

## Chapter 2

### Fractal Generation of the Classical Hexagonal Hierarchies

Concepts from fractal geometry embrace the entire geometry of classical central place theory. One type of fractal iteration sequence alone, a self-similarity transformation, produces all possible central place nets. This alignment presents more than mere technique for a verification of a geometry of central places (although it does do that, too); the geometry of the central place model is well-known and has been clearly, and comprehensively, discussed in Michael Dacey's 1965 article ([Dacey 1965](#)). The conceptual material here reveals an exact procedure for the merging of two separate theories: one from pure mathematics and one from economics and geography. In so doing, it suggests, in general, the power of one to enrich the other through the lodging of one discipline in the house of the other, and in particular, a theory in its own right derived from associating mathematically-specialized central place concepts with mathematically-broader fractal concepts.

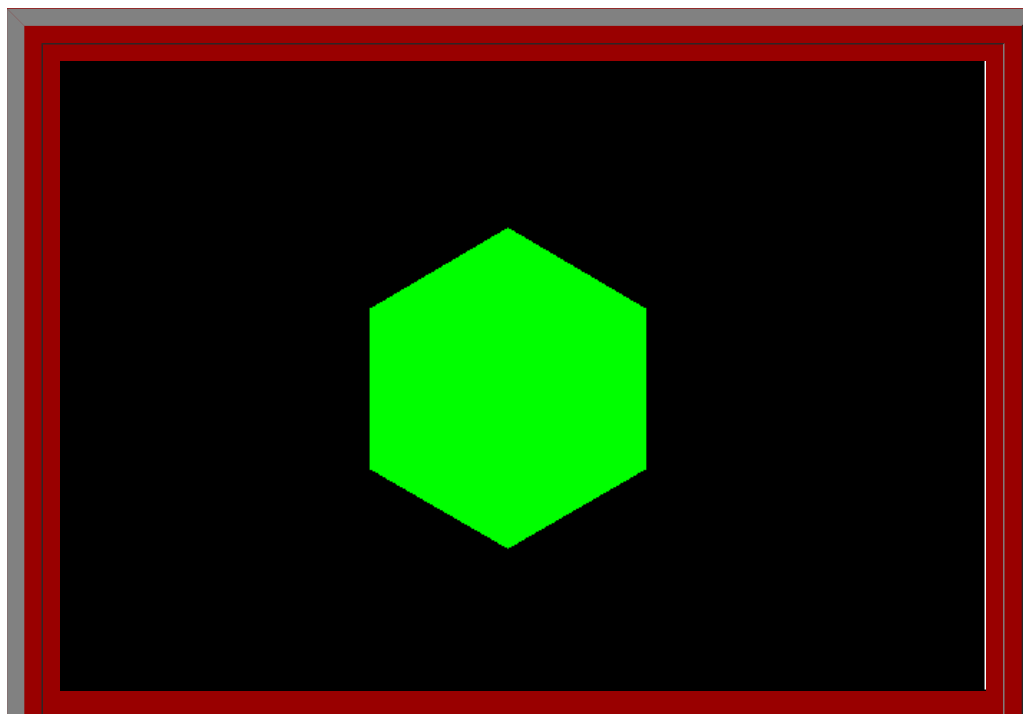
In this chapter, we show in detail the alignment of fractal geometry with classical central place theory. In subsequent work, we illustrate how this geometry extends into far-flung realms to solve existing unsolved problems of classical central place theory and probe directions for further theoretical discovery and empirical application. The alignment of the existing classical material with fractal geometry is of interest in its own right; far more important, however, is the benchmark that this alignment creates--classical central place geometry serves as a field check for the fractal maps prior to the use of these maps as guides in the realm of theoretical geography.

#### "Fractals Take a Central Place"

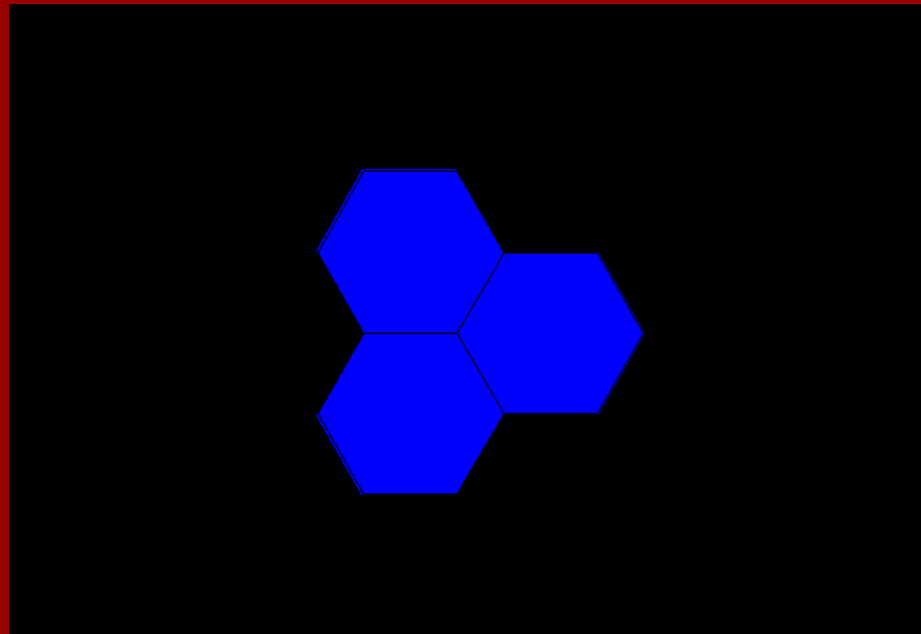
In the material below, we illustrate, using a hexagon as an initiator, different selections of generators applied to the initial hexagon, to produce the different hexagonal hierarchies of classical central place theory (based on original concept and work of S. Arlinghaus (1985); see [link](#)). In the previous chapter we formed central place landscapes by moving from small hexagons to large ones; here, we reverse the process and dissect, using the self-similarity transformation, a large hexagon to create the smaller ones. In both processes, the results correspond exactly. The art is in generator selection, and it is simply that art that is presented in this chapter. Later work will delve into the mathematical foundations of that art.

#### *The K=3 Hierarchy*

When an hexagonal initiator is chosen and a two-sided generator, with included angle of 120 degrees, is used to make successive replacement of the sides of the hexagon (as in the animated Figure 2.1 a), the outline of the next layer of the  $K=3$  central place hierarchy is generated (the black lines in Figure 2.1a suggest interior connections). The replacement sequence applies the generator in an alternating pattern to the outside and then to the inside of the initiator. When the original generator is scaled down, with shape preserved, and applied in the outside/inside sequence to the newly formed blue polygon, the next lower level central place  $K=3$  hierarchy is formed (as in the animated Figure 2.1b). The second, blue polygon contains three scaled-down hexagons, self-similar to the first hexagon (Figure 2.1a); the red polygon in the animation sequence contains three shapes self-similar to the blue polygon (Figure 2.1b), and 27 (or 3 cubed) hexagons self-similar to the original hexagon (Figure 2.1b). The invariant of 3, in the  $K=3$  hierarchy, is replicated in this particular fractal iteration sequence.

