Chapter 4
Geometric Representations for Arbitrary Hierarchies

In the last chapter, we saw the complex mechanics of classical central place theory come alive as a single dynamic system when viewed through the lens of fractal geometry. The fit of the classical and fractal geometric hierarchies is exact. Thus, as one might use a carefully surveyed topographic map, with field-checked spot elevations, as a guide into dense jungle or other unsurveyed landscapes, so too we use our carefully field checked alignment of the classical and the fractal as a guide into unseen or unproven areas of theoretical geography. The difference is that the "field" tests in one case occur in "terrestrial space" while in the other the "field" tests occur in "geometry, number theory, and pure mathematics."

We saw hexagonal hierarchies, of different orientation, cell size, and stacking characteristics, arise from the same base of unit hexagons. These were associated with three integers: 3, 4, and 7. The thoughtful reader might naturally ask a number of questions, such as:

- are there other numbers that would serve as *K* values or are 3, 4, and 7 the only such values?
    - are 5 or 6 possible *K* values?
    - are there *K* values larger than 7?
    - how many *K* values are there?

  Dacey considered these issues.
- How does one determine the number of sides in a fractal generator that will generate a correct hierarchy for arbitrary *K* values?
- How does one determine fractal generator shape that will generate a correct hierarchy for arbitrary *K* values?

In this chapter we offer geometric representations for higher numerical *K* values. Subsequent chapters will develop the number theory required to execute these constructions.

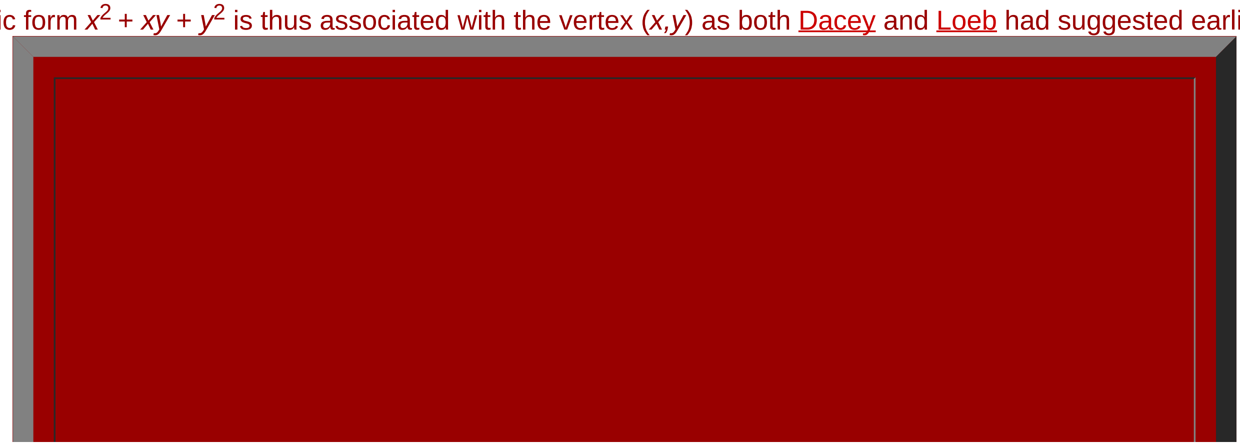
Coordinatization of the triangular lattice

Previous chapters employed a lattice of points to represent central places as a synthetic (coordinate-free) system. It is straightforward to assign a coordinate system, as well. If a point is given the usual Euclidean coordinate (x,y), based on perpendicular axes, its distance from the origin is easily calculated to be $(x^2 + y^2)^{1/2}$. Thus, there is a natural association of the quadratic expression $x^2 + y^2$ with the point (x,y). Such an expression is sometimes called a quadratic form. Quadratic forms have been studied extensively and material about them will appear in Chapter 5.

When the plane is occupied by a set of hamlets, each equidistant from its nearest neighbors, it seems more reasonable to pick a coordinate system which gives coordinates (0,0), (1,0), and (0,1) to three hamlets which are vertices of an equilateral triangle. Thus it is natural to have a horizontal x-axis and a y-axis at angle 60° to the x-axis, as illustrated in Figure 4.1. Now, however, the calculation of distance between a point with coordinates (x,y) and the origin (0,0) is more complicated. Figure 4.1 illustrates the situation when x and y are both positive. The right triangle whose hypotenuse has length this distance now has vertical side of length $\sqrt{3}y/2$ and horizontal side of length $y/2+x$. Thus the distance, d, is

$$\sqrt{(\sqrt{3}y/2)^2 + (y/2+x)^2} = \sqrt{3y^2/4 + y^2/4 + yx + x^2} = \sqrt{x^2 + xy + y^2}$$

The quadratic form $x^2 + xy + y^2$ is thus associated with the vertex (x,y) as both Dacey and Loeb had suggested earlier.



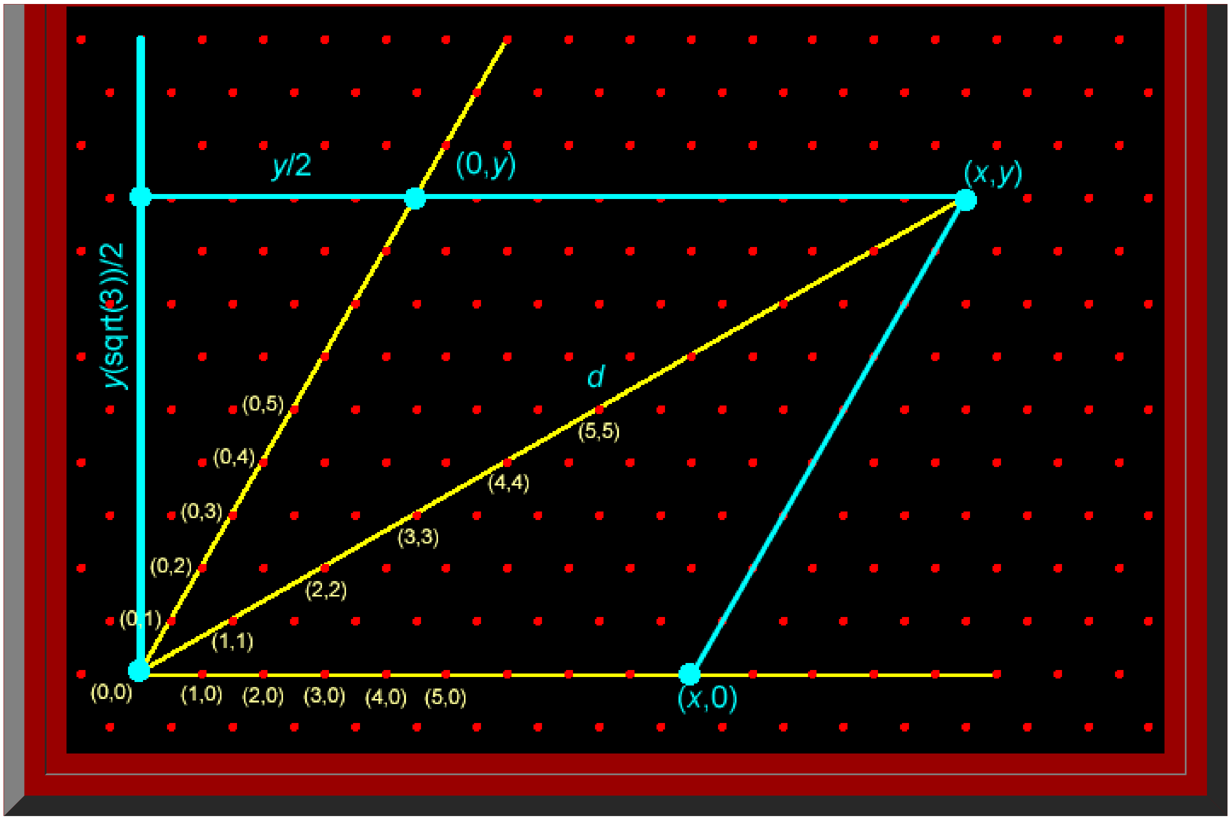


Figure 4.1. The standard *y*-axis of Cartesian coordinates is shown in cyan (turquoise). The yellow lines show the oblique axes. The value *d* represents the distance between the origin and an arbitrary point, (*x*,*y*), drawn in this case on the line $y = x$ of the oblique axes.

