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An Atlas of Steiner Networks

Sandra Lach Arlinghaus

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IMaGe

AN ATLAS OF STEINER NETWORKS

Sandra Lach Arlinghaus

Institute of Mathematical Geography
Ann Arbor, MI

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To William E. Arlinghaus and Andrea L. Voorhees,

on the occasion of their marriage,

February 25, 1989.

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AN ATLAS OF STEINER NETWORKS

CHAPTER I: INTRODUCTION

Statement of Steiner's Problem

The Steiner problem is that of finding a shortest path joining an arbitrary number of points. The solution is a graph-theoretic tree of minimum length, and it may include points not given in the original set. It is precisely this latter possibility that distinguishes the Steiner problem from others, such as the traveling salesman problem, although both of these, and other network optimization problems, are NP-complete. Any new point, chosen to reduce total network length joining the original set of points, may be chosen from an infinite set of positions. Hence, the enormity of the problem.

An efficient algorithm to find Steiner trees has long been sought. Much of the research on it appears in the mathematics literature and in the applied mathematics and telephone engineering literature [1]. A Steiner network is a network of minimum construction cost. Because it also tends to focus congestion, it is economically desirable as a routing strategy when the cost of network construction outweighs other network costs.

Potential arenas in which Steiner networks might prove useful are in the theoretical underpinnings of Geographic Information Systems [2]. If centers of gravity are used as a centering scheme in a triangulated irregular network, then it is desired to have no centroid lie outside a triangular cell. Thus, no cell should have angle greater than 120° and so the Steiner network (where all angles are exactly 120°) will serve as an outer edge (a limiting position) for acceptable triangles to be induced on a surface. It is at this point that the geometry and graph theory of the minimal tree come face-to-face with the fundamental triangular units (simplexes) of combinatorial topology.

Conceptual Background

To find a path from one place to another is a problem of enduring interest that has stimulated a variety of literary and mathematical response. Literary support for the permanence of human concern with this problem may be drawn from Greek mythology as well as from traditional children's stories. The thread Ariadne gave to Theseus provided him a solution for escape from the labyrinth after he had slain the Minotaur; Hansel and Gretel hoped to trace their path into the forest from a trail of bread crumbs, while Alice tried in vain to discover a path to a hilltop so she might see the garden of live flowers [3]. The maze characterizes a simple expression of this mathematical problem: to move from location A to location B along a path that twists around a set of barriers with fixed spatial position [4]. Mazes and labyrinths have survived as popular forms of entertainment and as interesting hedge formations in formal gardens. More rigorous analysis of mazes has produced varying sets of mathematical rules for their solution [5]. However, with any style of solution, the puzzle is considered solved once the existence of a path from A to B, satisfying the given spatial constraints, has been determined. If that solution is unique then the maze is completely solved in a formal sense and therefore generates no immediate questions for further development.

If the maze admits more than one path as a solution then the following question and extensions of it arise naturally and lead to solutions for classes of problems more complex than the original one. Of all the paths through the maze, which is the shortest (or, dually, the longest)? If the physical landscape represents the spatial constraints of the maze, then this question might be interpreted as that of finding the shortest route for traveling from one location to another, with distance measured in time, cost, Euclidean space or whatever seems appropriate. If each route is to include a specified set of intervening locations as part of the given spatial structure, then the original problem leads to the "traveling salesman" problem, which in theory can be solved, but which requires large numbers of calculations when even only a relatively small number of intervening locations are included [6].

None of these concerns requires the path to branch. If the path connecting two or more places branches in order to pass through intervening locations, the problem remains to determine which of all these tree-like paths is shortest; this is the minimal spanning tree problem. The earliest efficient algorithms for the solution of this problem were given by Kruskal and Prim; they rely on indexing the edge set according to length and on replacing, recursivly, edges in circuits of any candidate tree [7].

Both traveling salesman and minimum spanning tree paths in a finite distribution of points generally assume uniformity of carrying capacity among all possible routes. If varying carrying capacities are assigned to alternate routes then economic considerations may be imposed on the fundamental geometric and spatial structure of this problem. The general diagrammatic exposition of such a structure is as a directed graph with paths from A to B assigned numbers representing carrying capacities of various segments of each path. If A is viewed as a source of supply for a set of locations and if B is viewed as a collecting point, or sink, of demand for another set of places, then one can ask how to transmit maximal flow across this network in order to satisfy the demand at the second set of locations. In this way the simple maze problem is transformed into the "transportation problem" [8]. Gaspard Monge, the French descriptive/projective geometer of the eighteenth century was apparently the first to tackle it [9]. More recent interest dates to the early 1940's with Hitchcock's determination of the optimal economic distribution of flow from a set of sources to a set of sinks, and in the late 1940's to Koopmans' and Dantzig's use of linear programming [10]. Dantzig's simplex method of linear programming is effective for solving the transportation problem; however, it is not always efficient computationally for it may involve rejecting a large number of choices in order to find optimal routing [11]. In the case of Steiner's problem, the objective function fails to be well-defined when there are more than three points in the prescribed, initial, set of n locations. And, even if topological tree type is specified, the linear programming process itself is NP complete, producing unwieldy sets of equations for even fairly small values of n. Further, the rate at which the complexity increases, as n increases, parallels the NP completeness of the Steiner problem, itself. In contrast to the approach of linear programming of maximizing an objective function over an entire set of feasible solutions determined at the outset, Bellman's dynamic programming permits decisions regarding optimum solution to be made at a set of stages during the actual solution process, often permitting reduction in numbers of calculations to be made [12]; again though, it does not appear well-suited to Steiner's problem. Ford and Fulkerson's max-flow, min-cut theorem permits solution of the transportation problem by observing that flow along a given path of the network is constrained by the link in that path with minimum carrying capacity [13]. In contrast to the two previous approaches, Ford and Fulkerson use economic constraints on the spatial structure of the network (represented as a graph), to determine optimal economic gain from the network. Modifications and generalizations of Ford and Fulkerson's theorem range from extension of it to a set of sources and sinks, to modification of carrying capacities of the links to accommodate a set of different capacities on each link [14].

The entire class of problems discussed above, from paths through a maze to various solutions of the transportation problem, are such that an appropriate small shift in intermediate vertices of the graph, or in basic spatial constraints, could produce an improvement in optimal path length or flow through that network. This sort of observation leads naturally to a search for a set of intervening points, such that a network (of given topological form) is minimal with respect to the metric under consideration and is such that *no* shift in any of the intermediate points could improve path length. This notion is a generalization of Steiner's problem [15]. Steiner's original problem consists of determining the path of shortest length in the Euclidean plane, linking three vertices, v_1, v_2, and v_3. If v_1, v_2, v_3 form a triangle with one angle greater than or equal to $120°$, then the shortest path is along the two sides of the triangle that form that angle. Otherwise the minimal network is formed by locating a point S_p, (a *Steiner* point), in such a way that $|S_pV_1| + |S_pV_2| + |S_pV_3|$ is minimized; the lines S_pV_1, S_pV_2 and S_pV_3 will be said to form the Steiner network on $(V_1V_2V_3)$ and these lines form angles of $120°$ at S_p, relative to each other.

The generalized Steiner problem consists of finding a set of points $S_1, \ldots, S_p$ such that the network connection among a given set of points $V_1, \ldots, V_n$, $p < n$, is minimized [16]. The network that minimizes such connections will be called a Steiner network; its general form is such that every set of three edges incident with S_i, $1 \leq i \leq p$ will form angles of $120°$ relative to each other. In the general case the Steiner network will be an absolute minimum, selected from a set of relative minima, each of which is minimal for a particular style of connection (i.e., is minimal for a particular topological type). Each of the relative minima is a candidate for the Steiner network.

Sequential questioning of the original problem of finding a path from one place to another led along a path, through a class of problems, in which the given spatial constraints were such that the network improvement was possible by altering those constraints; that questioning jumped to the generalized Steiner question as a basis for generating a set of questions in which spatial barriers are to be determined in such a way that absolute optimal path selection is forced. This also follows from comparison of the simplest form of Steiner's problem with the simplest form of the original problem (the maze).

Because all improvements in any network approaching the Steiner network lead to the same Steiner network within the given set of points, the Steiner network might be viewed as a spatial invariant *within* the set of points it connects. Consequently, it would be useful as a standard against which networks to be built could be measured.

Whether or not the Steiner network connecting a set of points $\{V_1, \ldots, V_n\}$, using a set of Steiner points $\{S_1, \ldots, S_p\}$, is invariant under transformations of $\{V_1, \ldots, V_n\}$ is unclear. That is, if τ is a mapping such that

$$\tau : \{V_1, \ldots, V_n\} \to \{V_1', \ldots, V_m'\}, \quad n, m \in \mathcal{Z}^+$$

is defined by

$$V_i\tau = V_j', \quad 1 \leq i \leq n; \quad 1 \leq j \leq m,$$

then conditions under which an appropriate extension of τ, applied to $\{S_1, \ldots, S_n\}$, lead to the Steiner network of $\{V_1', \ldots, V_m'\}$, appear unclear.

This sort of consideration appears to be of fundamental importance, for identification of spatial network invariants leads to principles on which to determine geographical network location. The Steiner network is a spatial invariant around which general principles of network location might be built; it is the purpose of this work to examine in detail the formal, rather than the applied, nature of Steiner networks and to present the results in a way that permits the non-mathematician to replicate these results.

Thus, the following chapters present detailed treatments of aspects of the Steiner problem which, hopefully, suggest how to deal with those specific cases not covered here. Chapter II presents a complete analysis of Steiner's original problem for the triangle. Chapter III presents an algorithm exhibiting the detailed geometric construction of candidates for the Steiner network of the generalized Steiner problem. The proof of the algorithm is by mathematical induction; a variety of candidate trees will be exhibited as Figures (in place of the more traditional "Plates") in this Atlas. Chapter IV discusses the case for six points, showing how a wide variety of different candidate networks, including degenerate forms, come into being. Chapter V presents directions for some extensions of Steiner structures. Enumeration problems, including criteria for the selection of topological structure for Steiner networks, are considered in some detail.

Considerable effort has been made to illustrate steps of the proofs. The reason for this is that although procedures for locating candidates for the generalized Steiner network are available within the mathematical literature, the history of the problem is one that is riddled with difficulties. Melzak published an algorithm in 1961 showing the existence of a solution to the generalized Steiner problem, but did not exhibit it in detail [17]. Gilbert and Pollak later state that the generalized Steiner problem is solvable and extend Melzak's procedure in an effort to deal with degenerate Steiner networks (paths that follow, in some part, the set of linkages available along the convex hull of the original set of vertices) [18]. In articles post-dating Melzak's work, geographer Werner claims to make progress at solving this problem. However, his reasoning appears circular although he, himself, can evidently figure the positions of Steiner networks, and his grasp of the degenerate appears superficial [19]. In a later article, Cockayne states that Melzak's procedure is incorrect but does not say why it is [20]. The detailed proof presented here is compatible with that of Cockayne. Textbooks written in the 1970's, and in use and cited in the 1980's, perpetuate these difficulties. In geography, Abler, Adams, and Gould (1971) give no key references in their undergraduate text (perhaps understandably) to the literature surrounding this problem and rate it simply as "unsolved" [21]; nor do Haggett, Cliff, and Frey (1977), in their graduate text [22]. In the engineering/applied mathematics literature, Lawler (1976) cites both Melzak and Cockayne in his references, but does not mention the content of either article specifically, nor of the difficulty in dealing only with Melzak's approach [23].

Development of the solution for locating the Steiner network parallels that of the solution for the maze; once existence is determined the problem is abandoned. As with the maze, examination of alternate paths could lead to different sets of questions. In this case, the alternate paths might be considered candidate Steiner networks that were not the Steiner network. Adopting Courant and Robbins' idea of viewing a Steiner network as a boundary of a surface suggests that rejected Steiner candidate networks, as well as the Steiner network itself, may be minimal forms as boundaries of a variety of geographical surfaces. In particular Courant and Robbins view a Steiner network in the plane as the trace formed in a plane by a soap film stretched among n vertical rods connecting two parallel glass plates, each with boundary the convex hull of the n points [24]. Experiments with soap film by the Belgian physicist Plateau, in which the soap film is viewed as a boundary, led to determining the shape of a minimal surface from a given boundary [25]. Viewing a candidate for a Steiner network as a boundary of an area, one can ask what the minimal spatial form for such an area should be. Classification of areas served by networks might be achieved by grouping areas into a particular taxon if they are minimal areas bounded by a candidate Steiner network of a given topological type. In this way the problem of topological form associated with enumeration of Steiner networks would be turned into an asset, for it could be used to uncover similarity in spatial structure of disparate regions. This sort of abstract approach is precisely that which underlies the application of using Steiner trees to generate a triangulation each of whose cells contains its centroid.

This sort of general, abstract viewpoint is consistent with Werner's concern for topological considerations in network development and with Bunge's view of topology as fundamental to the study of spatial relations [26]. Further, it is consistent with the position of the study of topology within mathematics. In fact, if the prefixes of the pair of words 'topography' and 'geology' are switched, the pair 'geography' and 'topology' is obtained, suggesting that knowledge of topological structure underlying human "landforms," such as transportation networks, is as vital to understanding their evolution, spatial form and relations to one another, as is knowledge of geological structure to understanding associations of landforms composing the topographic surface of a region.

REFERENCES

1. W. M. Boyce and J. B. Seery, "An improved version of Cockayne and Schiller's program STEINER for the minimal network problem." *Bell Laboratories: Computing Science Technical Report #35* (1975)

2. P. A. Burrough, *Principles of Geographical Information Systems for Land Resources Assessment* (Oxford: The Clarendon Press, 1987.)

3. Thomas Bulfinch, *Mythology* abridgment by Edmund Fuller, (New York: Dell Publishing, 1959), p. 126. E. Humperdinck, "Hansel and Gretel." Lewis Carroll, *Through the Looking-Glass and What Alice Found There*.

4. The sequence of moving from the problem of finding a path from one place to another, to the puzzle of the maze and a further set of problems is not uncommon. Claude Berge, *The Theory of Graphs*, Alison Doig, translator, (New York: John Wiley and Sons, Inc., 1966), p. 65.

5. G. Tarry, "Le problème des labyrinthes," *Nouvelles Annales de Mathématiques*, 14, (1895), p. 187.

6. G. Dantzig; D. Fulkerson and S. Johnson, "Solution of a large-scale traveling salesman problem," *Operations Research*, 2, 1954, pp. 393-410. J. Little, K. Murty, D. Sweeney, and C. Karel, "An algorithm for the traveling salesman problem," *Operations Research*, 11, (1963), 972-989.

7. J. B. Kruskal, Jr., "On the shortest spanning subtree of a graph and the traveling salesman problem," *Proceedings, American Mathematical Society*, 7, (1956), 48-50. R. C. Prim, "Shortest connection networks and some generalizations," *Bell Systems Technical Journal*, 36, (1957), 1389-1401.

8. The transportation problem is discussed in a variety of places: e.g., Eugene L. Lawler, *Combinatorial Optimization: Networks and Matroids* (New York: Holt, Rinehart and Winston, 1976).

9. Gaspard Monge, "Sur les déblais et les remblais," *Histoire de L'Académie Royale des Sciences* (Paris: Imprimerie Royale, 1781), pp. 34-38.

10. George B. Dantzig, "Application of the simplex method to a transportation problem" in *Activity Analysis of Production and Allocation*, Cowles Comm.

Monograph, 13, New York, 1951, W. Wiley and Sons, p. 359. F. L. Hitchcock, "The distribution of a product from several sources to numerous localities," *Journal of Mathematics and Physics*, 20, (1941), p. 224, cited in T.C. Koopmans and S. Reiter, "A model of transportation" in *Activity Analysis of Production and Allocation*, Cowles Comm. Monograph, 13, New York, 1951, J. Wiley and Sons, p. 222.

11. Dantzig, *op. cit.*

12. R. Bellman, "The theory of dynamic programming," *Bulletin, American Mathematical Society*, 60, 1954, pp. 503-515.

13. L. R. Ford and D. R. Fulkerson, "Maximal flow through a network," *Canadian Journal of Mathematics*, 8, 1956, p. 399.

14. D. Gale, "A theorem on flows in networks" *Pacific Journal of Mathematics*, 7, 1957, p. 1073.

15. H. S. M. Coxeter, *Introduction to Geometry* (New York: John Wiley and Sons, 1961), p. 21.

16. E. J. Cockayne, "On the Steiner problem," *Canadian Mathematical Bulletin*, 10, (1967), pp. 431-50.

17. Z. A. Melzak, "On the problem of Steiner," *Canadian Mathematical Bulletin*, 4 (1961), pp. 143-48.

18. E. N. Gilbert and H. O. Pollak, "Steiner minimal trees," *SIAM Journal of Applied Mathematics*, 16, (1968), pp. 1-29.

19. C. Werner, "The Role of Topology and Geometry in Optimal Network Design," *Papers of the Regional Science Association*, 1968. C. Werner, "Networks of Minimum Length," *The Canadian Geographer*, Vol.XIII, No. 1, 1969, pp. 47-69.

20. Cockayne, *op. cit.*, p. 431.

21. R. Abler, J. S. Adams, and P. Gould, *Spatial Organization: The Geographer's View of the World*, (Englewood Cliffs: Prentice-Hall, 1971).

22. P. Haggett, A. D. Cliff, A. Frey, *Locational Analysis in Human Geography*, Vol. I: "Locational Models" (New York: John Wiley and Sons, 1977).

23. E. L. Lawler, *Combinatorial Optimization: Networks and Matroids* (New York: Holt, Rinehart, and Winston, 1976).

24. R. Courant and H. Robbins, *What Is Mathematics?* (London: Oxford

University Press, 1958).

25. F. J. Almgren Jr., *Plateau's Problem: An Introduction to Varifold Geometry* (New York: W. A. Benjamin, Inc., 1966).

26. W. W. Bunge Jr., "Theoretical Geography," Ph.D. dissertation, Department of Geography, University of Washington, 1960. C. Werner, "The role of topology and geometry in optimal network design," *Papers of the Regional Science Association*, XXI, p. 176.

CHAPTER II: NETWORKS OF MINIMAL TOTAL LENGTH IN THE TRIANGLE

For Jacob Steiner, the geographical challenge of connecting three villages, V_1, V_2, V_3, by a network of roads of minimum total length produced a geometrical response [1]. In Steiner's solution there are two logical possibilities; either the network of minimal total length consists of only two sides of the triangle $(V_1V_2V_3)$ (Figure II.1.a), or it does not (Figure II.1.b). If it does not, then there exists an interior intersection point S_p (the Steiner point), within the triangle $(V_1V_2V_3)$, such that the sum

$$V_1S_p + V_2S_p + V_3S_p \tag{1}$$

is minimal. The network consisting of vertices V_1, V_2, V_3 and S_p, linked by edges V_1S_p, V_2S_p, V_3S_p, is the Steiner network (Steiner tree) in the triangle $(V_1V_2V_3)$, induced by the Steiner point, S_p. This Steiner tree spans the vertices V_1, V_2, V_3, S_p and is the minimal spanning tree. The set of trees spanning V_1, V_2, V_3, that consist of two edges of the triangle $(V_1V_2V_3)$, as in Figure II.1.a, will be called degenerate networks connecting those vertices. Thus a network of minimal total length connecting the vertices V_1, V_2, V_3 is either

a) the Steiner tree on V_1, V_2, V_3; or,

b) a degenerate network.

The cases where a degenerate network appears in a triangle are known [2] and occur when one of the angles of the triangle is greater than or equal to $120°$.

As in the case with three vertices, the general problem of connecting n vertices, $V_1, V_2, \ldots, V_n$ by a network of minimal total length will respond, as well, to geometrical solution. The general solution is more complex in structure than is the solution for $n = 3$. In order to expose the reader to the style of procedure to be developed in the general case, the solution for $n = 3$ will be presented in great detail.

For $n = 3$, the development of the network of minimal total length and of the conditions under which that network is a Steiner tree is reproduced below and is based on the style of proof of J. E. Hoffmann, as found in Coxeter [3]. It is this style that motivated the generalization that is to follow.

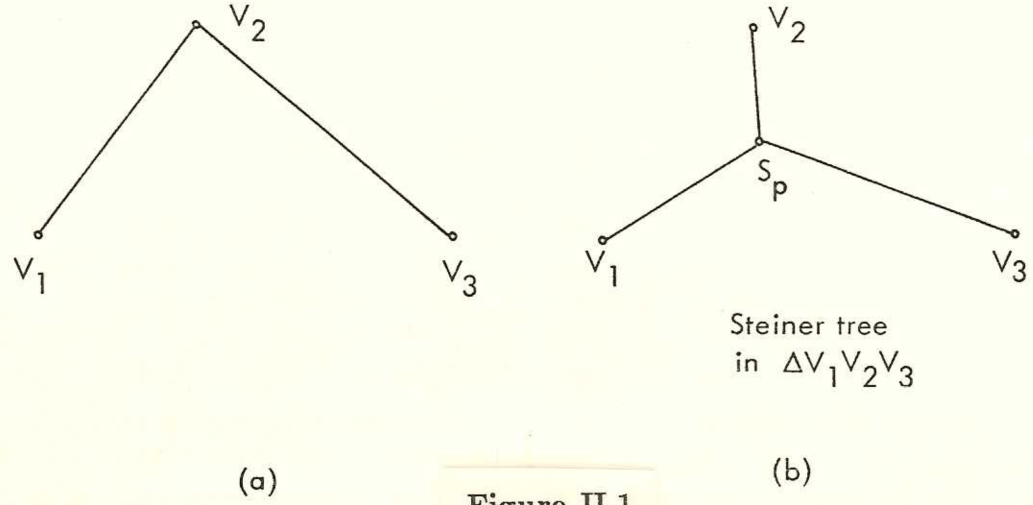


Figure II.1

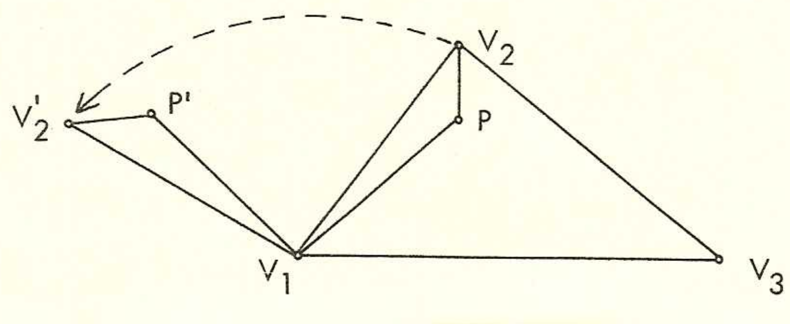


Figure II.2

Let P be *any* point in the interior of the triangle $(V_1V_2V_3)$. It is desired to locate P so that the sum

$$V_1P + V_2P + V_3P \tag{2}$$

is minimized. The sum (2) will be transformed into a sum of the same total magnitude representing a different geometrical configuration that is easy to minimize. Suppose that the triangle (V_1PV_2) is rotated [4] about the point V_1, where the direction of rotation is chosen in such a way that the side V_1V_2 of the triangle (V_1PV_2) never passes through the interior of the triangle $(V_1V_2V_3)$ (Figure II.2). This rotation takes place within the plane containing the triangle $(V_1V_2V_3)$. The motion of the triangle (V_1PV_2) resulting from this rotation will be said to be 'away from' the triangle $(V_1V_2V_3)$. Since it follows that (Figure II.2)

$V_2'V_1 = V_2V_1$ where V_2' represents V_2 rotated about V_1;

$V_2'P' = V_2P$ where P' represents P rotated about V_1;

$V_1P' = V_1P.$

As a direct result of this rotation, it follows that the length in (2) is the same as the length represented by the sum

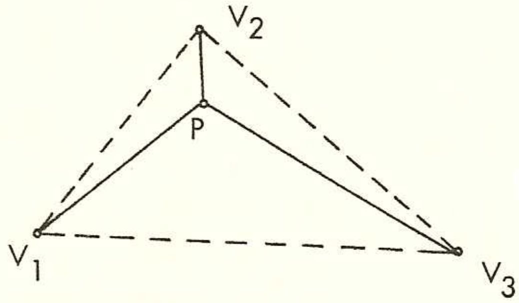
$$V_1P + V_2'P' + V_3P. \tag{3}$$

The transformation of rotation has not changed the Euclidean length, but it has altered the structure of incidence relations in the original network (Figure II.3).

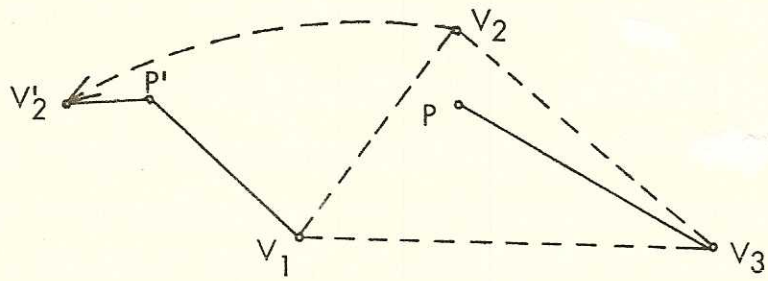
Suppose that the points P and P' are connected (Figure II.4), forming the triangle (V_1PP'). This triangle is isosceles, for, $V_1P' = V_1P$ since length is preserved by the rigid motion of rotation. We have not yet chosen the angle through which the triangle (V_1PP') is to be rotated. Since this flexibility in selection is available, the choice will be made to transform the sum (3) into one that is easy to minimize. If equation (3) can be transformed into

$$PP' + V_2'P' + V_3P, \tag{4}$$

this sum is minimized when V_2', P', P and V_3 (all in the Euclidean plane) are collinear, as is obvious from Figure II.4, although it is not apparent in the



——— network determined by V_1P, V_2P, V_3P.