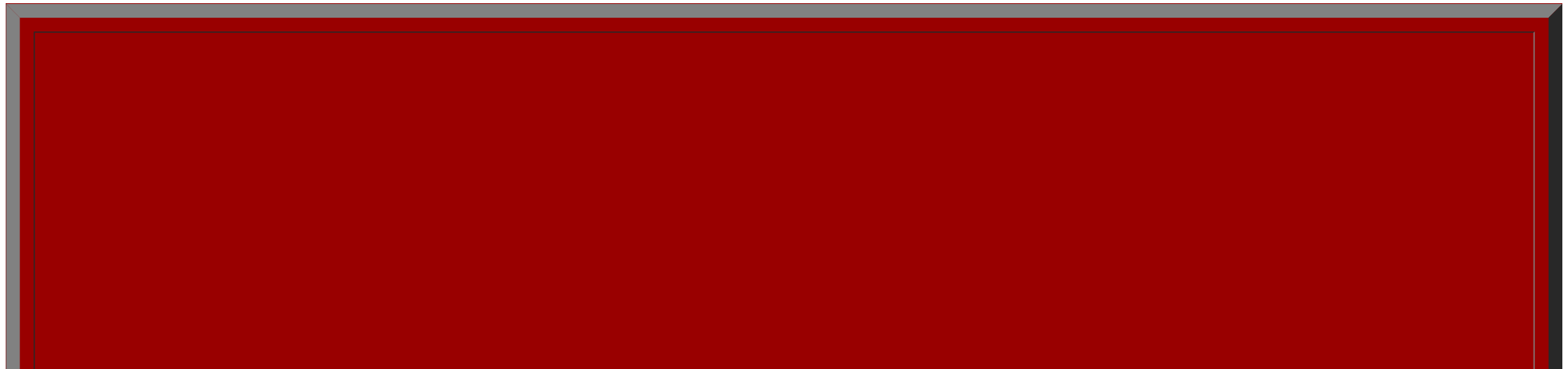


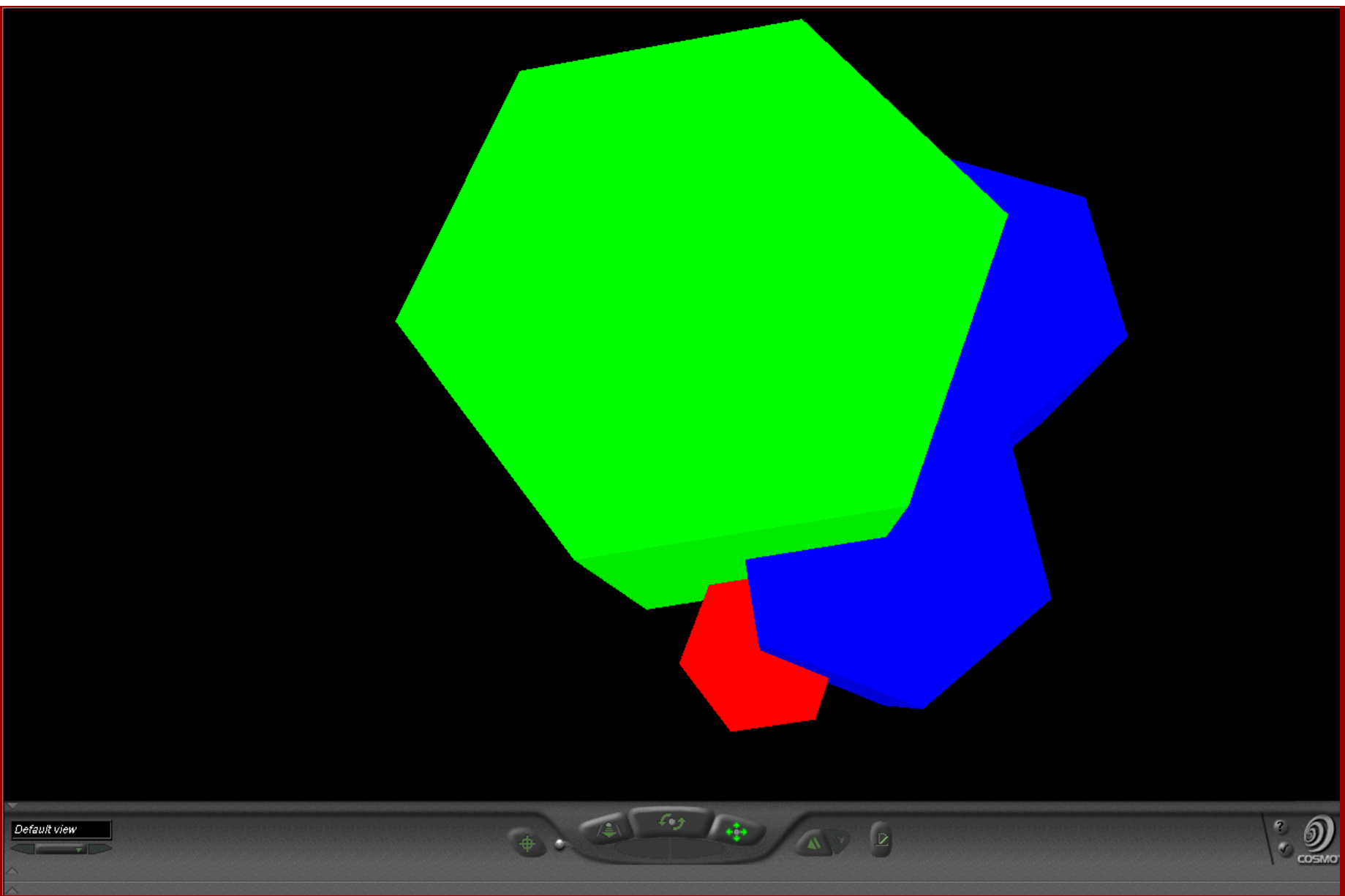
**Figure 2.1a.** Animated  $K=3$  fractal iteration sequence: first transformation using a two-sided fractal generator applied successively to sides of the hexagonal initiator.



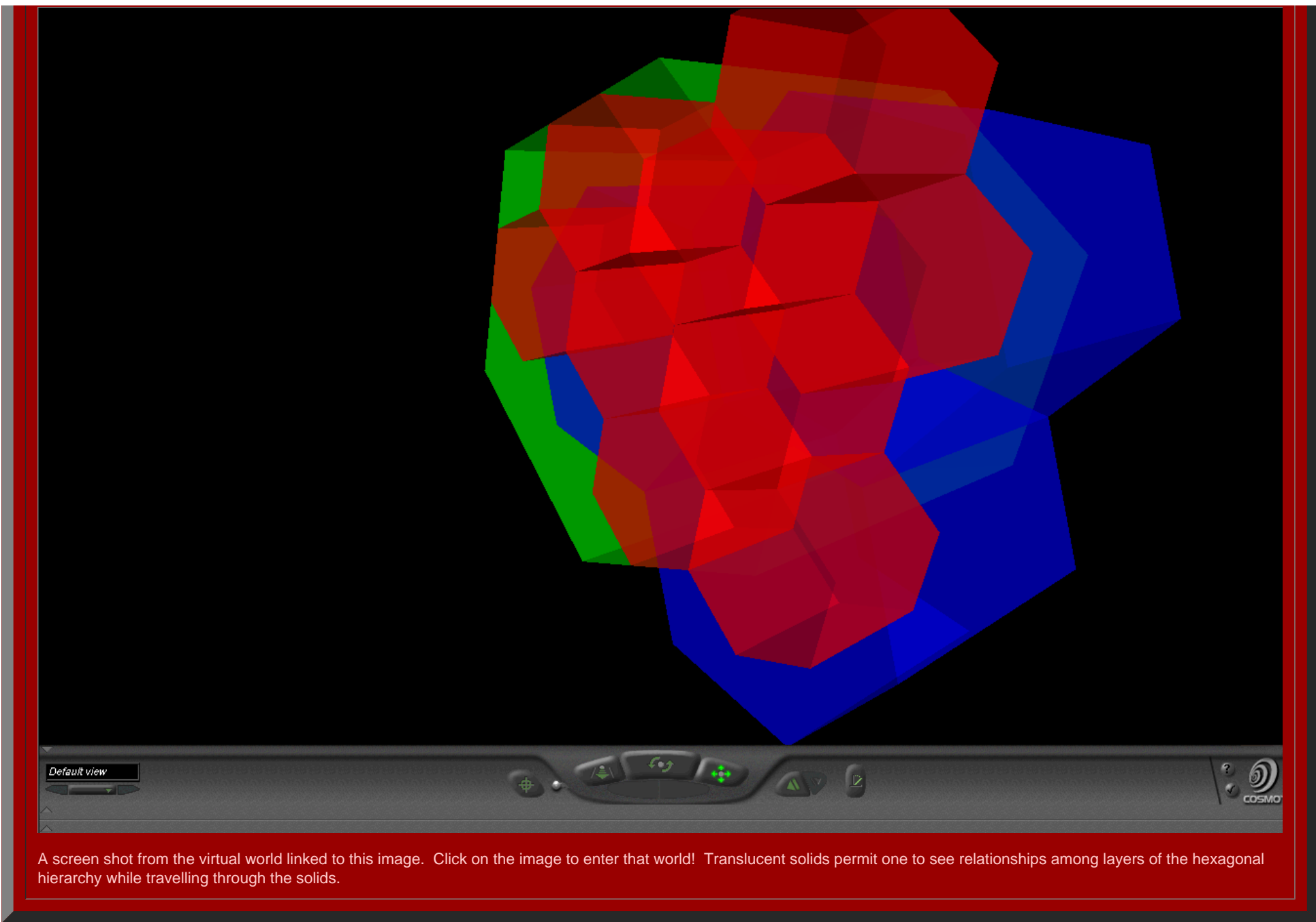
**Figure 2.1b.** Animated  $K=3$  fractal iteration sequence: second transformation using a scaled-down two-sided fractal generator applied successively to sides of the blue polygon generated in Figure 2.1a.

It remains to determine if the polygons generated in Figure 2.1 will in fact fit together to form the broad central place landscape of arbitrary size suggested in Figure 1.5. To that end, we stack the layers generated above using the fractal iteration sequence to form a tile of layers centered on the single polygonal initiator (Figure 2.2). Click [here](#) to see a virtual solid model of the tile with which the reader can interact. Click [here](#) to see a virtual translucent model of the tile with which the reader can interact. Consider the screen captures below from those virtual reality models.



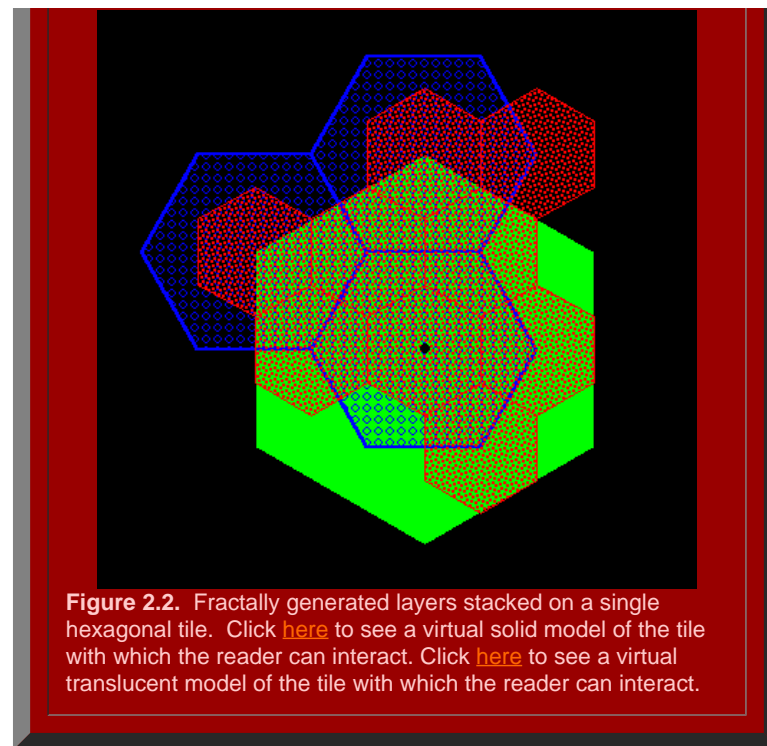


A screen shot from the virtual world linked to this image. Click on the image to enter that world!



