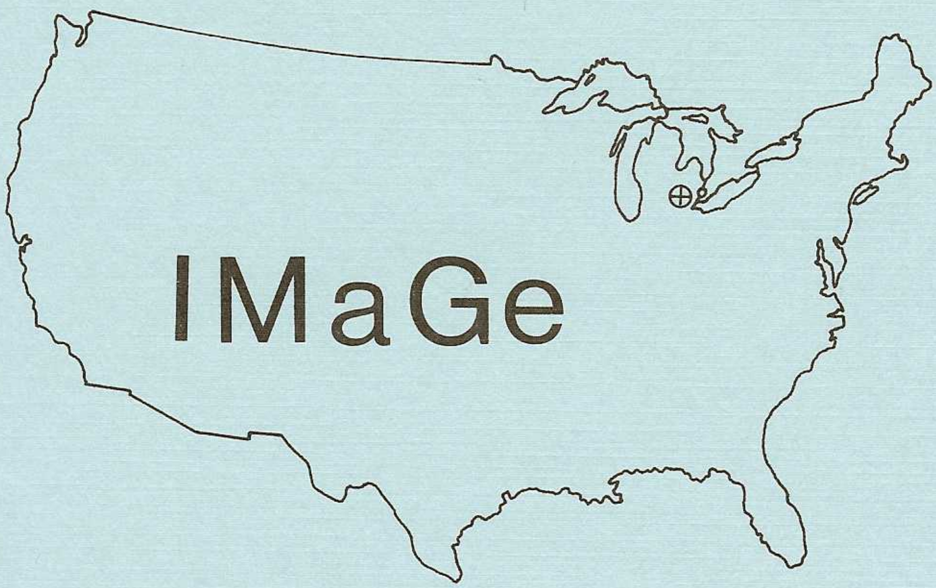
Institute of Mathematical Geography

MONOGRAPH SERIES

TITLE

? IMaGe



"IMaGe-in-nation"

An MDS Publication

MATHEMATICAL GEOGRAPHY AND GLOBAL ART:

THE MATHEMATICS OF DAVID BARR'S "FOUR CORNERS PROJECT."

by

SANDRA L. ARLINGHAUS, Ph.D.
Director,
Institute of Mathematical
Geography.

JOHN D. NYSTUEN, Ph.D.
Professor, Urban Geography
and Planning,
The University of Michigan.

INSTITUTE OF MATHEMATICAL GEOGRAPHY

MONOGRAPH #1

IMAGE IMAGINATION

A MICHIGAN DOCUMENT SERVICES PUBLICATION

ANN ARBOR, MICHIGAN

1986

MATHEMATICAL GEOGRAPHY AND GLOBAL ART:

THE MATHEMATICS OF DAVID BARR'S "FOUR CORNERS PROJECT."

Library of Congress Identification Number: DCLC86188295-B
Library of Congress Card Number: 86188295-B
Library of Congress Call Number: GA23.H75 1986.

ISBN: 1-877751-02-2

EDITOR'S NOTE

The primary contribution of this monograph is to place in the literature a complete analysis of the actual calculations employed by artist David Barr in constructing his global sculpture, "The Four Corners Project." The physical positioning of this sculpture is chronicled in the film, "In celebration: David Barr's 'Four Corners Project'," completed in July 1985 for The Smithsonian Institution.

Beyond that, this analysis reveals the central role that classical mathematical geography played in this contemporary art form. From map projection selection, to rotation matrices, to spherical trignometry, all are accessible to the reader who is well-trained in high school mathematics. Thus, this document may also serve as an instructional tool in the application of mathematics to solving empirically-oriented geographical questions.

PREFACE

"The Four Corners Project" is the construction of the world's largest sculpture using the least amount of material. It consists of an invisible tetrahedron spanning the inside of the earth with the outer four corners just protruding from the crust of the earth. These visible corners are located in Easter Island, South Africa, New Guinea, and Greenland, with imaginary planes extending through the earth from each corner to the other three. The corner is a pinnacle of marble (a four inch tetrahedron) barely emerging from the ground like a sprouting plant. The sculpture is a colorform of such vastness that, like the planet itself, it is impossible to perceive it as an entity. Its viability resides in its being collectively constructed and collectively experienced. Its process of construction and communication will be a cultural/geographic/ spiritual/esthetic metaphor.

This project, as an act of constructive creation, has at its core a faith in humanity and a faith that when we are united by the arts, the world community is most loving, most sane, and most human. The structures that bind humans through interests, needs, relationships, heritage, intellect, faith, and understanding, are an invisible filigree of connective links, a filigree that embraces innumerable people in the past, present, and future. When our spirit is enlarged, enriched, and enjoyed through connection, it moves quite naturally toward the celebration of that reality. The culmination of connection is celebration in its most transcendent form.

In 1976 I designed "The Four Corners Project," and constructed an instrument to rotate a small globe model inside four equidistant points to locate general coordinates for placement of a tetrahedron. In 1981, I was able to begin the actual construction of the project, and therefore I required two factors . . . verification that my placement in an obviously approximate scale model would correspond when actualized in world scale, and two, the precise coordinates of that placement. John Nystuen supplied both, indeed offered several choices within the original constraints and calculated the precise coordinates. In January of 1985, after complex negotiations with the Indonesian government, the final corner was implanted in Irian Jaya. The sculpture was complete.

During this time, John Nystuen and Sandra Arlinghaus have continued to explore the geo-graphy (-metry) and other implications of the project. For their stimulating and crucial contributions, their care, their insights, and their creativity, I am forever grateful.

David Barr
Northville, Michigan
1985.

ACKNOWLEDGEMENTS

The authors wish to express their gratitude to David Barr, Professor of Art at Macomb Community College in Michigan, for his continuing, friendly, interest in the development of this material. His desire to create a global sculpture stimulated our study.

TABLE OF CONTENTS

1: INTRODUCTION

Fascination with the regular polyhedra (tetrahedron, cube, octahedron, dodecahedron, and icosahedron) dates from antiquity and emerges at various times in a wide range of disciplines (Figure 1.1). The architecture of the ancient Egyptians is often characterized by the pyramid (a half octahedron), while some of the mathematics and philosophy of ancient Greece was centered on the whole set of regular polyhedra. Euclid's Elements, the great compendium of Greek geometry, is thought by some to be a treatise devoted to the development of theorems about these solids.[1] Plato, in the dialogue "Timaeus," linked these polyhedra with the basic components from which he believed the universe was formed. The tetrahedron was to correspond to fire, the cube to earth, the octahedron to air, the icosahedron to water, and the dodecahedron to his idea of the universe as a whole. Since he used this correspondence as part of his natural philosophy, the adjective "Platonic" is often employed to describe this set of solids.[2]

Evidence that this particular classical idea went through the resurrection characteristic of the Renaissance in Western Europe is present in both

---

[1]H. S. M. Coxeter, Regular Polytopes, 2nd ed., (New York and London: Macmillan, 1963), p. 13. Hermann Weyl, Symmetry, (Princeton: Princeton University Press, 1952), p. 74. According to Coxeter, Proclus was the first to suggest such an idea. In the modern view of the history of mathematics, Heath, according to Coxeter, does not support this view of the Elements (Heath, A History of Greek Mathematics, Oxford, 1921), while according to Weyl, A. Speiser does support it.

[2]Plato, "Timaeus." Also reference to this is made in Weyl, Symmetry, p. 74, in H. S. M. Coxeter, Introduction to Geometry, (New York: John Wiley and Sons, 1961), p. 4, and in Coxeter, Polytopes, p. 13.

scientific and literary works of that time. Platonic solids captured the imagination of the young Kepler (1571-1630), who used them, and spheres inscribed in and circumscribed about them, to try to describe the harmonious patterns of symmetry he observed in the spacing of the planets about the sun.[3] Literary reference to such ideas is present in the works of William Shakespeare (1564-1616),

> "The heavens themselves, the planets, and this center
>
> Observe degree, priority, and place,
>
> Insisture, course, proportion, season, form
>
> Office, and custom, in all line of order;"[4]

of Sir Thomas Browne (1605-1682, physician and author),

> "For there is a music wherever there is a harmony, order or proportion; and thus we may maintain the music of the spheres; for those well ordered motions, and regular paces, though they give no sound unto the ear, yet to the understanding they strike a note most full of harmony;"[5]

and of John Milton (1608-1674),

> "Yonder starry sphere
>
> Of planets and of fixed in all her wheels
>
> --mazes intricate
>
> Eccentric, intervolved, yet regular
>
> Then most, when most irregular they seem;
>
> And in their motions harmony divine

---

[3]Weyl, Symmetry, p. 74; Johannes Kepler, Prodromus Dissertationem Mathematicarum continens Mysterium Cosmographicum, (Tubingen, 1596).

[4]William Shakespeare, Troilus and Cressida, Act I, Scene iii.

[5]Sir Thomas Browne, Religio Medici, 1643, part ii.

So smoothes her charming tones that God's own ear
Listens delighted."[6]

In the twentieth century, Platonic solids appear as shapes useful in organizing the beginnings of one approach to the spatial design of structural material in biology, geophysics, and art. D'Arcy Thompson observes that while only the tetrahedron, cube, and octahedron appear naturally in the mineral world, all five Platonic solids are present in the living world in the skeletal structure of (among others) various Radiolarians.[7] Further, a particular solid may appear in the same type of substance that takes on disparate forms; the tetrahedron lends its shape, at the scale of the whole, to the entire diamond crystal, and its symmetry of structure, at the molecular level, to the carbon ring.

In the geophysics literature, Athelstan Spilhaus inscribes a tetrahedron inside a cube in a dodecahedron, all within the globe, and observes that "not only do the(se) solids provide a framework for patterns that appear in surface plates, but their own common vertices mark the locations of greatest tectonic activity on the Earth's surface."[8] Of the four common vertices, three are coincident with tectonic hot-spots in the North Atlantic,

---

[6]John Milton, Paradise Lost, book v.

[7]Sir D'Arcy Wentworth Thompson, On Growth and Form, (Abridged edition, edited by John Tyler Bonner), (Cambridge: The University Press, 1961), pp. 159-169, passim. The diagrams of Radiolarians that appear here are based on Haeckel's Monograph of the Challenger Radiolaria. Also of interest, Arthur L. Loeb, Space Structures: Their Harmony and Counterpoint, Foreword by Cyril Stanley Smith (metallurgist), (Reading, Mass.: Addison-Wesley, 1976).

[8]Athelstan Spilhaus, "New look in maps brings out patterns of plate tectonics," The Smithsonian, (Vol. 7, No. 5, Aug. 1976), p. 60. Athelstan Spilhaus, "Geo-Art: Tectonics and Platonic Solids," Transactions of the American Geophysical Union, 56. No. 2, 1975.

the South Atlantic, and the Indian Ocean near Sumatra, and so are fixed relative to underlying crustal plate movement.[9] Then appropriate projection of an inscribed icosahedron onto the globe produces curved faces that approximate the shapes of the tectonic plates underlying the surface of the earth, thereby revealing "regular patterns (that) result from the harmony of dynamical laws."[10]

In art, David Barr, a contemporary artist at Macomb Community College in Michigan, is in the process of constructing a global sculpture in the shape of a regular tetrahedron entitled "Four Corners Project."[11] The sculpture will consist of four separate marble 'corners' of a tetrahedron imbedded in the surface of the earth at locations representing vertices of an abstract tetrahedron inscribed within the earth. While it might seem appropriate, for the sake of structural permanence over geological time, to place these corners at Spilhaus's hot-spots of tectonic activity, pragmatics forced Barr to seek locations on land. Thus Barr's problem was to find locations on the earth's land masses that were vertices of a regular tetrahedron inscribed in a sphere. He specified further that one vertex be on Easter Island, as the 'navel of the earth' and the point of attachment for his sculpture within the earth. These practical constraints, together with Barr's recognition of the geographical character of this problem and of the associated one of dealing with the distribution of land masses across the surface of the earth, led him to John Nystuen of the Department of

---

[9]Spilhaus, "New look," p. 61.

[10]Ibid. p. 60.

[11]Others apparently also share Barr's enthusiasm for creating sculpture of this sort. Thomas Shannon of New York claims to be constructing a complete set of Platonic solids in the earth, in his project "Corners of the World," <u>The CoEvolution Quarterly</u>, Spring 1982, p. 122.

Geography at The University of Michigan. Nystuen's algorithm for verifying Barr's proposed solution for land-based positions for each of the four corners is presented below, and its use resulted in the selection of locations in Easter Island, the Kalahari Desert, Greenland, and New Guinea. This location set not only satisfies Barr's land-based criterion and choice of Easter Island, but also reflects a certain aesthetic quality in the placement of the vertices in terms of the diversity of substance available on earth: from the igneous rock deep within the earth below Easter Island, to the diamond crystals at Kimberley, to the ice fields of Greenland, and the mangrove swamps, composed of carbon-rich organic material, of New Guinea.[12]

The material that follows contains Nystuen's solution to Barr's problem that was used in the actual positioning of the sculpture in the earth, and Arlinghaus's solutions to (i) the natural extension of Barr's problem for the remaining Platonic solids, and (ii) questions surrounding the uniqueness of Barr's choice of a tetrahedron for his sculpture. This is done with an eye to viewing Barr's sculpture as a monument linking the intellectual traditions of man's past (from Egyptian architecture to Kepler's cosmos), with the spherical surface man inhabits. Or, as Barr himself puts it:

---

[12]David Barr--To emphasize that his sculpture unifies these widely separated substances and locations, Barr has put igneous rock from Easter Island into the Kalahari Desert (along with the tetrahedral vertex), has repeated this ceremony in Greenland, and will repeat it in New Guinea with an accumulation of physical material from the three other locations.

"When I think about space and time, my concepts of them tend to coalesce, for one seems to be the means of establishing my experience of the other. . . . Time is the exhibitor of space. Space demonstrates time."[13]

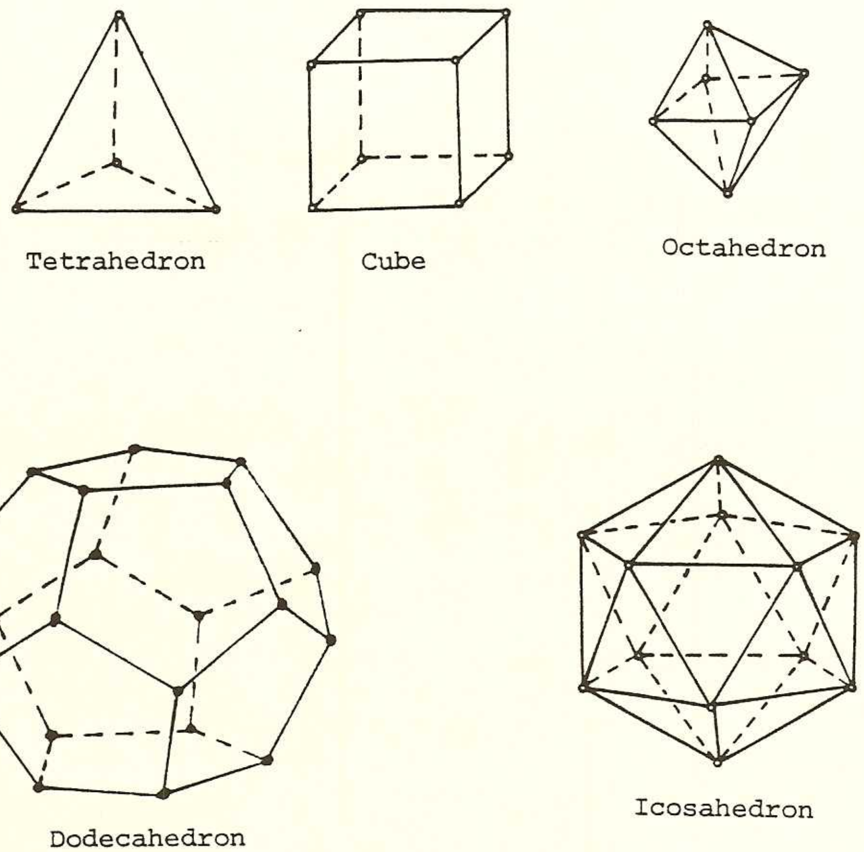


Figure 1.1

[13]David Barr, "Notes on Space/Time," The Structurist, (No. 15/16, 1975/76), p. 42.

2: FOUR CORNER SITES FOR THE TETRAHEDRON SCULPTURE[14]

David Barr's choice of a tetrahedron to be inscribed in the earth fixes on a pleasingly symbolic sculpture. The tetrahedron is the simplest of the Platonic solids and possesses elegant symmetries which permit several different approaches to the problem of finding the location of its four corners on the earth. In one approach, Nystuen chose to consider the tetrahedron to be defined by a set of vectors issuing from an origin, O, such that the heads of the vectors are on the surface of a sphere and where each vector is as far from the others as possible. This condition results in an equal spacing of points on the sphere (Figure 2.1). One attribute of this view of the tetrahedron is that the algebraic sum of vectors issuing from the center of the solid (considered as the origin) is zero, as will be shown below. (For this sort of view of other solids, see Appendix A.1.) This symmetry provided the key to Nystuen's solution of the theoretical portion of Barr's problem.

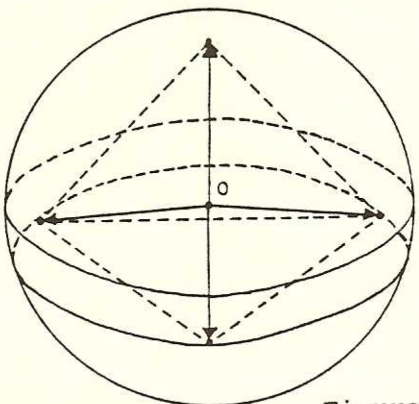


Figure 2.1

However, Barr's problem is more than a theoretical exercise. It has an empirical component that requires positions of the land-based vertices described in terms of earth-coordinates; thus the entire project has even greater scope. For ultimately

[14]From material in, John D. Nystuen, "Notes on David Barr's Four Corners of the Earth," unpublished Exhibit Notes, from Barr's museum exhibit, Meadowbrook Hall, Oakland University, Michigan, November 1982.

Barr must move from thought to action and visit his four corners of the earth. The logistics of this undertaking call for great effort on his part and for help from many others. The project takes on heroic proportions.

Barr specified that one vertex of his tetrahedron be located on Easter Island, a very prescient choice since the probability of locating the other three points on land are enchanced. By his own work, Barr was fairly certain of where the other three points lay. Nystuen's procedure of 1980, presented in this section and in Appendices A.1 and A.2, confirmed his analysis. This was the procedure actually used in placing elements of the sculpture on the earth.[15] More efficient means for determining suitable locations were found later, and these are described following Nystuen's original solution. This section concludes with a discussion of the problems of precision in location of earth-coordinates that result from assuming the earth to be spherical.

LOCATION OF THE TETRAHEDRON IN A SPHERE

In the Unit Sphere

The first task was to find the size and location of the small circle S_1, inscribed on the surface of the globe created by rotating a tetrahedron about the axis passing through Easter Island (Figure 2.2). Introduce a right-handed (x,y,z)-coordinate system into the sphere in such a way that the positive half of the z-axis passes through

---

[15]The complete procedure is explained below; Nystuen explained this process briefly in a film about Barr's sculpture, for the Archives of American Art, Smithsonian Institution, September 1983.

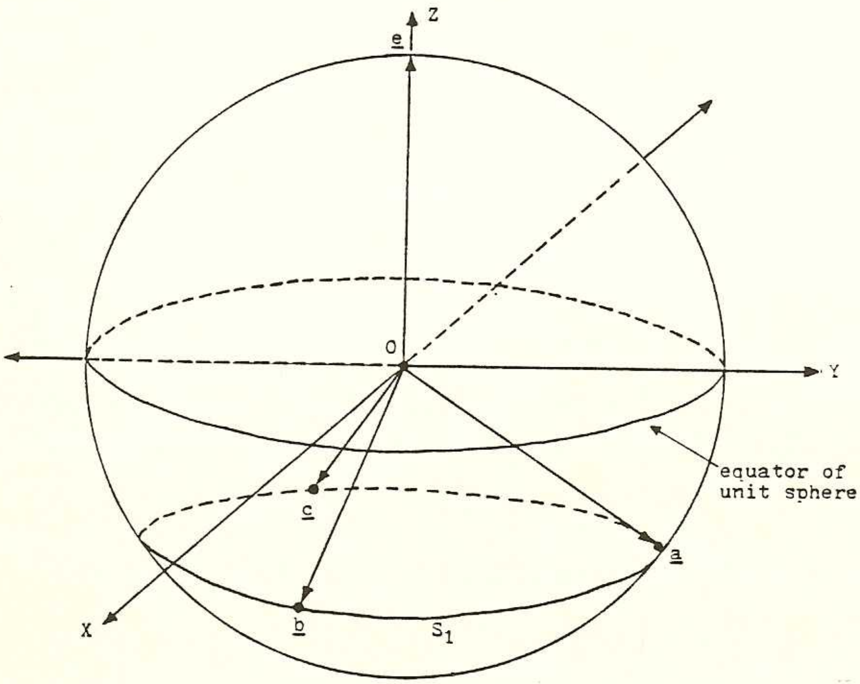


Figure 2.2

Easter Island (*e*). The origin, O, of this coordinate system is the center of the tetrahedron and of the sphere it is inscribed within. The small circle S_1 containing the other three vertices (as heads of vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$) will lie in a plane parallel to the (*xy*)-plane and in the hemisphere opposite from Easter Island (Figure 2.2).