The illustration in Figure 4.1 shows both values as positive. Thus, it must be established that this distance is the same even when *x* and *y* are not both positive. When *x* and *y* are both negative. $|x|$ must be substituted for *x*, and $|y|$ must be substituted for *y*, but the calculation remains the same.

When exactly one of *x* or *y* becomes negative, the geometry is different. Now the vertical side of the relevant right triangle has side of length

$$\sqrt{3}\,|y|/2$$

but the horizontal side is now of length x-$|y|/2$.

Thus the appropriate distance is

$$\sqrt{(-\sqrt{3}y/2)^2+(x-(-y)/2)^2} = \sqrt{x^2+xy+y^2}$$

When *x* is negative and *y* is positive, a similar calculation results.

Thus the form $x^2 + xy + y^2$ is indeed associated with the vertex (*x*,*y*) in this coordinate system. Any positive integer that is equal to $x^2 + xy + y^2$ for some integers *x* and *y* is called
Löschian, after August Lösch. One mathematical question that then occurs is: what integers are Löschian? The next chapter (Chapter 5) develops the number theory necessary to answer that question. Other issues involve values of *K* greater than 7 as well as questions involving number of generator sides and generator shape mentioned above. The rest of this chapter deals with these items.

Higher *K*-values: An Infinite Number

Earlier research, by August Lösch, Michael Dacey, and others shows illustrations of *K*-values greater than 7. Indeed, research by Arthur Loeb, in crystallography, and Dacey, in geography, led to independent discovery that the Diophantine equation, $x^2+xy+y^2 = K$ would generate all *K* values when pairs of positive integers were substituted for *x* and for *y*. Thus, when $(x,y)=(1,1)$ the equation $x^2+xy+y^2 = K$ yields a value of K=3; when $(x,y)=(0,2)$, it follows that K=4; and, when $(x,y)=(1,2)$, it follows that K=7. Pairs such as (0,0) and (1,0) yield only trivial results so that the values of 3, 4, and 7 are the three smallest *K*-values. There are no other *K* values less than 7.

The result of Loeb/Dacey is important because it shows

- that there is an infinite number of possible *K* values
- that this infinity of values is in one-to-one correspondence with the integral lattice points in the plane
- that one can give a numerical generating function to create *K* values

Thus, a graph of lattice points in the plane offers a convenient method of visualizing *K*-values larger than 7 (Figure 4.2).

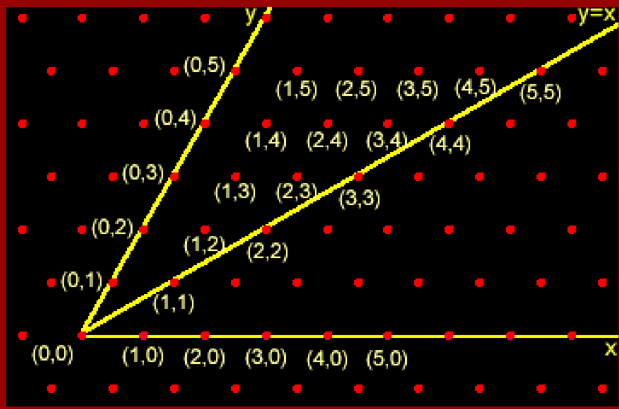


Figure 4.2. The coordinatized lattice points in yellow transform into *K*-values in cyan using the Diophantine equation $K=x^2+xy+y^2$

The Diophantine equation, $x^2+xy+y^2 = K$, is symmetric: the points (1,2) and (2,1) represent different locations in the plane. They both generate the same *K* value. Thus, it suffices to consider only a portion of the available lattice points: those in the first quadrant on one side of the line y=x. Because the pattern of points associated with central place hierarchies is based on a triangular lattice, the graph of lattice points displayed in Figure 4.2 is based on an oblique, rather than on a rectangular, coordinate system with angles at the origin of 60 and 120 degrees. Combining considerations of symmetry and oblique coordinates led to Figure 4.2. In that figure, both coordinate pairs, and corresponding *K*-value generated by the Diophantine equation, are shown.

With an infinite number of *K*-values available, it is a daunting consideration to try to figure out how to create a nested hierarchy of hexagonal trade areas suited to each value of *K*. Simple drawing skills do not suffice. One needs a formal strategy that can be replicated at will. Fractal geometry will offer that capability.

In the previous chapter, fractal generators appear simply to be plucked out of thin air: there is art in generator selection. Actually, the generator for the *K*=7 hierarchy suggested itself naturally with observations of overfit/underfit of the six small hexagons in relation to the boundary of the next larger one (Figure 4.3). From there, it was simply a matter of educated guessing and trial and error to find generators for *K*=3 and *K*=4 hierarchies. Someone with considerable experience working with this approach might, for example, figure out a fractal generator for a *K*=76 hierarchy, associated with the lattice point (4,6) (Figure 4.4). For replication by arbitrary individuals to be successful, however, one needs to capture the art in the formalized theory and language of mathematics: to transform art into science.

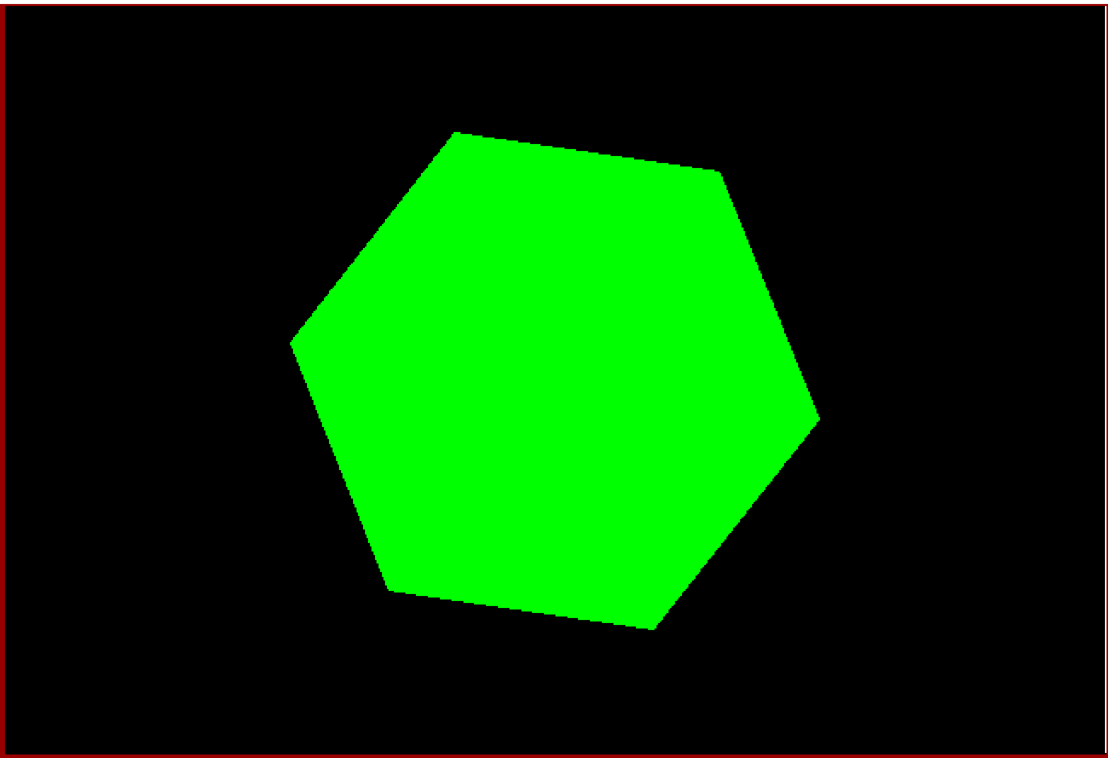


Figure 4.3. Overfit and underfit of small to larger hexagons emphasized: fractal generator, a zig-zag three sided shape, emerges from such observation.