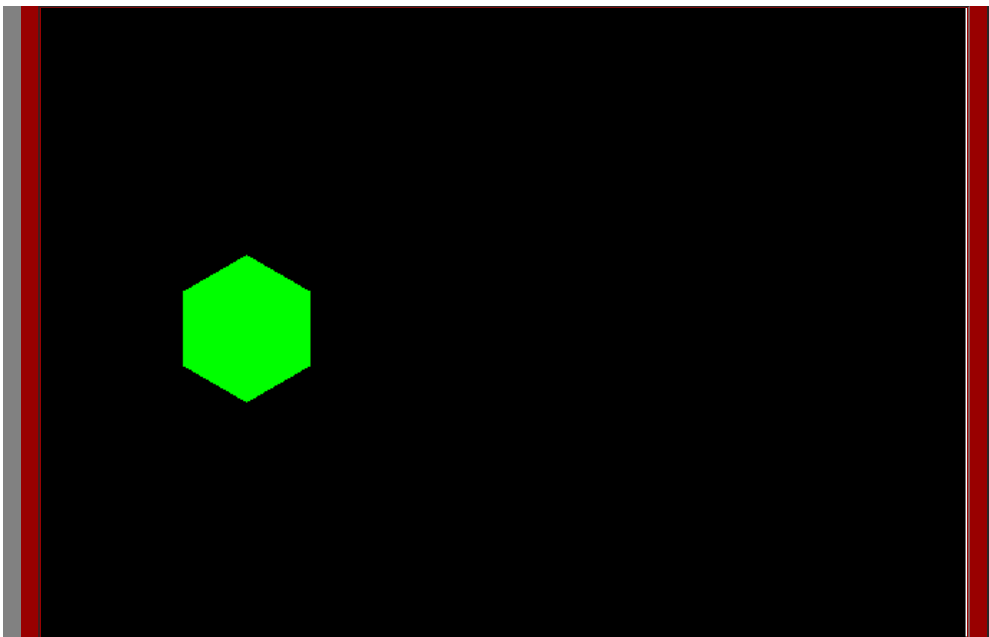
A screen shot from the virtual world linked to this image. Click on the image to enter that world! Translucent solids permit one to see relationships among layers of the hexagonal hierarchy while travelling through the solids.



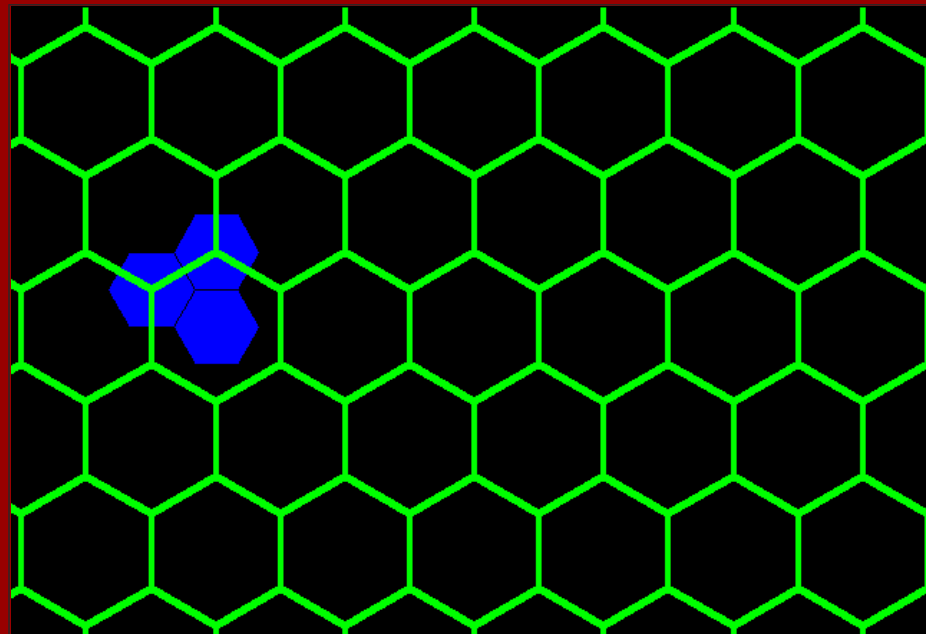


Finally, we tile the plane using the hexagonal initiators, from Figure 2.2 with the attached stack of smaller hexagons, to discover if the superimposed structure also fits together perfectly (Figure 2.3). Hexagonal tiles are used to cover the plane without gaps, as is the case with the sample of green hexagons in Figure 2.3a. The hexagons mesh perfectly to cover the plane (Theorem of [Gauss](#)). In Figure 2.3b, the green outline of the hexagons remains. Each of the solid green hexagons has had the fractal generator above applied and the consequent superimposed blue tiles come into view sequentially in this animation. Again, the fit is exact, as we had hoped it might be. Finally, in Figure 2.3c, the blue outline only is retained from Figure 2.3b (along with the green outline from Figure 2.3a). The final fractally generated layer derived from the blue polygons of Figure 2.3b comes into view in shades of red (or yellow/gold for contrast). The final layer of hexagonal base of unit hexagons appears last. The fit is perfect: each green hexagon contains the equivalent of four blue hexagons and each blue hexagon contains the equivalent of four red hexagons. The fractal generation procedure created exactly the classical central place landscape of [Figure 1.5](#) (return here using the Back button on your browser). As the animation proceeds in Figure 2.3, further layers of the fractally generated hierarchy, attached to the tile in Figure 2.2, come into view illustrating an exact meshing of tiles at all levels to form a  $K=3$  hierarchy.

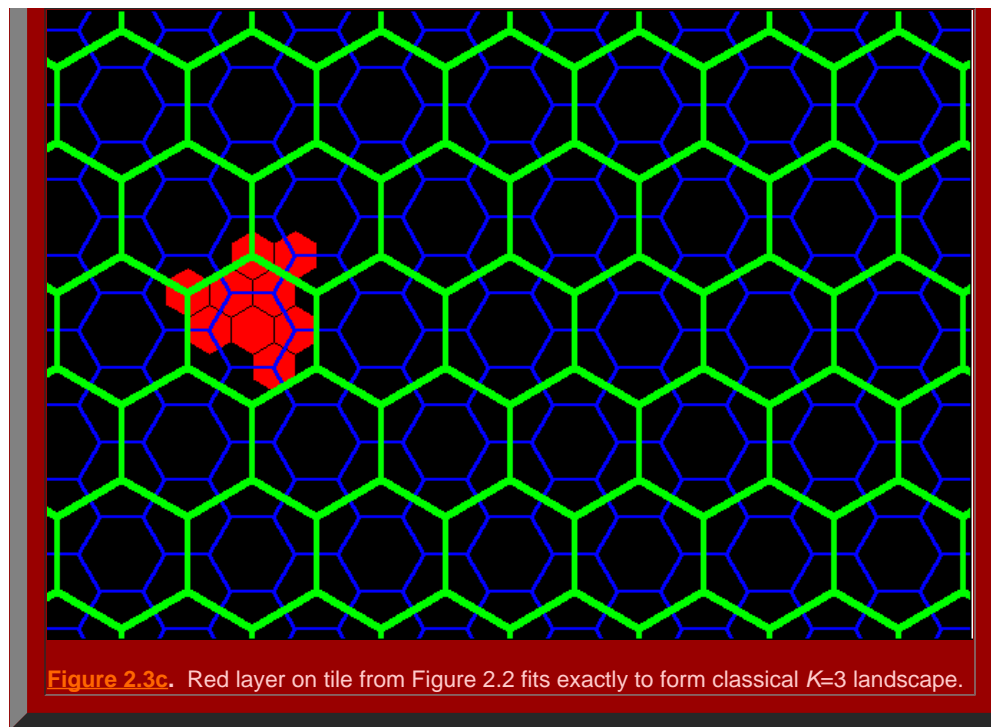




**Figure 2.3a.** Green layer on tile from Figure 2.2 fits exactly to tile the plane (Theorem of Gauss).



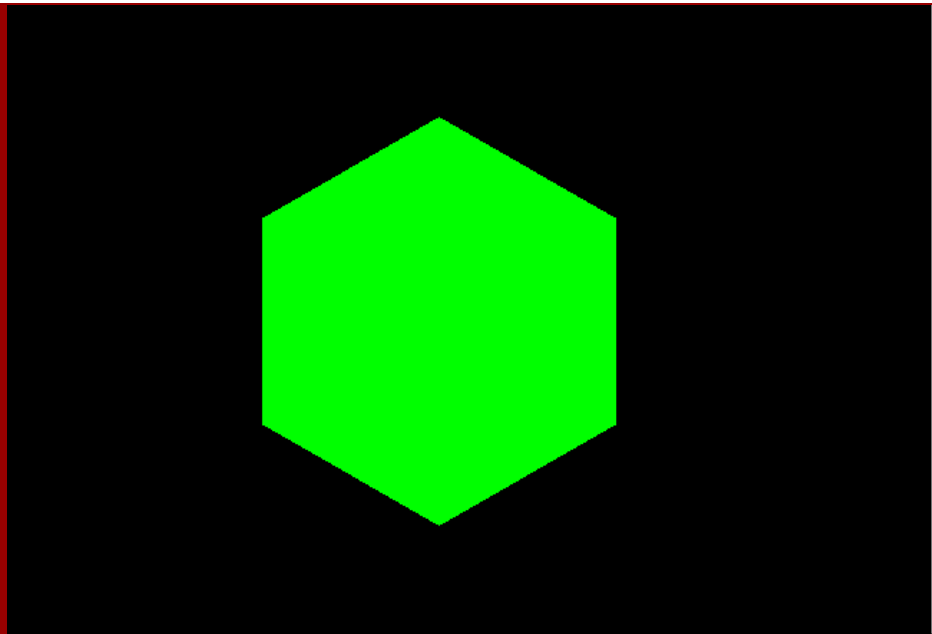
**Figure 2.3b.** Blue layer on tile from Figure 2.2 fits exactly to form classical  $K=3$  landscape.



