Introduction

The brain is wider than the sky,

For put them side by side;
The one the other will include,
With ease, and You beside.

Emily Dickinson

Synthesis begins with a global view and dissects that view to consider individual components, as they contribute to that whole: the whole is often more than the sum of the individual parts considered. In contrast, analysis often begins with a sequence of local views and assembles a global view from these, as a sum of parts: the assembled view may or may not fit "reality." These two different approaches to scholarship yield different results and employ different tools. In this work, we take a synthetic approach to looking at real world issues that have mathematics as a critical, but often unseen, component. This approach permits one, also, to learn the associated mathematics as it naturally unfolds in real world settings, often in a relatively "painless" manner. The reader learns mathematics required to consider the broad problem at hand, rather than learning mathematics according to the determination of a (perhaps) artificial curriculum. The underlying philosophy is different, as is the format. The eBook is required, however, given the interactive and animated features that enliven the book and motivate the reader to explore diverse realms in the worlds of geography and mathematics and in their interface.

Figure 0.1 offers an abstract view of the idea of "spatial synthesis." Each large hexagon represents a single scholarly discipline: mathematics on the left and geography on the right (as an example). A single field (geometry, for example) is identified (deep red trapezoid) from one the broad discipline of mathematics, and is aligned with a counterpart (deep red trapezoid) field (cartography) from the other broad discipline of geography. Despite the alignment of appropriate entities, a gap remains. In institutional settings reliant on predetermined disciplinary boundaries, interdisciplinary research often founders in this gap for lack of support of various kinds. What is required, therefore, is to seal the gap and to unify aligned, related structure (represented as a single deep red hexagon formed from two trapezoids in Figure 0.1).

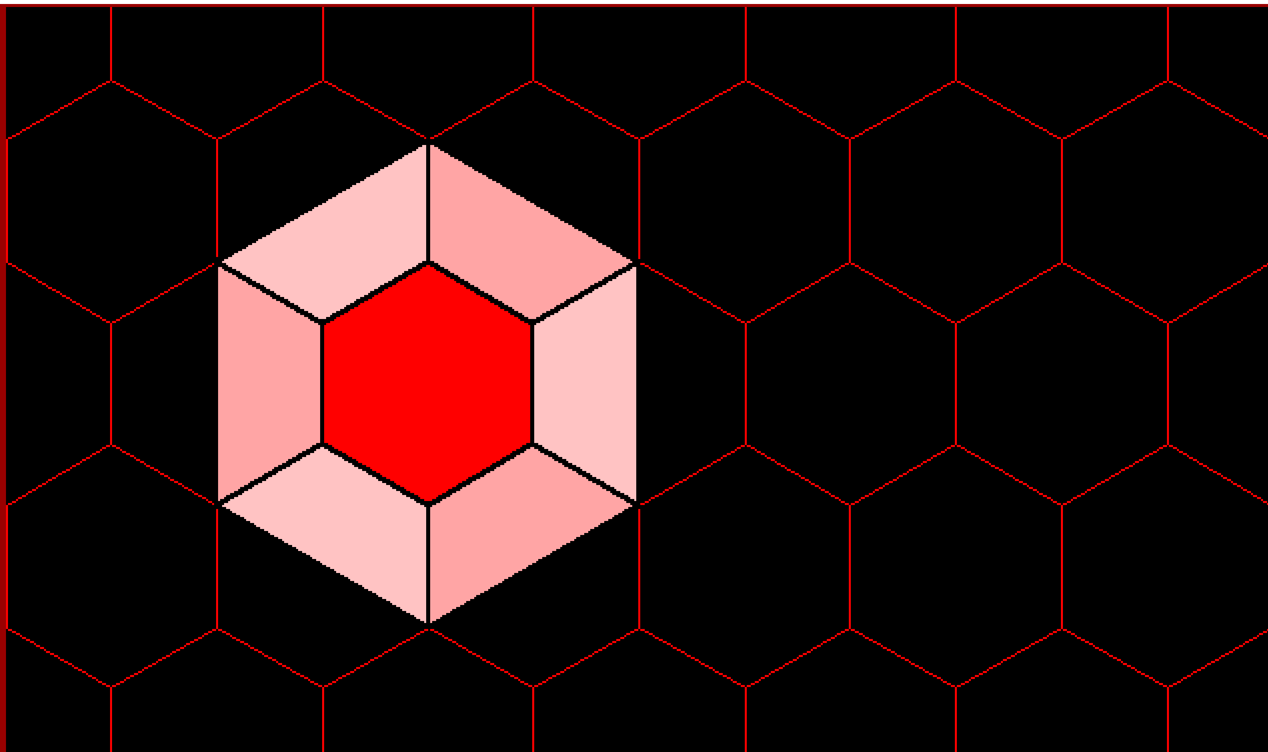

Figure 0.1. Spatial synthesis: abstract representation.

One well known classical approach to this style of scholarship appears in the work of Eratosthenes of Alexandria (c. 276-194 B.C.). Eratosthenes made critical discoveries in both mathematics and geography that might seem unrelated. His prime number sieve (see Chapter 3 of this work for more detail) permitted the complete characterization of all whole numbers and offered, therefore, an understanding of the whole number system. The idea of understanding a whole, that one could never see, also applies to his measurement of the circumference of the Earth. In his role as Librarian of the great library of Alexandria, he had access to works of his many outstanding predecessors. He worked with spherical geometry, latitude and longitude, earth-sun relations, and a variety of geometric and trigonometric ideas. Eratosthenes sealed the gap between number theory and the real world with the idea of using abstract tools to understand more than what one could see, be that an infinity of whole numbers or an Earth larger than a hometown; his brain was wider than his sky.

The astonishing scholarship of Eratosthenes serves as a model, in a variety of ways, for this effort. The body of the text draws from disciplines in pure mathematics and aligns them with real world material. The underlying mathematics is critical to the effort. Beyond the synthesis of apparently disparate disciplines, however, is the model of Eratosthenes the librarian. In our present world, each of us is an amateur librarian in the great virtual library of the Internet. Direct access to materials from all over the world is available on our desktops. Thus, where possible, we assemble a subset of this library in this electronic book. The eBook, itself, serves therefore not only as an interactive forum employing state-of-the-art techniques for communicating scientific information, but also as a library on the topic it covers. Thus, we dedicate this effort to the spirit of Eratosthenes of Alexandria: may his research strategy endure for many more millennia!

Spatial Synthesis: Centrality and Hierarchy, Volume I, Book 1.
Sandra Lach Arlinghaus and William Charles Arlinghaus

ERATOSTHENES'S MEASUREMENT OF THE CIRCUMFERENCE OF THE EARTH

Eratosthenes of Alexandria (appointed Director of the Great Library at Alexandria in 236 B.C.) was an innovator in measurement. Not only did he create a prime number sieve, but also he figured out how to measure the circumference of the Earth. To do so, he used Euclidean Geometry and simple measuring tools. Below, we show the style of measurement that he is said to have made (different accounts give different details).