Determine the components of the vector $\underline{a}$ in the (*yz*)-plane, issuing from the origin with head on the small circle S_1 (Figure 2.2). From Appendix A.2, the *z*-component, a_z, of $\underline{a}$ is $a_z=-\frac{1}{3}r$; since $r=1$ in the unit sphere, $a_z=-\frac{1}{3}$. Using the Pythagorean theorem, the *y*-component a_y is found as

$$a_y{}^2 + (-\tfrac{1}{3})^2 = 1.0, \text{ so } a_y = \pm\frac{2\sqrt{2}}{3}$$

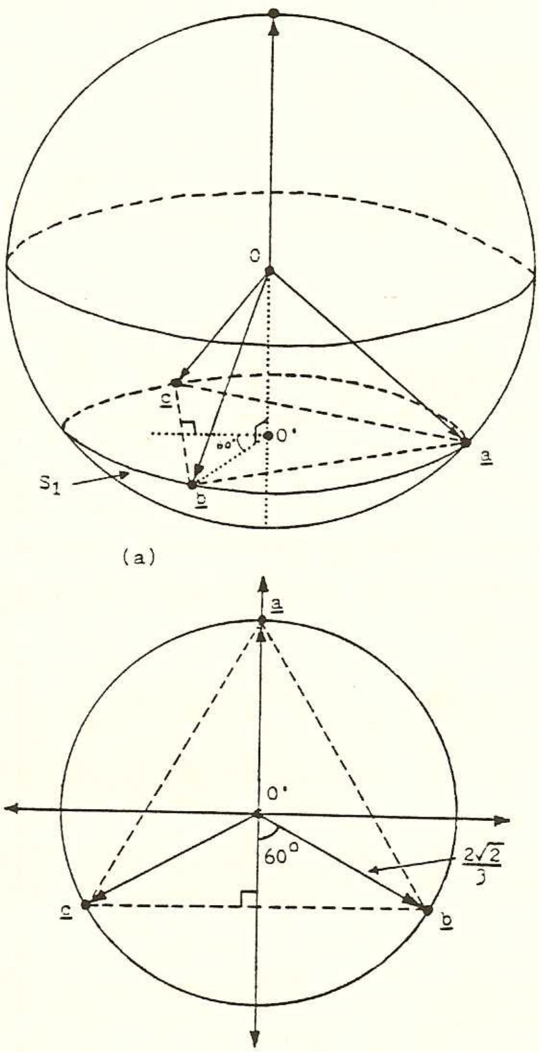
and the *x*-component, a_x, is obviously $a_x=0$, since $\underline{a}$ lies in the (*yz*)-plane. Thus $\underline{a} = [0, \frac{2\sqrt{2}}{3}, -\frac{1}{3}]$ where the positive direction is chosen for the *y*-component (Figure 2.2).

The components of the vectors $\underline{b}$ and $\underline{c}$, which determine the positions on S_1 for the remaining two vertices of the tetrahedron, may be found in a similar way. Since $\underline{a}$, $\underline{b}$, and $\underline{c}$ all lie on S_1, and since the plane containing S_1 is parallel to the (xy)-plane, the equation of this plane is $z=-\frac{1}{3}$ since $a_z=-\frac{1}{3}$. Thus $a_z=b_z=c_z=-\frac{1}{3}$. Since the triangle described on S_1 by these three vectors is equilateral, it follows that

$$b_y + c_y = -a_y \text{ and } b_y = c_y.$$

$$\text{Thus } 2b_y = -\frac{2\sqrt{2}}{3} \text{ (Figure 2.3), so } b_y = c_y = -\frac{\sqrt{2}}{3}.$$

Then use the Pythagorean Theorem to find b_x and c_x:



(a)

(b) Polar view of (a)

Figure 2.3

$$b_x^{\,2} + \left(-\frac{\sqrt{2}}{3}\right)^2 = \left(\frac{2\sqrt{2}}{3}\right)^2$$

$$c_x^{\,2} + \left(-\frac{\sqrt{2}}{3}\right)^2 = \left(\frac{2\sqrt{2}}{3}\right)^2$$

Thus $b_x = \pm\frac{\sqrt{6}}{3}$,

$c_x = \pm\frac{\sqrt{6}}{3}$; here

$b_x = \frac{\sqrt{6}}{3}$, $c_x = -\frac{\sqrt{6}}{3}$ (Figure 2.2) and the radius of S_1 is $\frac{2\sqrt{2}}{3}$.

Thus the set of vectors determining the positions for vertices of the tetrahedron inscribed in the unit sphere is:

$\underline{e}=[0,0,1]$

$\underline{a}=[0, \frac{2\sqrt{2}}{3}, -\frac{1}{3}]$ [1]

$\underline{b}=[\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}]$

$\underline{c}=[-\frac{\sqrt{6}}{3}, -\frac{\sqrt{2}}{3}, -\frac{1}{3}]$

To this point, only geometric properties of coordinate systems and of various plane and solid figures have been used; the power of adopting the vector approach to this problem has not yet been exhibited. The remainder of the analysis in this section will use results from linear algebra; the reader unfamiliar with them is referred to Appendix B for derivations. Vector notation reveals the symmetry in structure that was exhibited visually in the derivation of vector set [1] since the sum of the vectors in that set is zero.

The vector set [1] may be used to determine the lengths of the edges of the inscribed tetrahedron by viewing those edges as vectors [*e*-*a*], [*e*-*b*], [*e*-*c*], [*a*-*b*], [*a*-*c*], and [*b*-*c*]. Since the tetrahedron is regular, each of these vectors has the same length. The length of a typical vector, [*b*-*c*], is

$$|[b-c]| = \left(\left(\frac{2\sqrt{6}}{3}\right)^2\right)^{1/2} = \frac{2\sqrt{6}}{3} \cong 1.632993162. \qquad [2]$$

The same set, [1], of vectors may also be used to determine the angle θ between any two vectors of [1]; then θ may be used to measure the distance between vector heads on the earth's surface since this distance is the arc length of the central angle θ measured on the sphere. Using the symmetry of the tetrahedron, it is clear that the angles between each pair of vectors in [1] are the same size. The angle θ between two typical vectors $\underline{e}$ and $\underline{a}$ is found as follows (where "$\cdot$" denotes the dot product of vectors):

$$\cos\theta = \frac{e\cdot a}{|\underline{e}||\underline{a}|} = \underline{e}\cdot\underline{a} = -\frac{1}{3} \qquad [3]$$

since $|\underline{a}| = |\underline{b}| = |\underline{c}| = |\underline{e}| = 1$.

Thus $\theta \cong 109.4712206°$, or converted to radian measure, the angle contains $\frac{(\pi x \theta)}{180} = 1.910633203$ radians.

In the Earth

Using $\bar{R}$="the radius of the international sphere" for Clarke's ellipsoid of 1866 as the radius of a sphere with the same volume as the reference ellipsoid, $\bar{R}$=6,370,997 meters, or $\bar{R}$=3958.7461 miles.[16] Then all of the measurements found for the unit sphere may be calculated for the earth.

The radius, $r_{S_1}=\frac{2\sqrt{2}}{3}\text{x}\bar{R}$, of the small circle S_1 containing the three vertices determined by a, b, and c, is (from [1]),

$$r_{S_1} = \frac{2\sqrt{2}}{3} \text{ x } \bar{R} = \begin{cases} 6{,}006{,}633.576 \text{ meters} \\ 3732.3416 \text{ miles.} \end{cases}$$

Using [2], the length of the edges of the tetrahedron within the earth is,

$$\frac{2\sqrt{6}}{3} \text{ x } \bar{R} = \begin{cases} 10{,}403{,}794.54 \text{ meters} \\ 6464.6053 \text{ miles} \end{cases}$$

all of which would be underground. From [3], the distance on the sphere between two vertices is the arc length,

$$\frac{\pi \text{x} \theta}{180} \text{ x } \bar{R} = 1.910633203 \text{ x } \bar{R} = \begin{cases} 12{,}172{,}638.4 \text{ meters} \\ 7131.1358 \text{ miles.} \end{cases}$$

These measurements are intended to give some appreciation for the magnitude of this project.

---

[16] Simo H. Laurila, Electronic Surveying and Navigation (N.Y.: John Wiley & Sons, 1976), p. 193.

LOCATION OF THE TETRAHEDRON VERTICES IN EARTH-COORDINATES

Easter Island

Barr specified that one vertex of the tetrahedron be on Easter Island, the "navel of the earth." Easter Island is located between 109°16'30"W and 109°27'05"W and 27°03'00"S and 27°11'00"S. A convenient spot for one vertex of the sculpture might be at Tahae on the western shore of the island. The name signifies "the place of the setting sun."[17]

The site proposed for the first element of the sculpture is located at

109°25'30"W
27°06'20"S an approximation to -109.4305555°W
-27.10925°S.

A better approximation (but not the one used originally) is given by

(27°06'33"S, 109°25'50"W) or (-27°06'33", -109°25'50").

For more precision, but still not exactness, one can convert the decimal approximation by partitioning seconds of latitude/longitude into 60ths, and so on, to obtain earth coordinates:

$$(-27°06'33''18''', -109°25'49''59'''59^{(iv)}16^{(v)}48^{(vi)}).$$

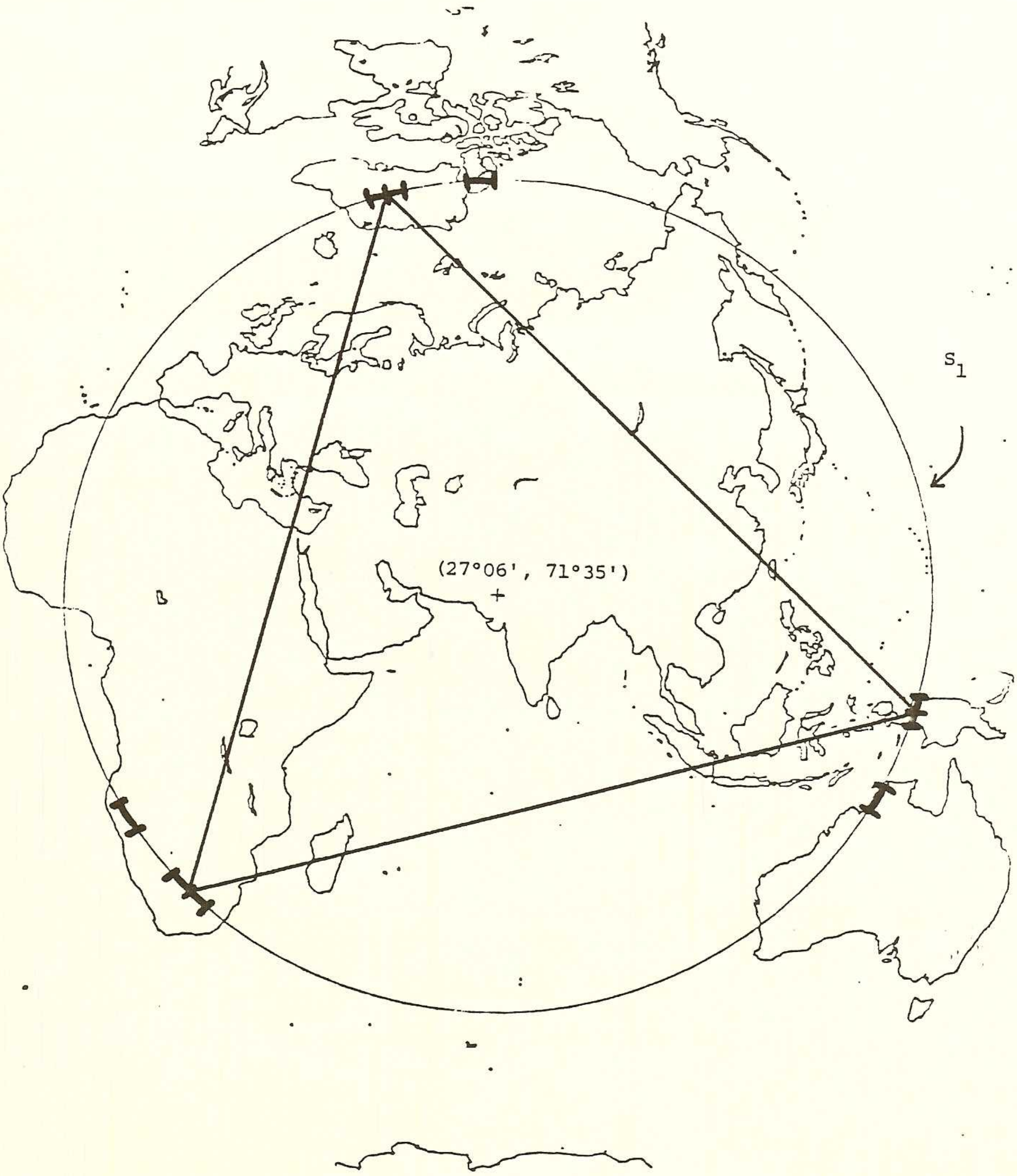
It remains to calculate the earth-coordinates associated with vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ whose heads lie on S_1. Professor Richard Taketa, formerly of the Department of Geography of the University of Michigan, used computer graphics to construct an azimuthal map projection centered

---

[17] John Dos Passos, Easter Island, Island of Enigma, Garden City, (New York: Doubleday & Co., 1971), illustrations 9 and 10. Also, Carlos Charlia Ojeda, Geo-etimologia de la Isla de Pascua, (Santiago, Chile: Instituto Geografico Militar, 1947), pp. 190-191, and map facing p. 208.

on the point antipodal to Easter Island, (27°06',71°35'), in the Thar Desert of India (based on a program of Waldo Tobler). This map is the critical tool in solving the empirical problem. Distances measured from the center of an azimuthal projection are in true proportion to distances on the sphere. Therefore a circle of radius r_{S_1} centered on this point in India, drawn to scale on the map, will contain the points sought. An equilateral triangle cut from a piece of paper was used as an aid in locating the collection of points which could serve as vertices of the proposed sculpture (Map 2.1). By placing its corners on the circle S_1 drawn on the azimuthal map, one can rotate the triangle and discover the collection of points which satisfy the condition that they all fall on dry land. This procedure confirmed that Barr's own discovery of the location set {Easter Island, Kalahari Desert, Greenland Ice Cap, and New Guinea} was feasible. Keeping in mind Barr's desire to have one of the vertices at the Kimberley diamond mines, the following procedure was used to find earth-coordinates for $\underline{a}$, $\underline{b}$, and $\underline{c}$ on S_1, and coordinates in R^3 for $\underline{e}$, $\underline{a}$, $\underline{b}$, and $\underline{c}$.

The actual calculation of the latitude and longitude of the points sought proceeded in two broad parts. First the relative location of the vertices was established by finding vector positions on a unit sphere endowed with a right-handed (x,y,z)-coordinate system. The second step was to move a unit vector by rotation (a rigid motion preserving length) to a position on Easter Island. This process simultaneously moved the three vectors from O to S_1 in a like manner. Three transformations were required; one in azimuth (longitudinal shift), one in elevation (latitudinal shift), and finally one rotation around the axis centered, in the first step, on Easter Island.



Map 2.1

AZIMUTHAL EQUIDISTANT PROJECTION

Coordinate Transformations

We know the relative positions of the vertices of the tetrahedron for an arbitrary unit sphere in the (x,y,z)-coordinate system. We now relate this system to the earth and rotate a unit vector to the location of Easter Island by a shift in longitude followed by a shift in latitude. This can be accomplished by multiplying each vector by a transformation matrix which combines the two rotations.

Longitude Shift: Let rotation through angle a, about the z-axis, be represented by the rotation matrix R_a,

$$R_a = \begin{bmatrix} \cos a & -\sin a & 0 \\ \sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ to be applied to a column vector.}$$

Latitude Shift: Let rotation through angle β, about the x-axis, be represented by the rotation matrix R_β,

$$R_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \text{ to be applied to a column vector.}$$

Product of transformations: Since these two matrices are conformable for multiplication their product

$$R_a R_\beta = R_{a\beta} = \begin{bmatrix} \cos a & -\sin a\cos\beta & \sin a\sin\beta \\ \sin a & \cos a\cos\beta & -\cos a\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

represents the product of two rotations, or rotation through a, followed by rotation through β.

To execute the desired coordinate transformations, let T be the tetrahedron determined by four vectors $\underline{e}$, $\underline{a}$, $\underline{b}$, $\underline{c}$, with $\underline{e}$ on the positive half of the axis, with head at longitude 0, latitude 0 (Figure 2.4). Here $\underline{e}$ is taken along the y-axis to aid in earth-based visualization; previously it was taken along the z-axis to facilitate visualization in the right-handed (x, y, z)-coordinate

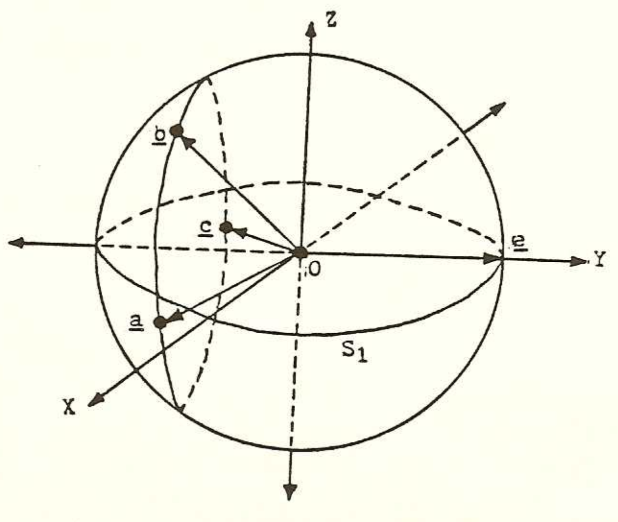


Figure 2.4

system. To distinguish any rotation of a vector in the unit sphere from rotations that lead to geographic coordinates of longitude and latitude, symbols such as α, β, γ will represent rotations of the former type while the symbols λ and ϕ will represent rotations of the latter sort.

Find Coordinates of e, a, b, c in R^3 on the Unit Sphere

Rotate T to center e=[0,1,0] on Easter Island at e'=(-27.10833°, -109.43056°); or, in terms of coordinates in R^3, at $e' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ (Fig. 2.5)

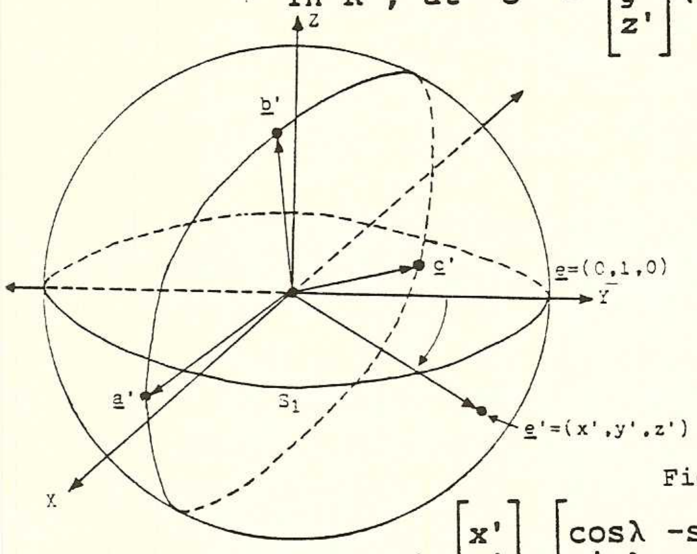


Figure 2.5

The vector e=[0,1,0] is to be rotated through ϕ=-27.10833° in latitude in the negative direction; followed by λ = -109.43056° in longitude in the negative direction; use $R_{\lambda\phi}$ applied to e to obtain e'. Thus:

$$\underline{e'} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\lambda & -\sin\lambda\cos\phi & \sin\lambda\sin\phi \\ \sin\lambda & \cos\lambda\cos\phi & -\cos\lambda\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin\lambda\cos\phi \\ \cos\lambda\cos\phi \\ \sin\phi \end{bmatrix}$$

Equating corresponding components,

$$\begin{aligned} x' &= -\sin\lambda\cos\phi \\ y' &= \cos\lambda\cos\phi \\ z' &= \sin\phi \end{aligned} \qquad [4]$$

Evaluating $[x',y',z']$ at $\lambda = -109.43056°$, $\phi = -27.10833°$, gives

$$\underline{e'} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0.83944859 \\ -0.29611987 \\ -0.45567433 \end{bmatrix}$$

Similarly, use of the rotation matrix $R_{\lambda\phi}$ applied to $\underline{a}$, $\underline{b}$, $\underline{c}$, will give coordinates in R^3 for $\underline{a'}$, $\underline{b'}$, $\underline{c'}$ (Figure 2.5):

$$\underline{a'} = R_{\lambda\phi} \times \underline{a} = R_{\lambda\phi} \begin{bmatrix} 0 \\ -\frac{1}{3} \\ -\frac{2\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} -.6849612799 \\ .2416182677 \\ -.6873467876 \end{bmatrix}$$

$$\underline{b'} = R_{\lambda\phi} \times \underline{b} = R_{\lambda\phi} \begin{bmatrix} \frac{\sqrt{6}}{3} \\ -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} -.348862384 \\ -.7427453601 \\ .5715105574 \end{bmatrix} \qquad [5]$$

$$\underline{c'} = R_{\lambda\phi} \times \underline{c} = R_{\lambda\phi} \begin{bmatrix} -\frac{\sqrt{6}}{3} \\ -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} .194375937 \\ .7972412609 \\ .5715105574 \end{bmatrix}$$

Note that if the vector $\underline{e}$ had been rotated from position $(0,r,0)$ within a sphere of radius r to position $\underline{e}_r'$, rather than in the unit sphere, it follows from [4] that,

$$\underline{e}_r' = \begin{bmatrix} x_r' \\ y_r' \\ z_r' \end{bmatrix} = \begin{bmatrix} \cos\lambda & -\sin\lambda\cos\phi & \sin\lambda\sin\phi \\ \sin\lambda & \cos\lambda\cos\phi & -\cos\lambda\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -r\sin\lambda\cos\phi \\ r\cos\lambda\cos\phi \\ r\sin\phi \end{bmatrix}$$ producing a more general form of [5]

as

$$x_r' = -r\sin\lambda\cos\phi$$
$$y_r' = r\cos\lambda\cos\phi$$
$$z_r' = r\sin\phi$$

But to find latitude and longitude, we may assume $r=1$, for latitude and longitude are the same on any sphere, regardless of radius.