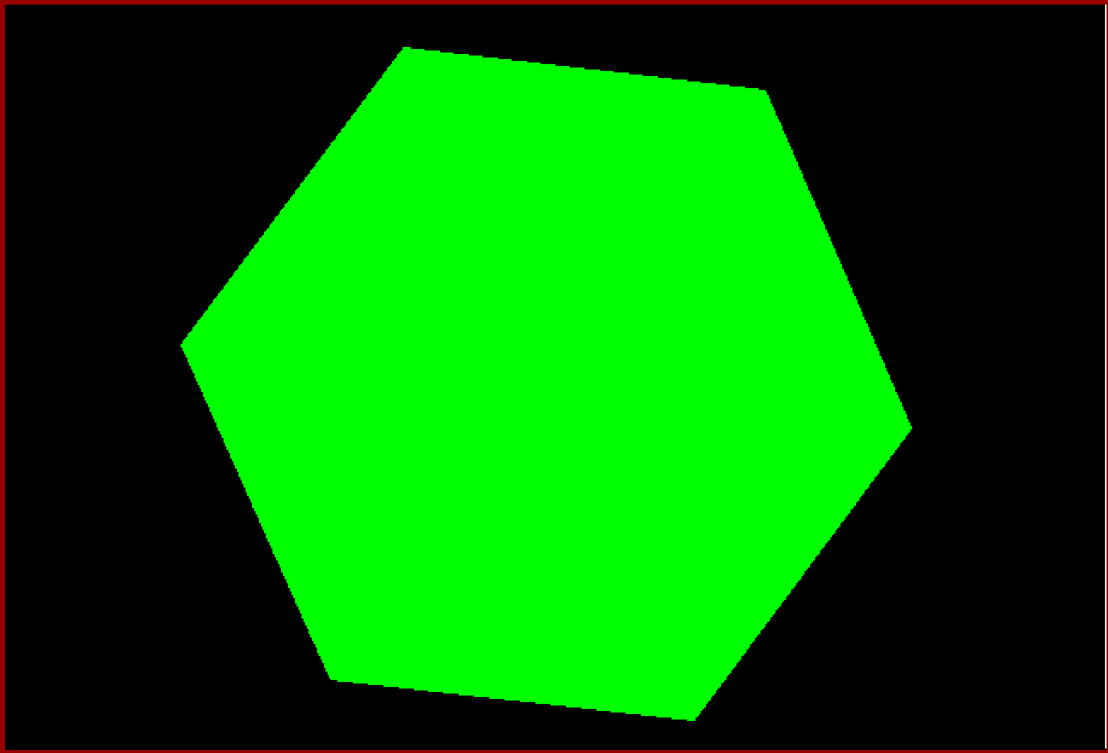


Figure 4.4. *K*=76 fractal generation; corresponding lattice point has coordinates (4,6). The generator shape appears at the edge.

Fractal Generation of Arbitrary K-Values: Geometric Evidence

Once one generator shape is known, can it be used to determine other generator shapes? The answer is yes. The table in Table 4.1 illustrates that the K=3 generator, with a half a "hexstep" added, leads to the K=7 generator.