——— network determined by V'_2P, V_1P', V_3P.

Figure II.3

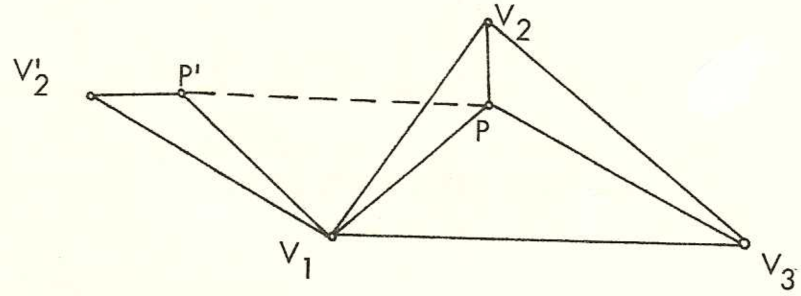


Figure II.4

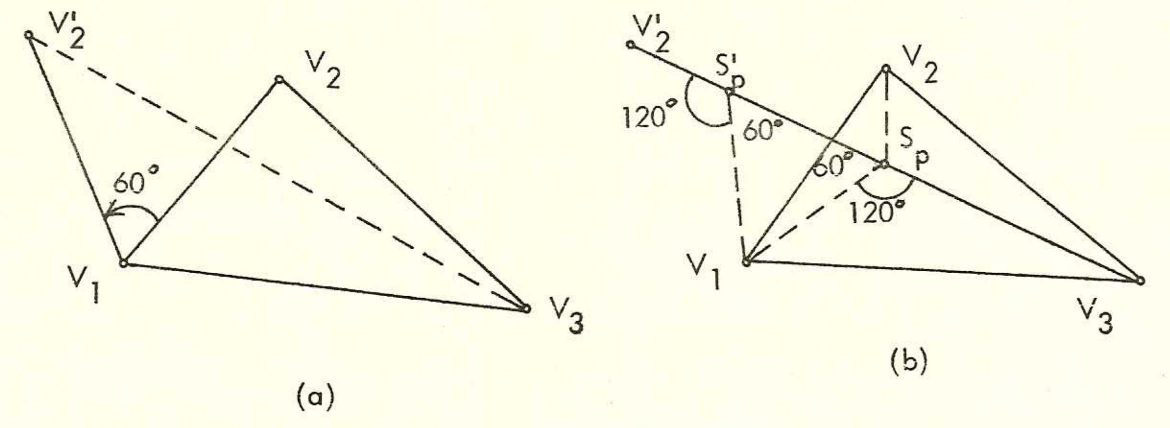


Figure II.5

equation itself. Choosing the angle of rotation to be 60° permits the desired transformation of (3) into (4). For, if $\angle P'V_1P = 60°$, then the isosceles triangle (V_1PP') is forced to have each angle equal to 60°, since the base angles of an isosceles triangle are equal ($\angle V_1P'P = \angle V_1PP' = 60°$). Since all three angles of the triangle (V_1PP') are equal, so are all three sides; the triangle (V_1PP') is an equilateral triangle. Therefore, as lengths, $PV_1 = PP'$. Thus, (3) can be written as

$$(4) \qquad PP' + V_2'P' + V_3P.$$

That is, S_p must lie on the line $V_2'V_3$ (Figure II.5.a). To locate S_p on $V_2'V_3$ another configuration which also contains S_p must be found. By construction comparable to the above, one such geometric configuration is the line $V_3'V_2$ where V_3V_1 is rotated about V_1 through an angle of 60° away from the triangle $(V_1V_2V_3)$. In this case, the Steiner point S_p is the intersection of these two lines. However, this somewhat obvious solution does not generalize in an obvious way, and so was rejected.

As an aid in the search for another geometrical configuration containing S_p, the following additional observations based on the minimization procedure given above (Figure II.5.b) are useful [5]. Since the points V_2', S_p', (corresponding to P' in Figure II.4) S_p, V_3 are collinear, it follows that $\angle S_p'S_pV_3 = 180°$. Because $\angle S_p'S_pV_1 = 60°$, it follows that $\angle V_1S_pV_3 = 120°$. Also, since $\angle V_2'S_p'S_p = 180°$ and $\angle V_1S_p'S_p = 60°$, then $\angle V_1S_p'V_2' = 120°$. Because angular measure is invariant under rotation, it follows that $\angle V_1S_pV_2 = 120°$. Thus, $\angle V_2S_pV_3 = 120°$. Hence, S_p is that point where

$$(5). \qquad \angle V_1S_pV_2 = \angle V_2S_pV_3 = \angle V_3S_pV_1 = 120°$$

From (5) the point S_p can lie anywhere along an arc of a circle, between V_1 and V_2, (Figure II.6.a), and be such that $\angle V_1S_pV_2 = 120°$. Because an angle inscribed in a circle subtends an arc of twice its angular measure, it follows that the long arc from V_1 to V_2 has measure 240°, for it is subtended by $\angle V_1S_pV_2$. Thus the short arc from V_1 to V_2 has measure 120° and is therefore subtended by an angle

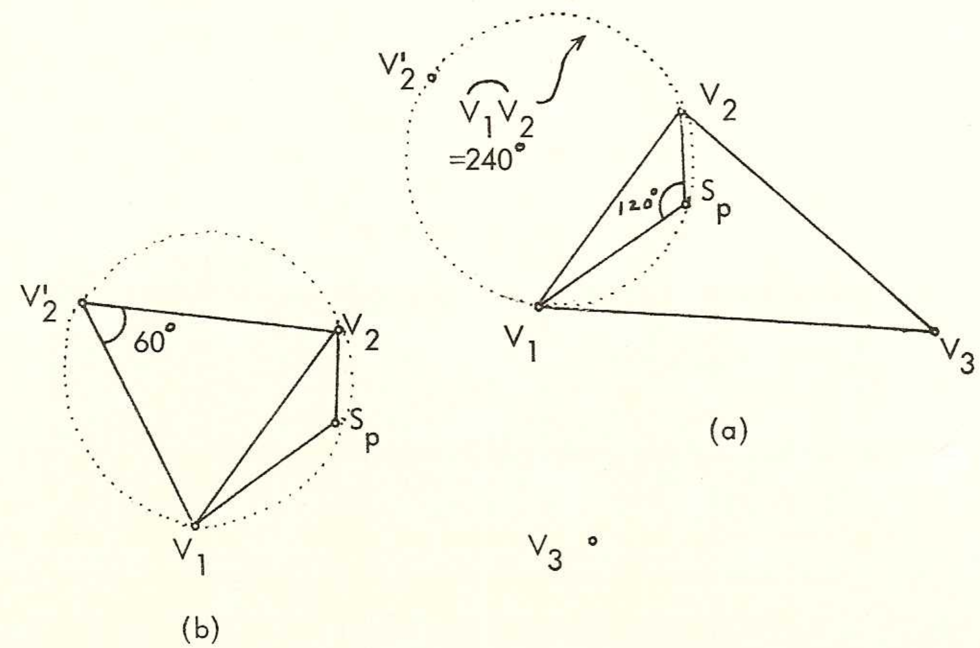


Figure II.6

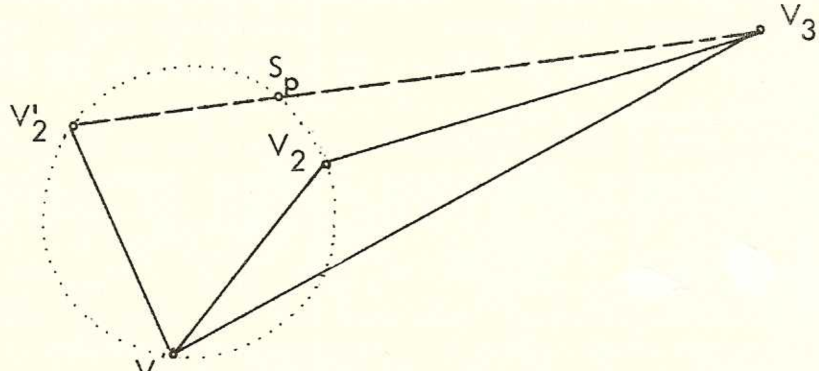


Figure II.7

of 60° (Figure II.6.b); and, in particular, it is subtended by $\angle V_1V_2'V_2$. Thus S_p lies on the circumscribed circle of the equilateral triangle $(V_1V_2'V_2)$. Therefore, the circumcircle of the triangle $(V_1V_2'V_2)$ intersected with $V_2'V_3$ gives S_p. Or,

$$(6) \qquad (circumcircle\ \triangle V_1V_2'V_2) \wedge (V_2'V_3) = S_p.$$

In the construction described above for determining the location of S_p on the line $V_2'V_3$, it is possible that the intersection of the segment $V_2'V_3$, may intersect the circumcircle of the triangle $(V_1V_2'V_2)$ outside the triangle $(V_1V_2V_3)$ (Figure II.7). Such a situation could occur only when $V_2'V_3$ does not pass through the interior of the triangle $(V_1V_2V_3)$. This can happen only if the measure of one angle of the triangle $(V_1V_2V_3)$ is greater than or equal to 120° [6]. For, if $\angle V_1V_2V_3 \geq 120°$, then $\angle V_2'V_2V_3 \leq 180°$ so that $V_2'V_3$ does not pass through the interior of the triangle $(V_1V_2V_3)$. In this case, the network of minimal total length connecting the vertices V_1, V_2, V_3 consists of two sides of the triangle $(V_1V_2V_3)$; that is, it is one that is degenerate.

Thus a network of minimal total length among three vertices V_1, V_2, V_3, is:

a) the Steiner tree on V_1, V_2, V_3 if the polygon containing these vertices has no angle with measure greater than or equal to 120°.

b) a degenerate network, if the polygon containing V_1, V_2, V_3 has some angle with measure greater than or equal to 120°.

It is important to notice that for degenerate networks, there are three possible candidates for *the* network of minimal total length:

a) V_1V_2, V_2V_3

b) V_1V_3, V_1V_2

c) V_3V_1, V_3V_2

The shortest of this set of three is the network of minimal total length, to be used whenever the degenerate network is appropriate. The construction for producing the Steiner tree on the vertices V_1, V_2, V_3 gives *the* network of minimal total length directly, in the case of a triangle.

REFERENCES

1. R. Courant and H. Robbins, *What Is Mathematics?* (London: Oxford University Press, 1941), p. 354.

2. *Ibid.*, p. 358. Also, C. Werner, "The Role of Topology and Geometry in Optimal Network Design," *Papers of the Regional Science Association*, XXI, p. 176.

3. H. S. M. Coxeter, *Introduction to Geometry* (New York: John Wiley and Sons, 1961), p. 21, citing J. E. Hoffmann, "Elementare Lösung einer Minimumsaufgabe," Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, 60 (1929), pp. 22-23.

4. Coxeter, *op. cit.*, p. 22. (The use of such a rotation provided the key to generalization).

5. *Ibid.*

6. Courant and Robbins, *loc. cit.*

CHAPTER III: NETWORKS OF MINIMAL TOTAL LENGTH, IN GENERAL

The observations made in Chapter II for the three vertex case motivate the following outline of solution for the general problem of finding the network of minimal total length linking a finite set of vertices, $V_1, V_2, \ldots, V_n$.

1) Connect $V_1, V_2, \ldots, V_n$ to each other to form an n-sided polygon (n-gon), $\mathcal{P}$.

a) if $\mathcal{P}$ is convex, then it is unique.

b) if $\mathcal{P}$ is concave, then there exist a finite number of other concave polygons distinct from $\mathcal{P}$, that can be formed on $V_1, V_2, \ldots, V_n$.

2) Within $\mathcal{P}$, be it convex or concave, the following considerations show that for $n > 3$ it will generally be necessary to have more than one Steiner point within $\mathcal{P}$ whenever the network of minimal total length is other than a set of $(n-1)$ edges of $\mathcal{P}$.

When $n = 1$, the minimal total network length is the distance between a point and itself, or, zero.

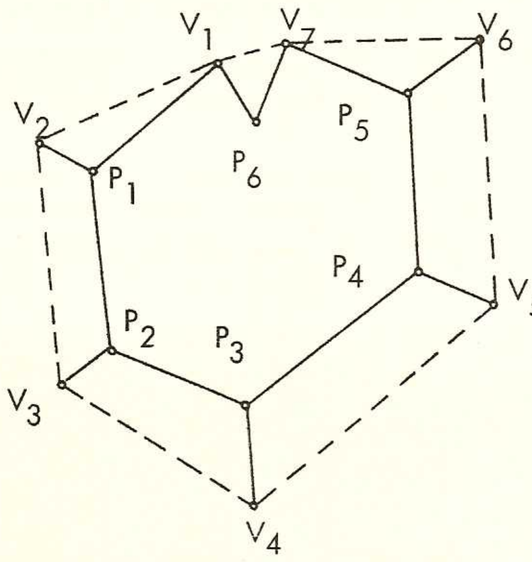
When $n = 2$, the minimal total network length is the distance between two points (e.g. a straight line segment in the plane, an arc of a great circle on a sphere, etc.). The connection, here, is drawn as a straight line in a plane.

When $n = 3$, the minimal total network length is determined as above and the network has the general structure of the graph in Figure II.1.b when every angle has measure less than 120°, and has the structure of the graph in Figure II.1.a otherwise.

As n increases, the level of complexity of the procedure increases; $n = 1$ is a less developed form of the next higher case, as is $n = 2$. Thus, one might suspect that similar constructions for higher values of n could introduce new problems in construction. In particular, for $n > 3$ how many Steiner points interior to the n-gon should be used when the total network length is not minimized by a set of $(n-1)$ consecutive edges of the n-gon? In the case $n = 3$ it is clear that more than one interior point S_p cannot produce a shorter Steiner tree than the one associated with S_p. Extra point(s) introduce alternate paths linking the same places and consequently the total network length is not minimal. In the same way, in any larger n-gon, $(n-2)$ is the largest number of interior points that can lead to a Steiner network [1]; more points lead to redundant edges (Figure III.1). Fewer than $(n-2)$ points can minimize total network length; these cases are either degenerate (with part or all of the network consisting of sides of $\mathcal{P}$) or are compressed (with one or more Steiner points superimposed (Figure III.2)) forms of the case with $(n-2)$ interior points. The most powerful form of proof will be obtained by considering the most general case, so it will be assumed, that when interior intersection points are required, there are $(n-2)$ of them labeled $P_1, \ldots, P_{(n-2)}$ within an n-gon $V_1, \ldots, V_n$, and that they are linked together as a cubic tree [2]. (A tree that is not cubic, is not disconnected, does not contain loops, and is based on $(n-2)$–interior multiple intersections can exist; however, such a tree contains at least one vertex of degree two and is therefore homeomorphic to a cubic tree.)

3) To determine the network of minimal total length in $\mathcal{P}$ (assuming that it is other than a sequence of $(n-1)$ edges of $\mathcal{P}$) separate the (finite) set of all possible paths into a set of classes based on the possible topologically–different forms that the minimal spanning tree can take; label the set $T_1, \ldots, T_m$. Find the minimal length of each representative for each class. The minimal tree for each class will be a "first-level candidate" Steiner tree; *the* Steiner tree will be the absolute minimum of this set of "first-level candidate" relative minima (Figure III.3). Each "first level candidate" will be the minimum of a set of "second level candidates" derived from each topological class, as shown below.

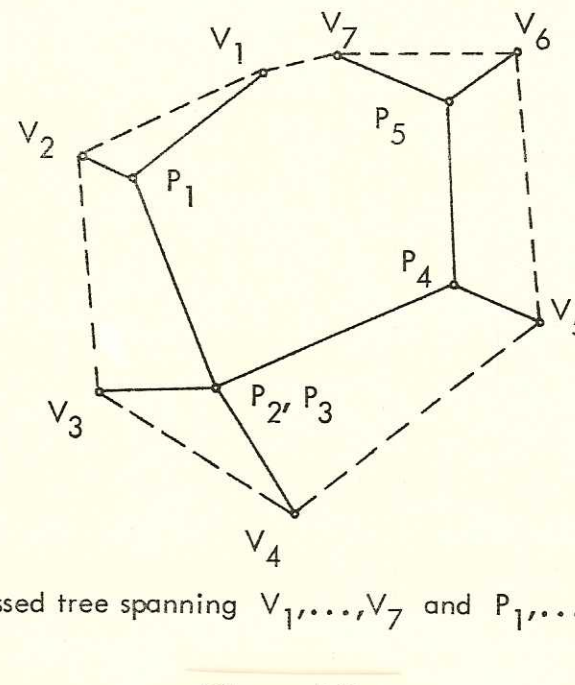
Thus, it remains to show how to derive the set of second level candidates for an n-sided polygon $\mathcal{P}$. Each second level candidate is a tree that is minimal relative to some connection pattern within $\mathcal{P}$. An algorithm for locating second level candidates is proved below using procedures of mathematical induction. Constructions following parts of the proof, exhibited as Figures of the Atlas, will illustrate how the proof leads to the actual location of these second level candidate-networks. A typical second level candidate is a tree connecting the



There is more than one path from V_1 to V_7 so total network length is not minimized:

$$\text{Card}(P) > \text{Card}(V) - 2$$

Figure III.1



A compressed tree spanning $V_1, \ldots, V_7$ and $P_1, \ldots, P_5$.

Figure III.2

n vertices of $\mathcal{P}$, $V_1, \ldots, V_n$ to the $(n-2)$ interior intersection points $P_1, \ldots, P_{(n-2)}$. The sub-tree which links the $(n-2)$ interior points will be referred to as an $(n-2)$ spanning tree (Figure III.4).

To exhibit a wide range of technique in proof, vital for adapting the general procedure to variety in topological structure of the $(n-2)$–spanning tree, we proceed by proving the general construction for n odd and assume that the $(n-2)$–spanning tree has no interior intersection points of degree greater than two (see Appendix to this chapter). The case of n even will then be proved and it will be assumed in that case that the $(n-2)$–spanning tree has the maximal number of interior points of degree three and has maximal branching. Thus, procedure for either extreme should suggest how to deal with the intermediate cases. The next (odd–numbered) case occurs when

$\underline{n=5}$.

Let P_1, P_2, P_3 be *any* three points that are in the interior of the pentagon $(V_1, V_2, V_3, V_4, V_5)$ and are linked as a cubic tree with V_1, V_2, V_3, V_4, V_5 as shown in Figure III.5.1. Then it is desired to minimize the sum

$$V_1P_1 + V_5P_3 + \sum_{i=1}^{3} V_{(i+1)}P_i + \sum_{j=1}^{2} P_jP_{(j+1)} \tag{7}$$

the total length of the cubic tree. Rotate the triangles $(V_1V_2P_1)$ and $(V_4P_3V_5)$ through 60°, about V_2 and V_4 respectively, toward the exterior of the polygon (Figure III.5.2). Under this rotation, the triangle $(V_2P_1V_1)$ is transformed into the triangle $(V_2P_1'V_1')$ and the triangle $(V_4P_3V_5)$ is transformed into the triangle $(V_4P_3'V_5')$. It follows that

$V_1P_1 = V_1'P_1'$

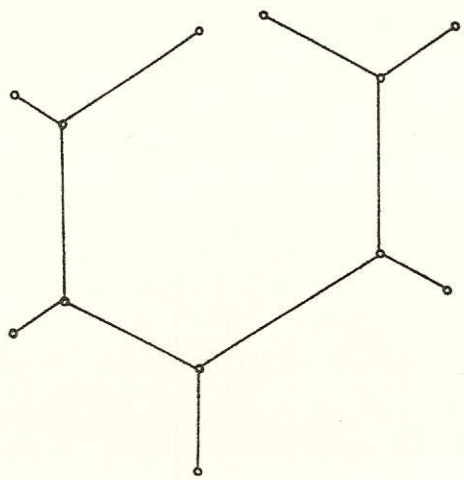
$V_2P_1 = P_1P_1'$ since $\triangle V_2P_1P_1'$ is equilateral

and

$V_5P_3 = V_5'P_3'$

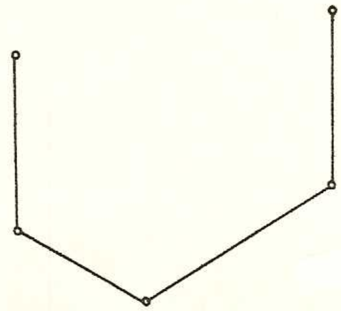
$P_3V_4 = P_3P_3'$ since $\triangle V_4P_3P_3'$ is equilateral.

Thus,



A Steiner tree

Figure III.3



The (n-2)-spanning tree

Figure III.4

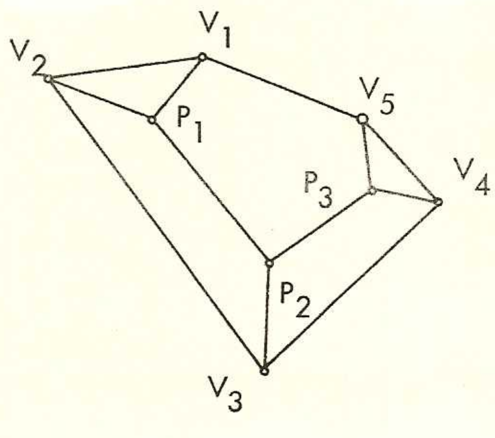


Figure III.5.1

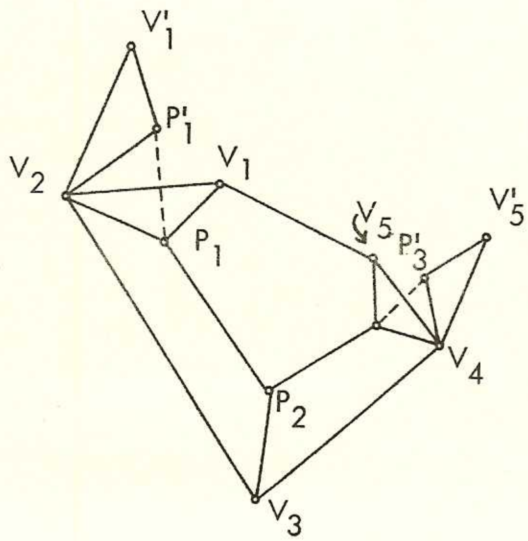


Figure III.5.2

$$min(V_1P_1 + P_3P_5 + \sum_{i=1}^{3} V_{(i+1)}P_i + \sum_{j=1}^{3} P_jP_{(j+1)})$$

$$(8) \qquad = min(V_1'P_1' + P_1'P_3 + P_1P_2 + P_2V_3 + P_2P_3 + P_3P_3' + P_3'V_5').$$

Retrieving the graph theoretical structure of this tree, it is clear that the network represented by the edges in (8) (Figures III.5.3.a and III.5.3.b) has the same basic topological form as (is homeomorphic to) the general cubic tree with three endpoints and one interior intersection (Figure III.5.3.c). Thus the sum is minimized by the minimal total network length connecting the vertices V_1', V_3, V_5' and this can be found using the procedure of the case $n = 3$ on the triangle $(V_1'V_3V_5')$. This triangle will be referred to as the polygon of the first rotation associated with this pentagon [3].

Because the network on the pentagon is minimized by the minimal tree on $\triangle V_1'V_3V_5'$ it follows that

$\angle V_1'P_2V_3 = \angle P_1P_2P_3 = \angle V_5'P_2V_3 = 120°$

and that

$\angle V_1P_1V_2 = \angle V_1'P_1'V_2$ **(by rotation)** $= 180° - 60° = 120°$

and $\angle V_5P_3V_4 = 120°$ by a similar argument.

Rotate V_1V_2 and V_4V_5 through $60°$ to produce V_1' and V_5', the vertices forming the polygon of the second rotation associated with this pentagon, (Figure III.6.1). Then, using the construction for the case $n = 3$ find the Steiner point of $\triangle V_1'V_3V_5'$, (Figure III.6.2). This will be S_{p_2}. Linking S_{p_2} to the vertices of the polygon of the first rotation produces a second-level candidate for the Steiner network of the polygon of the first rotation (in this case uniquely). Then, points S_{p_1} and S_{p3} will be determined as the intersections of $S_{p_2}V_1'$ with the circumcircle of $\triangle V_1V_1'V_2$ (Figure III.6.3), and of $S_{p_2}V_5'$ with the circumcircle of $\triangle V_4V_5V_5'$, forming a second level candidate for the Steiner network in the original polygon—some of which is incident with that of the Steiner network in the polygon of first rotation.