Find the Earth-Coordinates of $\underline{a}'$

From [5], $$\underline{a}' = \begin{bmatrix} -\sin\lambda\cos\phi \\ \cos\lambda\cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} -.6849612799 \\ .2416182677 \\ -.6873467876 \end{bmatrix}$$

Equating like components,

$$\sin\phi = -.6873467876, \quad \phi = -43.4204490°,$$

$$\cos\phi = .72632941.$$

$$\text{and } \sin\lambda = \frac{-x'}{\cos\phi} = \frac{.6849612799}{.72632941} = .94304494, \text{ so}$$

$$\lambda = 70.569368° \text{ or } \lambda = 160.569368°,$$

$$\text{and } \cos\lambda = \frac{+y'}{\cos\phi} = \frac{.2416182677}{.72632941} = .33265658, \text{ so}$$

$$\lambda = 70.569901° \text{ or } \lambda = -70.569901°$$

Thus $\lambda = 70.569°$.

Therefore, when $\underline{e}$ is rotated to $\underline{e}'$, $\underline{a}$ in the (yz)-plane is forced to $\underline{a}' \cong (-43.42044934°, 70.569°)$ which is in the South Indian Ocean (Map 2.2).

Rotate $\underline{a'}$ to Kimberley

Find the angle γ through which to rotate $\underline{a'}$ (-43.4°, 70.6°) to approximately the location of the Kimberley diamond mine in South Africa $\underline{a''}$ (-28.8°, 24.8°). Here λ=24.8°, ϕ=-28.8°.

$$\text{Therefore, } \underline{a''}=\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}=\begin{bmatrix} -\sin\lambda\cos\phi \\ \cos\lambda\cos\phi \\ \sin\phi \end{bmatrix}=\begin{bmatrix} -.3675686618 \\ .7954914684 \\ -.4817536741 \end{bmatrix}$$

To find γ, use [3], so

$$\cos\gamma=\frac{\underline{a'}\cdot\underline{a''}}{|\underline{a'}||\underline{a''}|}=\underline{a'}\cdot\underline{a''}=\begin{bmatrix} -.6849612799 \\ .2416182677 \\ -.6873467876 \end{bmatrix}\cdot\begin{bmatrix} -.3675686618 \\ .7954914684 \\ -.4817536741 \end{bmatrix}$$

$$=(-.6849612799)\times(-.3675686618)+(.2416182677)\times(.7954914684)$$
$$+(-.6873467876)\times(-.4817536741)=.7751074119.$$

Using inverse functions, $\gamma=\pm 39.185228°$. To shift $\underline{a'}$ to South Africa requires choosing the negative sign (Map. 2.2). In the original calculation Nystuen used $\gamma=\pm 40°$ as an approximation, because it appeared to be easier to work with.

Rotate $\underline{a'}$, $\underline{b'}$, $\underline{c'}$ through $\gamma\cong-40°$ about the Easter Island Axis

The rotation matrix R_γ, representing rotation about the Easter Island axis through (-40°) is :

$$R_\gamma=\begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix}=\begin{bmatrix} .7660444431 & 0 & .6427876097 \\ 0 & 1 & 0 \\ -.6427876097 & 0 & .7660444431 \end{bmatrix}.$$

Use this, together with $R_{\lambda\phi}$, to determine the rotation matrix $R_{\lambda\phi\gamma}$, representing rotation through λ, followed by rotation through ϕ and then through γ:

$$R_{\lambda\phi\gamma}=R_{\lambda\phi}R_{\gamma}=\begin{bmatrix} -.3326641738 & .8394485865 & .4297215587 \\ -.9430453581 & -.2961198717 & -.1515865234 \\ 0 & -.4556743268 & .8901465654 \end{bmatrix}$$

$$\cdot\begin{bmatrix} .7660444431 & 0 & .6427876097 \\ 0 & 1 & 0 \\ -.6427876097 & 0 & .766044431 \end{bmatrix}$$

$$=\begin{bmatrix} -.5310552353 & .8394485865 & .115353403 \\ -.624976717 & -.2961198717 & -.722299855 \\ -.57221751831 & -.455674368 & -.68189183 \end{bmatrix}$$

Thus $\underline{a''} = R_{\lambda\phi\gamma} \text{ x } \underline{a} = R_{\lambda\phi\gamma}\begin{bmatrix} 0 \\ -\frac{1}{3} \\ -\frac{2\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} -.3885721403 \\ .7796973856 \\ -.4910023404 \end{bmatrix}$

$\underline{b''} = R_{\lambda\phi\gamma} \text{ x } \underline{b} = R_{\lambda\phi\gamma}\begin{bmatrix} \frac{\sqrt{6}}{3} \\ -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} -.6590425792 \\ -.7520802613 \\ .006159253 \end{bmatrix}$

(See Map 2.2)

Convert a'' and b'', as Expressed Above, to Latitude and Longitude

$\underline{a''}$: From [5], equating corresponding components,

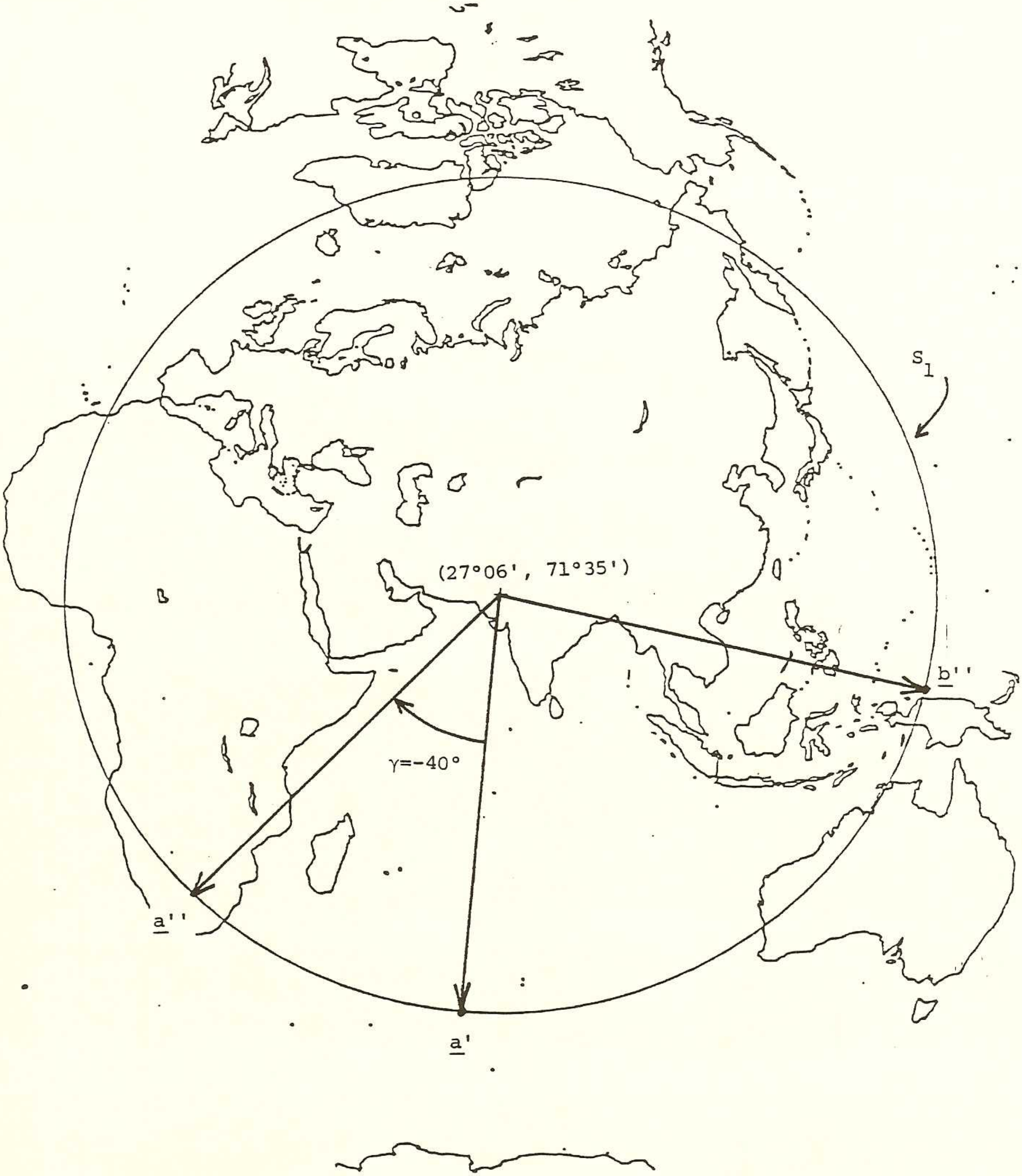
$\sin\phi = -.4910023404$, so $\phi=-29.406484°$

(or $\phi = -119.406484°$)

$\cos\phi = .87115825$

and $\sin\lambda = \dfrac{.3885721403}{\cos\phi} = .44604082$, so $\lambda = \{ \begin{matrix} 26.489949° \\ \text{or} \\ 116.489949° \end{matrix}$

and $\cos\lambda = \dfrac{.7796973856}{\cos\phi} = .89501233$, so $\lambda = \{ \begin{matrix} 26.489986° \\ \text{or} \\ -26.489986° \end{matrix}$



Map 2.2

AZIMUTHAL EQUIDISTANT PROJECTION

Thus the earth-coordinates of <u>a"</u> are (-29.406484°, 26.489°)

<u>*b*"</u>: From [5], equating corresponding components,

$\sin\phi = .006159253$, so $\phi = 0.35290121°$

$\cos\phi = .99998103$

and $\sin\lambda = \dfrac{.6590425792}{\cos\phi} = .65905507$, so $\lambda = \begin{cases} 41.227847° \\ \text{or} \\ 138.77215° \end{cases}$

and $\cos\lambda = \dfrac{-.7520802613}{\cos\phi} = -.75209453$, so $\lambda = \begin{cases} 138.77214° \\ \text{or} \\ -138.777214° \end{cases}$

Thus earth-coordinates of <u>*b*"</u> are (0.35290121°, 138.7721°)

The <u>a"</u> location falls in the Kalahari Desert, and the <u>*b*"</u> location is close to the equator, about 5° north of New Guinea (Map 2.2).

<u>**Make a Small Adjustment in γ to Fix Site in New Guinea**</u>

Since the value γ=-40° produced a value for *b*" that was about 5° north of New Guinea, use γ=-45° to obtain a value for *b*" on New Guinea. Here,

$$R_{\lambda\phi(\gamma=-45°)} = R_{\lambda\phi}R_{(\gamma=-45°)} = R_{\lambda\phi}\cdot\begin{bmatrix}\cos(-45°) & 0 & -\sin(-45°)\\ 0 & 1 & 0\\ \sin(-45°) & 0 & \cos(-45°)\end{bmatrix}$$

$$= \begin{bmatrix}-.3326641738 & .8394485865 & .4297215587\\ -.9430453581 & -.2961198717 & -.1515865234\\ 0 & -.4556743268 & .8901465654\end{bmatrix}$$

$$\cdot\begin{bmatrix}.70710678 & 0 & .70710678\\ 0 & 1 & 0\\ -.70710678 & 0 & .70710678\end{bmatrix}$$

$$= \begin{bmatrix} -.5390881213 & .8394485865 & .068629935 \\ -.559645909 & -.2961198717 & -.7740216263 \\ -.6294708468 & -.4556743268 & .6294249829 \end{bmatrix}$$

$$\text{Thus } \underline{b}''=R_{\lambda\phi(\gamma=-45°)} \times \underline{b}'=R_{\lambda\phi(\gamma=-45°)} \cdot \begin{bmatrix} \frac{\sqrt{6}}{3} \\ -\frac{1}{3} \\ \frac{\sqrt{2}}{3} \end{bmatrix}$$

$$=R_{\lambda\phi(\gamma=-45°)} \begin{bmatrix} .81649658 \\ -.33333333 \\ .47140452 \end{bmatrix} = \begin{bmatrix} -.68762734 \\ -.72311963 \\ -.06535557 \end{bmatrix}$$

<u>Convert to latitude and longitude</u>:

$\sin\phi = -.06535557$, so $\phi = -3.7472692°$

$\cos\phi = \ .99786204$

and $\cos\lambda = \dfrac{-.72311963}{\cos\phi} = -.72466894$, so $\lambda = \left\{ \begin{matrix} 136.44131° \\ \text{or} \\ -136.44131° \end{matrix} \right.$

So, coordinates for b'' are (-3.7472692°, 136.44131°). Since this point is in the interior of New Guinea, on a high plateau that is relatively inaccessible, coastal sites to the north and south of this were sought; on the north when γ = -43° and on the south when γ =-46°.

<u>If γ = -43°</u>: Here,

$$R_{\lambda\phi(\gamma=-43°)}=R_{\lambda\phi}R_{(\gamma=-43°)}=R_{\lambda\phi}\begin{bmatrix} .7313537016 & 0 & .6819983601 \\ 0 & 1 & 0 \\ -.6819983601 & 0 & .7313537016 \end{bmatrix}$$

$$= \begin{bmatrix} -.5363645732 & .8394485866 & .0874929316 \\ -.5863179531 & -.296119817 & -.7540197527 \\ -.6070784979 & -.4556743268 & .6510119856 \end{bmatrix}$$

Thus, under the condition that γ=-43°,

$$\underline{b}''=R_{\lambda\phi(\gamma=-43^\circ)} \cdot \underline{b'} = R_{\lambda\phi(\gamma=-43^\circ)}\begin{bmatrix}\frac{\sqrt{6}}{3}\\ -\frac{1}{3}\\ \frac{-\sqrt{2}}{3}\end{bmatrix}=\begin{bmatrix}-.6765543228\\ -.7354678289\\ -.0368960825\end{bmatrix}$$

<u>Convert to latitude and longitude</u>:

$\sin\phi = -.0368960825$, so $\phi = -2.114469738$

$\cos\phi = \ .99931911$

and $\cos\lambda = \dfrac{-.7354678289}{\cos\phi} = -.73596893$, so $\lambda = 137.3891542^\circ$

So, coordinates for $\underline{b}''$ when $\gamma = -43^\circ$

are $(-2.114469738^\circ, 137.3891542^\circ)$.

<u>If $\gamma = -46^\circ$</u> :Here,

$$R_{\lambda\phi(\gamma=-46^\circ)}=R_{\lambda\phi}R_{(\gamma=-46^\circ)}=R_{\lambda\phi}\begin{bmatrix}.6946583705 & 0 & .7193398003\\ 0 & 1 & 0\\ -.7193398003 & 0 & .6946583705\end{bmatrix}$$

$$=\begin{bmatrix}-.5402037731 & .8394485866 & .0592110974\\ -.5460521323 & -.2961198717 & -.7836709069\\ -.6403178526 & -.4556743268 & .6183477626\end{bmatrix}$$

Thus, under the condition that $\gamma = -46^\circ$,

$$\underline{b}'' = R_{\lambda\phi(\gamma=-46^\circ)} \times \underline{b'} = R_{\lambda\phi(\gamma=-46^\circ)}\begin{bmatrix}\frac{\sqrt{6}}{3}\\ -\frac{1}{3}\\ \frac{-\sqrt{2}}{3}\end{bmatrix}=\begin{bmatrix}-.6929783502\\ -.7165690834\\ -.079433964\end{bmatrix}$$

<u>Convert to latitude and longitude</u>:

$\sin\phi = -.079433964$, so $\phi = -4.556030743^\circ$

$\cos\phi = \ .99684013$

and $\cos\lambda = \dfrac{-.7165690834}{\cos\phi} = -.71884052$, so $\lambda = 135.9588349^\circ$

So coordinates for $\underline{b}''$ when $\gamma = -46^\circ$

are $(-4.556030743, 135.9588349^\circ)$.

Barr chose the position on New Guinea associated with γ=-43°; it remains to determine values for <u>a"</u>, and <u>c"</u>, given this choice of <u>b"</u>.

$$\underline{a''}=R_{\lambda\phi(\gamma=-43°)} \times \underline{a'} = R_{\lambda\phi(\gamma=-43°)}\begin{bmatrix} 0 \\ -\frac{1}{3} \\ -\frac{2\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} -.3622196211 \\ .8096023215 \\ -.4618885439 \end{bmatrix}$$

$$\underline{c''}=R_{\lambda\phi(\gamma=-43°)} \times \underline{c'} = R_{\lambda\phi(\gamma=-43°)}\begin{bmatrix} -\frac{\sqrt{6}}{3} \\ -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} \end{bmatrix} = \begin{bmatrix} .1993253574 \\ .2219853791 \\ .954458933 \end{bmatrix}$$

<u>Convert to latitude and longitude</u>:

For <u>a"</u>:

$\sin\phi = -.4618885439$, so $\phi = -27.509039°$

$\cos\phi = .88693798$

and $\cos\lambda = \dfrac{.809602315}{\cos\phi} = .912806$, so $\lambda = 24.103953°$

So, coordinates for <u>a"</u> when γ=-43°

are (-27.509039°, 24.103953°).

For <u>c"</u>:

$\sin\phi = .954458933$, so $\phi = 72.641933$

$\cos\phi = .29834234$

and $\cos\lambda = \dfrac{.2219853791}{\cos\phi} = .74406257$, so $\lambda = 41.921356$

So, coordinates for <u>c"</u> when γ=-43°

are (72.64193326°, 41.921356°).

Converting approximations of two significant figures when γ=-43°,

a" =	(27°30'36"S, 24°06' E)	Kalahari Desert
b" =	(2°06'36"S, 137°23'24"E)	New Guinea
c" =	(72°38'24"N, 41°55'12"E)	Greenland
e" =	(27°06'20"S, 109°25'30"W)	Easter Island

Checking the Placement of the Vertices

On the Unit Sphere

The distances between the pairs of vectors *a"*, *b"*, *c"*, *e"* are given as follows:

|[e"-a"]|= 1.632991083	
|[e"-*b*"]|= 1.632992981	Range:0.000005093
|[e"-*c*"]|= 1.632992981	Mean: 1.632991519
|[a"-*b*"]|= 1.632989906	"Correct" distance:
|[a"-*c*"]|= 1.632994999	1.632993162
|[*b*"-*c*"]|= 1.632993162	established by [2].

Variation in lengths is due to approximation in numerical evaluation of trigometric functions.

MORE EFFICIENT USE OF THIS APPROACH TO BARR'S PROBLEM

Using the values for *e*, *a*, *b*, *c* found in [1] and the locations *e'* (-27.10833°, -109.43056°) for Easter Island and *a'*(-43.42044934°, 70.569°) in the South Indian Ocean, together with the azimuthal map, values for *a"*, *b"*, *c"* may be found more efficiently and precisely than in the preceding material, as follows (Barr's interest in Kimberley will still be addressed).

Map all intervals on S_1 which satisfy Barr's conditions (Map 2.3). This results in two disjoint intervals, I_1 and I_2, for each of the locations for the heads of *a"*, *b"*, *c"*. Choose, as the head of *a"*, the point in I_1 closest to Kimberley. This forces the sites for the heads of *b"*, and *c"*. The length of all intervals designated I_1 is determined by the length of the shortest interval that satisfies Barr's land-basing criterion; in this case, the New Guinea interval I_1 determines the length of I_1 in Africa and I_1 in Greenland. [The length of I_2 is determined by the I_2 for *c"*].

To complete the problem, find earth-based coordinates for the heads of <u>*a*"</u>, <u>*b*"</u>, <u>*c*"</u>. To do this, use suitable rotation matrices as illustrated in Nystuen's procedure above.

DETERMINATION OF ALL OTHER TETRAHEDRA WITH ONE VERTEX AT EASTER ISLAND

There are an infinite number of tetrahedra $0_0 0_1 0_2 0_3$ that could have satisfied Barr's conditions; however, the positions for their vertices are within tightly bounded intervals leading to tetrahedra 'close' to each other within the sphere. Let 0_0 denote Easter Island. Use an equilateral triangle within the circle on Map 2.3 to determine intervals for positions of the vertices 0_1, 0_2, 0_3. Since there are an infinite number of points in any single interval, since the choice of any single point, A_1, within such an interval forces the positions of the remaining two points, A_2, A_3, and since the choice of a different single point, A_1', would force two different points A_2', A_3', it follows that there are an infinite number of tetrahedra available that satisfy Barr's conditions. (Sample positions for the equilateral triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ are exhibited in Map 2.3. The reader wishing to verify this, should cut out an equilateral triangle this size and rotate it within this circle). Each of these triangles would serve as a base for a tetrahedron with apex at Easter Island.