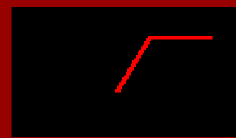
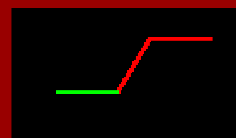
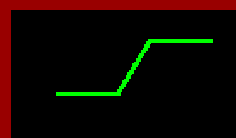
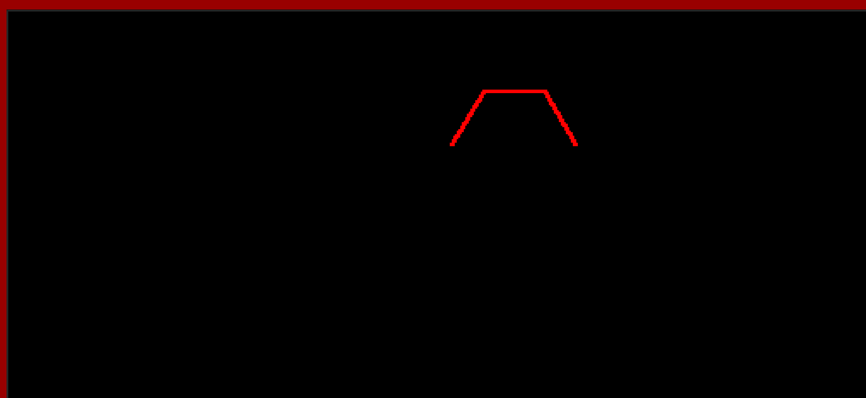
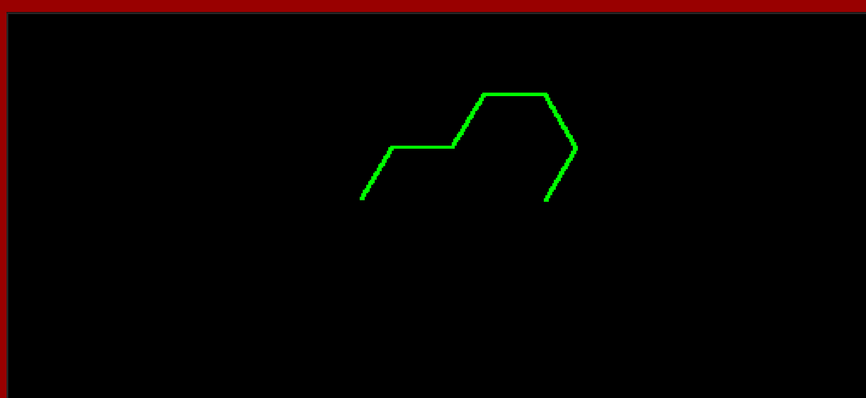
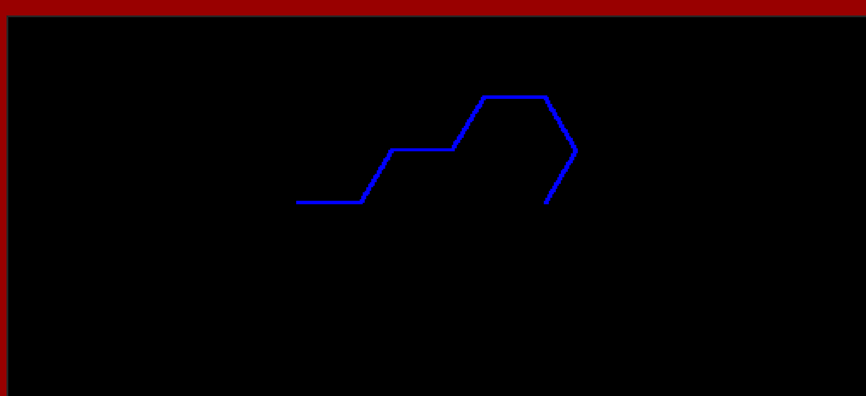
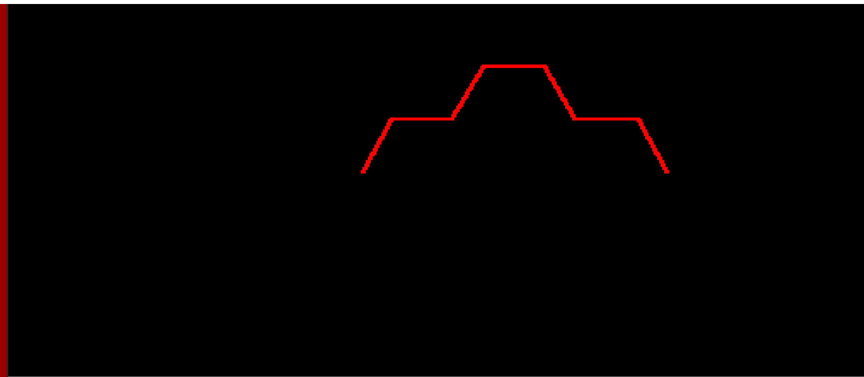
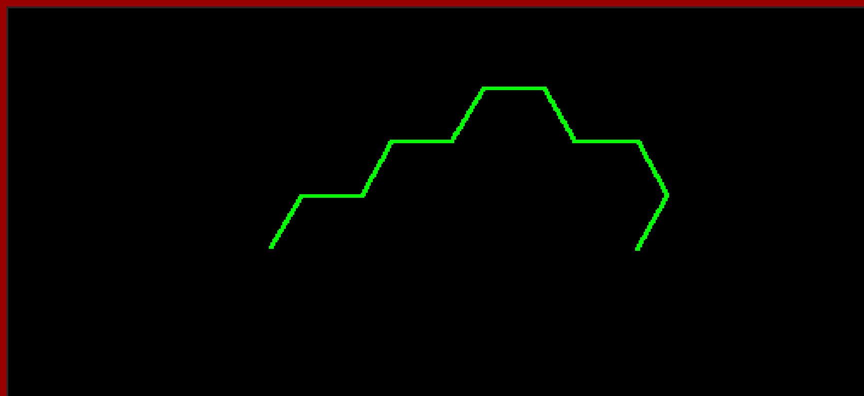
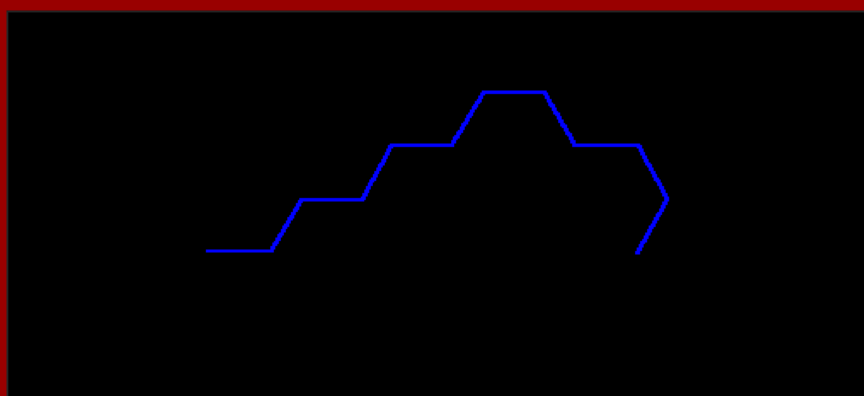
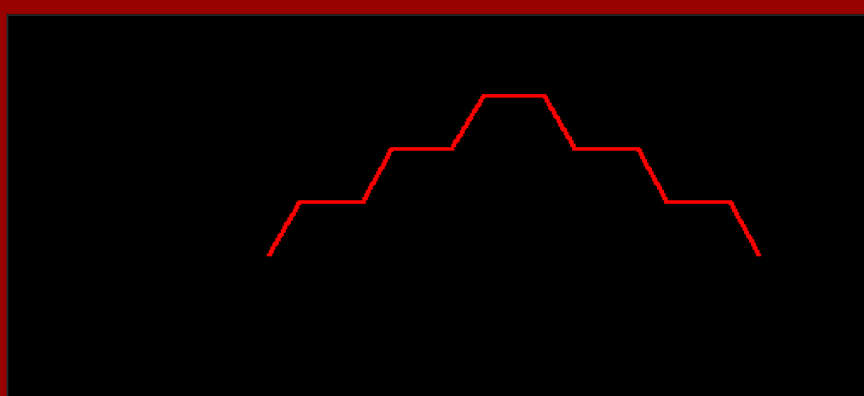
K=3 generator.	
K=3 generator (red) with one half hexstep added (green).	
K=7 generator	

Table 4.1. The K=3 generator leads to the K=7 generator.

To try to uncover pattern useful in creating fractal generators for arbitrary K values, we begin by looking at a subset of K values: those along the y-axis, only. In that case, because $x^2+xy+y^2 = 0 + 0 + y^2 = K$, it follows that the square root of K is an integer and that the lattice points (0,y) on the y-axis may be rewritten as (0, $\sqrt{K}$). Trial and error with finding fractal generators for selected values along the y-axis produced the table of generators shown in Figure 4.5. The trivial value of K=1, associated with the point (0,1) is not shown in this figure. Within that "trial and error" process a pattern emerged, demonstrated in the animated Figure 4.5.

- Begin with the K=4 generator (lowest value on the y-axis): generator in red.
  - To the previous generator, add a full hexstep to the left of it and a half hexstep, curled under, to the right: generator in green. This new generator produces the K=9 hierarchy.
  - To the previous generator, add a half hexstep to the left (flat portion of the step): generator in blue. This new generator produces the K=16 hierarchy.
- Add two full hexsteps, one on each side, to the K=4 generator, creating a symmetric generator that produces the K=25 hierarchy.
  - To the previous generator, add a full hexstep to the left of it and a half hexstep, curled under, to the right: generator in green. This new generator produces the K=36 hierarchy.
  - To the previous generator, add a half hexstep to the left (flat portion of the step): generator in blue. This new generator produces the K=49 hierarchy.
- Add two full hexsteps, one on each side, to the K=25 generator, creating a symmetric generator that produces the K=64 hierarchy.
  - To the previous generator, add a full hexstep to the left of it and a half hexstep, curled under, to the right: generator in green. This new generator produces the K=81 hierarchy.
  - To the previous generator, add a half hexstep to the left (flat portion of the step): generator in blue. This new generator produces the K=100 hierarchy.