Three assumptions underlie this procedure: first, the Steiner point S_{p_2} of $\triangle V_1'V_3'V_5$ is inside the triangle; second, S_{p_2} is contained in the original pentagon

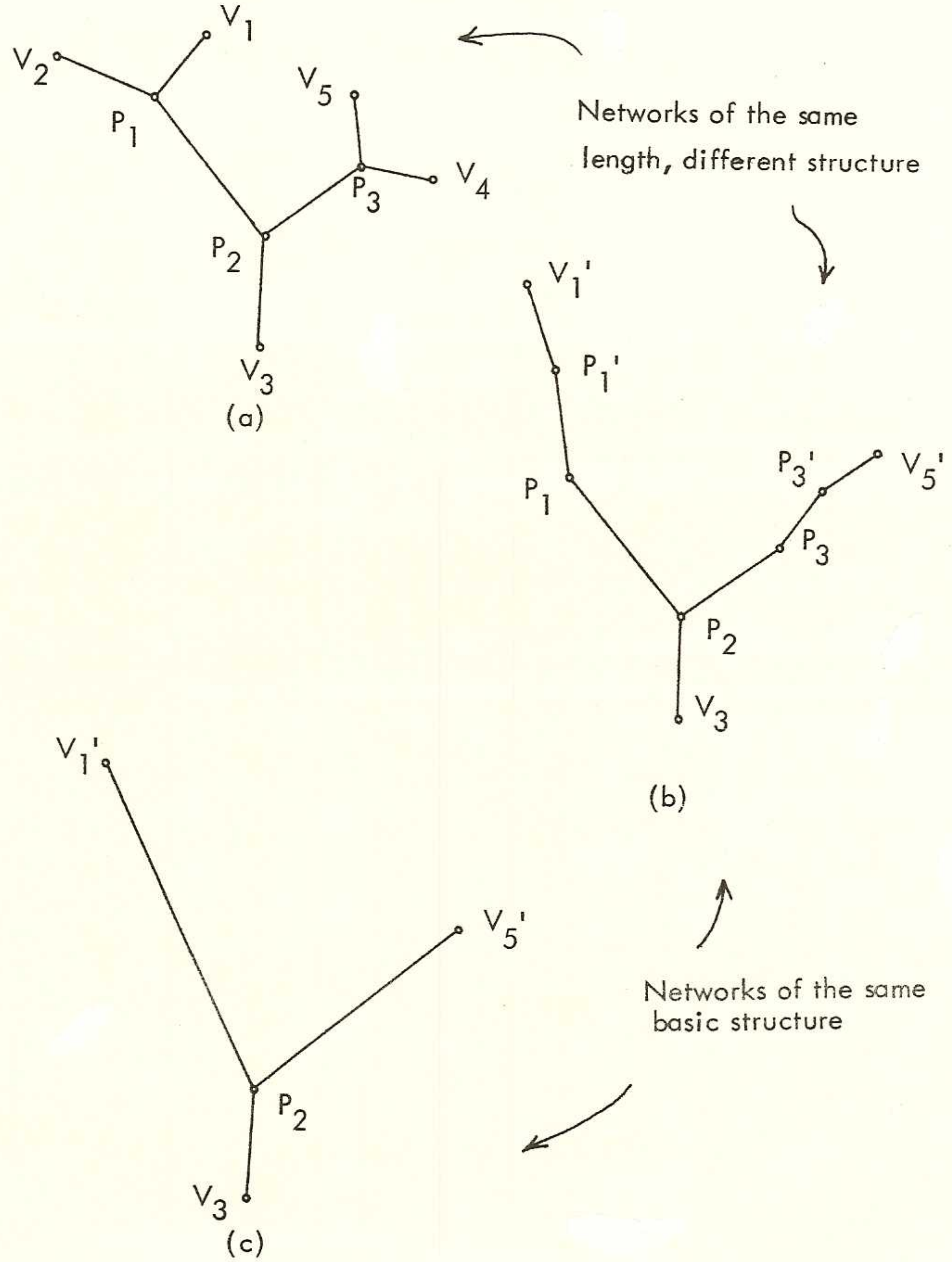


Figure III.5.3

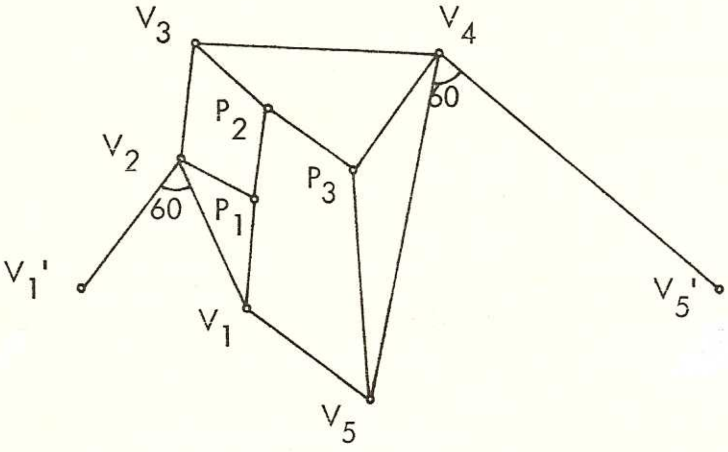


Figure III.6.1

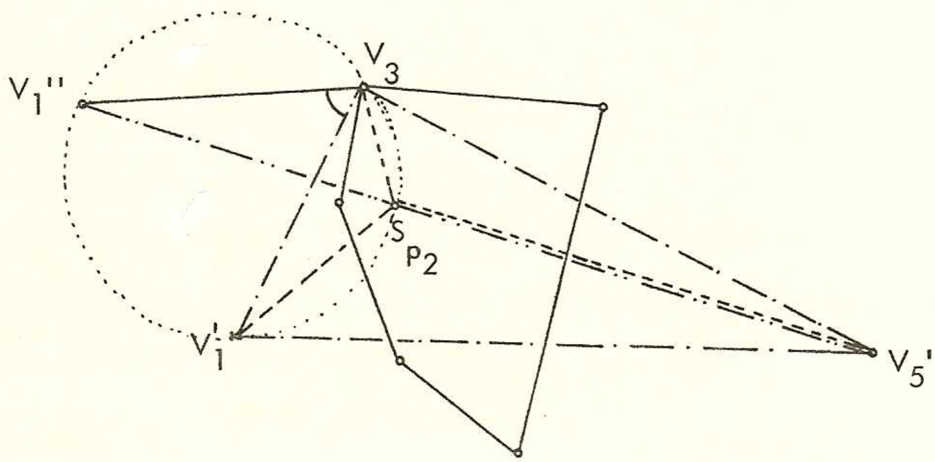


Figure III.6.2

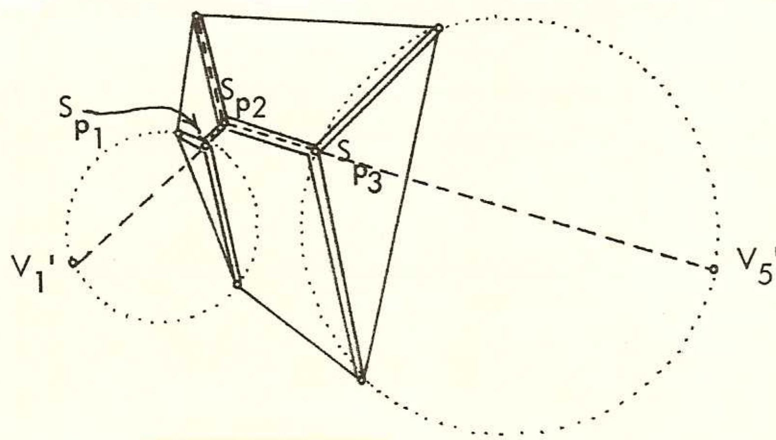


Figure III.6.3

(it could conceivably lie in that part of the triangle which does not overlap the pentagon), and third the intersections $((S_{p_2}V_1') \wedge (circumcircle\ \triangle V_1V_2V_1'))$ and $((S_{p_2}V_5') \wedge (circumcircle\ \triangle V_4V_5V_5'))$ lie within the pentagon. Figure III.6.4 illustrates a case where the third assumption is violated, but the first two hold, producing a degenerate second level candidate for the network of minimal total length in this pentagon. Notice that the two pentagons in figures III.5 and III.6 are identical; only the labeling of the vertices has been changed and, therefore, the sides that are rotated to form a different polygon of first rotation. Thus, from the same pentagon one construction yields a degenerate case while another does not.

An even more degenerate case is produced in the pentagon $V_1V_2V_3V_4V_5$ (Figure III.6.5) when V_1V_2 and V_3V_4 are rotated through $60°$ to produce $\triangle V_1'V_3'V_5$ in which $\angle V_1'V_5V_3' > 120°$. Thus $V_5 = S_{p_2}$, and $V_1'V_5$ intersects circumcircle $V_1V_1'V_2$ outside the pentagon so that $S_{p_1} = V_1$. Thus, this second level candidate for the Steiner network consists of V_2V_1, V_1V_5, $V_5S_{p_3}$, $S_{p_3}V_3$, $S_{p_3}V_4$. These three networks are three distinct second level candidates for the Steiner network on the original pentagon.

Retrieve the graph theoretic structure of the $(n-2)$–spanning tree and note that in the case $n = 3$, the $(n-2)$–spanning tree was a point. For $n = 5$ the $(n-2)$–spanning tree was homeomorphic to a straight line, and was unique. This will not continue to be true for higher values of n and, indeed, is one of the complexities arising from generalization. In the case $n = 7$ there will be two homeomorphically irreducible $(n-2)$–spanning trees (see Appendix).

<u>*Suppose* $n = 7$.</u>

In this proof choose the $(n-2)$–spanning tree that has no interior intersection points of degree three. Therefore, let $P_1, \ldots, P_5$ be $(n-2)$ points interior to a heptagon, $V_1, \ldots, V_7$ that are connected into a cubic tree where the $(n-2)$–spanning tree has no interior intersection points of degree three. Discarding the graph theoretic structure and emphasizing the geometrical content of the network, the quantity to minimize is:

$$(9) \qquad V_1P_1 + V_7P_5 + \sum_{i=1}^{5} V_{(i+1)}P_i + \sum_{j=1}^{4} P_jP_{(j+1)}$$

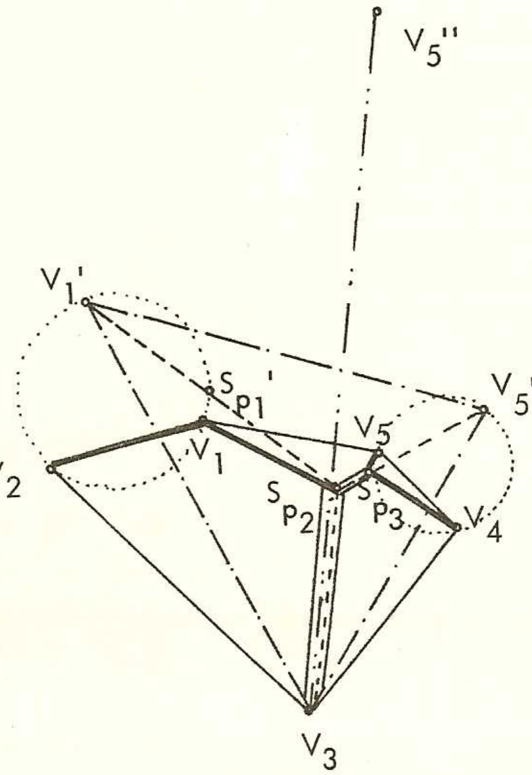


Figure III.6.4

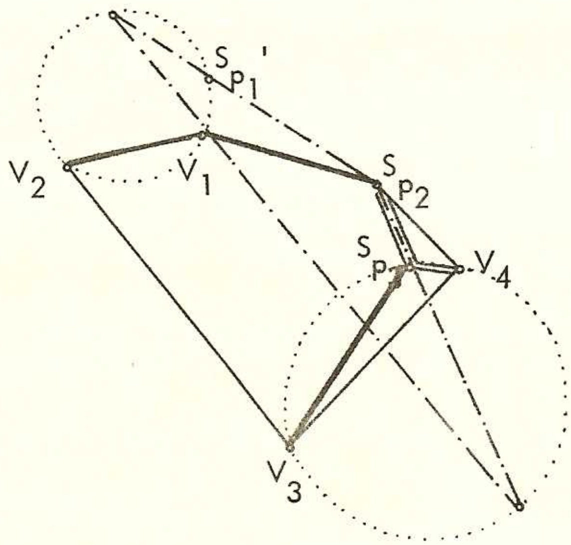


Figure III.6.5

the length of the cubic tree (Figure III.7.1). Again, by rotating triangles $(V_1V_2P_1)$ and $(V_7V_6P_5)$ through $60°$, and noting equalities resulting from congruences as before, the problem is reduced to finding the network of minimal length on the pentagon $V_1'V_3V_4V_5V_7'$, a polygon of the first rotation, or it is reduced to the case $n = 5$.

The Steiner points, S_{p2}, S_{p3}, S_{p4} of the pentagon $V_1'V_3V_4V_5V_7'$ (Figure III.7.2) are determined by locating S_{p3} as the Steiner point of $\Delta V_1''V_7''V_4$, a polygon of the second rotation, where V_1'' is produced by rotating $V_1'V_3$ through $60°$ about V_1', and V_7'' is produced by rotating V_5V_7' through $60°$ about V_7' (Figure III.7.3). Thus, S_{p3} is determined as the intersection of $V_7''V_1'''$, the Steiner network of a polygon of the second rotation, and the circumcircle of $V_1''V_4V_1'''$, where V_1''', a vertex of a polygon of the third rotation, is produced by rotating $V_3'V_4$ through $60°$ about V_4, so that the construction for $n = 3$ can then be used. Thus, (Figure III.7.4)

$$S_{p2} = (S_{p3}V_1'') \wedge (\textit{circumcircle}\ \Delta V_1'V_3V_1'')$$

$$S_{p4} = (S_{p3}V_7'') \wedge (\textit{circumcircle}\ \Delta V_5V_7''V_7')$$

giving Steiner points and network (for this second level candidate) of the polygon of first rotation $V_1'V_3V_4V_5V_7'$ under assumptions similar to those mentioned in the construction for $n = 5$, and

$$S_{p1} = (S_{p2}V_1') \wedge (\textit{circumcircle}\ \Delta V_1V_1'V_2)$$

$$S_{p5} = (S_{p4}V_7') \wedge (\textit{circumcircle}\ \Delta V_6V_7V_7')$$

(Figure III.7.5) completing a set of Steiner points for a particular second level candidate for the Steiner tree of the original heptagon with no assumptions violated. Degenerate networks can arise in a variety of ways, each a different second level candidate, and the same sort of analysis, extended, works for the general case, as it did for the case $n = 5$. In the above case the $(n-2)$–spanning tree was linear in form (i.e. had no interior intersection point of degree 3 (Figure III.7.5)). However, it could have had an interior intersection point of degree 3 and have been connected as in Figure IV.T_1.1. Thus, in this case and cases for

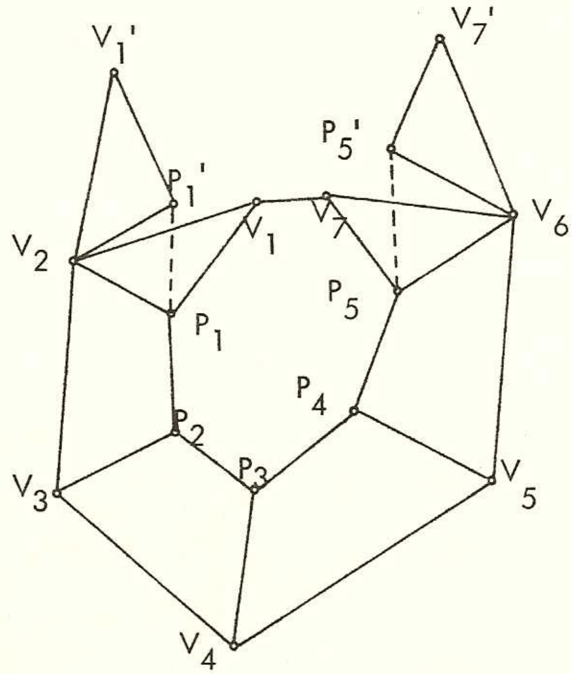


Figure III.7.1

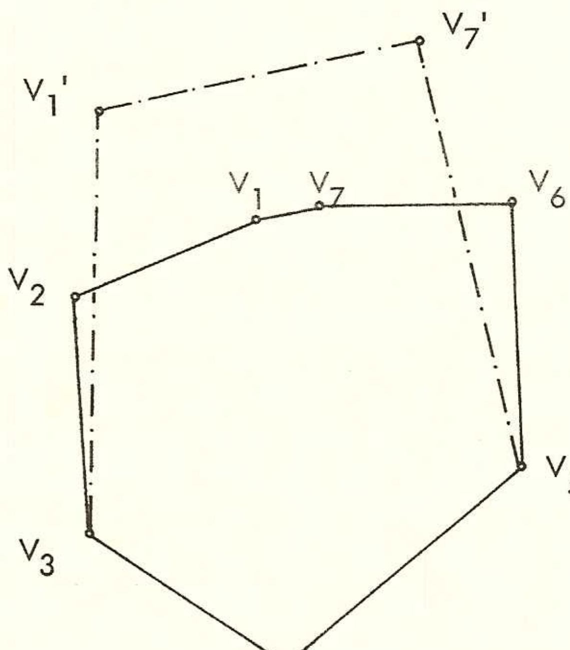


Figure III.7.2

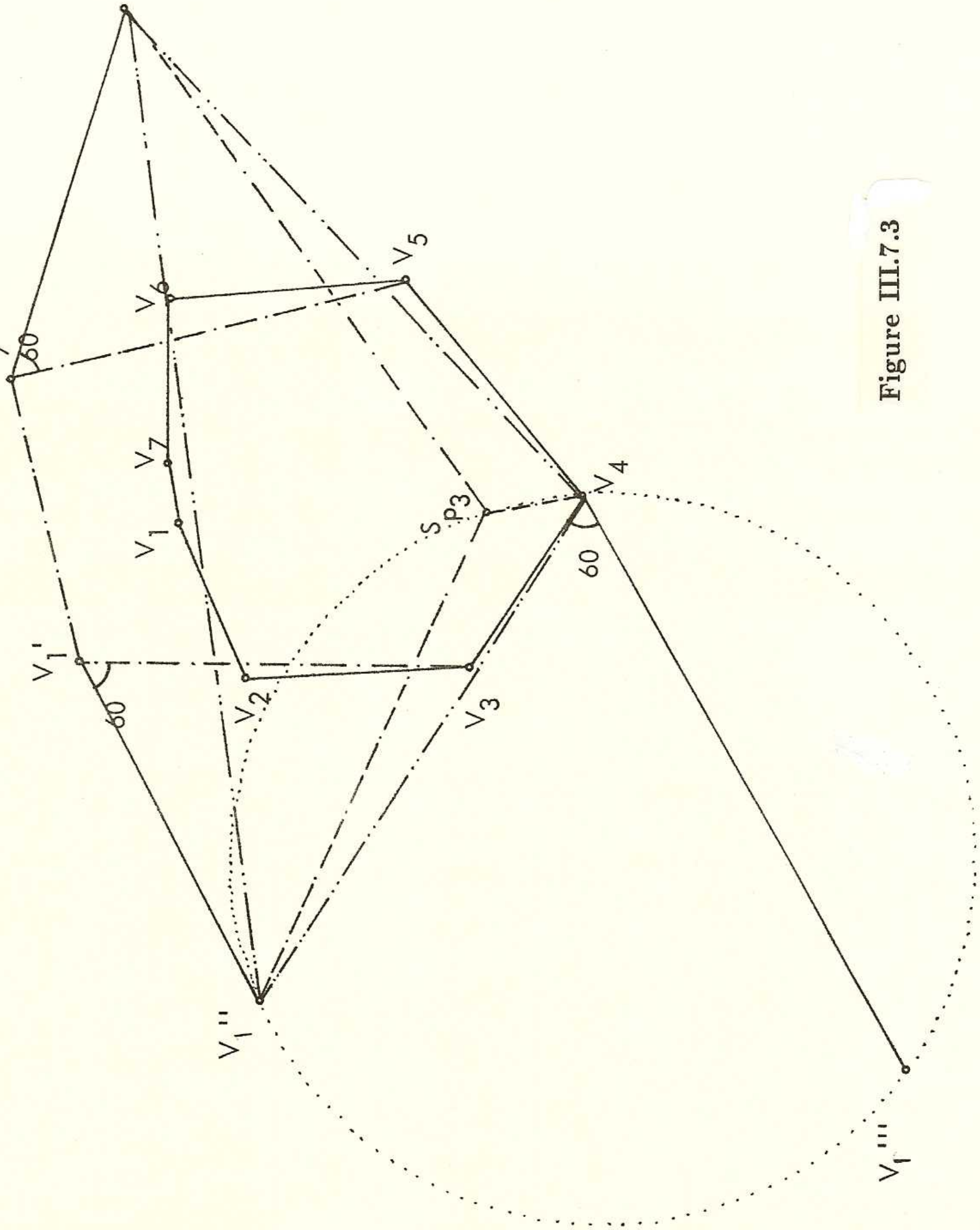


Figure III.7.3

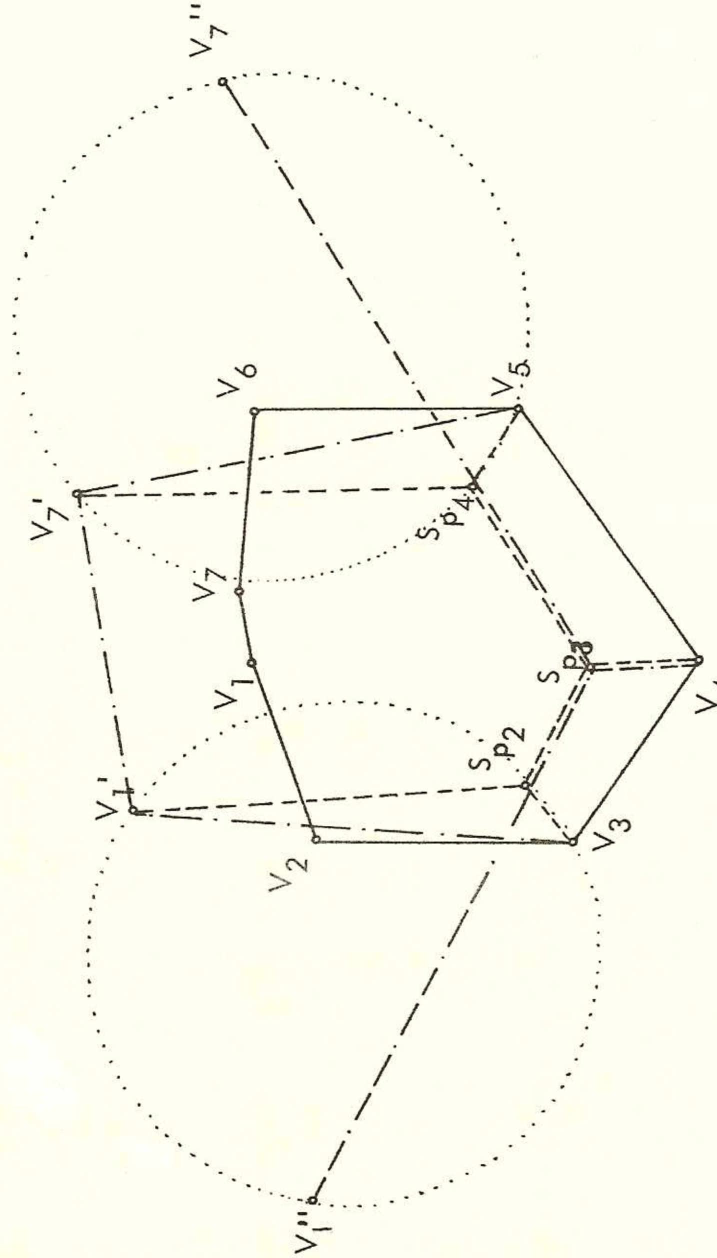


Figure III.7.4

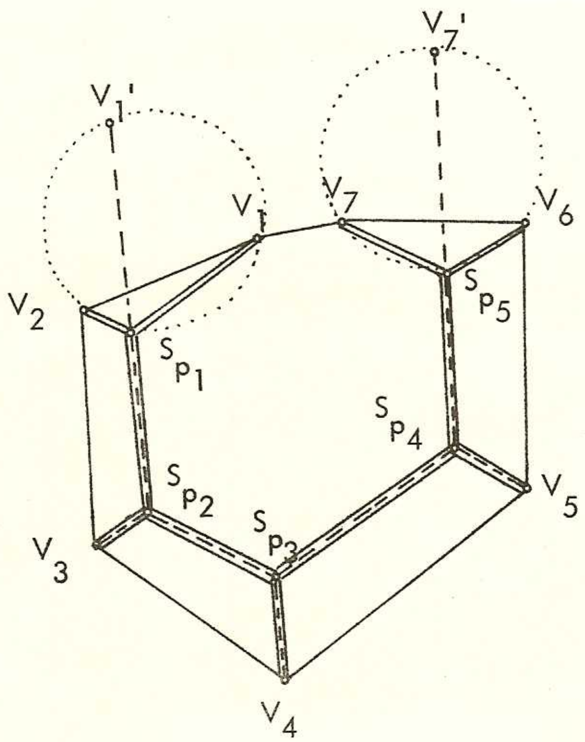


Figure III.7.5

n higher than this, basic topological structure of the $(n-2)$–spanning tree must be specified (see Appendix) before minimization of tree type within an n-gon can occur; that is, for $n > 5$, there will be more than one first level candidate. Additionally it will be necessary to specify how such an $(n-2)$–spanning tree is to be hooked into the original n-gon.