PROBLEMS IN LOCATIONAL PRECISION ARISING FROM THE ASSUMED SPHERICITY OF THE EARTH

Replication of Nystuen's procedure to locate positions on the earth can lead to varying amounts of accumulated error based on

1) problems associated with assuming the earth to be a perfect sphere, and
2) problems associated with assuming all locations to

Map 2.3

AZIMUTHAL EQUIDISTANT PROJECTION

be at the same level on the earth's surface. Consequently, use of this tool as a locational guide should take into account

1) the amount the curvature of the earth at a given location deviates locally from the curvature of a sphere, and
2) the amount the elevation at a given location deviates above or below global mean sea level if extremely precise locational criteria are necessary.

3: EXTENSION OF BARR'S PROBLEM TO THE SET OF PLATONIC SOLIDS

General procedure for determining the location of circles that contain the vertices of a Platonic solid inscribed in an arbitrary sphere will be developed below. To do this, it will be necessary to determine the dihedral angles of these solids.[18] Means for calculating dihedral angles is available in the mathematics literature; Coxeter's derivation for the measure of the dihedral angle of a Platonic solid, expressed only in terms of the number of edges in a face (denoted p) and the number of edges incident with a vertex of that solid (denoted q), is included in Appendix A.[19] Steps from that derivation used in the material below, will be referenced by bracketed number to appropriate statements in Appendix A.

The material in this Chapter, together with the antipodal landmass map (Terrae Antipodum), will form the foundation for the proofs of Arlinghaus's uniqueness theorems in the next Chapter. As in Chapter 2, the calculations that follow assume the earth to have a radius of 3958.7461 miles; they will be made compatible for use with a standard 12-inch globe, and will be summarized in table form at the end of the chapter.

Both the procedure to be exhibited below, and Nystuen's procedure of Chapter 2, contain error resulting from using approximations in the numerical evaluation of trigonometric

---

[18]A dihedral angle in a polyhedron is an angle between planes containing adjacent faces of the solid. Coxeter, Introduction, p. 150.

[19]The Schläfli symbol for a Platonic solid, $\{p,q\}$, denotes a regular polyhedron with p edges in a face and q edges incident with a vertex. Hence $\{3,3\}$ denotes a tetrahedron, $\{4,3\}$ a cube, $\{3,4\}$ an octahedron, $\{5,3\}$ a dodecahedron, and $\{3,5\}$ an icosahedron.

functions. To give some idea of its extent, we will begin with analysis of the tetrahedron and compare the two answers.

THE TETRAHEDRON: $\{p,q\}=\{3,3\}$

Suppose a tetrahedron is inscribed in a sphere as shown in Figure 3.1.

Find : (i) the radius $r(S_1)$ of the small circle S_1 containing three vertices of one face of the tetrahedron

(ii) the latitude $\phi(S_1)$ of S_1 relative to the equator of this sphere.

(iii) the length of a side of the tetrahedron

(iv) procedure for determining latitude and longitude, relative to the earth, for vertices of the tetrahedron on S_1.

In this case, $p=3$, $q=3$, and the dihedral angle is 70.6°.

(i) Find $r(S_1)=O_0O_2$; refer to Figure 3.1.

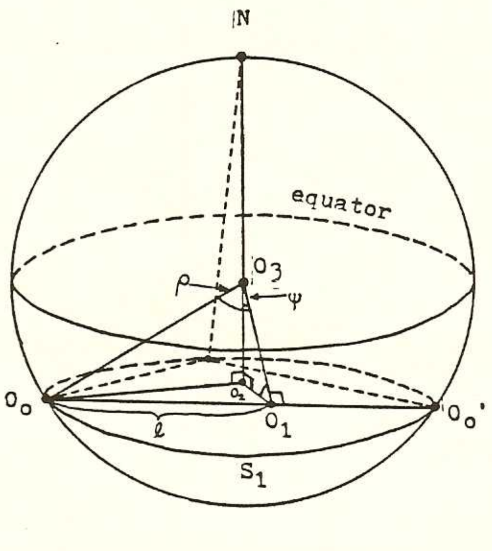


Figure 3.1

By Appendix A, [4],

$$k^2=\sin^2(\tfrac{\pi}{q})-\cos^2(\tfrac{\pi}{p})$$

$$=\sin^2(\tfrac{\pi}{3})-\cos^2(\tfrac{\pi}{3})=\tfrac{1}{2}.$$

Thus $k\cong.7071$. Therefore, $\sin\rho=k \times \csc(\frac{\pi}{3})\cong.8164$, from Appendix A, [5]. Thus $\rho=54.7°$. From [6a], $O_0O_3=l\csc\rho$ and since O_0O_3=radius of earth=3958.7461, it follows, from equating left-hand sides of these equations, that

$l=3958.7461 \times .8164$

$= 3231.9203.$

From Appendix A, [0], $<\psi_C=<O_2O_1O_3$=(dihedral angle)/2=35.3°.

But from [6b], $\cos\psi_c=(O_1O_2)/(O_1O_3)=(O_1O_2)/(l \cot\rho)$. Therefore, $O_1O_2=l \cot\rho\cos\psi_c=1867.5898$. Then, by the Pythagorean Theorem, $(O_0O_2)^2=(O_0O_2)^2 + l^2 = 13{,}933{,}200$. Therefore, $r(S_1)= O_0O_2= 3732.7202$ and the unit is 'miles' since that was the unit used in the measurement of the radius of the sphere.

(ii) Find $\phi(S_1)$; refer to Figure 3.2

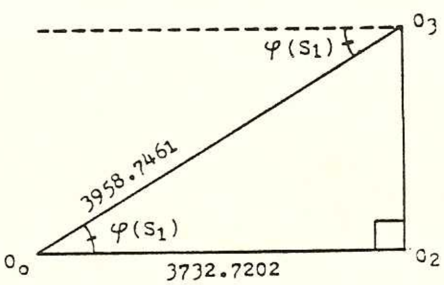


Figure 3.2

Within the right triangle $(O_0O_2O_3)$,

$$\cos(\phi(S_1))=\frac{3732.7202}{3958.7461} =.94290469$$

Therefore, using inverse functions, $\phi(S_1)\cong 19.454776°$ (or $19°27'17"$).

Nystuen's answer of 109.47128° for $<O_0O_3N$ produces a value of $109.47128° -90° = 19.47128°$ (or $19°28'16"$) for $\phi(S_1)$.

These two answers are indistinguishable on the globe, but not on the earth.

(iii) The length of a side of the tetrahedron is $2l=6463.8406$ miles.

The difference between this answer, and that in Chapter 2, is .7647 miles which is negligible for all but very small scale problems.

(iv) Procedure

The following procedure provides a quick, but crude, approximation of latitude and longitude for the vertices of the tetrahedron on S_1. Cut out an annulus, from rigid cardboard, with inner radius 3732.7202 miles, scaled to a 12-inch globe; thus, the inner radius, x, of the annulus is found from the ratio

$$\frac{3732.7202}{3958.7461} = \frac{x}{6} \text{ so } x = 5.657428 \text{ inches.}$$

Place the annulus over the globe so that it follows a small circle on the globe that is parallel to the equator. Stick a pin in the North Pole and join the pin to the annulus via three evenly spaced strings tied to the pin and the annulus. Now remove the pin-annulus-string structure and replace it on the globe with the pin stuck into the point in the Thar Desert that is antipodal to Easter Island. Now sets of points that are simultaneously land-based maybe marked off on the annulus, since it is in the position of S_1, and their earth-coordinates, in latitude and longitude, may be read from the globe.

This procedure is not very accurate; Nystuen's procedure is much more precise though more tedious. For Barr's purposes the added precision was vital; for the theoretical arguments that follow, it is not.

THE CUBE: $\{p,q\}=\{4,3\}$

Suppose a cube is inscribed in a sphere as shown in Figure 3.3. Find:

(i) the radius $r(S_1)$ of the small circle S_1 containing four vertices of one face of the cube.

(ii) the latitude $\phi(S_1)$ of S_1 relative to the equator of this sphere

(iii) the length of a side of the cube.

(iv) procedure for determining latitude and longitude, relative to the earth for the vertices of the cube on S_1.

(i) **Find $r(S_1)=O_0O_2$; refer to Figure 3.3.**

By Appendix A, [4],

$$k^2=\sin^2\frac{\pi}{q}-\cos^2\frac{\pi}{p}= \sin^2\frac{\pi}{3}-\cos^2\frac{\pi}{4}=\frac{1}{4}.$$

Thus, $k=\frac{1}{2}$.

Therefore,

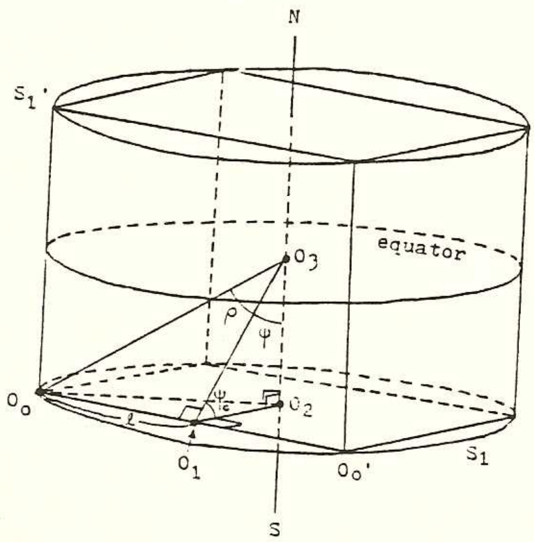


Figure 3.3

$\sin\rho = kx \csc \frac{\pi}{3} \cong .57735027$, from [5]. Thus $\rho = 35.26439°$. From [6a], $O_0O_3 = l \csc\rho$ and since $O_0O_3 = 3958.7461$, it follows that $l = 3958.7461 \times \sin\rho \cong 2285.5831$. From Appendix A, [0], $<\psi_c =$ (dihedral angle)/2 = 45°. But, from [6b], $\cos\psi_c = \dfrac{O_1O_2}{O_1O_3} = \dfrac{O_1O_2}{l\cot\rho}$.

Therefore,

$O_1O_2 = l \cot\rho \cos\psi_c = 2285.5831$. Then, by the Pythagorean Theorem, $(O_0O_2)^2 = (O_1O_2)^2 + l^2 = 10{,}447{,}780$ miles. Therefore, $r(S_1) = O_0O_2 = 3232.3026$ miles.

(ii) Find $\phi(S_1)$; refer to Figure 3.4

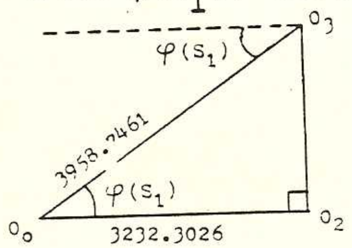


Figure 3.4

Within the right triangle $(O_0O_2O_3)$, $\cos(\phi(s_1)) = \dfrac{3232.3026}{3958.7461} = .81649658$. Therefore, using inverse functions, $\phi(S_1) \cong 35.26439°$

(iii) The length of a side of the cube is $2l$ = 4571.1662 miles (Figure 3.1).

(iv) The inner radius in inches, x, of an annulus suitable for approximating latitude and longitude, on a 12-inch globe, of vertices of the cube on S_1 is given by $\dfrac{3232.3026}{3958.7461} = \dfrac{x}{6}$ so $x = 4.8989794$ inches.

Pass the annulus through Easter Island, O_0, and determine coordinates of three other land-based positions on S_1; do the same for S_1'.

THE OCTAHEDRON: $\{p,q\} = \{3,4\}$

Suppose an octahedron is inscribed in a sphere as in Figure 3.5 Find

(i) the radius $r(S_1)$

(ii) $\phi(S_1)$

(iii) the length of a side

(iv) inner radius of an annulus suitable for determining earth-based positions.

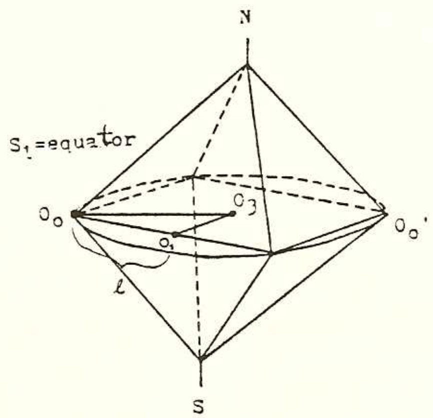


Figure 3.5

(i) Since S_1 is the equator, $r(S_1) = 3958.7461$

(ii) For the same reason, $\phi(S_1) = 0°$

(iii) To find l, first find k as above: $k^2 = \sin^2\frac{\pi}{4} - \cos^2\frac{\pi}{3} = \frac{1}{4}$. Thus $k = \frac{1}{2}$. Then $\sin\rho = k \csc \frac{\pi}{4} \cong .70710678$, so $\rho = 45°$.

Since $O_0O_3 = l \csc \rho$, $l = 2799.2562$, so the length of a side of the octahedron is $2l = 5598.5124$ miles.

(iv) The radius of an annulus suitable for locating vertices of the octahedron on S_1 is 6 inches. Then follow the procedure outlined in step (iv) for the tetrahedron using four evenly-spaced strings.

THE DODECAHEDRON: $\{p,q\} = \{5,3\}$

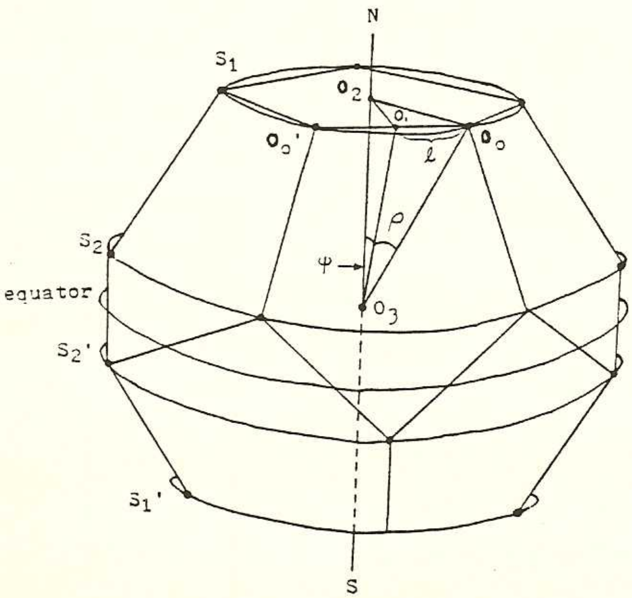


Figure 3.6

Suppose a dodecahedron is inscribed in a sphere as in Figure 3.6. Find

(i) the radius $r(S_1)$

(ii) $\phi(S_1)$

(iii) the length, $2l$, of a side of the dodecahedron.

(iv) the inner radius of an annulus for locating earth-coordinates for vertices on S_1.

(v) the radius $r(S_2)$

(vi) $\phi(S_2)$

(vii) inner radius of an annulus for locating earth-coordinates for vertices on S_2.

(i) Find $r(S_1) = O_0O_2$; refer to Figure 3.6. By Appendix A, [4], $k^2 = \sin^2\frac{2\pi}{q} - \cos^2\frac{2\pi}{p} = \sin^2\frac{2\pi}{3} - \cos^2\frac{2\pi}{5} = .0954915$. Thus $k = .30901699$. Therefore, $\sin\rho = k\csc\frac{\pi}{3}$ $= .35682208$ from[5]. Thus $\rho = 20.905157°$. From [6a], $O_0O_3 = l\csc\rho$ and since $O_0O_3 = 3958.7461$, it follows that $l = 3958.7461 \times \sin\rho = 1412.568$. From [0], $<\psi_c =$ $<O_2O_1O_3 = \frac{\text{dihedral angle}}{2} = 58.3°$. But, from [6b], $\cos\psi_c = \frac{O_1O_2}{O_1O_3} = \frac{O_1O_2}{l\cot\rho}$. Therefore, $O_1O_2 =$ $l\cot\rho\cos\psi_c = 1943.2736$. Then, by the Pythagorean Theorem, $(O_0O_2)^2 = (O_1O_2)^2 + l^2 = 5771660.8$. Therefore $r(S_1) = O_0O_2 = 2402.4281$ miles.

(ii) Find $\phi(S_1)$; refer to Figure 3.7.

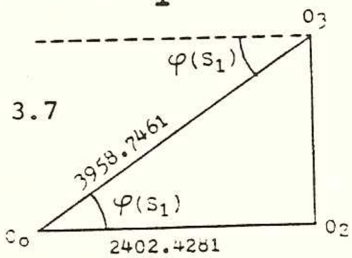


Figure 3.7

Within the right triangle $(O_0O_2O_3)$, $\cos(\phi(S_1)) = \dfrac{2402.4281}{3958.7461} = .60686592$. Therefore, using inverse functions, $\phi(S_1) = 52.636768°$.

(iii) The length of a side of the dodecahedron is $2l$=2825.136 miles.

(iv) The inner radius in inches, x, of an annulus suitable for approximating latitude and longitude, on a 12-inch globe, of vertices of the dodecahedron on S_1 is given by $\dfrac{2402.4281}{3958.7461} = \dfrac{x}{6}$ so x = 3.64119552 inches. Pass the annulus through Easter Island, at O_0, and determine coordinates for four other land-based positions on $S_1(S_1')$.

(v) Find $r(S_2) = O_0O_2'$; refer to Figure 3.8.

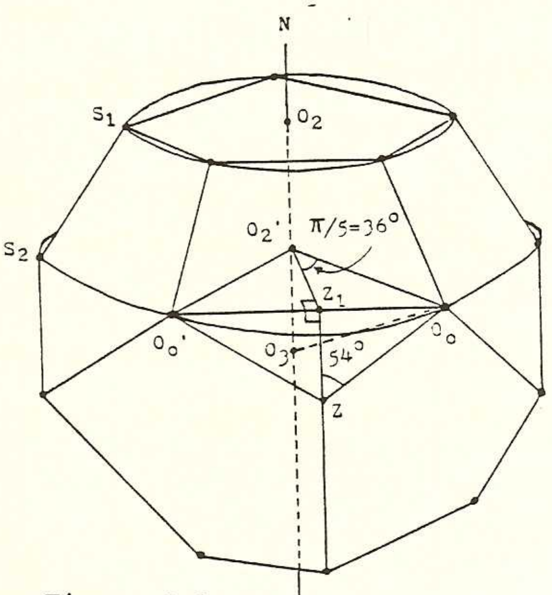


Figure 3.8

From (i) above, l = 1412.568. To find O_0O_2', first find O_0z'. Using triangle (O_0zz'), $\sin 54° = \dfrac{O_0z'}{2l}$. Thus $O_0z' = 2l \sin 54°$ = 2285.583. Using triangle $(O_0O_2'z')$, $\sin 36° = \dfrac{O_0z'}{O_0O_2'}$. Thus $r(S_2) = O_0O_2' = \dfrac{O_0z'}{\sin 36°}$ = 3888.4661 miles.

(vi) Find $\phi(S_2)$; refer to Figure 3.9.

Within the right triangle $(O_0O_2'O_3)$, $\cos(\phi(S_2)) = \dfrac{3888.4661}{3958.7461} = .98224691$. Therefore using inverse functions, $\phi(S_2) = 10.812329°$.

(vii) The inner radius in inches, x, of an annulus

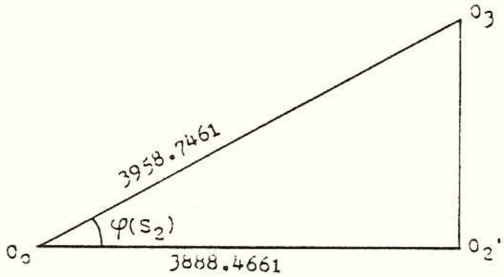


Figure 3.9

suitable for approximating latitude and longitude, on a 12-inch globe, of vertices of the dodecahedron on S_2, is given by $\frac{3888.4661}{3958.7461} = \frac{x}{6}$ so $x = 5.8934814$ inches. Pass the annulus through Easter Island, at O_0, and determine coordinates for four other land-based positions on $S_2(S_2')$.

THE ICOSAHEDRON: $\{p,q\} = \{3,5\}$

Suppose an icosahedron is inscribed in a sphere as in Figure 3.10. Find

(i) the radius $r(S_1)$

(ii) $\phi(S_1)$

(iii) the length, $2l$, of a side of the icosahedron

(iv) the inner radius of an annulus for locating earth-coordinates for vertices on S_1.