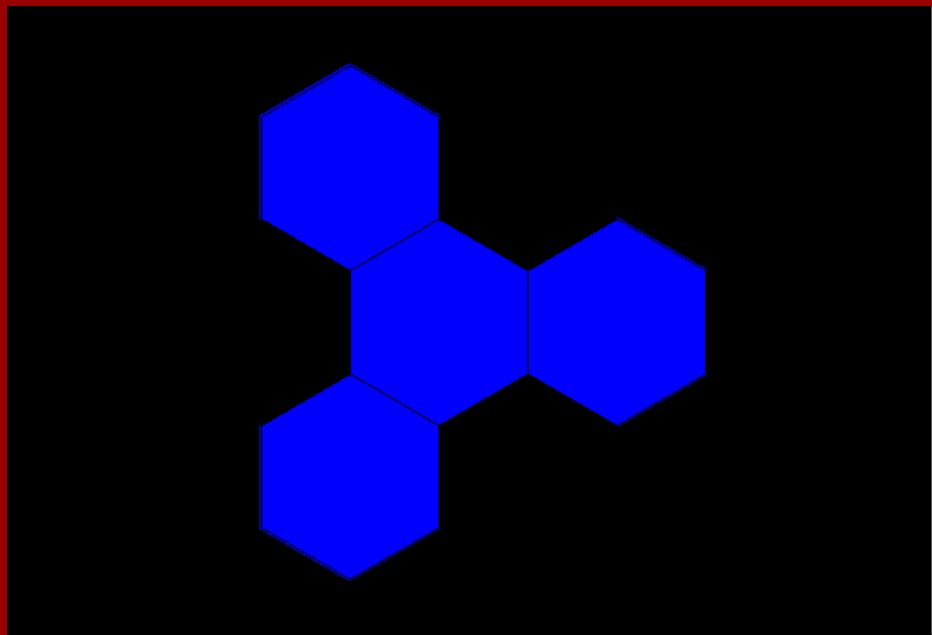
#### *The $K=4$ Hierarchy*

When an hexagonal initiator is chosen and a three-sided generator, with included angles of 120 degrees and shaped in the form of an isosceles trapezoid, is used to make successive replacement of the sides of the hexagon (as in the animated Figure 2.4a), the outline of the next layer of the  $K=4$  central place hierarchy is generated (the black lines in Figure 2.4a suggest interior connections). The replacement sequence applies the generator in an alternating pattern to the outside and then to the inside of the initiator. When the original generator is scaled down, with shape preserved, and applied in the outside/inside sequence to the newly formed blue polygon, the next lower level central place  $K=4$  hierarchy is formed (as in the animated Figure 2.4b). The second, blue polygon contains four scaled-down hexagons, self-similar to the first hexagon (Figure 2.4a); the red polygon in the animation sequence contains four shapes self-similar to the blue polygon (Figure 2.4b), and 64 (or 4 cubed) hexagons self-similar to the original hexagon (Figure 2.4b). The invariant of 4, in the  $K=4$  hierarchy, is replicated in this particular fractal iteration sequence.





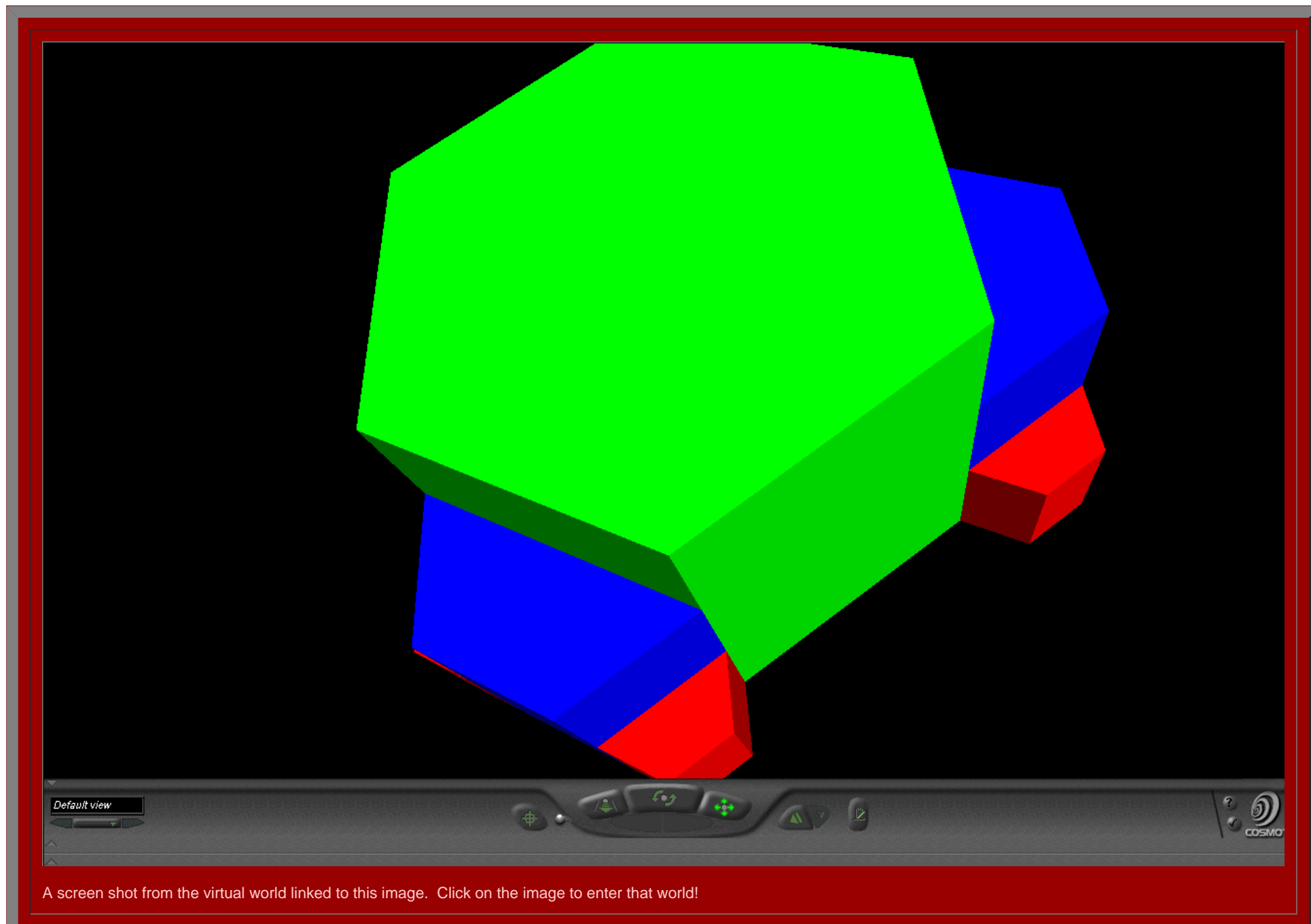
**Figure 2.4a.** Animated  $K=4$  fractal iteration sequence: first transformation using a three-sided fractal generator applied successively to sides of the hexagonal initiator.

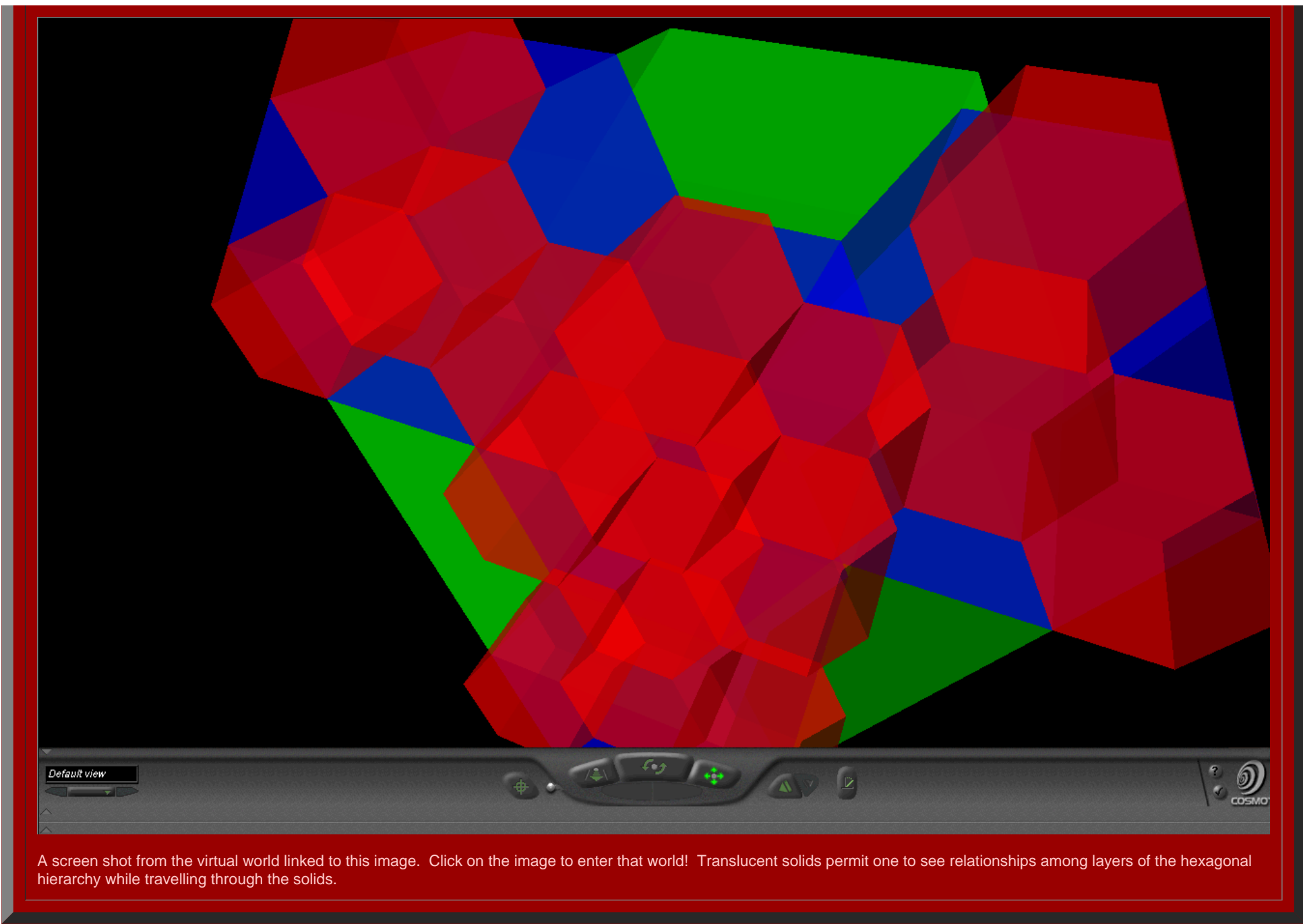


**Figure 2.4b.** Animated  $K=4$  fractal iteration sequence: second transformation using a scaled-down three-sided fractal generator applied successively to sides of the blue polygon generated in Figure 2.4a.