- Assume the Earth is a sphere.
- The circumference of the sphere is measured along a great circle on the sphere.
- Find the circumference of the Earth by finding the length of intercepted arc of a small central angle.
- Find two places on the surface of the Earth that lie on the same meridian (or close to it): meridians are halves of great circles.
- Eratosthenes chose Alexandria and Syene, near contemporary Aswan (Figure 1).

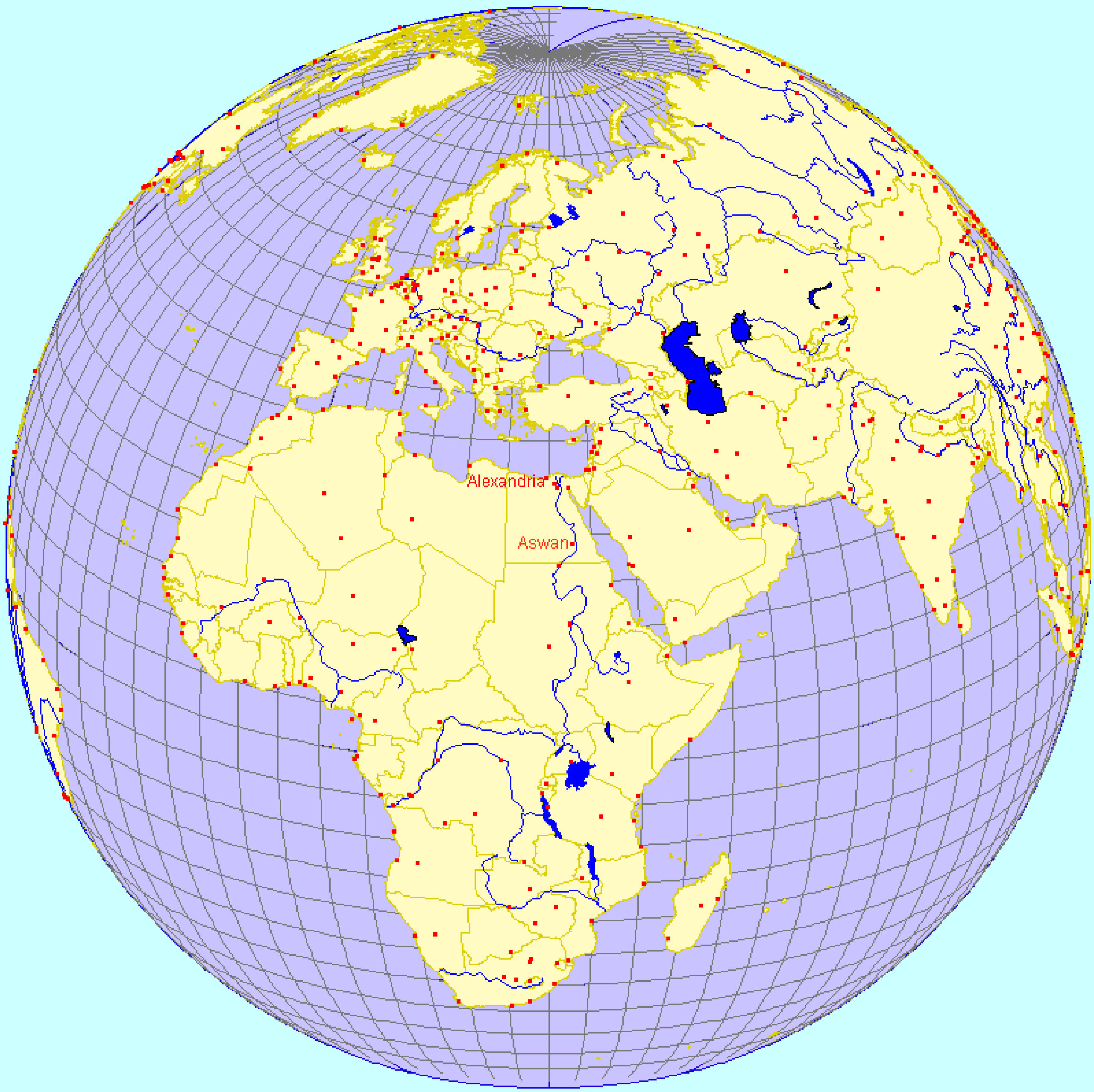

Figure 1. Relative location of Alexandria and Aswan. They are close to lying on the same meridian (half of a great circle).

- Assume that the rays of the Sun are parallel to each other.
- The Sun's rays are directly overhead, on the Summer Solstice (c. June 21), at 23.5 degrees N. Latitude.
- Syene is located at about 23.5 degrees N. Latitude. Hence, on the Summer Solstice, the reflection of the sun will appear in a narrow well (and it will not on other days). Eratosthenes apparently understood this idea.
- Alexandria is north of Syene. Thus, on June 21, objects at Alexandria will cast shadows whereas those at Syene will not.
- Eratosthenes focused on an obelisk or post located in an open area. He measured the shadow that the obelisk cast (A'A"), functioning in the manner of a gnomon on a sundial, and then measured the height of the obelisk (AA') (perhaps using a string anchored to the tip of the obelisk).

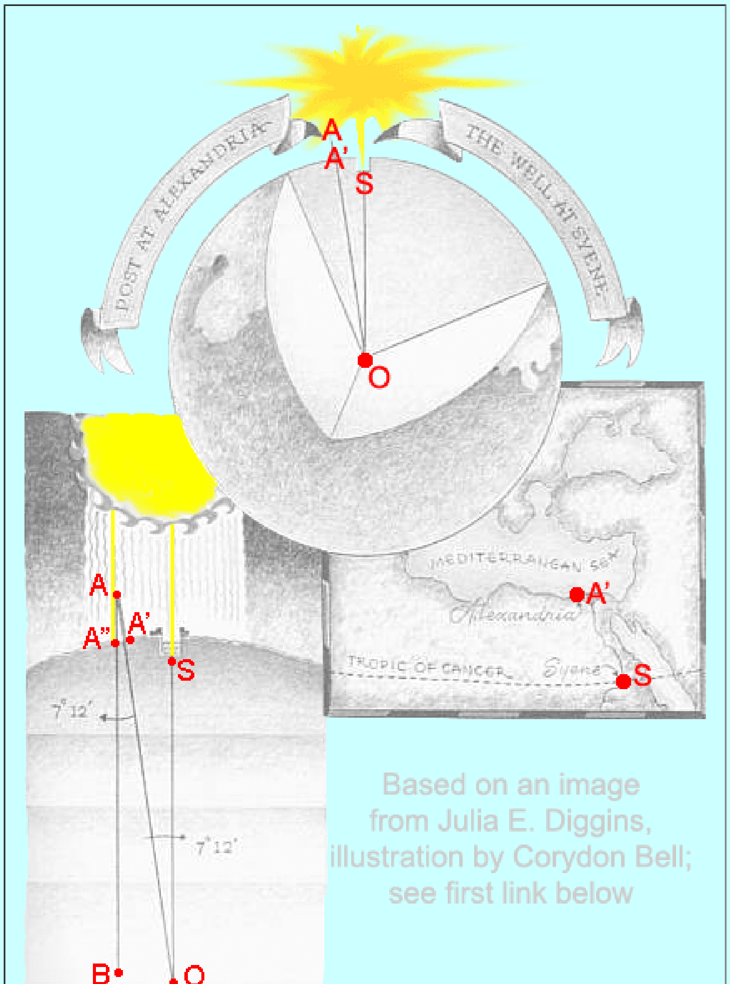

Figure 2. Eratosthenes's measurement of the circumference of the Earth, based on a Theorem of Euclid.