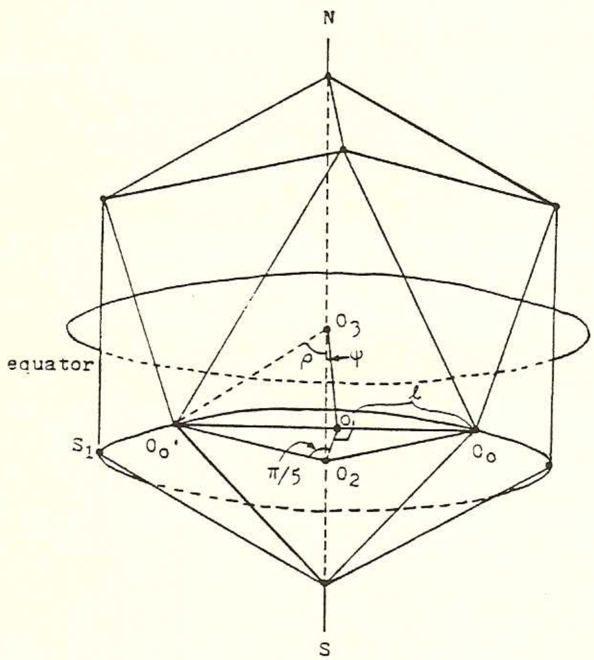


Figure 3.10

(i) Find $r(S_1) = O_0O_2$; refer to Figure 3.10. By [4], $k^2 = \sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p} = \sin^2\frac{\pi}{5} - \cos^2\frac{\pi}{3} = .0954915$. Thus $k = .30901699$. Therefore, $\sin\rho = k\csc\frac{\pi}{5} = .52573111$ from [5]. Thus $\rho = 31.717474°$ (where $\rho = \angle O_0O_3O_1$). From [6a], $O_0O_3 = l\csc\rho$ and since $O_0O_3 = 3958.7461$, it follows that l

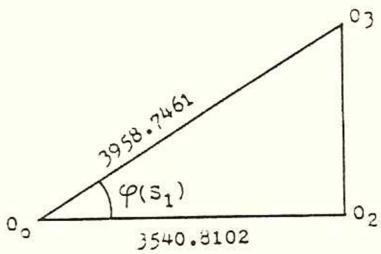


Figure 3.11

$= 3958.7461 \times \sin \rho$ $\cong 2081.236$. Use l to find O_0O_2 from triangle $(O_0O_1O_2)$; $\sin\frac{\pi}{5} = \sin 36° = \frac{l}{O_0O_2}$ so $O_0O_2 = \frac{l}{\sin 36°}$. Therefore $r(S_1) = O_0O_2 = 3540.8102$.

(ii) **Find $\phi(S_1)$; refer to Figure 3.11.** Within the right triangle $(O_0O_2O_3)$, $\cos(\phi(S_1)) = \frac{3540.8102}{3958.7461} = .89442719$. Therefore, using inverse functions, $\phi(S_1) = 26.565051°$.

(iii) **The length of a side of the icosahedron is $2l$ = 4162.472 miles.**

(iv) **The inner radius in inches, x, of an annulus suitable for approximating latitude and longitude, on a 12-inch globe, of vertices of the icosahedron on S_1 is given by** $\frac{3540.8102}{3958.7461} = \frac{x}{6}$ so $x = 5.3665632$ inches. Using this, repeat the procedure in Step (iv) for the tetrahedron, using five, instead of three, evenly-spaced strings.

The table that follows (Table 3.1) summarizes the computations of this chapter, and includes (in addition) a section that contains values for the same measures scaled to the azimuthal projection of Chapter 2. The latter section will be of use, in addition to the others, in Chapter 4.

Table 3.1
Measurements Associated with Platonic Solids

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
$\phi(S_1)$ (degrees) [$\phi(S_2)$]	19.454776°	35.26439°	0°	S_1 | 52.636768° S_2 | 10.812329°	26.565051°
$r(S_1)$ on the globe (inches) [$r(S_2)$]	5.657428"	4.8989794"	6.0"	S_1 | 3.64119552" S_2 | 5.8934814"	5.3666895"
$r(S_1)$ on the earth (miles) or $r(S_2)$	3732.7202mi.	3232.3026mi.	3958.7461mi.	S_1 | 2402.4281mi. S_2 | 3888.4661mi.	3540.8102mi.
length of side, $2l$, in earth (miles)	6463.8406mi.	4571.1662mi.	5598.5124mi.	S_1 | 2825.136mi. S_2 | 2825.136mi.	4162.472mi.
$r(S_1)$ on the map (inches) or $r(S_2)$	2.9"	2.5"	3.07"	S_1 | 1.86" S_2 | 3.02"	2.75"
length of side, $2l$, on the map (inches)	5.02"	3.5"	4.34"	2.18"	3.2"

4: UNIQUENESS QUESTIONS

Beyond utilizing the Platonic association of 'tetrahehdron' with 'fire' to claim some sort of unique natural correspondence of this tetrahedral sculpture within the earth to the fiery core within the earth, it seems appropriate to ask

(a) about natural generalizations of Barr's problem

(b) if the tetrahedron is the only Platonic solid that could have fulfilled Barr's conditions

(i) that all vertices be land-based

(ii) that one of the vertices be at Easter Island.

GENERALIZATION OF BARR'S PROBLEM

Condition (i) is easy to generalize; all vertices must lie on land, no matter how many there are, in order to retain the character of Barr's original problem. Condition (ii), however, leads to ambiguity in generalization, as will be seen.

Lemma 1: The tetrahedron is the only Platonic solid in which the points antipodal to each of its vertices are not also vertices of that solid (e.g., Easter Island is a vertex of the tetrahedron but its antipodal point in the Thar Desert is not).

Proof: Proof is by examination of each Platonic solid (Figure 4.1).

On the sphere P_0 and P_0', P_1 and P_1', P_2 and P_2', P_3 and P_3' are antipodal, but only P_0, P_1, P_2, P_3 lie on the tetrahedron; for all other Platonic solids P_0 and P_0' are antipodal on the sphere and are both vertices of the Platonic solid as well (and so also for for P_1, P_1', etc.). Q.E.D.

Each Platonic solid exhibits bilateral symmetry with respect to two axes, A_1 and A_2, as follows:

A_1: with respect to an axis of the sphere which passes through a vertex of the solid.

A_2: with respect to an axis of the sphere which passes through the center of a face of the solid.

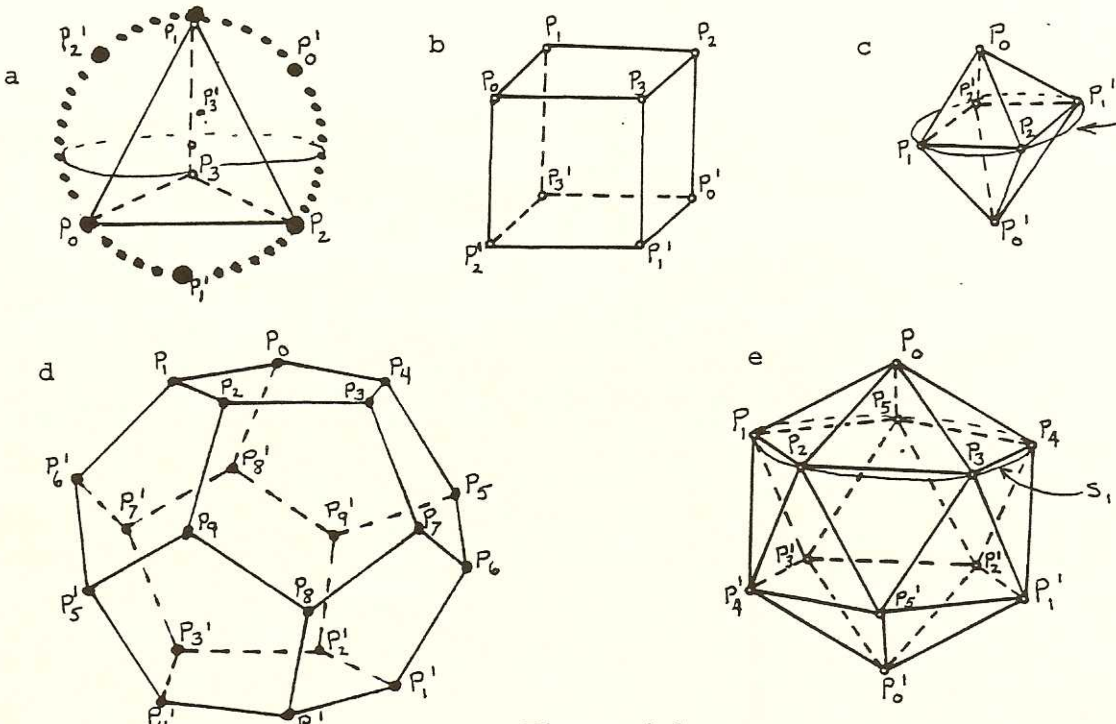


Figure 4.1

Lemma 2: The tetrahedron is the only Platonic solid in which the axes of symmetry, A_1 and A_2, coincide.

Proof: This is a direct consequence of Lemma 1 (Figure 4.2) and reflects the fact that the graph of the tetrahedron is self-dual.[20] Q.E.D.

Using Lemma 2, generalize condition (ii) which states "one of the vertices is at Easter Island" to "the axis of the solid passes through Easter Island." There are accordingly two distinct positions, for each of the remaining Platonic solids, relative to Easter Island, depending on whether A_1

---

[20]Harary, Graph Theory, (Reading, Mass.:Addison-Wesley, 1969)

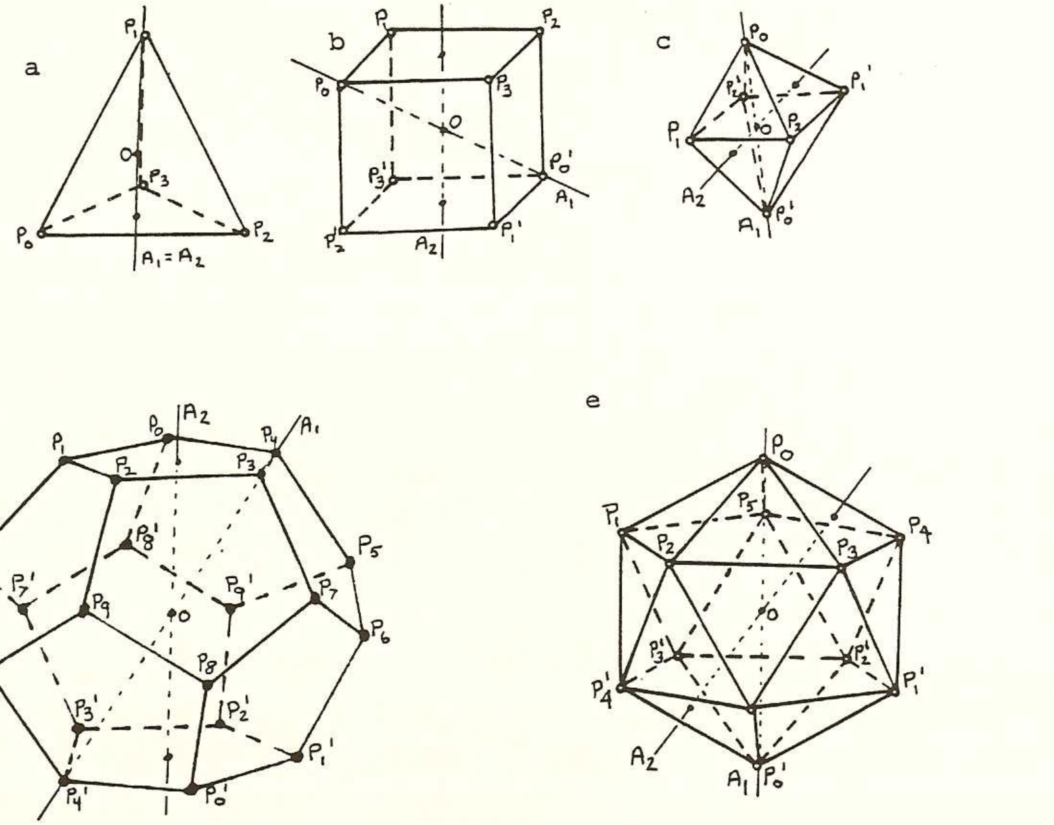


Figure 4.2

or A_2 passes through it. Since Barr's problem is stated for the simplest, self-dual, case, its full structure is not unfolded and hence there is ambiguity in generalization to cases whose graphical duals are not themselves (the cube and octahedron are graphical duals, as are the dodecahedron and icosahedron)[21]

UNIQUENESS THEOREMS

Uniqueness Theorem 1 (Arlinghaus)

Barr's choice of a tetrahedron for his sculpture, relative to the axis A_1, and the land basing criterion (i), is unique within the set of Platonic solids.

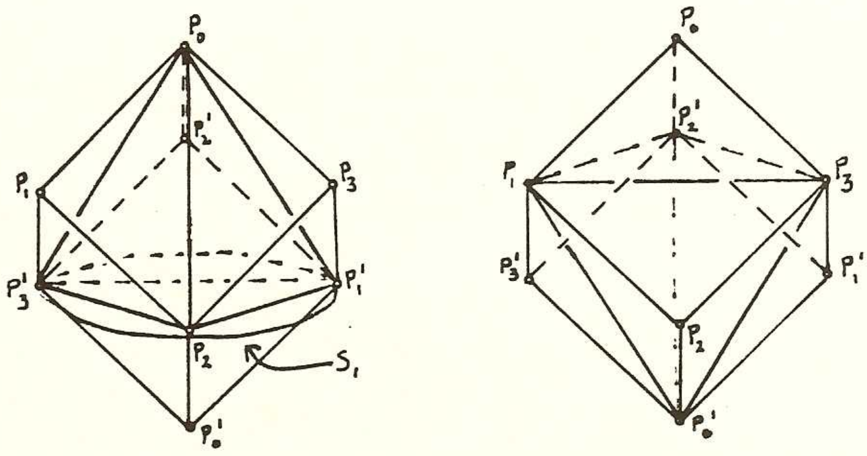
Proof: Proof will be by exhaustion of all

---

[21] Magnus J. Wenninger, Spherical Tesselations, Cambridge University Press, 1979, and Coxeter, Introduction to Geometry and Harary, Graph Theory.

possibilities, and will rely on the earlier construction for the tetrahedron.

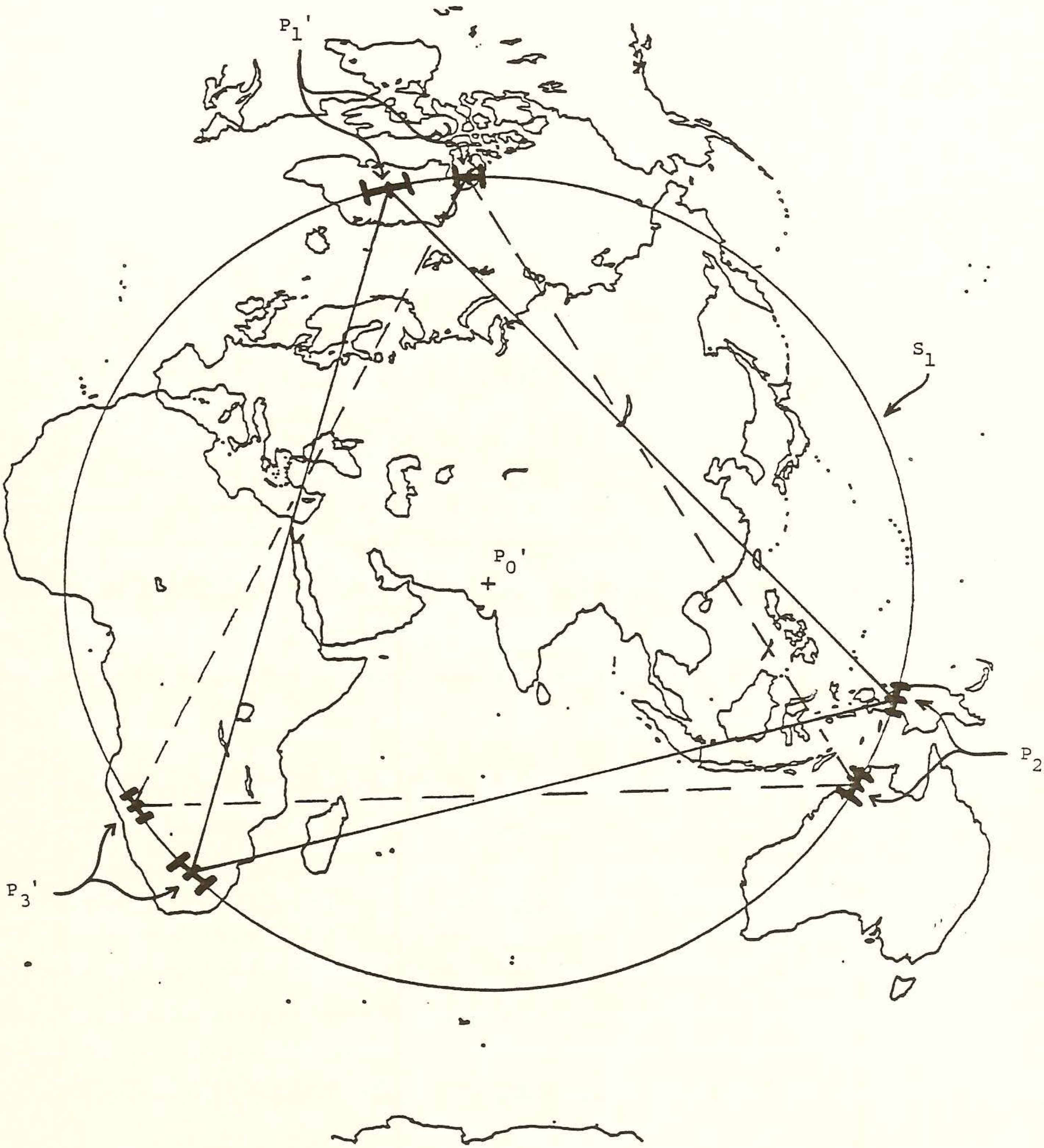
Cube: Within a cube with vertices P_0, P_1, P_2, P_3 on one face (top) and antipodal vertices P_0', P_1', P_2', P_3' on the opposite face (bottom), the points P_0, P_1', P_2, P_3' will form one regular tetrahedron while the vertices P_0', P_1, P_2', P_3 will form another (Figure 4.3, a, b).



a b

Figure 4.3

Thus $P_1'P_2P_3'$ will lie on a small circle at 19°25'50" South Latitude on a unit sphere where P_0 is the north pole. In the case when P_0 is taken to be Easter Island, the axis A_1 is P_0P_0', and the intervals of possible location, satisfying condition (i), are near the Arctic for P_1, in New Guinea/ Australia for P_2, and in South Africa for P_3(Map 4.1). So four of the eight vertices may be based on land; use the antipodal land mass map to determine if the points antipodal to the ones determined are also land-based. The New Guinea/ Australia intervals that contain P_2 have intervals antipodal to them that are not land-based (as seen from the map of antipodal land masses). Thus P_2' cannot be land-based, and so there is no cube that can be inscribed in the earth with one vertex at Easter Island.



Map 4.1

AZIMUTHAL EQUIDISTANT PROJECTION

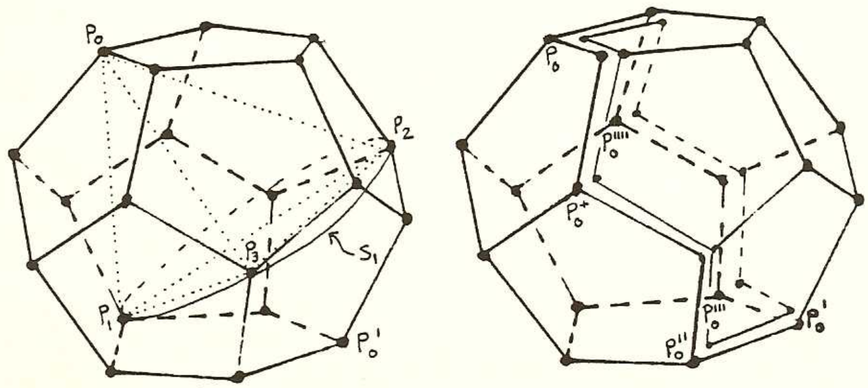
Octahedron

Using Figure 4.1 (c), the equatorial circle, S_1 must contain the four vertices P_1, P_1', P_2, P_2' when P_0 is Easter Island, and P_0P_0' is the axis A_1 (Figure 4.2(c)). The radius of S_1 is 3.0755302 on Map 4.2. Since S_1 is a great circle in this case, points that are diametrically opposed on S_1 are antipodal on the sphere. Taking advantage of this, Map 4.2 can be used to try to determine positions for P_1, P_2', P_1', P_2 which are all land based as follows. Examine all possible positions for P_1 that are land-based, by rotating it along S_1 until P_2' is reached, (clockwise). Land-based intervals of candidates for P_1 are the intervals $[C',B']$, and $[A',P_2']$. Intervals for candidates for P_1', antipodal to P_1, may be found by determining intervals $[C,B]$ and $[A,P_2]$, diametrically opposed to those for P_1. Within the intervals $[C,B]$ and $[A,P_2]$, only the points in the interval $[D,E]$ are land-based; thus P_1' must fall in $[D,E]$. This narrows the set of candidates for P_1 to those points in the interval $[D',E']$. Then rotating $[D,E]$ through 90° clockwise and counterclockwise produces the intervals $[D''',E''']$ and $[D'',E'']$ as intervals of points that are candidates for the location of P_2 and P_2' respectively. Since no points in $[D'',E'']$ or in $[D''',E''']$ are land-based, no octahedron with symmetry with respect to A_1 and one vertex at Easter Island can be inscribed in the sphere.

Dodecahedron

Within a dodecahedron (with 20 vertices, Figure 4.1(d)) a sequence of five positions of a tetrahedron ($P_0P_1P_2P_3$) may be used to locate the vertices by moving the positions of P_0 as follows (Figure 4.4a). A dodecahedron is formed from two bowls with scalloped edges; let P_0 occupy each 'low' point in the edge of the left bowl (as P_0^+, P_0'', P_0''', P_0''''

successively) (Figure 4.4b). The associated tetrahedron will cover the remaining vertices (as P_1 covers one vertex in the bottom of the left bowl; P_2 covers one vertex in the bottom of the right bowl; and P_3 covers one 'high' point on the edge of the bowl (Figures 4.4 a and b).