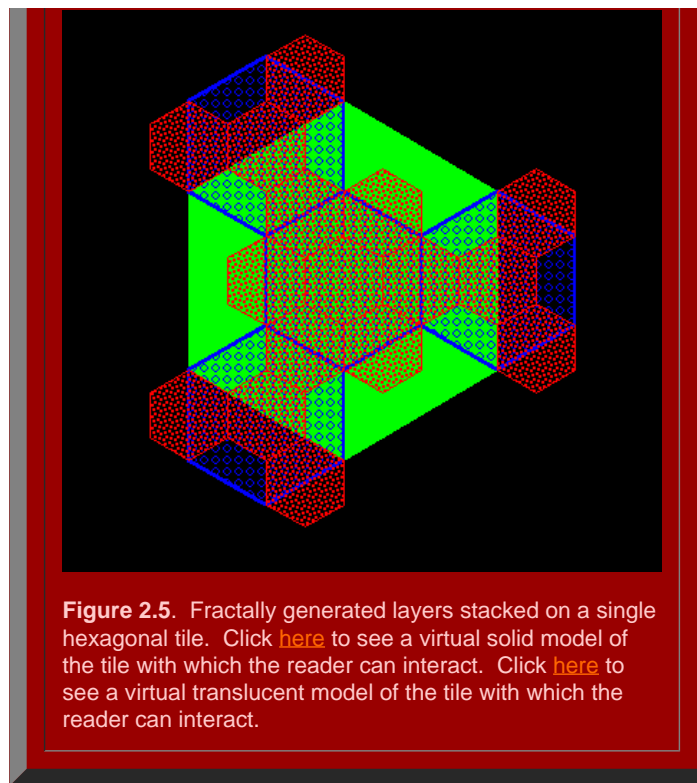
It remains to determine if the polygons generated in Figure 2.4 will in fact fit together to form the broad central place landscape of arbitrary size suggested in [Figure 1.7](#). To that end, we stack the layers generated above using the fractal iteration sequence to form a tile of layers centered on the single polygonal initiator (Figure 2.5). Click [here](#) to see a virtual solid model of the tile with which the reader can interact. Click [here](#) to see a virtual translucent model of the tile with which the reader can interact. Consider the screen captures below from those virtual reality models.



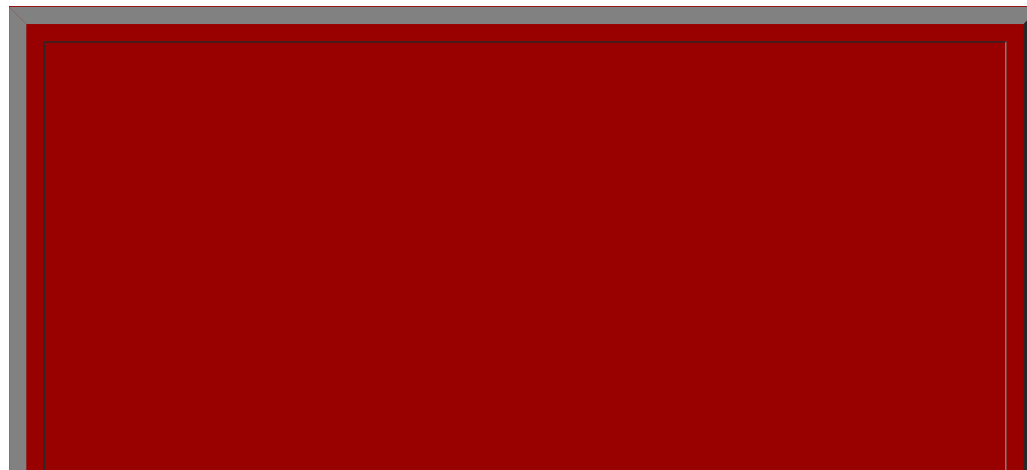


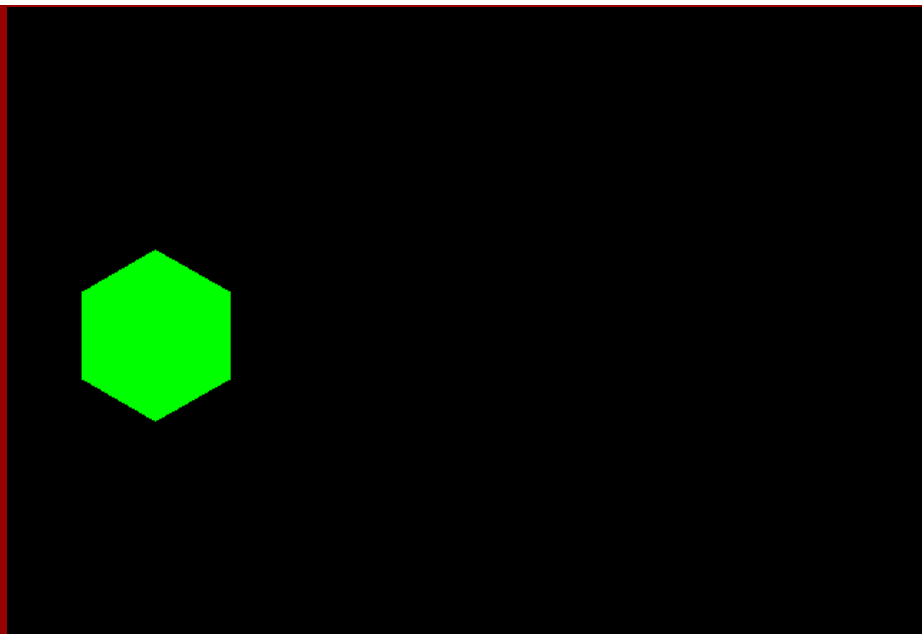
A screen shot from the virtual world linked to this image. Click on the image to enter that world! Translucent solids permit one to see relationships among layers of the hexagonal hierarchy while travelling through the solids.



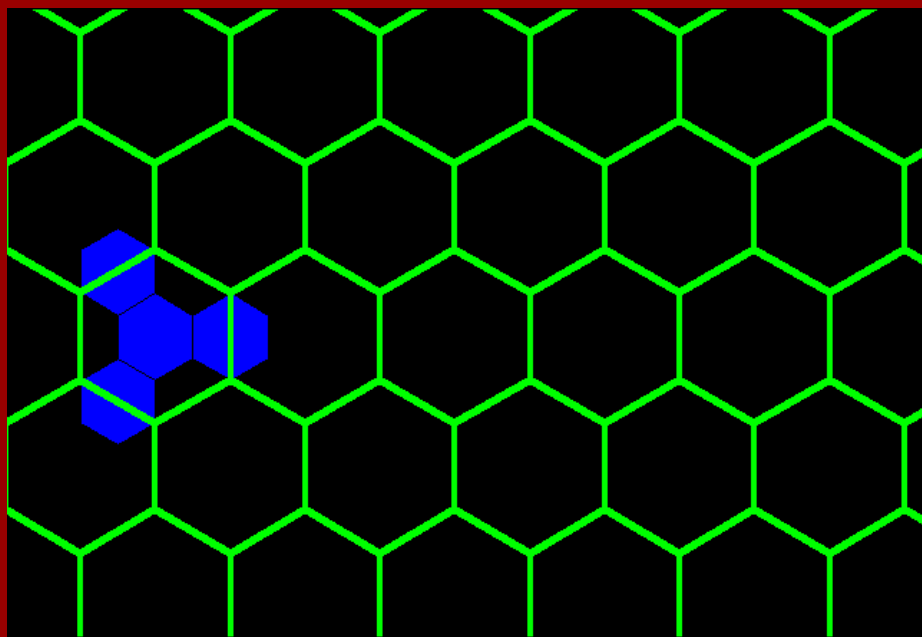


Finally, we tile the plane using the hexagonal initiators, from Figure 2.5 with the attached stack of smaller hexagons, to discover if the superimposed structure also fits together perfectly (Figure 2.6). Hexagonal tiles are used to cover the plane without gaps, as is the case with the sample of green hexagons in Figure 2.6a. The hexagons mesh perfectly to cover the plane (Theorem of [Gauss](#)). In Figure 2.6b, the green outline of the hexagons remains. Each of the solid green hexagons has had the fractal generator above applied and the consequent superimposed blue tiles come into view sequentially in this animation. Again, the fit is exact, as we had hoped it might be. Finally, in Figure 2.6c, the blue outline only is retained from Figure 2.6b (along with the green outline from Figure 2.6a). The final fractally generated layer derived from the blue polygons of Figure 2.6b comes into view in shades of red (or yellow/gold for contrast). The final layer of hexagonal base of unit hexagons appears last. The fit is perfect: each green hexagon contains the equivalent of four blue hexagons and each blue hexagon contains the equivalent of four red hexagons. The fractal generation procedure created exactly the classical central place landscape of [Figure 1.7](#) (return here using the Back button on your browser). As the animation proceeds in Figure 2.6, further layers of the fractally generated hierarchy, attached to the tile in Figure 2.5, come into view illustrating an exact meshing of tiles at all levels to form a  $K=4$  hierarchy.

