies at the University of Toronto. Because of this interest, I continued to study both pure and (mostly) applied mathematics, as well as theoretical and (mostly) applied statistics, during my graduate tenures at Indiana University of Pennsylvania, the University of Toronto, and the Pennsylvania State University. In many ways the contents of this monograph typify the geography/mathematics interface, and in a number of ways represent the application of skills whose attainment is a culmination of my hybrid education. This research project has drawn extensively on my background in and knowledge of numerical analysis, computer programming, calculus, applied statistics, and urban geography.

The final encouragement for taking another step in this research endeavor came in the form of a grant from the SUNY/Buffalo Office of Teaching Effectiveness; this grant provided funding for development of the PC computer program, whose code is present here in Appendix A. The PC programming was done in cooperation with Darren Griffith; this joint venture was a fringe benefit of our enrolling together in the evening continuing education BASIC programming sequence taught at Kenmore East High School (Tonawanda, New York) during 1983-86. A PC demonstration of the computer program described here will be presented at the 1989 annual meeting of the Association of American

Geographers, in Baltimore. I am sincerely appreciative of the research opportunity given to me through the funding provided by SUNY/Buffalo.

Because external funding supported the development of this computer program, I consider the Version 2 code to be in the public domain; hence, a free digital copy of it may be obtained by sending a blank, formatted high density diskette to me. Since the code appearing in Appendix A is printed directly from an ASCII version of the BASIC source code, it should be free of typing errors; but users may find securing a digital copy more convenient, and certainly such a copy would help eliminate any typing errors that may be introduced by re-keying the code.

A Word of Caution

The computer code appearing in Appendix A was devised for IBM-compatible PCs. Version 1 was developed on an IBM-PC and IBM-AT using BASICA, whereas Version 2 is an updated, debugged and modified variant of Version 1, composed using a NEC PowerMate 386 PC with GW-BASIC. Programming results have been verified using FORTRAN and the IMSL 10.0 library on a mainframe DEC VAX. One of the major drawbacks uncovered in this developmental progression has to do with rounding error problems; these complications arise from the utilization of different computer algorithms, of different programming languages, and of different hardware. Those numerical algorithms whose performance consistently appeared to be most efficient were selected for implementation here; perhaps more powerful ones can be found in IMSL subroutines, but they did not always perform well on the PC. BASIC carries fewer significant digits that does especially double precision FORTRAN, yielding a rounding error inconsistency problem when results from both are compared; for the most part, the large volumes of output inspected were identically the same. And, NEC employs words having more bits than do those found on the aforementioned IBM PCs; unfortunately this apparently spurious precision tends to confuse the BASIC computer language. So, the convergence criteria introduced in the computer code have been judiciously calibrated in order to minimize rounding error calamities (see lines 3030, 3330, and 3740); this disagreement between the mainframe FORTRAN and the PC BASIC codes renders noticeable but probably acceptable rounding errors. Successful implementation of the source code appearing in Appendix A may require additional modifications of these convergence criteria, particularly when other IBM-compatible machines are employed; one prudent way of improving upon the convergence tests performed in the computer program would be to convert each number being tested to a percentage of its corresponding target value, rather than using an absolute value (as is done here).

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January 11, 1989
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1.0. INTRODUCTION

The principal objective of this monograph is to present a description of and source PC computer code for, together with its affiliated salient geographic theory and mathematical underpinnings, a program that simulates K = 3 Christaller central place structures using a constant elasticity of substitution consumption function. Primarily this program is designed to serve pedagogic purposes. It furnishes a vehicle for operationalizing a particular type of central place structure in experimental Social Science classroom environments. The philosophy behind it is similar to that for the spatial autocorrelation simulation game reported on in Griffith (1987), the Von Thuenen agricultural land use computer simulation program reported on by Marble and Anderson (1972), and the location-allocation game reported on by Rushton (1975). This present effort builds upon classroom and professional educational experiences of the author, in terms of the aforementioned spatial autocorrelation simulation game, as well as spatial diffusion and industrial location games [see Griffith (1982, 1984)]. It also builds upon the author's knowledge and research experiences in the fields of quantitative urban and economic geography.

This program, suited for either introductory or advanced economic or urban geography courses, focuses on exploration of selected parameters of the ideal K = 3 Christaller central place structure. It exploits fundamental concepts of central place theory, and uses a deterministically oriented approach rather than the more traditional inductive statistical perspective underlying much of the research summarized in textbooks. One aim is to allow students to alter parameters of a central place system, and then view graphic and tabular results attributable to these changes, without experiencing the tedium of massive calculations. The program is organized in such a way that students can execute selected contrived examples whose results already are known; these examples also facilitate implementation testing. Students are able to vary single or multiple parameters of the problem, too, permitting a study of how certain changes manifest themselves within the context of a theoretical central place structure. Hierarchical classification criteria may be changed, demand elasticities may or may not vary and can take on a wide range of non-negative values, the uniform transport cost may be set at any positive level, assorted fixed costs and variable costs may be introduced, again within a rich range of non-negative possibilities, and the number of commodities can be altered. Past experience with these types of simulation exercises suggests that students gain more insight into, and better comprehend mathematical spatial theory when it is presented in a simulation format. In addition, flexibilities embraced by this program should promote a deeper understanding of central place theory, permit an inspection of the impact of relaxing assumptions of central place theory, and help achieve a better comprehension of the complex role distance plays in the space-economy.

The BASIC computer programming language was selected for composition purposes here because it is widely available on PCs, whose presence has proliferated in educational institutions during the past decade.

2.0. BACKGROUND

Mathematical analyses associated with central place theory are replete in the geography literature. Dacey (1965) has sketched properties of the resulting geometric map patterns. Hudson (1967) has discussed number theory properties of central place nestings that, treating Christaller's K values as though they were prime numbers, allow many Loeschian structures to be interpreted within the context of the three Christallerian organizing principles (i. e., marketing, transportation, and administration). Marshall (1975, 1977) and Tinkler (1978) have refined this mathematical analysis, again within the context of number theory, in various ways. Alao et al. (1977) have produced a systematic mathematical treatment of abstract central place theory. Mulligan (1981, 1982) has addressed the spatial economics of central place theory. And, Gahitte (1987) has criticized Dacey's earlier work, claiming to prove that Dacey failed to build a mathematical central place system, and proposing a new method of construction of the Loeschian landscape. This paper builds upon these sundry traditions.

2.1. Results for the linear consumption function

Much of the background literature to date focuses on linear demand functions [see Greenhut and Hwang (1979), Mulligan (1981, 1982)]; one exception is Griffith (1986). Although the research described in this monograph is concerned with non-linear (specifically negative exponential) demand functions, its predecessors merit some attention in order to establish an historical perspective.

2.1.1. Important attributes of spatial demand

Throughout this monograph the standard assumptions of central place theory, such as an isotropic surface, a uniform distribution of consumers, identical demand functions, and completely rational behavior, will be made. Further, for the sake of simplicity the linear demand function posited will be of the form [after Greenhut and Hwang (1970)]

$$D = A - \eta(p_{fob} + t\,\delta) \quad , \qquad (2.1)$$

where D = the actual density of demand for a given commodity,

A = the maximum density of demand for a given commodity,

p_{fob} = the per unit free-on-board price of a given commodity,

t = the transport rate,

δ = the distance separating some specified consumer and a central place, and

η = the elasticity of demand parameter.

If demand is inelastic, then $\eta = 0$; otherwise $\eta > 0$.

Given Equation (2.1), three features of spatial demand can be addressed, namely the ideal range of a commodity, the threshold of a commodity, and the real range of a commodity [see Griffith (1986) for a somewhat detailed discussion of these concepts within the present context]. The notion of an ideal range of a commodity, which Saey (1973) points out is the basis of a Christallerian landscape, refers to the distance from a central place at which demand falls to zero, or

$$0 = A - \eta(p_{fob} - t\delta)$$

$$\delta = (A - \eta p_{fob}) / (\eta t) \quad . \qquad (2.2)$$

Because the surface involved is isotropic, this range creates a circle, centered on the central place, whose radius is given by δ of Equation (2.2).

The threshold of a commodity is some distance δ_* that defines a circle circumscribing the minimum demand necessary for a central place to survive, which Saey (1973) points out is the basis of a Loeschian landscape. Moreover, since the area of this circle is $\pi\delta_*^2$, for a uniform spatial distribution of demand the problem reduces to one of finding δ_* such that excess profits are zero, or

$$(\pi\delta_*^2)\,(p_{fob} - c_v) \;-\; c_f = 0 \quad ,$$

where c_v = the per unit variable costs for a given commodity, and

c_f = the fixed costs for a given commodity that are to be covered during a specified time period.

The concept of a circular threshold holds only when spatial competition is absent.

When spatial competition is present, the ideal range of a commodity takes on meaning. Because a hexagonal market area results from spatial competition, the threshold demand demarcated by the foregoing circle now must be captured by a hexagonal market area. Superimposing this hexagon on the corresponding circle results in the six corners of the hexagon extending beyond the circumference of this circle, and the sides of this hexagon cutting off six segments of this circle (see Diagram 1). Each circle segment that extends beyond the perimeter of the hexagon is equal in area to each corner segment of the hexagon that extends beyond the perimeter of the circle. Dimensions of these two geometric figures can be determined by equating their respective

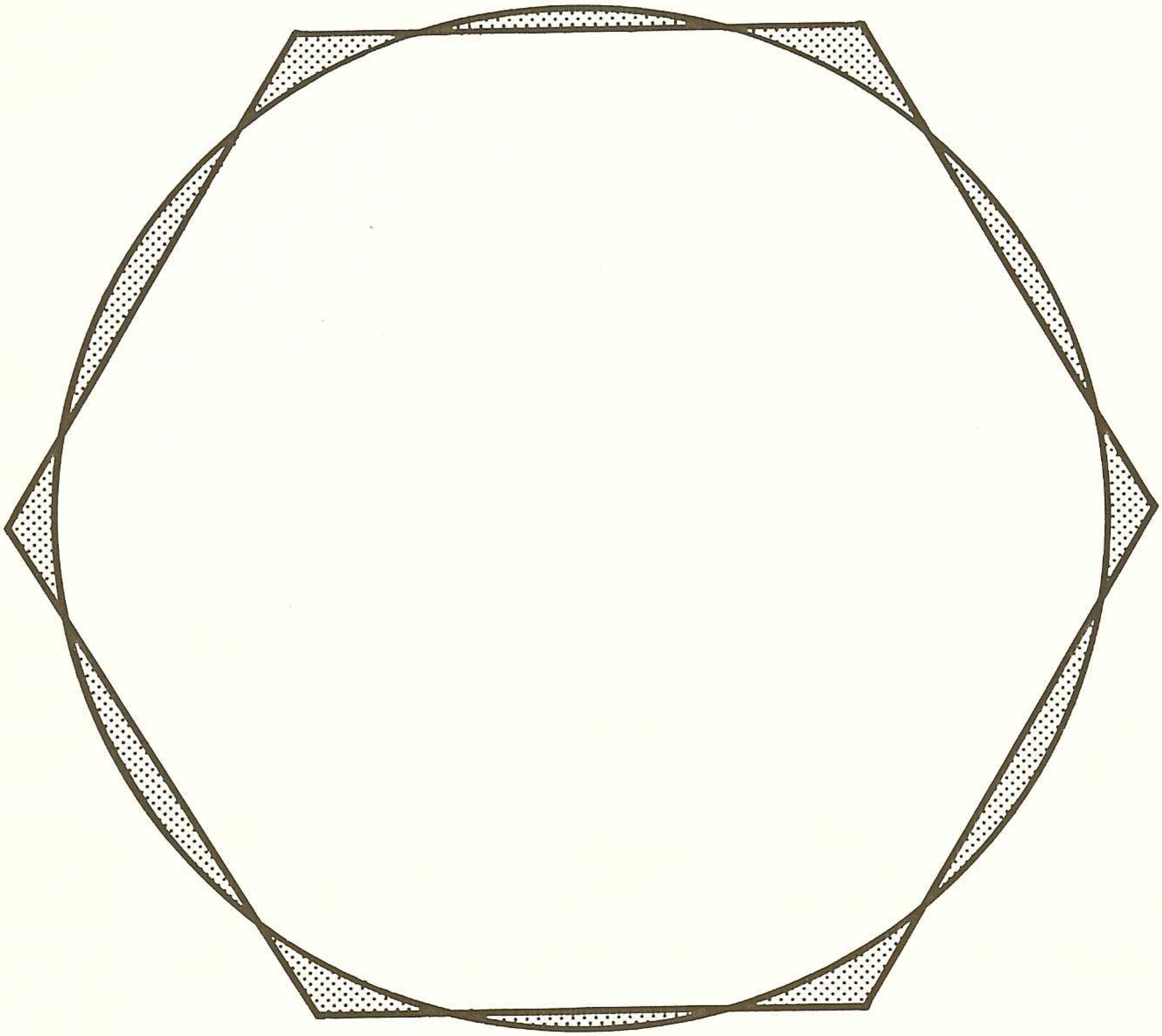


DIAGRAM 1. A hexagon superimposed upon a circle of equal area.

areas. In other words,

$$3b^2\cos(\pi/6)/[2*\sin(\pi/6)] = \pi r^2 ,$$

where b = the length of each side of the hexagon, and

r = the radius of the circle.

Solving for b yields

$$b = r[2\pi/(3*3^{1/2})]^{1/2} \approx 1.1 r .$$

Accordingly, the distance to the midpoint of any side of the hexagon is $[\pi/(2*3^{1/2})]^{1/2} \approx 0.95 r$. Hence the real range of a commodity, or the maximum distance a consumer is willing to travel when competition is present, must be at least a distance of b. This conclusion implies that the following relationship must prevail for spatial competition situations:

$$\text{threshold distance} < \text{real range} \leq \text{ideal range} .$$

Clearly the threshold and range can never coincide in a competitive geographic landscape.

The area of a circle that is transferred to the corners of its superimposed hexagon is not considerable, though. For each of the six circle segments this area is defined as

$$r^2[\pi*\sin^{-1}[(2*3^{1/2} - \pi)/(2*3^{1/2})]/90 - \sin\{\pi*\sin^{-1}[(2*3^{1/2} - \pi)/(2*3^{1/2})]/90\}]/2$$

$$\approx (1.7*10^{-4})\pi r^2 .$$

One salient drawback affiliated with the planar perspective employed in this section is that it overlooks the possibility of elastic demand. As soon as η of Equation (2.1) becomes positive, even though the spatial distribution of customers is uniform, the spatial distribution of demand varies over the isotropic surface. This variation requires the addition of a third dimension, representing demand, to the analysis, and then working with volumes rather than simply areas.

2.1.2. The case of inelastic demand

The simplest case to consider is the counterpart to the foregoing planar analysis. Suppose $\eta = 0$ in Equation (2.1). Then the demand measured on a third axis, say z, that is orthogonal to the (x,y) planar surface, would equal A everywhere. Thus, the spatial distribution of demand is uniform. The only non-trivial solutions here occur in the presence of spatial competition.

Total demand in a market area is represented by a parallelepiped having a hexagonal base. The volume of this solid is $3*3^{1/2}b^2A/2$. Accordingly, the threshold hexagonal market area is determined by solving the following equation:

$$(3*3^{1/2}b^2A/2)(p_{fob} - c_v) - c_f = 0 . \tag{2.3}$$

The real range for this commodity in question is obtained by solving Equation (2.3) for b, yielding

$$b = \{2*c_f/[3*3^{1/2}A(p_{fob} - c_v)]\}^{1/2} \quad .$$

Consequently, total threshold demand is defined as $3*c_f/(p_{fob} - c_v)$, and the spacing of higher order centers is defined as $2*c_f/[3^{1/2}A(p_{fob} - c_v)]$. Since consumers are perfectly rational, and t is constant over the surface, these consumers necessarily are distance minimizers. This particular feature forces p_{fob} to be fixed across the set of central places, and at a minimum.

2.1.3. The case of elastic demand

In the presence of competition the resulting solid becomes quite complex in form. Consider the case of just two central places that are in competition. Then their cones overlap, resulting in a truncated cone of the type appearing in Diagram 2. In order to determine the amount of demand lost through this competition, a polar coordinate system must be employed. Because p_{fob}, η and t are spatially invariant in the spatial economic landscape, the market boundary will be a straight line.

From an aerial viewpoint, in which the (x,y)-axis system is rotated so that the market boundary in question is perpendicular to the x-axis, this boundary line will have the polar coordinates equation

$$x = r*\cos\theta \quad , \qquad (2.4)$$

where x is constant, and θ is the angle between the x-axis and the circle radius that intersects the line described by Equation (2.4) at the circumference of the circle. For a circle radius of $A_1/(\eta t)$,

$$x = 3^{1/2}A_1/(2\eta t) = r*\cos\theta \quad , \text{ and so}$$

$$r = 3^{1/2}A_1/(2\eta t*\cos\theta) \quad .$$

Therefore, the quantity of demand lost by each central place due to competition is given by

$$2\int_0^{\pi/6}\int_{3A_1/(2\eta t*\cos\theta)}^{A_1/(\eta t)} \delta(A_1 - \eta t\delta)\, d\delta\, d\theta = \{3\pi - 3^{1/2}[8 - 3*\ln(3)]\}A_1^3/(48\eta^2 t^2) \quad .$$

Furthermore, if the length of the market boundary is $A_1/(\eta t)$, and complete competition is present over the planar surface so that regular hexagonal market areas result, then the total demand captured by a single central place is defined as

$$12\int_0^{\pi/6}\int_0^{3A_1/(2\eta t*\cos\theta)} \delta(A_1 - \eta t\delta)\, d\delta\, d\theta = 3^{1/2}[8 - 3*\ln(3)]A_1^3/(8\eta^2 t^2) \quad .$$

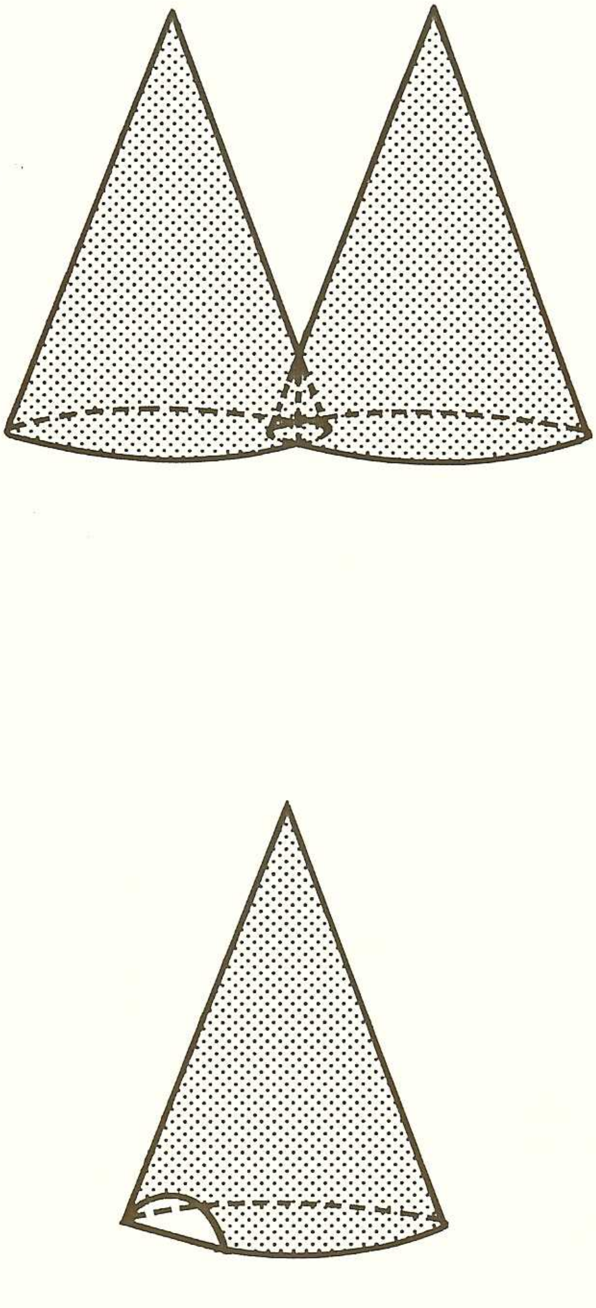


DIAGRAM 2. Truncation of overlapping demand cones as a result of spatial competition.

Suppose that a threshold exists at a hexagon radius δ, where demand has fallen to A_2. Now the solid is a cone that has had six cuts made through its sides, at 120° angles to one another, setting on top of a parallelepiped having a hexagonal base (see Diagram 3). The volume of this solid is the sum of its two component solid parts, namely

$$3^{1/2}[8 - 3*\ln(3)](A_1 - A_2)^3/(8\eta^2t^2) + 3*3^{1/2}(A_1 - A_2)^2A_2/(2\eta^2t^2) \quad .$$

The corresponding free-on-board price for a given A_2 can be obtained from the following fourth-degree polynomial:

$$\{3^{1/2}[8 - 3*\ln(3)](A - \eta p_{fob} - A_2)^3/(8\eta^2t^2) +$$
$$3*3^{1/2}(A - \eta p_{fob} - A_2)^2A_2/(2\eta^2t^2)\}(p_{fob} - c_v) - c_f = 0 \quad ,$$

or

$$p^4_{fob} - \{[3(A - A_2) + \eta c_v\}p^3_{fob} + \{[3(A - A_2)^2 + 3(A - A_2)\eta c_v]/\eta^2\}p^2_{fob} -$$
$$\{[(A - A_2)^3 + 3(A - A_2)^2\eta c_v]/\eta^3\}p_{fob} +$$
$$c_v(A - A_2)^3/\eta^3 + 8c_ft^2/\{3^{1/2}\eta[8 - 3*\ln(3)]\} = 0 \quad . \qquad (2.5)$$

The four roots of this equation may be extracted numerically, with the root that equals p_{fob} satisfying the inequality

$$c_v < R_i < (A - A_2)/\eta \quad .$$

At most two of the four roots will satisfy this inequality, since at least two of them will be imaginary. If the remaining two are real, the one closest to c_v should be equated with p_{fob}, since this value will permit the demand in question to be maximized. The fourth root may permit the size of a market area to be maximized, but the accompanying price would be higher. Cases in which all four roots are imaginary pertain to commodities whose cost structure is such that it is infeasible to supply them in the given spatial economic landscape.

When multiple commodities are present, and are divided into bundles in accordance with their temporal frequency of demand, for the lowest level of the central place hierarchy Equation (2.5) must be solved for each commodity, with $A_2 = 0$ in order to maximize the corresponding market area. That commodity having the shortest range, defined by $(A_3 - \eta p_{fob})/(\eta t)$, is the marginal commodity. Accordingly, it establishes a value for A_2 that must be used for all other commodities at this hierarchical level. This new value of A_2 then can be substituted into Equation (2.5), and prices generated for all commodities in a given bundle.

If a K=3 central place hierarchy is being constructed, then the spacing between central places is given by $3(A - \eta p_{fob})/(\eta t)$, where p_{fob} is the price of the marginal commodity at the lowest level of the hierarchy. The spacing at the second-lowest level [Beavon (1977)] will be $3[3^{1/2}(A - \eta p_{fob})]/(\eta t)$. Setting this latter value equal to A_2, and solving for p_{fob}, will yield prices at this second hierarchical level. In this second instance, if $A_2 = 0$ then a

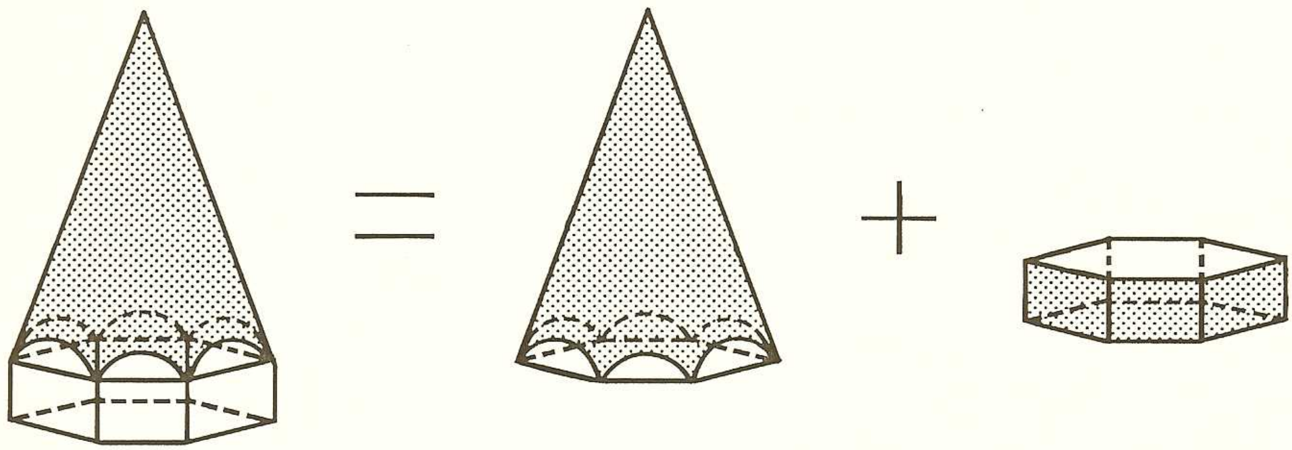


DIAGRAM 3. A truncated demand cone, sitting on a hexagonal base solid, resulting from demand spatial competition in a central place system.

Loeschian type of system can be generated that may well differ from one of the three Christallerian landscapes. Marshall (1975) has demonstrated that a Loeschian number always will be produced. In addition, with this second scheme each hierarchical level would have a marginal commodity. If a Christallerian landscape is desired, though, this constraint could lead to either the presence of excess profits, or the failure for provision of some commodities (demand going unsatisfied), depending upon the values of the roots of Equation (2.5). It is unlikely that, as is contended by Greenhut and his co-authors (1973, 1975, 1979), hexagons would be replaced by pentagons, or any other type of higher-order alphagons.

2.2. Results for the CES consumption function

Griffith (1986) presents the mathematical and spatial theoretical overview for using a constant elasticity of substitution (CES) consumption function in order to represent demand in the space-economy. To summarize (since anything more would be superfluous), historically the problem of negative demand has been encountered in constructing spatial demand cones when using linear demand functions, like those discussed in Section 2.1. The CES demand function avoids this problem; the single most serious trade-off is a dramatic increase in complexity of the mathematics involved, primarily due to the non-linear nature of all equation reformulations for characterizing the central place system. Further, a feedback mechanism is introduced between price and demand, analogous to what was done in Section 2.1. For the purposes of this paper, only the case of competition will be addressed, either with elastic or inelastic demand; Griffith's discussion of non-competitive situations is irrelevant to this present work.

One noteworthy error has been uncovered in Table 1 of Griffith (1986) during the course of the research reported on here. For Good #1, the margin of consumption should be 97.17 rather than the reported value of 183.96; this former value is the true minimum non-negative root of the polynomial in question. This mistake makes little difference to Griffith's discussion, but is noted here because of the discrepancy it introduces into Table 1 of Section 5.2.

Section 3.0 presents a detailed, step-by-step presentation of the derivation of mathematical findings summarized in Griffith (1986).

2.3. The K = 3 central place structure map

The simplest Christallerian central place structure is nested, based upon the marketing principle (the need for central places to be as near as possible to the customers they serve), and has a nesting value of 3. The adaption of his scheme here involves a four-tier hierarchy, consisting of hamlets (Level 4), villages (Level 3), towns (Level 2), and cities (Level 1). Excess profits are set equal to zero [see Equation (5.1) of Section 5.0; this condition could be changed without great difficulty]. The minimum non-negative threshold value for the lowest level appearing in a particular problem determines the spacing of centers. The map that is graphically displayed on a cathode-ray tube (CRT) constitutes a half-hexagon market area for the city level, and is composed of 22 central places; it is based upon the map appearing in Griffith (1976), which is reproduced here in a modified form in Diagram 4. Examples of the computer version of this map are presented in the sample output appearing in Section 5.2. Both the original and the modified maps depict the equal spacing of central places having the lowest level of specialization (referred to here as villages), the hexagonally shaped hinterlands, the central location of a more specialized village for every ring of six hamlets located at each corner of its hexagonal trade area, the equal spacing of villages, and the trade area of a village being three times larger than that of a hamlet. Once the basic commodity at Level 4 is identified, distance between centers at a given level is set equal to $3^{1/2}$ times the distance between the centers immediately below them in the hierarchy. Since a hexagonal radius is equivalent to the distance between a given pair of central places and the perpendicular bisector of a straight line connecting these competitors (the hexagonal market areas may be demarcated by constructing Thiessen polygons), the reported spacing distances equal twice the radius distances appearing in the corresponding table.

The accompanying tabular output defines the total number of commodities provided, the pricing of these commodities, the types of commodities provided, and the levels of specialization and size of trade areas served. Students can obtain a copy of a table or a map by printing the screen, and should transfer their graphical results to a more comprehensive central place landscape map provided by their instructor.

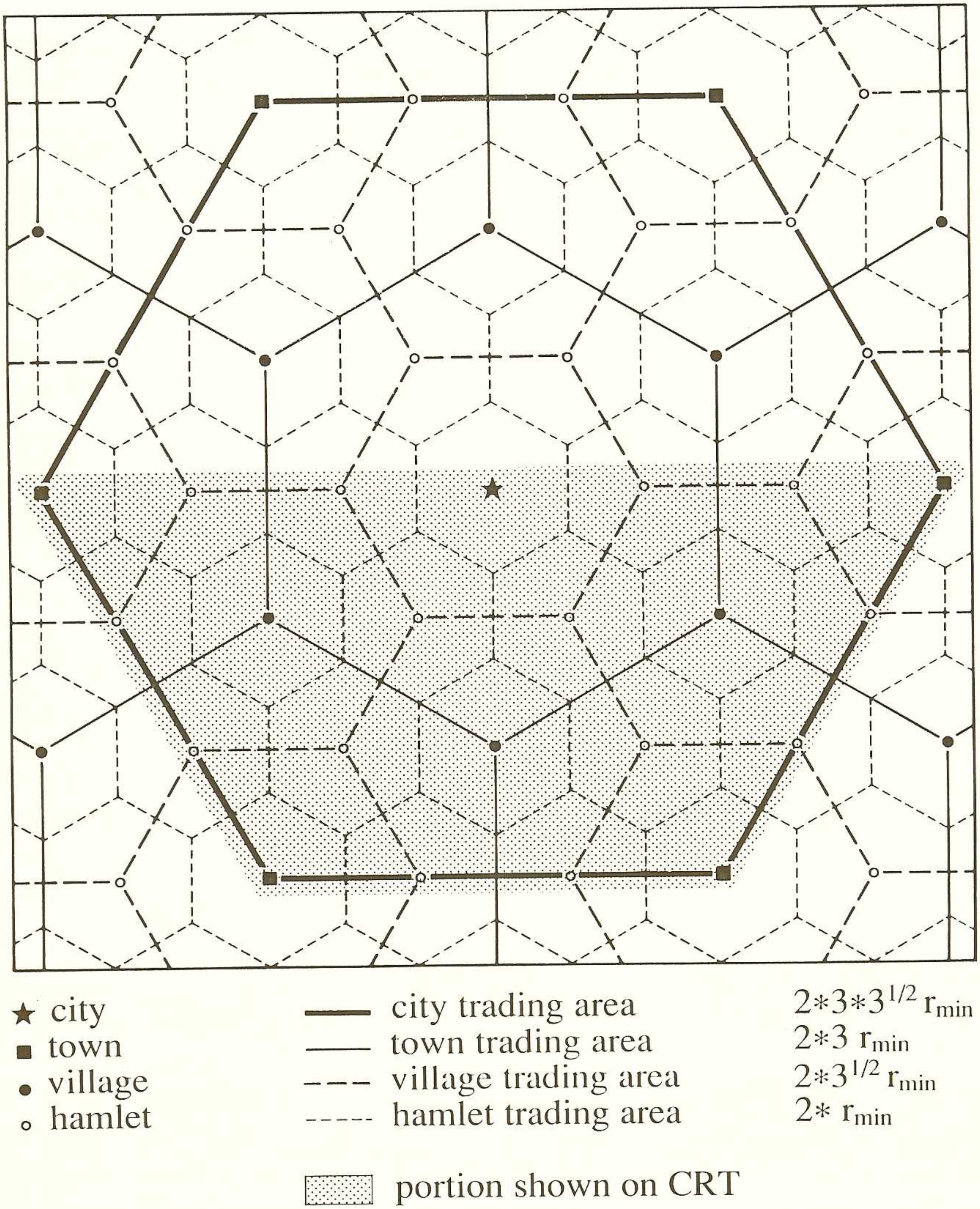


DIAGRAM 4. Central Place Structure Map

3.0. MATHEMATICAL AND NUMERICAL SOLUTIONS TO COMPONENTS OF THE K = 3 CENTRAL PLACE STRUCTURE PROBLEM

Analytical and approximate solutions to the central place equations will be discussed in this section; analytical solutions were derived with the aid of Beyer (1978) and Bois (1961). Because non-linear equations are being dealt with, the possibility of multiple solutions for any given problem frequently exists. In addition, since single equations often contain two unknowns, namely price and radius, unique solutions usually are elusive. Basic principles will be invoked here in order to circumvent these complications. Each of these principles relates to the profit equation,

$$\text{Demand}^* (p_{fob} - c_v) - c_f = 0 \quad , \qquad (3.1)$$

where demand will be cast as a function of free-on-board price, p_{fob} (hereafter to be denoted p), c_v denotes variable cost, and c_f denotes fixed cost.