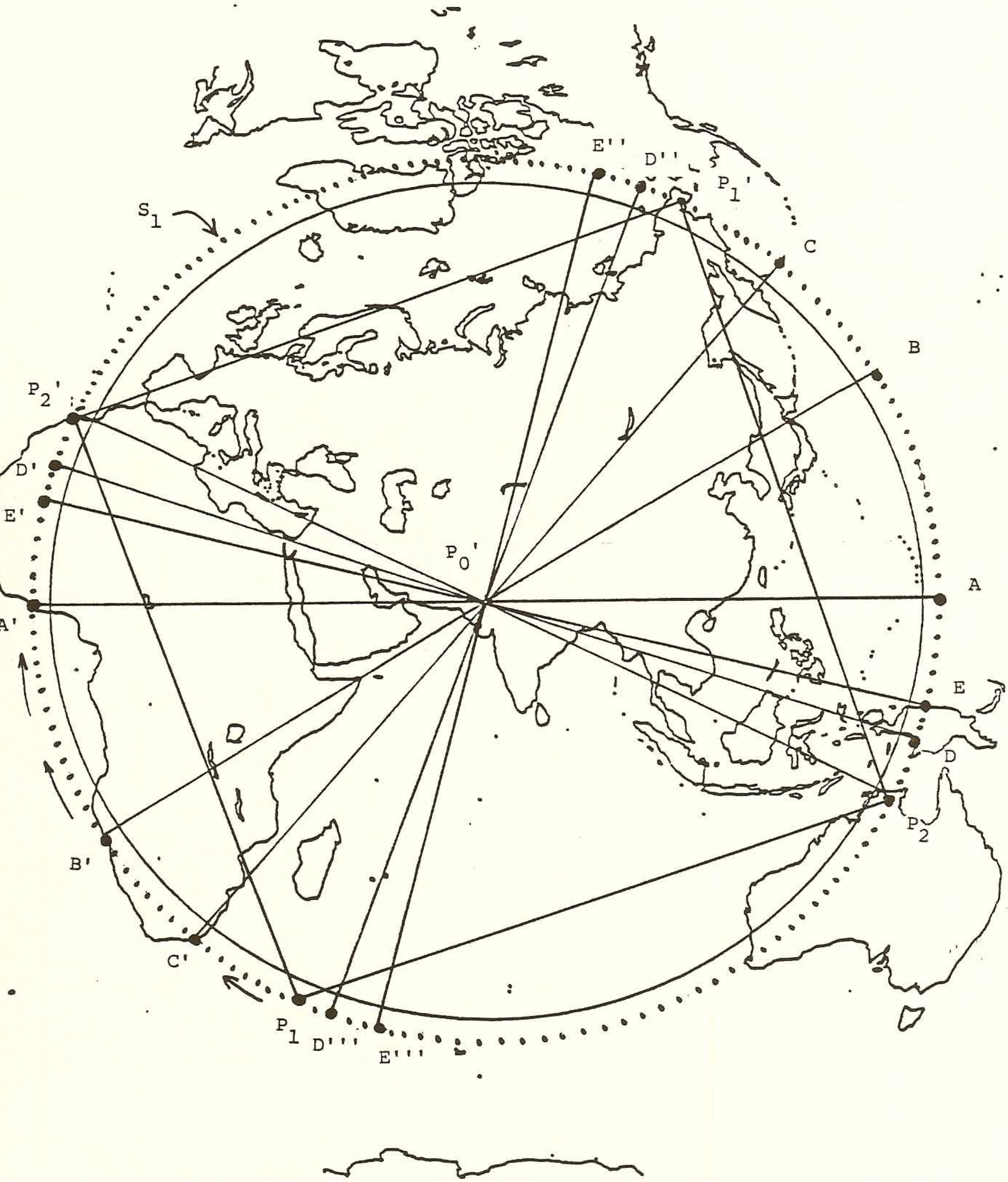
a b

Figure 4.4

Place the dodecahedron in the globe so that P_0 is coincident with Easter Island, and P_0' is coincident with the point antipodal to Easter Island in the Thar Desert, on axis A_1. The points P_1, P_2, P_3, forming the base of a tetrahedron with apex P_0, lie on small circle S_1. To base these vertices on land, P_1 must be in the intervals near the Arctic, P_2 in the New Guinea/Australia intervals, and P_3 in the South Africa intervals (Map 4.3). But since the points P_1', P_2', P_3', antipodal to these, also lie on the solid (by Lemma 1), they must be land-based (condition (i)). Using the map of antipodal land masses, it is clear that none of the points antipodal to the New Guinea/Australia intervals are land-based, and so P_2' cannot be land-based. Thus a dodecahedron cannot be inscribed in the earth under the given conditions.

Icosahedron

Using Map 4.4, the small circle S_1 (Figure 4.1e) must contain five vertices P_1, P_2, P_3, P_4, P_5 of the icosahedron with P_0 at Easter Island and P_0P_0' determining axis A_1. An inscribed pentagon rotated in S_1 cannot produce any positions for P_1, P_2, P_3, P_4, P_5 that are simultaneously



Map 4.2

AZIMUTHAL EQUIDISTANT PROJECTION

land-based since one end of one edge (P_1P_2 in Map 4.4) necessarily falls in the Indian Ocean. Thus no land-based icosahedron with one vertex at Easter Island can be inscribed in the spherical earth. Q.E.D.

<u>Uniqueness Theorem 2</u> (Arlinghaus)

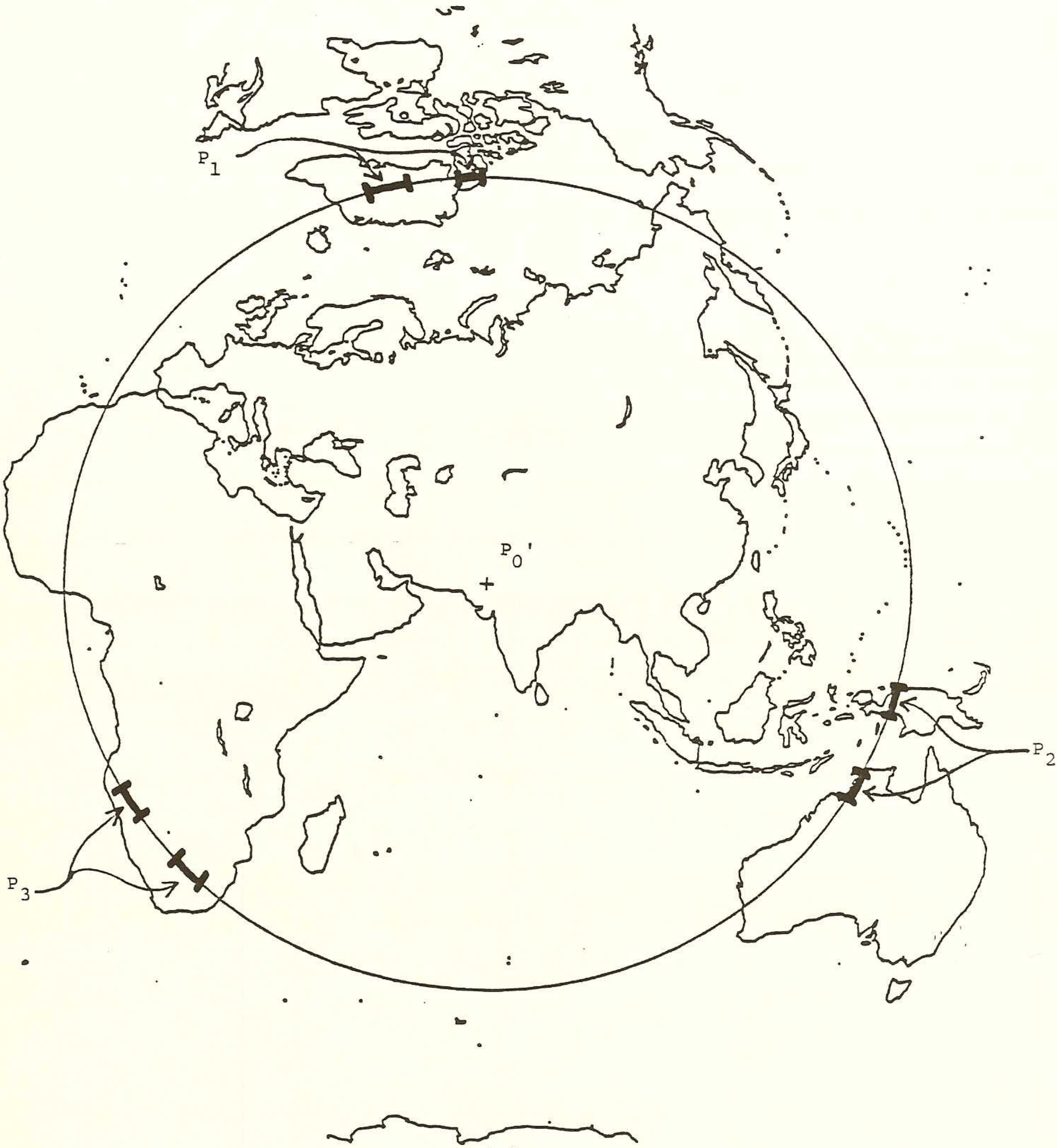
Barr's choice of a tetrahedron for his sculpture, relative to axis A_2 and the land-basing criterion, is unique within the set of Platonic solids.

<u>Proof</u>: Proof will be by exhaustion of all possibilities.

<u>Cube</u>: Using Figure 4.2(a & b), the small circle S_1 must contain the four vertices P_0, P_1, P_2, P_3 of the cube when Easter Island and its antipodal point form an axis in position A_2 relative to the cube (Figure 4.2b). An inscribed square rotated in S_1 cannot produce any positions for P_0, P_1, P_2, P_3 that are simultaneously land-based since one end of one edge (P_0P_3 in Map 4.5) necessarily falls in the Indian Ocean. No cube can be inscribed in the earth satisfying the given conditions.

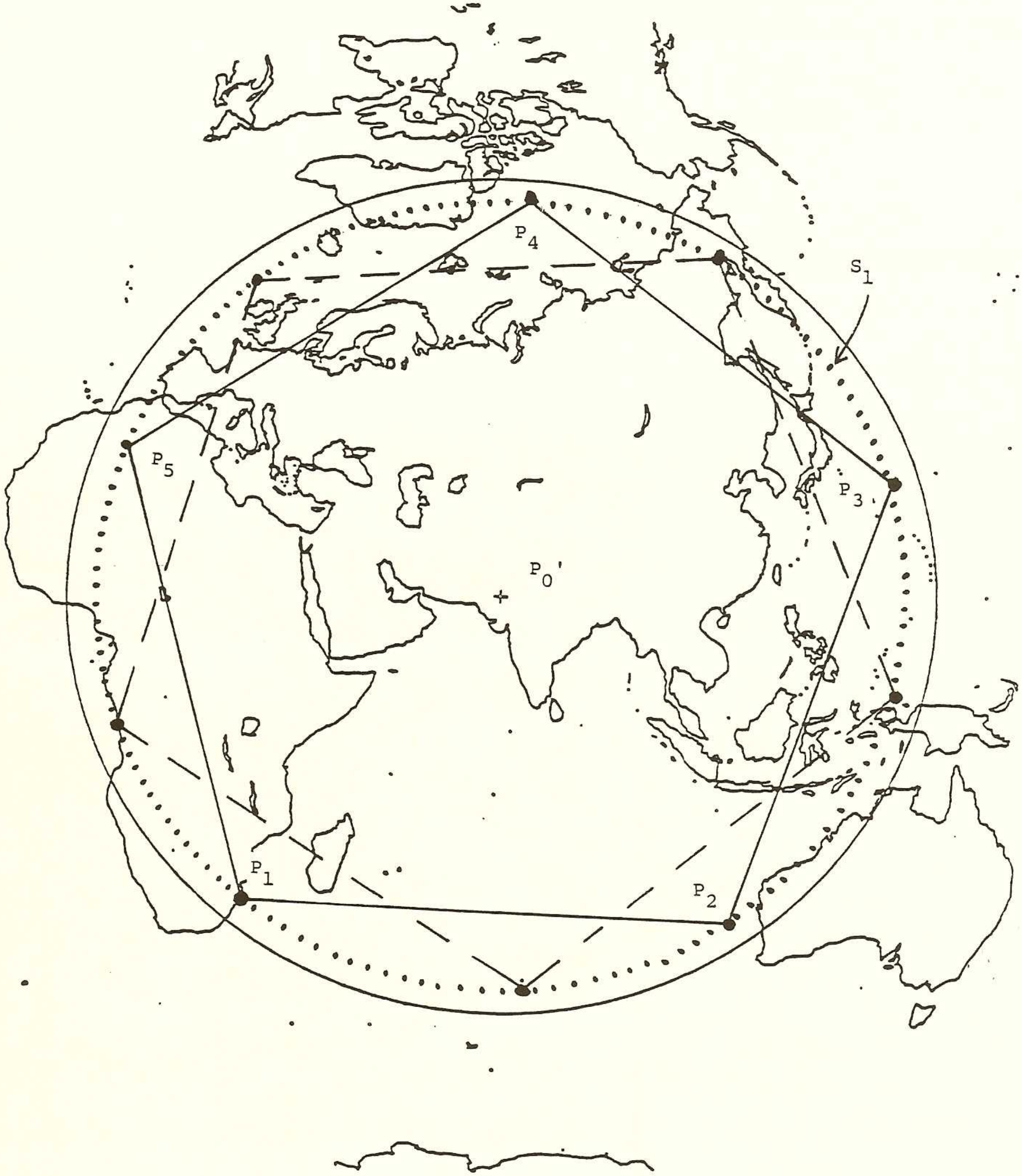
<u>Octahedron</u>

Position an octahedron in the globe in such a way that axis A_2, determined by Easter Island and its antipodal point, pierces faces ($P_0P_1'P_2$) and ($P_0'P_1P_2'$) (Figures 4.1c and 4.2c). Then the azimuthal projection will be used to determine whether simultaneous land-basing of three of these six vertices, (P_0P_1' P_2) lying on small circle S_2, is possible (Figure 4.5).



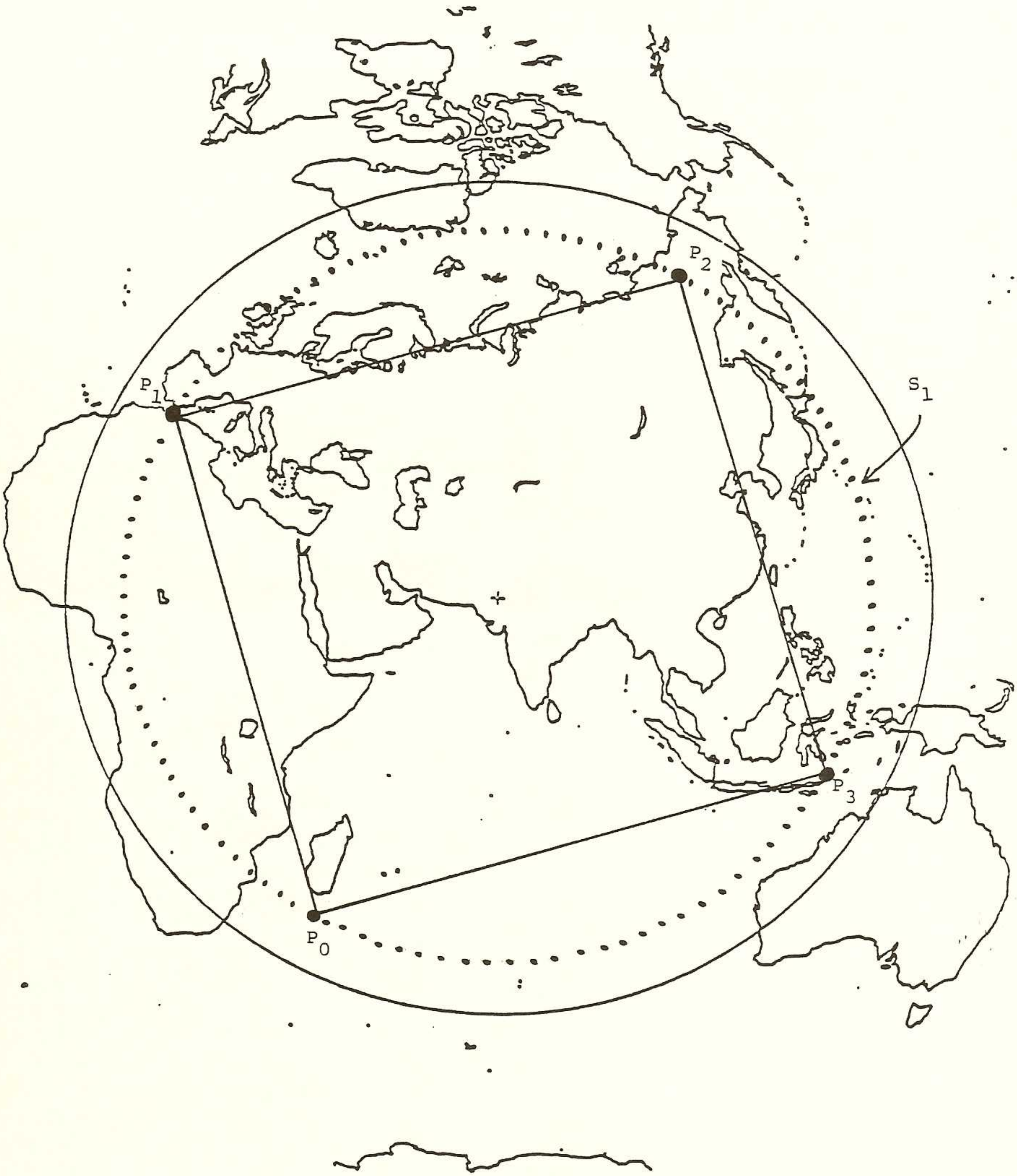
Map 4.3

AZIMUTHAL EQUIDISTANT PROJECTION



Map 4.4

AZIMUTHAL EQUIDISTANT PROJECTION



Map 4.5

AZIMUTHAL EQUIDISTANT PROJECTION

Calculate the Radius r_2 of S_2

In the triangle (P_0QR) (Figure 4.5), the segment P_0Q has length

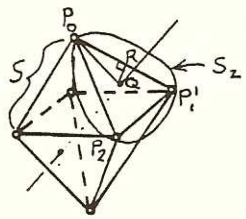


Figure 4.5

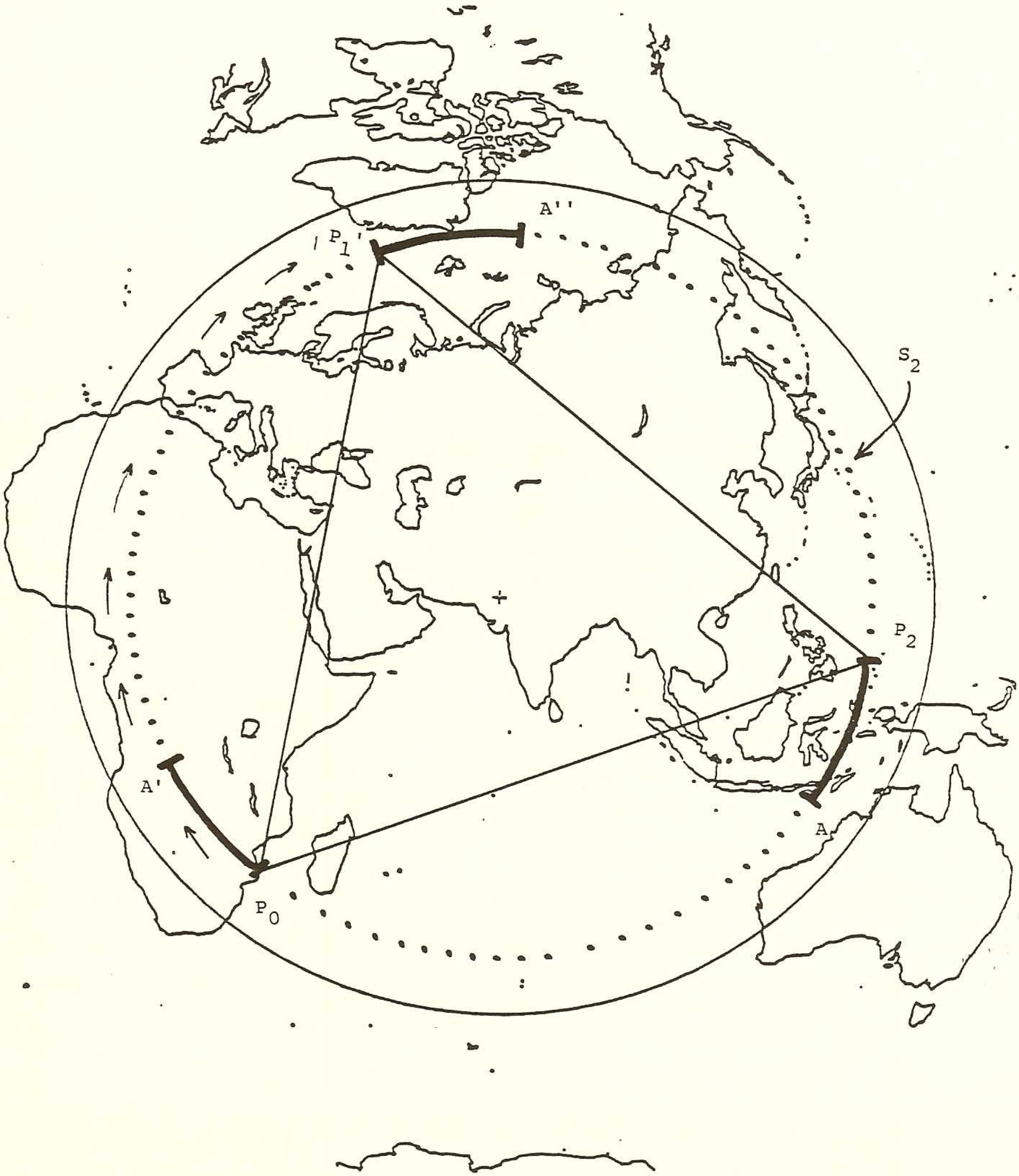
r_2 while P_0R has length $l = \frac{4.3495589}{2} = 2.1747795$ (from Table 3.1). Further $\angle P_0QR = 60°$ since $\Delta P_0QP_1'$ is isosceles and since $\angle P_0QP_1 = 120°$. Thus $\frac{l}{\sqrt{3}} = \frac{r_2}{2}$, so $r_2 = \frac{2l}{\sqrt{3}} = 2.511219$

Determine Land-based Positions for P_0, P_1', P_2

Construct a circle of radius r_2 as shown in Map 4.6. Rotate an inscribed triangle $(P_0P_1'P_2)$ within S_2 so that P_0 slides along S_2 to position P_1'. As P_0 slides along this arc, P_2 is forced clockwise to P_0. The only land-based positions possible for P_2 along this arc are in the interval $[P_2, A]$. This restricts the intervals for possible land-basing of P_0 to $[P_0, A']$, and of P_1' to the interval $[P_1', A'']$. But the interval $[P_1', A'']$ contains no land, hence an octahedron cannot be inscribed in the earth under the given conditions (Map 4.6).