Higher values of n do not produce more new situations to be dealt with; $n = 7$ is general—none of the essential structures is degenerate or folded up. The form of the $(n + (n-2))$ Steiner candidate tree is in full view as is that of the $(n-2)$–spanning tree. Proceed to the induction hypothesis, in which it is assumed that the construction holds for $n = 2q - 1$ or for the kth step. It will be proven that the construction holds for the $(k+1)$st case, or when $n = 2q + 1$. thus, the construction will be valid for any odd n.

Induction Hypothesis :

Suppose $n = 2q - 1$, q a positive integer, and that the constructions analogous to the earlier ones hold for all odd n up to this point: that the following quantity has been minimized

$$V_1P_1 + V_{2q-1}P_{2q-3} + \sum_{i=1}^{n-2} V_{(i+1)}P_i + \sum_{j=1}^{n-3} P_jP_{(j+1)}$$

for all odd $n = 2q - 1$, $n > 2$, $n \in \mathcal{Z}^+$.

The $(k+1)$ case : $n = 2q + 1$

Let $P_1, \ldots, P_{2q-1}$ be $(n-2)$ points in the interior of the n–gon, $V_1, \ldots, V_{2q+1}$ that are connected as a cubic tree with $V_1, \ldots, V_{2q+1}$, in such a way that the $(n-2)$– spanning tree has no vertices of degree three, consistent with decisions made in earlier cases. Ignoring the graph theoretical structure and focusing on geometrical form, it is required, therefore, to minimize

$$V_1P_1 + V_{2q+1}P_{2q-1} + \sum_{i=1}^{n-2} V_{(i+1)}P_i + \sum_{j=1}^{n-3} P_jP_{(j+1)}.$$

Rotating the triangles $(V_1V_2P_1)$ and $(V_{2q+1}V_{2q}P_{2q-1})$ through 60° toward the exterior of the polygon shows that (Figure III.8)

$$V_1P_1 = V_1'P_1'$$

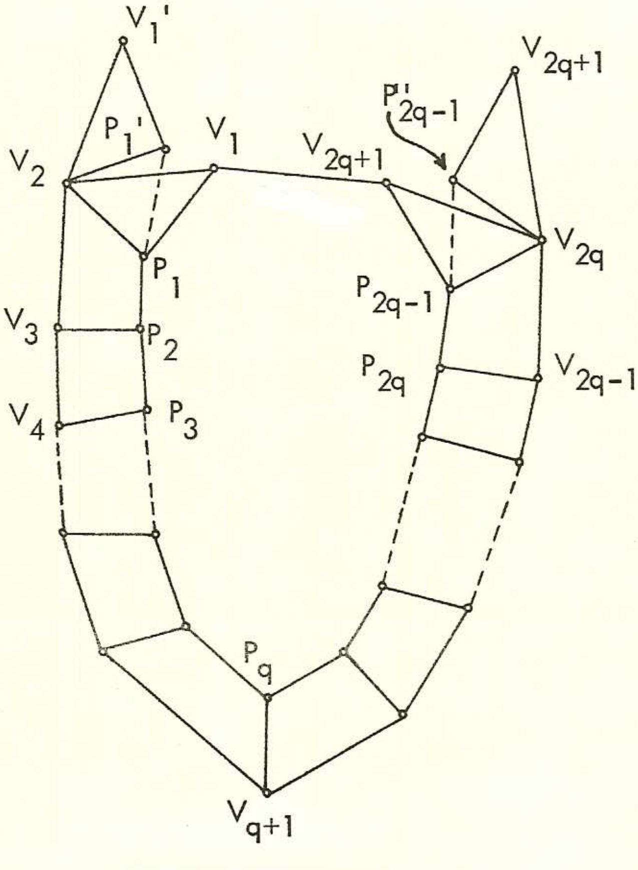


Figure III.8

$$V_2P_1 = P_1P_1'$$

and

$$V_{(2q+1)}P_{(2q-1)} = V'_{(2q+1)}P'_{(2q-1)}$$

$$V_{2q}P_{(2q-1)} = P_{(2q-1)}P'_{(2q-1)}$$

Thus,

$$min(V_1P_1 + V_{2q+1}P_{2q-1} + \sum_{i=1}^{n-2} V_{(i+1)}P_i + \sum_{j=1}^{n-3} P_jP_{(j+1)})$$

$$= min(V_1P_1' + V'_{2q+1}P_{2q-1} + P_1P_1' + P_{2q-1}P'_{2q-1} + \sum_{i=2}^{2q-2} V_{i+1}P_i + \sum_{j=1}^{n-3} P_jP_{(j+1)}).$$

The network represented by this last sum is homeomorphic to the cubic tree with $(2q-3)$ vertices of degree three and $(2q-1)$ vertices of degree one. Thus, the right hand side is minimized by the minimal cubic tree on the points

$$P_2, \dots, P_{2q-2}, V_1', V_3, V_4, \dots, V_{2q-1}, V'_{2q+1}$$

and the corresponding construction can be executed using the induction hypothesis on the polygon

$$V_1', V_3, V_4, \dots, V_{2q-1}, V'_{2q+1}$$

where (retrieving the graph theoretic structure and using the homeomorphism) $V_1'P_1'+P_1P_1'$ is replaced by $V_1'P_1'$ and $V'_{2q+1}P_{2q-1}+P_{2q-1}P'_{2q-1}$ is replaced by $V'_{2q+1}P'_{2q-1}$. Thus the sum becomes

$$(V_1'P_1' + V'_{2q+1}P'_{2q-1} + \sum_{i=2}^{2q-2} V_{(i+1)}P_i + \sum_{j=1}^{n-3} P_jP_{(j+1)})$$

and this sum is minimized by the induction hypothesis.

Q.E.D., n odd.

The actual construction of such a network is conceptually identical to earlier cases (all angles between pairs of lines intersecting at an interior point are 120°) although it is much more complex mechanically.

Consider the case where n is even. In the case where n was odd, it was assumed that the $(n-2)$–spanning tree contained no interior intersection points of degree 3 (i.e. was 'linear'). The same proof holds when n is even, if the $(n-2)$–spanning tree is linear. However, since a wide variety of $(n-2)$–spanning trees is available for given n, and since the $(n-2)$–spanning tree, of maximal branching, with a maximal number of interior intersection points of degree 3 is unique for n even, (see Appendix), it seems productive to show how the basic proof should be modified in order to accommodate networks of this type of different basic topological structure. So, suppose n is even and that the $(n-2)$–spanning tree has the maximal number of interior intersections of degree three and has maximal branching.

Suppose $n = 4$

Let P_1 and P_2 be points in the interior of quadrangle $(V_1V_2V_3V_4)$ that are connected in a cubic tree with V_1, V_2, V_3, V_4. It is desired to minimize

$$P_1V_1 + P_1V_2 + P_2V_3 + P_2V_4 + P_1P_2$$

the total length of the cubic tree (Figure III.9.1). From the rotation of triangles $(V_1V_2P_1)$ and $(V_3V_4P_2)$ through 60° toward the exterior of the polygon, it follows that

$$V_2P_1 = V_2'P_1'$$

$$V_1P_1 = V_1P_1' = P_1'P_1$$

since $\triangle V_1P_1P_1'$ is equilateral, and

$$V_4P_2 = V_4'P_2'$$

$$V_3P_2 = P_2P_2'$$

Thus,

$$min(P_1V_1 + P_1V_2 + P_2V_3 + P_2V_4 + P_1P_2) = min(V_2P_1' + P_1'P_1 + P_1P_2 + P_2P_2' + P_2'P_4')$$

and this right hand sum is minimized when $V_2'P_1'P_1P_2P_2'V_4'$ is a straight line.

Then it follows by rotation that

$\angle V_1P_1V_2 = \angle V_1'P_1'V_2 = 180° - \angle V_2P_1'P_1 = 180° - 60° = 120°.$

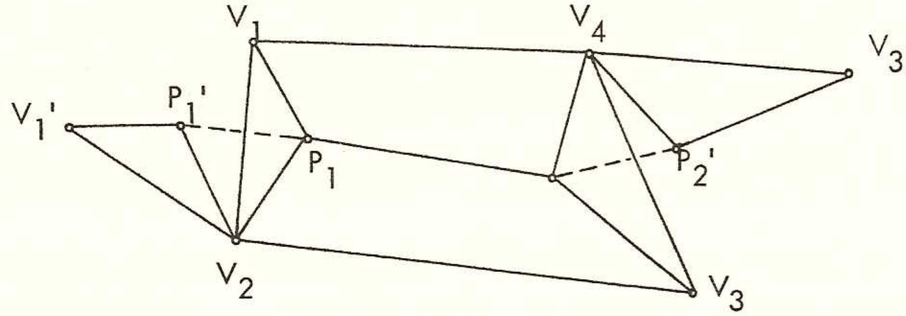


Figure III.9.1

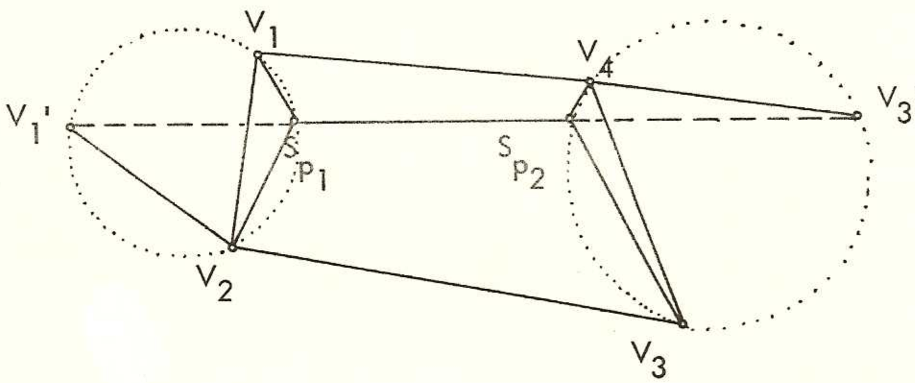


Figure III.9.2

Also, $\angle V_2P_1P_2 = 180° - \angle V_2P_1P_1' = 120°$. Similarly, $\angle V_4P_2V_3 = \angle P_1P_2V_3 = 120°$. To locate S_{p1} and S_{p2}, (Figure III.9.2), rotate V_1V_2 and V_3V_4 through angles of $60°$ about an endpoint away from the interior of the quadrangle to V_1' and V_3', vertices of the polygon of the first rotation. Then construct the circumcircles of $\triangle V_1V_2V_1'$ and $\triangle V_3V_4V_3'$. The intersections of the two circumcircles with $V_1'V_3'$ will produce two Steiner points for this second level candidate (again assuming that these intersections exist within the quadrangle) and these induce a second level candidate for the Steiner network in the original quadrangle.

<u>*Suppose* $n = 6$</u>

Let P_1, P_2, P_3, P_4 be points in the interior of the hexagon $(V_1, V_2, \ldots, V_6)$ (Figure III.10.1) that are joined as a cubic tree with $V_1, \ldots, V_6$. It is desired to minimize

$$V_1P_1 + V_2P_1 + V_3P_2 + V_4P_2 + V_5P_3 + V_6P_3 + P_1P_4 + P_2P_4 + P_3P_4.$$

From the rotation of triangles $(V_1P_1V_2)$, $(V_3P_2V_4)$, $(V_5P_3V_6)$ through $60°$ toward the exterior of the polygon, it follows that

$$V_1P_1 = V_1'P_1'; \quad V_2P_1 = V_2P_1' = P_1P_1'$$

$$V_3P_2 = V_3'P_2'; \quad V_4P_2 = V_4P_2' = P_2P_2'$$

$$V_5P_3 = V_5'P_3'; \quad V_6P_3 = V_6P_3' = P_3P_3'$$

Thus,

$$min(V_1P_1 + V_2P_1 + V_3P_2 + V_4P_2 + V_5P_3 + V_6P_3 + P_1P_4 + P_2P_4 + P_3P_4)$$

$$= min(V_1'P_1' + P_1'P_1 + P_1P_4 + V_3'P_2' + P_2P_2' + P_2P_4 + V_5'P_3' + P_3P_3' + P_3P_4)$$

The network represented by the edges in this last sum is homeomorphic to the cubic tree with one vertex of degree three and three vertices of degree one. Thus, the right hand side is minimized by the Steiner tree on the triangle $(V_1'V_3'V_5')$, a polygon of the first rotation, in which alternate sides of the hexagon are rotated through $60°$ in the manner described above, to produce vertices V_1', V_3', V_5' (Figure III.10.2). The Steiner point, S_{p4}, of this triangle of the first rotation is determined

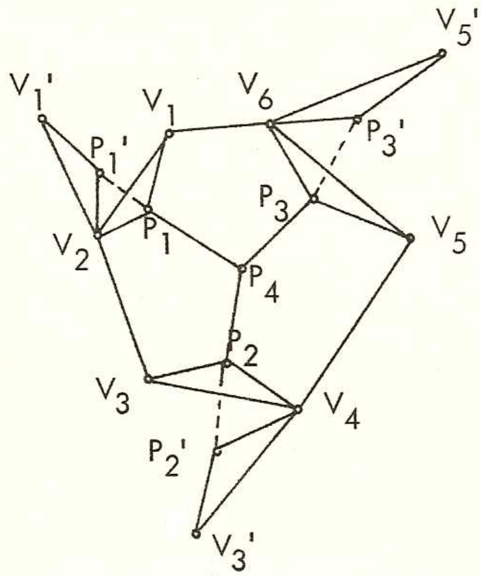


Figure III.10.1

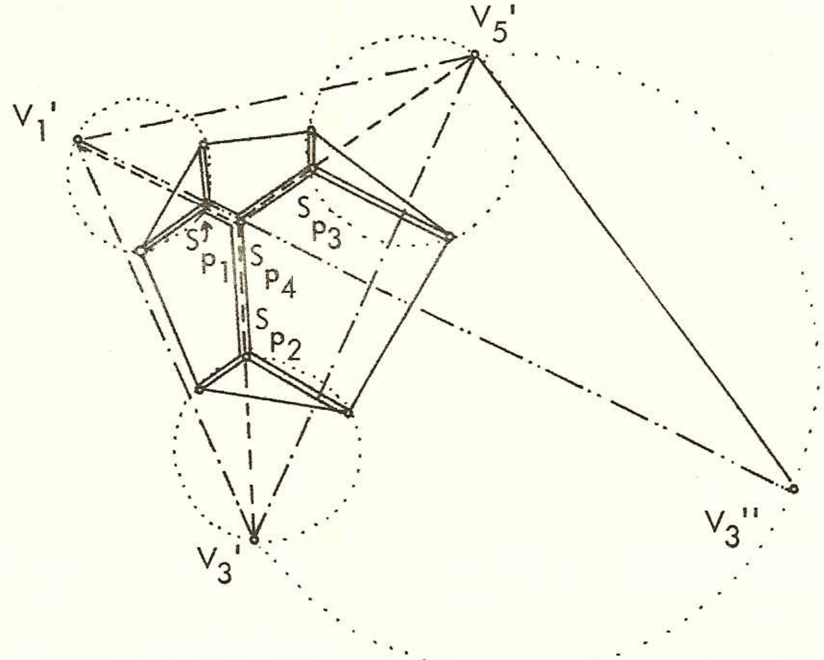


Figure III.10.2

by rotating one of its sides ($V_3'V_5'$ for example) through $60°$ about V_5', producing V_3'', a vertex of a polygon of the second rotation. Then the intersection

$$(V_1V_3'') \wedge (circumcircle\ \triangle V_3'V_3''V_5')$$

will produce the Steiner point of $\triangle V_1'V_3'V_5'$, or S_{p4}, inducing a Steiner network in the polygon of the first rotation. Then,

$$S_{p1} = (S_{p4}V_1') \wedge (circumcircle\ \triangle V_1V_1'V_2)$$

$$S_{p2} = (S_{p4}V_3') \wedge (circumcircle\ \triangle V_3V_4V_3')$$

$$S_{p3} = (S_{p4}V_5') \wedge (circumcircle\ \triangle V_5V_5'V_6)$$

This second level candidate for the Steiner network induced in this hexagon is non-degenerate.

<u>*Suppose* $n = 8$</u>

Let $P_1, \ldots, P_6$ be $(n-2)$ points interior to an octagon that are connected in the desired way. Minimize

$$\sum_{i=1}^{4}(P_iV_{2i-1} + P_iV_{2i-1}) + P_1P_5 + P_2P_5 + P_3P_6 + P_4P_6 + P_5P_6.$$

Again, from the rotation of the triangles $(V_1V_2P_1)$, $(V_3V_4P_2)$, $(V_5V_6P_3)$, $(V_7V_8P_4)$ through $60°$, and from equalities resulting from congruent triangles, the problem is reduced to finding the network of minimum length on quadrangle $V_1'V_3'V_5'V_7'$ (Figure III.11.1).

The Steiner points, S_{p5}, S_{p6}, of this quadrangle are located as in the case $n = 4$ (Figure III.11.2). The vertices V_1'' and V_5'' are produced by rotating $V_1'V_3'$ and $V_5'V_7'$ through $60°$. Then,

$$S_{p5} = (V_1''V_5'') \wedge (circumcircle\ \triangle V_1'V_1''V_3')$$

$$S_{p6} = (V_1''V_5'') \wedge (circumcircle\ \triangle V_5'V_5''V_7')$$

determining the Steiner points for this second level candidate in the quadrangle derived from the octagon (under assumptions leading to a non-degenerate network) and

$$S_{p1} = (S_{p5}V_1') \wedge (circumcircle\ \triangle V_1V_1'V_2)$$

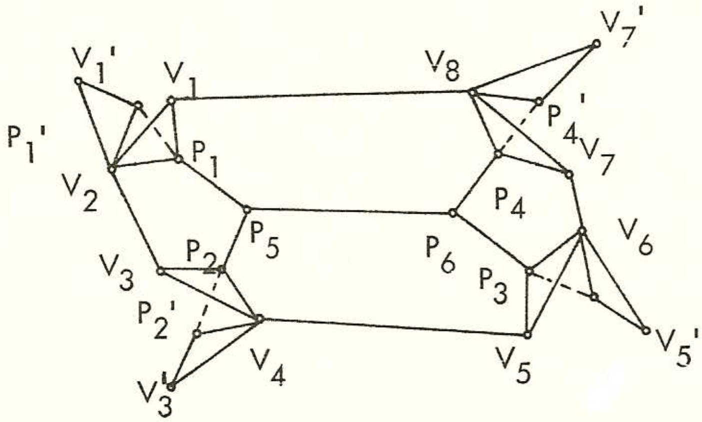


Figure III.11.1

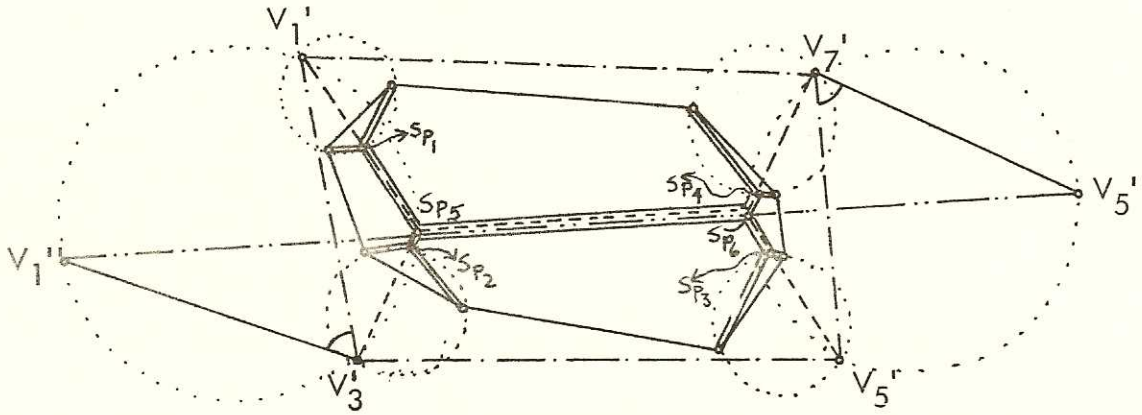


Figure III.11.2

$$S_{p_2} = (S_{p_5}V_3') \wedge (circumcircle\ \triangle V_3V_3'V_4)$$

$$S_{p_3} = (S_{p_6}V_5') \wedge (circumcircle\ \triangle V_5V_5'V_6)$$

$$S_{p_4} = (S_{p_6}V_7') \wedge (circumcircle\ \triangle V_7V_7'V_8)$$

producing a complete set of Steiner points for this second level candidate in the octagon $V_1, V_2, \ldots, V_8$.

The above examples provide motivation for the following proof. Generally, when n is even, the unique factorization [4] of n into powers of primes may be written

$$n = 2^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}, \quad p_1, \ldots, p_r \quad primes$$

$\alpha_1, \ldots, \alpha_r$ integers greater than or equal to zero, $\alpha_0 \neq 0$, a positive integer. Notice that $p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ is an odd number, for it contains no powers of 2 since n was uniquely factored. The general proof for n even will be by induction on the size of α_0.

Suppose $\alpha_0 = 1$

Then $n = 2 \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ for at least one of $\alpha_i \neq 0$. In this case, the general cubic tree contains $n/2$ or $p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ "small triangles." By rotating these outward from the polygon through $60°$, the desired minimum will be represented by the length of the minimal tree on the polygon with $p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ vertices, as determined above. The cubic tree on this polygon can be minimized by procedures from the case where n is odd.

Induction hypothesis

Suppose a minimal cubic tree can be constructed in any polygon with n sides, where $n = 2^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ and $0 < \alpha_0 \leq s$, s is a positive integer.

Suppose $\alpha_0 = s + 1$ **Then,**

$$n = 2^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r} = 2^{s+1} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}.$$

The general cubic tree on a polygon with this number of sides contains $n/2$ or $2^s \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ "small triangles." Rotate these outward from the polygon, producing a set of vertices for a polygon with $2^s \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ sides. The desired

minimum will be represented as the length of the minimal cubic tree in the polygon defined by these $2^s \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ vertices. This tree can be found by the induction hypothesis.

Q.E.D., n even.

The procedure presented above allows us to determine that path of minimal total length among a set of n points, $V_1, V_2, \ldots, V_n$. Assuming that these points are connected to form a polygon $\mathcal{P}$, the proof provides a method for calculating the shortest form of a network, with given structure of the $(n-2)$–spanning tree, in $\mathcal{P}$, for a particular choice of m edges of $\mathcal{P}$ $(m < n)$; m = number of vertices of degree 1 in the $(n-2)$–spanning tree. Each of these shortest networks was called a second level candidate and we can now completely determine the set of second level candidates for any set of n points. The enumeration problems that remain, in moving from the set of second level candidates to the network of minimal total length in $\mathcal{P}$ may be unwieldy, for n large. Lacking specific theorems for removing these problems, it may be useful to use local geographical limits in particular cases and to find mathematical solution within geographical constraints.

REFERENCES

1. R. Courant and H. Robbins, *What Is Mathematics?* (London: Oxford University Press, 1941), pp 360-361.

2. A cubic tree is a tree in which each interior intersection point has degree three, where the degree of a point refers to the number of edges of the graph incident with it. Refer to: F. Harary, *Graph Theory* (Reading: Addison-Wesley, 1969), pp. 14-15.

3. Suppose that $\mathcal{P}$ is an n-sided polygon with vertices $V_1, \ldots, V_n$. Suppose that a subset of distinct sides of $\mathcal{P}$, $\{(V_{ai}V_{bi})|1 \leq i \leq r\}$ are rotated as follows $(r < n)$: each $V_{ai}V_{bi}$ is rotated about V_{ai} through an angle of $60°$, in the plane. Under this rotation $V_{ai}V_{bi}$ becomes $V_{ai}V'_{bi}$. The set of vertices of the polygon of the first rotation consists of $\{(V'_{bi})|2 \leq i \leq r\}$ $\{(V_{bj})|V'_{bj} \neq Vbi', \quad 1 \leq j \leq r\}$ and the polygon of the first rotation, $\mathcal{P}'$, is formed by linking these vertices in the obvious way. A polygon of the second rotation $\mathcal{P}''$ of $\mathcal{P}$ is the polygon of the first rotation of $\mathcal{P}'$. Higher levels are defined by similar recursion of definition.