- According to Euclid, two parallel lines cut by a transversal have alternate interior angles that are equal. The Sun's rays are the parallel lines. One ray, at Alexandria, touches the tip of the obelisk and extends earthward toward the tip of the shadow of the obelisk, AA". It is extended to AB in Figure 2. The other ray, SO, at Syene, goes into the well and extends abstractly to the center of the Earth, O. The obelisk, AA', also extends abstractly to the center of the Earth, O; thus, the line, AO, determined by the tip of the obelisk and the center of the Earth is a transversal cutting the two parallel rays, SO and AB, of the sun.
- Angles (BAO) and (SOA) are thus alternate interior angles in geometric configuration described above; therefore, they are equal.
- Use the length of the obelisk shadow and the height of the obelisk to determine angle BAO; triangle AA'A" is a right triangle with the right angle at A'. Thus, we would note, tan (A'AA") = (length of shadow)/(height of obelisk). Eratosthenes's measurements of these values led him to conclude that the measure of angle (A'AA") was 7 degrees and 12 minutes.
- The value of 7 degrees and 12 minutes is $1/50^{th}$ of the degree measure of a circle. Since he assumed that Alexandria and Syene both lay on a meridian (half a great circle), it followed that the distance between these two locations was $1/50^{th}$ of the circumference of the Earth.
- Eratosthenes calculated the distance between Alexandria and Syene using records involving camel caravans. The distance he

used was 5000 stadia. Thus, the circumference of the Earth is 250,000 stadia, which translates to somewhat less than 25,000 miles (depending on how ancient units convert to modern units). This value is remarkably close to current values used.

Many of the assumptions made by Eratosthenes were not accurate; apparently, however, underfit and overfit of error balanced out to produce a good result. For example, Syene and Alexandria are not on the same meridian; Syene is not at exactly 23.5 degrees N. Latitude, and so forth. See the link to Astronomy Online for more discussion of historical and astronomical matters.

J. E. Diggins, The Whole Round Earth, http://www.anselm.edu/homepage/dbanach/erat.htm
Astronomy Online: http://www.algonet.se/~sirius/eaae/aol/market/collabor/erathost/

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Latitude and Longitude

Figures and text based on images and text from a GeoSystems webpage which no longer exists.
Permission was granted for an earlier use of selected images when the previous site was contacted.

Consider the Earth to be modeled as a sphere: the Earthsphere. The Earth is not actually a sphere, but a sphere is a good approximation to its shape and the sphere is easy to work with using classical mathematics of Euclid and others.

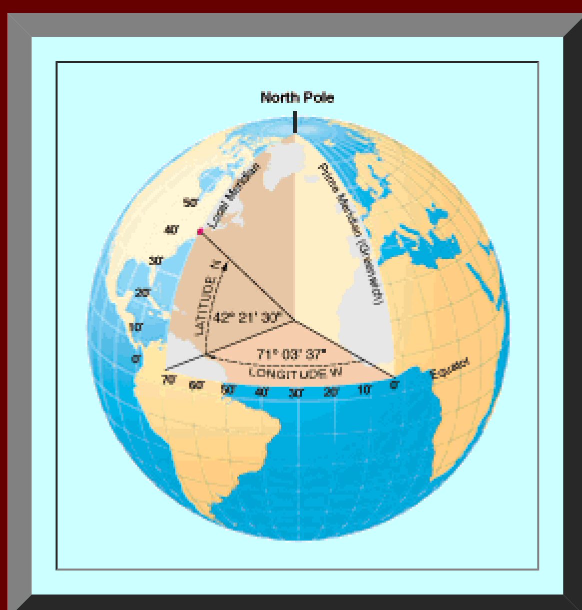
- Given a sphere and a plane. There are only a few logical possibilities about the relationship between the plane and the sphere.
  - The sphere and the plane do not intersect.
  - The plane touches the sphere at exactly one point: the plane is tangent to the sphere.
  - The plane intersects the sphere.
    - and does not pass through the center of the sphere: in that case, the circle of intersection is called a small circle.
    - and does pass through the center of the sphere: in that case, the circle of intersection is as large as possible and is called a great circle.
- Great circles are the lines along which distance is measured on a sphere: they are the geodesics on the sphere.
  - In the plane, the shortest distance between two points is measured along a line segment and is unique.
  - On the sphere, the shortest distance between two points is measured along an arc of a great circle.
    - If the two points are not at opposite ends of a diameter of the sphere, then the shortest distance is unique.
    - If the two points are at opposite ends of a diameter of the sphere, then the shortest distance is not unique: one may traverse either half of a great circle. Diametrally opposed points are called antipodal points: anti+pedes, opposite+feet, as in drilling through the center of the Earth to come out on the other side.
- To reference measurement on the Earthsphere in a systematic manner, introduce a coordinate system.
  - One set of reference lines is produced using a great circle in a unique position (bisecting the distance between the poles): the Equator. A set of evenly spaced planes, parallel to the equatorial plane, produces a set of evenly spaced small circles, commonly called parallels. They are called that because it is the planes that are parallel to each other.
  - Another set of reference lines is produced using a half of a great circle, joining one pole to another, that has a unique position: the half of a great circle that passes through the Royal Observatory in Greenwich, England (three points determine a circle). Here it is historical consideration that produces the uniqueness in selection. Choose a set of evenly spaced halves of great circles obtained by rotating the diametral plane along the polar axis of the Earth. These lines are called meridians: meri+dies=half day, the situation of the Earth at the equinoxes (see page on seasons). The unique line is called the Prime Meridian; other halves of great circles are called meridians.

  This particular reference system for the Earth is not unique; an infinite number is possible. There is abstract similarity between this particular geometric arrangement and the geometric pattern of Cartesian coordinates in the plane.
  - To use this arrangement, one might describe the location of a point, *P*, on the Earthsphere as being at the 3rd parallel north of the Equator and at the 4th meridian to the west of the Prime Meridian. While this might serve to locate *P* according to one reference system, someone else might employ a reference system with a finer mesh (halving the distances between success lines) and for that person, a correct description of the location of *P* would be at the 6th parallel north of

the Equator and at the 8th meridian to the west of the Prime Meridian. Indeed, an infinite number of locally correct designations might be given for a single point: an unsatisfactory situation in terms of being able to replicate results. The problem lies in the use of a relative, rather than an absolute, locational system.