The first principle, alluded to in Sections 2.1 and 2.2, conjectures that a useful single commodity spatial equilibrium exists when $p = rt$, the threshold hexagon radius, r, times the transport cost, t. Accordingly, each non-nested radius can be determined by setting price equal to the product rt. This principle is consistent with the result based upon a linear demand function, for which the margin of consumption is $r = p/t$. It is further reinforced by the general solution reported in this section for $\eta = 2$, where $p = rt$ is a critical point beyond which one solution holds, and prior to which another solution holds; at $p = rt$ yet a third, and very simple, solution holds. This critical point is far less conspicuous but also is uncovered for $\eta = 1$ and $\eta = 3$, when the partial derivative with respect to r is calculated; this certainly is one of the conditions for establishing an equilibrium.

The second principle has to do with the notion of maximum packing of hexagons on a planar surface. Because the Christallerian market principle maintains that central places should be as close as possible to consumers, the smallest radius of the lowest order commodity seems most appropriate to select from a multiple roots solution. This radius, called r_{min}, then can be used to generate the central place spacings at each hierarchy level. Since multiple solutions almost always exist for Equation (3.1), for the demand term is a higher order function of price, then the smallest non-negative radius should be selected for nesting purposes; moreover, larger radii solutions still allow the equation to be equal to zero, but result in a lower density of hexagonal market areas. In addition, since each entrepreneur wants to maximize demand, the equation holds for multiple solutions only because increases in price are offset by decreases in demand. With regard to this situation, then, if a higher price and larger radius are selected (remembering that in spatial equilibrium p = rt), then an incentive exists for competing entrepreneurs to enter a central place and offer the same commodity at the lower price, without sacrificing normal profit.

The third principle pertains to profit. Intuitively speaking, Equation (3.1) should be nearly satisfied if price equals variable cost. So, the search for a feasible root (i. e., a positive real rather than a negative or complex number) for either radius or price respectively is restricted to one that is as close as possible to, but exceeding, variable cost divided by transport cost, or variable cost. Again, while other real roots may exist for this equation, the affiliated price would be higher, the accompanying demand would be lower, and hence an incentive would exist for competitors to enter the market.

3.1. Determining the margins of consumption and affiliated demand

For a given commodity, the margin of consumption is defined here, for a competitive hexagonal tessellation, as the hexagon radius at which p = rt. Calculating this radius becomes a function of forcing the excess profits to zero for the prevailing elasticity of demand associated with the commodity in question. These calculations are characterized by various cases. The first case is for inelastic demand, and may be summarized as follows, where demand is denoted by D:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\,(p + t\delta)^0\, d\delta\, d\theta = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\, d\delta\, d\theta$$

$$= 12A \int_0^{\pi/6} (1/2)\,(r/\cos\theta)^2\, d\theta = 6A \int_0^{\pi/6} r^2/\cos^2\theta\, d\theta$$

$$= 6Ar^2 \int_0^{\pi/6} 1/\cos^2\theta\, d\theta$$

$$= 6Ar^2 \tan\theta \Big|_0^{\pi/6} = 6Ar^2/3^{1/2} = 2*3^{1/2}Ar^2 \quad .$$

Equation (3.1) becomes

$$2*3^{1/2}Ar^2(rt - c_v) - c_f = 0$$

$$r^3 - (c_v/t)r^2 - c_f/(2*3^{1/2}At) = 0 \quad .$$

The relevant radius for this equation is, from the general solution to cubic equations [see Uspensky (1948)]

$$r = [-[- c_f/(2*3^{1/2}At) - 2c_v^3/(27t^3)]/2 + \{[- c_f/(2*3^{1/2}At) - 2c_v^3/(27t^3)]^2/4 + [- c_v^2/(3t^2)]^3/27\}^{1/2}]^{1/3} + [-[- c_f/(2*3^{1/2}At) - 2c_v^3/(27t^3)]/2 - \{[- c_f/(2*3^{1/2}At) - 2c_v^3/(27t^3)]^2/4 + [- c_v^2/(3t^2)]^3/27\}^{1/2}]^{1/3} .$$

This radius is guaranteed to be real and positive; the first term in each cube root always is positive (a negative times a negative), the quantity whose square root is calculated always is positive [the squared expression contains a positive version of $-c_v^6/(27^2t^6)$], and the square root expression always is less than the term that precedes it.

The second case is where $\eta = 1$, and may be summarized as follows:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\ (p + t\delta)^{-1}\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} (1/t)\ \delta/(r + \delta)\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} (1/t)\ [\delta - r \ln(r + \delta)] \Big|_0^{r/\cos\theta} d\theta$$

$$= (12A/t) \int_0^{\pi/6} [r/\cos\theta - r \ln(r + r/\cos\theta) - 0 + r \ln(r)]\ d\theta$$

$$= (12Ar/t) \int_0^{\pi/6} [1/\cos\theta - \ln(1 + \cos\theta) + \ln(\cos\theta)]\ d\theta$$

$$= (12Ar/t)\{\ln[\tan(\pi/3)] - \ln(1) - (\pi/6)\ln(1 + 3^{1/2}/2) + (\pi/6)\ln(3^{1/2}/2) + \int_0^{\pi/6} \theta[\tan\theta - \tan(\theta/2)]\ d\theta\} = (12Ar/t)(0.1738034) .$$

Equation (3.1) becomes

$$(12Ar/t)(0.1738034)(rt - c_v) - c_f = 0$$

$$r^2 - (c_v/t)r - c_f/[12A(0.1738034)] = 0 .$$

The relevant radius for this equation is, from the general solution to quadratic equations

$$r = [c_v/t + \{(c_v/t)^2 + c_f/[3A(0.1738034)]\}^{1/2}]/2 \quad .$$

Clearly a non-negative radius always exists for this case.

The third case is where $\eta = 2$, and may be summarized as follows:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\ (p + t\delta)^{-2}\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} (1/t^2)\ \delta/(r + \delta)^2\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} (1/t^2)\ [r/(\delta + r) + \ln(r + \delta)] \Big|_0^{r/\cos\theta} d\theta$$

$$= (12A/t^2) \int_0^{\pi/6} [r/(r/\cos\theta + r) + \ln(r/\cos\theta + r) - 1 - \ln(r)]\ d\theta$$

$$= (12A/t^2) \int_0^{\pi/6} [\cos\theta/(1 + \cos\theta) + \ln(1 + \cos\theta) - \ln(\cos\theta) - 1]\ d\theta$$

$$= (12A/t^2)\{[\theta - \tan(\theta/2)] \Big|_0^{\pi/6} - \theta\ln(\cos\theta) \Big|_0^{\pi/6} + \theta\ln(1 + \cos\theta) \Big|_0^{\pi/6} + \int_0^{\pi/6} \theta[\tan(\theta/2) - \tan\theta]\ d\theta - \pi/6\}$$

$$= (12A/t^2)\{\pi/6 - \tan(\pi/12) - (\pi/6)\ln(3^{1/2}/2) + (\pi/6)\ln(1 + 3^{1/2}/2) - \pi/6 + \int_0^{\pi/6} \theta[\tan(\theta/2) - \tan\theta]\ d\theta\} = (12A/t^2)(0.1075536) \quad .$$

Equation (3.1) becomes

$$(12A/t^2)(0.1075536)(rt - c_v) - c_f = 0 \quad .$$

The relevant radius for this equation is

$$r = c_v/t + c_f t/[12A(0.1073556)] \quad .$$

Again a non-negative radius always exists.

The fourth case covers all other values of η ($\neq$ 0, 1, 2), and may be summarized as follows:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\ (p + t\delta)^{-\eta}\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\ (rt + t\delta)^{-\eta}\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} (1/t^{\eta})\ \delta/(r + \delta)^{\eta}\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} (1/t^{\eta})\ [(\delta + r)^{-\eta+2}/(-\eta + 2) - r(r + \delta)^{-\eta+1}/(-\eta + 1)] \Big|_0^{r/\cos\theta} d\theta$$

$$= (12A/t^{\eta}) \int_0^{\pi/6} [(r/\cos\theta + r)^{-\eta+2}/(2 - \eta) - r(r/\cos\theta + r)^{-\eta+1}/(1 - \eta) - r^{2-\eta}/(2 - \eta) + r(r^{1-\eta})/(1 - \eta)]\ d\theta$$

$$= r^{2-\eta}[12A/(t^{\eta})] \int_0^{\pi/6} \{(1 + 1/\cos\theta)^{2-\eta}/(2 - \eta) - (1 + 1/\cos\theta)^{1-\eta}/(1 - \eta) + [-(1 - \eta) + (2 - \eta)]/[(1 - \eta)(2 - \eta)]\}\ d\theta$$

$$= r^{2-\eta}[12A/(t^{\eta})]\{\int_0^{\pi/6} (1 + 1/\cos\theta)^{2-\eta}/(2 - \eta)\ d\theta - \int_0^{\pi/6} (1 + 1/\cos\theta)^{1-\eta}/(1 - \eta)\ d\theta + \pi/[6(1 - \eta)(2 - \eta)]\}\ .$$

The two right-hand-side integrals contained in this last step must be solved in general using numerical integration methods, which will be discussed briefly in the next section; special cases can be solved analytically, though. Three of these particular cases will be treated here, because they represent the final three integral elasticity of demand parameter values employed in this study. Hence, Case IVa is for $\eta = 3$, and may be characterized as follows:

$$D = [12A/(rt^3)]\{\int_0^{\pi/6} -\cos\theta/(1 + \cos\theta)\ d\theta + \int_0^{\pi/6} \cos^2\theta/[2(1 + \cos\theta)^2]\ d\theta + \pi/12\}$$

$$= [12A/(rt^3)]\{\tan(\pi/12) - \pi/6 + (1/2)[\int_0^{\pi/6} 1/(1 + \cos\theta)^2\ d\theta - \int_0^{\pi/6} 1/(1 + \cos\theta)\ d\theta + \int_0^{\pi/6} \cos\theta/(1 + \cos\theta)\ d\theta] + \pi/12\}$$

$$= [12A/(rt^3)])\{\tan(\pi/12) - \pi/12 + (1/2)[(1/2)\tan(\pi/12) + (1/6)\tan^3(\pi/12) - \tan(\pi/12) - \tan(\pi/12) + \pi/6]\}$$

$= [12A/(rt^3)](1/2)[(1/2)\tan(\pi/12) + (1/6)\tan^3(\pi/12)]$

$= [6A/(rt^3)](0.1371809)$.

Equation (3.1) becomes

$[6A/(rt^3)](0.1371809)(rt - c_v) - c_f = 0$

$[t - (1/r)c_v] - c_f t^3/[6A(0.1371809)] = 0$.

The relevant radius for this equation is

$r = c_v/\{t - c_f t^3/[6A(0.1371809)]\}$.

This radius is positive when $c_f t^2/A < 0.8230854$.

Case IVb is for $\eta = 4$, and may be characterized as follows:

$$D = [12A/(r^2t^4)]\{\int_0^{\pi/6} - \cos^2\theta/[2(1 + \cos\theta)^2]\, d\theta + \int_0^{\pi/6} \cos^3\theta/[3(1 + \cos\theta)^3]\, d\theta + \pi/36\}$$

$= [12A/(r^2t^4)])\{(1/3)(1/4)[(2/3)\tan^3(\pi/12) - (1/5)\tan^5(\pi/12) - 5\tan(\pi/12) + \pi/2] - (1/2)[-(3/2)\tan(\pi/12) + (1/2)(1/3)\tan^3(\pi/12) + \pi/6] + \pi/36\}$

$= [12A/(r^2t^4)][-\pi/72 - (1/60)\tan^5(\pi/12) - (1/36)\tan^3(\pi/12) + (1/3)\tan(\pi/12)]$

$= [12A/(r^2t^4)](0.0451258)$.

Equation (3.1) becomes

$[12A/(r^2t^4)](0.0451258)(rt - c_v) - c_f = 0$

$r^2 - [12A/(t^3c_f)](0.0451258)r + [12Ac_v/(t^4c_f)](0.0451258) = 0$.

The relevant radius for this equation is, from the general solution to quadratic equations

$$r = [12A(0.0451258)/(t^3c_f) \pm \{[12A(0.0451258)/(t^3c_f)]^2 - 4[12Ac_v/(t^4c_f)](0.0451258)\}^{1/2}]/2 \quad .$$

One should note that (a) not all roots are guaranteed to be real (i. e., the discriminant may be negative), and, as mentioned in the introduction to this section, (b) r should be chosen such that MIN: $(r - c_v/t)^2$, subject to $r \geq c_v/t$.

Finally, Case IVc is for $\eta = 5$, and may be characterized in the following way:

$$D = [12A/(r^3t^5)]\{\int_0^{\pi/6} - \cos^3\theta/[3(1+\cos\theta)^3]\, d\theta + \int_0^{\pi/6} \cos^4\theta/[4(1+\cos\theta)^4]\, d\theta + \pi/72\}$$

$$= [12A/(r^3t^5)])\{(1/4)(1/8)[16\pi/12 - 15\tan(\pi/12) + (11/3)\tan^3(\pi/12) - (21/5)\tan^5(\pi/12) + (1/7)\tan^7(\pi/12)] - (1/3)(1/4)[(2/3)\tan^3(\pi/12) - (1/5)\tan^5(\pi/12) - 5\tan(\pi/12) + \pi/2] + \pi/72\}$$

$$= [12A/(r^3t^5)][\pi/72 + (1/224)\tan^7(\pi/12) - (11/96)\tan^5(\pi/12) + (17/288)\tan^3(\pi/12) - (5/96)\tan(\pi/12)] = [12A/(r^3t^5)](0.0306553) \quad .$$

Equation (3.1) becomes

$$[12A/(r^3t^5)](0.0306553)(rt - c_v) - c_f = 0$$

$$r^3 - [12A/(t^4c_f)](0.0306553)r + [12Ac_v/(t^5c_f)](0.0306553) = 0 \quad .$$

The relevant radius for this equation is, from the general solution to cubic equations

$$r = [12A(0.0306553)/(t^4c_f)\{- c_v/(2t) + [c^2_v/(4t^2) - 12A(0.0306553)/(27t^4c_f)]^{1/2}\}]^{1/3} + [12A(0.0306553)/(t^4c_f)\{- c_v/(2t) - [c^2_v/(4t^2) - 12A(0.0306553)/(27t^4c_f)]^{1/2}\}]^{1/3} \quad .$$

Here one should note that (a) at least one root is guaranteed to be real, and again (b) r should be chosen such that MIN: $(r - c_v/t)^2$, subject to $r \geq c_v/t$, in order to maximize hexagon packing of the surface.

As an aside, from the theory of equations [see Uspensky (1948)], which has been used here to determine the relevant roots for integral elasticity of demand parameter values, one can determine whether or not roots x_1, x_2 and x_3 have desirable characteristics by studying three properties they display for cubic equations of the form

$$x^3 + ax^2 + bx + c = 0 \quad ,$$

namely

(1) $x_1 + x_2 + x_3 = a$,
(2) $x_1*x_2 + x_2*x_3 + x_1*x_3 = b$, and
(3) $x_1*x_2*x_3 = c$.

A further auxiliary consideration has to do with restrictions needed for the fixed and variable production costs. If both $c_v = 0$ and $c_f = 0$, then the solution to the equation

$$r^{2-\eta}(12A/t^{\eta})*\text{integral}*(rt - c_v) \quad - \quad c_f \quad = \quad 0$$

is r = 0, which makes no sense in the space-economy; hence either fixed or variable cost must be positive. If $c_f = 0$, then $r = c_v/t$. If $c_v = 0$, then the numerical integration and root extraction methods to be discussed in the succeeding section still must be resorted to.

3.2. Numerical methods for integration and root extraction

The numerical methods resorted to in this project were selected after a careful review of Press et al. (1986), Sedgewick (1983), and Kohn (1987). The extended Simpson's Rule, a composite integration approximation formula obtained by adding twice the formula for the rectangle method to the formula for the trapezoid method, and then dividing this sum by three, was decided upon; in both of these cases height is measured at the midpoint of each sub-interval (leading to quadrature formulae). This formula is exactly equivalent to a piecewise interpolation obtained from connected parabolic segments passing through successive, non-overlapping pairs of sub-intervals covering the limits of integration; there need to be an even number of these sub-intervals. It is one of the most heavily used of the existing formulae employed for numerical integration, and virtually always uses a uniform sub-interval, or step, size for partitioning the interval over which integration is to occur.

The smaller the step size, and thus the more base points used between the integration limits, the smaller the error tends to be that is attributable to piecewise approximation. But, increasing the number of sub-intervals increases the magnitude of round-off error. At some point, then, accumulated round-off error exceeds approximation error and actually makes the calculated integral value less accurate. In this problem, the limits of integration are 0 and $\pi/6$. The best agreement between selected analytical results and their corresponding numerical approximations with BASIC subroutines using this rule occur when the number of sub-intervals is set at 20. Consequently, Simpson's Rule has been written here as

$$\int_0^{\pi/6} y(\theta)\, d\theta \approx \{[\pi/(6*20)]/3\}\{y(0) + y(\pi/6) + 2\sum_{i=1}^{19} y(\pi i/120) + 2\sum_{i=0}^{9} y[\pi(2i + 1)/120]\} \quad .$$

This rule has been used in Section 3.1 to determine numerical integration results for the Case IV situations in which there are non-integral elasticity of demand parameter values. Its step size has been calibrated for this project using the integral parameter values found in Cases I, II, III, IVa, IVb, and IVc. One also should consult Section 5.0 to inspect relevant contrived test data set verifications.

In all cases of p = rt, the versions of Equation (3.1) are polynomials in the radius variable r. The single relevant root of each of these polynomials has been obtained with a modified form of the *regula falsi* method. This root extraction technique is an iterative procedure, and is based upon linear interpolation between two previous approximations to the root being converged upon. For some polynomial equation f(x) = 0, this method's formula may be stated as

$$x_\tau = x_{\tau-1} - (x_{\tau-1} - x_{\tau-2}) f(x_{\tau-1}) / [f(x_{\tau-1}) - f(x_{\tau-2})] \quad . \qquad (3.2)$$

This particular numerical technique allows one to restrict attention to a specific interval known to contain the root [here that interval is $(c_v/t, c_v/t + \varepsilon)$, $\varepsilon > 0$] for the radius, and allows the root to be calculated to within some prespecified degree of accuracy. The straight line joining the endpoints of the interval in question is called a secant; the slope of this secant is $[f(x_{\tau-1}) - f(x_{\tau-2})]/(x_{\tau-1} - x_{\tau-2})$. Since the equation f(x) must equal zero at its root, the relative change in f(x) at this root becomes $f(x_{\tau-1})$. The ratio of this change in f(x) to this secant slope is the right-hand term of Equation (3.2). The search using this technique begins with $r = c_v/t$, and continues as long as the left-hand side of Equation (3.1) continues to converge upon zero. If the iterative value of r_τ causes Equation (3.1) to begin to move away from zero, then the previous value $r_{\tau-1}$ is returned to, and the increment is modified by dividing it by 10. The algorithm is terminated after several sequential modifications occur, resulting in a predetermined level of accuracy.

3.3. Price calculations and demand calculations for nested structures

Recalling that the mathematical problem here involves one equation, which specifies that price and demand values must lead to zero excess profits, in two unknowns, namely the hexagon radius and price, obtaining solutions requires that one of these two unknowns be fixed. In Section 3.1, price was fixed by setting it equal to rt. Once the real radii for all commodities are calculated, if they exist, using this constraint, then the nesting radius can be fixed and commodity prices calculated. Moreover, $p \neq rt$, except for that commodity positioned at the lowest level of the hierarchy and having the smallest radius. All other radii are written as functions of this single marginal commodity, and then prices are determined. Either the radius is equated with this minimum radius, if a commodity is positioned at the same hierarchical level, or some appropriate multiplicative factor times this minimum radius, if a commodity is positioned at a higher hierarchical level (see Sections 2.3 and 5.0).

Once more various cases must be considered. First, Case V is for η = 0, and may be summarized as follows:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta \, (p + t\delta)^0 \, d\delta \, d\theta$$

$$= 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta \, d\delta \, d\theta = 12A \int_0^{\pi/6} (1/2) \, (r/\cos\theta)^2 \, d\theta$$

$$= 6A \int_0^{\pi/6} r^2/\cos^2\theta \, d\theta \;=\; 6Ar^2 \int_0^{\pi/6} 1/\cos^2\theta \, d\theta$$

$$= 6Ar^2 \tan\theta \Big|_0^{\pi/6} \;=\; 6Ar^2/3^{1/2} \;=\; 2*3^{1/2}Ar^2 \quad .$$

Equation (3.1) becomes

$$2*3^{1/2}Ar^2(p - c_v) \;-\; c_f \;=\; 0 \quad .$$

The solution to this linear equation is

$$p \;=\; c_v \;+\; c_f/(2*3^{1/2}Ar^2) \quad .$$

Case VI is where $\eta \;=\; 1$, and may be summarized as follows:

$$D \;=\; 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta \; (p + t\delta)^{-1} \, d\delta \, d\theta$$

$$= 12A \int_0^{\pi/6} [\delta/t \;-\; (p/t^2) \; \ln(p + \delta t)] \Big|_0^{r/\cos\theta} d\theta$$

$$= (12A/t) \int_0^{\pi/6} [r/\cos\theta \;-\; (p/t) \; \ln(p + rt/\cos\theta) \;+\; (p/t) \; \ln(p)] \; d\theta$$

$$= (12A/t)\{r \int_0^{\pi/6} 1/\cos\theta \, d\theta \;+\; (p/t) \int_0^{\pi/6} \ln[1 + rt/(p*\cos\theta)] \; d\theta\}$$

$$= (12A/t)\{r*\ln[\tan(\pi/3)] \;+\; (p/t) \int_0^{\pi/6} \ln(p*\cos\theta + rt) \; d\theta\} \quad .$$

Applying the extended Simpson's Rule outlined in Section 3.2. to the integral appearing in the second term of this last result yields

$$\int_0^{\pi/6} \ln[1 + rt/(p*\cos\theta)] \; d\theta \;\approx\; [(\pi/120)/3]\{\ln(1 + rt/p) \;+\; \ln[1 + 2rt/(3^{1/2}p)] \;+\; 2\sum_{i=1}^{19} \ln\{1 + rt/[p*\cos(\pi i/120)]\} \;+\; 2\sum_{i=0}^{9} \ln[1 + rt/\{p*\cos[\pi(2i + 1)/120]\}]\} \quad .$$

Because p is embedded in this numerical integration, demand must be calculated for each value of the price root that is obtained for Equation (3.1). In other words, at each demand approximation iteration numerical root extraction is undertaken, with convergence occurring when demand converges.

The seventh case is for $\eta = 2$, and may be summarized as follows:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\,(p + t\delta)^{-2}\,d\delta\,d\theta$$

$$= 12A \int_0^{\pi/6} [p/[t^2(\delta t + p) + (1/t^2)\ \ln(p + t\delta)]\ \Big|_0^{r/\cos\theta}\ d\theta$$

$$= (12A/t^2) \int_0^{\pi/6} [p/(rt/\cos\theta + p) + \ln(rt/\cos\theta + p) - 1 - \ln(p)]\ d\theta$$

$$= (12A/t^2)\,[\int_0^{\pi/6} \{1/[1 + (rt/p)\sec\theta] + \ln(rt + p{*}\cos\theta) - \ln(\cos\theta)\}\ d\theta - (\pi/6)[1 + \ln(p)]]$$

$$= (12A/t^2)\{\pi/6 - rt \int_0^{\pi/6} 1/(rt + p{*}\cos\theta)\ d\theta + \int_0^{\pi/6} \ln(rt + p{*}\cos\theta)\ d\theta + (-0.02461715) - (\pi/6)[1 + \ln(p)]\}$$

$$= (12A/t^2)\,[- rt \int_0^{\pi/6} 1/(rt + p{*}\cos\theta)\ d\theta + \int_0^{\pi/6} \ln(rt + p{*}\cos\theta)\ d\theta + (-0.02461715) - (\pi/6)\ln(p)]\ .$$

This last form of the solution has three particular instances that need to be considered, namely

if $rt/p = 1$, then $\int_0^{\pi/6} 1/(rt + p{*}\cos\theta)\ d\theta = \tan(\pi/12)$,

if $rt/p > 1$, then $\int_0^{\pi/6} 1/(rt + p{*}\cos\theta)\ d\theta =$

$$\{2/[(rt)^2 - p^2]^{1/2}\}\tan^{-1}\{[(rt - p)/(rt + p)]^{1/2}\tan(\pi/12)\}\ ,$$

and

if $rt/p < 1$, then $\int_0^{\pi/6} 1/(rt + p{*}\cos\theta)\ d\theta =$

$$\{1/[p^2 - (rt)^2]^{1/2}\}\ln[\{[(rt + p)/(rt - p)]^{1/2} + \tan(\pi/12)\}/\{[(rt + p)/(rt - p)]^{1/2} - \tan(\pi/12)\}]\ .$$

Again iterative root extraction is embedded in iterative demand approximation, in order to obtain a solution to Equation (3.1), except for the case of $p = rt$, which is not relevant here (see Section 3.1).

The eighth, and final, case is for $\eta \neq 0, 1, 2$, and may be summarized as follows:

$$D = 12A \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta\ (p + t\delta)^{-\eta}\ d\delta\ d\theta$$

$$= 12A \int_0^{\pi/6} \{ (\delta t + p)^{-\eta+2}/[(-\eta+2)t^2] - p(\delta t + p)^{-\eta+1}/[(-\eta+1)t^2] \} \Big|_0^{r/\cos\theta} d\theta$$

$$= (12A/t^2) \int_0^{\pi/6} \{ [1/(2 - \eta)][1/(rt/\cos\theta + p)^{\eta-2}] - [p/(1 - \eta)][1/(rt/\cos\theta + p)^{\eta-1}] - [1/(2 - \eta)](1/p^{\eta-2}) + [p/(1 - \eta)](1/p^{\eta-1}) \}\ d\theta$$

$$= (12A/t^2)\{ [1/(2 - \eta)] \int_0^{\pi/6} 1/(rt/\cos\theta + p)^{\eta-2}\ d\theta + [p/(1 - \eta)] \int_0^{\pi/6} 1/(rt/\cos\theta + p)^{\eta-1}\ d\theta + (\pi/6)(-1 + \eta + 2 - \eta)/[(2 - \eta)(1 - \eta)p^{\eta-2}] \}$$

$$= (p^{2-\eta})(12A/t^2)\{ [1/(2 - \eta)] \int_0^{\pi/6} 1/[1 + rt/(p*\cos\theta)]^{\eta-2}\ d\theta + [p/(\eta - 1)] \int_0^{\pi/6} 1/[1 + rt/(p*\cos\theta)]^{\eta-1}\ d\theta + \pi/[6(\eta - 2)(\eta - 1)] \} \quad .$$

Applying the extended Simpson's Rule to the integrals appearing in the second and third terms of this last result yields

$$\int_0^{\pi/6} 1/[(1 + rt/(p*\cos\theta)]^{\eta-2}\ d\theta \approx [(\pi/120)/3] \Big[(1 + rt/p)^{2-\eta} + [1 + 2rt/(3^{1/2}p)]^{2-\eta} + 2 \sum_{i=1}^{19} \{1 + rt/[p*\cos(\pi i/120)]\}^{2-\eta} + 2 \sum_{i=0}^{9} \{1 + rt/\{p*\cos[\pi(2i + 1)/120]\}^{2-\eta} \Big]$$

and

$$\int_0^{\pi/6} 1/[(1 + rt/(p*\cos\theta)]^{\eta-1}\ d\theta \approx [(\pi/120)/3] \Big[(1 + rt/p)^{1-\eta} + [1 + 2rt/(3^{1/2}p)]^{1-\eta} + 2 \sum_{i=1}^{19} \{1 + rt/[p*\cos(\pi i/120)]\}^{1-\eta} +$$

$$2 \sum_{i=0}^{9} \{1 + rt/\{p*\cos[\pi(2i + 1)/120]\}^{1-\eta}] \quad .$$

Again iterative root extraction is embedded in iterative demand approximation, in order to obtain a solution to Equation (3.1).

Insights into properties of these price roots can be achieved through mathematical analysis of Equation (3.1). Letting (where "integral" refers to the appropriate integration result for D from above)

$$Z = r^{2-\eta}(12A/t^{\eta})*\text{integral}*(rt - c_v) - c_f$$

$$\partial Z/\partial r \Rightarrow r_{optimal} = [(2 - \eta)/(3 - \eta)]c_v/t$$

$$\partial^2 Z/\partial r^2 = (2 - \eta)\{[(3 - \eta)/(2 - \eta)](t/c_v)\}^{\eta} c_v (12A/t^{\eta})*\text{integral}$$

if $0 < \eta < 2$, $\eta \neq 1, 2$, then

$\partial^2 Z/\partial r^2 > 0 \Rightarrow r_{optimal}$ is a minimum

$$[(2 - \eta)/(3 - \eta)]^{2-\eta}(12A/t^2)*\text{integral}*c_v^{3-\eta}/(\eta - 3) - c_f > 0 \Rightarrow$$

no real roots exist.

if $\eta > 3$, then

$\partial^2 Z/\partial r^2 < 0 \Rightarrow r_{optimal}$ is a maximum

$$[(2 - \eta)/(3 - \eta)]^{2-\eta}(12A/t^2)*\text{integral}*c_v^{3-\eta}/(\eta - 3) - c_f < 0 \Rightarrow$$

no real roots exist.

Next, letting

$$Z = (12A/t^{\eta-1})*\text{integral}*r - c_f r^{\eta-2} - c_v(12A/t^{\eta})*\text{integral}$$

$$\partial Z/\partial r^{\eta-2} \Rightarrow r_{optimal} = \{12A*\text{integral}/[(\eta - 2)c_f t^{\eta-1}]\}^{1/(\eta-3)}$$

$$\partial^2 Z/\partial (r^{\eta-2})^2 =$$

$$[(3 - \eta)/(\eta - 2)^2][(\eta - 2)c_f t^{\eta-1})/(12A*\text{integral})]^{(5-2\eta)/(3-\eta)}12A*\text{integral}/t^{\eta-1}$$

$\partial^2 Z/\partial (r^{\eta-2})^2 > 0 \Rightarrow r_{optimal}$ is a minimum

$$[(\eta - 2)c_f t^{\eta-1}/(12A*\text{integral})]^{(\eta-2)/(3-\eta)}(\eta - 3)c_f - c_v 12A*\text{integral}/t^{\eta} < 0$$

$\Rightarrow$ at least one real root always exists.

As an aside, if $c_f = 0$, then $p = c_v$ for all values of r. If $c_v = 0$, then the numerical integration and root extraction methods still must be resorted to.

4.0. A MULTINOMIAL LOGIT CLASSIFICATION MODEL FOR DETERMINING CENTRAL PLACE HIERARCHICAL LEVELS FOR COMMODITIES

One of the problems faced when dealing with the simulation of central place structures is how to relate a particular good to a specific hierarchical level. In this paper each good is characterized by three prominent attributes, namely its variable cost component (c_v), its fixed cost component (c_f), and its elasticity of demand parameter (η). These three variables have been used in the formulation of a classification function for determining which of four hierarchical levels any given good should be allocated to; the central place system employed here is restricted to four hierarchy levels. A multinomial logit model formulation was selected because it yields a set of four probabilities for each good; the level having the highest probability is the one a good is allocated to here. Two data sets have been analyzed in order to establish estimates for the parameters of this model [the interested reader should consult Anas (1982) and/or Domencich and McFadden (1975) to review methods of estimating parameters of a multinomial logit model]. The first data set comes from the hypothetical data presented in Griffith (1986), and the second is compiled from empirical observations that are in keeping with the results summarized in Yeates and Garner (1980). The function embedded in the algorithm is based upon a pooling of these two data sets. In part this pooling is permissible because all data are transformed to their z-score counterparts.