Lattice Coordinate	*K*-value (y-axis values)	Generator Shape	Hierarchy Type
(0,2)	*K*=4		4
(0,3)	*K*=9		3
(0,4)	*K*=16		7

(0,5)	K=25		4
(0,6)	K=36		3
(0,7)	K=49		7
(0,8)	K=64		4
(0,9)	K=81		3

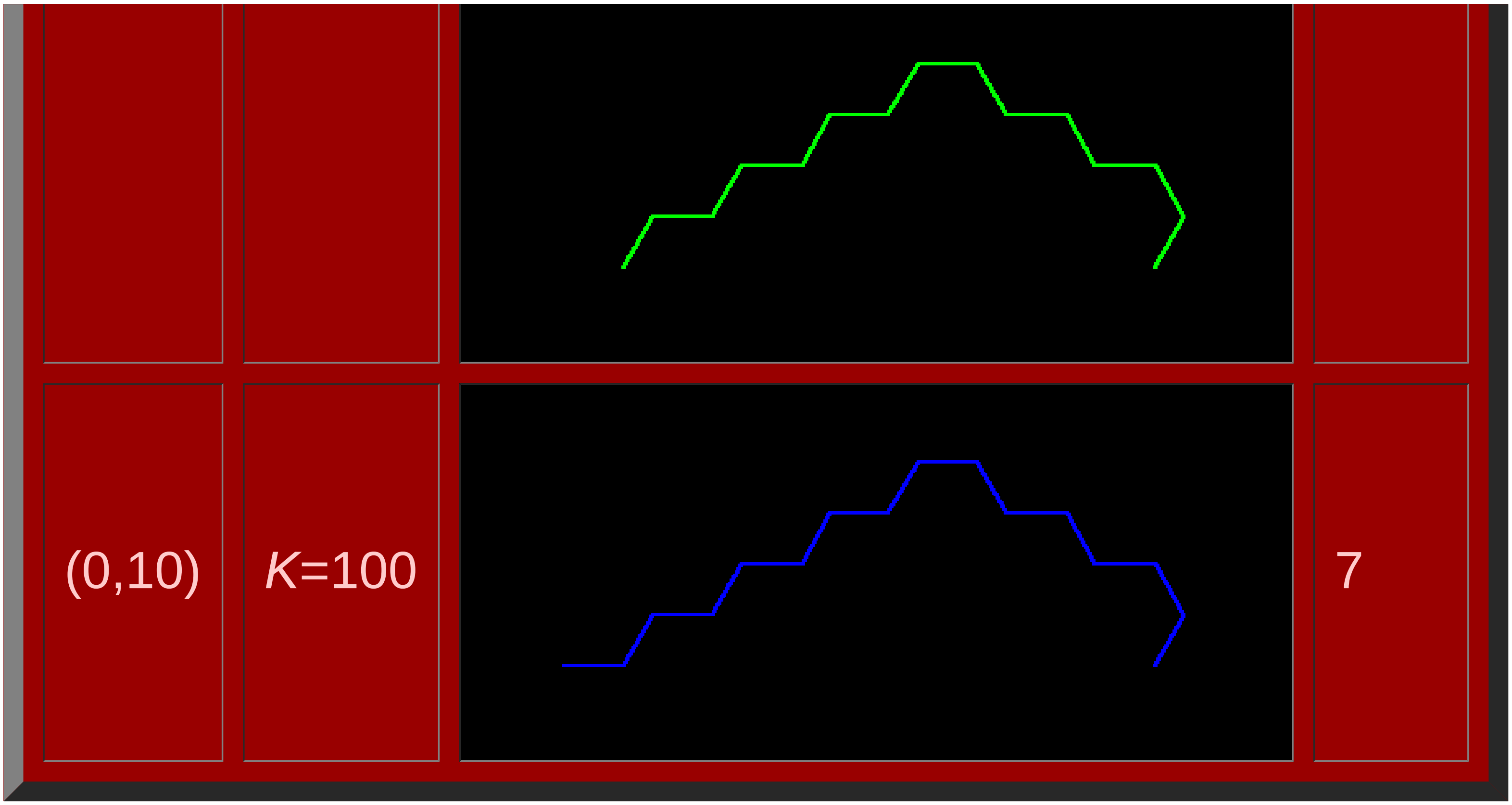


Figure 4.5. Fractal generators for selected values on the *y*-axis of lattice points generated by $x^2+xy+y^2 = K$. Animation and static versions.

Broadly speaking, we observe the following pattern (refer for this discussion to the static portion of Figure 4.5; the colors match).

- **(0,2), *K*=4.** Begin with the isosceles trapezoidal shape of the *K*=4 hierarchy.
  - Note its bilateral symmetry about a vertical line.
  - Note that it takes two sides to climb the right side to the top of the generator
- **(0,3), *K*=9.** For *K*=9, begin with the *K*=4 generator and
  - add one full hex-step to the left
  - add one half hex-step to the right, tucked under at 120 degrees
  - note that it takes three sides to climb the right side to the top of the generator, one more than it did for the "begin" value
- **(0,4), *K*=16.** For *K*=16, begin with the *K*=9 generator and
  - add one half level part of a hex-step to the left
  - note that it takes the same number of sides to climb the right side to the top of the generator as it did in the previous case.
- **(0,5), *K*=25.** For *K*=25, begin with the *K*=4 generator and
  - add a full hex-step to the left
  - add a full hex-step to the right
  - note that it takes four sides to climb the right side to the top of the generator
  - note the bilateral symmetry about a vertical line
- **(0,6), *K*=36.** For *K*=36, begin with the *K*=25 generator and
  - add one full hex-step to the left
  - add one half hex-step to the right, tucked under at 120 degrees
  - note that it takes one more side to climb the right side to the top of the generator than it did for the "begin" value
- **(0,7), *K*=49.** For *K*=49, begin with the *K*=36 generator and
  - add one half level part of a hex-step to the left
  - note that it takes the same number of sides to climb the right side to the top of the generator as it did in the previous case.
- **(0,8), *K*=64.** For *K*=64, begin with the *K*=25 generator and
  - add a full hex-step to the left
  - add a full hex-step to the right
  - note that it takes six sides to climb the right side to the top of the generator

note the bilateral symmetry about a vertical line

- **(0,9), *K*=81.** For *K*=81, begin with the *K*=64 generator and
    - add one full hex-step to the left
    - add one half hex-step to the right, tucked under at 120 degrees
    - note that it takes one more side to climb the right side to the top of the generator than it did for the "begin" value
- **(0,10), *K*=100.** For *K*=100, begin with the *K*=81 generator and
    - add one half level part of a hex-step to the left
    - note that it takes the same number of sides to climb the right side to the top of the generator as it did in the previous case.

continue, using the pattern without Figure 4.5

- **(0,11), *K*=121.** For *K*=121, begin with the *K*=64 generator and
    - add a full hex-step to the left
    - add a full hex-step to the right
    - note that it takes eight sides to climb the right side to the top of the generator
    - note the bilateral symmetry about a vertical line
- **(0,12), *K*=144.** For *K*=144, begin with the *K*=121 generator and
    - add one full hex-step to the left
    - add one half hex-step to the right, tucked under at 120 degrees
    - note that it takes one more side to climb the right side to the top of the generator than it did for the "begin" value

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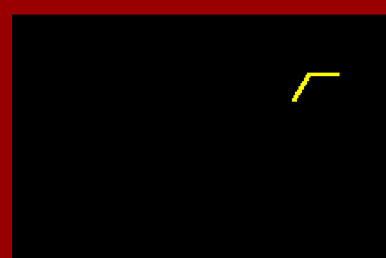
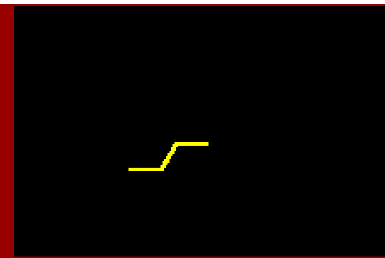
After a number of cases, there appears to be a clear pattern, color-coded as red-green-blue. If that pattern is correct, then the reader could, through successive (but perhaps tedious) application of this pattern, produce a fractal generator for an arbitrary value on the *y*-axis. We will show a more systematic procedure later in this Chapter for finding generator size and shape; it will, however, be based on the observations above, so it is important that the reader have a clear understanding of them.