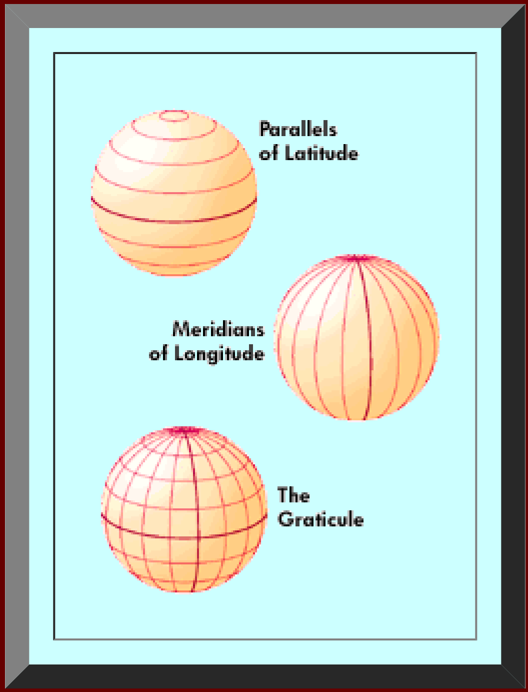
- To convert this system to an absolute system, that is replicable, employ some commonly agreed upon measurement strategy to standardize measurement. One such method is the assumption that there are 360 degrees of angular measure in a circle.
  - Thus, *P* might be described as lying 42 degrees north of the equator, and 71 degrees west of the Prime Meridian. The degrees north are measured along a meridian; the degrees west are measured along the Equator or along a parallel (the one at 42 north is another natural choice). The north/south angular measure is called Latitude; the east/west angular measure is called Longitude.
  - The use of standard circular measure creates a designation that is unique for *P*; at least unique to all whose mathematics rests on having 360 degrees in the circle.

    Parts of degrees may be noted as minutes and seconds, or as decimal degrees. A degree (°) of latitude or longitude can be subdivided into 60 parts called minutes ('). Each minute can be further subdivided into 60 seconds ("). Thus, 42 degrees 30 minutes is the same as 42.50 degrees because 30/60 = 50/100. Current computerized mapping software often employs decimal degrees as a default; older printed maps may employ degrees, minutes, and seconds. Thus, the human mapper needs to take care to analyze the situation and make appropriate conversions prior to making measurements of position. Such conversion is simple to execute using a calculator. For example, 42 degrees 21 minutes 30 seconds converts to 42 + 21/60 +30/3600 degrees = 42.358333 degrees; powers of ten replace powers of 60.

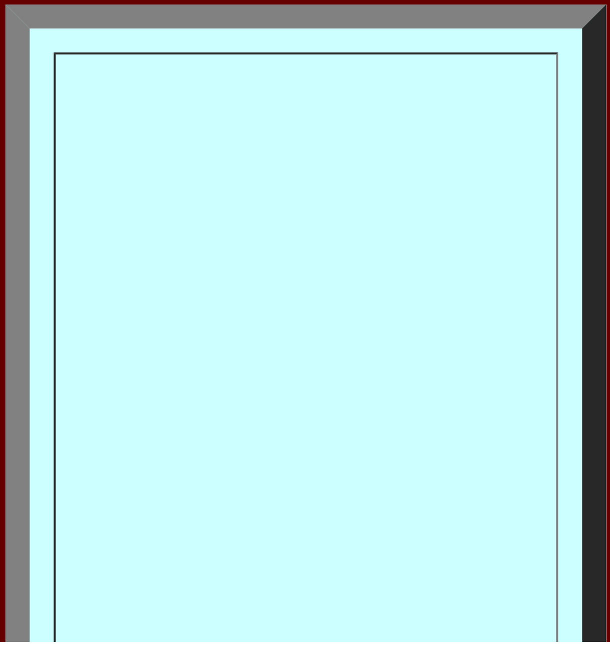


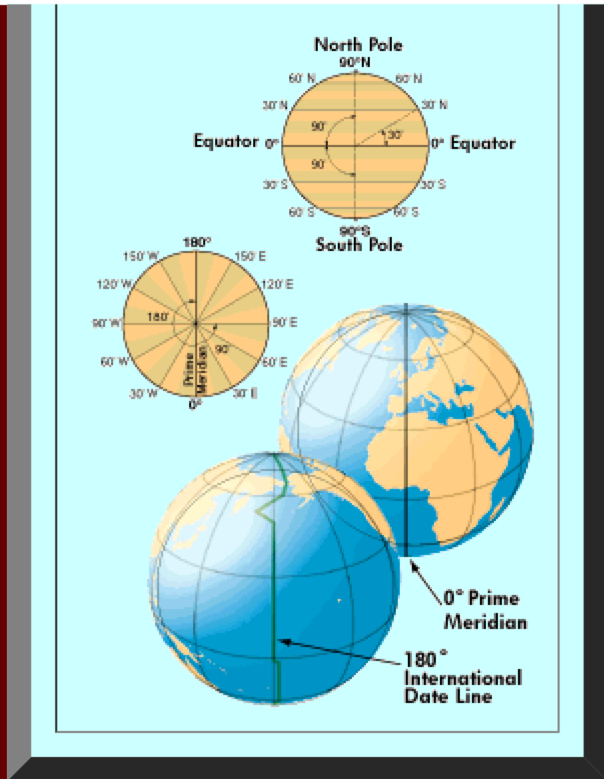
- The figure below shows the reference system described above placed on a sphere. What might be called a Cartesian grid in the plane is called a graticule on the sphere.
  - All parallels have the same latitude; they are the same distance above or below, north of or south of, the Equator.
  - All meridians have the same longitude; they are the same distance east or west of the Prime Meridian.

Spacing between successive parallels or meridians might be at any level of detail; however, when circular measure describes the position of these lines, that description is unique up to agreement to use 360 degrees in a circle. One spacing for the set of meridians that is convenient on maps of the world, is to choose spacing of 15 degrees between successive meridians. The reason for this is that since the meridians converge at the ends of the polar axis, that each meridian then represents the passage of one hour of time. Given that we agree to partition a day into 24 hours, 24 times 15 is 360, meridians may also mark time.