4.1. The hypothetical data used by Griffith

Griffith (1986) employed artificial data that he felt reflected the relative attributes of goods that are grouped into bundles in accordance with central place hierarchical levels. In this case the elasticity parameters selected for each level were constant; this lack of variation eliminated any within-groups variation. To alleviate the statistical analysis problems arising from this lack of variation, the elasticity of demand parameters had a stochastic error term added to them for individual commodities; this error term was drawn from a population that is normally distributed, has a mean of 0.0, and has a variance of 0.0001. Not surprisingly, then, the resulting data are indistinguishable from the original data. These modified data are as follows:

level	c_v	c_f	η	level	c_v	c_f	η
4	7	400	4.25001	3	20	1600	4.00001
4	2	900	4.24998	3	12	1800	4.00001
4	7	500	4.25000	3	2	400	4.00001
4	10	100	4.25001	3	20	1400	3.99999
4	2	500	4.25000	3	16	1200	4.00000
4	10	300	4.25000	3	8	1000	3.99999
4	9	1000	4.25000	2	45	1500	3.50002
4	8	500	4.24999	2	9	1500	3.50001
4	7	900	4.25001	2	81	2100	3.50000
4	8	300	4.24999	1	312	3600	2.75002
4	9	900	4.25000	1	208	1600	2.75000
4	8	100	4.24999				

The sample statistics for these variables, regardless of hierarchical level, are as follows:

variable	mean	standard deviation
c_v	35.6522	74.4731
c_f	1047.8261	799.9260
η	3.9565	0.4563

A priori probabilities were calculated from the relative frequency of goods associated with each hierarchical level. Hence these probabilities are

hierarchical level	1	2	3	4
a prior probability	0.0870	0.1304	0.2609	0.5217

Because inferences are not going to be drawn from these data, a test of homogeneity of within-groups covariance matrices was not conducted. The resulting multinomial logit model in this instance yielded posterior probabilities in which the probability of being classified into the correct hierarchical class always was 1.0. Thus, a perfect classification result was obtained here.

4.2. Empirical data for Snohomish County, Washington

Empirical data were collected in accordance with central place hierarchy results summarized in Yeates and Garner (1980). The 1977 Census of Service Industries, the 1977 Census of Retail Trade, and Tables 7.1 (p. 154) and 7.4 (p. 160) of Yeates and Garner rendered the following data:

level	commodity	threshold	sales (1000s)	# establishments
4	gas service station	196	53748847	146523
	auto supplies	488	12389968	36807
	drinking places	282	6901388	70886
	eating places	276	54405754	237728
	grocery	254	143937637	126635
	hardware	431	5695140	19351

level	commodity	threshold	sales (1000s)	# establishments
4	bulk fuel	419	9761902	14655
	barber	386	432915	11539
	beauty	480	3392412	74549
3	furniture	546	19344276	53413
	appliances	385	4519502	12242
	radio,tv,music*	385	7704611	26087
	variety store	549	6948314	14152
	drug store	458	23057288	47169
	building materials	598	26804291	33985
	funeral home	1214	2651966	15106
	physicians	380	26948835	136164
	dentists	426	9528655	82739
2	auto/new	398	121883263	30793
	auto/used*	398	5186357	13287
	women's clothing*	590	13163828	44374
	men's clothing*	590	6838807	20643
	shoes*	590	5517220	24450
	jewelry	827	5010477	19670
	florist	729	2177051	20092
	bakery*	610	2140723	15949
	liquor store*	610	12190536	35144
	motel	430	75778837	26711
	cleaners/laundry	754	6326263	46549
	self-service laundry	1307	844916	12446
1	hotel	846	9837126	10443
	shoe repair	896	150081	3112
	department store	1083	76909452	8807

* denotes an estimate from Table 7.1, Yeates and Garner (1980)

The County and City Data Book reported that in 1960 the population density of Snohomish County was 172199/2098 persons per square mile. The national Consumer Price Index value at the end of 1977 was 186.1, and in September of 1987 was 344.4; the difference between these two figures was used here to inflate the figures appearing above to estimated current dollar amounts. The 1979 FTC Quarterly Financial Report for Manufacturing, Mining and Trade Corporations reported that the aggregate proportion of depreciation equaled 1976/173722, while the aggregate proportion of operating costs equaled 163943/173722, for the economy as a whole. These figures have been used to approximate fixed cost and variable cost terms. The 1978 Survey of Current Business, reported that profit in the retail sector was 24%. This figure was used to modify the variable cost estimate. All figures were prorated to daily values per threshold unit.

First the fixed cost term for each commodity was calculated as follows:

$$c_f = (1976/173722)*\text{sales}*(1.0 + 1.583)/[(\#\ \text{establishments})*365] \quad ,$$

where the first fraction equals the national proportion of depreciation, 1.583 is the increase in the CPI between 1977 and 1987, and 365 is the number of days in a calendar year. Then the variable cost term for each commodity was calculated as follows:

$$c_v = \text{sales}*(1.0 + 1.583)*(1.0 - 1976/173722 - 0.24)/[(\#\ \text{establishments})*\text{threshold}*365] \quad ,$$

where 0.24 refers to the rate of profit in the retail sector. Next, the elasticity of demand parameter was estimated in a sequence of steps. Since "self-service laundry" had the largest threshold value (i. e., 1307), its elasticity of demand was set equal to 0.0. This restriction allowed the demand equation to be solved for r:

$$(2*3^{1/2})*(172199/2098)*r^2 = 1307 \quad ,$$

$$r = 2.1440 \text{ miles} \quad ,$$

where 172199/2098 is the density of demand. Since "gas service station" had the lowest threshold value (i. e., 196), its elasticity of demand was set equal to 5.0. A transport rate of 0.225 was used in calibrating its demand equation, in accordance with the current viewpoint held by the IRS regarding cost of travel by car. Since r has been determined in the preceding step, then the demand equation can be solved for the price variable p:

$$12*(172199/2098) \int_0^{\pi/6} \int_0^{r/\cos\theta} \delta(p + t\delta)^{-5}\, d\delta\, d\theta = 196 \quad ,$$

$$p = 1.1554 \quad .$$

These two end-point estimates then were used to calibrate all intermediate commodity elasticity of demand values. Essentially the question addressed asked what the elasticity of demand for a given commodity would be if its price was equal to that for the commodity with the lowest threshold, while its margin of consumption was equal to that for the commodity with the largest threshold; although this perspective is a very narrow one, which ignores nesting constraints, among other considerations, it does allow relative elasticity parameter estimates to be determined. The ensuing calibrations produced the following empirical data set:

level	commodity	c_v	c_f	η
4	gas service station	9.9152	29.528	5.0000
4	auto supplies	3.6544	27.096	2.5330
4	drinking places	1.8290	7.837	4.0000
4	eating places	4.3929	18.422	4.0580
4	grocery	23.7073	91.492	4.2850
4	hardware	3.6176	23.690	2.8610
4	bulk fuel	8.4223	53.618	2.9360
4	barber	0.5149	3.020	3.1540
4	beauty	0.5023	3.663	2.5760

level	commodity	C_v	C_f	η
3	furniture	3.5141	29.153	2.2380
3	appliances	5.0801	29.717	3.1610
3	radio,tv,music	4.0641	23.773	3.1610
3	variety store	4.7379	39.521	2.2230
3	drug store	5.6543	39.347	2.7000
3	building materials	6.9873	63.486	2.0000
3	funeral home	0.7661	14.131	0.1856
3	physicians	2.7592	15.931	3.1960
3	dentists	1.4322	9.270	2.8920
2	auto/new	52.6872	318.607	3.0730
2	auto/used	5.1958	31.419	3.0730
2	women's clothing	2.6638	23.879	2.0350
2	men's clothing	2.9748	26.667	2.0350
2	shoes	2.0262	18.164	2.0350
2	jewelry	1.6318	20.504	1.1613
2	florist	0.7874	8.722	1.4860
2	bakery	1.1657	10.804	1.9480
2	liquor store	3.0126	27.921	1.9480
2	motel	34.9531	228.361	2.8670
2	cleaners/laundry	0.9549	10.940	1.3990
2	self-service laundry	0.2752	5.464	0.0000
1	hotel	5.8989	75.824	1.1031
1	shoe repair	0.2852	3.882	0.9562
1	department store	42.7188	702.935	0.4740

The sample statistics for these variables, regardless of hierarchical level, are as follows:

variable	mean	standard deviation
C_v	7.5389	12.4909
C_f	61.7209	131.7887
η	2.3865	1.1527

A priori probabilities were calculated from the relative frequency of goods associated with each hierarchical level. Hence these probabilities are

hierarchical level	1	2	3	4
a prior probability	0.0909	0.3636	0.2727	0.2727

Because inferences are not going to be drawn from these data, a test of homogeneity of within-groups covariance matrices was not done. The resulting multinomial logit model in this instance yielded the following posterior probabilities:

Actual Level	Commodity	Classified into Level	1	2	3	4
4	gas service station	4	0.0000	0.0033	0.0429	0.9538
4	auto supplies	3 *	0.0018	0.3042	0.4513	0.2427
4	drinking places	4	0.0000	0.0140	0.1384	0.8476
4	eating places	4	0.0000	0.0170	0.1384	0.8446
4	grocery	4	0.0000	0.1281	0.1697	0.7022
4	hardware	3 *	0.0008	0.1942	0.4211	0.3839
4	bulk fuel	3 *	0.0009	0.2503	0.3975	0.3513
4	barber	4	0.0003	0.0871	0.3485	0.5641
4	beauty	3 *	0.0012	0.2376	0.4667	0.2945
3	furniture	3	0.0035	0.4099	0.4389	0.1477
3	appliances	4 *	0.0004	0.1333	0.3598	0.5065
3	radio,tv,music	4 *	0.0003	0.1212	0.3577	0.5208
3	variety store	2 *	0.0042	0.4358	0.4248	0.1352
3	drug store	3	0.0014	0.2805	0.4320	0.2861
3	building materials	2 *	0.0090	0.5451	0.3679	0.0780
3	funeral home	2 *	0.0512	0.8558	0.0918	0.0012
3	physicians	4 *	0.0003	0.1000	0.3449	0.5548
3	dentists	4 *	0.0006	0.1551	0.4167	0.4276
2	auto/new	2	0.0017	0.9496	0.0387	0.0101
2	auto/used	4 *	0.0005	0.1570	0.3799	0.4627
2	women's clothing	2	0.0049	0.4704	0.4204	0.1043
2	men's clothing	2	0.0051	0.4750	0.4172	0.1028
2	shoes	2	0.0045	0.4608	0.4271	0.1076
2	jewelry	2	0.0185	0.7217	0.2447	0.0152
2	florist	2	0.0096	0.6328	0.3232	0.0344
2	bakery	2	0.0048	0.4803	0.4210	0.0940
2	liquor store	2	0.0061	0.5062	0.4017	0.0860
2	motel	2	0.0038	0.8089	0.1389	0.0483
2	cleaners/laundry	2	0.0113	0.6594	0.3015	0.0278
2	self-service laundry	2	0.0509	0.8750	0.0734	0.0007
1	hotel	2 *	0.0528	0.7236	0.2123	0.0114
1	shoe repair	2 *	0.0177	0.7655	0.2074	0.0094
1	department store	1	1.0000	0.0000	0.0000	0.0000

* denotes a misclassified commodity

A summary of the affiliated discriminant analysis classification for these data is as follows:

From Level	To: 1	2	3	4	Total
1	1	2	0	0	3
2	0	11	0	1	12
3	0	3	2	4	9
4	0	0	4	5	9
Total	1	16	6	10	33

This classification table illustrates that (a) roughly 58% of the commodities are classified correctly into their corresponding hierarchical levels, and that (b) 39% of the commodities are misclassified into an adjacent hierarchical level. Overall, then, this classification function appears to be reasonable.

4.3. Results for the pooled data set

The two preceding data sets then were merged in order to establish a multinomial logit function for commodity classification purposes. The sample statistics for these variables, regardless of hierarchical level, are as follows:

variable	mean	standard deviation
c_v	19.0854	50.0404
c_f	466.7286	711.1160
η	3.0313	1.2099

These sample statistics are used in the algorithm to convert input data to z-scores. These standardized scores allow only the relative values of commodity attributes to be considered when classifying commodities into hierarchical levels. In the case of the joint data set, these z-scores are as follows:

Data Set	Commodity	Level	c_v	c_f	η
Snohomish County	1	4	-0.18326	-0.61481	1.6272
	2	4	-0.30837	-0.61823	-0.4119
	3	4	-0.34485	-0.64531	0.8006
	4	4	-0.29361	-0.63043	0.8486
	5	4	0.09236	-0.52767	1.0362
	6	4	-0.30911	-0.62302	-0.1408
	7	4	-0.21309	-0.58093	-0.0788
	8	4	-0.37111	-0.65209	0.1014
	9	4	-0.37136	-0.65118	-0.3763
	10	3	-0.31117	-0.61534	-0.6557
	11	3	-0.27988	-0.61454	0.1072
	12	3	-0.30018	-0.62290	0.1072
	13	3	-0.28672	-0.60076	-0.6681
	14	3	-0.26841	-0.60100	-0.2738
	15	3	-0.24177	-0.56706	-0.8524
	16	3	-0.36609	-0.63646	-2.3520
	17	3	-0.32626	-0.63393	0.1361
	18	3	-0.35278	-0.64330	-0.1151
	19	2	0.67149	-0.20829	0.0345
	20	2	-0.27757	-0.61215	0.0345
	21	2	-0.32817	-0.62275	-0.8235
	22	2	-0.32195	-0.61883	-0.8235
	23	2	-0.34091	-0.63079	-0.8235
	24	2	-0.34879	-0.62750	-1.5456
	25	2	-0.36566	-0.64407	-1.2772

Data Set	Commodity	Level	c_v	c_f	η
Snohomish County	26	2	-0.35810	-0.64114	-0.8954
	27	2	-0.32120	-0.61707	-0.8954
	28	2	0.31710	-0.33520	-0.1358
	29	2	-0.36232	-0.64095	-1.3491
	30	2	-0.37590	-0.64865	-2.5054
	31	1	-0.26352	-0.54971	-1.5937
	32	1	-0.37570	-0.65087	-1.7151
	33	1	0.47229	0.33216	-2.1137
Griffith	34	4	-0.24151	-0.09384	1.0073
	35	4	-0.34143	0.60928	1.0073
	36	4	-0.24151	0.04679	1.0073
	37	4	-0.18156	-0.51571	1.0073
	38	4	-0.34143	0.04679	1.0073
	39	4	-0.18156	-0.23446	1.0073
	40	4	-0.20155	0.74991	1.0073
	41	4	-0.22153	0.04679	1.0073
	42	4	-0.24151	0.60928	1.0073
	43	4	-0.22153	-0.23446	1.0073
	44	4	-0.20155	0.60928	1.0073
	45	4	-0.22153	-0.51571	1.0073
	46	3	0.01828	1.59365	0.8006
	47	3	-0.14159	1.87490	0.8006
	48	3	-0.34143	-0.09384	0.8006
	49	3	0.01828	1.31240	0.8006
	50	3	-0.06166	1.03116	0.8006
	51	3	-0.22153	0.74991	0.8006
	52	2	0.51787	1.45303	0.3874
	53	2	-0.20155	1.45303	0.3874
	54	2	1.23729	2.29677	0.3874
	55	1	5.85357	4.40613	-0.2325
	56	1	3.77524	1.59365	-0.2325

A priori probabilities were calculated from the relative frequency of goods associated with each hierarchical level. Hence these probabilities are

hierarchical level	1	2	3	4
a prior probability	0.0893	0.2679	0.2679	0.3750

Because inferences are not going to be drawn from these data, a test of homogeneity of within-groups covariance matrices was not conducted. The pooled covariance matrix $\underline{C}$ is as follows:

Variable	$z(c_v)$	$z(c_f)$	$z(\eta)$
$z(c_v)$	0.67283893	0.56314922	0.27452309
$z(c_f)$	0.56314922	0.93394799	0.46484397
$z(\eta)$	0.27452309	0.46484397	0.58025381

The multinomial logit equation defines probabilities, using this covariance matrix, as follows:

$$P(\text{hierarchy level } j|\underline{z}_i) = \exp[-(\underline{z}_i - \bar{\underline{z}}_j)^T \underline{C}^{-1}(\underline{z}_i - \bar{\underline{z}}_j)/2 - \ln(\text{prior}_j)] / \sum_{k=1}^{k=4} \exp[-(\underline{z}_i - \bar{\underline{z}}_k)^T \underline{C}^{-1}(\underline{z}_i - \bar{\underline{z}}_k)/2 - \ln(\text{prior}_j)] \quad , \qquad (4.1)$$

where prior_j denotes the aforementioned prior probabilities of being assigned to hierarchy level j. The form of this multinomial logit equation requires the covariance matrix be inverted; in this example the inverted matrix is as follows:

Variable	$z(c_v)$	$z(c_f)$	$z(\eta)$
$z(c_v)$	3.00140	-1.83447	0.04961
$z(c_f)$	-1.83447	2.90199	-1.45690
$z(\eta)$	0.04961	-1.45690	2.86704

The exponentiation argument for each of the posterior probability values is as follows:

Level	1	2	3	4
1	4.83173156	12.06559897	18.53581071	27.42483294
2	14.26274888	2.63458165	3.86701519	7.98239605
3	20.73296062	3.86701519	2.63458165	4.23108266
4	30.29490599	8.65531919	4.90400580	1.96165851

And, the individual hierarchy level means for each variable are as follows:

Level	1	2	3	4
$z(c_v)$	1.89238	-0.05722	-0.23086	-0.24479
$z(c_f)$	1.02627	-0.10964	0.06219	-0.21046
$z(\eta)$	-1.17750	-0.65622	0.01581	0.73779

The resulting multinomial logit model in this instance yielded the following posterior probabilities:

Actual Level	Commodity	Classified into Level	1	2	3	4
4	1	4	0.0000	0.0004	0.0118	0.9878
4	2	3 *	0.0002	0.3789	0.4086	0.2123
4	3	4	0.0000	0.0101	0.0849	0.9051
4	4	4	0.0000	0.0087	0.0763	0.9149
4	5	4	0.0000	0.0057	0.0494	0.9448
4	6	4	0.0000	0.2256	0.3829	0.3915
4	7	4	0.0000	0.2113	0.3656	0.4231
4	8	4	0.0000	0.1139	0.3056	0.5806
4	9	3 *	0.0001	0.3429	0.4148	0.2422
3	10	2 *	0.0005	0.5198	0.3739	0.1058

Actual Level	Commodity	Classified into Level	1	2	3	4
3	11	4 *	0.0000	0.1217	0.3046	0.5737
3	12	4 *	0.0000	0.1195	0.3044	0.5761
3	13	2 *	0.0006	0.5329	0.3673	0.0993
3	14	3	0.0001	0.3070	0.4024	0.2905
3	15	2 *	0.0014	0.6375	0.3099	0.0512
3	16	2 *	0.1101	0.8517	0.0379	0.0002
3	17	4 *	0.0000	0.1071	0.2930	0.5999
3	18	4 *	0.0000	0.2051	0.3774	0.4174
2	19	4 *	0.0017	0.3189	0.3083	0.3711
2	20	4 *	0.0000	0.1496	0.3310	0.5193
2	21	2	0.0009	0.6035	0.3330	0.0626
2	22	2	0.0009	0.6050	0.3320	0.0621
2	23	2	0.0008	0.6005	0.3352	0.0635
2	24	2	0.0100	0.8435	0.1420	0.0045
2	25	2	0.0038	0.7756	0.2077	0.0130
2	26	2	0.0010	0.6309	0.3173	0.0509
2	27	2	0.0012	0.6393	0.3109	0.0486
2	28	2	0.0008	0.3552	0.3508	0.2933
2	29	2	0.0049	0.7967	0.1886	0.0099
2	30	2	0.1606	0.8111	0.0283	0.0001
1	31	2 *	0.0168	0.8573	0.1228	0.0031
1	32	2 *	0.0154	0.8690	0.1131	0.0024
1	33	1	0.6899	0.3014	0.0087	0.0000
4	34	4	0.0000	0.0126	0.1257	0.8618
4	35	4	0.0000	0.0354	0.3422	0.6224
4	36	4	0.0000	0.0159	0.1559	0.8282
4	37	4	0.0000	0.0060	0.0613	0.9327
4	38	4	0.0000	0.0153	0.1626	0.8221
4	39	4	0.0000	0.0100	0.0978	0.8922
4	40	4	0.0000	0.0450	0.3806	0.5744
4	41	4	0.0000	0.0161	0.1546	0.8293
4	42	4	0.0000	0.0370	0.3308	0.6321
4	43	4	0.0000	0.0099	0.0996	0.8905
4	44	4	0.0000	0.0377	0.3263	0.6360
4	45	4	0.0000	0.0059	0.0625	0.9316
3	46	3	0.0000	0.1578	0.6914	0.1508
3	47	3	0.0000	0.1579	0.7513	0.0908
3	48	4 *	0.0000	0.0258	0.1980	0.7763
3	49	3	0.0000	0.1379	0.6322	0.2299
3	50	3	0.0000	0.1099	0.5624	0.3277
3	51	3	0.0000	0.0797	0.4850	0.4354
2	52	3 *	0.0008	0.3698	0.5569	0.0724
2	53	3 *	0.0000	0.2545	0.6831	0.0623
2	54	2	0.0273	0.5505	0.4059	0.0163
1	55	1	1.0000	0.0000	0.0000	0.0000
1	56	1	0.9997	0.0003	0.0000	0.0000

* denotes a misclassified commodity

A summary of the affiliated discriminant analysis classification for these data is as follows:

From Level	To: 1	2	3	4	Total
1	3	2	0	0	5
2	0	11	2	2	15
3	0	4	6	5	15
4	0	0	2	19	21
Total	3	17	10	26	56

This classification table illustrates that (a) roughly 70% of the commodities are classified correctly into their corresponding hierarchical levels, and that (b) 27% of the commodities are misclassified into an adjacent hierarchical level. Overall, then, this classification function appears to be reasonable.

4.4. Data dimensions for hierarchical classification

The preceding results present at least one interesting puzzle that needs to be solved at this point. While the simulated data produce a perfect classification function, whereas the empirical data produce a reasonably good classification function, the sizeable differences in sample means and variances for the three variables in question here between these two data sets are marked. Nevertheless, the pooled data set continues to yield quite good classification results. An important question asks "along which discriminant function dimensions do the hierarchical levels differ most conspicuously?" One answer to this question is given by the normalized eigenvectors of the product of the inverse of the within-levels SSCP matrix multiplied by the between-levels SSCP matrix, which yield an orthogonal partitioning of total variance. The importance of each of these dimensions is indicated by their associated eigenvalues. For the three data sets analyzed here, these eigenfunctions are as follows:

Data set	Variable	DF1	DF2	DF3
Griffith	c_v	0.0000	0.8201	0.2237
	c_f	0.0000	-0.2882	0.6563
	η	-1.0000	0.4943	0.7206
	λ_j	$2.5631*10^9$	5.2823	0.0628
	%	100.00	0.00	0.00
Snohomish County	c_v	0.2475	-0.6138	0.9455
	c_f	0.1911	0.7485	-0.3036
	η	-0.9499	0.2511	0.1181
	λ_j	1.3281	0.3171	0.0290
	%	79.33	18.94	1.73
Joint	c_v	-0.3775	0.7700	-0.4118
	c_f	-0.3235	-0.5101	0.9079
	η	0.8677	0.3833	0.0785
	λ_j	2.2850	0.2539	0.0237
	%	89.17	9.91	0.92

These eigenfunction results demonstrate that only one pronounced conspicuous discriminatory dimension is latent in each of these data sets; the first dimension accounts for an overwhelming percentage of the variance (respectively, 100%, 79.33%, and 89.17%). Because the pattern of signs is arbitrary, by a multiplicative factor of -1, all three data sets exhibit the same loading pattern for this first dimension, which refers to the elasticity of demand parameter. For the simulated data set this dimension is purely the elasticity of demand parameter, since the other two variables have loading coefficients of 0.0 on it; the empirical and joint data sets are not as pure, since each of the remaining two variables has a low (but not zero) loading on it. The remaining two dimensions isolate fixed cost and variable cost impacts, which are inversely related in terms of the second discriminant dimension in this rotated variable space. But, so little variance is accounted for by the second dimension (respectively 0.0%, 18.94%, and 9.91%), while virtually no variance is accounted for by the third dimension (respectively 0.0%, 1.73%, and .92%), that these two variables will tend not to be effective discriminators between hierarchical levels. Consequently, the classification results tend to be consistent across the three data sets because in all three cases a commodity is allocated to a hierarchical level almost strictly in accordance with its elasticity of demand parameter. Inspection of the sample statistics from the simulated and empirical data sets for this parameter alone reveal that the means and variances are not dramatically different. However, they are significantly different ($F = 6.3816$ for the variance ratios, and $t = 7.07$ for the difference of means).

5.0. RESULTS FOR CONTRIVED TEST DATA SETS

Twenty-six test data sets have been employed here in an attempt to subject the algorithm to a comprehensive evaluation. All results have been verified by replicating them using the IMSL subroutine package as well as implementing the numerical approximations in FORTRAN code on a mainframe VAX. Output from each of these contrived data sets will be described here; each file is available with the digital version of the source code.

Table 1 and Figure 1 present results for the data used in Griffith (1986); again, one should note the correction to this earlier finding (Good #1 has a non-nested radius of 97.17, not the mistakenly reported radius of 183.26). The data for this example are housed in an ASCII file entitled PAPER. Whereas in Griffith (1986) the hierarchy levels for commodities were prespecified, here the hierarchical levels are determined by the discriminant function allocation mechanism embedded in the algorithm; this change has resulted in Good #15 being classified as a Level 4 rather than Level 3 commodity, and Goods #19 and #20 being classified as a Level 3 rather than Level 2 commodity. Because of the re-classification of some of the commodities, the radius value upon which the central place hierarchy is generated becomes 20.63 (for Good #15), rather than 21.09 (for Good #5). Because of constraints placed on the system by nesting, one should note (not surprisingly) that nested demand is always less than or equal to non-nested demand, when supplying a commodity is feasible; infeasibility is denoted in the table by an asterisk, and represents equation solutions that are strictly complex numbers (none of the roots of the equation in question are real numbers). This change in the spacing of central places has resulted in changes in the prices at which an equilibrium occurs (e. g., Good #2 now sells for 2.24, rather than 2.01); one should note, too, that a computer program coding error was uncovered in the price equations used in Griffith (1986). For these data only three commodities would be sold in the central place system (#2, #5, and #15), all central places would occur at the village level, and nearest neighbor villages would be spaced 41.26 units apart (twice the distance of the radius, which is measured to the perpendicular bisector of a straight line connecting nearest neighbor locations).

Table 2 and Figure 2 present results for a data set that has been constructed in a way that ensures all commodities are offered, and all hierarchical levels are present. This end was attained by lowering the transport cost and revising the elasticity of demand parameters used in Griffith's data set. The data for this example are embedded in the computer program. Now the marginal commodity becomes Good #5, which is classified into the lowest hierarchical level and has the smallest non-nested radius. In this situation most commodities are sold by setting their prices equal to their respective variable costs; the few exceptions to this tendency have prices slightly above their corresponding variable costs. Nested demand varies considerably, with very sizeable amounts being achieved in the upper levels of the hierarchy. The spacing of centers at progressively higher hierarchical levels is equal to $2*(200.09) = 400.1875$, $2*3^{1/2}*(200.09) = 693.1451$, $2*3*(200.09) = 1200.563$, and $2*3^{3/2}*(200.09) = 2079.435$. One should be able to easily trace the hexagonal market areas on the map; they are very conspicuous.

Tables 3-9 and their affiliated Figures 3-9 present results for a sequence of data sets in which a single elasticity of demand value is shared by all commodities in each set. The program allows this parameter to take on values between 0 and 5, which seems to be a reasonable range since values beyond 5 tend to eliminate virtually all possibilities of obtaining feasible solutions; closed form solutions exist to certain integration problems for integer values of the elasticity parameter, allowing the algorithm to reduce execution time for these cases. The sequence of six data sets uses the elasticity parameter values of 0, 1, 2, 3, 4.25, and 5; the data for these examples are housed in ASCII files entitled NU0, NU1, NU2, NU3, NU4.25, and NU5. When all commodities have a parameter value of zero (inelastic demand), all commodities are offered at either hierarchical Levels 1 or 2, price becomes equal to variable cost, and central place spacings are relatively small. This scenario changes very little when demand becomes more sensitive or responsive to changes in price, and hence elasticity is increased by raising the parameter value to unity. Fewer commodities are classified into hierarchical Level 1, and some prices begin to slightly exceed the affiliated variable costs. Spacing distances are incremented by an almost negligible amount. An additional increase in the elasticity of demand parameter value, to 2, further reinforces the pattern of change occurring between the values of 0 and 1; yet again fewer commodities are classified into hierarchical Level 1, more commodity prices slightly exceed their corresponding variable costs, and central place spacings are scarcely incremented. But, now a commodity becomes infeasible in the nested hierarchical central place system, even though it is feasible in a non-nested system (e. g., a Loeschian system). A further increase in the elasticity parameter value, to 3, creates considerable differentiation in the results. Commodities now are classified into all four hierarchical levels, the offering of more commodities becomes infeasible within the nested system, and several commodities begin to deviate noticeably from their respective variable costs (e. g., Good #19 has a price of 49.06 and a variable cost of 45). Central places are spaced further apart, by a considerable amount, too. Incrementing the elasticity parameter to a value of 4 begins the elimination of upper hierarchical levels; Level 2 disappears when commodities are allocated to levels. Now a goodly number of commodities no longer can be offered; in many instances, supplying them would be infeasible even in a non-nested system. The spacing of the lowest level of central places decreases dramatically, because a new marginal commodity has emerged; this is a facet of the restructuring of the commodities-to-hierarchical levels allocation in the commodities feasibility state space. These tendencies are further reinforced as the elasticity parameter value is increased to 4.25, and then to 5; the basic non-nested radius increases from 20.63 to 20.85, and then to 22.47, while more commodities are allocated to Level 4, and more commodities attain the status of being infeasible.

Table 8 and Figure 8 have a non-integer elasticity parameter, in order to test the numerical integration approximations routines in the computer code. Tables 10-11 and their accompanying Figures 10-11 expand upon this theme, for the special cases of zero fixed cost and zero variable cost. The data for these examples are housed respectively in ASCII files entitled NU04.25 and NU4.250. If the fixed cost becomes zero, then as the constraint equation

$$\text{Demand}*(p - c_v) - 0 = 0 \tag{5.1}$$

suggests, price (recall that this actually is p_{fob}) will exactly equal variable cost. Removing variation in fixed cost from the hierarchical level classification function yields a further reduction in the differentiation of commodities by level. The offering of all commodities becomes feasible in a non-nested system, whereas those commodities having a high variable cost cannot be offered in a nested system. The marginal commodity radius exactly equals c_v/t, and is noticeably smaller than when fixed costs are non-zero. In contrast, if the variable cost becomes zero, although the offering of most commodities remains feasible in a non-nested system, most commodities cannot be offered in a nested system. A less homogeneous classification of commodities to hierarchical levels is obtained as variation attributable to variable cost is removed from the hierarchical level classification function. The spacing between neighboring central places positioned at the same level of the hierarchy substantially increases. Not surprisingly, prices far exceed variable costs. And, non-nested demand is reduced by a noticeable order of magnitude.