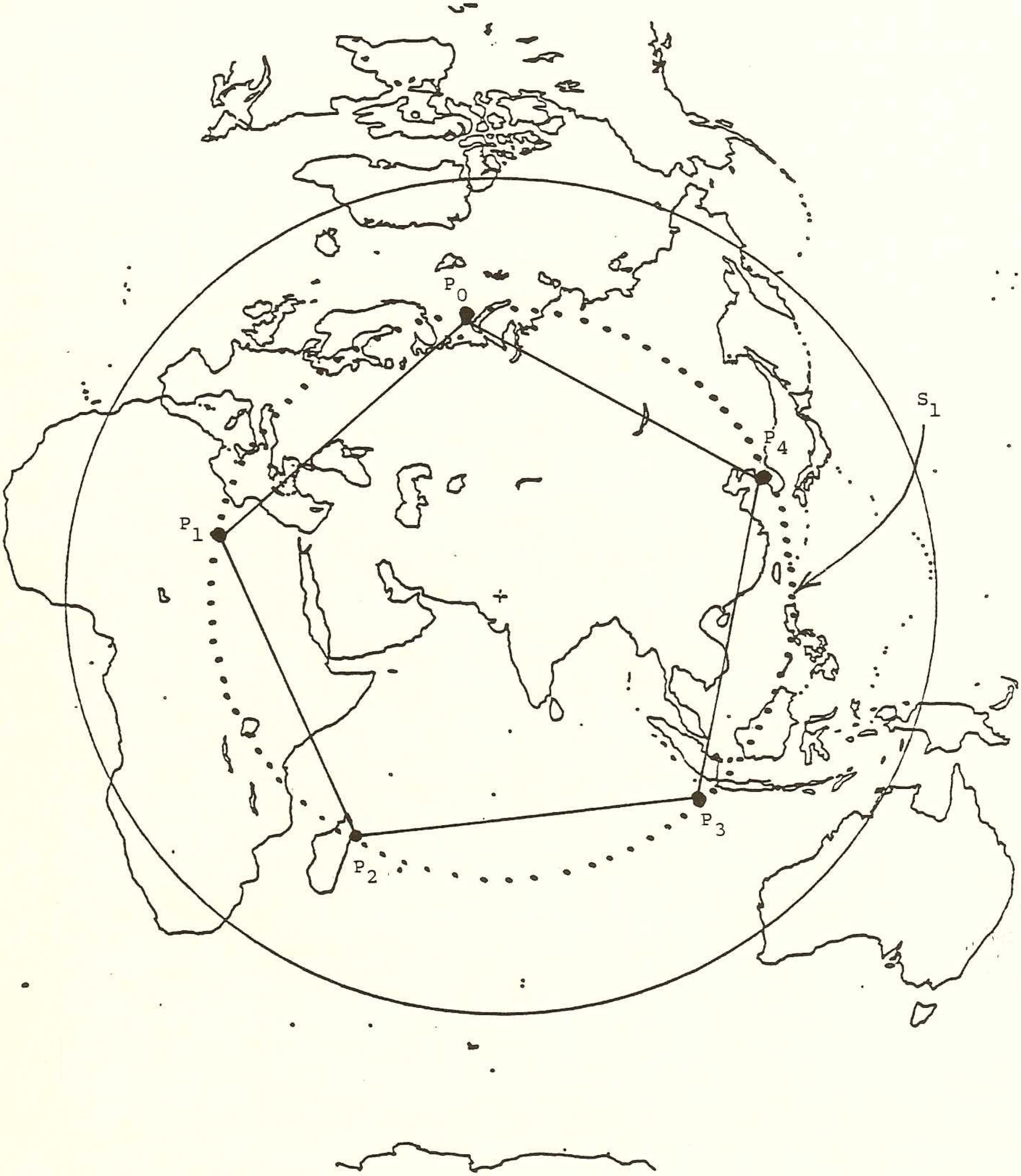
Dodecahedron

Position a dodecahedron within the earth so that axis A_2, through Easter Island and its antipodal point, pierces faces $(P_0P_1P_2P_3P_4)$ and $(P_0'P_1'P_2'P_3'P_4')$ (Figure 4.1d). Small circle S_1 contains P_0, P_1, P_2, P_3, P_4. An inscribed pentagon rotated in S_1 shows no positions that are simultaneously land-based for P_0, P_1, P_2, P_3, P_4 (Map 4.7). One end of one edge (P_2P_3 in Map 4.7) necessarily falls in the Indian Ocean. No dodecahedron can be constructed under the given conditions.



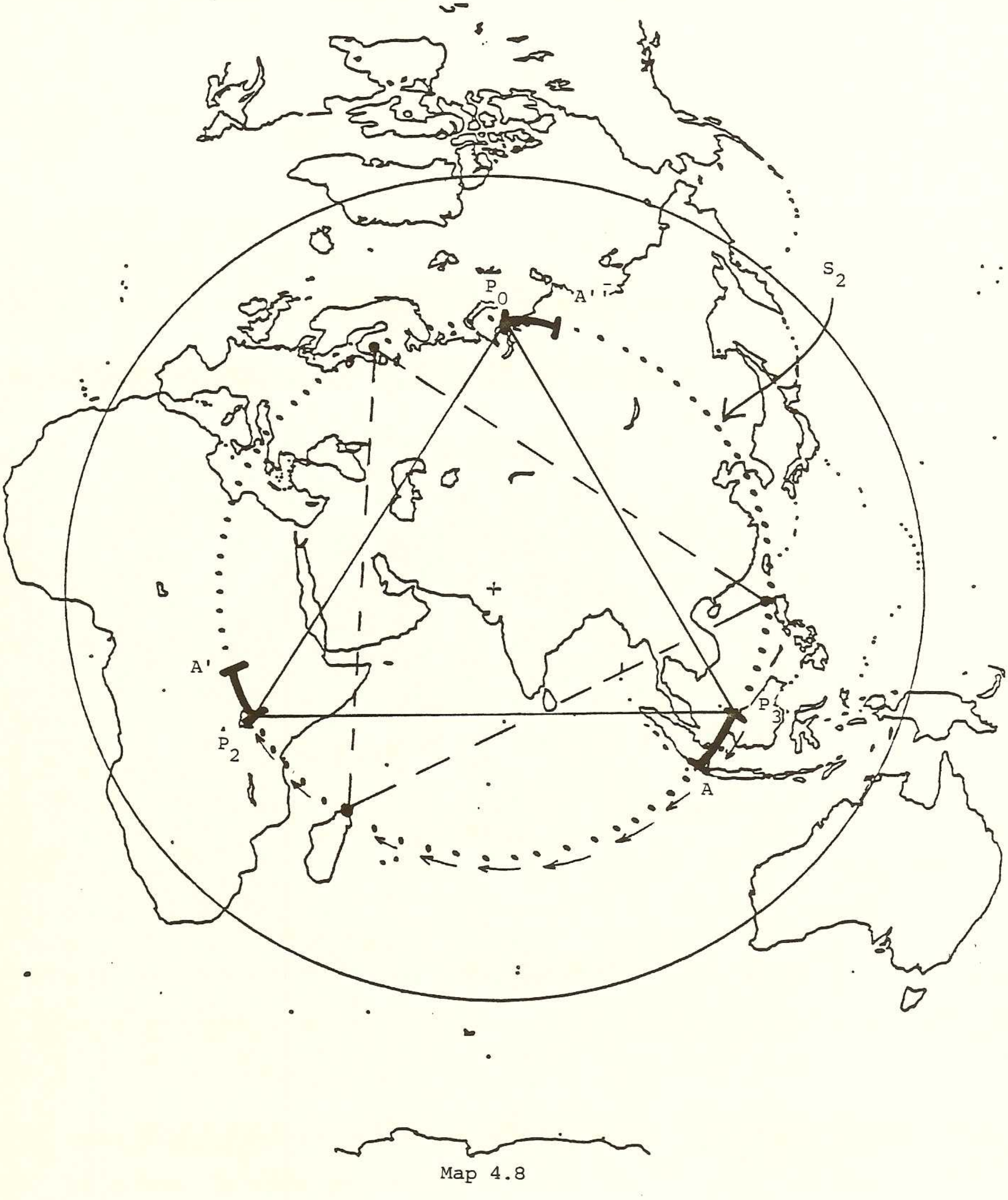
Map 4.6

AZIMUTHAL EQUIDISTANT PROJECTION



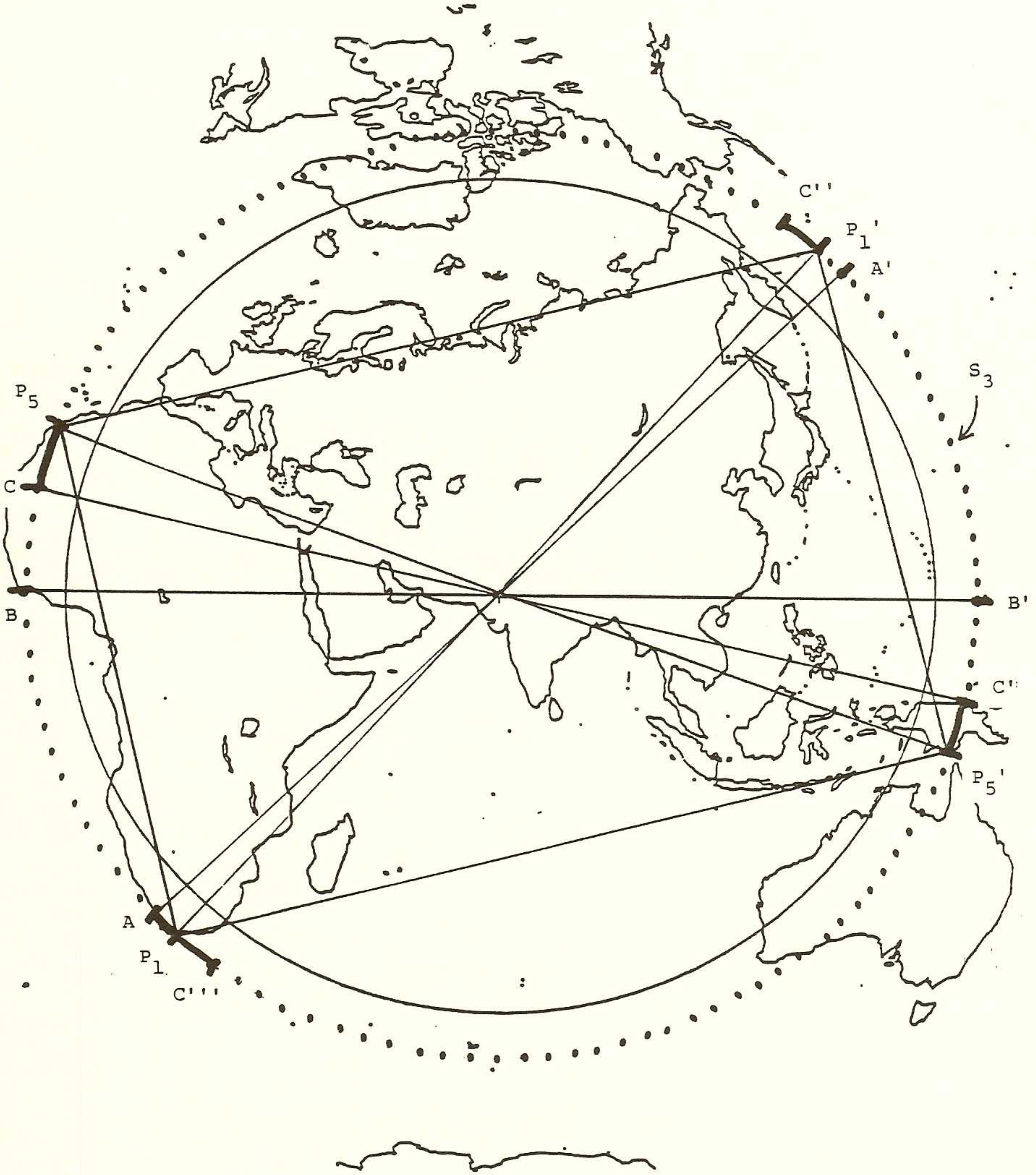
Map 4.7

AZIMUTHAL EQUIDISTANT PROJECTION



Map 4.8

AZIMUTHAL EQUIDISTANT PROJECTION



Map 4.9

AZIMUTHAL EQUIDISTANT PROJECTION

Icosahedron

Position an icosahedron in the earth in such a way that axis A_2, determined by Easter Island and its antipodal point, pierces faces $(P_0P_2P_3)$ and $(P_0P_2'P_3')$ (Figures 4.1e and 4.2e). Then the azimuthal projection will be used to determine whether simultaneous land-basing of the three vertices P_0, P_2, P_3, on circle S_2, is possible (Figure 4.6).

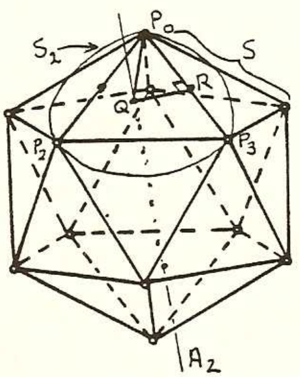


Figure 4.6

Calculate the radius r_2 of S_2

In the triangle (P_0QR) (Figure 4.6), the segment P_0Q has length r_2 while P_0R has length $l = \frac{3.23388}{2} = 1.61694$ (from Table 3.1). Further, $<P_0QR = 60°$ since ΔP_0QP_3 is isosceles and since $<P_0QP_3 = 120°$. Thus, $r_2 = \frac{2l}{\sqrt{3}} = 1.8670815$.

Determine Land-based Positions for P_0, P_2, P_3

Construct a circle of radius r_2 as in Map 4.8. Rotate an inscribed triangle $(P_0P_2P_3)$ within S_2 so that P_3 slides along S_2 to position P_2. The arcs $[P_3,A]$, $[P_2,A']$, and $[P_0,A'']$ give land based intervals suitable for the simultaneous location of P_3, P_2, P_0 respectively (Map 4.8).

Land-based Positions for P_0', P_2', P_3'

Use of the map of antipodal land masses suggests that P_0', P_2', P_3' may also be based on land; P_0' in the Antarctic, P_2' on an island in the Pacific, and P_3' in South America. Thus 6 of the twelve vertices may be land-based. It remains to determine if the rest can be so based.

Determine Land-based Positions for P_1, P_5, P_1', P_5'

The vertices $P_1P_5P_1'P_5'$ lie on the equator of the sphere with axis A_2 (Figures 4.1e and 4.6). That this is so may be seen as follows: any three of these four points determine a circle, S_3. By the symmetry of the solid, the center of this circle S_3 will be on the polar axis of the solid. Also by symmetry the fourth point, antipodal to one of the three already on the circle, S_3 will also lie on S_3. That S_3 is a great circle follows from the fact that the intersection point of P_1P_1' with P_5P_5' is the center of S_3 and is also the center O of the sphere with radius $\bar{R}$ =3.0755302 inches on the azimuthal map. Angle $(P_1P_0P_5)$ will be found; then using the azimuthal projection, simultaneous land-basing of P_1, P_5, P_1', P_5' will be attempted (Figure 4.7)

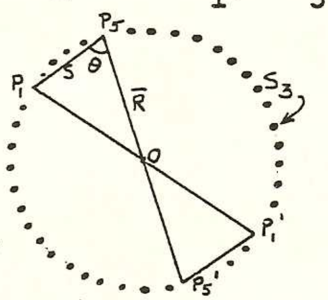


Figure 4.7

Calculate $<P_1OP_5$

Use the Law of Cosines to find θ: $OP_1 = OP_5 = \bar{R} = 3.0755302$, and $P_1P_5 = 2l = 3.23388$ (Table 3.1). Thus $(OP_1)^2 = (P_1P_5)^2 + (OP_5)^2 - 2(P_1P_5)(OP_5)\cos\theta$

$$\text{Therefore } \cos\theta = \frac{(P_1P_5)^2+(OP_5)^2-(OP_1)^2}{2(P_1P_5)(OP_5)} = \frac{P_1P_5}{2(OP_5)} = \frac{2l}{2\bar{R}} = .5257435.$$

Thus $\theta = 58.281691°$ and so $<P_1OP_5 = 180° - 2\theta = 63.436618°$.

Then, using a rectangle, of approximate dimensions, rotated within S_3 determine land-based positions for P_1. Slide P_1 along S_3 to position P_5. As with the octahedron, use the fact that S_3 is a great circle and so is centered on O (Map 4.9). The intervals $[P_1,A]$ and $[B,P_5]$ contain possible locations for P_1. Since S_3 is a great circle, the intervals antipodal to these, $[P_1',A']$ and $[B',P_5']$, contain the only possible locations for P_1'. Within those intervals, only $[P_5',C']$ contains land-based locations, so the set of candidates for the location of P_1 is narrowed to $[C,P_5]$. This forces P_5 to be located in $[P_1',C'']$ and P_5' to be located in $[P_1,C''']$. But $[P_1,C''']$ contains no land.

Thus an icosahedron cannot be inscribed in the spherical earth under the given conditions. (Map 4.9) Q.E.D.

The proofs of theorems 1 and 2 suggest the following algorithm for earth-basing of Platonic solids.

Corollary: Algorithm for basing an inscribed Platonic solid in earth-coordinates:

1) Choose one location on which to center an azimuthal map (so far only the point antipodal to Easter Island has been used as this location)

2) Use this map to show the content of one hemisphere of the globe.

3) Using a circle (or circles) of approximate radius and position, determine intervals of possible locations for land-based positions for vertices of the solid.

4) Using the antipodal landmass map, determine intervals of possible land-based locations for vertices antipodal to those in step (3).

5) If steps (3) and (4) produce locations which are all land-based, find the earth coordinates of these locations
   a) using Nystuen's procedure, for a relatively precise answer.
   b) using coordinates read from the globe, via an annulus of proper radius, for an approximate answer.

APPENDIX A: SOME SOLID GEOMETRY

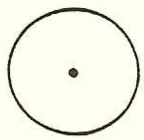
1: SOLIDS IN A SPHERE[22]

The tetrahedron may be considered as one element in a series of objects defined by sets of vectors issuing from the origin O of the coordinate system (and center of a sphere) such that the heads of the vectors are on the surface of the sphere and where each vector is as 'far' from the others as possible. This results in distributing the heads of the vectors across the surface of the sphere; the pattern of linking these vertices, to enclose a volume inscribed in the sphere, produces a variety of solids. The simplest cases are shown below; the Platonic solids will emerge as a subset of several sequences of solids. Other solids that are not Platonic solids, but that are generated below, will be referred to in A.3.

Let *n* denote the number of vectors issuing from O.

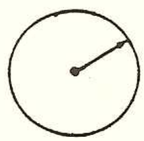
Figure A.1

<u>*n*=0</u>: The trivial case of the sphere and its center, O, is generated



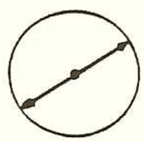
O (a)

<u>*n*=1</u>: A single vector emerges from the center, O.



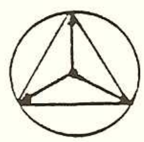
1 (b)

<u>*n*=2</u>: Vectors issuing from O have heads at antipodal points.



2 (c)

<u>*n*=3</u>: Three vectors issuing from O determine a plane whose intersection with the sphere is a great circle containing an inscribed equilateral triangle

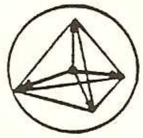


3 (d)

---

[22]From Nystuen, "Notes," p. 2., expanded form.

n=4: Four vectors issuing from O generate a volume which is tetrahedral



(e)

For larger values of *n*, the spacing of points is difficult to visualize, but when *n*=6 an octahedron is generated, when *n*=8 a cube emerges, when *n*=12 an icosahedron is generated, and when *n*=20 a dodecahedron arises. Whether or not the solids generated by this process are unique representations for a given value of *n* is beyond the scope of this work, as are existence conditions for higher values of *n*.