4. S. Mac Lane and G. Birkhoff, *Algebra* (New York: The Macmillan Company, 1967), pp. 154-55.

APPENDIX: ENUMERATION OF (N − 2)-SPANNING TREES

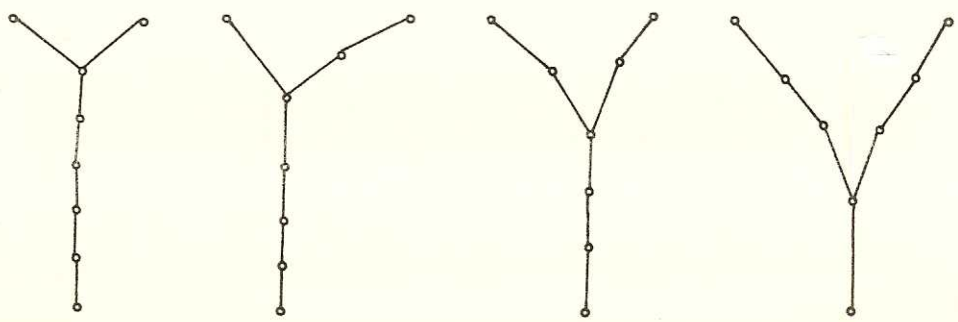
The following examples suggest the variety of types of $(n-2)$–spanning trees available for a given n.

Example 1 :

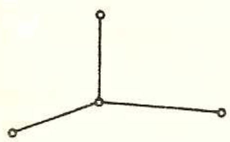
Suppose $n = 10$ (and, $(n-2) = 8$). Below are the $(n-2)$–spanning trees, grouped according to number of vertices of degree 3.

0 VERTICES OF DEGREE 3

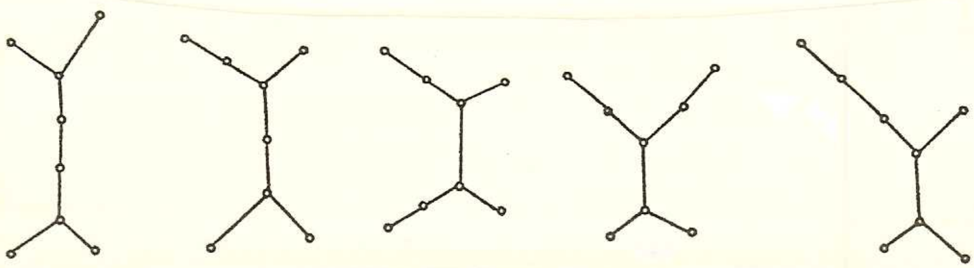
1 VERTEX OF DEGREE 3



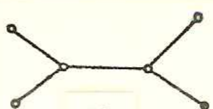
All of which have the same basic structure as the graph below.

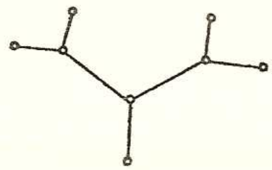


2 VERTICES OF DEGREE 3



All of which have the same basic structure as the graph below.





And, this is a different structure.

In order to find all the $(n-2)$–spanning trees, references that enumerate some tree-types [1] are useful, although hopefully the geographic conditions of the problem will suggest how many vertices of degree 1, how many of degree 2 and how many of degree 3 are desirable.

Example 2

Suppose $(n-2) = 9$. There are:

a) 1 tree with 0 vertices of degree 3;

b) 5 trees with 1 vertex of degree 3;

c) 9 trees with 2 vertices of degree 3;

d) 3 trees with 3 vertices of degree 3.

Example 3

Suppose $(n-2) = 10$. There are:

a) 1 tree with 0 vertices of degree 3;

b) 7 trees with 1 vertex of degree 3;

c) 17 trees with 2 vertices of degree 3;

d) 10 trees with 3 vertices of degree;

e) 2 trees with 4 vertices of degree 3.

In the method of construction in Chapter III it was the 'small triangles' that were of concern. Therefore it is important to note that the number of 'small triangles' in the cubic n-tree is equal to the number of vertices of degree 1, ($|V(deg1)|$), in the $(n-2)$–spanning tree. And, this number is related to the

number of vertices of degree 3, $|V(deg3)|$), chosen to be in the $(n-2)$–spanning tree:

$$|V(deg1)| = 2 + |V(deg3)|.$$

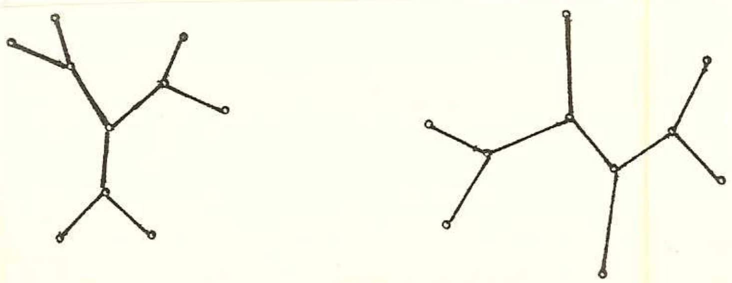
Thus when n is even, $(n-2) = 2q$, the largest number of vertices of degree three that the $(n-2)$–spanning tree may contain is $(q-1)$, for from above

$$|V(deg1)| = 2 + (q-1) = q+1$$

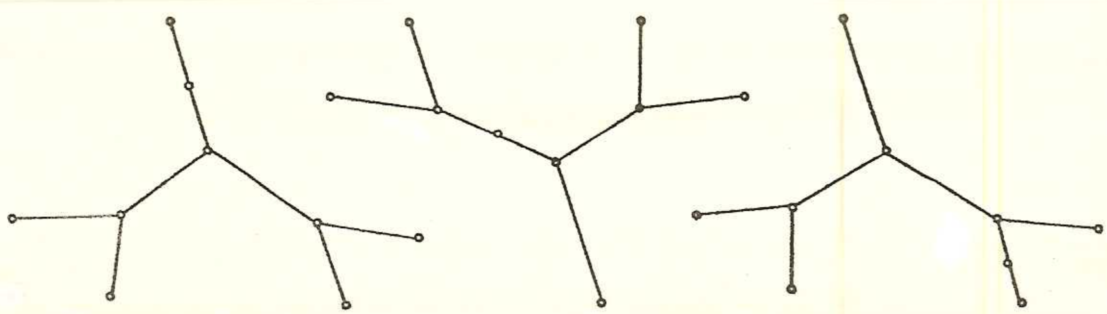
and

$$|V(deg1)| + |V(deg3)| = (q+1) + (q-1) = 2q$$

the number of vertices of the $(n-2)$–spanning tree. When there are $(q-1)$ such vertices the spanning tree has maximal branching and is unique. To illustrate the idea of maximal branching, note that the 10-spanning tree on the left (below) possesses this characteristic while that on the right does not.



When n is odd, the uniqueness criterion does not hold. For, when $(n-2) = 2q+1$, the maximal number of vertices of degree 3 is still $(q-1)$, for, if the added vertex were of degree 3, then $|V(deg3)| = q$, so $|V(deg1)| = q+2$ and therefore the number of vertices of the $(n-2)$–spanning tree is greater than or equal to $q + (q+2) = 2q+2$, contradicting $(n-2) = 2q+1$.



Thus, in order to preserve connectedness of the spanning tree, there is exactly one vertex of degree 2. This vertex may be placed in a variety of locations, so that the $(n-2)$–spanning tree of maximal branching with $|V(deg3)|$ maximal is not unique when n is odd.

REFERENCES

1. Harary, F. *Graph Theory*, Reading: Addison–Wesley, 1969.

CHAPTER IV

GEOMETRIC CONSTRUCTIONS OF SECOND LEVEL CANDIDATE STEINER NETWORKS: THE SIX POINT CASE

The set of figures that follows shows most of the constructions available for deriving second level candidates for a polygon linking six vertices. The algorithm of Chapter III is used to find these and the following legend refers to all the figures. Definition of terms in the legend is available in Chapter III.

A hexagon linking the six vertices is drawn as a solid line.

The polygon of the first rotation is drawn as a dashed line with single dots between the dashes.

The polygon of the second rotation is drawn as a dashed line with double dots between the dashes.

The second level candidate for the network of minimal total length is drawn as two parallel solid lines.

In what follows, discussion will be given to illustrate how to enumerate potential second level candidates, and then the figures will show how to derive them geometrically, using the algorithm of Chapter III.

There are two types of 4-spanning trees in the hexagon $V_1V_2V_3V_4V_5V_6$:

Type 1 has one vertex of degree three and three vertices of degree 1 (i.e., it exhibits maximal branching).

Type 2 has two vertices of degree two and two vertices of degree 1 (i.e., it is linear).

These are the only 4–spanning trees, for according to the Appendix to Chapter III, the largest number of vertices of degree 3 that a 4–spanning tree may contain is 1, for $n = 6$ so $(n-2) = 2 \cdot 2 = 2q$ and $(q-1) = 1$.

Thus, there are two distinct topological classes, T_1 and T_2 associated with the Type 1 and the Type 2 4–spanning trees, respectively.

Within T_1 the only way to hook the 4–spanning tree into the hexagon is so that the three small triangles are each separated by one side of the hexagon.

There are two ways to do this: when the 4-tree is linked to V_2V_3, V_4V_5, and V_1V_6 the second level candidate of Figure IV.T_1.1 is generated. Alternately, when the 4-tree is linked to V_1V_2, V_3V_4 and V_5V_6, a totally degenerate form arises (Figure IV.T_1.2). Figure IV.0 shows the different connection patterns ("topological" types), and among those, that in frame (e) is the only representative of T_1.

Within T_2 (according to an enumeration rule to be given in Chapter V) when $n = 6$, there will be six ways of hooking a linear 4–spanning tree into a hexagon, in such a way that the 'small triangles' so formed are separated by exactly one side of the hexagon (Figure V.2.a and .b). This gives rise to six ways to form a 4–spanning tree in this style of hooking the tree into the polygon. Since the polygon of the first rotation is a quadrangle, there are two ways to find Steiner points within it. Therefore two second level candidates arise for each of the six forms. Or, this style of hooking a 4-tree into the polygon gives rise to twelve second level candidates. A second, and final, style of hooking the 4-spanning tree into the polygon is to have the 'small triangles' separated by two edges of the polygon. There are three distinct ways in which this can occur and there are two separate forms of each of these three ways for within a given form, each of the two vertices of degree two can be linked to one of two remaining vertices of the polygon. This generates six ways of hooking the 4-spanning tree into the polygon. The polygon of the first rotation in each of these is a quadrangle; thus there are two ways to find its Steiner points, and so there are a total of 12 second level candidates derived from this style of hooking a 4–tree into the hexagon. Thus, 24 second level candidates can be derived from class T_2.

The outline below shows which of the following figures is representative of the various forms described above, within class T_2:

1) the two 'small triangles' are separated by one side of the polygon.

a) V_1V_2 separates the two 'small triangles'

i) sides one and three of the polygon of the first rotation are rotated; that is $V_1'V_3'$ and V_4V_5 are rotated (Figure IV.T_2.1.a.i).

ii) sides two and four of the polygon of the first rotation are rotated (Figure

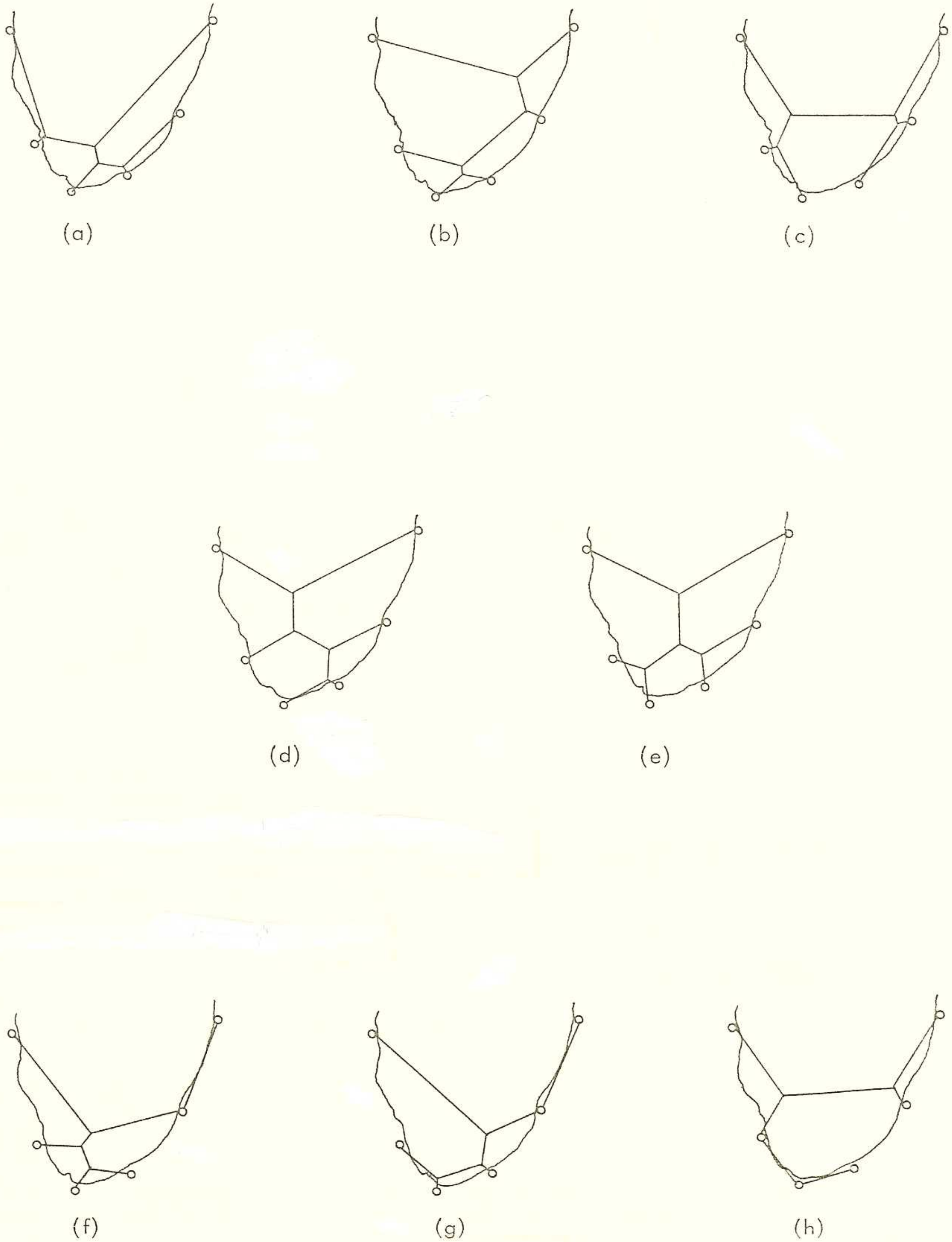


Figure IV.0: vertices shown as locations around a lake's perimeter.

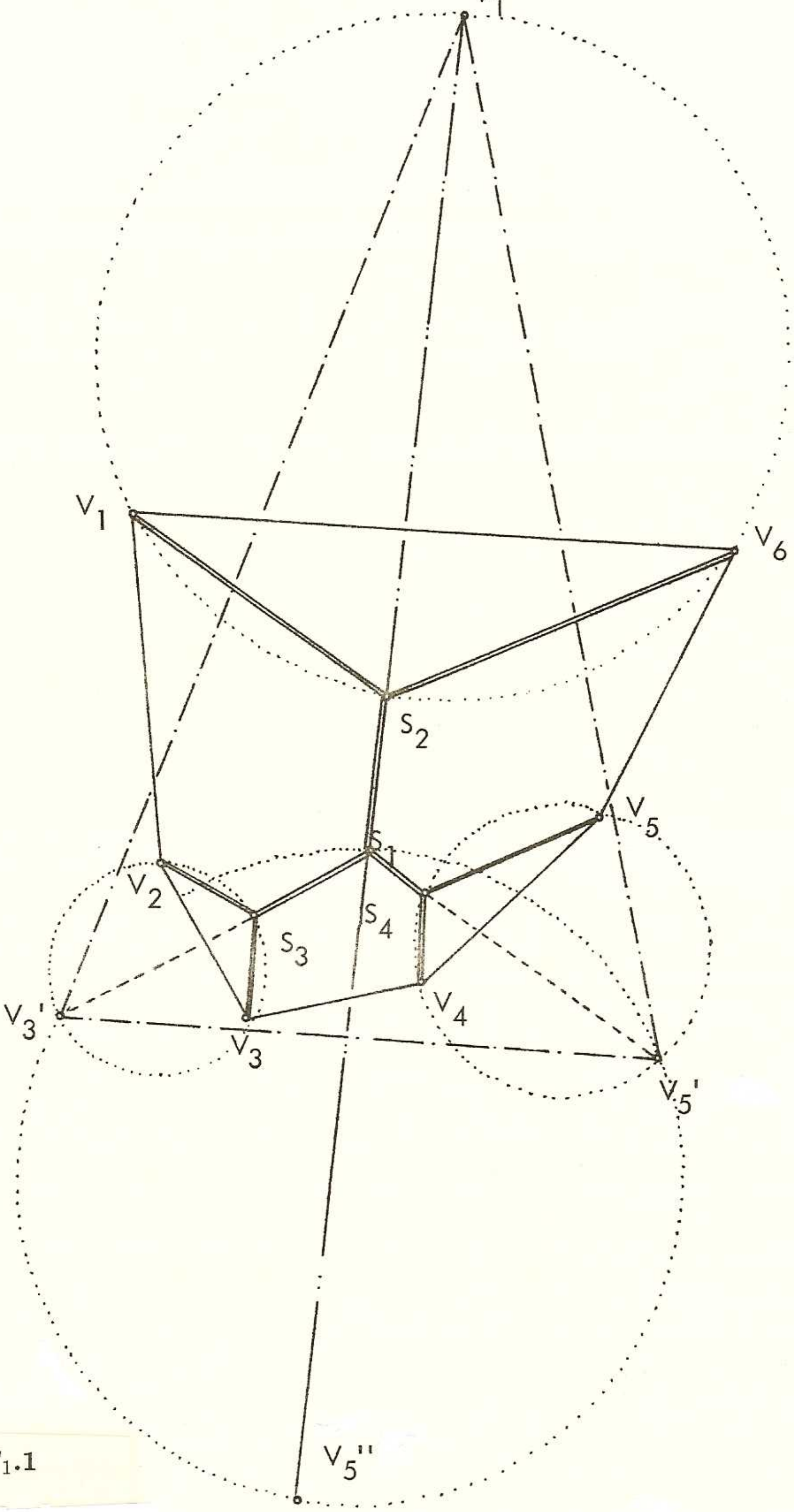


Figure IV.T_1.1

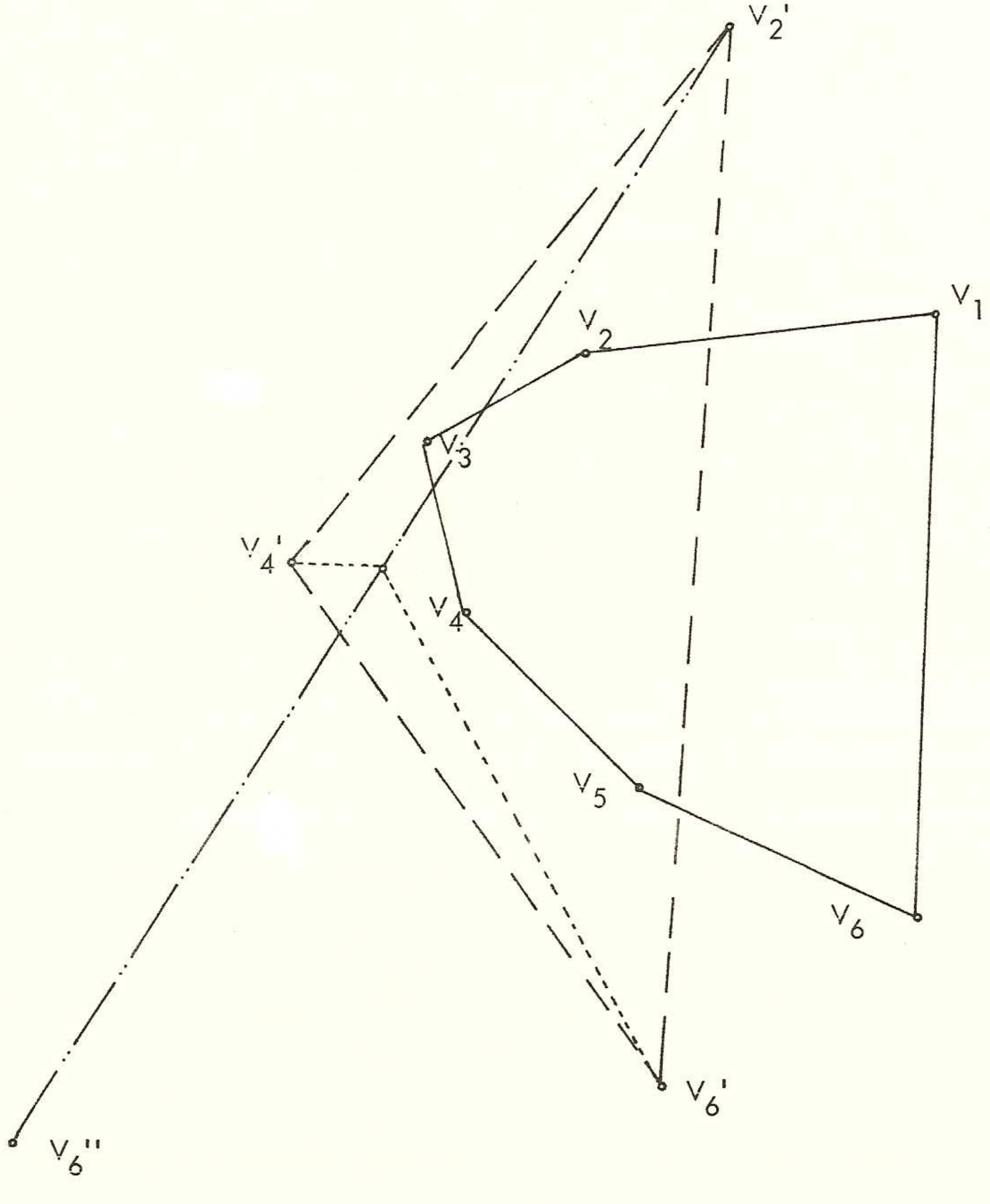


Figure IV.T_1.2

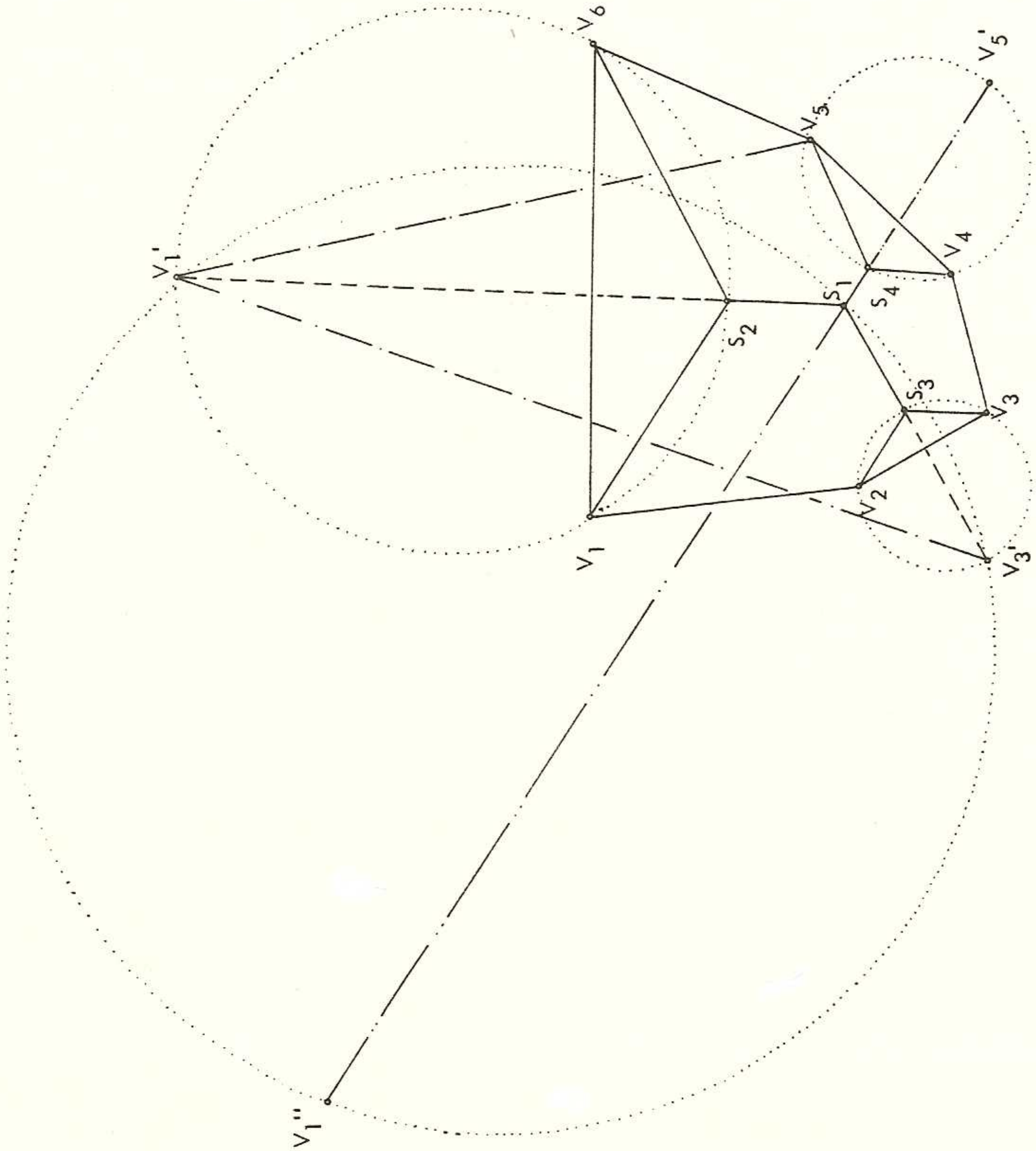


Figure IV.T_2.1.a.i

IV.T_2.1.a.ii).

b) V_2V_3 separates the two 'small triangles'

i) $V_1'V_4'$ and V_5V_6 are rotated (Figure IV.T_2.1.b.i).

ii) $V_1'V_6$ and $V_4'V_5$ are rotated (Figure IV.T_2.1.b.ii).

c) V_3V_4 separates the two 'small triangles'

i) $V_3'V_5'$ and V_1V_6 are rotated (Figure IV.T_2.1.c.i).

ii) V_1V_3' and $V_5'V_6$ are rotated (Figure IV.T_2.1.c.ii).

d) V_4V_5 separates the two 'small triangles'

i) V_1V_2 and $V_4'V_6'$ are rotated (Figure IV.T_2.1.d.i).

ii) V_2V_4' and V_1V_6' are rotated (Figure IV.T_2.1.d.ii).

e) V_5V_6 separates the two 'small triangles'

i) V_2V_3 and $V_1'V_5'$ are rotated (Figure IV.T_2.1.e.i).

ii) $V_1'V_2$ and V_3V_5' are rotated (Figure IV.T_2.1.e.ii).

f) V_6V_1 separates the two 'small triangles'

i) $V_2'V_6'$ and V_3V_4 are rotated (Figure IV.T_2.1.f.i).

ii) V_4V_6' and $V_2'V_3$ are rotated (Figure IV.T_2.1.f.ii).