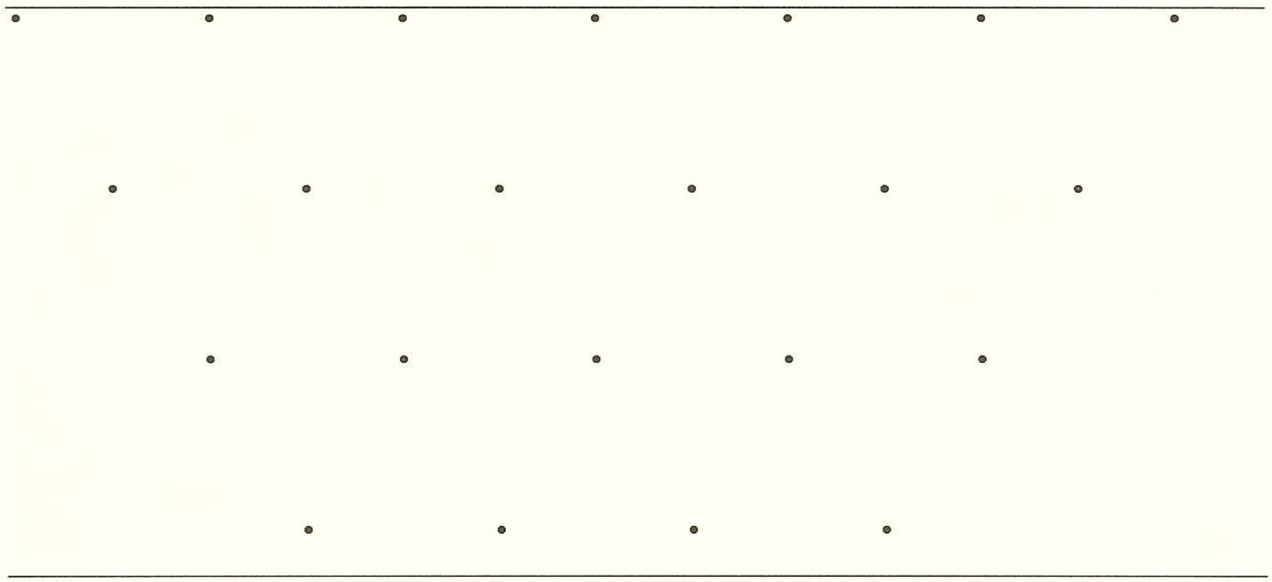
Tables 12-26 coupled with Figures 12-26 present results for a sequence of data set that has been constructed in a way that ensures only pre-selected combinations (in fact, all possible combinations, of which there are $2^4 - 1$) of hierarchical levels are present. The data for these examples are housed in ASCII files entitled 1, 2, 3, 4, 1&2, 1&3, 1&4, 2&3, 2&4, 3&4, 1&2&3, 1&2&4, 1&3&4, 2&3&4, and 1&2&3&4; they were judiciously selected from the embedded data set (but now letting t = 0.1) in order to guarantee the presence of the pre-selected levels that are indicated in their corresponding file names. The only real variations in tabular output across this set of results is attributable to the nesting of levels. In other words, Table 12 has unique nested results, Tables 13 and 16 have common Level 2 nested results, Tables 14, 17, 19 and 22 have common Level 3 nested results, and Tables 15, 18, 20, 21, 23, 24, 25 and 26 have common Level 4 results. The maps accompanying these groupings of results illustrate impacts made upon a central place structure by the addition of another hierarchical level. Within each group the spacing of central places at the lowest hierarchical level remains constant across the combinations involved. Prices and nested demand remain constant, too, since the basic radius for nesting goes unaltered. A comparison of results for upper levels of the hierarchy reveals that prices tend to remain unchanged when lower levels materialize, while nested demand displays considerable variation (e. g., Level 1, Tables 12, 16, 17 and 18).

These test data sets suggest a number of potential experiments that students can undertake. For instance, given a certain cost structure, identifying what sort of central place system would unfold (which hierarchical levels would exist, which commodities would be economically feasible, what prices would be attached to commodities, what demand would materialize, how central places would be spaced). Or, given a certain set of fixed costs and variable costs, identifying structural shifts in the central place system as demand becomes increasingly/decreasingly elastic. Or, given a set of commodities whose offerings would be infeasible, identifying the necessary shifts in costs structures that would allow the commodities to be supplied. Or, what are the marginal effects on the central place hierarchy of a given commodity. Moreover, there is a richness of themes that can be explored in an experimental manner with this computer program. Furthermore, the industrious and creative instructor could modify the algorithm's code in order to address other themes, such as the minimum magnitude of excess profits necessary to ensure the offering of a given economically infeasible commodity when excess profits are zero.

Table 1
(<u>Economic Geography</u> Paper) File PAPER (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	400	4.25	97.17	147.2	20.63	*	*
4	2	900	4.25	22.21	4076.04	20.63	3789.15	2.24
4	7	500	4.25	*	*	20.63	*	*
4	10	100	4.25	108.75	114.26	20.63	*	*
4	2	500	4.25	21.09	4577.27	20.63	4497.03	2.11
4	10	300	4.25	*	*	20.63	*	*
4	9	1000	4.25	*	*	20.63	*	*
4	8	500	4.25	*	*	20.63	*	*
4	7	900	4.25	*	*	20.63	*	*
4	8	300	4.25	103.47	127.79	20.63	*	*
4	9	900	4.25	*	*	20.63	*	*
4	8	100	4.25	85.03	198.76	20.63	*	*
3	20	1600	4	*	*	35.73	*	*
3	12	1800	4	*	*	35.73	*	*
4	2	400	4	20.63	6362.59	20.63	6369.31	2.06
3	20	1400	4	*	*	35.73	*	*
3	16	1200	4	*	*	35.73	*	*
3	8	1000	4	*	*	35.73	*	*
3	45	1500	3.5	733.05	52.99	35.73	*	*
3	9	1500	3.5	105.44	971.42	35.73	*	*
2	81	2100	3.5	*	*	61.89	*	*
1	312	3600	2.75	3179.03	609.9	107.19	*	*
1	208	1600	2.75	2099.22	832.59	107.19	*	*

Figure 1
Central Place Structure Map

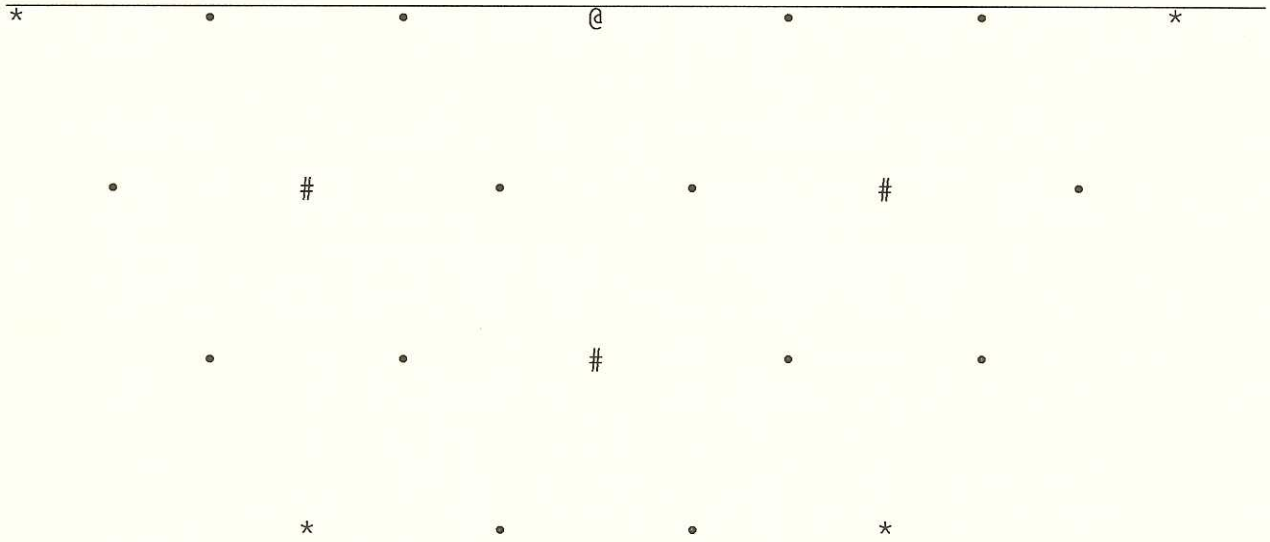


The distance between hamlets is: 41.25733 • - hamlet

Table 2
Embedded Example (t = .01, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	400	4.25	701.3	30661.71	200.09	8310.151	7.05
4	2	900	4.25	200.17	514891.3	200.09	514751.7	2
4	7	500	4.25	701.63	30629.49	200.09	8257.42	7.06
4	10	100	4	1000.38	27055.19	200.09	4147.65	10.02
4	2	500	4	200.09	676253	200.09	677092.4	2
4	10	300	4	1001.13	27014.67	200.09	4075.87	10.07
3	9	1000	3.5	900.81	123018.8	346.57	42948.67	9.02
4	8	500	3.5	800.34	146896.4	200.09	27647.22	8.02
3	7	900	3.5	700.5	179394.7	346.57	86607.32	7.01
3	8	300	2.75	800.03	965263.1	346.57	342240.2	8
3	9	900	2.75	900.1	883602.6	346.57	264114.6	9
3	8	100	2.75	800.01	965281.7	346.57	342293.6	8
2	20	1600	2	2000.02	6453216	600.28	1077366	20
2	12	1800	2	1200.03	6453216	600.28	2441402	12
2	2	400	2	200.01	6453216	600.28	2.084653E+07	2
2	20	1400	2	2000.02	6453216	600.28	1077396	20
2	16	1200	2	1600.02	6453216	600.28	1554574	16
2	8	1000	2	800.02	6453216	600.28	4404456	8
1	45	1500	0	4500	3.507403E+10	1039.72	1.872368E+09	45
1	9	1500	0	900	1.402962E+09	1039.72	1.872368E+09	9
1	81	2100	0	8100	1.136398E+11	1039.72	1.872368E+09	81
1	312	3600	0	31199.99	1.686046E+12	1039.72	1.872368E+09	312
1	208	1600	0	2080	7.493546E+11	1039.72	1.872368E+09	208

Figure 2
Central Place Structure Map



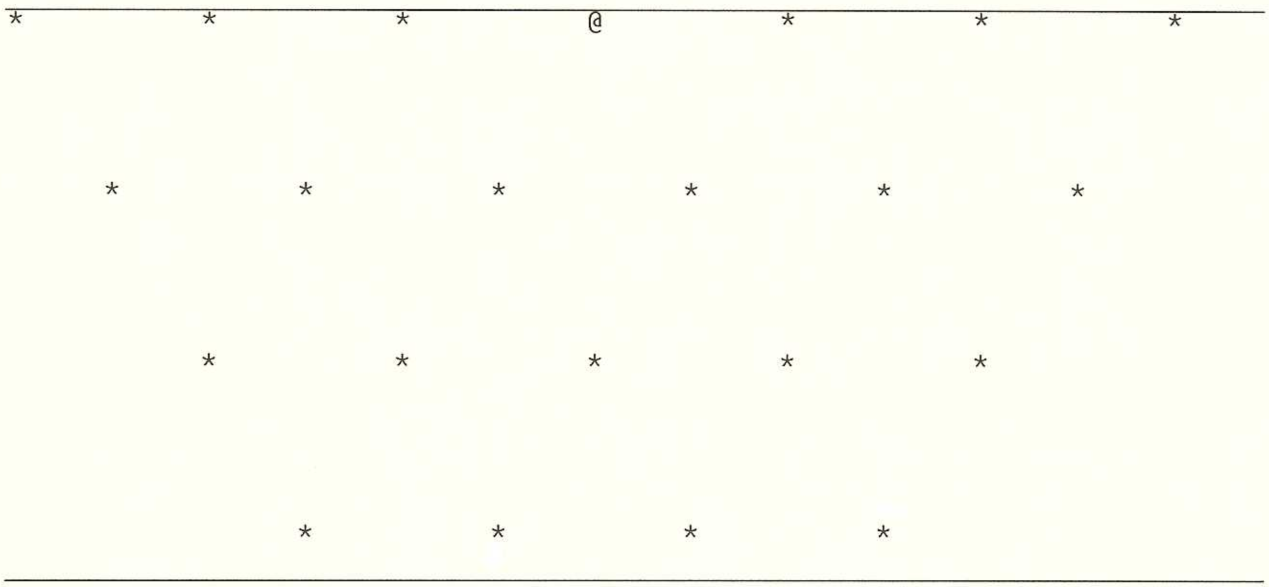
The distance between cities: 2079.435
The distance between towns: 1200.563
The distance between villages: 693.1451
The distance between hamlets: 400.1875

@ - city
* - town
# - village
• - hamlet

Table 3
File NU0 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	7	400	0	70	8487164	20.01	693220.3	7
2	2	900	0	20.01	693719.3	20.01	693220.3	2
2	7	500	0	70	8487189	20.01	693220.3	7
2	10	100	0	100	1.732052E+07	20.01	693220.3	10
2	2	500	0	20.01	693320	20.01	693220.3	2
2	10	300	0	100	1.732056E+07	20.01	693220.3	10
2	9	1000	0	90	1.402983E+07	20.01	693220.3	9
2	8	500	0	80	1.108525E+07	20.01	693220.3	8
2	7	900	0	70	8487308	20.01	693220.3	7
2	8	300	0	80	1.10852E+07	20.01	693220.3	8
2	9	900	0	90	1.402981E+07	20.01	693220.3	9
2	8	100	0	80	1.108515E+07	20.01	693220.3	8
1	20	1600	0	200	6.928218E+07	34.65	2079661	20
1	12	1800	0	120	2.494183E+07	34.65	2079661	12
2	2	400	0	20.01	693220.2	20.01	693220.3	2
1	20	1400	0	200	6.928216E+07	34.65	2079661	20
1	16	1200	0	160	4.434064E+07	34.65	2079661	16
2	8	1000	0	80	1.108537E+07	20.01	693220.3	8
1	45	1500	0	450	3.507404E+08	34.65	2079661	45
1	9	1500	0	90	1.402994E+07	34.65	2079661	9
1	81	2100	0	810	1.136399E+09	34.65	2079661	81
1	312	3600	0	3120	1.686048E+10	34.65	2079661	312
1	208	1600	0	2080	7.493543E+09	34.65	2079661	208

Figure 3
Central Place Structure Map



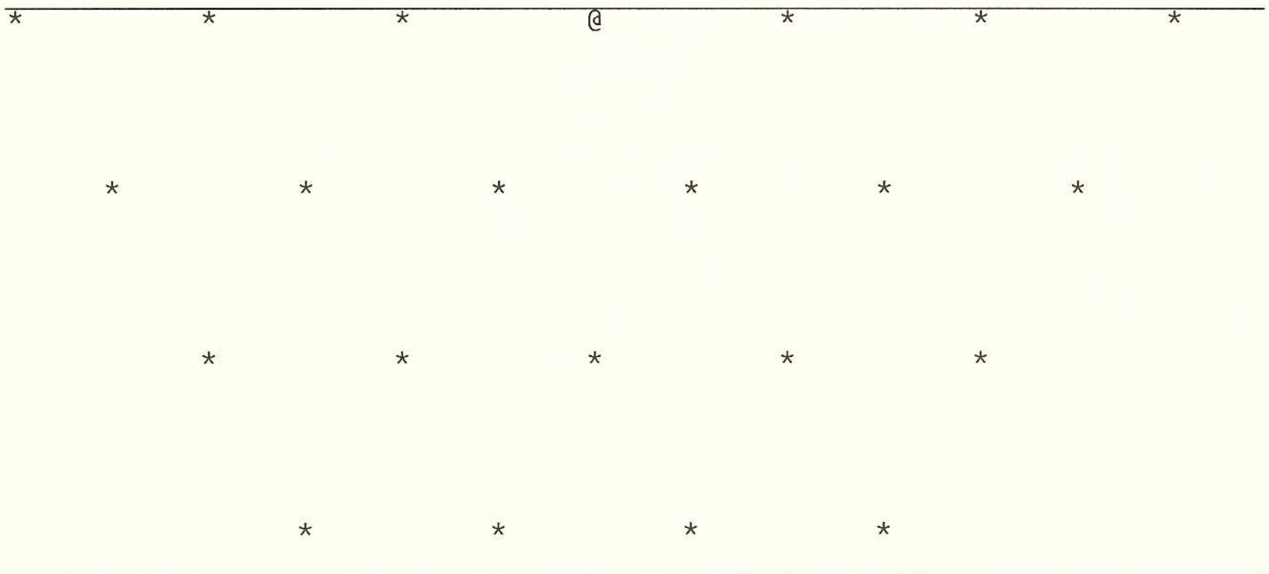
The distance between cities: 69.30203
The distance between towns: 40.01154

@ - city
* - town

Table 4
File NU1 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	7	400	1	70.01	730031.4	20.02	82841.82	7
2	2	900	1	20.04	209013.1	20.02	208609.9	2
2	7	500	1	70.01	730045.8	20.02	82829.96	7.01
2	10	100	1	100	1042830	20.02	60973.78	10
2	2	500	1	20.02	208813.8	20.02	208733.4	2
2	10	300	1	100	1042850	20.02	60956.49	10
2	9	1000	1	90.01	938649.4	20.02	66774.47	9.01
2	8	500	1	80.01	834318.8	20.02	73966.78	8.01
2	7	900	1	70.01	730102.8	20.02	82781.56	7.01
2	8	300	1	80	834293.8	20.02	73987.96	8
2	9	900	1	90.01	938638.4	20.02	66783.28	9.01
2	8	100	1	80	834268.9	20.02	74009.65	8
2	20	1600	1	200.01	2085721	20.02	32370.22	20.05
2	12	1800	1	120.01	1251534	20.02	51721.6	12.03
2	2	400	1	20.02	208763.9	20.02	208764.4	2
2	20	1400	1	200.01	2085711	20.02	32380.24	20.04
2	16	1200	1	160.01	1668588	20.02	39847.63	16.03
2	8	1000	1	80.01	834381.3	20.02	73913.48	8.01
1	45	1500	1	450	4692725	34.67	43892.85	45.03
2	9	1500	1	90.01999	938704.9	20.02	66726.46	9.02
1	81	2100	1	810	8446871	34.67	24937.42	81.08
1	312	3600	1	3120	3.253601E+07	34.67	6635.16	312.54
1	208	1600	1	2080	2.169067E+07	34.67	9893.76	208.16

Figure 4
Central Place Structure Map



The distance between cities: 69.34841
The distance between towns: 40.03832

@ - city
* - town

Table 5
File NU2 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	7	400	2	70.06	64532.16	20.06	9875.191	7.04
2	2	900	2	20.14	64532.16	20.06	64213.67	2.01
2	7	500	2	70.08	64532.16	20.06	9850.981	7.05
2	10	100	2	100.02	64532.16	20.06	5370.97	10.02
2	2	500	2	20.08	64532.16	20.06	64475.18	2.01
2	10	300	2	100.05	64532.16	20.06	5335.48	10.06
2	9	1000	2	90.15	64532.16	20.06	6286.66	9.16
2	8	500	2	80.08	64532.16	20.06	7838.5	8.06
2	7	900	2	70.14	64532.16	20.06	9753.99	7.09
2	8	300	2	80.05	64532.16	20.06	7881.62	8.04
2	9	900	2	90.14	64532.16	20.06	6306.29	9.14
2	8	100	2	80.02	64532.16	20.06	7924.74	8.01
2	20	1600	2	200.25	64532.16	20.06	1369.81	21.17
2	12	1800	2	120.28	64532.16	20.06	3619.67	12.5
2	2	400	2	20.06	64532.16	20.06	64284.75	2.01
2	20	1400	2	200.22	64532.16	20.06	1389.48	21.01
2	16	1200	2	160.19	64532.16	20.06	2165.77	16.55
2	8	1000	2	80.15	64532.16	20.06	7730.28	8.13
2	45	1500	2	450.23	64532.16	20.06	254.9	50.88
2	9	1500	2	90.23001	64532.16	20.06	6187.98	9.24
1	81	2100	2	810.33	64532.16	34.75	247.49	89.49
1	312	3600	2	3120.56	64532.16	34.75	*	*
1	208	1600	2	2080.25	64532.16	34.75	30.08	261.2

Figure 5
Central Place Structure Map

```
*         *         *         @         *         *         *


     *         *         *         *         *         *


          *         *         *         *         *


               *         *         *         *
```

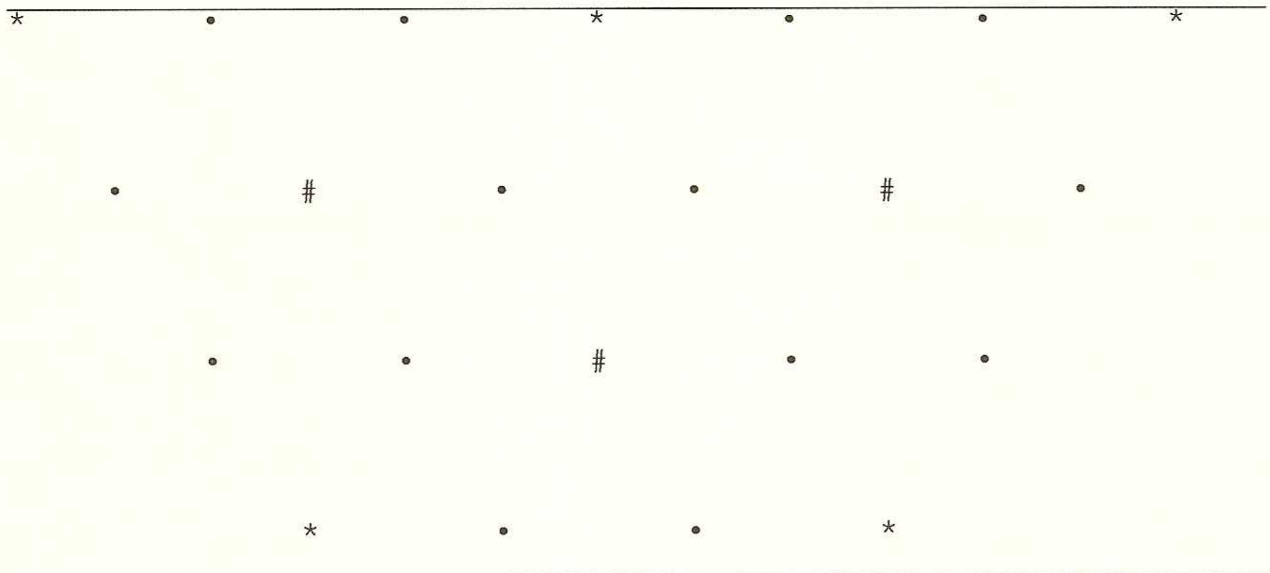
The distance between cities: 69.49675
The distance between towns: 40.12397

@ - city
* - town

Table 6
File NU3 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
3	7	400	3	70.69	5822.04	138.9	10138.83	7.04
3	2	900	3	20.45	20127.14	138.9	60145.18	2.01
3	7	500	3	70.86	5807.75	138.9	10115.13	7.05
4	10	100	3	100.24	4105.43	80.19	3257.19	10.03
3	2	500	3	20.25	20327.14	138.9	60394.11	2.01
3	10	300	3	100.73	4085.43	138.9	5466.53	10.05
3	9	1000	3	92.24	4461.59	138.9	6468.3	9.15
3	8	500	3	80.98001	5081.78	138.9	8071.14	8.06
3	7	900	3	71.57	5750.61	138.9	10020.04	7.09
3	8	300	3	80.59	5106.78	138.9	8114.12	8.04
3	9	900	3	92.01	4472.7	138.9	6488.23	9.14
4	8	100	3	80.19	5131.78	80.19	5131.79	8.02
3	20	1600	3	208.09	1977.71	138.9	1214.47	21.32
3	12	1800	3	125.49	3279.52	138.9	3642.25	12.49
3	2	400	3	20.2	20377.14	138.9	60456.28	2.01
3	20	1400	3	207.04	1987.71	138.9	1238.03	21.13
3	16	1200	3	164.81	2497.14	138.9	2072.45	16.58
3	8	1000	3	81.99	5019.28	138.9	7963.1	8.13
2	45	1500	3	467.02	881.21	240.58	369.47	49.06
3	9	1500	3	93.4	4406.03	138.9	6368.09	9.24
2	81	2100	3	853.55	482.15	240.58	*	*
1	312	3600	3	3419.09	120.37	416.7	*	*
1	208	1600	3	2164.14	190.16	416.7	*	*

Figure 6
Central Place Structure Map



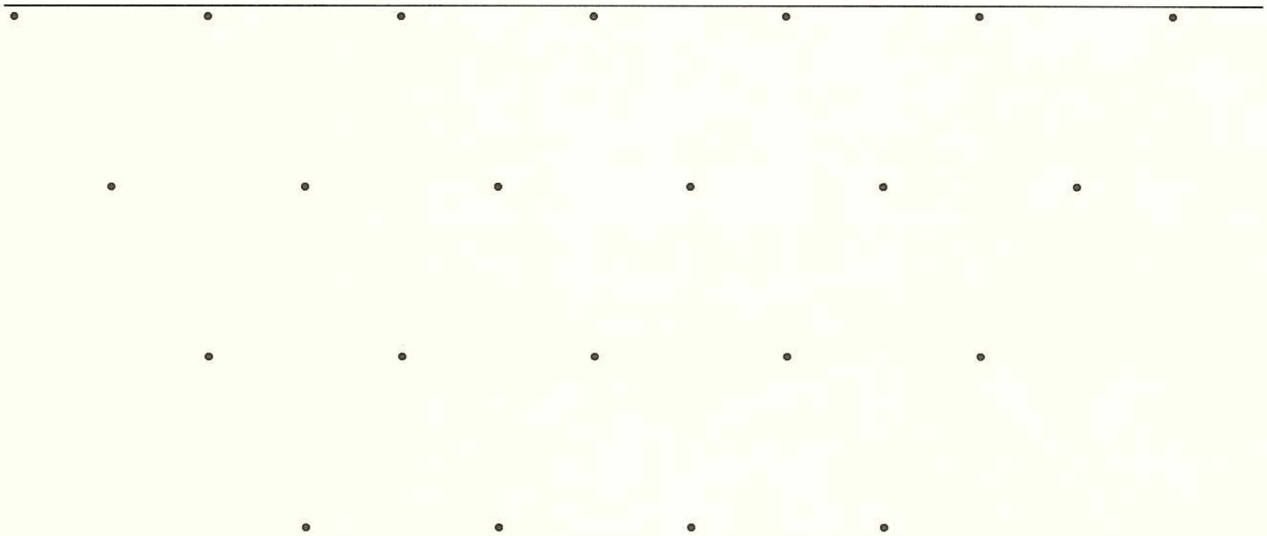
The distance between towns: 481.1692
The distance between villages: 277.8032
The distance between hamlets: 160.3897

* - town
\# - village
• - hamlet

Table 7
File NU4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	400	4	79.29	430.69	20.63	*	*
4	2	900	4	21.54	5834.16	20.63	5613.01	2.16
4	7	500	4	82.6	396.85	20.63	*	*
4	10	100	4	103.99	250.36	20.63	*	*
4	2	500	4	20.8	6258.88	20.63	6224.08	2.08
4	10	300	4	114.54	206.39	20.63	*	*
3	9	1000	4	*	*	35.73	*	*
4	8	500	4	97.58999	284.32	20.63	*	*
4	7	900	4	110.83	220.42	20.63	*	*
4	8	300	4	88.72	343.97	20.63	*	*
4	9	900	4	*	*	20.63	*	*
4	8	100	4	82.51	397.66	20.63	*	*
3	20	1600	4	*	*	35.73	*	*
3	12	1800	4	*	*	35.73	*	*
4	2	400	4	20.63	6362.59	20.63	6369.31	2.06
3	20	1400	4	*	*	35.73	*	*
3	16	1200	4	*	*	35.73	*	*
3	8	1000	4	*	*	35.73	*	*
3	45	1500	4	*	*	35.73	*	*
3	9	1500	4	*	*	35.73	*	*
3	81	2100	4	*	*	35.73	*	*
1	312	3600	4	*	*	107.19	*	*
1	208	1600	4	*	*	107.19	*	*

Figure 7
Central Place Structure Map

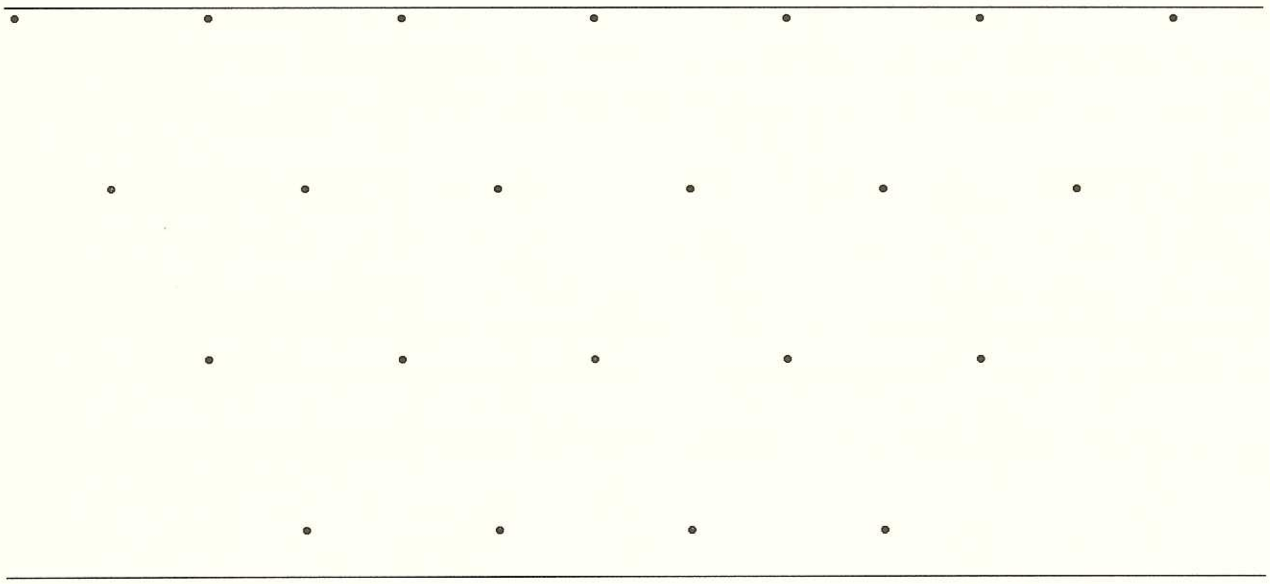


The distance between hamlets is: 41.25733 • - hamlet

Table 8
File NU4.25 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	400	4.25	97.17	147.2	20.85	*	*
4	2	900	4.25	22.21	4076.04	20.85	3831.46	2.23
4	7	500	4.25	*	*	20.85	*	*
4	10	100	4.25	108.75	114.26	20.85	*	*
4	2	500	4.25	21.09	4577.27	20.85	4535.88	2.11
4	10	300	4.25	*	*	20.85	*	*
4	9	1000	4.25	*	*	20.85	*	*
4	8	500	4.25	*	*	20.85	*	*
4	7	900	4.25	*	*	20.85	*	*
4	8	300	4.25	103.47	127.79	20.85	*	*
4	9	900	4.25	*	*	20.85	*	*
4	8	100	4.25	85.03	198.76	20.85	*	*
3	20	1600	4.25	*	*	36.12	*	*
3	12	1800	4.25	*	*	36.12	*	*
4	2	400	4.25	20.85	4697.03	20.85	4697.02	2.09
3	20	1400	4.25	*	*	36.12	*	*
3	16	1200	4.25	*	*	36.12	*	*
4	8	1000	4.25	*	*	20.85	*	*
3	45	1500	4.25	*	*	36.12	*	*
3	9	1500	4.25	*	*	36.12	*	*
3	81	2100	4.25	*	*	36.12	*	*
1	312	3600	4.25	*	*	108.35	*	*
1	208	1600	4.25	*	*	108.35	*	*

Figure 8
Central Place Structure Map

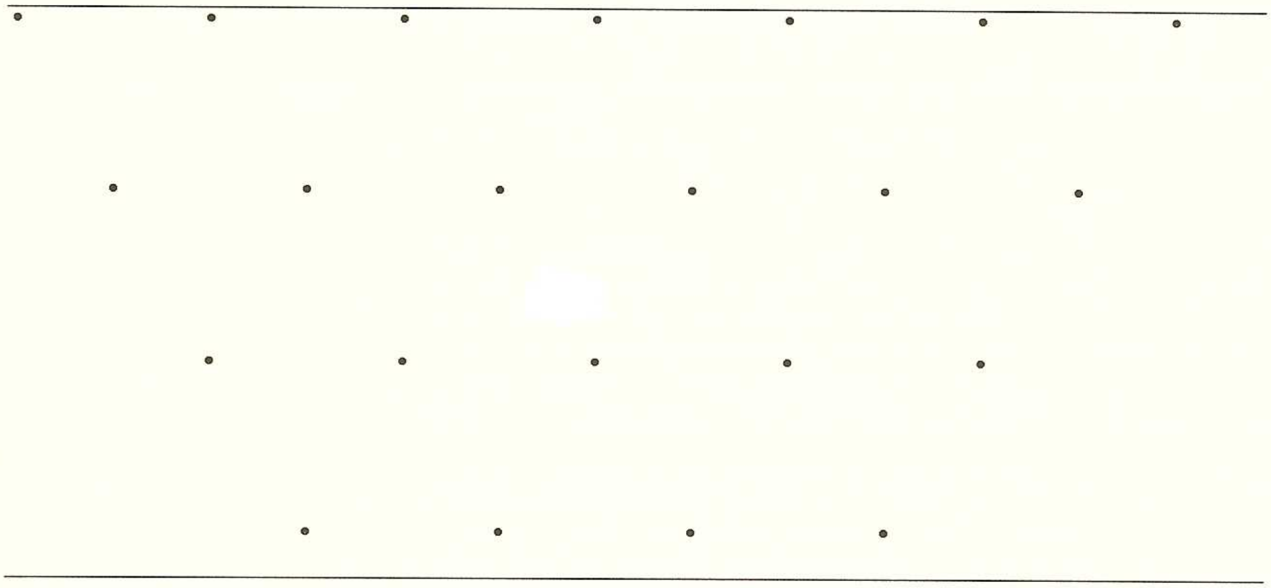


The distance between hamlets is: 41.70317 • - hamlet

Table 9
File NU5 ($t = .1$, $A = 500$)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	400	5	*	*	22.47	*	*
4	2	900	5	*	*	22.47	*	*
4	7	500	5	*	*	22.47	*	*
4	10	100	5	*	*	22.47	*	*
4	2	500	5	23.55	1408.11	22.47	1341.49	2.37
4	10	300	5	*	*	22.47	*	*
4	9	1000	5	*	*	22.47	*	*
4	8	500	5	*	*	22.47	*	*
4	7	500	5	*	*	22.47	*	*
4	8	300	5	*	*	22.47	*	*
4	9	900	5	*	*	22.47	*	*
4	8	100	5	*	*	22.47	*	*
4	20	1600	5	*	*	22.47	*	*
3	12	1800	5	*	*	38.91	*	*
4	2	400	5	22.47	1622.13	22.47	1630.27	2.25
4	20	1400	5	*	*	22.47	*	*
4	16	1200	5	*	*	22.47	*	*
4	8	1000	5	*	*	22.47	*	*
4	45	1500	5	*	*	22.47	*	*
4	9	1500	5	*	*	22.47	*	*
3	81	2100	5	*	*	38.91	*	*
1	312	3600	5	*	*	116.74	*	*
1	208	1600	5	*	*	116.74	*	*