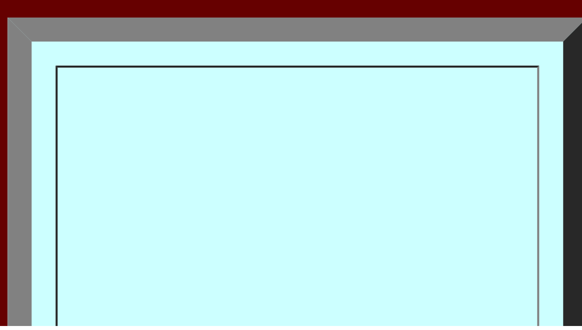


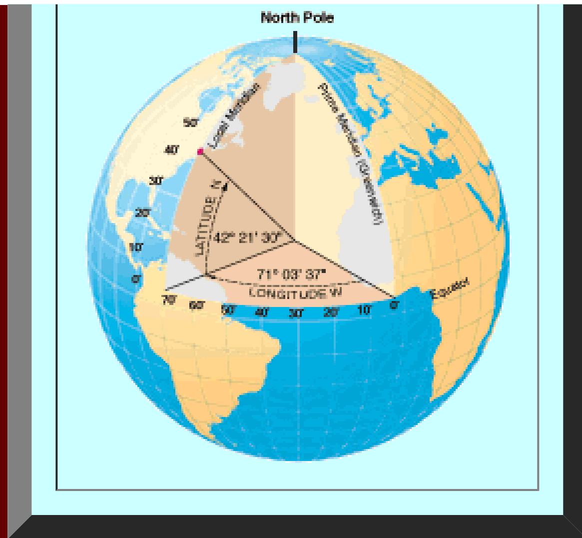
- Bounds of measurement (see the figure below).
  - Latitude runs from 0° at the equator to 90°N or 90°S at the poles.
  - Longitude runs from 0° at the prime meridian to 180° east or west, halfway around the globe. The International Date Line follows the 180° meridian, making a few jogs to avoid cutting through land areas.





- Length of one degree on the Earthsphere.
  - One degree of latitude, measured along a meridian or half of a great circle, equals approximately 69 miles (111 km). One minute is just over a mile, and one second is around 100 feet (a pretty precise location on a globe with a circumference of 25,000 miles). Calculation: 25,000/360 = 69.444.
  - Because meridians converge at the poles, the length of a degree of longitude varies, from 69 miles at the equator to 0 at the poles (longitude becomes a point at the poles). Calculation: at latitude theta, find the radius, r, of the parallel, small circle, at that latitude. The radius, R, of the Earthsphere is R = 25,000 / (2*pi)=3978.8769 miles. Thus, cos theta = r/R (using a theorem of Euclid that alternate interior angles of parallel lines cut by a transversal are equal). Therefore, r=R cos theta. Then, the circumference of the small circle is 2r*pi and the length of one degree at theta degrees of latitude is 2r*pi / 360. For another application of this particular theorem of Euclid, see the linked page concerning Eratosthenes measurement of the Earthsphere.
    - For example, at 42 degrees of latitude, r = 2956.882. Thus, the circumference of the parallel at 42 degrees north is approximately 18578.6205 miles. Thus, the length of one degree of longitude, measured along the small circle at 42 degrees of latitude, is: 51.607 miles.
    - This particular calculation scheme is a rich source of elementary problems using geometry and trigonometry. Consider the following question: at what latitude is the length of one degree on longitude exactly half the value of one degree of longitude at the equator?
    - Readers wishing a visual review of trigonometry may find this link to be of use.

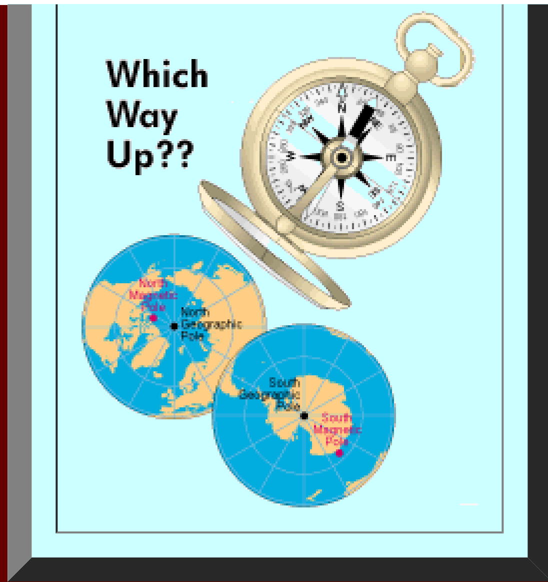




- The position of the sun in the sky. On June 21, the direct ray of the sun is overheard, or perpendicular to a plane tangent to the Earthsphere, at 23.5 degrees north latitude (link to page about seasons). The angle of the sun in the sky at noon on that day is 90 degrees. What is the angle of the sun in the sky, at noon on June 21, at 42 degrees north latitude? Again, simple geometry and trigonometry solve the problem for this value and for any other. Use the fact that 42-23.5=18.5 degrees; that there are 180 degrees in a triangle (look for a right triangle with the right angle at 42 degrees north latitude); and that corresponding angles of parallel lines cut by a transversal are equal. The answer works out to be 71.5 degrees. Thus, on June 21 at local noon, in the northern hemisphere at 42 degrees north latitude the sun will appear in the south at 71.5 degrees above the horizon; in the southern hemisphere at 42 degrees south it will appear in the northern sky at 71.5 degrees above the horizon. Between the tropics, some interesting situations prevail (link to Parallels between Parallels, pages 74-86). Use of this technique is important in calculating shadow and related matters in electronic mapping: it was employed in making several virtual reality models of in this book.

Further Directions:

- The north and south poles are the earth's geographic poles, located at each end of its axis of rotation. All meridians meet at these poles. The compass needle points to either of the earth's two magnetic poles. The north magnetic pole is located in the Queen Elizabeth Islands group, in the Canadian Northwest Territories. The south magnetic pole lies near the edge of the continent of Antarctica, off the Adélie Coast. The magnetic poles are constantly moving. What are the implications of this fact for the stability of our graticule?



- All of our geometric analysis is based on Euclidean geometry, assuming Euclid's Parallel Postulate: given a line and a point not on the line--through that point there passes exactly one line that does not intersect the given line. Non-Euclidean geometries violate this Postulate. What does the geometry of the Earthsphere become in the non-Euclidean world?

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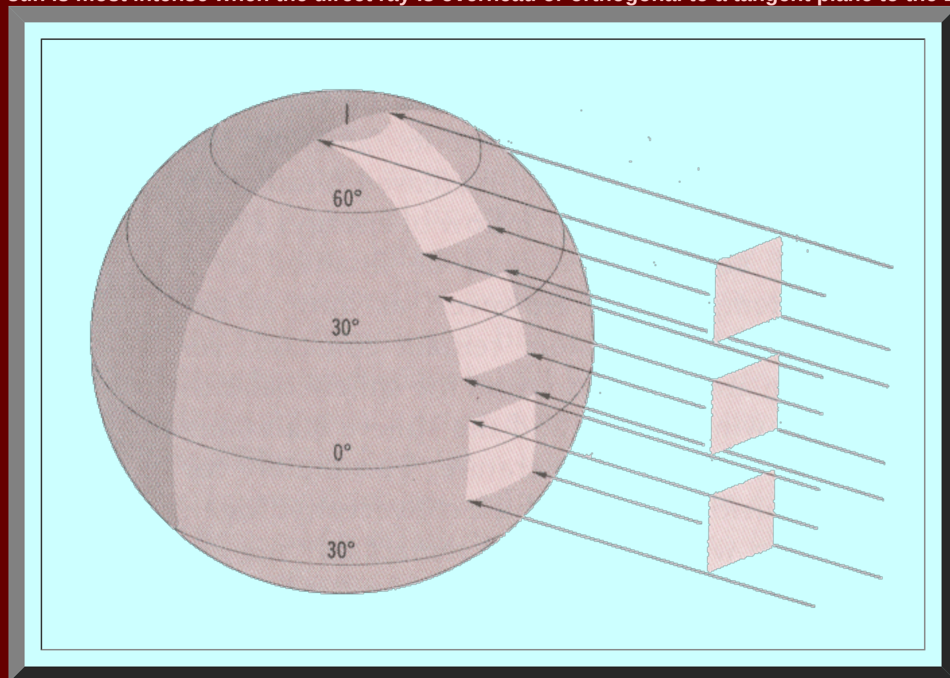
SEASONS: A VISUAL DISPLAY

Diagrams based on images from pages 64-67,
John F. Kolars and John D. Nystuen, drawings by Derwin Bell,
Physical Geography: Environment and Man, McGraw-Hill, 1975.
Figures produced here with permission.