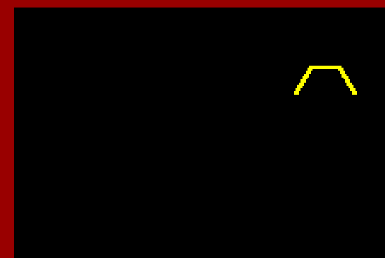
To produce generators for K-values not on the y-axis is also a simple matter of examining pattern. Figure 4.6 shows how to form generators for all other *K*-values (not on the *y*-axis) simply by adding hex-steps to the *y*-axis *K*-value generators as one moves to the right across lines of lattice points parallel to the line *y*=*x* (cyan lines in Figure 4.7). Thus, the reader could then form a fractal generator for an arbitrary *K*-value.

Y-axis value and line position	Fractal generators off the *y*-axis as determined by *y*-axis generator shape.
(0,0): the first generator appears for *K*=3 (not on the *y*-axis; *K*=0, on the *y*-axis, is disregarded) and subsequent generators appear for *K*-values of 12, 27, and 48 as one moves to the right across the line *y*=*x*.	
(0,1): the first generator appears for *K*=7 (not on the *y*-axis; *K*=1, on the *y*-axis, is disregarded) and subsequent generators appear for *K*-values of 19, 37, and 61 as one moves to the right across the line displaced by 1 from, and parallel to, the	

line $y=x$.



(0,2): the first generator appears for $K=4$ and subsequent generators appear for K-values of 13, 28, 49, and 76 as one moves to the right across the line displaced by 2 from, and parallel to, the line $y=x$.



(0,3): the first generator appears for $K=9$ and subsequent generators appear for K-values of 21, 39, 63, and 93 as one moves to the right across the line displaced by 3 from, and parallel to, the line $y=x$.

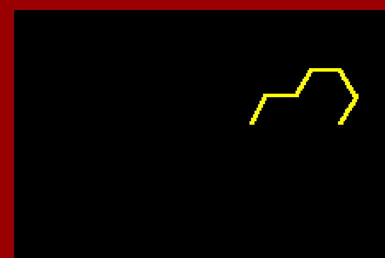


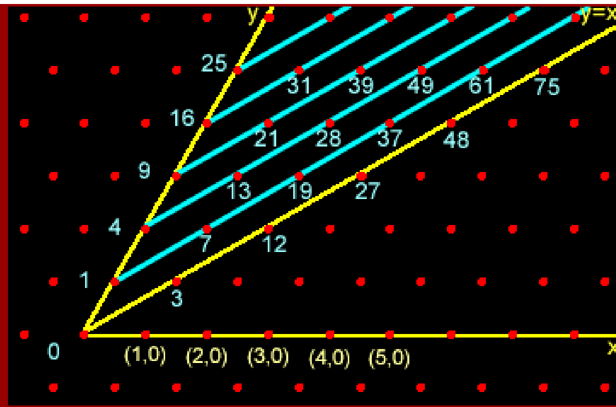
Figure 4.6. Fractal generators are produced for arbitrary K-values from y-axis K-values. See the cyan lines in Figure 4.7.

Fractal Generation of Arbitrary K-Values: the Cross-cut Equation as a Systematic Characterization of Generator Size and Shape

In Figure 4.5, the "red" category corresponds to a generator type similar to the $K=4$ generator: an isosceles trapezoid with some number of hex-steps on either side, producing a bilaterally symmetric form. The "blue" category corresponds to a generator type similar to the $K=7$ category, identical to the previous case but with one half level hex-step added to the left. If one looks at the trivial cases of $K=0$ and $K=1$ in Figure 4.6, this style of pattern is evident in moving from $K=3$ to $K=7$. Thus, it appears that there are really only three basic, or primitive, styles of generator type: $K=3$ (green), $K=4$ (red), and $K=7$ (blue). Because generator shape on the y-axis determines generator shape off the axis along rays parallel to the line $y=x$ (cyan lines in Figure 4.7), we next consider partitioning the set of all K values according to these three categories using a set of parallel rays (cyan lines in Figure 4.7) to create that partition. Thus, we seek another equation to represent that partition with the intent of using it to cut across the Diophantine equation and provide straightforward solution to finding the number of sides and the shape of fractal generators for arbitrary K-values.

The pattern of numerical K values along rays parallel to $y=x$, shown as cyan lines, is captured in Figure 4.7. Thus,

- along $y=x$, we have $K=3x^2$: the K values along that ray, only, are generated exactly by substituting 0, 1, 2, 3, 4,... into that equation yielding K values of 0, 3, 12, 27, 48, ...
- along the ray displaced 1 unit from the line $y=x$, we have $K=3x^2+3x+1$: the K values along that ray, only, are generated exactly by substituting 0, 1, 2, 3, 4,... into that equation yielding K values of 1, 7, 19, 37, 61,...
- along the ray displaced 2 units from the line $y=x$, we have $K=3x^2+6x+4$: the K values along that ray, only, are generated exactly by substituting 1, 2, 3, 4,... into that equation yielding K values of 4, 13, 28, 49, 76,...
- along the ray displaced 3 unit from the line $y=x$, we have $K=3x^2+9x+9$: the K values along that ray, only, are generated exactly by substituting 0, 1, 2, 3, 4,... into that equation yielding K values of 9, 21, 39, 63, 93,...



Equations for y=x and cyan rays parallel to y=x:

$K=3x^2$ for the line y=x
$K=3x^2 + 3x + 1$, for the cyan line displaced by 1
$K=3x^2 + 6x + 4$, for the cyan line displaced by 2
$K=3x^2 + 9x + 9$, for the cyan line displaced by 3

Figure 4.7. Pattern suggesting cross-cut equation. The equation for the cyan rays, parallel to $y=x$, generate exactly the K-values associated with lattice points along that ray (when integers are substituted into those equations..

Generalizing from the observed pattern, it appears that the equation $K=3x^2 + 3bx + b^2$, where b counts the number of units the ray is displaced from $y=x$, will serve to capture all the K-values associated with the lattice points along a single horizontal ray parallel to $y=x$ (cyan, Figure 4.7), and to capture no others elsewhere. This idea is stated in the following theorem; the next chapter will present some of the number theoretic infrastructure behind it.

Fundamental Theorem (S. Arlinghaus, 1985, and S. Arlinghaus in S. and W. Arlinghaus, 1989).

> In a triangular central place lattice, the central place K-values, lying along any single one of the rays parallel to the line $y=x$, are generated by substituting non-negative integral values into a quadratic equation $K=3x^2 + 3bx + b^2 = 3x^2 + 3(\sqrt{K})x + (\sqrt{K})^2$, where $b=0, 1, ..., n, ...$ counts the number of units the ray is translated from $y=x$.

Thus, any single K value has two quadratic equations that represent it. We now proceed to use properties of these quadratic equations to extract information from them sufficient to determine the number of fractal generator sides and number of hex-steps for arbitrary K values from algebraic information, alone. Thus, an informed reader will have the capability to replicate the results, independent of relative position on any image. As an analogy, this process is similar to using latitude and longitude as numerical measures to determine absolute position on the earth-sphere rather than using parallels and meridians as geometric measures to determine relative position on the earth-sphere.

Number of Generator Sides

The patterns observed appear often to involve partitions related to the number 3. Thus, we consider a value based on divisibility by 3 as one index.

Define an integral value j, using K-values along the y-axis, only, as:

- If K is congruent to 0(mod 3), then $j=(\sqrt{K})/3$;
- If K is not congruent to 0(mod 3), then $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$, which ever is an integer (clearly, both cannot be integers--K leaves either a remainder of 1 or of 2 when divided by 3).