Figure 9
Central Place Structure Map



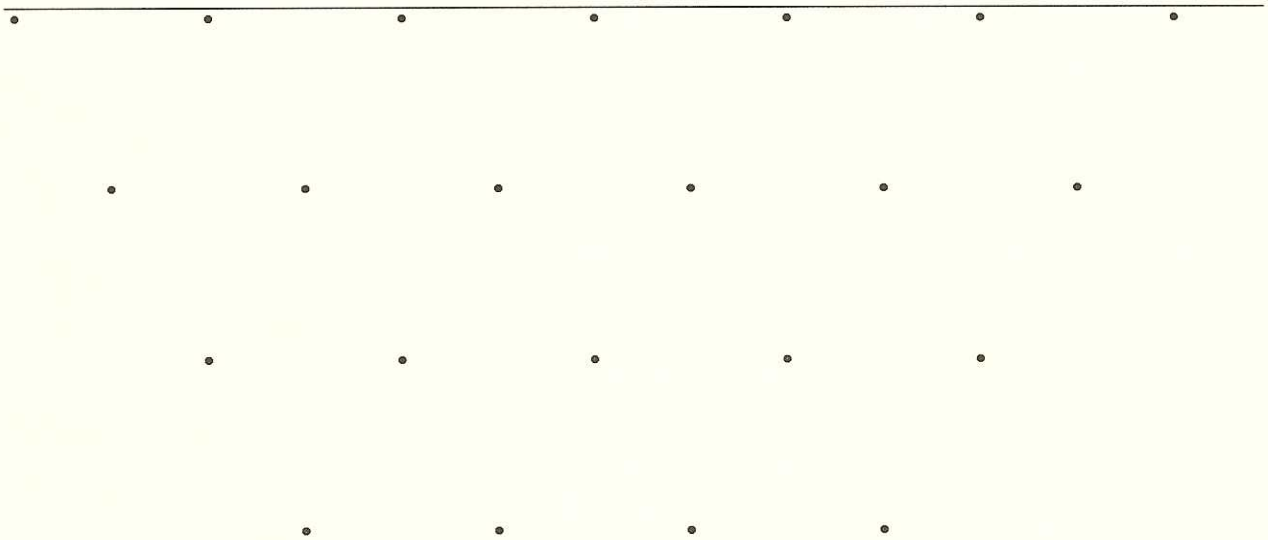
The distance between hamlets is: 44.93177

• - hamlet

Table 10
File NU0425 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	0	4.25	70	307.9	20	85.13	7
4	2	0	4.25	20	5159.04	20	5159.04	2
4	7	0	4.25	70	307.9	20	85.13	7
4	10	0	4.25	100	138	20	22.8	10
4	2	0	4.25	20	5159.04	20	5159.04	2
4	10	0	4.25	100	138	20	22.8	10
4	9	0	4.25	90	174.92	20	33.84	9
4	8	0	4.25	80	228	20	52.33	8
4	7	0	4.25	70	307.9	20	85.13	7
4	8	0	4.25	80	228	20	52.33	8
4	9	0	4.25	90	174.92	20	33.84	9
4	8	0	4.25	80	228	20	52.33	8
4	20	0	4.25	200	29.01	20	1.54	20
4	12	0	4.25	120	91.56	20	11.4	12
4	2	0	4.25	20	5159.04	20	5159.04	2
4	20	0	4.25	200	29.01	20	1.54	20
4	16	0	4.25	160	47.93	20	3.73	16
4	8	0	4.25	80	228	20	52.33	8
4	45	0	4.25	450	4.68	20	.06	45
4	9	0	4.25	90	174.92	20	33.84	9
4	81	0	4.25	810	1.25	20	*	*
1	312	0	4.25	3120	.06	103.92	*	*
4	208	0	4.25	2080	.15	20	*	*

Figure 10
Central Place Structure Map



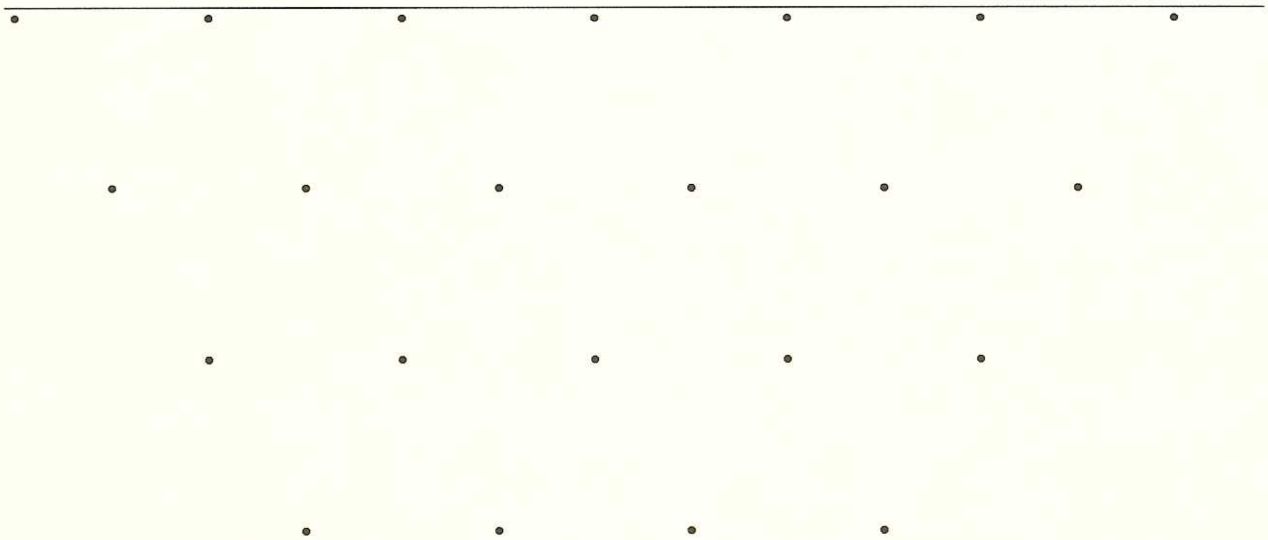
The distance between hamlets is: 40

• – hamlet

Table 11
File NU4250 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	0	400	4.25	269.32	14.85	129.39	*	*
4	0	900	4.25	140.77	63.93	129.39	*	*
4	0	500	4.25	225.29	22.19	129.39	*	*
4	0	100	4.25	816.41	1.22	129.39	*	*
4	0	500	4.25	225.29	22.19	129.39	*	*
4	0	300	4.25	339.01	8.850001	129.39	*	*
4	0	1000	4.25	129.39	77.28	129.39	77.28	12.94
4	0	500	4.25	225.29	22.19	129.39	*	*
4	0	900	4.25	140.77	63.93	129.39	*	*
4	0	300	4.25	339.01	8.850001	129.39	*	*
4	0	900	4.25	140.77	63.93	129.39	*	*
4	0	100	4.25	816.41	1.22	129.39	*	*
3	0	1600	4.25	88.83999	180.1	224.11	*	*
3	0	1800	4.25	80.85	222.63	224.11	*	*
4	0	400	4.25	269.32	14.85	129.39	*	*
3	0	1400	4.25	98.86	141.62	224.11	*	*
3	0	1200	4.25	111.83	107.3	224.11	*	*
4	0	1000	4.25	129.39	77.28	129.39	77.28	12.94
3	0	1500	4.25	93.55	160.34	224.11	*	*
3	0	1500	4.25	93.55	160.34	224.11	*	*
3	0	2100	4.25	71.47	293.82	224.11	*	*
3	0	3600	4.25	46.44	775.24	224.11	*	*
3	0	1600	4.25	88.83999	180.1	224.11	*	*

Figure 11
Central Place Structure Map

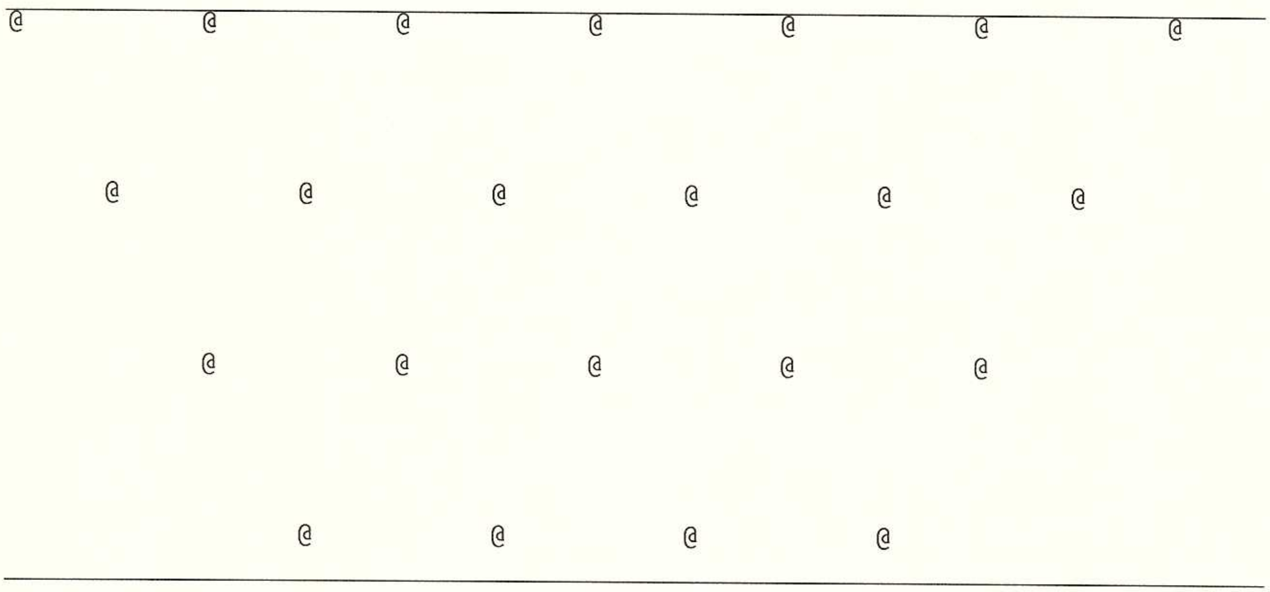


The distance between hamlets is: 258.785 • - hamlet

Table 12
File 1 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	450	3.507404E+08	45

Figure 12
Central Place Structure Map

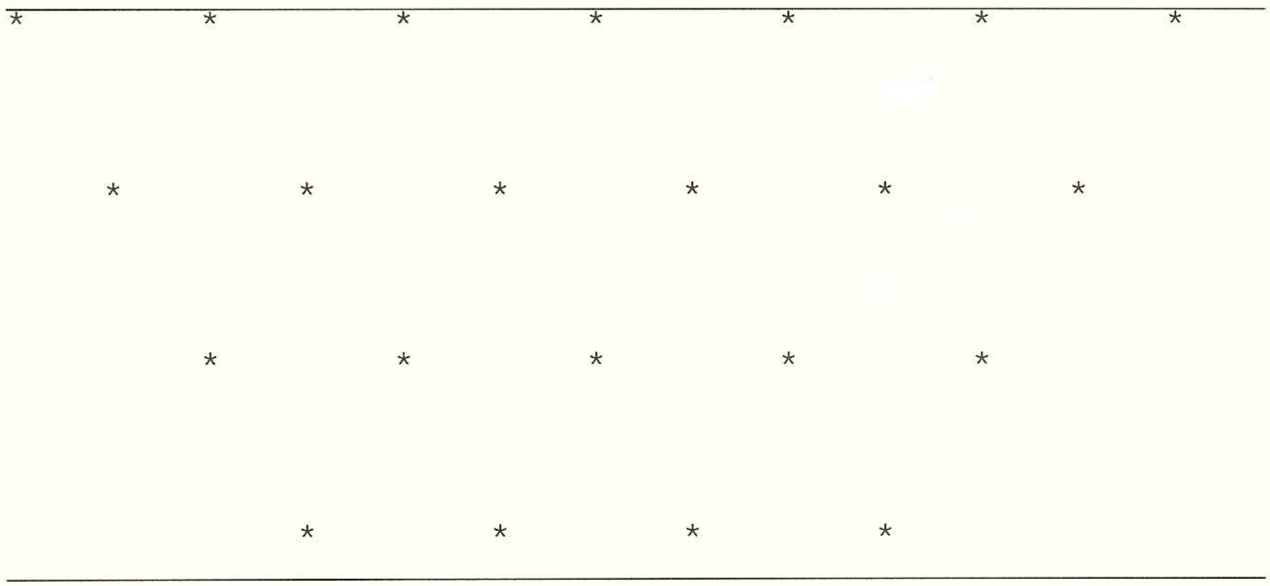


The distance between cities is: 900.0001 @ - city

Table 13
File 2 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	2	100	1	20	208614.1	20	208614.5	2

Figure 13
Central Place Structure Map

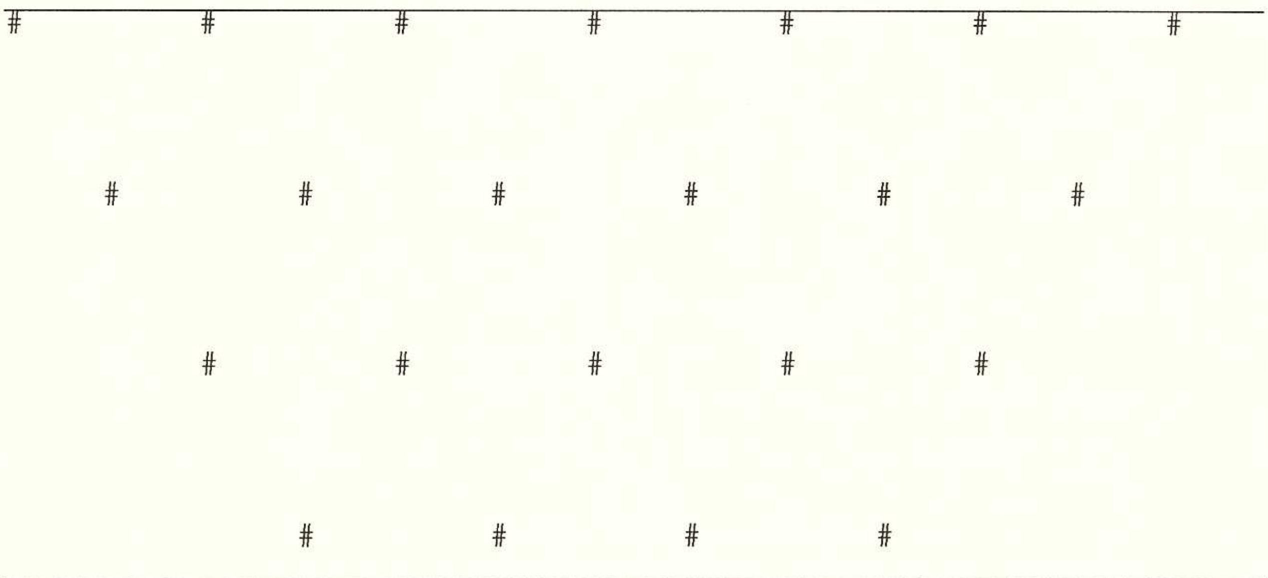


The distance between towns is: 40.00959   * - town

Table 14
File 3 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
3	7	900	3.5	75.63	1599.19	75.63	1599.19	7.56

Figure 14
Central Place Structure Map

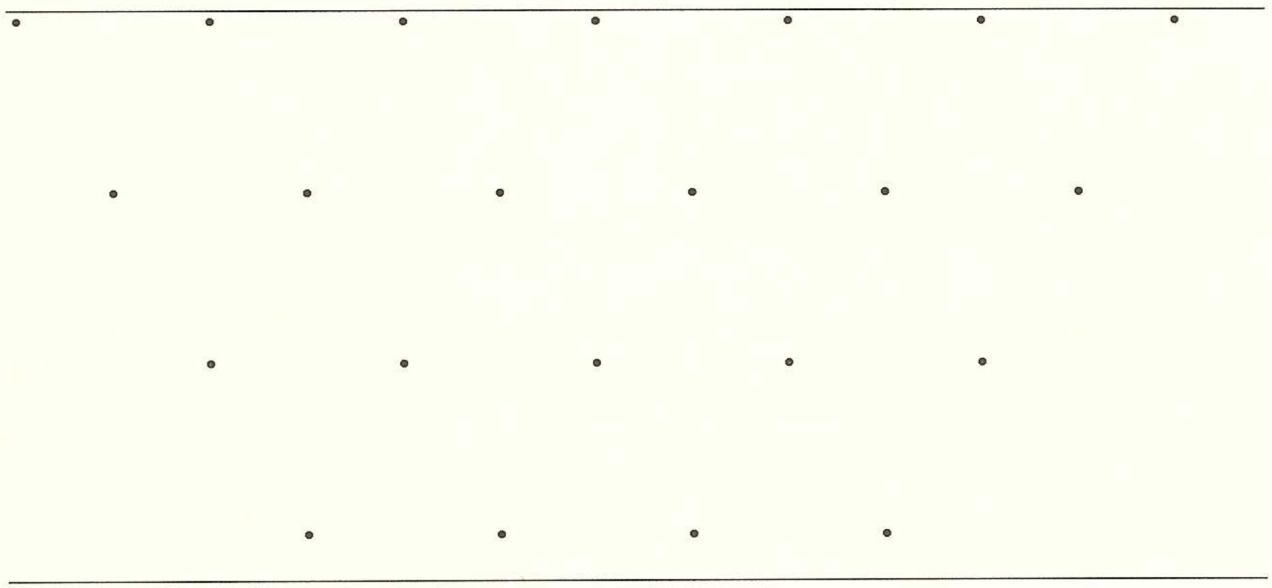


The distance between villages is: 151.2557 # - village

Table 15
File 4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 15
Central Place Structure Map



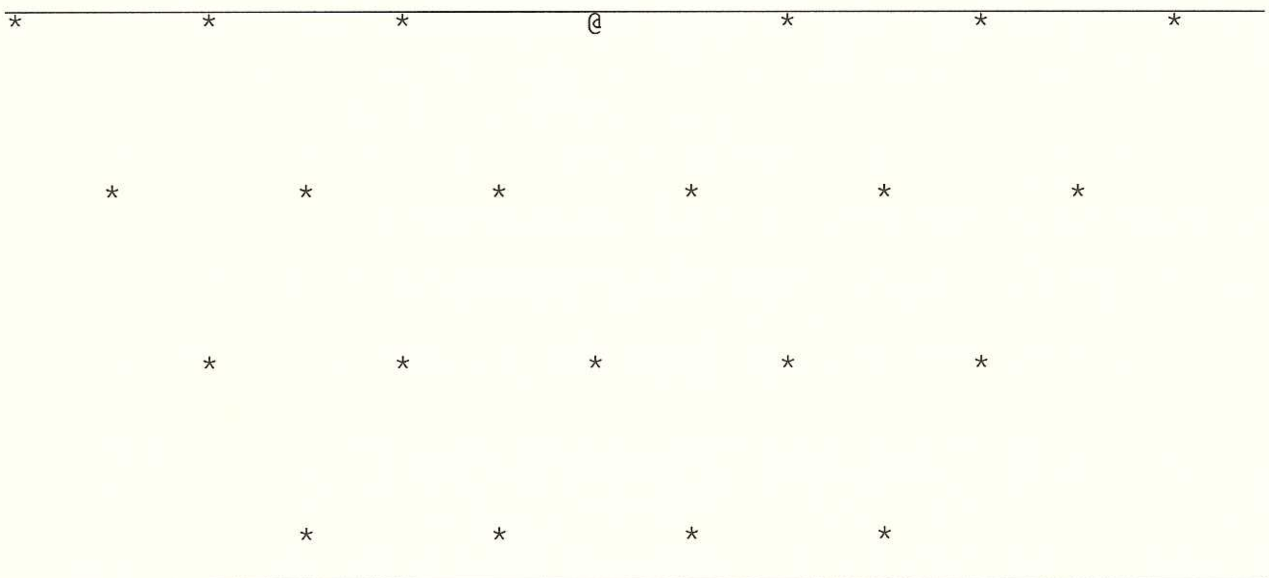
The distance between hamlets is: 158.5747

• - hamlet

Table 16
File 1&2 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	34.65	2079457	45
2	2	100	1	20	208614.1	20	208614.5	2

Figure 16
Central Place Structure Map



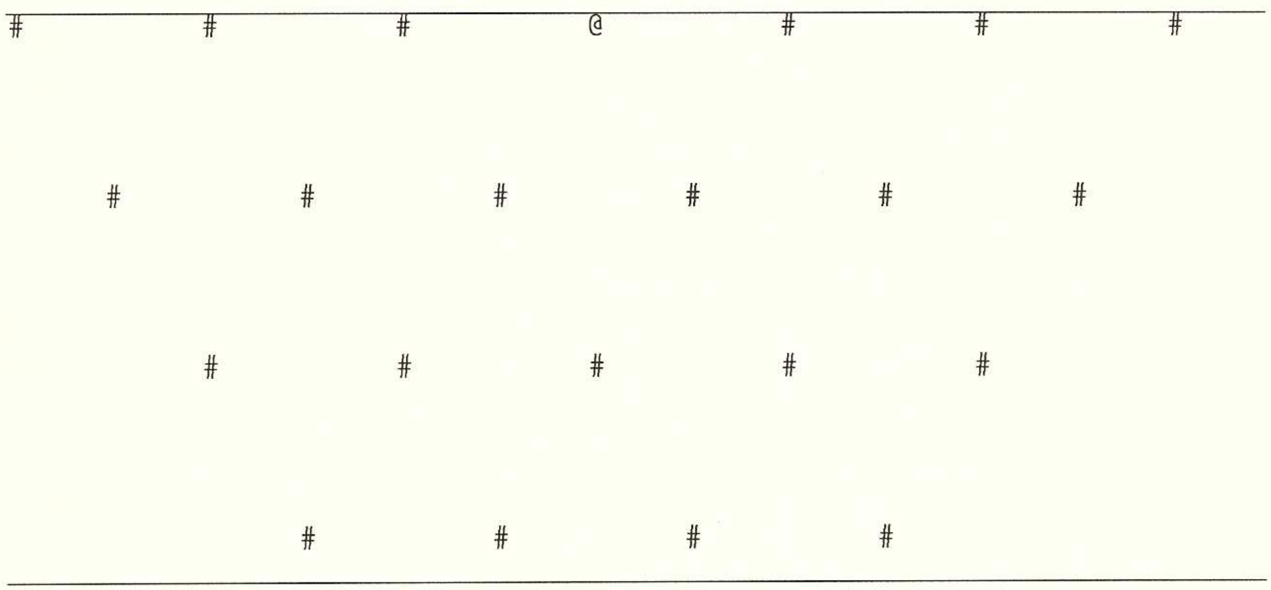
The distance between cities: 69.29864
The distance between towns: 40.00959

@ - city
* - town

Table 17
File 1&3 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	226.88	8.91593E+07	45
3	7	900	3.5	75.63	1599.19	75.63	1599.19	7.56

Figure 17
Central Place Structure Map



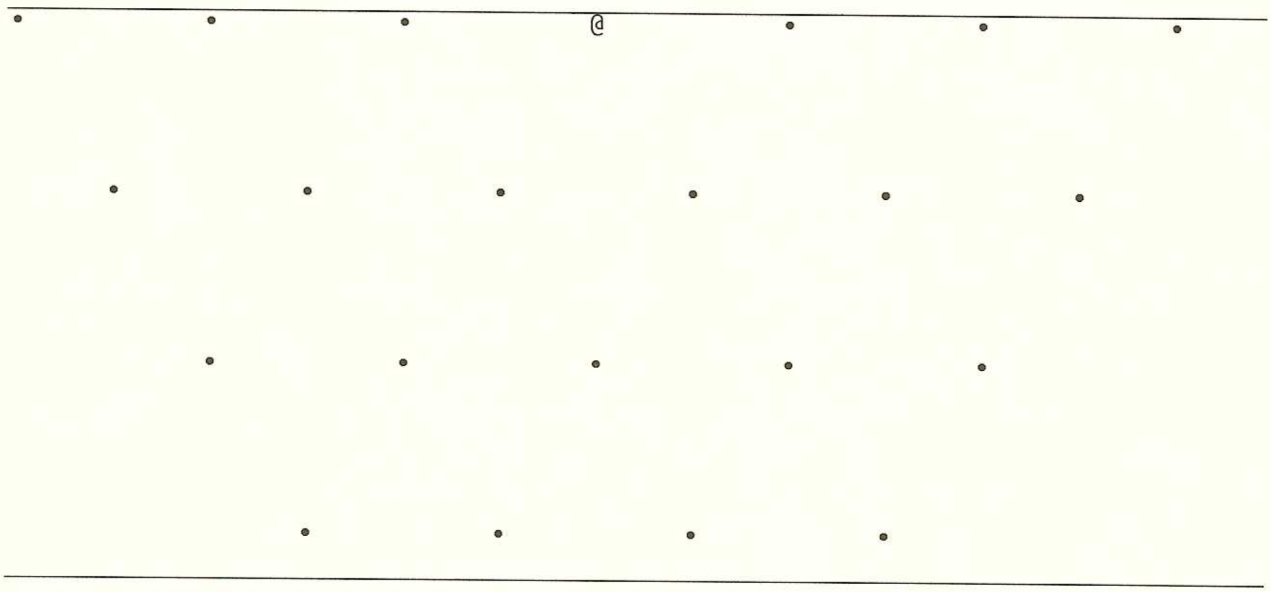
The distance between cities: 453.7672
The distance between villages: 151.2557

@ - city
# - village

Table 18
File 1&4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	411.99	2.939898E+08	45
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 18
Central Place Structure Map



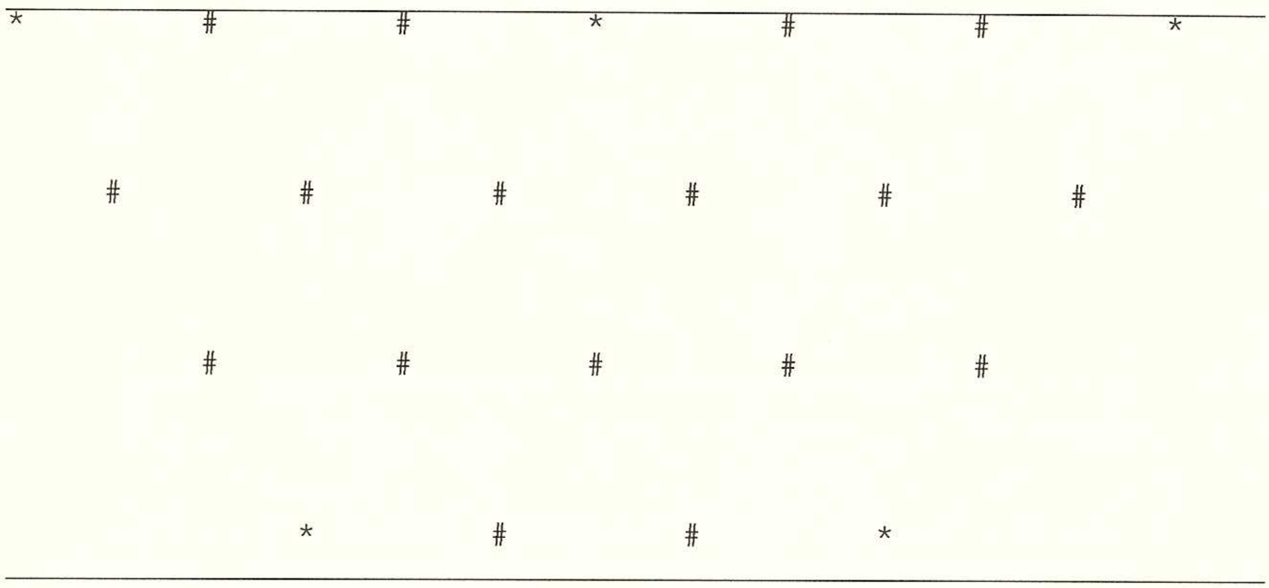
The distance between cities: 823.9783
The distance between hamlets: 158.5747

@ - city
• - hamlet

Table 19
File 2&3 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	2	100	1	20	208614.1	130.99	3021341	2
3	7	900	3.5	75.63	1599.19	75.63	1599.19	7.56

Figure 19
Central Place Structure Map



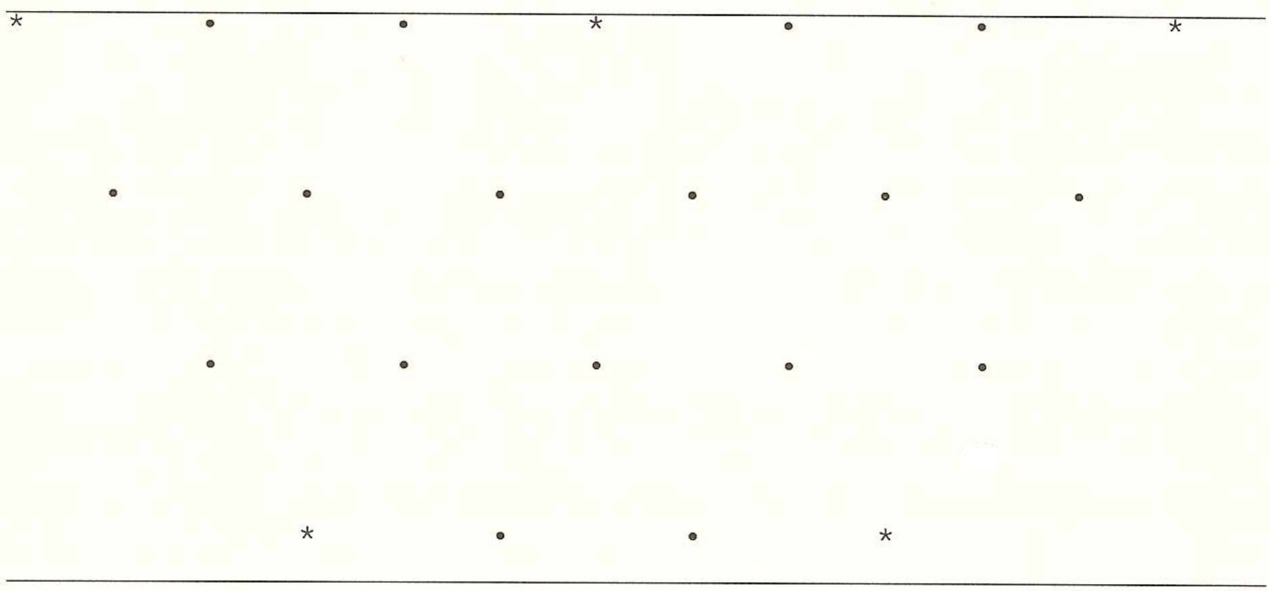
The distance between towns: 261.9826
The distance between villages: 151.2557

* - town
# - village

Table 20
File 2&4 ($t = .1$, $A = 500$)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	2	100	1	20	208614.1	237.86	6205788	2
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 20
Central Place Structure Map