2) the two 'small triangles' are separated by two sides

a) V_1V_2 and V_2V_3 separate the 'small triangles'

i) the vertex of degree two of the 4–spanning tree that is closest, within the network, to the vertex of the polygon with lowest subscript (V_1 in this case), is linked to V_5. This vertex will be called V_0.

α) V_2V_4' and $V_1'V_5$ are the sides of the polygon of first rotation that are rotated (Figure IV.T_2.2.a.i.α.)

β) $V_1'V_2$ and $V_4'V_5$ are rotated; this leads to a totally degenerate network and is not shown.

ii) V_0 is linked to V_2

α) $V_1'V_2$ and $V_4'V_5$ are rotated (Figure IV.T_2.2.a.ii.α.)

Figure IV.T_2.1.a.ii

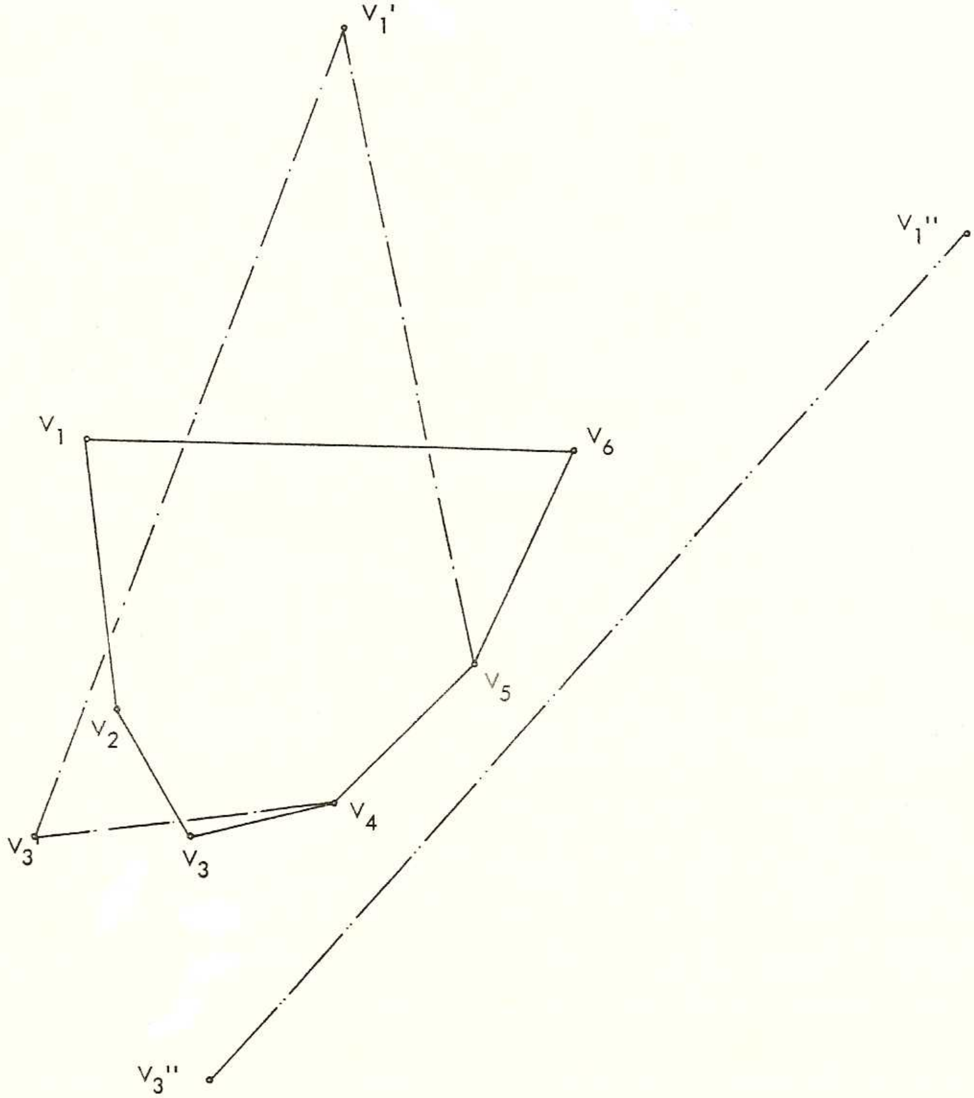
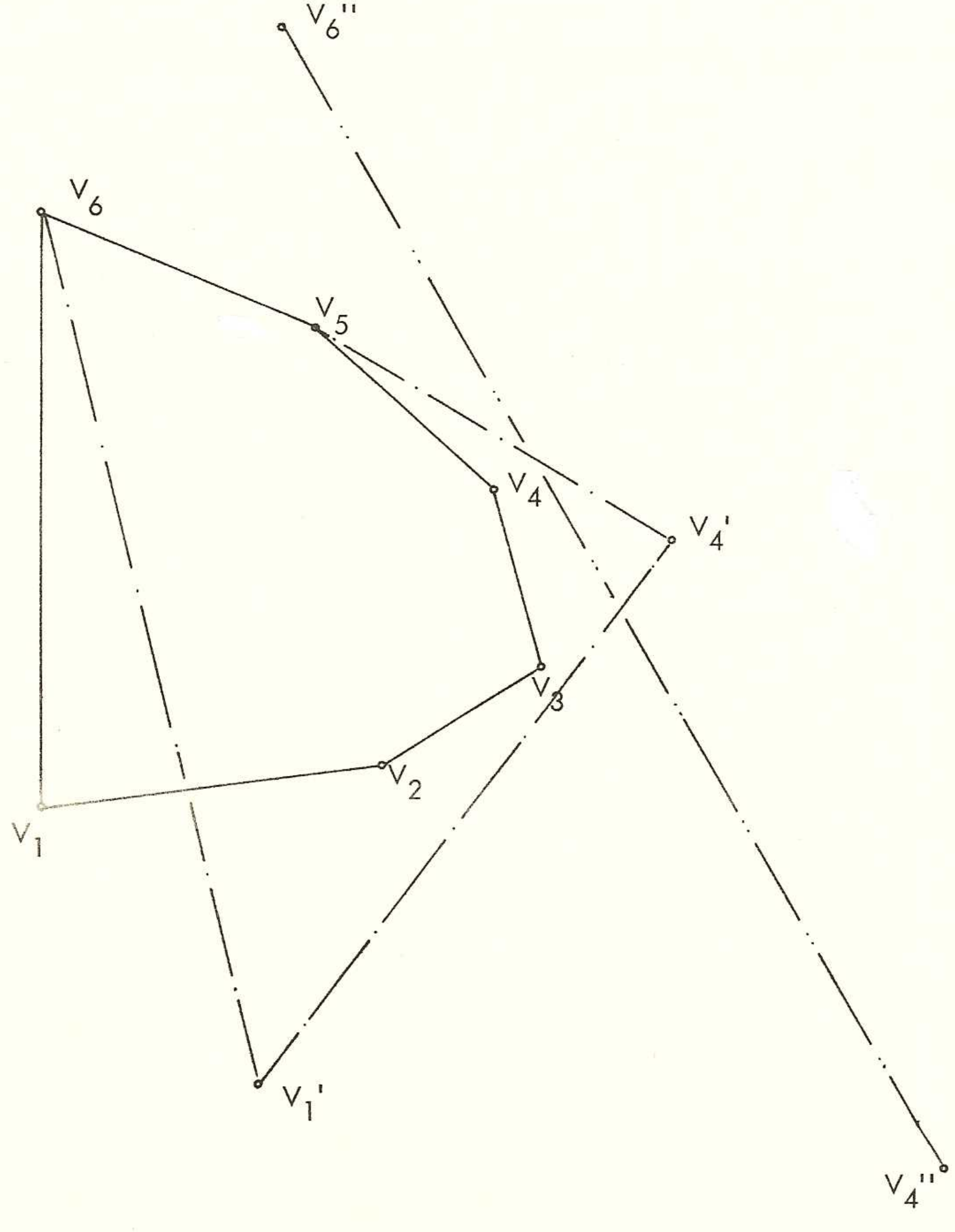


Figure IV.T_2.1.b.i



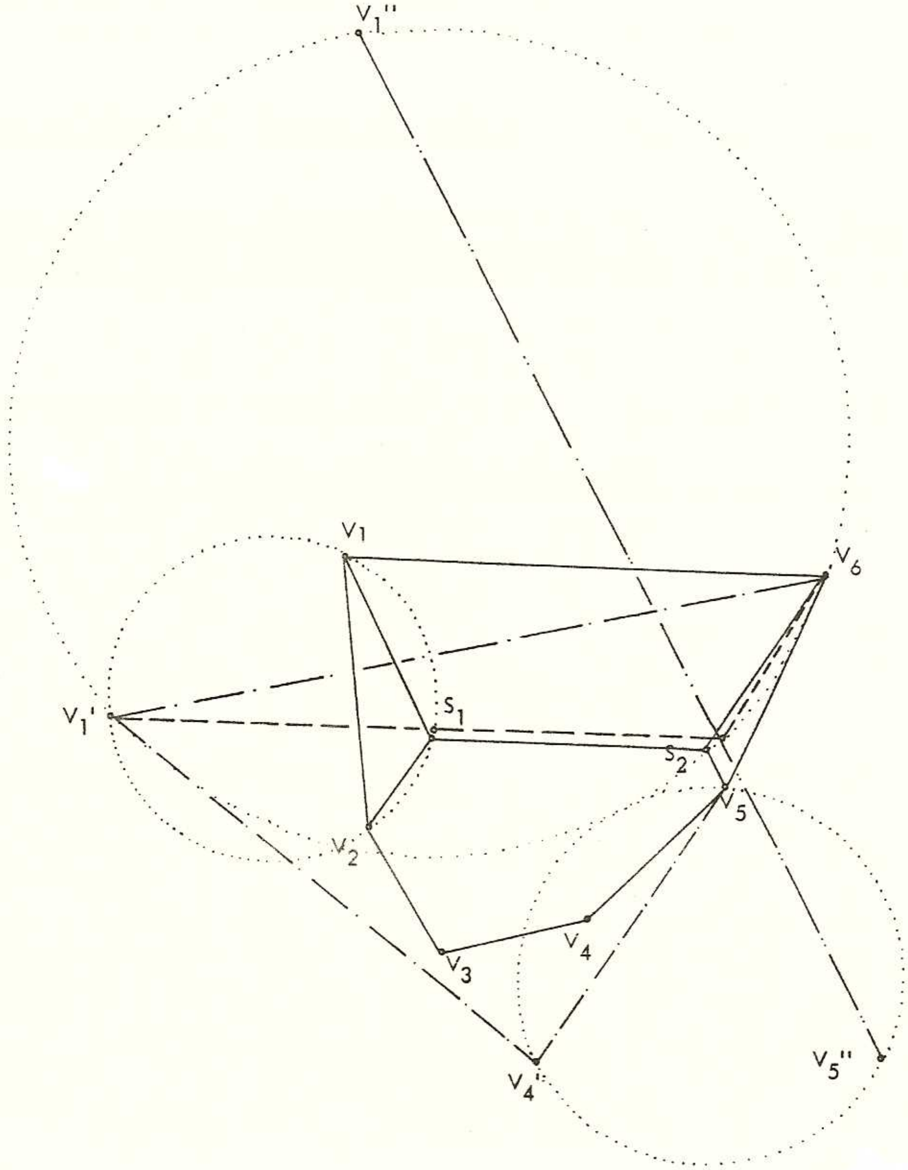


Figure IV.T_2.1.b. ii

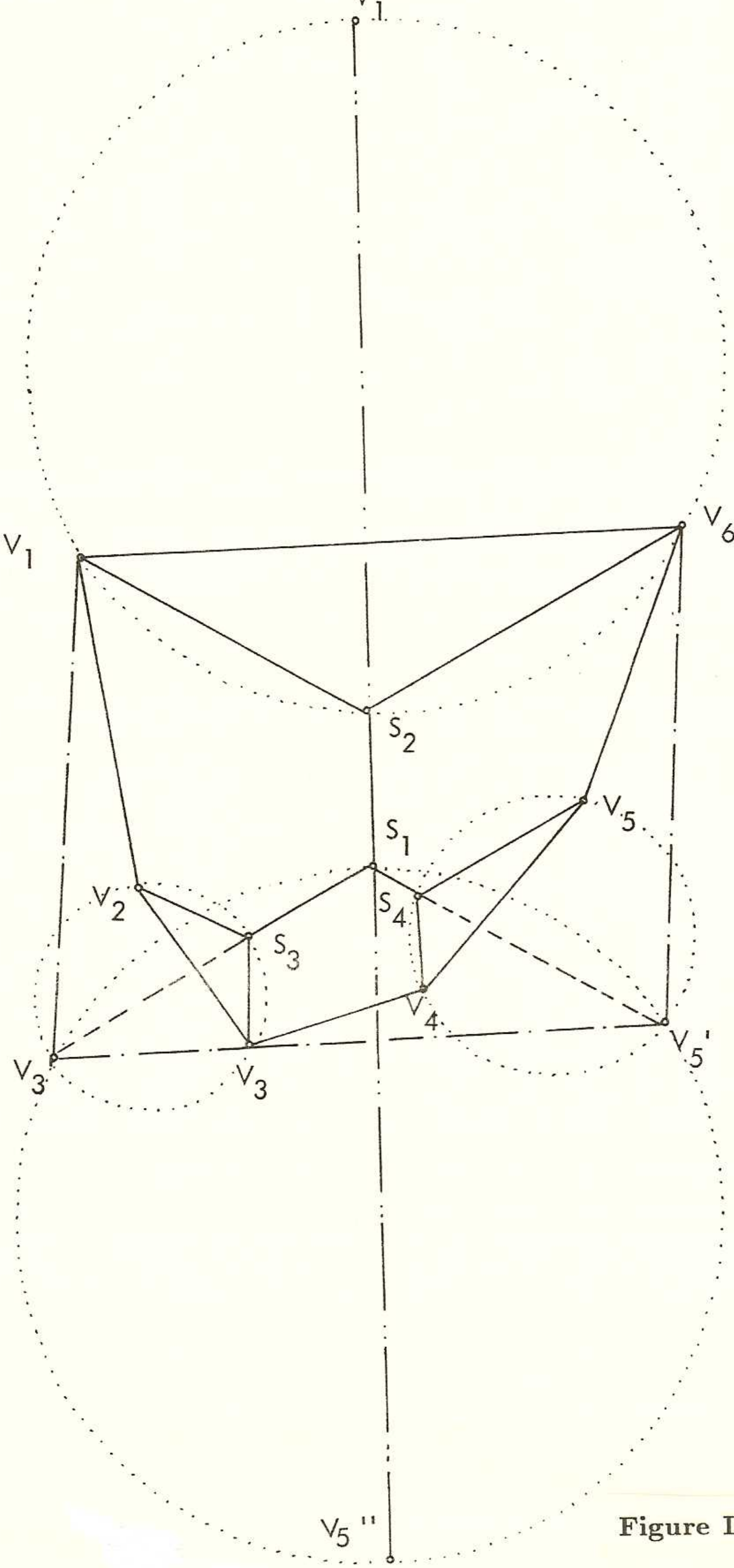


Figure IV.T_2.1.c.i

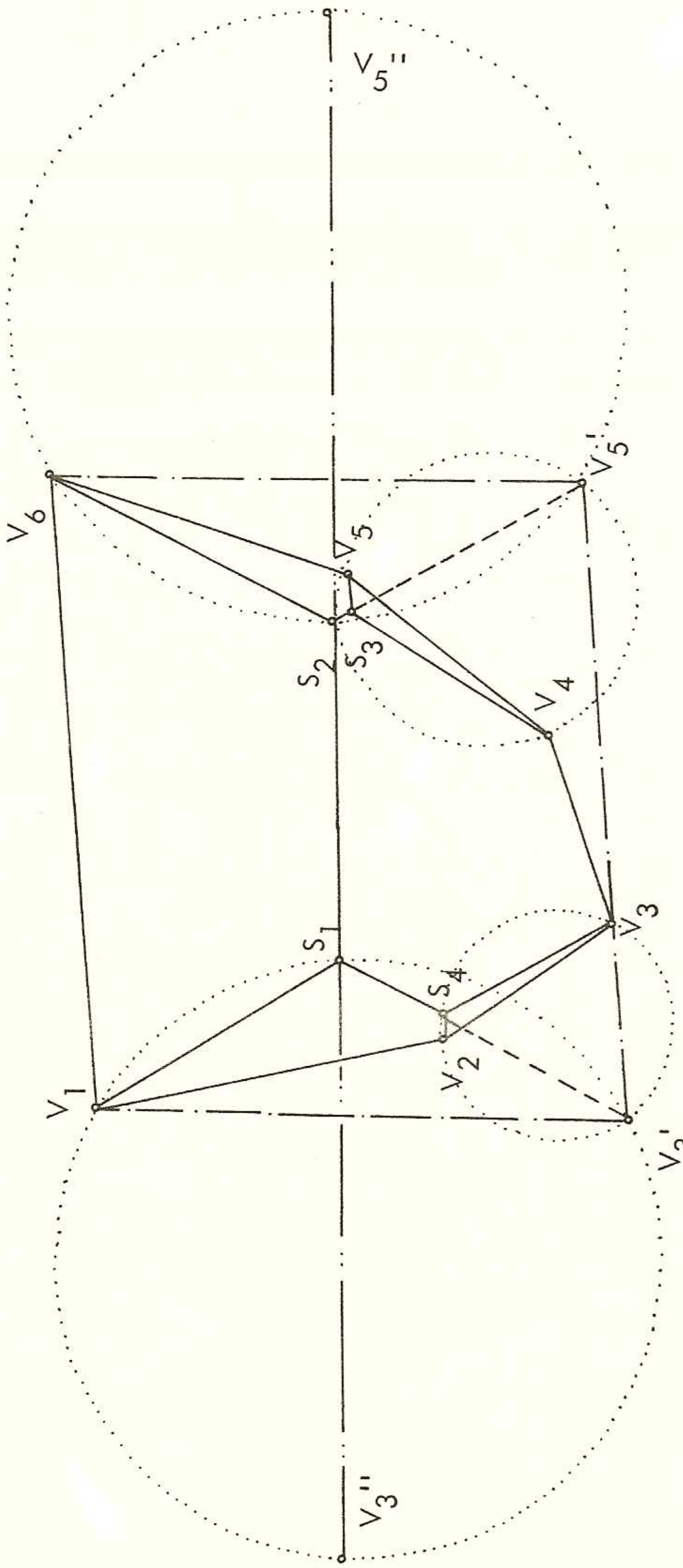
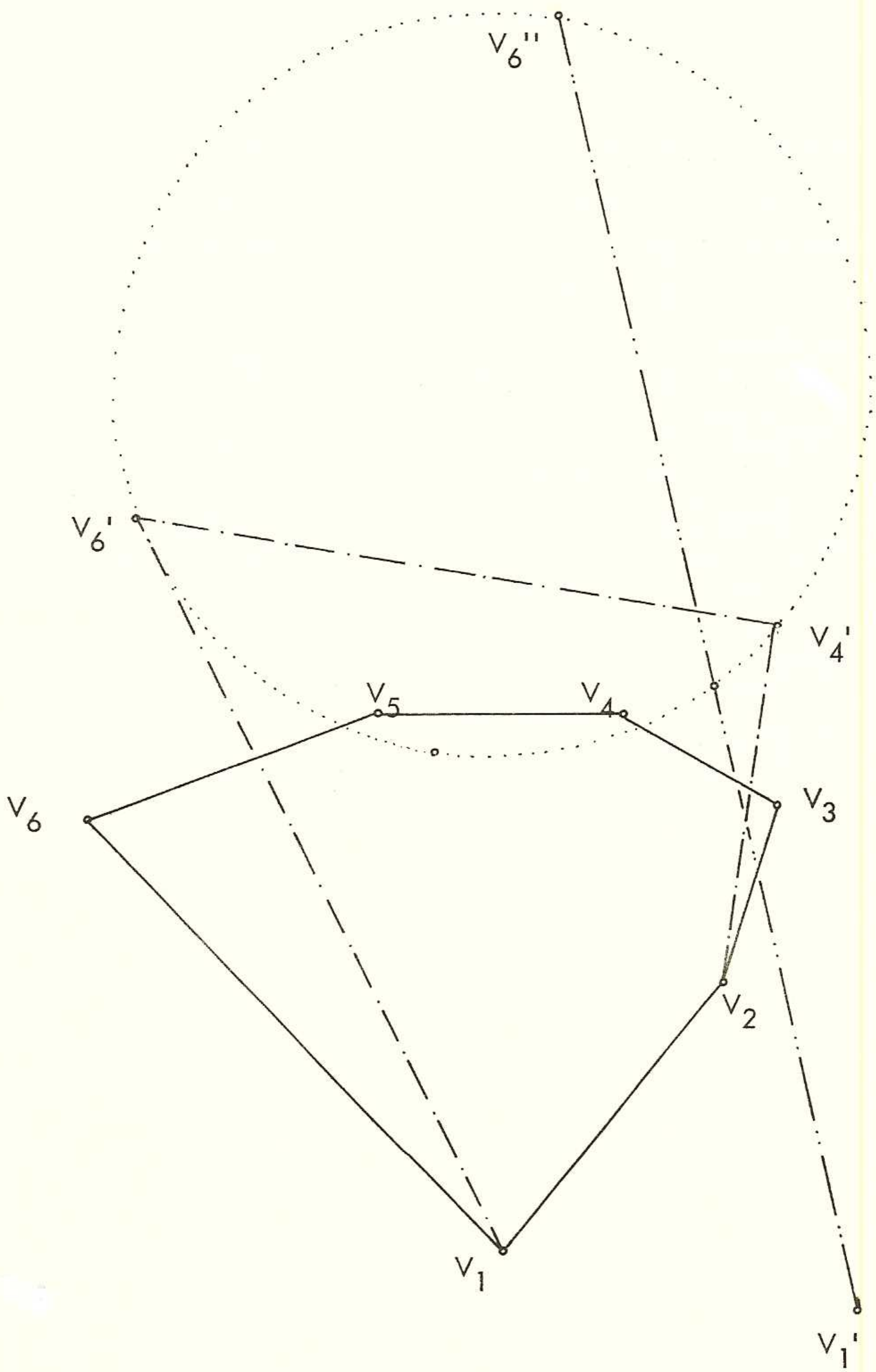