SUNLIGHT INTENSITY IS A FUNCTION OF LATITUDE

At higher latitudes, sunlight is spread out over a wider area and is thus less intense.
The sun is most intense when the direct ray is overhead or orthogonal to a tangent plane to the Earth.

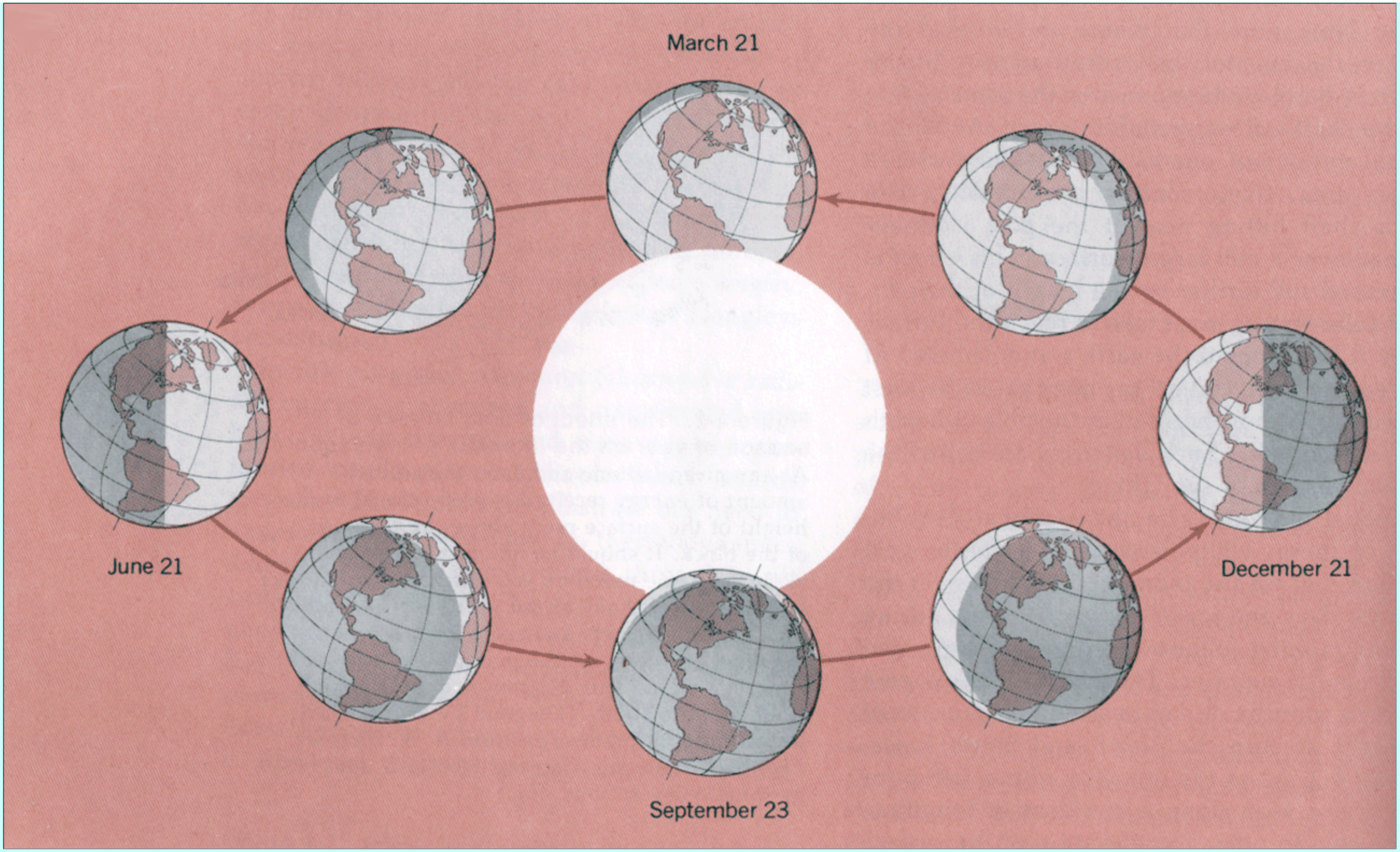


ANNUAL REVOLUTION OF THE EARTH AROUND THE SUN

Note the constant tilt of the Earth's polar axis.

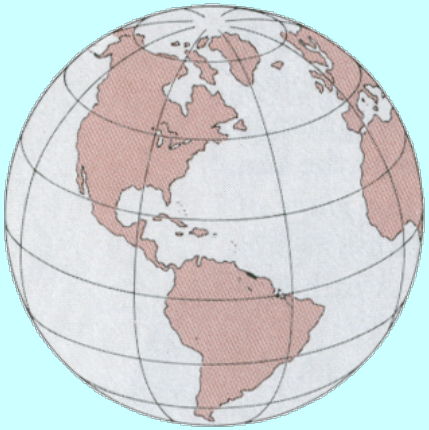
The circle of illumination, or terminator, marks the separation of day and night.

At the equinoxes (about March and September 21), the circle of illumination passes through both poles (the only time of the year at which it does so). Thus, at those times, the circle of illumination bisects all parallels of latitude; half the day is on the dark side, half the day is on the light side for all latitudes. Hence, equinox=equal night, all over the world.