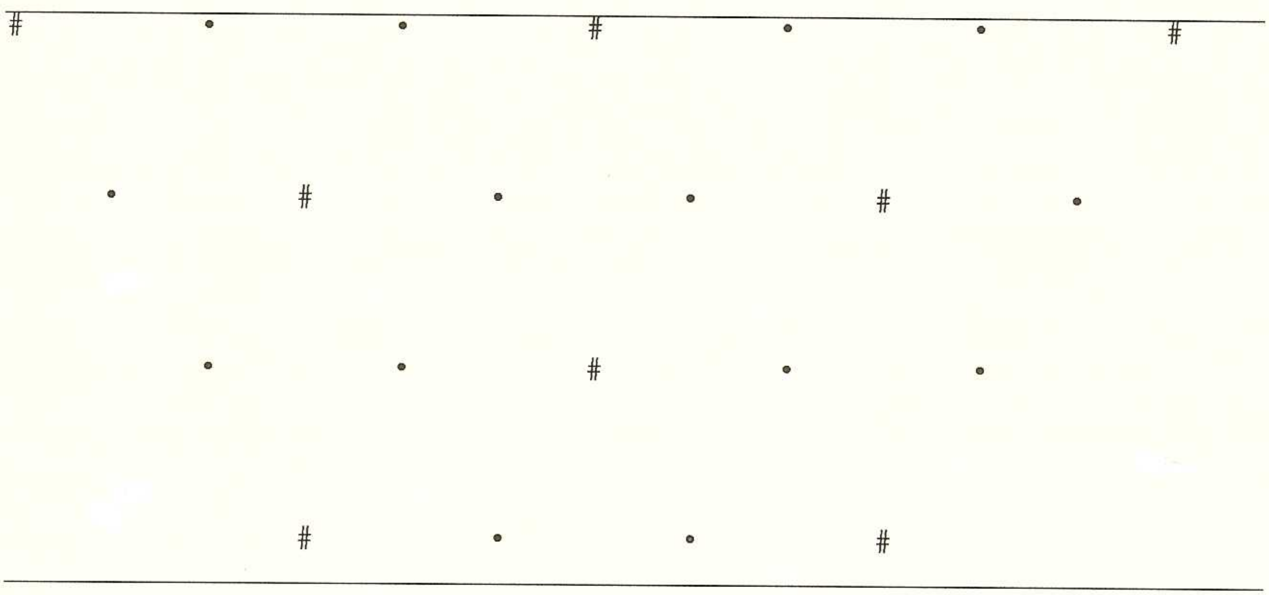
The distance between towns: 475.7241
The distance between hamlets: 158.5747

* - town
• - hamlet

Table 21
File 3&4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
3	7	900	3.5	75.63	1599.19	137.33	2555.64	7.35
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 21
Central Place Structure Map



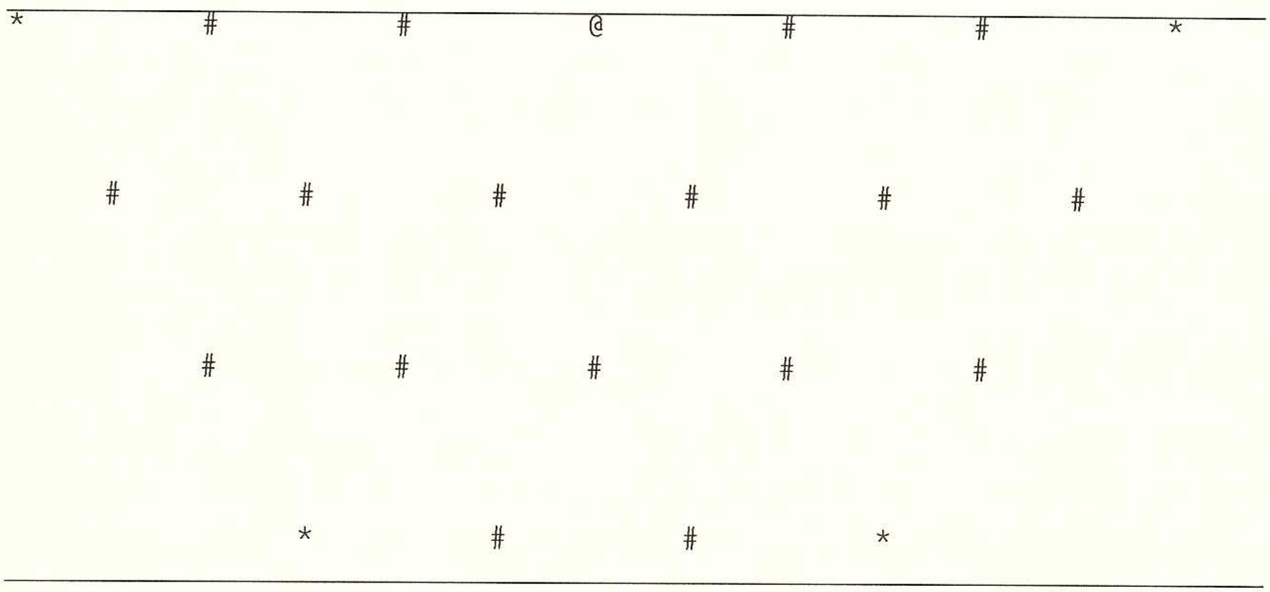
The distance between villages: 274.6595
The distance between hamlets: 158.5747

- village
• - hamlet

Table 22
File 1&2&3 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	226.88	8.91593E+07	45
2	2	100	1	20	208614.1	130.99	3021341	2
3	7	900	3.5	75.63	1599.19	75.63	1599.19	7.56

Figure 22
Central Place Structure Map



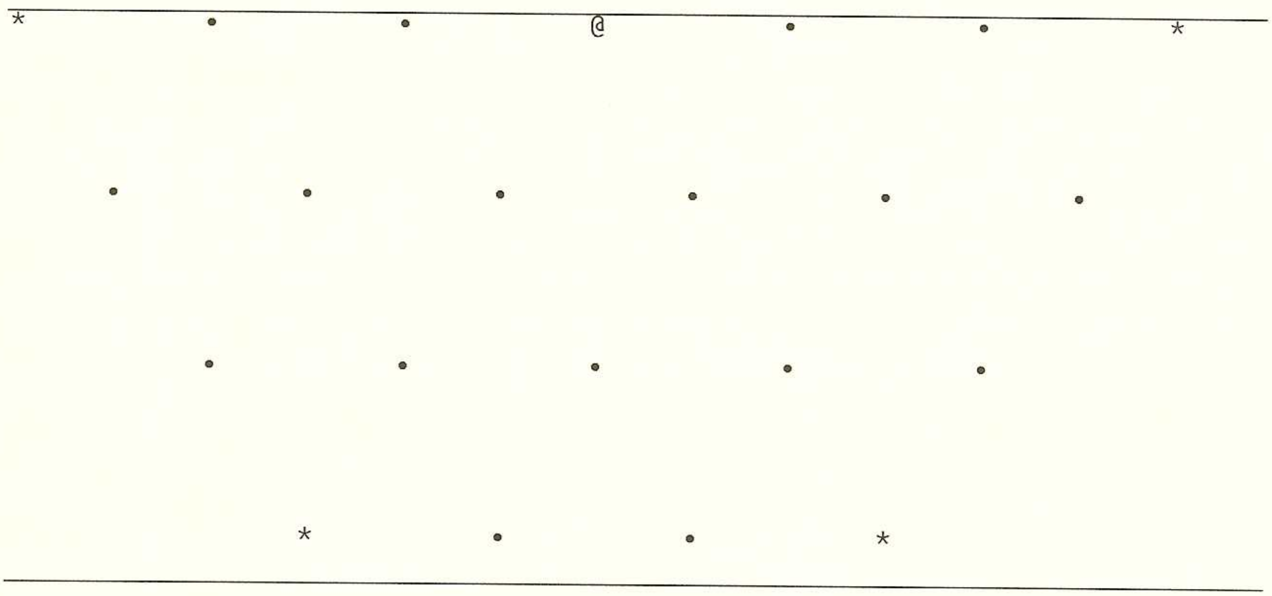
The distance between cities: 453.7672
The distance between towns: 261.9826
The distance between villages: 151.2557

@ - city
* - town
# - village

Table 23
File 1&2&4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	411.99	2.939898E+08	45
2	2	100	1	20	208614.1	237.86	6205788	2
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 23
Central Place Structure Map



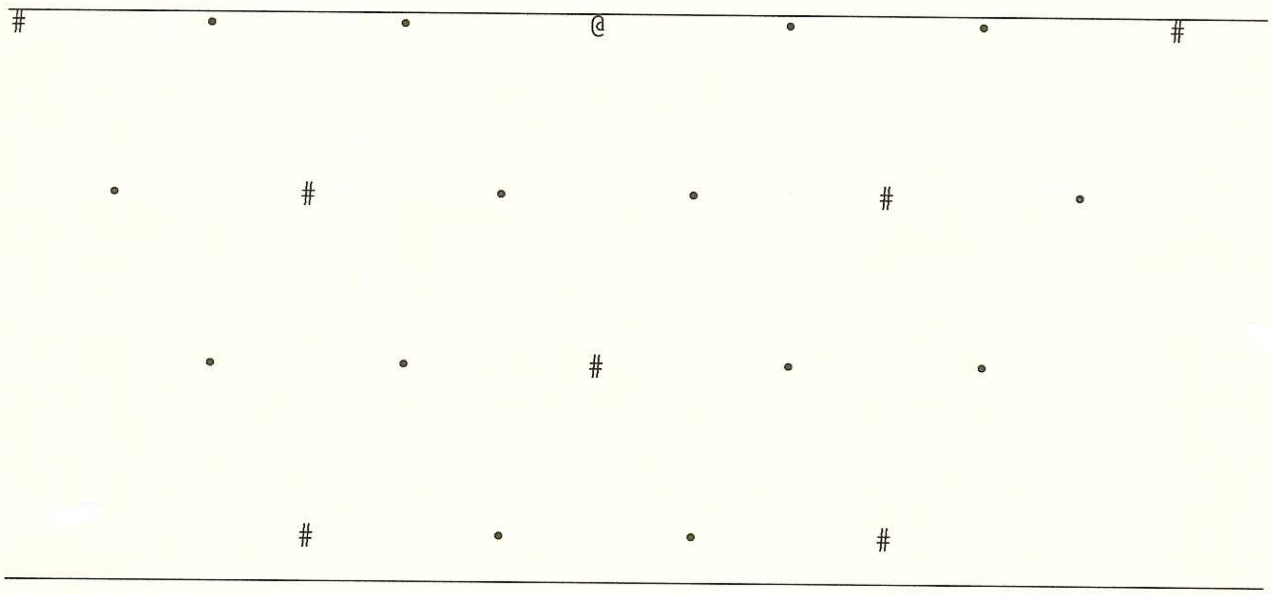
The distance between cities: 823.9783
The distance between towns: 475.7241
The distance between hamlets: 158.5747

@ - city
* - town
• - hamlet

Table 24
File 1&3&4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507404E+08	411.99	2.939898E+08	45
3	7	900	3.5	75.63	1599.19	137.33	2555.64	7.35
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 24
Central Place Structure Map



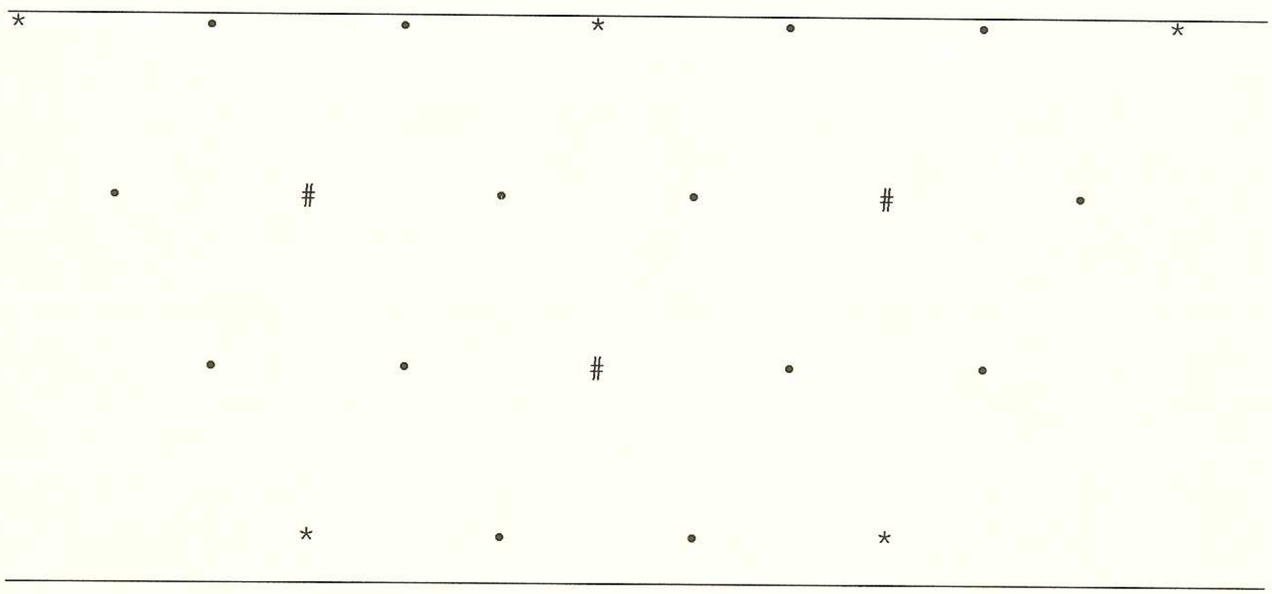
The distance between cities: 823.9783
The distance between villages: 274.6595
The distance between hamlets: 158.5747

@ - city
# - village
• - hamlet

Table 25
File 2&3&4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
2	2	100	1	20	208614.1	237.86	6205788	2
3	7	900	3.5	75.63	1599.19	137.33	2555.64	7.35
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 25
Central Place Structure Map



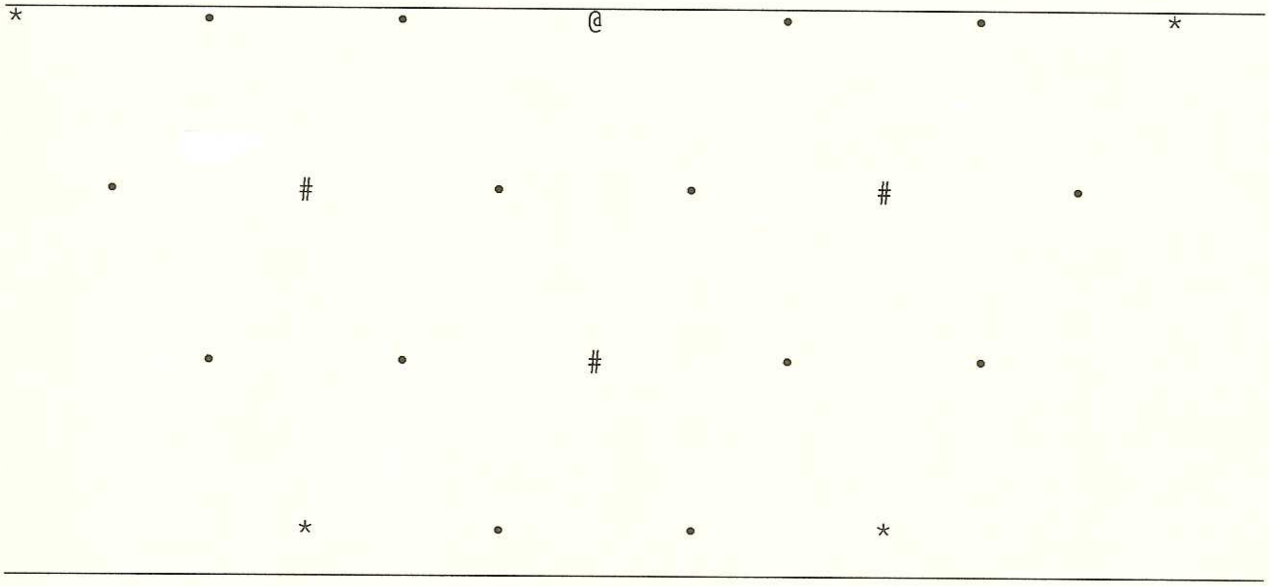
The distance between towns: 475.7241
The distance between villages: 274.6595
The distance between hamlets: 158.5747

* - town
\# - village
• - hamlet

Table 26
File 1&2&3&4 (t = .1, A = 500)

L	C_v	C_f	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price
1	45	1500	0	450	3.507402E+08	411.99	2.939898E+08	45
2	2	100	1	20	208614.1	237.86	6205788	2
3	7	900	3.5	75.63	1599.19	137.33	2555.64	7.35
4	7	400	4	79.29	430.69	79.29	431.31	7.93

Figure 26
Central Place Structure Map



The distance between cities : 823.9783
The distance between towns: 475.7241
The distance between villages: 274.6595
The distance between hamlets: 158.5747

@ - city
* - town
# - village
• - hamlet

6.0. ABOUT THE ALGORITHM

The algorithm is presented here in Appendix A. It is not a streamlined version, having some redundancies in sections of the code. In order to minimize running time, the special cases of integer elasticity of demand parameter values have been included; while the numerical approximation code could easily handle these six special cases, too, reducing the size of the source code, execution time for these parameter values would be significantly increased. Duplicate code exists, too, for the calculations of certain individual non-nested commodity radii before and after nesting occurs; this duplication has been included in order to best handle rounding error problems that were encountered during the development of the algorithm. And, in general "spaghetti" programming characterizes the code structure, mostly because initially a few selected sections were composed [development began with a reproduction of the results presented in Griffith (1986)], and then these sections were sequentially augmented by supplemental components as the complete program unfolded. Moreover, the concatenation of sections of code reflects the history and evolution of Version 1.0 of the algorithm's development. Certainly the structure of this code could be optimized, in order to enhance program efficiency. But, we were not overly concerned with this aspect of the program; rather, we wanted to make the program easy to read and easy to debug. One should realize that, regardless of its programming sloppiness, Version 2.0 presented here is known to work.

6.1. Summary and description

Brief comments included within the algorithm itself help to explain what various sections of computer code accomplish. A somewhat more thorough description and explanation of these lines of code should aid in achieving a deeper and more thorough understanding of the algorithm and its organization.

Lines 60-170 furnish a general description of the program. Lines 190-490 provide a set of reference definitions for the basic variables employed in the algorithm. Lines 40 and 5050-5460 generate the printing on a CRT of acknowledgements (this is the first observed CRT output). Lines 500-610 establish dimensions of working arrays for the algorithm. Lines 620-660 define the K = 3 spacing multiplier factors, q_l (l=1,2,3,4), for the four possible hierarchical levels (see Section 5.0). Lines 670-1320, 4970-5040, and 8260-8780 yield queries on a CRT [this is the second (and optionally the third or fourth) observed CRT output] soliciting a specification of initial conditions and problem parameter values, as well as they initialize working arrays. Lines 1330-1430 create a CRT notification (this is the fifth possible observed CRT output) that data processing is taking place, together with a caution concerning one potential inconsistency that may arise, due to rounding error, and hence may become conspicuous in tabular results. Lines 1440-1650 calculate discriminant function analysis multinomial logit probabilities as defined by Equation (4.1). Lines 1660-1770 use the probabilities calculated in Lines 1440-1650 to allocate commodities to hierarchical levels, based on the criterion that a commodity is allocated to that level having the maximum classification probability. Lines 1800-1840 generate a printing of the caption for tabular output on a CRT (this is part of the fifth possible observed CRT output). Lines 1850-2750 and 7680-8250 calculate the non-nested commodity threshold radii, and then select the minimum radius from that set of commodities that have been classified into the lowest existing hierarchical level. Lines 2751-3990 calculate the nested radius, demand, and price for each commodity. Lines 3995-4550 and 7680-8250 re-calculate the non-nested radius and demand for certain commodities. Lines 4570-4810 generate a printing of selected parameter values, and non-nested and nested results for a commodity on a CRT (this is part of the sixth possible observed CRT output). Lines 4820-4830 ask the user whether or not the printing of a CRT map is desired. Lines 4850-4950 permit the user to select either the undertaking of an additional program execution, or the termination of the program execution. Lines 4840 and 5470-7090 generate a printing of a map of the central place structure on a CRT, together with the relevant spacing distances and a map legend (this is the seventh possible observed CRT output). Lines 7100-7280 and 7425-7660 house the embedded example test data set. Lines 7290-7420 house the default discriminant function analysis parameter values, extracted with a SAS analysis of the pooled data set presented and described in Section 4.3. Lines 8530-8780 permit alternative discriminant function parameter values, which would replace the default values, to be introduced into a problem. Line 8790 allows a new input data file to be constructed by the user for employment in a problem. Lines 8800-8940 enable any of the 26 test data sets (see Section 5.0) housed on the diskette to be retrieved for employment in a problem. Finally, Lines 8950-9000 define the default density of demand and transport cost parameter values.

Solutions for the different classes of values of the elasticity of demand parameter are given in the following lines of computer code, for the equilibrium case of $p = rt$ (determining the threshold polygon distance radius for a single commodity, without a constraint of hierarchical nesting) and $r = q_{l}*r_{min}$ (determining results for the nested radius based upon r_{min}:

η	Non-nested Solution Code Lines	Nested Solution Code Lines
0	2020-2070	2870-2980
1	2090-2100	3020-3280
2	2130-2140	3320-3660
3	2170-2180; 4030-4080	3710-3990
4	2210-2270; 4110-4180	3710-3990
5	2300-2540; 4210-4450	3710-3990
decimal	7700-8250	3710-3990

Equation (5.1) translates into an n-th order polynomial in terms of radius for the non-nested solutions. When $\eta = 0$, Equation (5.1) is of degree 3, and the radius selected is the smallest non-negative real root of this equation, if one exists. When $\eta = 1$, Equation (5.1) is of degree 2, and the radius selected is the one associated with the positive discriminant of the quadratic equation solution, if this discriminant is not negative (thus yielding a complex number). When $\eta = 2$, Equation (5.1) is of degree 1, and the radius is obtained by solving the resulting linear equation. When $\eta = 3$, Equation (5.1) again is of degree 1, and the radius is obtained by solving the resulting linear equation, if this solution is non-negative. When $\eta = 4$, Equation (5.1) is of degree 2, and the radius selected is the one closest to but exceeding c_v/t, if one exists. When $\eta = 5$, Equation (5.1) is of degree 3, and the radius selected is the smallest non-negative real root of this equation; such a root is guaranteed to exist. No relatively simple solution exists when η is not an integer.

Equation (5.1) translates into an n-th order polynomial in terms of price for the nested solutions, since the radius is set equal to the hierarchical level distance multiplier factors (q_l) times the pre-determined global minimum (r_{min}). When $\eta = 0$, Equation (5.1) is of degree 1, and the price is obtained by solving the resulting linear equation. Otherwise, demand must be determined with the aid of numerical integration, and then appropriate polynomial roots must be determined with the aid of numerical root extraction.

Redundancies are apparent in the above tabular results. For $\eta = 3$, 4, and 5, a second set of computer codes is accessed for calculating non-nested radii and the affiliated non-nested demand after the corresponding nested values have been calculated. These very same values are calculated in the respective earlier lines of code.

Restrictions are imposed upon user-inputed parameter values, to (a) help ensure the existence of at least some feasible solutions, (b) prevent the introduction of nonsensical or physically meaningless values, and (c) prevent problems of an unmanageable or unreasonable size from being pursued. In Line 840, the number of commodities, denoted by CN in the algorithm, is constrained such that $1 \le CN \le 19$. In Line 990, each commodity is constrained to have a distinct numerical name. In Line 1020, the variable cost, c_v, is constrained such that $0 \le c_v \le 312$. In Line 1070, the fixed cost, c_f, is constrained such that $0 \le c_f \le 3600$. In Line 1120, the fixed and variable costs are constrained such that they cannot concomitantly be equal to zero; at least one of these two cost parameters must be positive. In Line 1130 the elasticity of demand parameter, η, is constrained such that $0 \le \eta \le 5$. In Line 1240 the density of demand, A, is constrained such that $A > 0$. And, in Line 1280, the transport rate, t, is constrained such that $t > 0$.

Numerical methods are relied upon heavily in order to derive the necessary solutions to a given problem; both numerical integration and numerical root extraction procedures have been implemented in the algorithm. In order to expedite numerical integration calculations, those special cases of $p = rt$ for which solutions are known in advance were calculated analytically (see Section 3.1) as well as using the IMSL 10.0 subroutine DTWODQ, which renders double-precision results, and iteratively computes a two-dimensional integral using a globally adaptive utilized Gauss-Kronrod rules [based upon Piessen, et al. (1983)]. The results for these special cases are as follows:

η	Numerical Integration Solution	Relevant Lines of Code
0	$1/3^{1/2}$	2910-2920, 2980
1	0.1738034	2100, 3060, 3170
2	0.1075536	2140, 3360, 3560
3	0.1371809/2	2180, 4060, 4070
4	0.0451258	2220-2250, 4120-4150
5	0.0306553	2310, 2330-2340, 2590, 2610, 4220, 4240-4250, 4440, 4490, 4510

When external numerical integration was not possible, Simpson's rule (see Section 3.2) was used to devise code for internal numerical integration purposes [see Press, et al. (1986), and Kohn (1987)]. The best agreement between IMSL 10.0 numerical integration results and those obtained from the algorithm presented here were achieved by dividing the circle into 240 intervals, each being 1.5° in size. Individual numerical integration subroutines occur in the following sets of lines: 7700-7830 (non-integer values of η, $p = rt$, non-nested solution), 3070-3190 ($\eta = 1$, $r = q_1*r_{min}$, nested solution), 3330-3570 ($\eta = 2$, $r = q_1*r_{min}$, nested solution), and 3750-3870 ($\eta \neq 0, 1, 2$, $r = q_1*r_{min}$, nested solution).

Numerical root extraction is achieved with a modified form of the *regula falsi* method [see Section 3.2; see Press, et al. (1986), and Kohn (1987)]. A particular root of the equation is being sought, namely one that is larger than but as close as possible to c_v/t when $p = rt$, and one that is larger than but as close as possible to c_v when the radius is a nesting function of r_{min}. Individual numerical root extraction subroutines occur in the following sets of lines: 7860-8250 (non-integer values of η, $p = rt$, non-nested solution), 3210-3280 ($\eta = 1$, $r = q_1*r_{min}$, nested solution), 3580-3660 ($\eta = 2$, $r = q_1*r_{min}$, nested solution), and 3910-3980 ($\eta \neq 0, 1, 2$, $r = q_1*r_{min}$, nested solution). Numerous results were verified with the IMSL 10.0 subroutine DZPORC, which renders double-precision results, and iteratively finds the roots of a polynomial with real coefficients using Laguerre's method [based upon Smith (1967)].

Although this program allows one to input various parameter values that then will override the embedded default values, checks have not been performed on this algorithm with regard to an evaluation of necessary properties of potential inputed data; there literally are an infinite number of combinations of input data that could be considered, making such a task impossible to complete. In keeping with this drawback, standard deviations are not tested to ensure that they are positive, the covariance matrix is not tested to ensure that it is symmetric, and means, standard deviations, and covariances are not tested to ensure that their accompanying classification scheme sensibly maps onto a four-tier hierarchy. Furthermore, it is conceivable that acceptable sets of values exist from which computer calculation errors could result; however, presumably this is not the case. Consequently, one should be careful when changing the nature and degree of the variable relationships; caution should be exercised, for those default values selected and included in this algorithm are based upon theoretical experience and empirical evidence (see Section 4.0).

6.2. Directions for Algorithm Execution

Execution of this algorithm, entitled CENTRALP.BAS on its source diskette, involves inserting the source diskette into an IBM-compatible PC, after the system has been booted, accessing either BASICA or GWBASIC (whichever is available), and then typing the following sequence of commands:

```
BASICA <enter>                 OR     GWBASIC <enter>
LOAD "CENTRALP.BAS" <enter>
RUN <enter>
```

An exact specification of the first of these three preceding lines of instruction may well vary, in accordance with the way in which a given PC's disk space is partitioned. Now the algorithm is executing in interactive mode, and a sequence of CRT screens will appear, systematically leading the user through the problem at hand.

The first screen image is the acknowledgement, noting the version of, the date of creation of, the funding source for development of, and credits for the architects who developed this algorithm, and appears as follows:

```
Math/debugging consultation by Daniel A. Griffith

A

D                               Created May 14,  1988.
Da                              Version 2
Dar
Darr
Darre
Darren                          Funded by:
Darren G
Darren Gr                       SUNY/Buffalo Office of Teaching Effectiveness
Darren Gri
Darren Grif
Darren Griff
Darren Griffi
Darren Griffit
Darren Griffith

Program.
```

This screen will appear for a few moments, and then automatically be cleared from the CRT.

The second screen image solicits information about the data set to be used in a problem. It appears as follows:

Do you want to:

1) Use the example data file contained in the program?
2) Enter your own data?
3) Use a data file available on the disk?

(NOTE: smaller integral values of nu reduce the run time.)

If the user selects Option #1, then the next CRT image to appear will be the fifth screen. If the user selects Option #2, then the next CRT image to appear will be a sequence of screens that will be referred to here as the third screen. And, if the user selects Option #3, then the next CRT image to appear will be the fourth screen.

The third CRT image actually is a sequence of separate set of screens posing questions with which the user is queried about data to be entered. Screen 3a states

Enter the number of commodities (between 1 and 19):

Screen 3b presents four successive questions about each commodity, and states, as each question is answered,

Enter commodity number:

Enter the variable cost (between 0 and 312):

Enter the fixed cost (between 0 and 3600):

Enter the elasticity of demand (between 0 and 5):

A new screen presenting these four questions appears for each commodity to be studied. Next, Screen 3c sequentially solicits the density of demand and transport cost data, stating

Enter the density:

Enter the transport rate:

Screen 3d asks about changing the parameters of the discriminant function, and states

Do you want to specify means and standard deviations of variables, matrix of z-score group means, and inverse z-score covariance matrix? (y/n)

If new function parameter values are to be inserted into the problem, then Screen 3e requests the mean and standard deviation for each of the three variables (variable cost, fixed cost, and elasticity of demand), stating

---

Input means and standard deviations of variables. (#,#)

This query is followed by Screen 3f, which solicits the variable means for the four hierarchical levels, and states

---

Input twelve z-score group means.

Finally, Screen 3g asks for the inverse pooled within-groups covariance matrix for the discriminant function, stating

---

```
Input inverse z-score covariance matrix.
In the form:  #,#,#
              #,#,#
              #,#,#
```

Screens 3e-3g should be avoided by all but the most knowledgeable program users.

The fourth screen image provides a user with the option of retrieving a file from the source diskette; there are 25 possibilities from which to choose. Screen 4 appears as follows:

NU0	NU1	NU2	NU3	NU4	NU5
1	2	3	4	1&2	1&3
1&4	2&3	2&4	3&4	1&2&3	1&3&4
2&3&4	1&2&4	1&2&3&4	PAPER	NU425	NU0425
NU4250					

Files named 'NU' have the test data, with all commodities having
 the specified nu value given in the filename.
File 'NU0425' has all cf values of zero, and file 'NU4250' has all
 cv values of zero.
File 'PAPER' has the original 'Economic Geography' article data.
Files named '_&_&_&_' have those levels given in the filename
 appearing on the map.
Which file?

The appearance of Screens 3 and 4 is determined by which option is selected from Screen 2. Regardless of whether or not either of these two screens has appeared, the next CRT image is Screen 5, which has the following display:

Density =
Transport rate =

PROCESSING; PLEASE WAIT!

THE NON-NESTED AND NESTED DEMANDS MAY NOT BE EQUAL BECAUSE THE FIRST WAS DONE BY EXTERNAL NUMERICAL INTEGRATION ON A VAX AND THE SECOND BY INTERNAL NUMERICAL INTEGRATION IN BASIC

Screen 6 exhibits the tabular results. Information about each commodity being studied in the problem is printed in the table. The format of this table is as follows:

L	Cv	Cf	Nu	non-nested radius	non-nested demand	nested radius	nested demand	price

.
.
.
.
Do you want a map? ('y' or 'n')?

Examples of this particular type of tabular output can be found in Tables 1-26, in Section 5.0.

A positive response to the question posed on Screen 6, regarding whether or not a map is desired, renders Screen 7. This CRT image is a central place structure, showing the locations of places positioned at different levels of the urban hierarchy, and the spacing of central places at each hierarchical level. Examples of this output can be found in Figures 1-26, in Section 5.0. This map and legend are followed by the prompt

<<< Press 'r' to run, or any other key to end. >>>

Typing 'r' results in another execution of the algorithm, beginning with Screen 1. Typing any other key yields Screen 8, causing to materialize on the CRT

Thank you; have a nice day!

With this final CRT image, the algorithm execution is completed.

7.0. CONCLUDING COMMENTS

In conclusion, the primary aim of this monograph has been to outline facets of a set of PC-BASIC computer code that simulates and constructs central place networks. This program focuses on the exploration of selected parameters of the ideal K = 3 Christaller network. Various numerical examples are presented in Section 5.0 which exemplify this goal. Although the potential short-term benefits of this exercise were intended to be and probably will be pedagogic in nature, long-term theoretical benefits may well accrue from it.