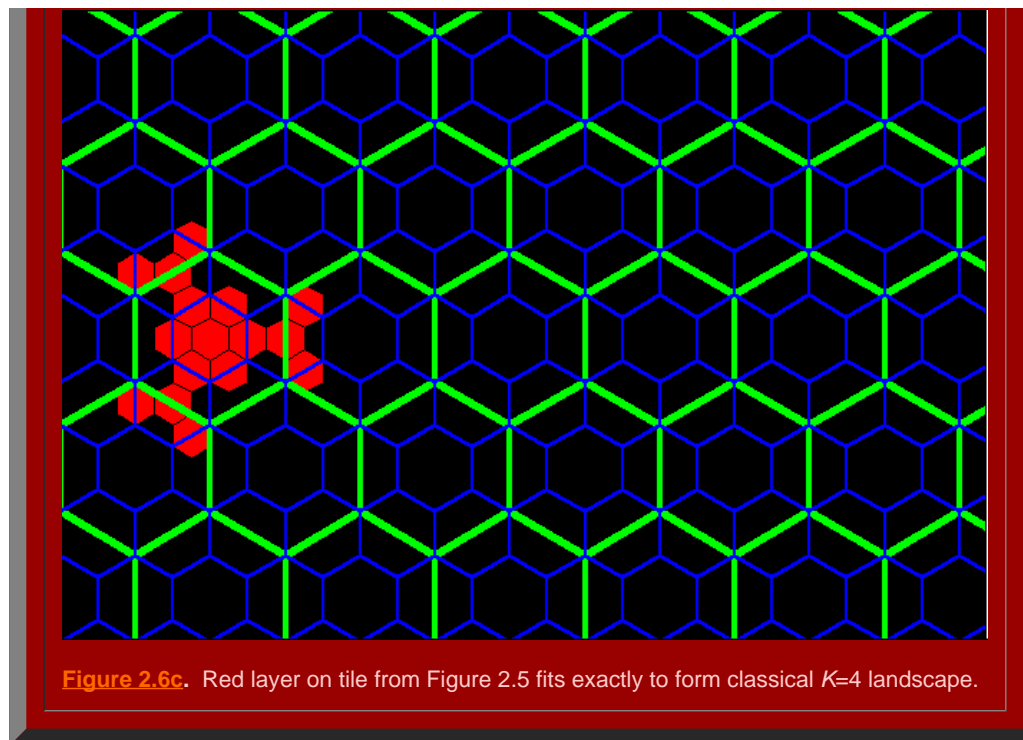




**Figure 2.6a.** Green layer on tile from Figure 2.5 fits exactly to tile the plane (Theorem of Gauss).



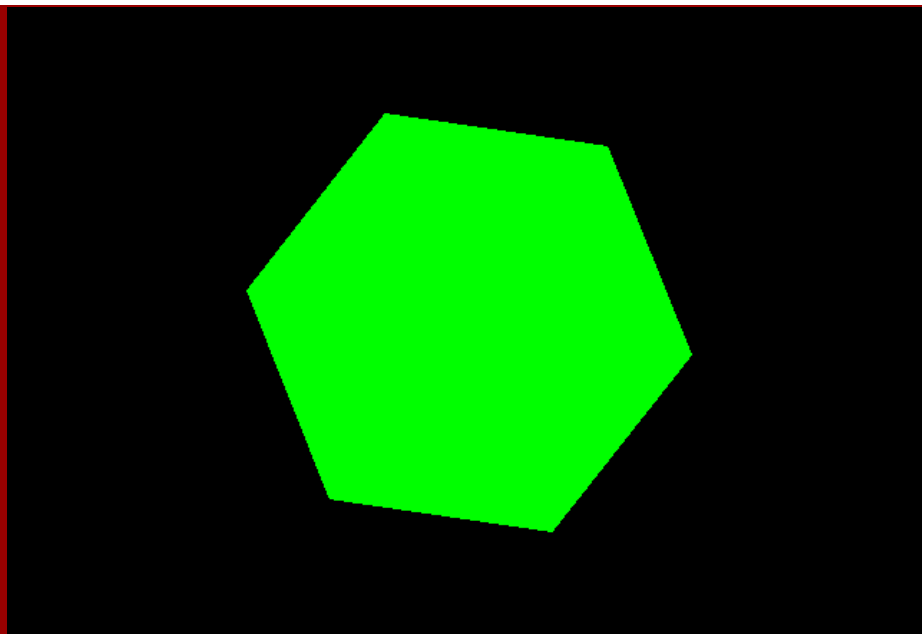
**Figure 2.6b.** Blue layer on tile from Figure 2.5 fits exactly to form classical  $K=4$  landscape.



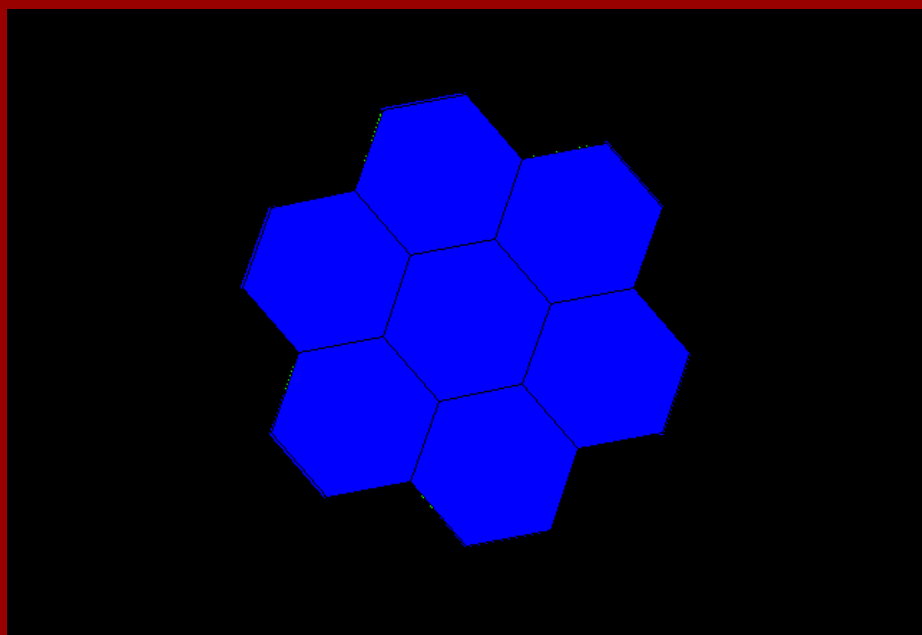
### *The $K=7$ Hierarchy*

When an hexagonal initiator is chosen and a three-sided generator, with included angles of 120 degrees and shaped in a zig-zag form, is used to make successive replacement of the sides of the hexagon (as in the animated Figure 2.7a), the outline of the next layer of the  $K=7$  central place hierarchy is generated (the black lines in Figure 2.7a suggest interior connections). The replacement sequence applies the generator in an alternating pattern to the outside and then to the inside of the initiator. When the original generator is scaled down, with shape preserved, and applied in the outside/inside sequence to the newly formed blue polygon, the next lower level central place  $K=7$  hierarchy is formed (as in the animated Figure 2.7b). The second, blue polygon contains seven scaled-down hexagons, self-similar to the first hexagon (Figure 2.7a); the red polygon in the animation sequence contains seven shapes self-similar to the blue polygon (Figure 2.7b), and 343 (or 7 cubed) hexagons self-similar to the original hexagon (Figure 2.7b). The invariant of 7, in the  $K=7$  hierarchy, is replicated in this particular fractal iteration sequence.