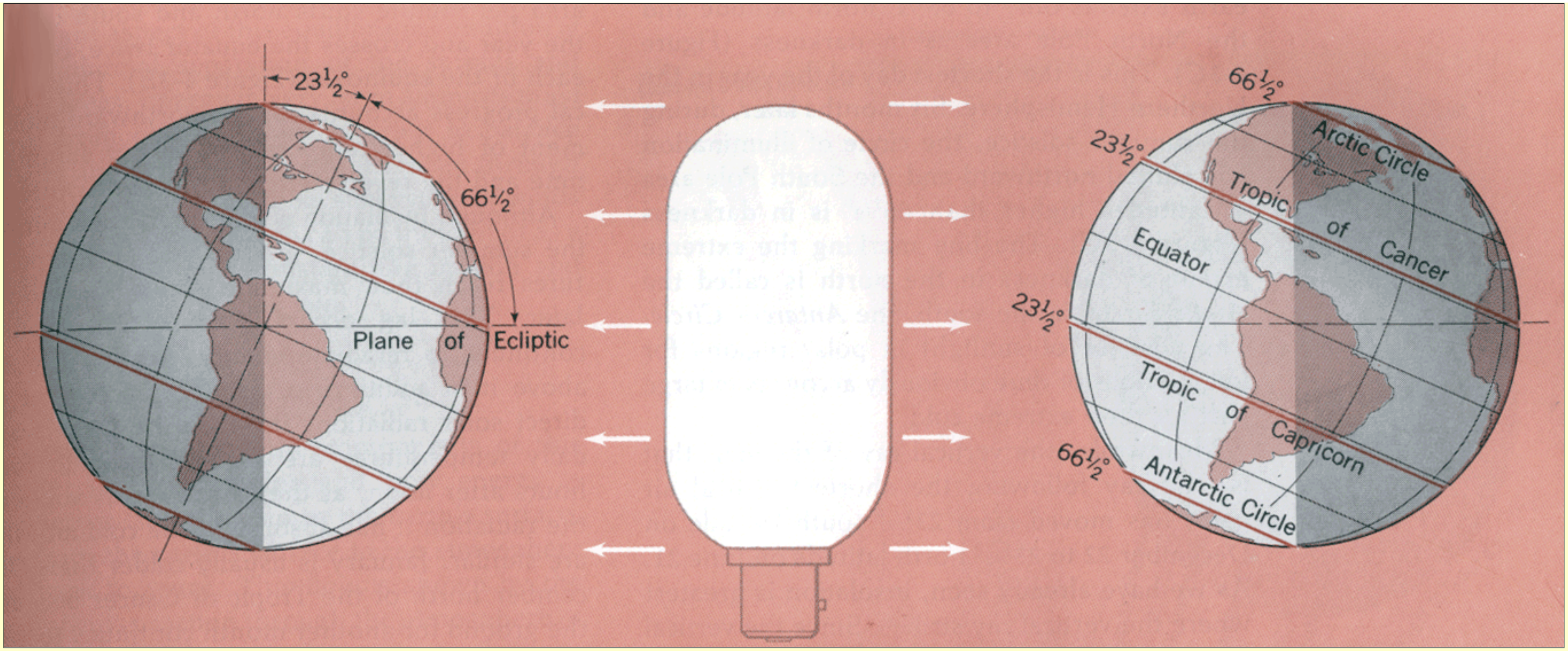
ANIMATION SHOWING THE EARTH VIEWED FROM THE SUN AT SOLSTICES AND EQUINOXES

June 21



LATITUDE AND LONGITUDE MEASURE POSITION OF SUN'S RAYS

The direct ray of the sun is overhead at 23.5 north on June 21 (as far north as it ever will be);
at 23.5 south on December 21 (as far south as it ever will be);
and at 0 degrees on the equinoxes.
The ecliptic is the Sun's diametral plane.



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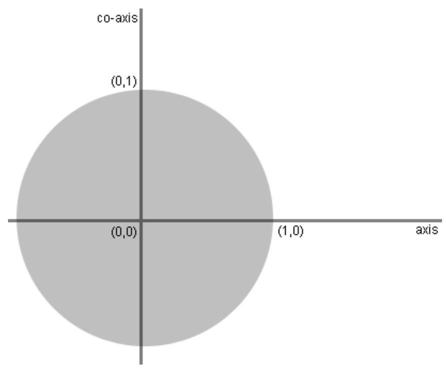
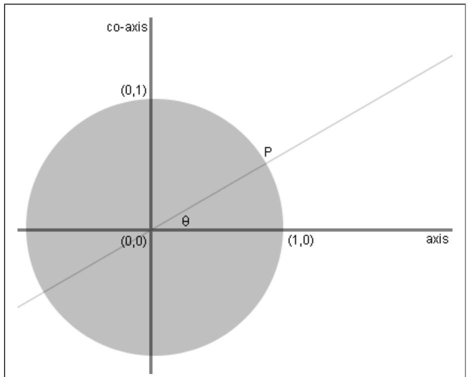
VISUAL TRIGONOMETRY REVIEW

Set screen to highest resolution; scroll to the right when needed. The big table has four columns.

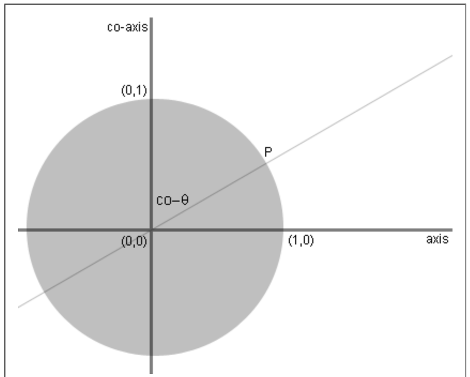
THE BASIC SETUP:



Unit circle, axis, and complementary (orthogonal) axis designated as co-axis.

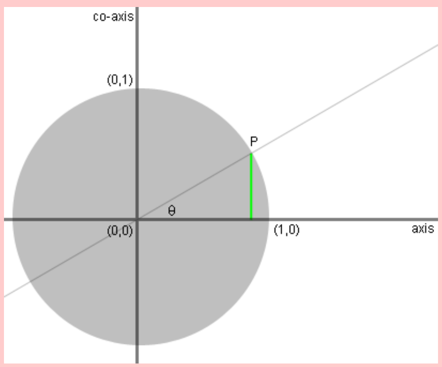


Secant line, a geometric object, cuts the circle at an angle of theta to the horizontal axis. The secant line intersects the circle at point P.

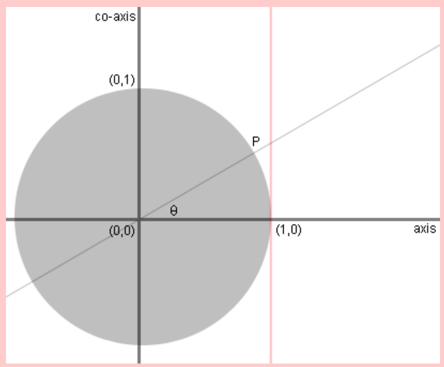


The same secant line also determines the complementary angle to theta (another geometric object), denoted co-theta.

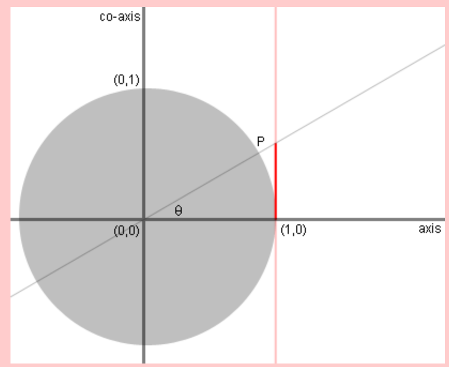
THE TOP ROW SHOWS DERIVATIONS OF TRIGONOMETRIC FUNCTIONS FOR ANGLE THETA; THE SECOND ROW SHOWS DERIVATIONS OF TRIGONOMETRIC FUNCTIONS FOR ANGLE CO-THETA.