Figure IV.T_2.1.c.ii

Figure IV.T_2.1.d.i



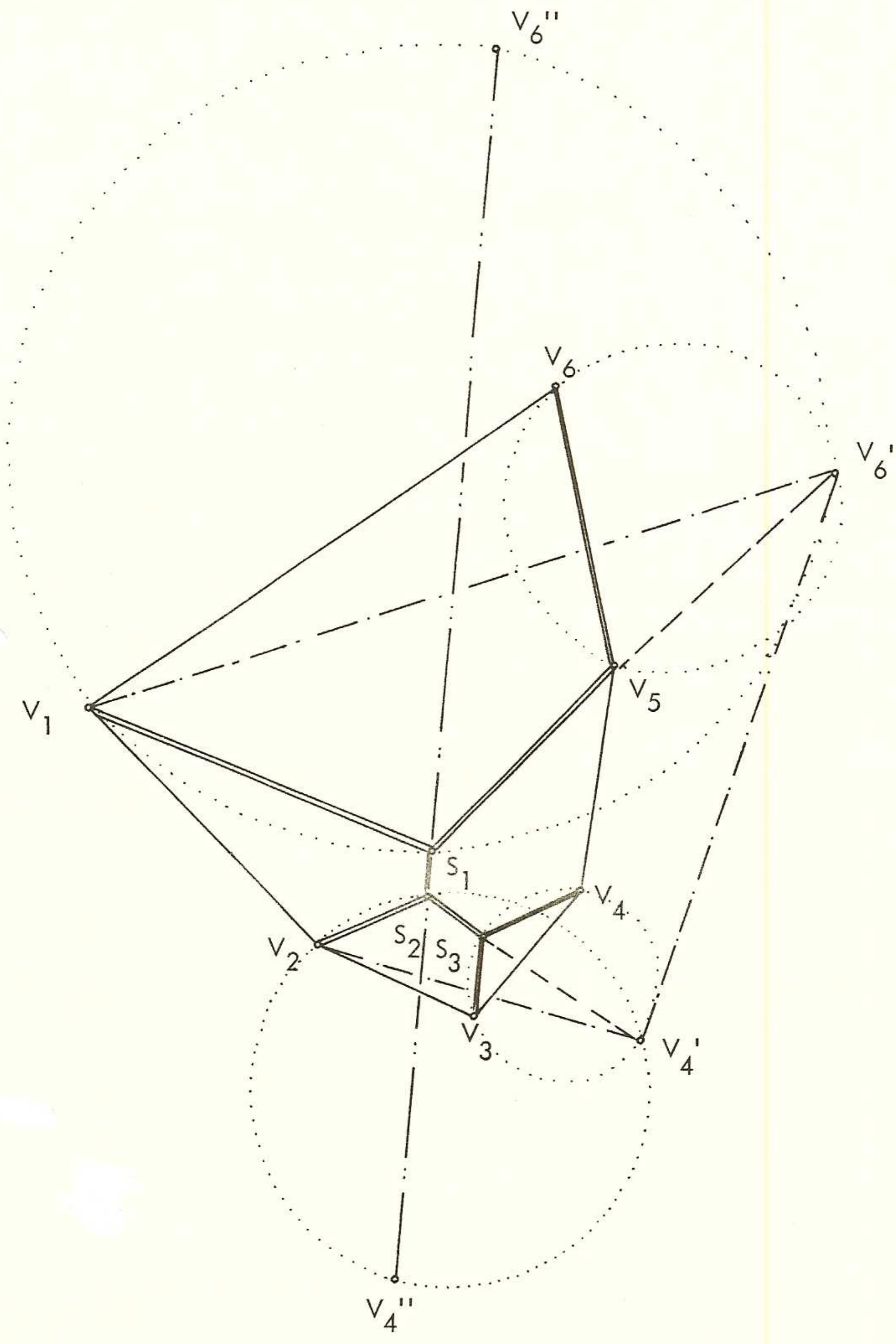


Figure IV.T_2.1.d.ii

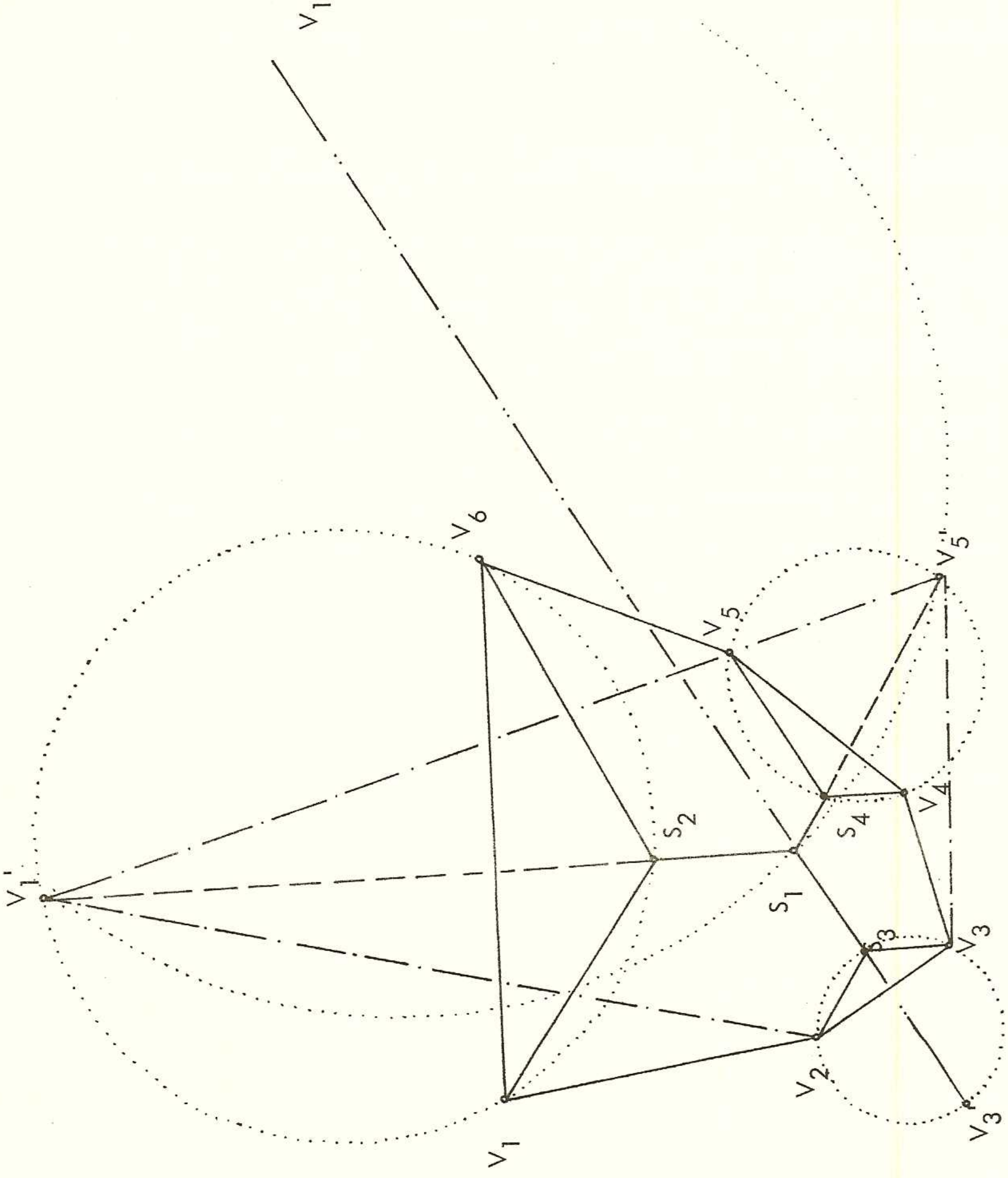


Figure IV.T_2.1.e.i

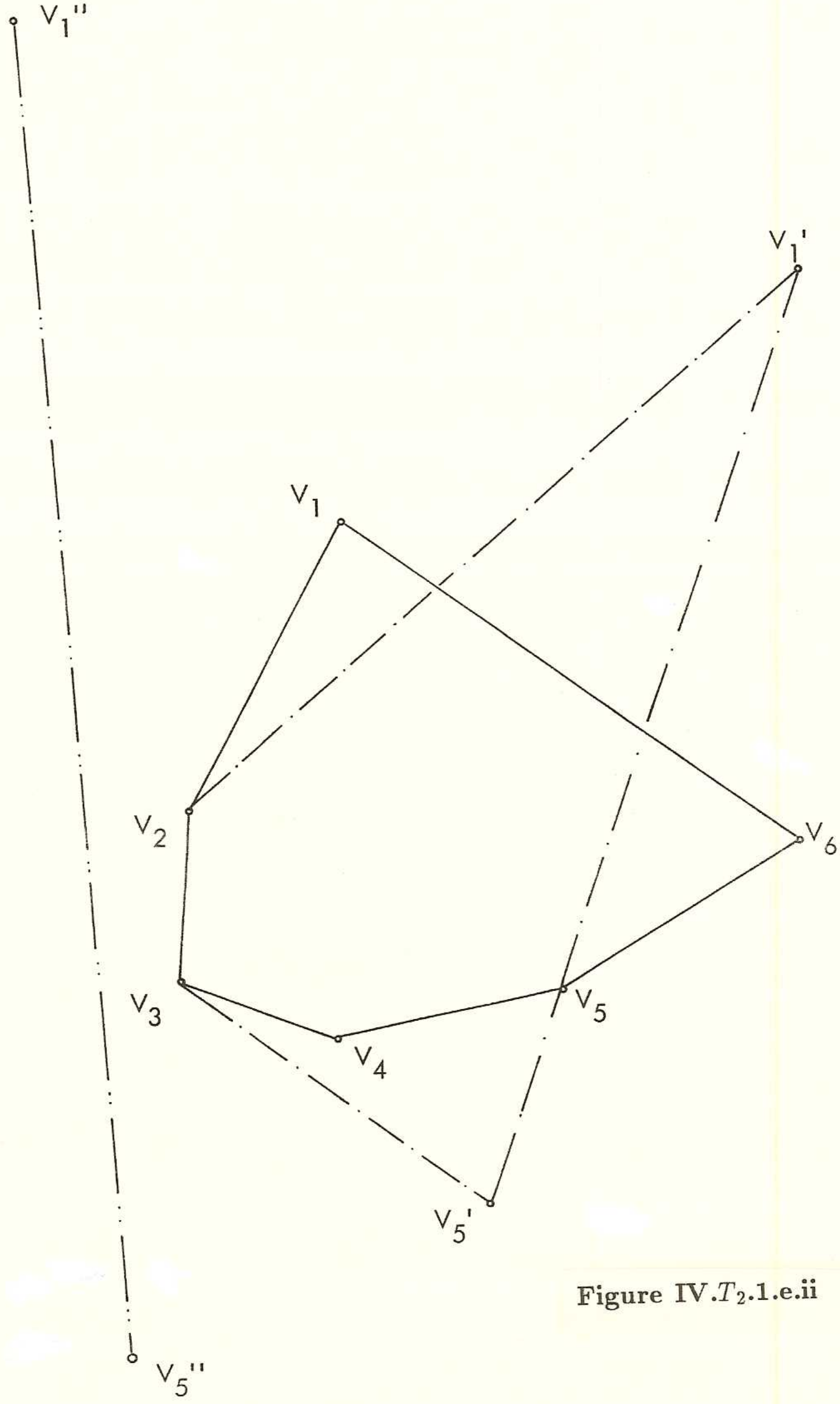


Figure IV.T_2.1.e.ii

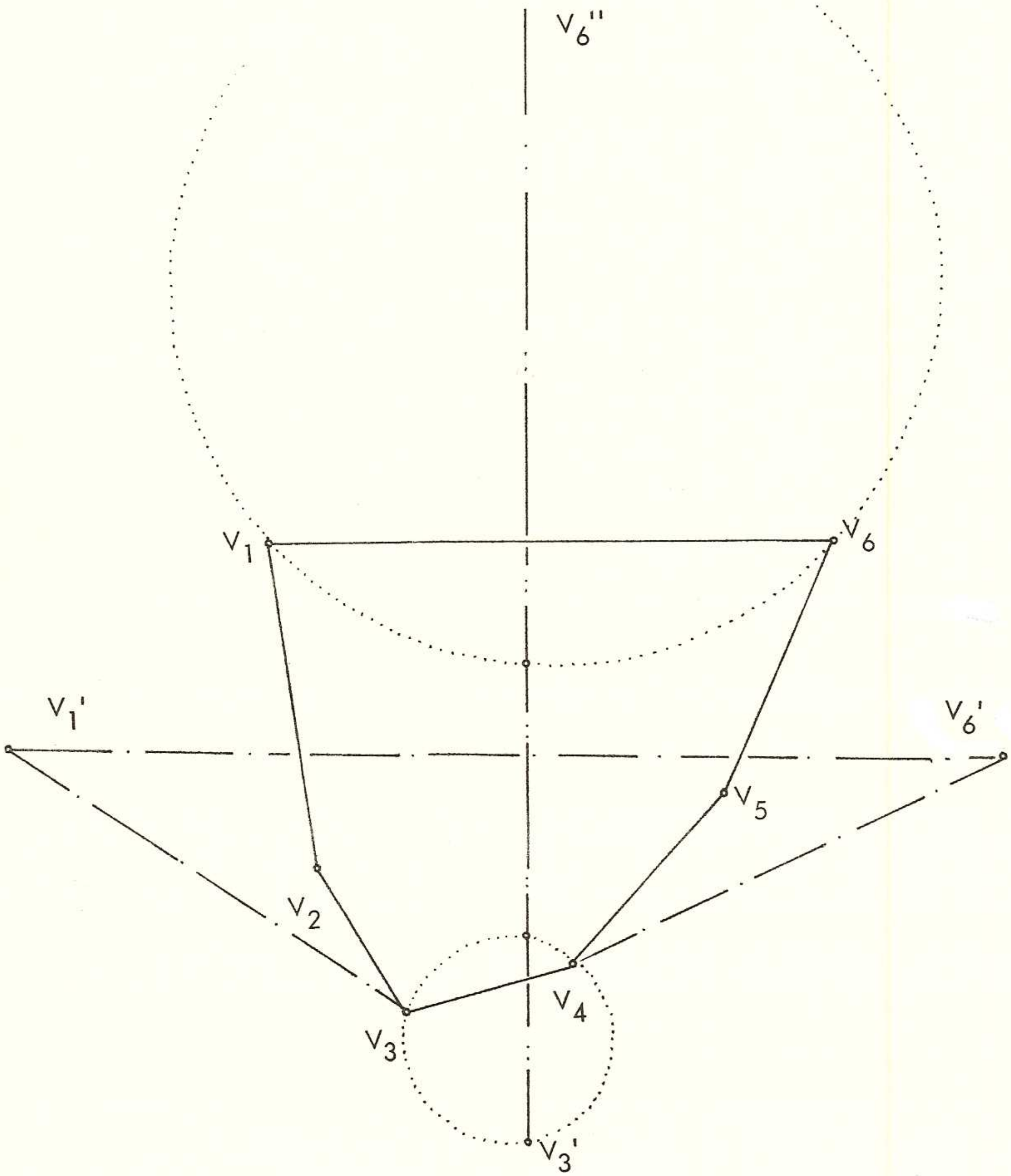
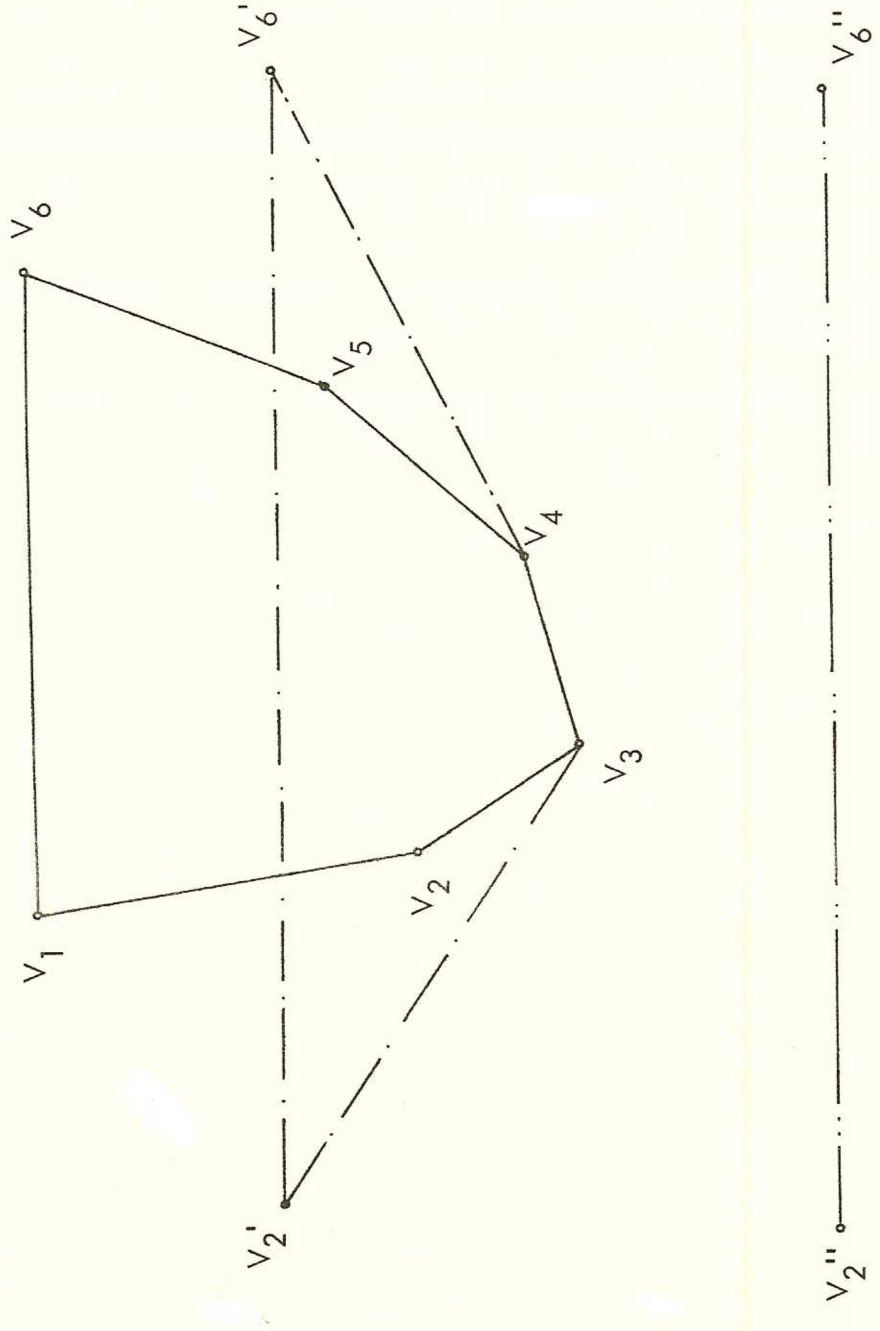


Figure IV.T_2.1.f.i

Figure IV.T_2.1.f.ii



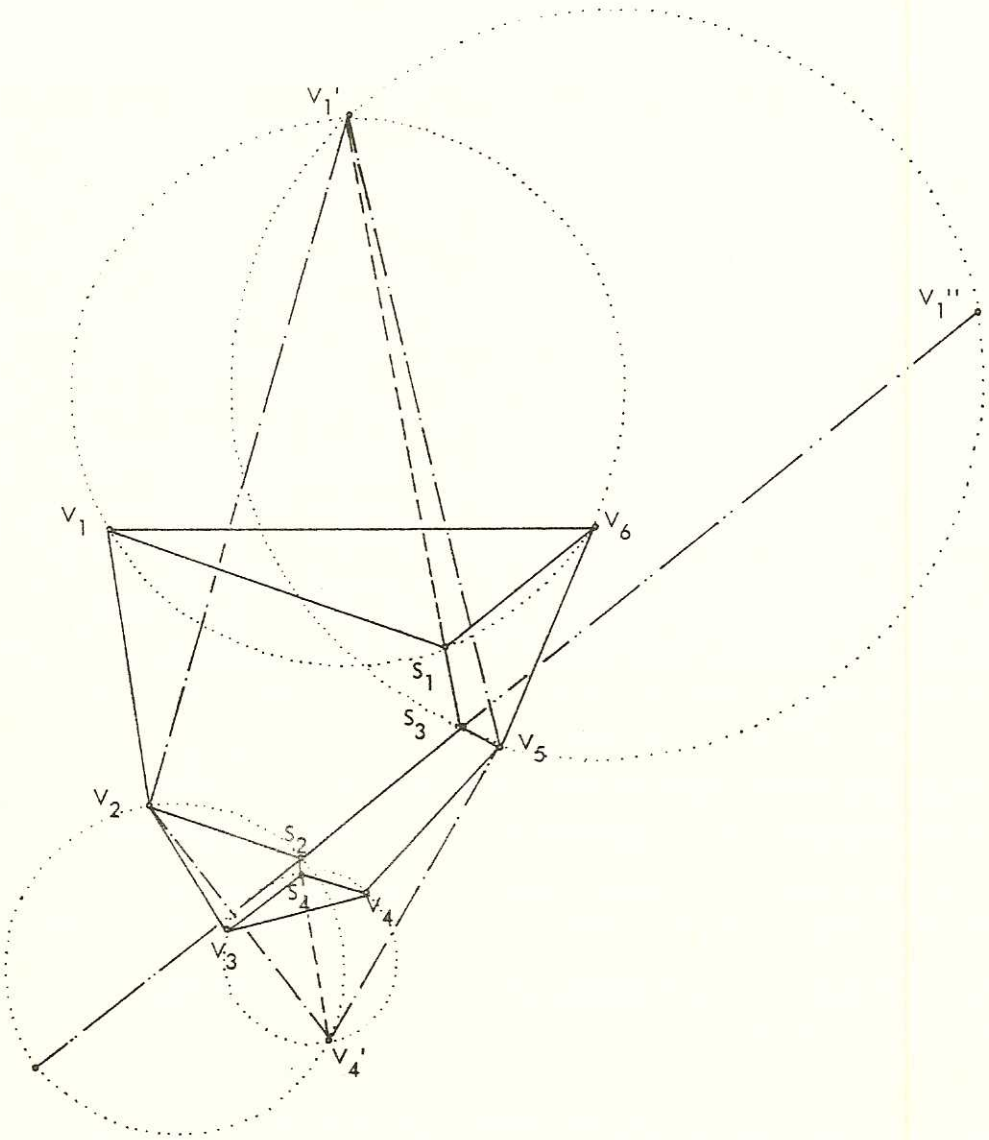


Figure IV.T_2.2.a.i.α

β) $V_1'V_5$ and V_2V_4' are rotated; this leads to a totally degenerate network and is not shown.

b) V_2V_3 and V_3V_4 separate the 'small triangles'

i) V_0 is linked to V_6

α) $V_2'V_6$ and $V_5'V_3$ are rotated (Figure IV.T_2.2.b.i.α.)

β) $V_2'V_3$ and $V_5'V_6$ are rotated (Figure IV.T_2.2.b.i.β.)

ii) V_0 is linked to V_3

This produces only degenerate networks and is not shown.

c) V_3V_4 and V_4V_5 separate the 'small triangles'

i) V_0 is linked to V_4

α) $V_3'V_4$ and V_1V_6' are rotated (Figure IV.T_2.2.c.i.α.)

β) V_1V_3' and V_4V_6' are rotated (Figure IV.T_2.2.c.i.β.)

ii) V_0 is linked to V_1

This produces only degenerate networks and is not shown.

Referring to Figure IV.0, it follows that:

Frame (a) is derived from Figure IV.T_2.2.b.i.α.

Frame (b) is derived from Figure IV.T_2.2.a.i.α.

Frame (c) is derived from Figure IV.T_2.1.c.ii.

Frame (d) is derived from Figure IV.T_2.2.a.ii.α.

Frame (e) is derived from Figure IV.T_1.1.

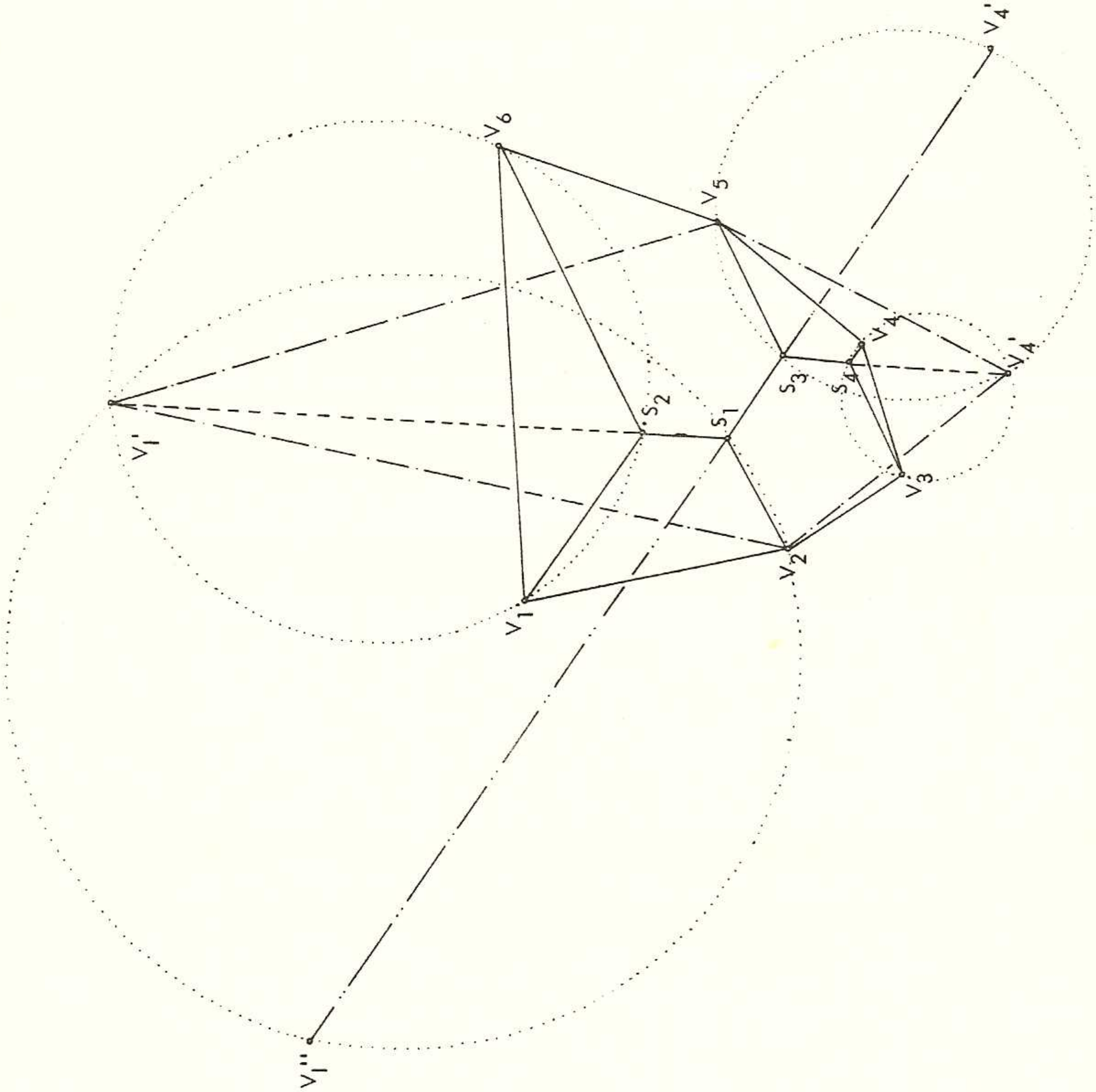
Frame (f) is derived from Figure IV.T_2.1.d.ii.