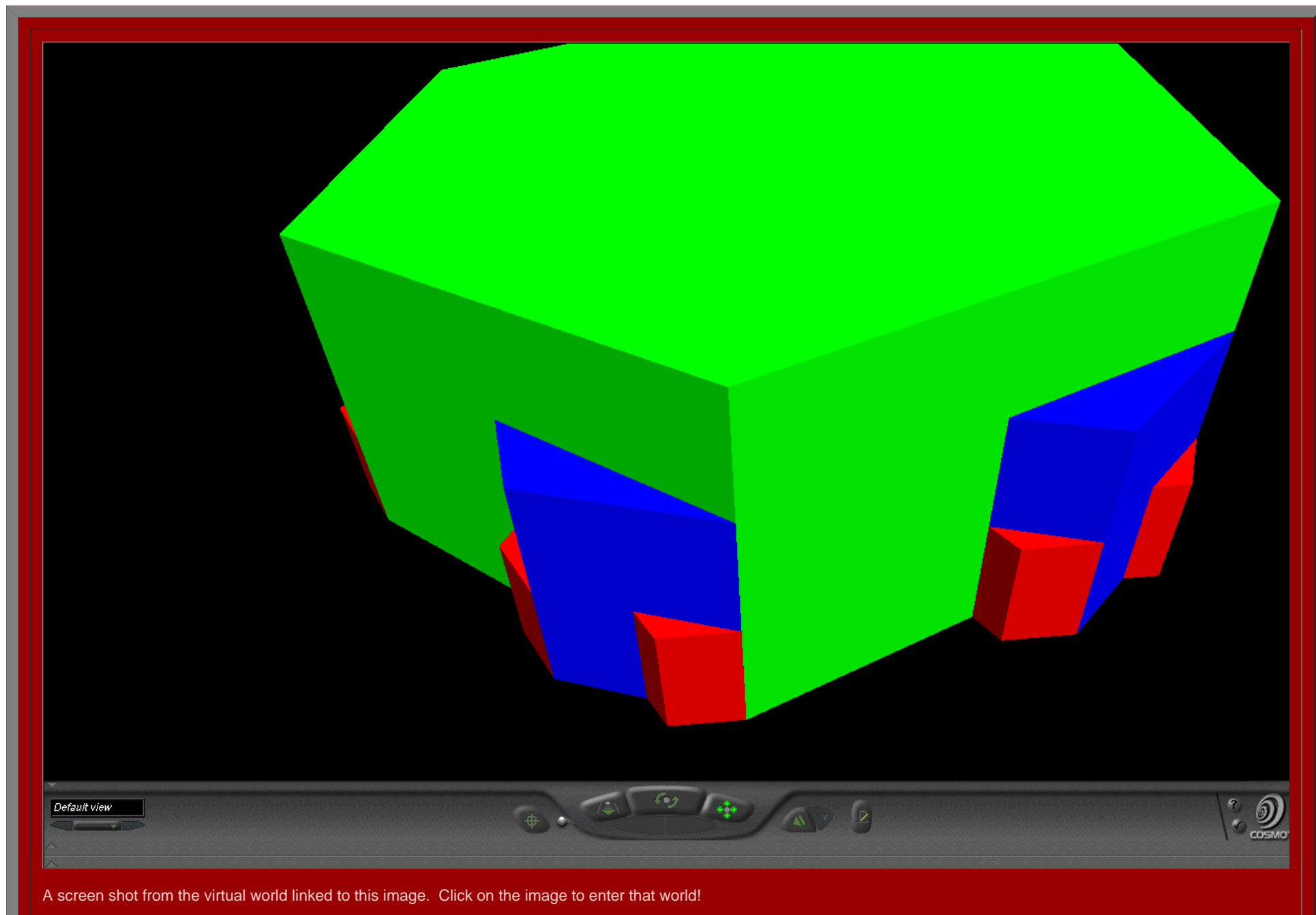


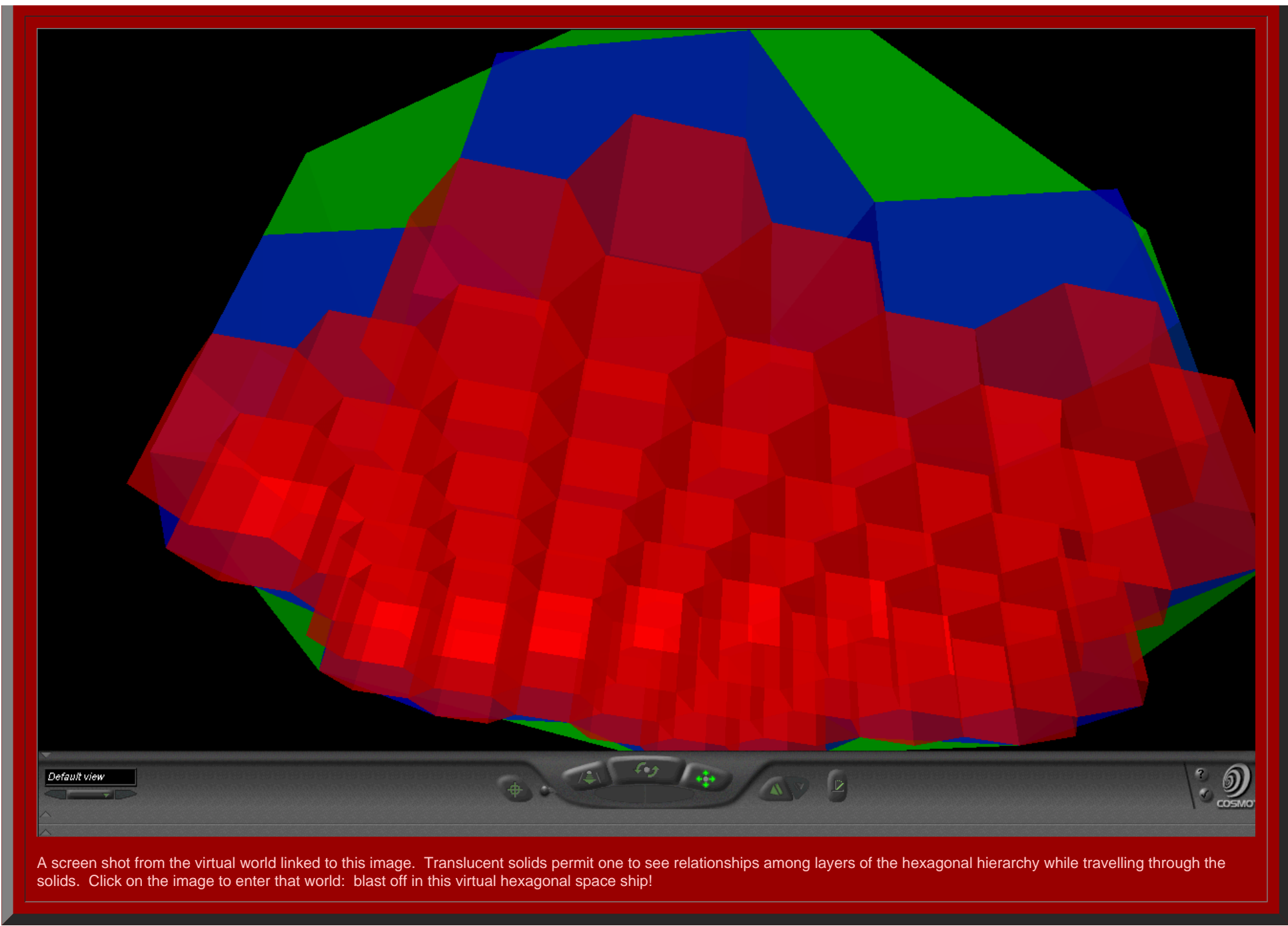
**Figure 2.7a.** Animated  $K=7$  fractal iteration sequence: first transformation using a three-sided fractal generator applied successively to sides of the hexagonal initiator.



**Figure 2.7b.** Animated  $K=7$  fractal iteration sequence: second transformation using a scaled-down three-sided fractal generator applied successively to sides of the blue polygon generated in Figure 2.7a.

It remains to determine if the polygons generated in Figure 2.7 will in fact fit together to form the broad central place landscape of arbitrary size suggested in [Figure 1.9](#). To that end, we stack the layers generated above using the fractal iteration sequence to form a tile of layers centered on the single polygonal initiator (Figure 2.8). Click [here](#) to see a virtual solid model of the tile with which the reader can interact. Click [here](#) to see a virtual translucent model of the tile with which the reader can interact. Consider the screen captures below from those virtual reality models.

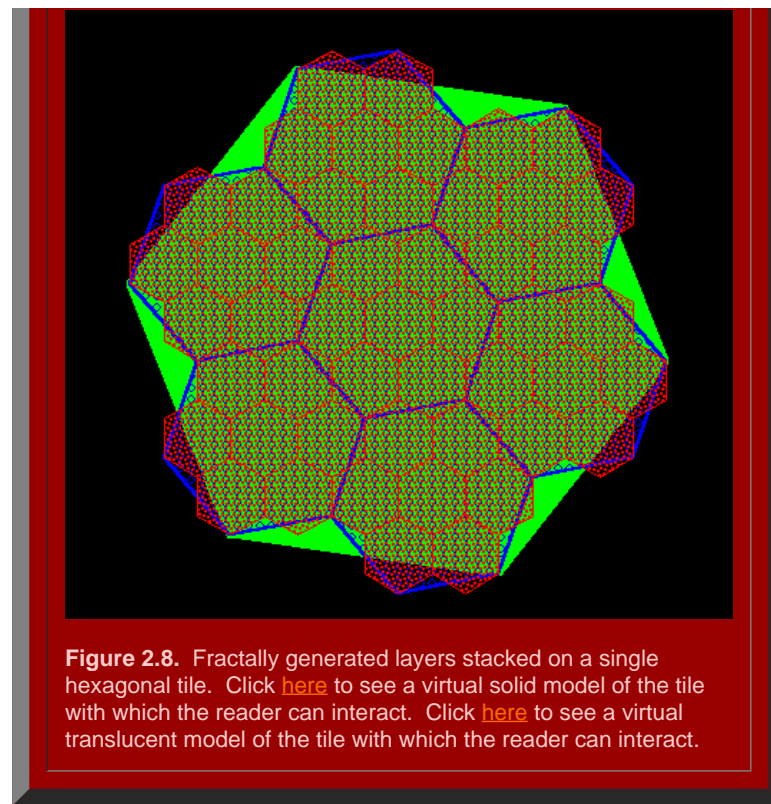




A screen shot from the virtual world linked to this image. Translucent solids permit one to see relationships among layers of the hexagonal hierarchy while travelling through the solids. Click on the image to enter that world: blast off in this virtual hexagonal space ship!

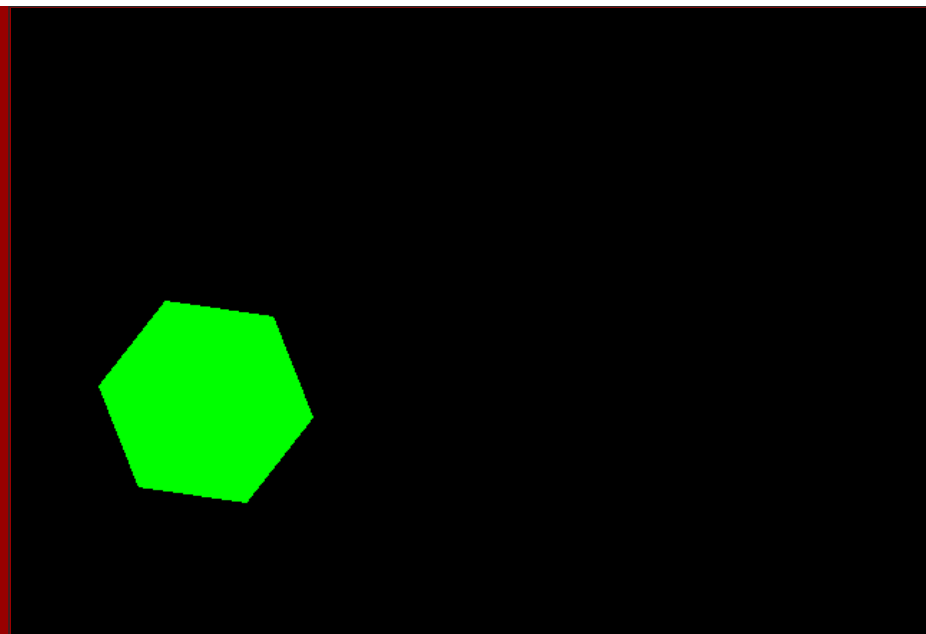




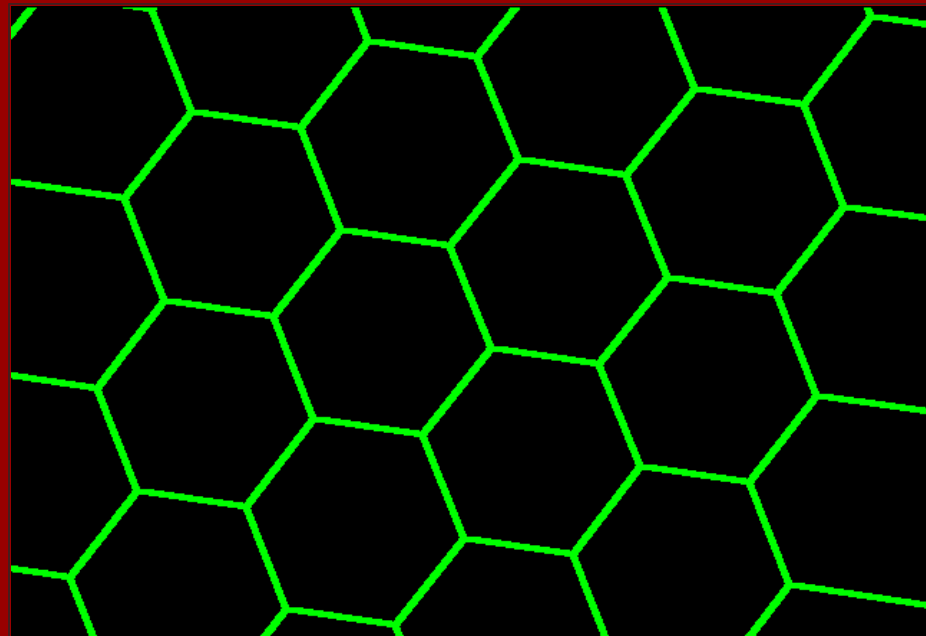


Finally, we tile the plane using the hexagonal initiators, from Figure 2.8 with the attached stack of smaller hexagons, to discover if the superimposed structure also fits together perfectly (Figure 2.9). Hexagonal tiles are used to cover the plane without gaps, as is the case with the sample of green hexagons in Figure 2.9a. The hexagons mesh perfectly to cover the plane (Theorem of [Gauss](#)). In Figure 2.9b, the green outline of the hexagons remains. Each of the solid green hexagons has had the fractal generator above applied and the consequent superimposed blue tiles come into view sequentially in this animation. Again, the fit is exact, as we had hoped it might be. Finally, in Figure 2.9c, the blue outline only is retained from Figure 2.9b (along with the green outline from Figure 2.9a). The final fractally generated layer derived from the blue polygons of Figure 2.9b comes into view in shades of red (or yellow/gold for contrast). The final layer of hexagonal base of unit hexagons appears last. The fit is perfect: each green hexagon contains the equivalent of four blue hexagons and each blue hexagon contains the equivalent of four red hexagons. The fractal generation procedure created exactly the classical central place landscape of [Figure 1.9](#) (return here using the Back button on your browser). As the animation proceeds in Figure 2.9, further layers of the fractally generated hierarchy, attached to the tile in Figure 2.8, come into view illustrating an exact meshing of tiles at all levels to form a  $K=3$  hierarchy.