Much of the literature to date has focused on the geometric patterns of market areas that are manifestations of spatial competition. Results reported here complement the literature on geometric patterns, especially by allowing specific prices and inter-central place distances to be calculated. The algorithm presented here furnishes a vehicle for more comprehensive numerical explorations of such identified and unidentified geometric patterns. Conspicuous drawbacks to this algorithm include the assumptions of uniform fob pricing and transport rates, and an infinite planar surface. Modification of each of these assumptions, in turn, should provide fertile themes for subsequent theoretical research efforts, using this algorithm as a blueprint for accompanying numerical work. A considerable literature exists on the geometric patterns of market areas for which p_{fob} and t vary over a spatial economic landscape. Hypercircles, rather than straight lines, describe market boundaries [see Boots (1980), Hyson and Hyson (1950)]. Hence, if the boundary between two central places follows a hyperbolic curve, for instance, then Equation (2.2) needs to be rewritten accordingly, in order to determine the domain of integration. Similarly, a bounded surface may be dealt with by permitting demand at the periphery to be inadequately satisfied. In addition, the outer hexagons will become partly circular along that portion of their respective perimeters facing the regional borders. As a result, some market areas will have three sides of a hexagon, and some will have four sides of a hexagon. Because this lack of complete truncation will cause the quantity demanded of an item to change, either p_{fob} will need to vary amongst central places, or excess profits will necessarily need to emerge. In this former situation, the aforementioned market area literature should furnish information regarding the geometric distortions that would result. In this latter situation, monopoly economic rents would provide an incentive to develop peripheral areas. Nevertheless, while demand cone boundaries would coincide for all commodities in a given bundle, when spatial competition prevails, different demand cone boundaries would characterize the periphery. Clearly, the boundary value problem takes on an interesting dimension within this context.

Two additional shortcomings that modifications to this algorithm could reflect upon are the static nature of classical central place systems, and the partial equilibrium nature of the solution achieved here. Integrating the construction technique outlined in this monograph with dynamic formulations should prove rewarding. As for a general equilibrium solution, if a budget constraint is imposed upon households, c_v is varied in accordance with delivery costs from factories to the individual retail outlets (a la Weberian industrial location theory), and factories are competing for resources, then a system of equations could be composed from which a general equilibrium solution would be forthcoming. The algebraic, calculus, and numerical analysis details furnished in Section 3.0 should be sufficient to support this line of work.

Within the context of classical theory, this algorithm could be modified so that optionally $K = 4$ and $K = 7$ structures are simulated. Or, any type of Loeschian structure could be generated by removing the sometimes severe restriction to commodity provision imposed by nesting, and introducing an identification of underlying convolutions of Christallerian systems [see Dacey (1964), Hudson (1967)]. In keeping with this theme, the conjecture of a spatial equilibrium when $p = rt$, for the constant elasticity of demand consumption function, merits thorough scrutiny. Empirically, additional data sets should be analyzed for comparative purposes, so that the discriminant function analysis results reported in Section 4.0 can be either fortified or revised.

Two technical tasks could be pursued here, too. First, the algorithm's computer code needs to be optimized. This would be a straight-forward, but tedious although probably not excessively time consuming, job to undertake. The documentation provided in Section 6.0 should be sufficient in detail to facilitate a timely and successful completion of this job. Second, the graphics need to be enhanced, especially in terms of the inclusion of color options and output; this enhancement would restrict the adaptability of this algorithm, however, because of the current absence of many high-resolution multiple-color CRTs in university classrooms and teaching laboratories (this situation should change as prices drop).

Since the prominent impetus for completing the research summarized in this monograph was educational in nature, certainly worthwhile and interesting research should be undertaken concerning how effective the algorithm presented here is in conveying a better, more comprehensive, and more insightful understanding of central place theory and concepts. Classroom evaluation of the algorithm, especially as a supplement to lecture and textbook materials, is warranted. Well tested experiments need to be formulated and evaluated. Displays presented in Section 6.0 need to be altered in order to maximize the user-friendly qualities.

Consequently, the simulation algorithm presented in this monograph is one of a number of possible experimental tools that can be used when teaching introductory courses on economic or urban geography, or when engaging in numerically intensive theoretical research concerning central place theory. Hopefully it will augment those simulation models presently in use, and in doing so further enrich the discipline of geography and geographical analysis.

8.0. BIBLIOGRAPHY

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APPENDIX A:
BASIC COMPUTER CODE
FOR IBM COMPATIBLE PCs

```
10 KEY OFF
20 MIN = 1000000000#
30 VERSION = 2
40 GOSUB 5050 ' Introduction.
50 '
60 ' This program has been created by
70 '
80 '        Darren L. Griffith
90 '
100 '              &
110 '
120 '       Daniel A. Griffith
130 '
140 ' This program will construct a k = 3 Christaller central place structure
       for a given set of commodities, and determine the corresponding demand,
       prices, market areas, and hierarchy levels.
150 ' A constant elasticity of demand consumption function is employed.
160 ' A test data set is included for implementation diagnostic checking.
170 ' Test data files are available for diagnostic checks.
180 '
190 ' List of variables :
200 ' CN                            ( number of commodities )
210 ' CM                            ( commodity matrix )
220 ' CM (CN,2)                     ( variable cost )
230 ' CM (CN,3)                     ( fixed cost )
240 ' CM (CN,4)                     ( elasticity of demand parameter nu)
250 ' MCV                           ( mean of: variable cost
260 ' MCF                                      fixed cost
270 ' MED                                      elasticity of demand )
280 ' SDCV                          ( standard deviation of: variable cost
290 ' SDCF                                                   fixed cost
300 ' SDED                                                   elasticity of demand
)
310 ' ZCV                           ( z - score of: variable cost
320 ' ZCF                                           fixed cost
330 ' ZED                                           elasticity of demand )
```

```
340 ' GROUP1                           ( arrays to hold the
350 ' GROUP2                              values put into hierarchical level groups
360 ' GROUP3                              1, 2, 3, and 4,
370 ' GROUP4                              respectively )
380 ' EXP(*)                           ( multinomial probability term
                                            calculated through z - scores )
390 ' G1, G2                             ( counters of numbers of commodities in each

400 ' G3, G4                                hierarchical group )
410 ' Y0, Y1, Y2                         ( variables for calculating a1, a2 )
420 ' A1, A2                             ( variables for calculating k )
430 ' PI                                 ( 3.141593 )
440 ' A                                  ( density of demand )
450 ' K                                  ( value of a complex number formula )
460 ' R0, R1                             ( variables for calculating formula )
470 ' RM                                 ( multiplicative factor for min and rmin )
480 ' HBAR                               ( hierarchical level means of each variable )

490 '
500 ' Dimensioning arrays.
510 '
520 DIM CM (25,5)
530 DIM GROUP1# (25)
540 DIM GROUP2# (25)
550 DIM GROUP3# (25)
560 DIM GROUP4# (25)
570 DIM R (25)
580 DIM STACK (3)
590 DIM STACK2 (25)
600 DIM HBAR(3,4)
610 PI = 3.141593
620 ' Weights for the hierarchical level geographic spacings (on the map).
630 RM(4) = 1
640 RM(3) = SQR(3)
650 RM(2) = 3
660 RM(1) = 3 * SQR(3)
670 '
680 ' Reading in data, and initial input and setup error checking.
690 '
700 GOSUB 8260      ' Initializing the variables with default values.
710 CLS
720 ' Questioning user for initial conditions information.
730 PRINT "Do you want to:"
740 PRINT "     1) Use the example data file contained in the program?"
750 PRINT "     2) Enter your own data?"
760 PRINT "     3) Use a data file available on the disk?"
770 PRINT
780 PRINT "(NOTE:  smaller integral values of nu reduce the run time.)"
790 INPUT OT
800 IF OT <> 1 AND OT <> 2 AND OT <> 3 THEN 730
810 IF OT = 1 THEN GOSUB 7110 : GOTO 1340     ' End of keyboard input;
                                                  7110 reads in the example
file.
```

```
820 IF OT = 3 THEN GOSUB 8790 : GOTO 1340      ' End of keyboard input;
                                                  8790 reads in a file from
disk.
830 CLS
840 PRINT "Enter the number of commodities (between 1 and 19): ";
850 INPUT CN
860 IF CN >= 1 AND CN <= 19 THEN 890
870 PRINT "Between 1 and 19"
880 GOTO 850
890 '
900 CLS
910 GOSUB 4960     ' Initializing the commodity matrix with default values.
920 FOR I = 1 TO CN
930    PRINT "Enter commodity number: ";
940    INPUT CC
950    IF CC <= 19 AND CC >= 1 THEN 980
960    PRINT "Between 1 and 19, please."
970    GOTO 930
980    IF CM (CC,1) = 0 THEN 1010
990    PRINT "That number has already been taken."
1000    GOTO 930
1010    CM (CC,1) = CC
1020    PRINT "Enter the variable cost (between 0 and 312): ";
1030    INPUT CM(CC,2)      ' cv
1040    IF CM (CC,2) <= 312 AND CM (CC,2) >=0 THEN 1070
1050    PRINT "Between 0 and 312, please."
1060    GOTO 1030
1070    PRINT "Enter the fixed cost (between 0 and 3600): ";
1080    INPUT CM(CC,3)      ' cf
1090    IF CM (CC,3) <= 3600 AND CM (CC,3) >= 0 THEN 1120
1100    PRINT "Between 0 & 3600, please."
1110    GOTO 1080
1120    IF CM(CC,2) = 0 AND CM(CC,3) = 0 THEN PRINT "Both cannot equal zero.
Please re-enter" : GOTO 1080
1130    PRINT "Enter the elasticity of demand (between 0 and 5): "
1140    INPUT CM(CC,4)      ' nu
1150    Z=CM (CC,4)
1160    IF Z => 0 AND Z <= 5 THEN 1190
1170    PRINT "Between 0 and 5, please."
1180    GOTO 1140
1190 CLS
1200 NEXT
1210 CLS
1220 INPUT "Enter the density: ";A
1230 IF A > 0 THEN 1260
1240 PRINT "The density must be > 0."
1250 GOTO 1220
1260 INPUT "Enter the transport rate: ";T
1270 IF T > 0 THEN 1300
1280 PRINT "The transport rate must be > 0."
1290 GOTO 1260
1300 CLS
1310 INPUT "Do you want to specify means and standard deviations of variables,
```

```
matrix of     z - score group means, and inverse z - score covariance matrix?
(y/n) ",YN$
1320 IF YN$ = "y" OR YN$ = "Y" THEN GOSUB 8530
1330 CLS
1340 LOCATE 12,29
1350 PRINT "PROCESSING; PLEASE WAIT!"
1360 FOR NOTE = 1 TO 10
1370 LOCATE 20,1
1380 PRINT "ThE nOn-NeStEd AnD nEsTeD DeMaNdS mAy NoT Be EqUaL bEcAuSE tHe
FiRsT wAs DoNe   By ExTeRnAl NuMeRiCaL InTeGrAtIoN oN a VaX aNd ThE SeCoNd By
InTeRnAl NuMeRiCaL InTeGrAtIoN iN BaSiC."
1390 LOCATE 20,1
1400 PRINT "tHe NoN-nEsTeD aNd NeStEd dEmAnDs MaY nOt bE eQuAl BeCaUse ThE
fIrSt WaS dOnE   bY eXtErNaL nUmErIcAl iNtEgRaTiOn On A vAx AnD tHe sEcOnD bY
iNtErNaL nUmErIcAl iNtEgRaTiOn In bAsIc!"
1410 NEXT
1420 LOCATE 20,1
1430 PRINT "THE NON-NESTED AND NESTED DEMANDS MAY NOT BE EQUAL BECAUSE THE
FIRST WAS DONE   BY EXTERNAL NUMERICAL INTEGRATION ON A VAX AND THE SECOND BY
INTERNAL NUMERICAL INTEGRATION IN BASIC."
1440 '
1450 ' Grouping based upon the attributes of the commodities; the hierarchical
        classification of commodities is conducted using a multinomial logit
         model.
1460 '
1470 G1 = 1 : G2 = 1 : G3 = 1 : G4 = 1
1480 FOR I = 1 TO 25
1490    IF CM (I,1) = 0 THEN 1770
1500 ' Calculating z - scores for the 3 variables based upon the sample data.

1510 Z(2) = (CM(I,3) - MEAN(2)) / SDV(2)
1520 Z(1) = (CM(I,2) - MEAN(1)) / SDV(1)
1530 Z(3) = (CM(I,4) - MEAN(3)) / SDV(3)
1540 ' Calculating the probability of hierarchical level membership using the
         multinomial logit model, the matrix of hierarchical level group
          means and the covariance matrix for the 3 variables.
1550    SUMT = 0
1560    FOR K = 1 TO 4
1570      SUM(K) = 0
1580      FOR ID = 1 TO 3
1590       FOR JD = 1 TO 3
1600        SUM(K) = SUM(K) + (Z(ID) - HBAR(ID,K)) * COV (ID,JD) * (Z(JD) -
HBAR (JD,K))
1610       NEXT
1620      NEXT
1630    SUM(K) = EXP (-1 * (SUM(K) - 2 * LOG (PROB (K))) / 2)
1640    SUMT = SUMT + SUM(K)
1650   NEXT
1660 ' Classification of a commodity into a hierarchical level group on the
         basis of the largest probability.
1670   IF SUMT = 0 THEN SUMT = 1
1680   BIG = -999999!
1690    FOR ID = 1 TO 4
```

```
1700      PHAT(ID) = SUM (ID) / SUMT
1710      IF PHAT(ID) < BIG THEN 1740
1720        BIG = PHAT(ID)
1730        CM(I,5) = ID
1740    NEXT
1750    BB = CM(I,5)
1760    IF BB = 1 THEN G1 = G1 + 1 ELSE IF BB = 2 THEN G2 = G2 + 1 ELSE IF BB
= 3 THEN G3 = G3 + 1 ELSE G4 = G4 + 1
1770 NEXT
1790 '
1800 ' Printing the caption for the commodity table.
1810 CLS
1820 PRINT "L" TAB(3) "Cv" TAB(7) "Cf" TAB(12) "Nu" TAB(17) "non-nested"
TAB(29) "non-nested" TAB(44) "nested" TAB(58) "nested" TAB(71) "price"
1830 PRINT TAB(17) "radius" TAB(29) "demand" TAB(44) "radius" TAB(58) "demand"

1840 PRINT
"-------------------------------------------------------------------------------
--"
1850 '
1860 ' Calculating the individual commodity thresholds and their global
minimum.
1880 TOP=0
1890 GR = 1     ' Picking the highest hierarchical level number with at least
                  one commodity in it.
1900 IF G2 > 1 THEN GREAT = G2 : GR = 2
1910 IF G3 > 1 THEN GREAT = G3 : GR = 3
1920 IF G4 > 1 THEN GREAT = G4 : GR = 4
1930 FOR I = CN TO 1 STEP - 1
1940 ' PRINT I,CM(I,2),CM(I,3),R
1950    N = CM(I,4)
1960    IF CM (I,5) <> GR THEN 2750
1970    IF CM(I,3) <> 0 THEN 2010
1980    R = CM(I,2) / T
1990    GOTO 2670
2010 IF N <> 0 THEN 2080
2015 '
2020 ' Finding the smallest real radius, if one exists, for nu = 0.
2030      Q = -1 * ( CM(I,3) / ( 2 * SQR (3) * A * T)) - ( 2 * CM (I,2)^3) /
(27 * T^3)
2040      P = -1 * CM (I,2)^2 / (3 * T^2)
2050      B = (-1 * Q / 2 - SQR (Q^2 / 4 + P^3 / 27))^(1/3)
2060      FISH = (-1 * Q / 2 + SQR ( Q^2 / 4 + P^3 / 27))^(1/3)
2070      R = FISH + B + CM(I,2) / (3 * T)
2080 IF N <> 1 THEN 2120
2085 '
2090 ' Finding the smallest real radius, if one exists, for nu = 1.
2100      R = ( CM(I,2) / T + SQR ((CM(I,2) / T)^2 + CM(I,3) / (3 * A *
.1738034))) / 2
2120 IF N <> 2 THEN 2160
2125 '
2130 ' Finding the smallest real radius, if one exists, for nu = 2.
2140      R = (CM(I,3) * T) / (12 * A * .1075536) + CM (I,2) / T
```

```
2160 IF N <> 3 THEN 2200
2165 '
2170 ' Finding the smallest real radius, if one exists, for nu = 3.
2180      R = CM(I,2) / (T - (CM(I,3) * T^3) / (6 * A * .1371809))
2200 IF N <> 4 THEN 2290
2205 '
2210 ' Finding the smallest real radius, if one exists, for nu = 4; the
solution        is from the "Theory of Equations."
2220      SQ = (12 * A / (T^3 * CM(I,3)) * .0451258)^2  - 4 * (12 * A *
CM(I,2) / (T^4 * CM (I,3))) * .0451258
2230      IF (3 * A * .0451258 / (T^2 * CM(I,3))) < CM(I,2) THEN R = 1000000!
: GOTO 2290
2240      R1 = (12 * A / (T^3 * CM(I,3)) * .0451258 + SQR (SQ)) / 2
2250      R2 = (12 * A / (T^3 * CM(I,3)) * .0451258 - SQR (SQ)) / 2
2260      IF R1 > CM(I,2) AND R1 < R2 THEN R = R1 ELSE R = R2
2270      IF CM(I,2) = 0 THEN R = (T^(N - 1) * D / CM(I,3)) ^ (1/3)
2290 IF N <> 5 THEN 2560
2295 '
2300 ' Finding the smallest real radius, if one exists, for nu = 5; the
solution        is from the "Theory of Equations."
2310      IF (27 * CM(I,2)^2 - 48 * A* .0306553 / (T^2 * CM(I,3))) > 0 THEN
2590
2320      PI = 3.14159265#
2330      E = -1 * 12 * A / (T^4 * CM (I,3)) * .0306553
2340      Q = 12 * A * CM(I,2) / (T^5 * CM (I,3)) * .0306553
2350      DELTA = -1 * E * SQR ( -1 * 4 * E - 27 * ( CM(I,2) / T)^2)
2360      PHI = ATN  (-1 * DELTA / (Q * SQR (27))) * (180 / PI)
2370      IF PHI < 0 THEN PHI = PHI + 360
2380      IF PHI < 90 THEN THETA = PHI + 90 ELSE IF PHI < 180 THEN THETA = PHI
ELSE IF PHI < 270 THEN THETA = PHI - 90 ELSE THETA = PHI + 180
2390      Y1 = 2 * SQR (-1 * E / 3) *  COS (THETA / 3 * (PI / 180))
2400      Y2 = 2 * SQR (-1 * E / 3) * COS ((THETA / 3 + 120) * (PI / 180))
2410      Y3 = 2 * SQR (-1 * E / 3) * COS ((THETA / 3 + 240) * (PI / 180))
2420      TEMP = CM(I,2) / T
2430      TOP = 0
2440      IF Y1 >= TEMP THEN TOP = TOP + 1 : STACK(TOP) = Y1
2450      IF Y2 >= TEMP THEN TOP = TOP + 1 : STACK(TOP) = Y2
2460      IF Y3 >= TEMP THEN TOP = TOP + 1 : STACK(TOP) = Y3
2470      TEMP = STACK (TOP)
2480      IF STACK (TOP) < TEMP THEN TEMP = STACK (TOP)
2490      TOP = TOP - 1
2500      IF TOP <> 0 THEN 2480
2510 R=TEMP
2520      IF R < 0 THEN R = 1000000!
2540      IF CM(I,2) = 0 THEN R = (T^(N - 1) * D / CM(I,3)) ^ (1/4)
2560      IF N <> INT(N) THEN GOSUB 7670 ' Determining if nu is an integer.
2570 GOTO 2650
2575 '
2580 ' The solution is from the "Theory of Equations."
2590 TERM1 = (12 * A * .0306553 / (T^5 * CM(I,3))) * (-1 * CM(I,2) / 2 + SQR
(CM(I,2)^2 / 4 - 12 * A * .0306553 / (27 * CM(I,3) * T^2)))
2600 MULT = SGN(TERM1)
2610 TERM2 = (12 * A * .0306553 / (T^5 * CM(I,3))) * (CM(I,2) / 2 + SQR
```

```
(CM(I,2)^2 / 4 - 12 * A * .0306553 / (27 * CM(I,3) * T^2)))
2620 MULT2 = SGN(TERM2)
2630 R = MULT * ((MULT * TERM1) ^ (1/3)) - MULT2 * ((MULT2 * TERM2) ^ (1/3))
2640      IF R < 0 THEN R = 1000000!
2650 '
2660 ' Picking the minimum radius for the lowest hierarchical level.
2670 TOP = TOP + 1
2680 STACK2(TOP) = R
2690 IF STACK2(TOP) < MIN AND STACK2(TOP) > 0 THEN MIN = STACK2(TOP)
2700 TOP = TOP - 1
2710 RMIN = MIN
2720 '
2750 NEXT
2751 ' Calculating the radius, demand, and price for a given minimum radius
        (rmin).
2755 '
2760 ' Calculating nested hierarchical distances based upon the minimum
radius.
2770 FOR I = 1 TO CN
2780   RMIN = MIN * RM(CM(I,5) + (4 - GR))
2790   N = CM(I,4)
2800   IF CM(I,3) <> 0 THEN 2830
2810     R = CM(I,2) / T
2820     P = CM(I,2)
2830 MULT = SQR (3) * SQR ((GR - CM(I,5))^2)
2840 IF MULT = 0 THEN MULT = 1
2850 IF N <> 0 THEN 3010
2855 '
2860 ' Finding radius, price, and demand for nu = 0 and the minimum radius.
2870      IF CM(I,3) = 0 THEN 2940
2880      Q = -1 * ( CM(I,3) / ( 2 * SQR (3) * A * T)) - ( 2 * CM (I,2)^3) /
(27 * T^3)
2900      P = -1 * CM (I,2)^2 / (3 * T^2)
2910      B = -1 * Q / 2 - (1/(6*T))*SQR(CM(I,3)/A)*SQR (3*CM(I,3)/(4*A) +
2*CM(I,2)^3/(3*SQR(3)*T^2))
2911      B = SGN(B)*(ABS(B))^(1/3)
2920      FISH = (-1 * Q / 2 + (1/(6*T))*SQR(CM(I,3)/A)*SQR (3*CM(I,3)/(4*A) +
2*CM(I,2)^3/(3*SQR(3)*T^2)))^(1/3)
2930      R = FISH + B + CM(I,2) / (3 * T)
2940      D = 2 * SQR (3) * A * R^2
2950      IF CM(I,2) = 0 THEN R = CM(I,3) / (T^(1 - N) * D)
2960      D2# = 2 * SQR (3) * A * RMIN^2
2970      IF CM(I,3) = 0 THEN 3010
2980      P = CM(I,2) + CM(I,3) / (2 * SQR(3) * A * RMIN^2)
3010 IF N <> 1 THEN 3300
3013 '
3016 ' Finding radius, price, and demand for nu = 1 and the minimum radius;
the        solution uses numerical root extraction.
3020      IF CM(I,3) = 0 THEN 3170
3030      DIF = .0001
3040      COU = 0
3050      IF CM(I,2) <> 0 THEN P = CM(I,2) ELSE P = 5
3060      R = ( CM(I,2) / T + SQR ((CM(I,2) / T)^2 + CM(I,3) / (3 * A *
```

```
.1738034))) / 2
3070       SUM# = 0
3080       FOR J = 1 TO 19
3090       SUM# = SUM# + LOG (P * COS (PI * J / 120) + RMIN * T)
3100       NEXT
3110       SUM2# = 0
3120       FOR J = 1 TO 19 STEP 2
3130       SUM2# = SUM2# + LOG (P * COS (PI * J / 120) + RMIN * T)
3140       NEXT
3150       SUM# = 2 * SUM#
3160       SUM2# = SUM2# * 2
3170       D = 12 * A * R / T * .1738034
3180       IF CM(I,2) = 0 THEN R = 51850!
3190       D2# = 12 * A / T * (RMIN * LOG (TAN (PI / 3)) + PI / 6 * (P / T) *
LOG (P) - P / T * 2.461715E-02 - P / T * (PI / 120 / 3) * (LOG (P + RMIN * T)
+ LOG (P * SQR (3) / 2 + RMIN * T) + SUM# + SUM2#))
3200       IF CM(I,3) = 0 THEN 3300
3210       PLAG# = P#
3220       P# = CM(I,3) /  D2# + CM(I,2)
3230       P = CSNG (P#)
3240       IF D2# < .00001 THEN P = -1
3250       IF P < 0 THEN 3070
3260       IF COU = 25 THEN DIF = DIF * 10 : COU = 0
3270       COU = COU + 1
3280       IF ABS (PLAG# - P#) >= DIF THEN 3070
3300 IF N <> 2 THEN 3690
3305 '
3310 ' Finding radius, price, and demand for nu = 2 and the minimum radius;
the          solution uses numerical root extraction.
3320       IF CM(I,3) = 0 THEN 3560
3330       DIF = .0001
3340       COU = 0
3350       P = CM(I,2)
3360       R = (CM(I,3) * T) / (12 * A * .1075536) + CM (I,2) / T
3370       IF CM(I,2) = 0 THEN R = T^(N - 1) * D / CM(I,3)
3380       D2 = D
3390       SUM# = 0 : SUM2# = 0
3400       FOR J = 1 TO 19
3410       SUM# = SUM# + LOG (P * COS (PI * J / 120) + RMIN * T)
3420       NEXT
3430       SUM2# = 0
3440       FOR J = 1 TO 19 STEP 2
3450       SUM2# = SUM2# + LOG (P * COS (PI * J / 120) + RMIN * T)
3460       NEXT
3470       SUM# = 2 * SUM#
3480       SUM2# = SUM2# * 2
3490       IF RMIN * T / P <> 1 THEN 3510
3500          TERM = TAN (PI / 12)
3510       IF RMIN * T / P >= 1 THEN 3530
3520          TERM = 1 / SQR (P^2 - (RMIN * T)^2) * LOG ((SQR ((P + RMIN * T) /
(P - RMIN * T)) + TAN (PI / 12)) / (SQR ((P + RMIN * T) / (P - RMIN * T)) -
TAN (PI / 12)))
3530       IF RMIN * T / P <= 1 THEN 3550
```

```
3540          TERM = 2 / SQR ((RMIN * T)^2 - P^2) * ATN (SQR ((RMIN * T - P) /
(RMIN * T + P)) * TAN (PI / 12))
3550 '
3560      D = 12 * A / T^2 * .1075536
3570      D2# = (12 * A / T^2) * (-1 * RMIN * T * TERM + (PI / 120 / 3) * (LOG
(P + RMIN * T) + LOG (P * SQR (3) / 2 + RMIN * T) + SUM# + SUM2#) +
2.461715E-02 - (PI / 6) * LOG (P))
3580      IF CM(I,3) = 0 THEN 3690
3590      PLAG# = P#
3600      P# = CM(I,3) /  D2# + CM(I,2)
3610      P = CSNG (P#)
3620      IF D2# < .00001 THEN P = -1
3630      IF P < 0 THEN 3670
3640      IF COU = 25  THEN DIF = DIF * 10 : COU = 0
3650      COU = COU + 1
3660      IF ABS(PLAG# - P#) >= DIF THEN 3390
3670 '
3680 ' Finding radius, price, and demand for nu = 3, 4, 5, or non-integer
      values, with the minimum radius; the solution uses numerical
integration       and numerical root extraction.
3690 IF N = 0 OR N = 1 OR N = 2 THEN 4570
3700 GOTO 4010
3710 ILC=0
3720      P = CM(I,2)
3730      IF CM(I,2) = 0 THEN P = R * T
3735 P# = P
3740      DIFF = .001
3750 SUM1 = 0 : SUM2 = 0
3760    FOR U = 1 TO 19
3770       SUM1 = SUM1 + (1 + RMIN * T / (P * COS (PI * U / 120))) ^ (2 - N)
3780       IF (U + 1) / 2 = INT ((U + 1) / 2) THEN SUM2 = SUM2 + (1 + RMIN * T
/ (P * COS (PI * U / 120))) ^ (2 - N)
3790    NEXT
3800       TERM1 = PI / 120 / 3 * ((1 + RMIN * T / P) ^ (2 - N) + (1 + 2 *
RMIN * T / (SQR(3) * P)) ^ (2 - N) + SUM1 * 2 + 2 * SUM2)
3810       SUM1 = 0 : SUM2 = 0
3820    FOR U = 1 TO 19
3830       SUM1 = SUM1 + (1 + RMIN * T / (P * COS (PI * U / 120))) ^ (1 - N)
3840       IF (U + 1) / 2 = INT ((U + 1) / 2) THEN SUM2 = SUM2 + (1 + RMIN * T
/ (P * COS (PI * U / 120))) ^ (1 - N)
3850    NEXT
3860       TERM2 = PI / 120 / 3 * ((1 + RMIN * T / P) ^ (1 - N) + (1 + 2 *
RMIN * T / (SQR(3) * P)) ^ (1 - N) + SUM1 * 2 + 2 * SUM2)
3870       D2# = P ^ (2 - N) * 12 * A / T^2 * (1 / (2 - N) * TERM1 + 1 / (N -
1) * TERM2 + PI / 6 * (1 / ((2 - N) * (1 - N))))
3880 IF D2# > DIFF THEN 3910
3890 P = -1
3900 GOTO 3990
3910 EQ# = D2# * (P - CM(I,2)) - CM(I,3)
3920 IF ABS (EQ#) < DIFF THEN 3990
3930     P# = CM(I,3) / D2# + CM(I,2)
3940     P = CSNG(P#)
3950 IF P < DIFF THEN 3890
```

```
3960 ILC = ILC + 1
3970 IF ILC > 10 THEN DIFF = DIFF * 10: ILC = 0
3980 GOTO 3750
3990 GOTO 4570
3995 '
4000 ' Finding radius, price, and demand.
4010 IF N <> INT (N) THEN GOSUB 7670     ' For decimal nu values.
4020 '
4030 ' Finding radius, price, and demand for nu = 3; unconstrained solution.
4040   IF N <> 3 THEN 4100
4050   IF CM(I,3) = 0 THEN 4070
4060     R = CM(I,2) / (T - (CM(I,3) * T^3) / (6 * A * .1371809))
4070     D = 6 * A / (R * T^3) * .1371809
4080     IF CM(I,2) = 0 THEN R = SQR (T^(N - 1) * D / CM(I,3))
4085 '
4090 ' Finding radius, price, and demand for nu = 4; unconstrained solution.
4100   IF N <> 4 THEN 4200
4110   IF CM(I,3) = 0 THEN 4170
4120     SQ = (12 * A / (T^3 * CM(I,3)) * .0451258)^2  - 4 * (12 * A * CM(I,2)
/ (T^4 * CM (I,3))) * .0451258
4130     IF (3 * A * .0451258 / (T^2 * CM(I,3))) < CM(I,2) THEN R = 1000000! :
GOTO 4170
4140     R1 = (12 * A / (T^3 * CM(I,3)) * .0451258 + SQR (SQ)) / 2
4150     R2 = (12 * A / (T^3 * CM(I,3)) * .0451258 - SQR (SQ)) / 2
4160     IF R1 > CM(I,2) AND R1 < R2 THEN R = R1 ELSE R = R2
4170     D = 12 * A / (T^4 * R^2) * .0451258
4180     IF CM(I,2) = 0 THEN R = (T^(N - 1) * D / CM(I,3)) ^ (1/3)
4185 '
4190 ' Finding radius, price, and demand for nu = 5; unconstrained solution;
     solution using numerical integration and numerical root extraction.
4200   IF N <> 5 THEN 4550
4210   IF CM(I,3) = 0 THEN 4440
4220     IF (27 * CM(I,2)^2 - 48 * A* .0306553 / (T^2 * CM(I,3))) > 0 THEN
4490
4230     PI = 3.14159265#
4240     E = -1 * 12 * A / (T^4 * CM (I,3)) * .0306553
4250     Q = 12 * A * CM(I,2) / (T^5 * CM (I,3)) * .0306553
4260     DELTA = -1 * E * SQR ( -1 * 4 * E - 27 * ( CM(I,2) / T)^2)
4270     PHI = ATN  (-1 * DELTA / (Q * SQR (27))) * (180 / PI)
4280     IF PHI < 0 THEN PHI = PHI + 360
4290     IF PHI < 90 THEN THETA = PHI + 90 ELSE IF PHI < 180 THEN THETA = PHI
ELSE IF PHI < 270 THEN THETA = PHI - 90 ELSE THETA = PHI + 180
4300     Y1 = 2 * SQR (-1 * E / 3) *  COS (THETA / 3 * (PI / 180))
4310     Y2 = 2 * SQR (-1 * E / 3) * COS ((THETA / 3 + 120) * (PI / 180))
4320     Y3 = 2 * SQR (-1 * E / 3) * COS ((THETA / 3 + 240) * (PI / 180))
4330     TEMP = CM(I,2) / T
4340     TOP = 0
4350     IF Y1 >= TEMP THEN TOP = TOP + 1 : STACK(TOP) = Y1
4360     IF Y2 >= TEMP THEN TOP = TOP + 1 : STACK(TOP) = Y2
4370     IF Y3 >= TEMP THEN TOP = TOP + 1 : STACK(TOP) = Y3
4380     TEMP = STACK (TOP)
4390     IF STACK (TOP) < TEMP THEN TEMP = STACK (TOP)
4400     TOP = TOP - 1
```

```
4410      IF TOP <> 0 THEN 4390
4420 R=TEMP
4430      IF R < 0 THEN R = 1000000!
4440      D = 12 * A / (T^5 * R^3) * .0306553
4450      IF CM(I,2) = 0 THEN R = (T^(N - 1) * D / CM(I,3)) ^ (1/4)
4470 GOTO 4550
4480 '
4490 TERM1 = (12 * A * .0306553 / (T^5 * CM(I,3))) * (-1 * CM(I,2) / 2 + SQR
(CM(I,2)^2 / 4 - 12 * A * .0306553 / (27 * CM(I,3) * T^2)))
4500 MULT = SGN(TERM1)
4510 TERM2 = (12 * A * .0306553 / (T^5 * CM(I,3))) * (CM(I,2) / 2 + SQR
(CM(I,2)^2 / 4 - 12 * A * .0306553 / (27 * CM(I,3) * T^2)))
4520 MULT2 = SGN(TERM2)
4530 R = MULT * ((MULT * TERM1) ^ (1/3)) - MULT2 * ((MULT2 * TERM2) ^ (1/3))
4540      IF R < 0 THEN R = 1000000!
4550 GOTO 3710
4570 '
4580 ' Printing a line of the table.
4600 D3 = D2#
4610 PRINT RIGHT$(STR$(CM(I,5)),1) TAB(2) STR$ (CM(I,2)) TAB(6) STR$ (CM(I,3))
TAB(11) STR$ (INT (CM(I,4) * 100 + .5) / 100) TAB(16)
4620 IF R = 51850! THEN PRINT "   arb" TAB(28) "     *"; : GOTO 4640
4630 IF R = 1000000! THEN PRINT "     *" TAB(28) "     *"; ELSE PRINT INT(R *
100 + .5) / 100 TAB(28) INT(D * 100 + .5)/ 100;
4640 IF RMIN <> 1E+09 AND RMIN <> 1000000! THEN PRINT TAB(43) INT (RMIN * 100
+ .5) / 100; ELSE PRINT TAB(43) "   *";
4650 IF D3 <.01 THEN P = -1
4660 IF P < 0 THEN PRINT TAB(57) "     *" TAB(70) "  *"
4670 IF P >= 0 THEN PRINT TAB(57) STR$ (INT (D3 * 100 + .5) / 100) TAB(70) INT
(P * 100 + .5) / 100 : LINN(CM(I,5)) = 1
4680 IF PUTZ > 18 THEN PRINT
4690 PUTZ = PUTZ + 1
4700 IF PUTZ < 19 THEN 4810 ELSE IF PUTZ > 19 THEN 4760
4710 LOCATE 23,18
4720 PRINT "<<< Press any key to continue. >>>"
4730 CARA$ = INKEY$
4740 IF CARA$ = "" THEN 4730
4750 LOCATE 23,18 : PRINT "                                        "
4760 LOCATE 1,1
4770 PRINT "L" TAB(3) "Cv" TAB(7) "Cf" TAB(12) "Nu" TAB(17) "non-nested"
TAB(29) "non-nested" TAB(44) "nested" TAB(58) "nested" TAB(71) "price"
4780 PRINT TAB(17) "radius" TAB(29) "demand" TAB(44) "radius        ";" demand
          ";
4790 PRINT
"--------------------------------------------------------------------------
--";
4800 LOCATE 23,1
4810 NEXT I
4820 INPUT "Do you want a map? ('y' or 'n')";YN$
4830 IF YN$ = "n" OR YN$ = "N" THEN 4850 ELSE IF YN$ <> "y" AND YN$ <> "Y"
THEN 4820      ' Mapping.
4840 GOSUB 5480
4850 LOCATE 23,1
```

```
4860 PRINT "                    <<< Press 'r' to run, or any other key to end. >>>"

4870 CARA$ = INKEY$
4880 IF CARA$ = "" THEN 4870
4890 CLS
4900 IF CARA$ = "r" OR CARA$ = "R" THEN RUN
4910 LOCATE 11,26
4920 PRINT "Thank you; have a nice day!"
4930 LOCATE 22,1
4940 KEY ON
4950 END
4960 '
4970 ' Initializing the commodity matrix.
4990 FOR T = 1 TO 25
5000    FOR I = 1 TO 5
5010            CM (T,I) = 0
5020    NEXT
5030 NEXT
5040 RETURN
5050 '
5060 ' Introduction.
5080 CLS
5090 PRINT : PRINT
5100 PRINT "A"
5110 PRINT
5120 CARA$ = "Darren Griffith"
5130 FOR OO = 1 TO 15
5140 PRINT LEFT$ (CARA$,OO)
5150 NEXT
5160 PRINT
5170 PRINT "Program."
5180 G2$ = "Math/debugging consultation by Daniel A. Griffith."
5190 LOCATE 1,1
5200 FOR I=1 TO 52
5210 PRINT MID$ (G2$,I,1);
5220 FOR CARA = 1 TO 25
5230 NEXT
5240 NEXT
5250 LOCATE 5,35
5260 PRINT "Created May 14, 1988."
5270 LOCATE 6,35
5280 PRINT "Version";VERSION
5290 Q = 35
5300 LOCATE 10,35
5310 PRINT "Funded by:  "
5320 FUN$ = "SUNY/Buffalo"
5330 FOR L = 1 TO 12
5340 IF Q / 2 = INT(Q / 2) THEN P = 2 ELSE P = 1
5350 FOR W = 78 TO Q STEP -P
5360   LOCATE 12,W
5370   PRINT MID$(FUN$,L,1)
5380   LOCATE 12,W + P
5390   PRINT " "
```

```
5400 NEXT
5410 Q = Q + 1
5420 NEXT
5430 LOCATE 12,48
5440 PRINT "Office of Teaching Effectiveness."
5450 FOR OO = 1 TO 1500 : NEXT
5460 RETURN
5470 '
5480 ' Mapping of central place structures.
5500 CLS
5510 IF LINN(1) = 0 THEN G1 = 0
5520 IF LINN(2) = 0 THEN G2 = 0
5530 IF LINN(3) = 0 THEN G3 = 0
5540 IF LINN(4) = 0 THEN G4 = 0
5550 GSUM=0
5560 IF G1 > 1 THEN GSUM = GSUM + 1
5570 IF G2 > 1 THEN GSUM = GSUM + 1
5580 IF G3 > 1 THEN GSUM = GSUM + 1
5590 IF G4 > 1 THEN GSUM = GSUM + 1
5600 '
5610 ' Setting values of central place map locations.
5630  CITY$ = ""
5640  TOWN$ = ""
5650  VILL$ = ""
5660  HAM$ = ""
5670 IF GSUM <> 4 THEN 5740
5680  CITY$ = "@"
5690  TOWN$ = "*"
5700  VILL$ = "#"
5710  HAM$ = "."
5720 GOTO 6250
5730 '
5740 IF GSUM <> 3 THEN 5970
5750 IF G1 > 1 THEN 5800
5760  CITY$ = "*"
5770  TOWN$ = "*"
5780  VILL$ = "#"
5790  HAM$ = "."
5800 IF G2 > 1 THEN 5850
5810  CITY$ = "@"
5820  TOWN$ = "#"
5830  VILL$ = "#"
5840  HAM$ = "."
5850 IF G3 > 1 THEN 5900
5860  CITY$ = "@"
5870  TOWN$ = "*"
5880  VILL$ = "."
5890  HAM$ = "."
5900 IF G4 > 1 THEN 5950
5910  CITY$ = "@"
5920  TOWN$ = "*"
5930  VILL$ = "#"
5940  HAM$ = "#"
```

```
5950 '
5960 GOTO 6250
5970 IF GSUM <> 2 THEN 6190
5980 IF G1 <= 1 THEN 6030
5990  CITY$ = "@"
6000  TOWN$ = "@"
6010  VILL$ = "@"
6020  HAM$ = "*"
6030 IF G2 <= 1 THEN 6080
6040  IF G1 <= 1 THEN CITY$ = "*"
6050  TOWN$ = "*"
6060  VILL$ = "*"
6070  HAM$ = "*"
6080 IF G3 <= 1 THEN 6130
6090  IF G1 <= 1 AND G2 <= 1 THEN CITY$ = "#"
6100  IF G2 <= 1 THEN TOWN$ = "#"
6110  VILL$ = "#"
6120  HAM$ = "#"
6130 IF G4 <= 1 THEN 6170
6140  IF G3 <= 1 AND G2 <= 1 THEN TOWN$ = "."
6150  IF G3 <= 1 THEN VILL$ = "."
6160  HAM$ = "."
6170 '
6180 GOTO 6250
6190 IF GSUM <> 1 THEN 6250
6200 IF G4 > 1 THEN THING$ = "." ELSE IF G3 > 1 THEN THING$ = "#" ELSE IF G2 >
1 THEN THING$ = "*" ELSE THING$ = "@"
6210 CITY$ = THING$
6220 TOWN$ = THING$
6230 VILL$ = THING$
6240 HAM$ = THING$
6250 ' End of IF statement.
6260 LOCATE 1,40
6270 PRINT CITY$
6280 LOCATE 1,1
6290 PRINT TOWN$
6300 LOCATE 1,79
6310 PRINT TOWN$
6320 LOCATE 16,16
6330 PRINT TOWN$
6340 LOCATE 16,62
6350 PRINT TOWN$
6360 LOCATE 6,19
6370 PRINT VILL$
6380 LOCATE 11,40
6390 PRINT VILL$
6400 LOCATE 6,60
6410 PRINT VILL$
6420 LOCATE 1,13
6430 PRINT HAM$
6440 LOCATE 1,26
6450 PRINT HAM$
6460 LOCATE 1,53                          ' Printing the map.
```

```
6470 PRINT HAM$
6480 LOCATE 1,66
6490 PRINT HAM$
6500 LOCATE 6,6
6510 PRINT HAM$
6520 LOCATE 6,33
6530 PRINT HAM$
6540 LOCATE 6,46
6550 PRINT HAM$
6560 LOCATE 6,74
6570 PRINT HAM$
6580 LOCATE 11,12
6590 PRINT HAM$
6600 LOCATE 11,25
6610 PRINT HAM$
6620 LOCATE 11,54
6630 PRINT HAM$
6640 LOCATE 11,67
6650 PRINT HAM$
6660 LOCATE 16,33
6670 PRINT HAM$
6680 LOCATE 16,46
6690 PRINT HAM$
6700 PRINT
"
______________________________________________________________________________"
6710 IF GSUM = 1 THEN 6980
6720 IF G1 <= 1 THEN 6740
6730 PRINT "The distance between cities: "; 2 * RM (1 + (4-GR)) * MIN
6740 IF G2 <= 1 THEN 6760
6750 PRINT "The distance between towns: "; 2 * RM (2 + (4 - GR)) * MIN
6760 IF G3 <= 1 THEN 6780
6770 PRINT "The distance between villages: "; 2 * RM (3 + (4 - GR)) * MIN
6780 IF G4 <= 1 THEN 6800
6790 PRINT "The distance between hamlets: "; 2 * RM (4 + (4 - GR)) * MIN
6800 LO = 19
6810 IF G1 <= 1 THEN 6850
6820  LOCATE LO,60
6830  PRINT "@";" - city"
6840  LO = LO + 1
6850 IF G2 <= 1 THEN 6890
6860  LOCATE LO,60
6870  LO = LO + 1
6880  PRINT "* - town"
6890 IF G3 <= 1 THEN 6930
6900  LOCATE LO,60
6910  PRINT "# - village"
6920  LO = LO + 1
6930 IF G4 <= 1 THEN 6960
6940  LOCATE LO,60
6950  PRINT ". - hamlet"
6960 RETURN
6970 '
```

```
6980 CLASS$(4) = "hamlets"
6990 CLASS$(3) = "villages"
7000 CLASS$(2) = "towns"
7010 CLASS$(1) = "cities"
7020 IF G4 > 1 THEN CLASS = 4 ELSE IF G3 > 1 THEN CLASS = 3 ELSE IF G2 > 1
THEN CLASS = 2 ELSE CLASS = 1
7030 PRINT "The distance between ";CLASS$(CLASS);" is: ";
7040 PRINT 2 * MIN
7050 GOTO 6800
7060 '
7070 LOCATE 20,60
7080 PRINT THING$;" - ";CLASS$(CLASS)
7090 GOTO 6960
7100 CLS
7110 '
7120 ' Input example file.
7140 GOSUB 4960
7150 CN = 23
7160 FOR CO1 = 1 TO CN
7170   CM (CO1,1) = CO1
7180   FOR CO2 = 2 TO 5
7190      READ NUMBER
7200      CM (CO1,CO2) = NUMBER
7210   NEXT
7220 NEXT
7230 CLS
7240 PRINT "Density:  500"
7250 PRINT "Transport rate:  .01"
7260 A = 500
7270 T = .01
7280 RETURN
7290 '
7300 ' <<<<<<<<<<<< Covariance matrix for variables by levels data.
>>>>>>>>>>>>
7320 DATA 3.00140, -1.83447,  .04961
7330 DATA -1.83447, 2.90199, -1.45690
7340 DATA   .04961, -1.4569, 2.86704
7350 '
7360 ' <<<<<<<<<<<<<<<<< Prior probabilities data. >>>>>>>>>>>>>>>>>
7370 DATA .08929, .26786, .26786, .375
7380 '
7390 ' <<<<<<<<<<<<<<< Matrix of variable means by levels data.
>>>>>>>>>>>>>>>
7400 DATA 1.89238,  -.05722,  -.23086,  -.24479
7410 DATA 1.02627,  -.10964,   .06219,  -.21046
7420 DATA -1.17750, -.65622,   .01581,   .73779
7425 '
7430 ' <<<<<<<<<<<<<<<<<<<<<<<< Example data. >>>>>>>>>>>>>>>>>>>>>>>>
7440 DATA 7,400,4.25,4
7450 DATA 2,900,4.25,4
7460 DATA 7,500,4.25,4
7470 DATA 10,100,4,4
7480 DATA 2,500,4,4
```

```
7490 DATA 10,300,4,4
7500 DATA 9,1000,3.5,4
7510 DATA 8,500,3.5,4
7520 DATA 7,900,3.5,4
7530 DATA 8,300,2.75,4
7540 DATA 9,900,2.75,4
7550 DATA 8,100,2.75,4
7560 DATA 20,1600,2,3
7570 DATA 12,1800,2,3
7580 DATA 2,400,2,3
7590 DATA 20,1400,2,3
7600 DATA 16,1200,2,3
7610 DATA 8,1000,2,3
7620 DATA 45,1500,0,2
7630 DATA 9,1500,0,2
7640 DATA 81,2100,0,2
7650 DATA 312,3600,0,1
7660 DATA 208,1600,0,1
7670 '
7680 ' Finding radius, demand, and price for non-integer nu values;
      unconstrained solution; solution using numerical integration and
       numerical root extraction.
7690 '
7700 DELTA = 120
7710 SUMA = 0
7720 SUMB = 0
7730 FOR ML = 1 TO 19
7740   SUMA = SUMA + (1 + 1 / COS (PI * ML / DELTA)) ^ (2 - N)
7750   SUMB = SUMB + (1 + 1 / COS (PI * ML / DELTA)) ^ (1 - N)
7760 NEXT
7770 SUMA2 = 0
7780 SUMB2 = 0
7790 FOR ML = 1 TO 19 STEP 2
7800 SUMA2 = SUMA2 + (1 + 1 / COS (PI * ML / DELTA)) ^ (2 - N)
7810 SUMB2 = SUMB2 + (1 + 1 / COS (PI * ML / DELTA)) ^ (1 - N)
7820 NEXT
7830 D = (12 * A / T ^ 2) * (PI / (6 * (2 - N) * (1 - N)) + (PI / 360) * ((2 ^
(2 - N) + (1 + 2 / SQR(3)) ^ (2 - N) + 2 * (SUMA + SUMA2)) / (2 -N) + (2 ^ (1
- N) + (1 +2 / SQR(3)) ^ (1 - N) + 2 * (SUMB + SUMB2)) / (N - 1)))
7840 IF CM(I,2) = 0 THEN R = (D * T ^ (3 - N) / CM(I,3)) ^ (1 / (N-3)) : GOTO
    8240
7850 IF CM(I,3) = 0 THEN 8240
7860 ML = I
7870   ICOUNT = 0
7880   IF (2 < N AND N < 3) THEN 7920
7890           TEST = (((2 - N) / (3 - N)) ^ (2 -N) * D * CM(ML,2) ^ (3 - N))
/ (N - 3) - CM(ML,3)
7900           IF TEST < 0 AND  N > 3 THEN 8200
7910           IF TEST > 0 AND  N < 2 THEN 8200
7920 R1 = .001
7930 R2 = CM (ML,2)
7940 R3 = R2 - (R2 - R1) * (R2 ^ (N - 2) - R2 * D / CM(ML,3) + D * CM(ML,2) /
CM(ML,3)) / (R2 ^ (N - 2) - R1 ^ (N - 1) - (D / CM (ML,3)) * (R2 - R1))
```

```
7950 IF (ABS (R3 - R2) < .0001) THEN 8020
7960 IF R3 > 0 THEN 7990
7970 R3 =  1000000!
7980 GOTO 8230
7990 R1 = R2
8000 R2 = R3
8010 GOTO 7940
8020 R3 = R3 / T
8030 RINC = 1
8040 R = R3
8050 TEST = 0
8060 TESTLAG = TEST
8070 TEST = ((R * T) ^ (2 - N)) * D * (R * T - CM (ML,2)) - CM (ML,3)
8080 ICOUNT = ICOUNT + 1
8090 IF ICOUNT > 500 THEN 8240
8100 IF ABS (TEST) < .01 THEN 8240
8110 IF ABS (TESTLAG / TEST - 1)  < .000001 THEN 8240
8120 IF RINC < .000001 THEN 8240
8130 IF TEST > 0 THEN 8160
8140 R = R + RINC
8150 GOTO 8060
8160 R = R - RINC
8170 RINC = RINC / 10
8180 R = R + RINC
8190 GOTO 8060
8200 R = 1000000!
8210 R3 = 1000000!
8220 GOTO 8240
8230 R = R3
8240 D = ((R * T) ^ (2 - N)) * D
8250 RETURN
8260 '
8270 ' Initializing variables (from line 700).
8290 FOR CARA = 1 TO 3
8300   FOR F = 1 TO 3
8310     READ DA
8320     COV (CARA,F) = DA
8330   NEXT
8340 NEXT
8350 FOR CARA = 1 TO 4
8360   READ DA
8370   PROB (CARA) = DA
8380   LINN(CARA) = 0
8390 NEXT
8400 FOR CARA = 1 TO 3
8410   FOR F = 1 TO 4
8420    READ DA
8430     HBAR (CARA,F) = DA
8440   NEXT
8450 NEXT
8460 MEAN(2) = 466.728
8470 MEAN(1) = 19.0854
8480 MEAN(3) = 3.03131
```

```
8490 SDV(2) = 711.116
8500 SDV(1) = 50.0404
8510 SDV(3) = 1.20989
8520 RETURN
8530 '
8540 ' Inputing data (from line 1310).
8550 RA = 1
8560 CLS
8570 FOR COUN = 1 TO 3
8580 PRINT "Input mean and standard deviation of variable ";COUN;" (#,#)"
8590 INPUT MEAN(COUN),SDV(COUN)
8600 NEXT
8610 CLS
8620 PRINT "Input twelve z-score group means."
8630 FOR CLF = 1 TO 4
8640 FOR CL = 1 TO 3
8650 INPUT HBAR(CL,CLF)
8660 NEXT
8670 NEXT
8680 CLS
8690 PRINT "Input inverse z-score covariance matrix."
8700 PRINT "In the form:  #,#,#"
8710 PRINT "              #,#,#"
8720 PRINT "              #,#,#"
8730 FOR CA = 1 TO 3
8740 RA = 1
8750 INPUT COV(CA,RA),COV(CA,RA + 1),COV(CA,RA + 2)
8760 NEXT
8770 CLS
8780 RETURN
8785 '
8790 ' To add a data file name it "(any name) .   "
8800 ' Read in data from disk.
8820 CLS
8830 FILES "*."
8840 PRINT "Files named 'NU' have the test data, with all commodities having
          the specified nu value given in the filename."
8845 PRINT "File 'NU0425' has all cf values of zero, and file 'NU4250' has all
         cv values of zero."
8846 PRINT "File 'PAPER' has the original 'Economic Geography' article data."

8850 PRINT "Files named '_&_&_&_' have those levels given in the filename
          appearing on the map."
8860 INPUT "Which file";FILE$
8870 OPEN "i",#1,FILE$
8880 INPUT# 1,CN
8890 FOR ELF = 1 TO CN
8900   CM(ELF,1) = ELF
8910   INPUT# 1,CM(ELF,2)
8920   INPUT# 1,CM(ELF,3)
8930   INPUT# 1,CM(ELF,4)
8940 NEXT
8950 A = 500
```

```
8960 T = .1
8970 CLS
8980 PRINT "Density = 500."
8990 PRINT "Transport rate = .1."
9000 RETURN
```