As some examples:

- **when K=4**, then 4 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-2)/3 = (2-2)/3=0$ is an integer.
- **when K=9**, then 9 is evenly divisible by 3 so that K is congruent to 0(mod 3). Thus, $j=(\sqrt{K})/3 = 3/3 = 1$.
- **when K = 16**, then 16 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-1)/3 = (4-1)/3=1$ is an integer.
- **when K=25**, then 25 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-2)/3 = (5-2)/3=1$ is an integer.
- **when K=36**, then 36 is evenly divisible by 3 so that K is congruent to 0(mod 3). Thus, $j=(\sqrt{K})/3 = 6/3 = 2$.
- **when K=49**, then 49 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-1)/3 = (7-1)/3=2$ is an integer.
- **when K=64**, then 64 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-2)/3 = (8-2)/3=2$ is an integer.
- **when K=81**, then 8 is evenly divisible by 3 so that K is congruent to 0(mod 3). Thus, $j=(\sqrt{K})/3 = 9/3 = 3$.
- **when K=100**, then 100 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-1)/3 = (10-1)/3=3$ is an integer.
- **when K=121**, then 121 is not evenly divisible by 3 so that K is not congruent to 0(mod 3). Thus, one of $j=((\sqrt{K})-1)/3$ or $j=((\sqrt{K})-2)/3$ is an integer. Indeed, $j=((\sqrt{K})-2)/3 = (11-2)/3=3$ is an integer.
- **when K=144**, then 144 is evenly divisible by 3 so that K is congruent to 0(mod 3). Thus, $j=(\sqrt{K})/3 = 12/3 = 4$.

The examples above are color-coded using the scheme of the geometric table to suggest pattern in the output of j-values. What the pattern suggests is, for K-values on the y-axis only,

- when K is congruent to 0(mod 3), then the number of sides for a fractal generator is given by $2+4j$
- when K is not congruent to 0(mod 3), then the number of sides for a fractal generator is given by $3+4j$.

Number of Generator Hex-steps: Generator Shape

To count the number of hex-steps for an arbitrary K-value on the y-axis, we employ not only the j-value calculated in the previous section, but also the well-known discriminant (square root of ($B^2 - 4AC$) in $Ax^2 + Bx + C$) of a quadratic form. Once again, we begin by considering a number of examples in order to study pattern and to piece it together with other pattern that is known. From the Fundamental Theorem, we know that $K=3x^2+3(\sqrt{K})x+(\sqrt{K})^2$. The discriminant of this equation is $D = (3(\sqrt{K}))^2 - 4\cdot3(\sqrt{K})^2= 9K - 12K = -3K$. To count the number of hex-steps for a fractal generator for a y-axis K-value, follow the strategy below:

- if K is congruent to 0(mod 3) and
  - if D is congruent to 1(mod 4), then the number of generator hex-steps is equal to the greatest integer in $(2+4j)/2 =1+2j$.
  - if D is congruent to 0(mod 4), then the number of generator hex-steps is equal to the greatest integer in $(2+4j)/2=1+2j$.
- if K is not congruent to 0(mod 3) and
  - if j is even (divisible by 2) then
    - if D is congruent to 1(mod 4), then the number of generator hex-steps is equal to the greatest integer in $(3+4j)/2$.
    - if D is congruent to 0(mod 4), then the number of generator hex-steps is equal to the greatest integer in $(5+4j)/2$.
  - if j is odd (not divisible by 2) then
    - if D is congruent to 1(mod 4), then the number of generator hex-steps is equal to the greatest integer in $(5+4j)/2$.
    - if D is congruent to 0(mod 4), then the number of generator hex-steps is equal to the greatest integer in $(3+4j)/2$.

Looking at some examples, again, the color-coding of red-green-blue shows that the number theoretic patterns provide an exact fit with the geometric pattern and they do so without requiring seeing previous cases: the number theoretic patterns offer absolute measures, replicable independent of examination of previous cases.

Thus, for K-values on the y-axis,

- **when K=4**, then $D = -3K = -12$ which is evenly divisible by 4. In addition, K is not congruent to 0(mod 3) and j=0 from above, which is even. Thus, the number of hex-steps is the greatest integer in $(5+4j)/2$ which is 2.
- **when K=9**, then $D = -3K = -27$ which is not evenly divisible by 4 and leaves a remainder of 1. In addition, K is congruent to 0(mod 3) and j=1 from above. Thus, the number of hex-steps is the greatest integer in $1+2j$ which is 3.
- **when K = 16**, then $D = -3K = -48$ which is evenly divisible by 4. In addition, K is not congruent to 0(mod 3) and j=1 from above, which is odd. Thus, the number of hex-steps is the greatest integer in $(3+4j)/2$ which is 3.
- **when K=25**, then $D = -3K = -75$ which is not evenly divisible by 4 and leaves a remainder of 1. In addition, K is not congruent to 0(mod 3) and j=1 from above, which is odd. Thus, the number of hex-steps is the greatest integer in $(5+4j)/2$ which is 4.
- **when K=36**, then $D = -3K = -108$ which is not evenly divisible by 4 and leaves a remainder of 1. In addition, K is congruent to 0(mod 3) and j=2 from above. Thus, the number of hex-steps is the greatest integer in $1+2j$ which is 5.
- **when K=49**, then $D = -3K = -147$ which is not evenly divisible by 4. In addition, K is not congruent to 0(mod 3) and j=2 from above, which is even. Thus, the number of hex-steps is the greatest integer in $(3+4j)/2$ which is 5.
- **when K=64**, then $D = -3K = -192$ which is evenly divisible by 4. In addition, K is not congruent to 0(mod 3) and j=2 from above, which is even. Thus, the number of hex-steps is the greatest integer in $(5+4j)/2$ which is 6.
- **when K=81**, then $D = -3K = -243$ which is not evenly divisible by 4 and leaves a remainder of 1. In addition, K is congruent to 0(mod 3) and j=3 from above. Thus, the number of hex-steps is the greatest integer in $1+2j$ which is 7.
- **when K=100**, then $D = -3K = -300$ which is evenly divisible by 4. In addition, K is not congruent to 0(mod 3) and j=3 from above, which is odd. Thus, the number of hex-steps is the greatest integer in (3+4j)/2 which is 7.
- **when K=121**, then $D = -3K = -363$ which is not evenly divisible by 4 and leaves a remainder of 1. In addition, K is not congruent to 0(mod 3) and j=3 from above, which is odd. Thus, the number of hex-steps is the greatest integer in $(5+4j)/2$ which is 8.
- **when K=144**, then $D = -3K = -432$ which is not evenly divisible by 4 and leaves a remainder of 1. In addition, K is congruent to 0(mod 3) and j=4 from above. Thus, the number of hex-steps is the greatest integer in $1+2j$ which is 9.