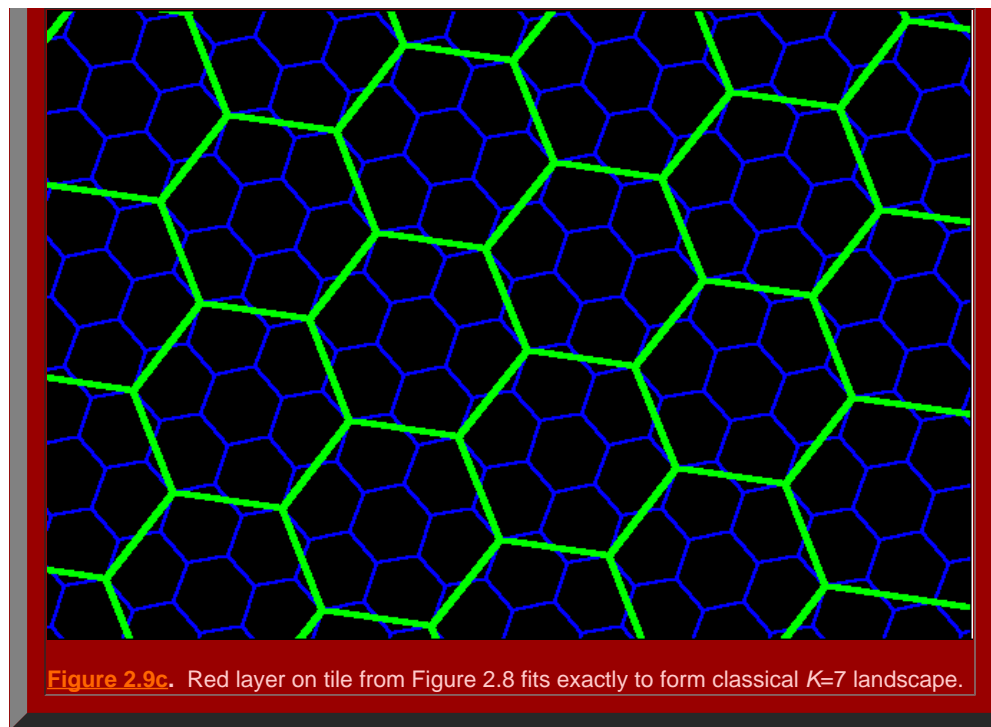




**Figure 2.9a.** Green layer on tile from Figure 2.8 fits exactly to tile the plane (Theorem of Gauss).



**Figure 2.9b.** Blue layer on tile from Figure 2.8 fits exactly to form classical  $K=7$  landscape.



**Figure 2.9c.** Red layer on tile from Figure 2.8 fits exactly to form classical  $K=7$  landscape.

Thus, the complex mechanics of classical central place theory come alive as a single dynamic system when viewed using fractal geometry. The fit is exact.

#### The added role of the fractional dimension

A fractal iteration sequence, such as those above but carried out infinitely, might be thought to increase the extent to which a line "fills" space. [Mandelbrot](#) captures this notion of space-filling with the concept of fractional dimension (hence "fractal"). He uses Hausdorff-Besicovitch dimension to measure the enduring mathematical concept of space-filling (see bibliography). We employ Mandelbrot's formulation for fractional dimension  $D$  as,  $D = \log(\text{number of generator sides}) / \log(\text{square root of } K)$ . Thus, the following values for fractally-generated central place hierarchies emerge:

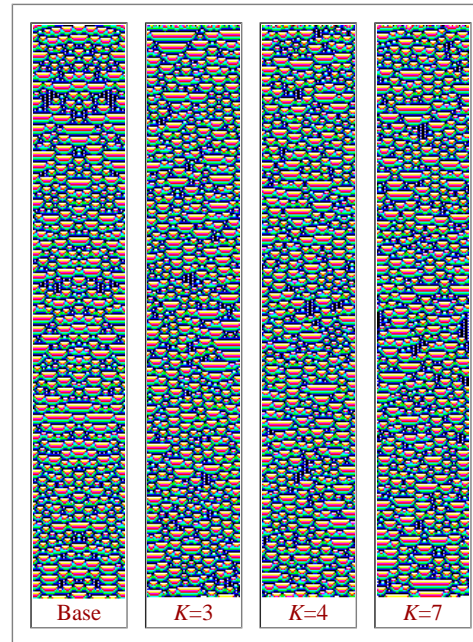
- $K=3$ ,  $D = \log 2 / \log \sqrt{3} = 1.2618595$
- $K=4$ ,  $D = \log 3 / \log 2 = 1.5849625$
- $K=7$ ,  $D = \log 3 / \log \sqrt{7} = 1.1291501$

The idea with the space-filling is to pick an arbitrary point in the bounded space containing the curve. Place a circle of arbitrarily small radius around that point. Does that circle contain a point on the curve as the fractal iteration sequence goes to infinity? If that is the case for any point, then the curve is said to fill space and have dimension 2. If not, then there are holes or gaps (perhaps of infinitesimal size) in the space and the curve fails to fill space completely and has fractional dimension between 1 and 2 (as a sort of Swiss cheese with holes). Thus, the  $K=4$  fractal iteration sequence, if permitted to repeat infinitely, has the highest fractional dimension of these three: this curve gets "closer" to arbitrary points in space than do the lines of the other hierarchies, as one might hope a hierarchy interpreted as a "transportation" hierarchy would. The fractional dimension of the fractal iteration sequence corresponds to the intuitive notion of scholars over time as to interpretation: another benchmark or field test of theory. The  $K=7$  fractal iteration sequence, if permitted to repeat infinitely, has the lowest fractional dimension of these three, keeping control from the center optimized and hence supporting the "administrative" or "separation" interpretation often given to the classical  $K=7$  hierarchy. Finally, the  $K=3$  falls between: marketing needs greater spatial penetration than does administration but less than does transportation. Here, the fit between classical interpretation and fractal calculation is reasonable (one could never say "exact" because the terms "marketing," "transportation," and "administrative" are inexact terms themselves).

What is difficult with fractals is to visualize the infinite process. Graphic color display, including three dimensional display, offers exciting strategies for visualization. Very quickly, however, it becomes difficult to draw the fine lines required by repeating the process at more and more local scales: physical lines have width. Electronic lines can be controlled and made finer than can pen lines, but eventually the line-width limits the capability to produce graphic images. Eventually, the mind's eye must take over and extrapolate the visual infinite process.

Another possibility might be to draw on the other human senses to aid in that extrapolation. Thus, Figure 2.10 shows figures generated by [Fractal Music 1.9](#); click on the images and hear the associated music. The left figure shows the cellular automata base generated by default--it is bilaterally symmetric about a central vertical line and was generated using a symmetrically

arranged initiator string of 64 digits ranging in value from 0 to 7 (one for each tone). The next figure,  $K=3$ , shows the cellular automata diagram (another sort of "bubble foam" in appearance) generated using the value for the fractional dimension of the  $K=3$  hexagonal hierarchy carried out to 64 decimal places as the initiator string for the music. The next figure,  $K=4$ , shows the cellular automata diagram generated using the value for the fractional dimension of the  $K=4$  hexagonal hierarchy carried out to 64 decimal places as the initiator string for the music. The final figure,  $K=7$ , shows the cellular automata diagram generated using the value for the fractional dimension of the  $K=7$  hexagonal hierarchy carried out to 64 decimal places as the initiator string for the music. Click on each figure to hear the music. Each musical sequence, of over 1000 steps, was created from the default base, changing only the initiator string, so that the fractional dimension is what operates on a "seed" value of basic notes. The listener should hear the basic pattern in all characterizations: great symmetry in the base value; abrupt changes of state in the  $K=3$  value; a smoother filling of musical space in the  $K=4$  music; and, gaps in the  $K=7$  musical characterization derived from the  $K=7$  fractional dimension. The bilaterally symmetric bubble-foam figure associated with the base has been skewed by each of the  $K$ -values, as has the corresponding music. Thus, we extend visualization from two dimensional graphical images to three dimensional graphical images to the mind's eye, and finally, to the mind's ear: capturing hierarchical pattern through 1000 steps or more is easy in the musical clips. Such characterization offers added capability to those of us with all of our senses that are functional: for those with limited visual sensory function, it offers a way to an auditory "visualization" of the beauty of geometry.



**Figure 2.10.** Fractal music connection. Click on the images generated by the fractional dimensions of the hexagonal hierarchies; compare these to the default base created by the software.

In subsequent chapters, we offer mathematical proof of these ideas and extensions of these ideas into new realms. The classical is used for alignment of new with the old: a strategy useful in a wide range of theoretical and applied research.

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Spatial Synthesis: Centrality and Hierarchy, Volume I, Book 1.  
Sandra Lach Arlinghaus and William Charles Arlinghaus