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"Imagination is more important than knowledge"
A. Einstein

MONOGRAPH SERIES

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1. Sandra L. Arlinghaus and John D. Nystuen. *Mathematical Geography and Global Art: the Mathematics of David Barr's "Four Corners Project,"* 1986.

This monograph contains Nystuen's calculations, actually used by Barr to position his abstract tetrahedral sculpture within the earth. Placement of the sculpture vertices in Easter Island, South Africa, Greenland, and Indonesia was chronicled in film by The Archives of American Art for The Smithsonian Institution. In addition to the archival material, this monograph also contains Arlinghaus's solutions to broader theoretical questions–was Barr's choice of a tetrahedron unique within his initial constraints, and, within the set of Platonic solids?

2. Sandra L. Arlinghaus. *Down the Mail Tubes: the Pressured Postal Era, 1853-1984*, 1986.

The history of the pneumatic post, in Europe and in the United States, is examined for the lessons it might offer to the technological scenes of the late twentieth century. As Sylvia L. Thrupp, Alice Freeman Palmer Professor Emeritus of History, The University of Michigan, commented in her review of this work "Such brief comment does far less than justice to the intelligence and the stimulating quality of the author's writing, or to the breadth of her reading. The detail of her accounts of the interest of American private enterprise, in New York and other large cities on this continent, in pushing for construction of large tubes in systems to be leased to the government, brings out contrast between American and European views of how the new technology should be managed. This and many other sections of the monograph will set readers on new tracks of thought."

3. Sandra L. Arlinghaus. *Essays on Mathematical Geography*, 1986.

A collection of essays intended to show the range of power in applying pure mathematics to human systems. There are two types of essay: those which employ traditional mathematical proof, and those which do not. As mathematical proof may itself be regarded as art, the former style of essay might represent "traditional" art, and the latter, "surrealist" art. Essay titles are: "The well-tempered map projection," "Antipodal graphs," "Analogue clocks," "Steiner transformations," "Concavity and urban settlement patterns," "Measuring the vertical city," "Fad and permanence in human systems," "Topological exploration in geography," "A space for thought," and "Chaos in human systems–the Heine-Borel Theorem."

4. Robert F. Austin, *A Historical Gazetteer of Southeast Asia*, 1986.

Dr. Austin's Gazetteer draws geographic coordinates of Southeast Asian place-names together with references to these place-names as they have appeared in historical and literary documents. This book is of obvious use to historians and to historical geographers specializing in Southeast Asia. At a deeper level, it might serve as a valuable source in establishing place-name linkages which have remained previously unnoticed, in documents describing trade or other communications connections, because of variation in place-name nomenclature.

5. Sandra L. Arlinghaus, *Essays on Mathematical Geography–II*, 1987.

Written in the same format as IMaGe Monograph #3, that seeks to use "pure" mathematics in real-world settings, this volume contains the following material: "Frontispiece–the Atlantic Drainage Tree," "Getting a Handel on Water-Graphs," "Terror in Transit: A Graph Theoretic Approach to the Passive Defense of Urban Networks," "Terrae Antipodum," "Urban Inversion," "Fractals: Constructions, Speculations, and Concepts," "Solar Woks," "A Pneumatic Postal Plan: The Chambered Interchange and ZIPPR Code," "Endpiece."

6. Pierre Hanjoul, Hubert Beguin, and Jean-Claude Thill, *Theoretical Market Areas Under Euclidean Distance*, 1988. (English language text; Abstracts written in French and in English.)

Though already initiated by Rau in 1841, the economic theory of the shape of two-dimensional market areas has long remained concerned with a representation of transportation costs as linear in distance. In the general gravity model, to which the theory also applies, this corresponds to a decreasing exponential function of distance deterrence. Other transportation cost and distance deterrence functions also appear in the literature, however. They have not always been considered from the viewpoint of the shape of the market areas they generate, and their disparity asks the question whether other types of functions would not be worth being investigated. There is thus a need for a general theory of market areas: the present work aims at filling this gap, in the case of a duopoly competing inside the Euclidean plane endowed with Euclidean distance.

(Bien qu'ébauchée par Rau dès 1841, la théorie économique de la forme des aires de marché planaires s'est longtemps contentée de l'hypothèse de coûts de transport proportionnels à la distance. Dans le modèle gravitaire généralisé, auquel on peut étendre cette théorie, ceci correspond au choix d'une exponentielle décroissante comme fonction de dissuasion de la distance. D'autres fonctions de coût de transport ou de dissuasion de la distance apparaissent cependant dans la littérature. La forme des aires de marché qu'elles engendrent n'a pas toujours été étudiée ; par ailleurs, leur variété amène à se demander si d'autres fonctions encore ne mériteraient pas d'être examinées. Il paraît donc utile de disposer d'une théorie générale des aires de marché : ce à quoi s'attache ce travail en cas de duopole, dans le cadre du plan euclidien muni d'une distance euclidienne.)

7. Keith J. Tinkler, Editor, *Nystuen—Dacey Nodal Analysis*, 1988.

Professor Tinkler's volume displays the use of this graph theoretical tool in geography, from the original Nystuen—Dacey article, to a bibliography of uses, to original uses by Tinkler. Some reprinted material is included, but by far the larger part is of previously unpublished material. (Unless otherwise noted, all items listed below are previously unpublished.) Contents: " 'Foreward' " by Nystuen, 1988; "Preface" by Tinkler, 1988; "Statistics for Nystuen—Dacey Nodal Analysis," by Tinkler, 1979; Review of Nodal Analysis literature by Tinkler (pre–1979, reprinted with permission; post—1979, new as of 1988); FORTRAN program listing for Nodal Analysis by Tinkler; "A graph theory interpretation of nodal regions" by John D. Nystuen and Michael F. Dacey, reprinted with permission, 1961; Nystuen—Dacey data concerning telephone flows in Washington and Missouri, 1958, 1959 with comment by Nystuen, 1988; "The expected distribution of nodality in random (p, q) graphs and multigraphs," by Tinkler, 1976.

8. James W. Fonseca, *The Urban Rank–size Hierarchy: A Mathematical Interpretation*, 1989.

The urban rank–size hierarchy can be characterized as an equiangular spiral of the form $r = ae^{\theta \cot \alpha}$. An equiangular spiral can also be constructed from a Fibonacci sequence. The urban rank–size hierarchy is thus shown to mirror the properties derived from Fibonacci characteristics such as rank–additive properties. A new method of structuring the urban rank–size hierarchy is explored which essentially parallels that of the traditional rank–size hierarchy below rank 11. Above rank 11 this method may help explain the frequently noted concavity of the rank–size distribution at the upper levels. The research suggests that the simple rank–size rule with the exponent equal to 1 is not merely a special case, but rather a theoretically justified norm against which deviant cases may be measured. The spiral distribution model allows conceptualization of a new view of the urban rank–size hierarchy in which the three largest cities share functions in a Fibonacci hierarchy.

9. Sandra L. Arlinghaus, *An Atlas of Steiner Networks*, 1989.

A Steiner network is a tree of minimum total length joining a prescribed, finite, number of locations; often new locations are introduced into the prescribed set to determine the minimum tree. This Atlas explains the mathematical detail behind the Steiner construction for prescribed sets of n locations and displays the steps, visually, in a series of Figures. The proof of the Steiner construction is by mathematical induction, and enough steps in the early part of the induction are displayed completely that the reader who is well–trained in Euclidean geometry, and familiar with concepts from graph theory and elementary number theory, should be able to replicate the constructions for full as well as for degenerate Steiner trees.

10. Daniel A. Griffith, *Simulating $K = 3$ Christaller Central Place Structures: An Algorithm Using A Constant Elasticity of Substitution Consumption Function*, 1989.

An algorithm is presented that uses BASICA or GWBASIC on IBM compatible machines. This algorithm simulates Christaller $K = 3$ central place structures, for a four–level hierarchy. It is based upon earlier published work by the author. A description of the spatial theory, mathematics, and sample output runs appears in the monograph. A digital version is available from the author, free of charge, upon request; this request must be accompanied by a 5.5–inch formatted diskette. This algorithm has been developed for use in Social Science classroom laboratory situations, and is designed to (a) cultivate a deeper understanding of central place theory, (b) allow parameters of a central place system to be altered and then graphic and tabular results attributable to these changes viewed, without experiencing the tedium of massive calculations, and (c) help promote a better comprehension of the complex role distance plays in the space–economy. The algorithm also should facilitate intensive numerical research on central place structures; it is expected that even the sample simulation results will reveal interesting insights into abstract central place theory.

The background spatial theory concerns demand and competition in the space–economy; both linear and non–linear spatial demand functions are discussed. The mathematics is concerned with (a) integration of non–linear spatial demand cones on a continuous demand surface, using a constant elasticity of substitution consumption function, (b) solving for roots of polynomials, (c) numerical approximations to integration and root extraction, and (d) multinomial discriminant function classification of commodities into central place hierarchy levels. Sample output is presented for contrived data sets, constructed from artificial and empirical information, with the wide range of all possible central place structures being generated. These examples should facilitate implementation testing. Students are able to vary single or multiple parameters of the problem, permitting a study of how certain changes manifest themselves within the context of a theoretical central place structure. Hierarchical classification criteria may be changed, demand elasticities may or may not vary and can take on a wide range of non–negative values, the uniform transport cost may be set at any positive level, assorted fixed costs and variable costs may be introduced, again within a rich range of non–negative possibilities, and the number of commodities can be altered. Directions for algorithm execution are summarized. An ASCII version of the algorithm, written directly from GWBASIC, is included in an appendix; hence, it is free of typing errors.

11. Sandra L. Arlinghaus and John D. Nystuen, *Environmental Effects on Bus Durability*, 1990.

This monograph draws on the authors' previous publications on "Climatic" and "Terrain" effects on bus durability. Material on these two topics is selected, and reprinted, from three published papers that appeared in the *Transportation Research Record* and in the *Geographical Review*. New material concerning "congestion" effects is examined at the national level, to determine "dense," "intermediate," and "sparse" classes of congestion, and at the local level of congestion in Ann Arbor (as suggestive of how one might use local data). This material is drawn together in a single volume, along with a summary of the consequences of all three effects simultaneously, in order to suggest direction for more highly automated studies that should follow naturally with the release of the 1990 U. S. Census data.

12. Daniel A. Griffith, Editor. *Spatial Statistics: Past, Present, and Future*, 1990.

Proceedings of a Symposium of the same name held at Syracuse University in Summer, 1989. Content includes a Preface by Griffith and the following papers:

Brian Ripley, "Gibbsian interaction models";
J. Keith Ord, "Statistical methods for point pattern data";
Luc Anselin, "What is special about spatial data";
Robert P. Haining, "Models in human geography:
problems in specifying, estimating, and validating models for spatial data";
R. J. Martin, "The role of spatial statistics in geographic modelling";
Daniel Wartenberg, "Exploratory spatial analyses: outliers, leverage points, and influence functions";
J. H. P. Paelinck, "Some new estimators in spatial econometrics";
Daniel A. Griffith, "A numerical simplification for estimating parameters of spatial autoregressive models";
Kanti V. Mardia "Maximum likelihood estimation for spatial models";
Ashish Sen, "Distribution of spatial correlation statistics";

Sylvia Richardson, "Some remarks on the testing of association between spatial processes";
Graham J. G. Upton, "Information from regional data";
Patrick Doreian, "Network autocorrelation models: problems and prospects."

Each chapter is preceded by an "Editor's Preface" and followed by a Discussion and, in some cases, by an author's Rejoinder to the Discussion.