Frame (g) is derived from Figure IV.T_2.2.c.i.α.

Frame (h) is derived from Figure IV.T_2.1.b.ii.

and that no other figures in this appendix give rise to any other second level candidates.

Figure IV.T_2.2.a.ii.α



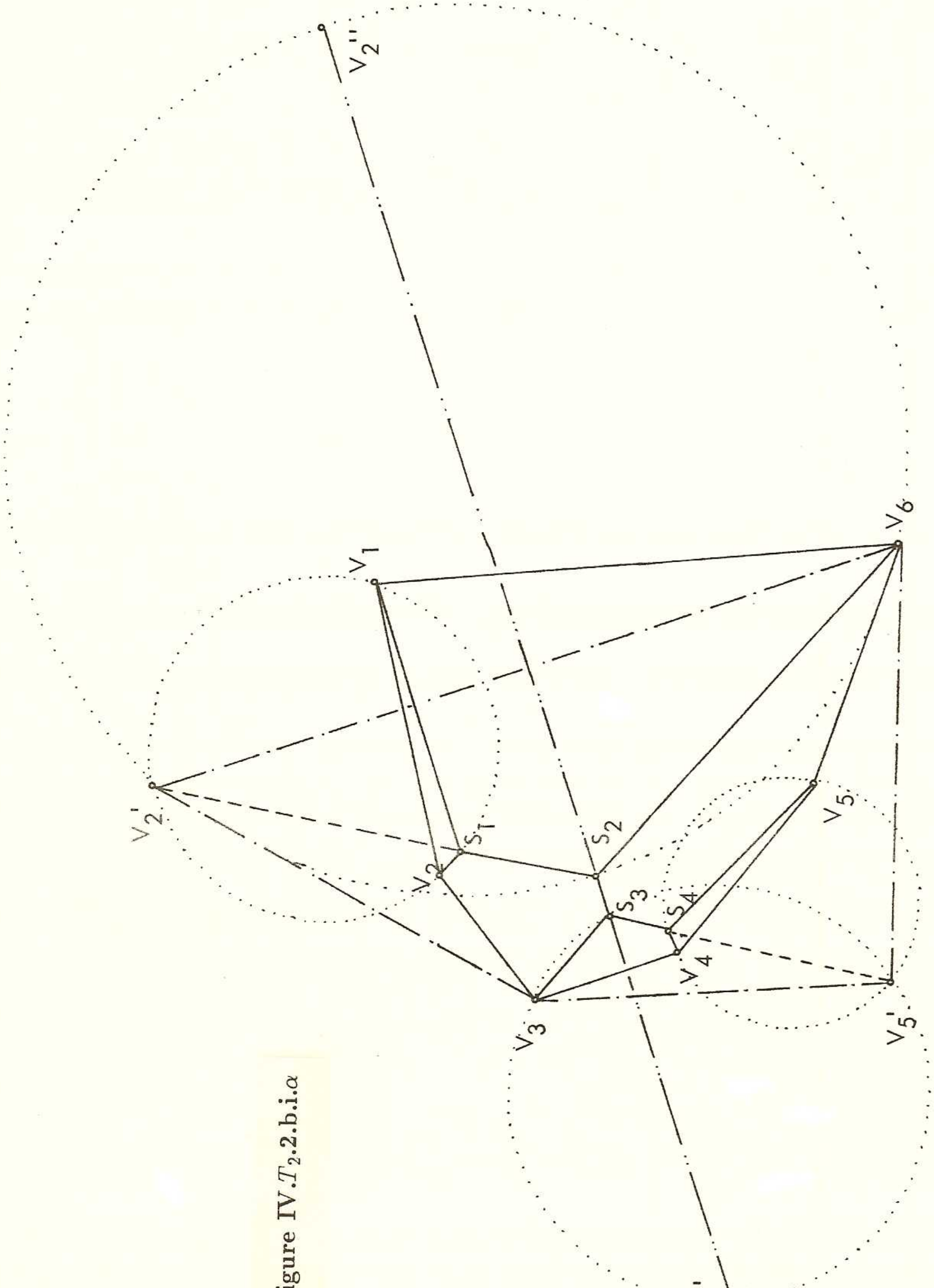


Figure IV.T_2.2.b.i.α

Figure IV.T_2.2.b.i.β

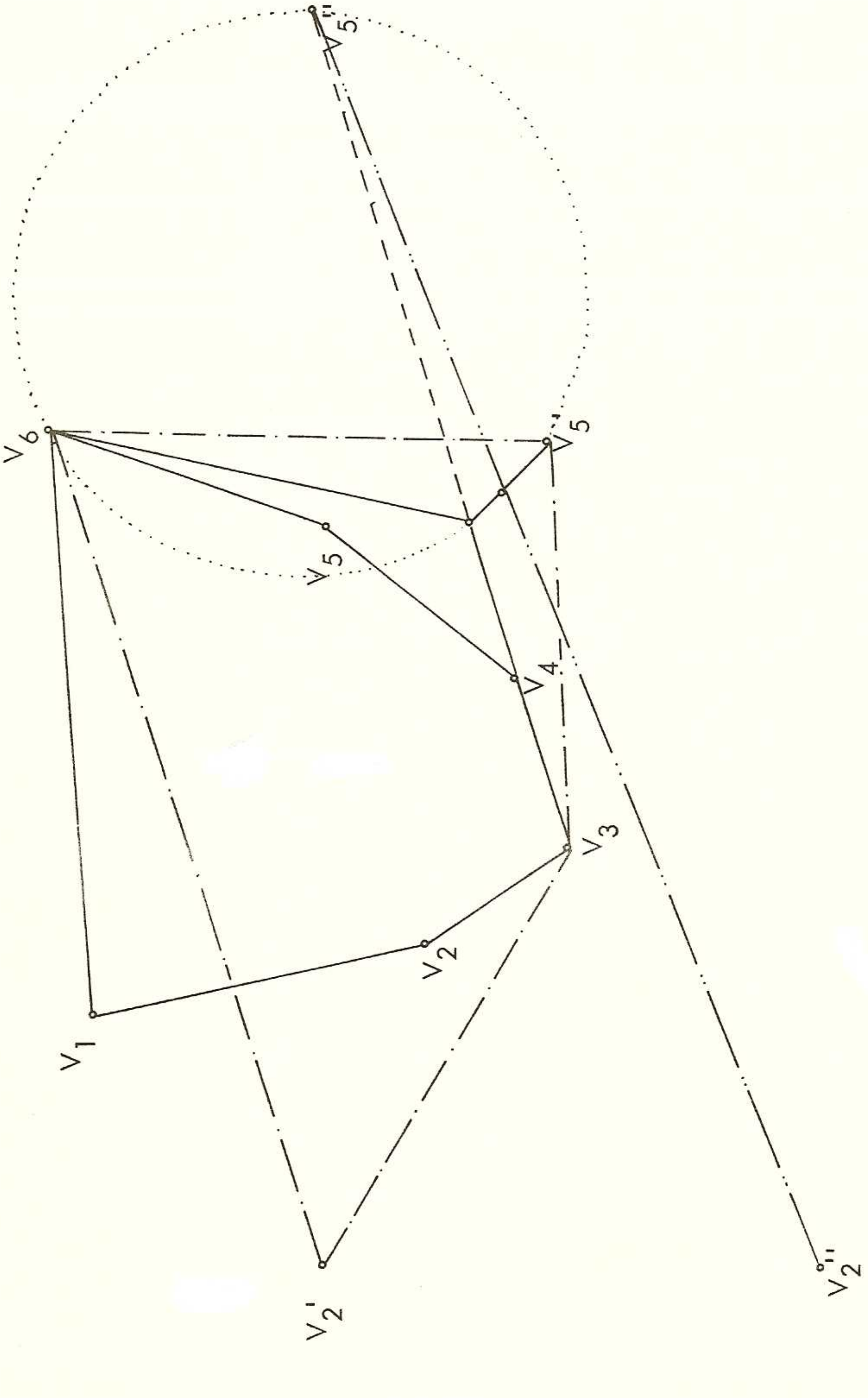


Figure IV.T_2.2.c.i.α

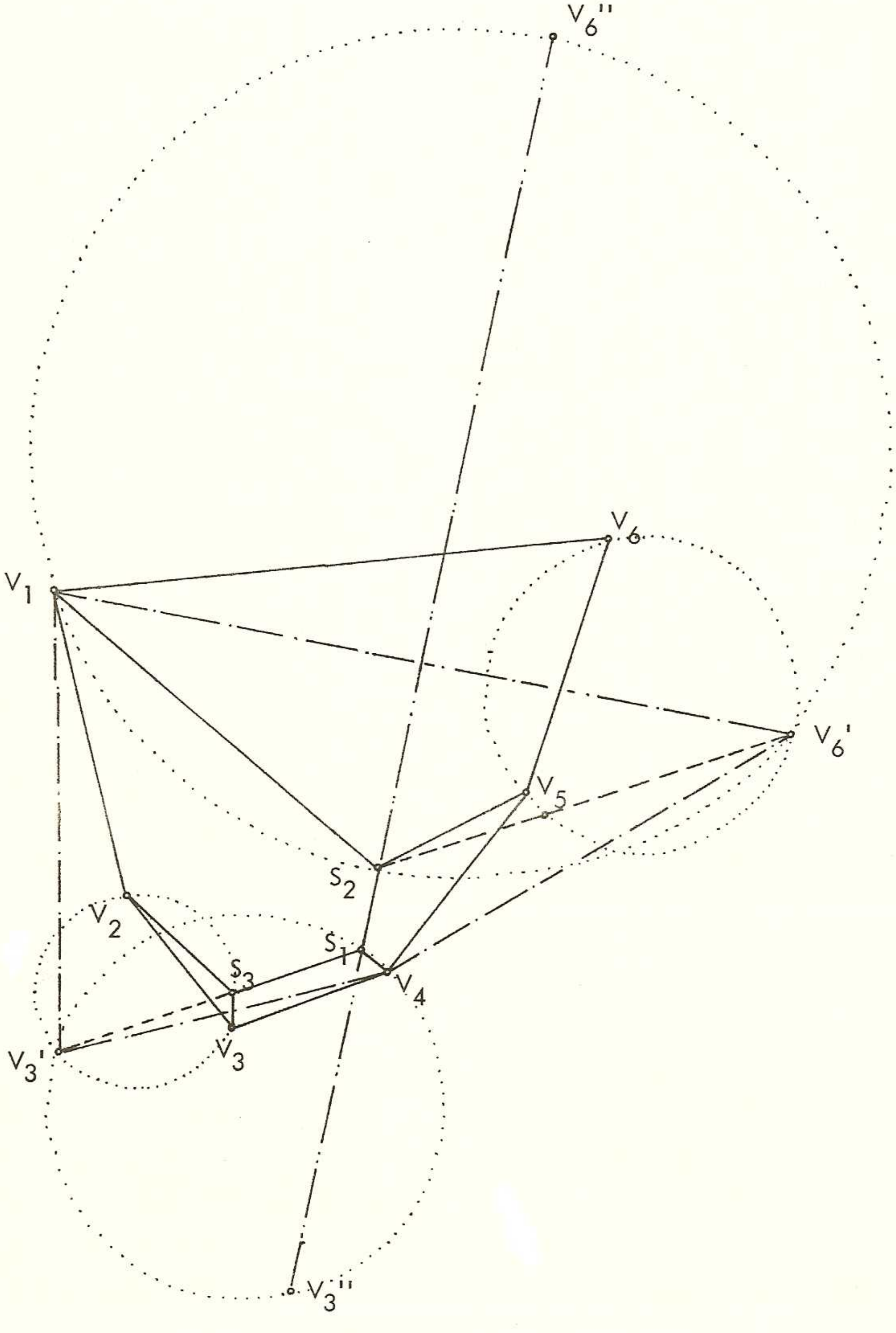
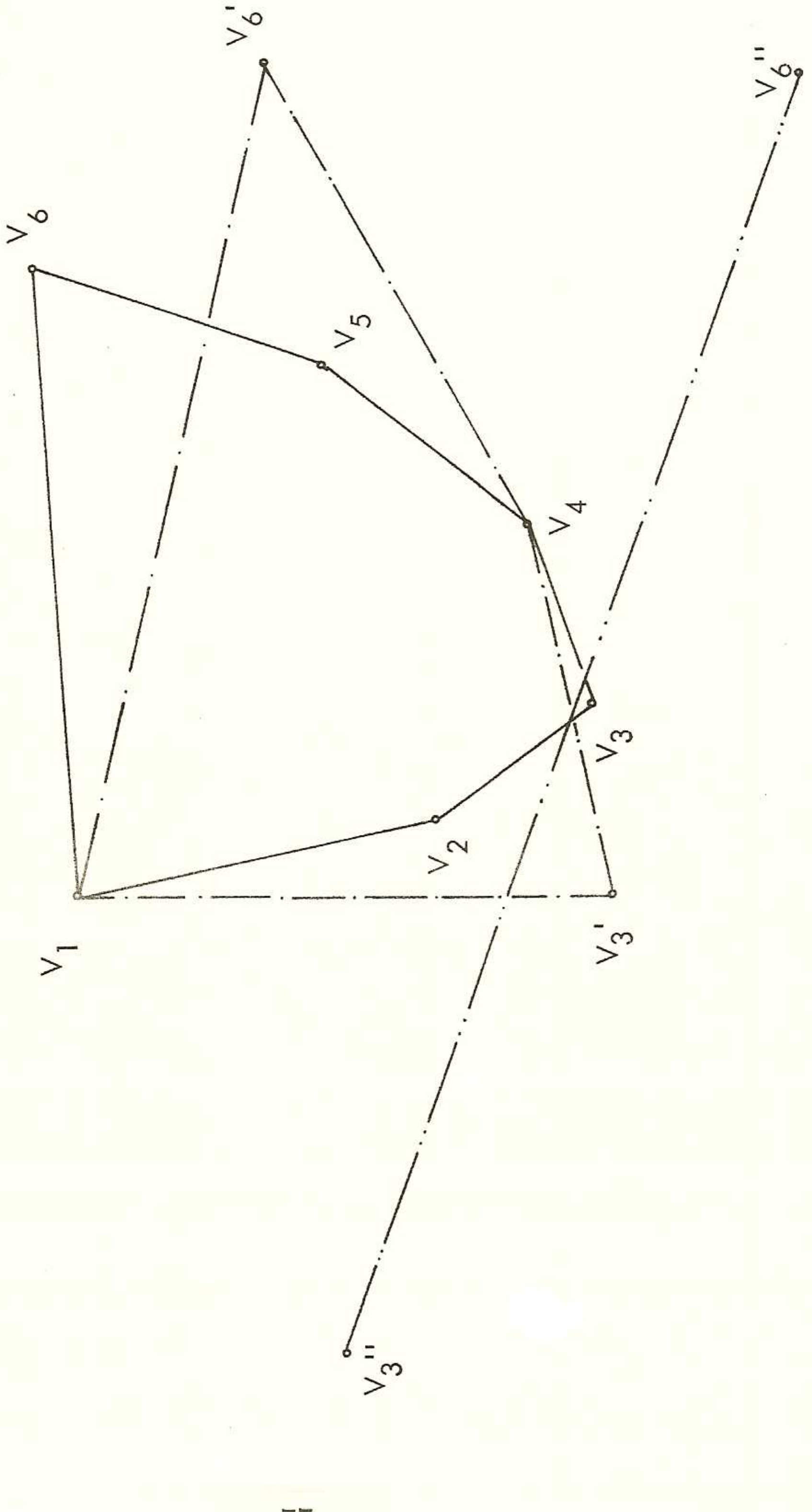


Figure IV.T_2.2.c.i.β



The theoretical material that follows is intended to be of eventual use in examining rejected candidate Steiner networks as boundaries of geographic spaces; the rejected network is a Steiner-like network for the geographic space containing it. To enumerate the number of possible candidates from which to choose the Steiner network (first level candidates) for a distribution of n points $V_1, \ldots, V_n$, count the number of separate classes of second-level candidates that can arise, (as each first level candidate is a representative of exactly one such class). It is assumed that the Steiner trees being counted are non-degenerate and that there are $(n-2)$ Steiner points $S_1, \ldots, S_{n-2}$ in each. Within any class of second level candidates, the $(n-2)$–spanning tree is of one topological type.

There are two different conditions that lead to different sorts of second-level candidates within a class: 1) the $(n-2)$–spanning tree (as in Figure V.1 for example) may be hooked into the distribution of points in a variety of distinct ways (Figure V.2); 2) the sides of the polygon that are chosen to be rotated through 60° may be selected in a variety of ways.

The procedure that follows will count the number of possible second-level candidates with an $(n-2)$–spanning tree of given topological form that is hooked into the set $V_1, \ldots, V_n$ in a pre-assigned manner. These constraints would be specified by the spatial relations of the problem under consideration.

Proposition V.1

Suppose that

1) $\mathcal{P}$ is a polygon with n vertices $V_1, \ldots, V_n$;

2) n is an odd positive integer;

3) the $(n-2)$–spanning tree is linear in topological form;

4) the $(n-2)$–spanning tree is to be hooked into $\mathcal{P}$ in such a way that

i) the two "small triangles" formed by connecting the two vertices of degree one of the linear $(n-2)$–spanning tree to $\mathcal{P}$ are separated by exactly one side of $\mathcal{P}$.

Figure V.1

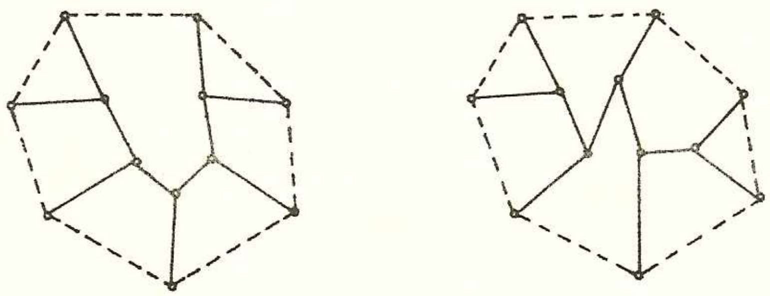


Figure V.2

ii) no two edges linking vertices of degree 2 or the $(n-2)$–spanning tree to $\mathcal{P}$ lie on opposite sides of the $(n-2)$–spanning tree.

Then there are $n \cdot (n-2) \cdot (n-4) \cdot \ldots \cdot 5$ possible distinct second–level candidates.

Proof:

There exist n different unordered pairs of sides of an n-gon with one side separating the two for if the sides of the n-gon are numbered clockwise, on alternate sides, then the desired set of unordered pairs is

$$\{(1,2),(2,3),\ldots,((n-1),n),(n,1)\}$$

(Figure V.3) which has n elements.

For each of the n choices, there are $(n-2)$ choices at the next level and then $(n-4)$ for each of the $(n-2)$ and so on, down to 5 (there is only one S_p in a given triangle) Q.E.D.

Proposition V.2 :

Suppose that (1) P is a polygon with n vertices, (2) n is an even number and (3) the $(n-2)$–spanning tree in P has maximal branching. Then there are $n \cdot (m-2) \cdot (m-4) \cdot \ldots \cdot 5$ possible distinct networks where $m = n/2^{\alpha_0}$ and α_0 is the exponent representing the maximum number of 2's that can be factored out of n.

Proof :

Decompose n into the unique product of prime factors as

$$n = 2^{\alpha_0} \cdot p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r},$$

where the p_i represent odd primes and the α_i represent integers greater than or equal to zero with $\alpha_0 \neq 0$. The possible number of networks is

$$2^{\alpha_0} \cdot m \cdot (m-2) \cdot (m-4) \cdot \ldots \cdot 5$$

where $m = n/2^{\alpha_0}$, for there are two choices for ways to select alternate sides as long as n is divisible by two (α_0 times), and, then we are left with a polygon with an odd number of sides; that is, one with $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$ sides. Then count

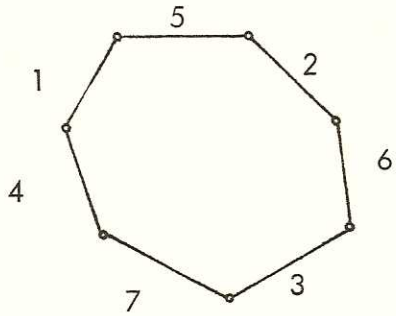
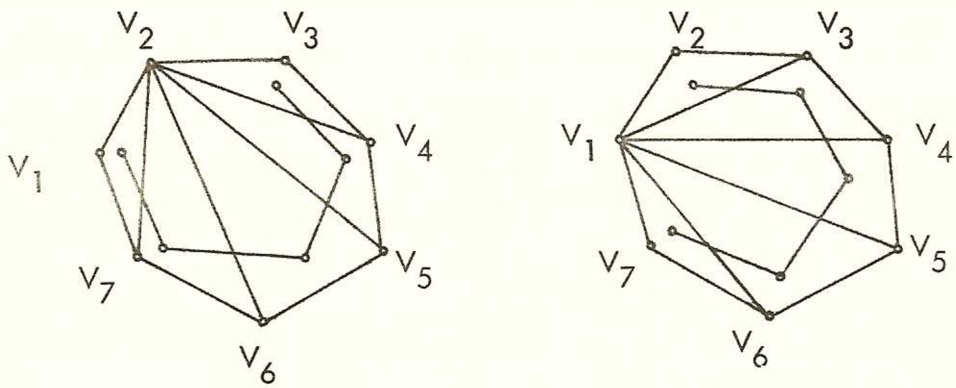


Figure V.3



(a)

(b)

Figure V.4

the rest of the networks using Proposition V.1, yielding the above formula which may be rewritten

$$n \cdot (m-2) \cdot (m-4) \cdot \ldots \cdot 5.$$

The hypotheses of these two propositions are highly restrictive. This suggests that if geographic constraints impose the restrictions in a natural way in a given problem, then the figure of 105 possible topologies for 6 vertices and 4 Steiner points, cited by Gilbert and Pollak, can be reduced to six possibilities using Proposition V.2 [1]. If a theorem that would remove whole sets of polygons from consideration could be developed, so that even as many second level candidates as those in Propositions V.1 and V.2 would not have to be counted, then we would be free to relax some of the other of those propositions constraints and move to reducing the figures given by Gilbert and Pollak [2]. Gradually, by making progress on each piece of this counting problem, it might be possible to efficiently execute Steiner network construction for large n in some reasonable number of steps. The conjecture below was suggested by numerous hand constructions of Steiner networks on 6 points with 4 Steiner points (see Chapter IV). It might lead to a theorem for eliminating large numbers of polygons from consideration, as stated above.

Definition V.1 :

The following construction will associate a number, called 3-length, to each vertex V_i of a convex polygon (Figure V.4). Connect vertex V_i to every point of the polygon, creating a partition of the polygon into $(n-2)$ triangles. Find the Steiner point of each of these $(n-2)$ triangles; link these together and measure the length, giving the 3-length associated with V_i.

Conjecture

In a polygon $\mathcal{P}$ with an odd number of sides, the minimal second level Steiner candidate network of a topological type that exhibits bilateral rather than radial symmetry will have a vertex of degree one that is incident with an axis of symmetry located at that vertex of $\mathcal{P}$ with which minimal 3-length is associated.

(There should be an extension of this to even numbers that are not congruent to 1(mod 3) and to even numbers that are not of the form $3 \cdot 2^{\alpha_0}$).

Material in the last chapter examined existence of constructions of networks and acknowledged that such networks are not unique; this chapter has responded to lack of uniqueness by indicating some procedure on how to enumerate minima.

REFERENCES

1. E. N. Gilbert and H. O. Pollak "Steiner minimal trees," *SIAM, Journal of Applied Mathematics*, 16, (1968), p. 11.

2. *Ibid.*