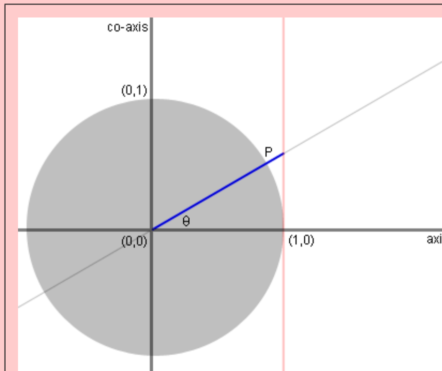
The length of the green line, dropping from P to the axis, measures the sine of theta: opposite side of a right triangle over a hypotenuse of the unit circle.



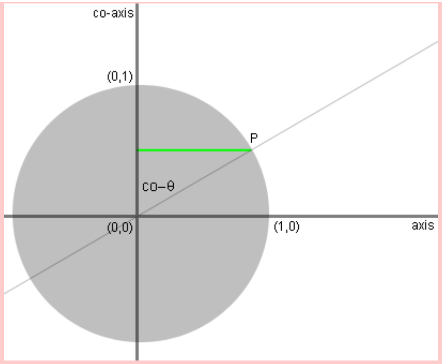
The pink line is a geometric line tangent to the unit circle at (1,0) on the horizontal axis.



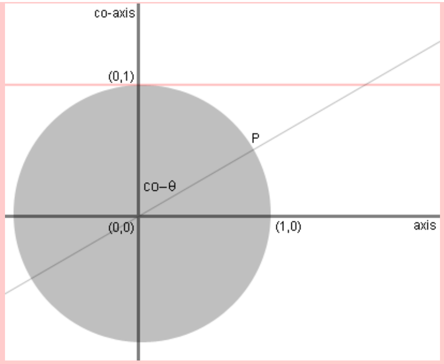
The length of the red line, intercepted by the secant line along the tangent line, measures the tanget of theta.



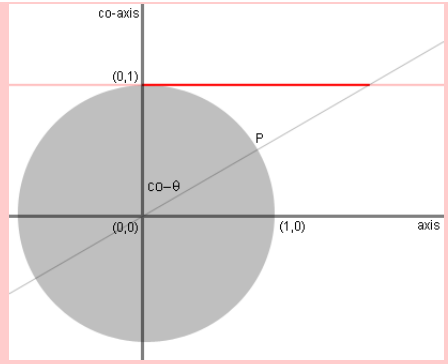
The length of the blue line, intercepted by the tangent line along the secant line, measures the secant of theta.



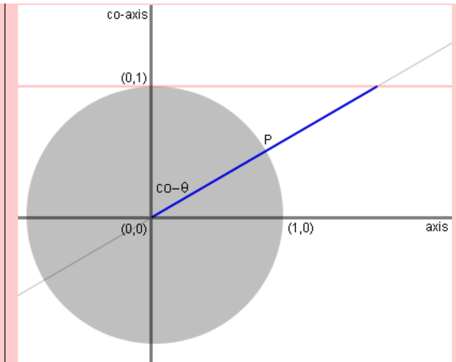
The length of the green line, dropping from P to the co-axis, measures the sine of co-theta: opposite side of a right triangle over a hypotenuse of the unit circle. Hence, cosine of theta.



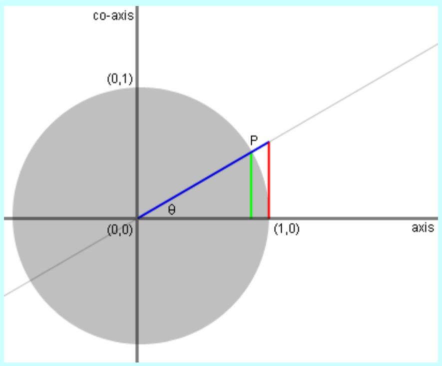
The pink line is a geometric line tangent to the unit circle at (1,0) on the co-axis.



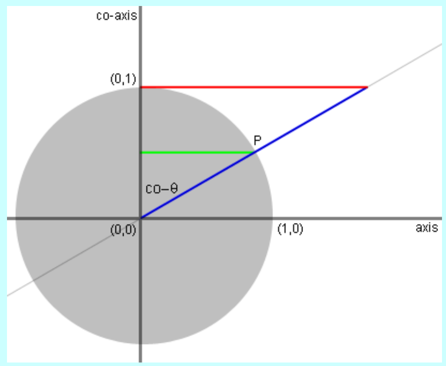
The length of the red line, intercepted by the secant line along the tangent line, measures the tanget of co-theta. Hence, cotanget of theta



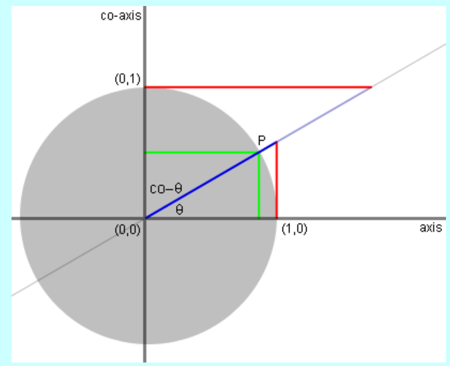
The length of the blue line, intercepted by the tangent line along the secant line, measures the secant of co-theta. Hence, cosecant of theta.



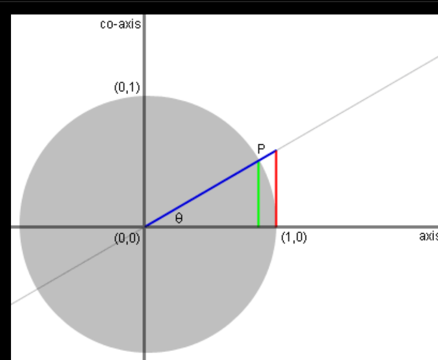
The three functions of theta measuring sine, tangent, and secant shown together.



The three functions of co-theta measuring sine of co-theta (cosine of theta), tangent of co-theta (cotangent of theta), and secant of co-theta (cosecant of theta).



All six functions of theta are shown in this image. The representations for secant lies on top of that for cosecant of theta. The latter is thus shaded a lighter shade.



An animation of the ideas shown as individual frames above.

A number of trigonmetric identities are evident from this visual approach.

From the Pythagorean Theorem, it follows that:

$\sin^2$ theta + $\cos^2$ theta = 1, the radius of the unit circle measured along the secant line;
$\sec^2$ theta = $\tan^2$ theta + 1, the radius of the unit circle measured along the horizontal axis;
$\csc^2$ theta = $\cot^2$ theta + 1, the radius of the unit circle measured along the co-axis.

What others do you note?

Based on an approach learned by S. Arlinghaus at The University of Chicago Laboratory Schools.