Use of the Fundamental Theorem

The following example shows how to determine the b-value corresponding to an arbitrary Löschian number and therefore create a second Diophantine expression of that number. Suppose it has been determined, using the sufficient condition given in the Fundamental Theorem, that the number 397 is a Löschian number. Express 397 as $3x^2 + 3bx + b^2$, for some b. To do so, consider values of b from 0 to $397^{1/2}$. The greatest integer in this square root is 19. Thus, possibilities for b range from 0 to 19. Because $3x^2 + 3bx = 397 - b^2$, it follows that $397 - b^2$ must be divisible by 3. This fact eliminates from further consideration, as b, any integer between 0 and 19 for which $397 - b^2$ is not a multiple of 3. These values are shown in the central column of Table 4.2. Additionally, the material associated with the Fundamental Theorem states that the discriminant of the quadratic form, $3x^2 + 3bx + b^2$ must be an integer. Thus, in the right-hand column of Table 4.2 we calculate these discriminants for each of the candidate b-values singled out in the central column. The only candidate b value that remains is b=1 and therefore the correct quadratic form to also associate with 397 is $3x^2 + 3x + 1$. Geometrically, the ordered pair that generates 397 is represented as a lattice point that lies along the line b=1, displaced one unit from the line y=x in the oblique coordinate system.

b	$(397-b^2)/3$	Square root of discriminant
0	132.333	
1	132	$4761^{1/2}=69$
2	131	$4752^{1/2}=68.934$
3	129.333	
4	127	$4716^{1/2}=68.673$
5	124	$4689^{1/2}=68.476$
6	120.333	
		1/2

7	116	4617 =67.948
8	111	$4572^{1/2}$=67.616
9	105.333	
10	99	$4464^{1/2}$=66.813
11	92	$4401^{1/2}$=66.340
12	84.333	
13	76	$4257^{1/2}$=65.245
14	67	$4176^{1/2}$=64.621
15	57.333	
16	47	$3996^{1/2}$=63.213
17	36	$3897^{1/2}$=62.425
18	24.333	
19	12	$3681^{1/2}$=60.671

Table 4.2. This table shows candidate values for b for K=397 in the left-hand column. It shows values of (397-b^2)/3 for each of those candidate values in the center column. The right-hand column shows the square root of the discriminant of the quadratic expression $3x^2 + 3bx + b^2$ for different values of b (from 0 to 19) that produce an integral value in the center column.

Now we have two quadratic equations to solve simultaneously to obtain the ordered pair that gives rise to the Löschian value of 397.

$$397 = 3x^2 + 3x + 1$$
$$397 = x^2 + xy + y^2$$

Solving the first equation using standard techniques from high school algebra, and choosing the positive value, gives $x = 11$. Use that value of x to find $y = 12$ in the second equation. There are no other lattice points that give rise to this particular Löschian number because there is no other integral discriminant of the quadratic form $3x^2 + 3bx + b^2$.

The geometric characterization of the central place lattice as a set of integral lattice points lying along a set of lines parallel to $y = x$ resulted, after some work, in a Fundamental Theorem that permits the algebraic determination of a quadratic form, $3x^2 + 3bx + b^2$, to generate a set of lattice points lying along a single line. When this second quadratic form was applied to the Diophantine equation, $K = x^2 + xy + y^2$, easy calculations permit the determination of the lattice coordinates associated with an arbitrary Löschian number and of whether or not those coordinates are unique.

Uniqueness of Löschian numbers: solutions to previously unsolved problems.

Solving the K-value Twin Hierarchy Problem

Dacey noticed that some K-values have more than one central place hierarchy associated with them. Using classical methods, however, it is difficult to determine what these hierarchies might look like. Indeed, Dacey noted this issue as an unsolved problem in the classical geometry of central place theory. The fractal approach, when extended to arbitrary K-values, permits solution of this unsolved problem using the Fundamental Theorem and associated constructions.

Suppose it has been determined that the number 49 is a Löschian number. Express 49 as $3x^2 + 3bx + b^2$, for some b. To do so, consider values of b from 0 to $49^{1/2}$. The greatest integer in this square root is 7. Thus, possibilities for b range from 0 to 7. Because $3x^2 + 3bx = 49 - b^2$, it follows that $49 - b^2$ must be divisible by 3. This fact eliminates from further consideration, as b, any integer between 0 and 7 for which $49 - b^2$ is not a multiple of 3. These values are shown in the central column of Table 4.3. Additionally, the material associated with the Fundamental Theorem states that the discriminant of the quadratic form, $3x^2 + 3bx + b^2$ must be an integer. Thus, in the right-hand column of Table 4.3 we calculate these discriminants for each of the candidate b-values singled out in the central column. The only candidate b values that remains are b=2 and b=7 and therefore the correct quadratic forms to also associate with 49 are $3x^2 + 6x + 4$ and $3x^2 + 21x + 49$. Geometrically, the ordered pairs that generate 49 are represented as a lattice points that lies along the line b=2, displaced two units from the line y=x in the oblique coordinate system, and along the line b=7, displaced seven units from the line y=x in the oblique coordinate system.

b	(49-b^2)/3	Square root of discriminant

0	16.333	
1	16	$585^{1/2}$=24.186
2	15	$576^{1/2}$=24
3	13.333	
4	11	$540^{1/2}$=23.237
5	8	$512^{1/2}$=22.627
6	4.333	
7	0	$441^{1/2}$=21

Table 4.3. This table shows candidate values for b for K=49 in the left-hand column. It shows values of $(49-b^2)/3$ for each of those candidate values in the center column. The right-hand column shows the square root of the discriminant of the quadratic expression $3x^2 + 3bx + b^2$ for different values of b (from 0 to 7) that produce an integral value in the center column.

Now we have two sets of two quadratic equations each to solve simultaneously to obtain the ordered pairs that gives rise to two different hexagonal hierarchies associated with the Löschian value of 49.

$$49 = 3x^2 + 6x + 4$$
$$49 = x^2 + xy + y^2$$

$$49 = 3x^2 + 21x + 49$$
$$49 = x^2 + xy + y^2$$

In the first set of equations, x=3 and y=5. In the second set, x=0 and y=7. There are no other lattice points that give rise to this particular Löschian number because there is no other integral discriminant, beyond these two, of the quadratic form $3x^2 + 3bx + b^2$. Thus, K=49 is produced either from the lattice point (0,7) on the y-axis or from the lattice point (5,3) not on the y-axis. Figure 4.8 illustrates the two different fractal generators that give rise to the two different hierarchies using the procedure based on the Fundamental Theorem in associating fractal generators with lattice points.

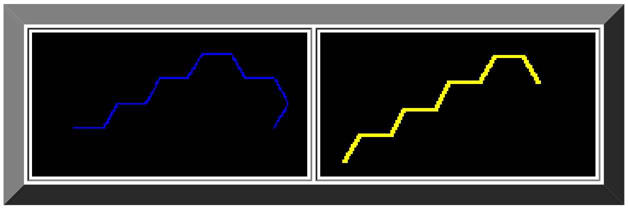


Figure 4.8. Geometric solution of the K=49 twin hierarchy puzzle. The left frame shows an 11-sided generator formed from the ordered pair (0,7) on the y-axis, yielding a K=7 type of hierarchy. The right frame shows a 9-sided generator formed from the ordered pair (5,3), not on the y-axis, yielding a K=4 type of hierarchy.

This example illustrates the fact that the Fundamental Theorem solves, completely, the previously unsolved problem involving uniqueness, and its lack, in calculating hexagonal hierarchies. The geometric characterization of the central place lattice as a set of integral lattice points lying along a set of lines parallel to $y = x$ resulted, after some work, in a Fundamental Theorem that permits the algebraic determination of a quadratic form, $3x^2 + 3bx + b^2$, to generate a set of lattice points lying along a single line. When this second quadratic form was applied to the Diophantine equation, $K = x^2 + xy + y^2$, easy calculations permit the determination of the lattice coordinates associated with an arbitrary Löschian number and also the determination of whether those coordinates are unique.

Putting the Pieces Together

The material in the previous sections indicates a plausible strategy for producing appropriate fractal generators that will create, through iteration, exact central place hierarchies for arbitrary K-values. It does so using only number theoretic properties of two quadratic forms. Thus, process is replicable and independent of previous cases. In the next chapter we consider some of the number theoretic infrastructure on which these methods rest and in a final chapter we present some real world considerations.

Spatial Synthesis: Centrality and Hierarchy, Volume I, Book 1.
Sandra Lach Arlinghaus and William Charles Arlinghaus