A related (but not identical) procedure, when applied to the hemisphere, produces a set of *p*-gonal pyramids (a pyramid with apex at the pole and base a polygon with *p* sides inscribed in the equatorial plane. (Figure A.2 shows vertices of a 6-gonal pyramid)). Thus an Egyptian pyramid is a 4-gonal pyramid (a half-octahedron). Gluing two of these hemispheres together, along the base of the *p*-gonal pyramid, produces a set of *p*-gonal crystals (for a given value of *p*, the equatorial cross-section of the crystal is a regular polygon of *p* sides). The octahedron is a 4-gonal crystal (Figure A.3).

Use of *p*-gonal pyramids shows that the octahedron, the cube, the icosahedron, and the dodecahedron are members of the first sequence of solids in an entire sphere for *n*=6, *n*=8, *n*=12, and *n*=20 respectively. For, the dodecahedron inscribed in sphere centered at O may be decomposed into 10 distinct 5-gonal congruent pyramids that, pairwise, share a common face. Join O to the vertices of one face, P_0, P_1, P_2, P_3, P_4, producing a 5-gonal pyramid. Since the dodecahedron is regular, all such 5-gonal pyramids are identical; there are 10 faces and so the desired result follows (Figure A.4).

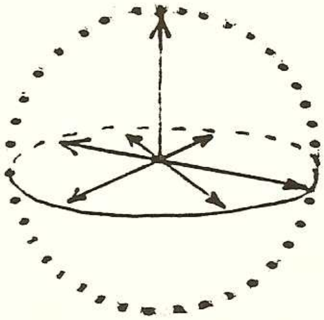


Figure A.2

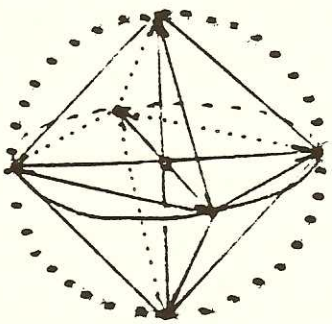


Figure A.3

Similarly, an octahedron may be decomposed into eight 3-gonal congruent pyramids, a cube into six 4-gonal congruent pyramids, and an icosahedron into twenty 3-gonal congruent pyramids. (A tetrahedron may be decomposed into four 3-gonal congruent pyramids).

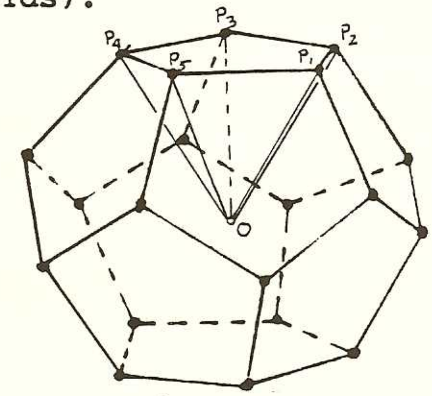


Figure A.4

2: VECTOR APPROACH TO FINDING THE LENGTH OF A SIDE OF THE TETRAHEDRON[23]

The goal here is to determine the *z*-components a_z, b_z, c_z of the vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ whose heads are three vertices of a tetrahedron opposite the apex, which is the head of vector $\underline{e}$. While it is not necessary to use vector notation to achieve this, it will be useful to do so to make the approach compatible with the approach in Chapter 2.

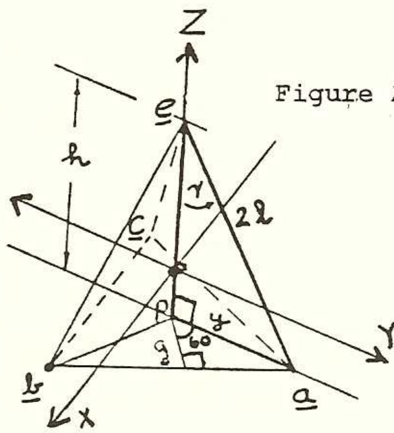


Figure A.5

In Figure A.4, the vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ lie in a plane parallel to the (*xy*)-plane. Let

[23]Ibid., pp. 17-18.

h denote the height of $\underline{e}$ above the base, let γ denote the angle between the polar axis and an edge of the tetrahedron, and let g
denote the length of the altitude of the isosceles triangle determined by vectors $\underline{a}$ and $\underline{b}$ in the base of the tetrahedron (thus the vertex angle of this triangle is 120°).

<u>Calculate g</u>

By the Pythagorean Theorem, $y^2 = g^2 + l^2$ (the altitude g is the perpendicular bisector of side of the tetrahedron, of length $2l$). Since the vertex angle of that isosceles triangle is 120°, half of it is 60°, so

$$\sin 60° = \sqrt{3}/2 = \frac{l}{y} \text{ and } \cos 60° = \frac{1}{2} = \frac{g}{y} \text{ (Figure A.5)}$$

Solving for y and g respectively, $y=\frac{2l}{\sqrt{3}}$, $g = \frac{2l}{2\sqrt{3}}$.

<u>Calculate h</u>

Using the Pythagorean Theorem applied to the right triangle with one leg, of length h, along the polar axis,

$$h^2 = (2l)^2 - y^2 = (2l)^2 - \frac{(2l)^2}{\sqrt{3}} = \frac{2 \cdot (2l)^2}{3} = \frac{8l^2}{3}$$

Thus $h = \frac{2\sqrt{2}l}{\sqrt{3}}$.

<u>Calculate γ</u>

Using the material above, $\cos \gamma = \frac{h}{2l} = \frac{\frac{2\sqrt{2}l}{\sqrt{3}}}{2l} = \frac{\sqrt{2}}{\sqrt{3}}$, and

$\sin \gamma = \frac{y}{2l} = \frac{\frac{2l}{\sqrt{3}}}{2l} = \frac{1}{\sqrt{3}}$. Thus $\gamma = 35.26435968°$.

<u>Calculate z</u>

Let r denote the radius of the sphere in which the tetrahedron is inscribed. Referring to Figure A.6, $a =$

tetrahedron is inscribed. Referring to Figure A.6, a = 180-2γ since a is the vertex angle of an isosceles triangle; the angle θ is supplementary to a, $a+\theta = 180°$. The length h may be partitioned into two pieces; the vector $\underline{e}$, of length r, and the length from the (xy)-plane to the plane containing the base of the tetrahedron (this length is measured by the z-component of any of $\underline{a}$, $\underline{b}$, $\underline{c}$). Thus $-z = r \cos\theta$, or $\cos\theta = -\frac{z}{r}$.

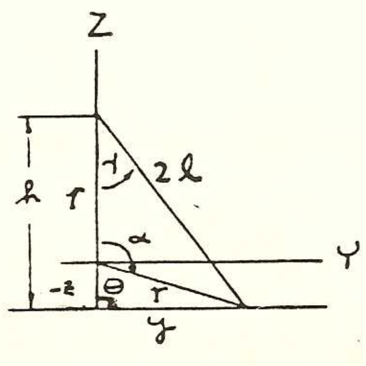


Figure A.6

Since $\theta = 180 - a$ $= 180 - (180-2\gamma) = 2\gamma$, it follows from the double-angle formula that $\cos\theta = \cos 2\gamma = \cos^2\gamma - \sin^2\gamma$ $=\frac{2}{2} - \frac{1}{3} = \frac{1}{3}$. Equating the two values for $\cos\theta$, it follows that $-\frac{z}{r} = \frac{1}{3}$, or $z = -\frac{1}{3}r$, giving a means of expressing the z-component of any of $\underline{a}$, $\underline{b}$, or $\underline{c}$ in terms of the radius, r, of the circumscribed sphere.

<u>Calculate $2l$</u>

Since $\cos\gamma = \frac{h}{2l}$, and $h = r-z$, then $\cos\gamma = \frac{r-z}{2l} = \frac{r+\frac{1}{3}r}{2l}$. Also, $\cos\gamma = \frac{\sqrt{2}}{\sqrt{3}}$. Thus, equating the two expressions for $\cos\gamma$, $\frac{\sqrt{2}}{\sqrt{3}} = \frac{\frac{4}{3}r}{2l}$ or $(2l)^2 = \frac{8r^2}{3}$, so that $2l = \frac{2r\sqrt{2}}{\sqrt{3}}$. This gives a formula for the length $2l$ of the side of a tetrahedron, inscribed in a sphere of radius r, expressed in terms of that radius.

3: THE CALCULATION OF DIHEDRAL ANGLES OF PLATONIC SOLIDS[24]

Definition: A dihedral angle of a solid is the angle between planes containing two adjacent faces of the solid (a "space" angle).

Notation: Schäfli symbol: $\{p,q\}$ p: # of edges in a face
q: # of edges incident with a vertex.

$\{3,3\}$: Tetrahedron
$\{4,3\}$: Cube
$\{3,4\}$: Octahedron
$\{5,3\}$: Dodecahedron
$\{3,5\}$: Icosahedron

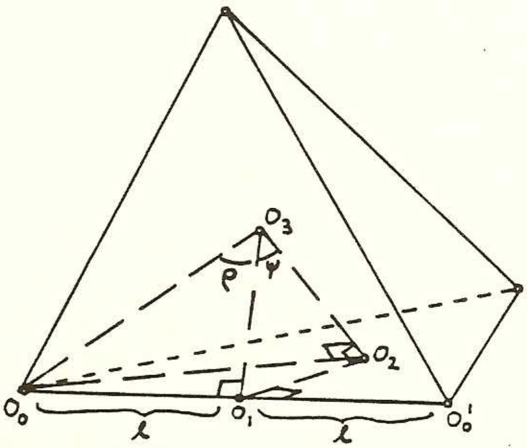


Figure A.7

The solid may be constructed from p-gonal congruent pyramids of suitable altitude with common apex at O_3. Further, O_3 is the center of the circumsphere passing through all vertices; of the midsphere touching all edges at their midpoints; and of the sphere touching all faces at their centers (Figure A.7). Thus:

$$\left.\begin{array}{l} O_0O_3 \text{ is a circumradius} \\ O_1O_3 \text{ is a midradius} \\ O_2O_3 \text{ is an inradius} \end{array}\right\} \rightarrow \begin{array}{l} O_3O_1 \perp O_0O_1,\ O_0O_1 \perp O_1O_2 \\ O_3O_2 \perp O_1O_2,\ O_3O_2 \perp O_0O_2 \end{array}$$

One such p-gonal pyramid is $O_0O_1O_2O_3$.

Let l denote $O_0O_1 = O_1O_0'$

Now $\measuredangle O_0O_2O_1 = \frac{\pi}{p}$ since $\Delta O_0O_2O_0'$ is isosceles and $\measuredangle O_0O_2'O_0' = \frac{2\pi}{p}$ (Figure A.7)

[24]From Coxeter, Introduction, pp. 155-157, 413.

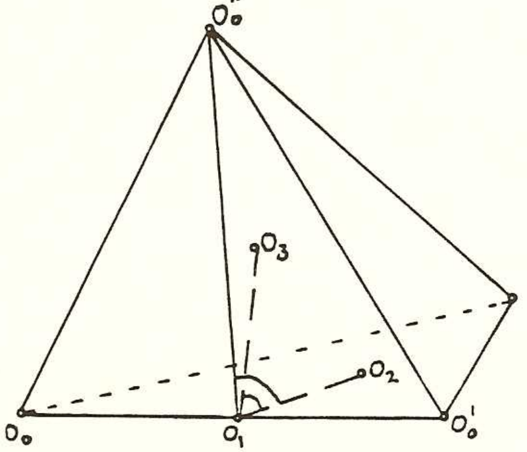


Figure A.8

Let $\rho = \sphericalangle O_0O_3O_1$, $\psi = \sphericalangle O_1O_3O_2$. Therefore $\frac{\pi}{2}-\psi = \sphericalangle O_3O_1O_2$ since $\Delta O_1O_2O_3$ is a right triangle (Figure A.7). Since O_3 is the center of the pyramid, $\sphericalangle O_3O_1O_2 = \frac{\pi}{2}-\psi = \frac{1}{2}\angle O_0''O_1O_2$ (0) (Figure A.8) But $\sphericalangle O_0''O_1O_2$ is the dihedral angle of the polyhedron. Therefore $\frac{\pi}{2}-\psi = \frac{1}{2}$ dihedral angle. Therefore dihedral angle $= \pi-2\psi$.

It remains to determine ψ in terms of known quantities, such as p and q.

<u>Definition:</u> The vertex figures of $\{p,q\}$ at vertex O_0 is the polygon formed by the midpoints of the q edges incident to O_0. E.g., the triangle $O_1O_1'O_1''$ is the vertex figure of $\{3,3\}$ at O_0 (Figure A.9)

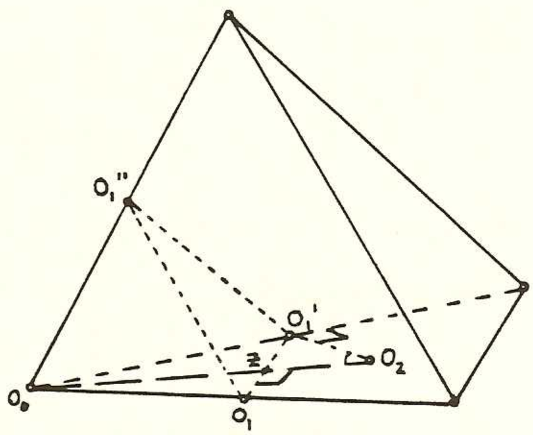


Figure A.9

$O_2O_0 \perp O_1O_1'$ since $\Delta O_1'O_2O_1$ is isosceles, and O_2 is the center of the base so O_2O_0 bisects the vertex angle of the isosceles triangle. $O_2O_1' \perp O_0O_1'$ since O_2O_1' is a midradius and O_0O_1' is tangent to the midsphere at O_1'.

Therefore $\Delta O_2O_1'O_0 \sim \Delta O_2ZO_1'$ since both contain $\sphericalangle O_0O_2O_1'$ whose measure $\frac{\pi}{p}$ (Figure A.10)

$$\text{Therefore } \frac{O_2O_1'}{O_2O_0} = \frac{O_2Z}{O_2O_1'} = \frac{ZO_1'}{O_1'O_0} \quad ((O_1'O_0 = 1)$$

$$\text{Therefore } O_1'Z = 1 \cdot \frac{O_2Z}{O_2O_1'} = 1 \cos\frac{\pi}{p}.$$

(1) Therefore $\boxed{O_1O_1' = 2l \cos \frac{\pi}{p}}$ or, the length of a side of the vertex figure = $2l \cos \frac{\pi}{p}$. (Figure A.9)

The plane $O_1O_1'O_1''$ of at the vertex figure is perpendicular to O_0O_3, and the center of $O_1O_1'O_1''$ is Q, the foot of the perpendicular from O_1 to O_0O_3 (Figure A.11).

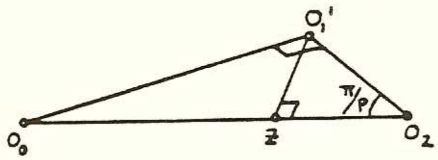
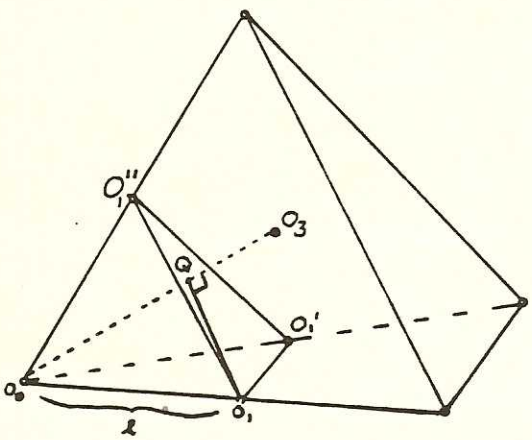


Figure A.10

$\Delta O_1O_3Q \sim \Delta O_0O_3O_1$. Therefore $\frac{O_1O_3}{O_0O_3} = \frac{QO_1}{l} = \frac{QO_3}{O_1O_3}$. Therefore

$O_1Q = l \cdot \frac{O_1O_3}{O_0O_3} = l \cdot \cos\rho$.

(2) Therefore $\boxed{O_1Q = l \cos\rho}$ Use $O_1Q = l \cos\rho$ to find alternate expression for O_1Q.



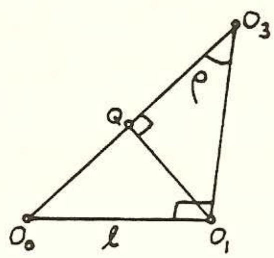
a

Since $\measuredangle\ O_1''QO_1 = \frac{2\pi}{q}$, therefore $\measuredangle Z'QO_1 = \frac{\pi}{q}$. Therefore $\sin \frac{\pi}{q} = \frac{l\cos\frac{\pi}{p}}{O_1Q}$. Therefore

$O_1Q = l \cos\frac{\pi}{p} \cdot \frac{1}{\sin \frac{\pi}{q}}$

$= l \cos\frac{\pi}{p} \csc \frac{\pi}{q}$. (Figure A.12).

Figure A.11



b

(3) Therefore $\boxed{\cos\rho = \cos\frac{\pi}{p}\csc\frac{\pi}{q}}$.

Summarizing:

$$O_0O_3 = l \csc\rho \text{ since } \sin\rho = \frac{l}{O_0O_3}$$

$$O_1O_3 = l \cot\rho \text{ since } \tan\rho = \frac{l}{O_1O_3}$$

$$(O_2O_3)^2 = (O_1O_3)^2 - (l \cot\frac{\pi}{p})^2$$

$$\text{since } \tan\frac{\pi}{p} = \frac{l}{(O_1O_2)}, \quad \cos\psi = \frac{(O_2O_3)}{(O_1O_3)}.$$

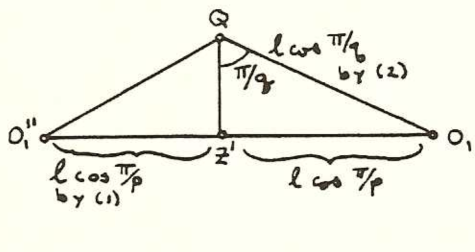


Figure A.12

(4) To find a value for ψ, introduce the following notation: let $k^2 = \sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p} = 1-\cos^2\frac{\pi}{q} - (1-\sin^2\frac{\pi}{p})$ $= \sin^2\frac{\pi}{p} - \cos^2\frac{\pi}{q}$.

Therefore since $\sin\frac{\pi}{q} = \dfrac{l\cos\frac{\pi}{p}}{l\cos\rho}$ (use figure), we have

$$k^2 = \sin^2\frac{\pi}{q} - \cos^2\frac{\pi}{p} = \frac{\cos^2\frac{\pi}{p}}{\cos^2\rho} - \cos^2\frac{\pi}{p} = \cos^2\frac{\pi}{p}\left(\frac{1}{\cos^2\rho} - 1\right)$$

$$= \cos^2\frac{\pi}{p}\left(\frac{1-\cos^2\rho}{\cos^2\rho}\right) = \cos^2\frac{\pi}{p}\left(\frac{\sin^2\rho}{\cos^2\rho}\right)$$

(5) Therefore $k = \sin\rho \cdot \dfrac{\cos\frac{\pi}{p}}{\cos\rho} = \dfrac{\sin\rho\cos\frac{\pi}{p}}{\dfrac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{q}}} = \sin\rho\sin\frac{\pi}{q}$.

Therefore $\boxed{\sin\rho = k\csc\frac{\pi}{q}}$.

(6a) Therefore $O_0O_3 = l\csc\rho = \frac{l}{k}\sin\frac{\pi}{q}$ (summary and (5)).

(6b) $O_1O_3 = l\cot\rho = \dfrac{l\cos\rho}{\sin\rho} = \dfrac{l\ \cos\frac{\pi}{p}\csc\frac{\pi}{q}}{k\csc\frac{\pi}{q}}$ (summary (3)(5)) $= \frac{l}{k}\cos\frac{\pi}{p}$

$$(O_2O_3)^2 = (O_1O_3)^2 - (l\ \cot\tfrac{\pi}{p})^2$$

$$= \frac{l^2}{k^2}\cos^2\frac{\pi}{p} - l^2\frac{\cos^2\frac{\pi}{p}}{\sin^2\frac{\pi}{p}} \text{ (summary \& (6b))} = \frac{l^2\cos^2\frac{\pi}{p}}{k^2\sin^2\frac{\pi}{p}}(\sin^2\frac{\pi}{p} - k^2)$$

$$= \frac{l^2\cos^2\frac{\pi}{p}}{k^2\sin^2\frac{\pi}{p}} \cdot \cos^2\frac{\pi}{q} \text{ (from (4)).}$$

(6c) Therefore $O_2O_3 = \frac{l}{k}\cot\frac{\pi}{p}\cos\frac{\pi}{q}$.

$$\cos\psi = \frac{O_2O_3}{O_1O_3} = \frac{\frac{l}{k}\cot\frac{\pi}{p}\cos\frac{\pi}{q}}{\frac{l}{k}\cos\frac{\pi}{p}} = \frac{\cos\frac{\pi}{p}}{\sin\frac{\pi}{p}} \cdot \frac{\cos\frac{\pi}{q}}{\cos\frac{\pi}{p}} = \frac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{p}} \text{(by 6b,6c)}$$

Since $\cos\psi = \sin(\frac{\pi}{2} - \psi) = \dfrac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{p}}$, $\frac{\pi}{2} - \psi = \sin^{-1}(\dfrac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{p}})$

or $\boxed{\pi - 2\psi = 2\ \sin^{-1}(\dfrac{\cos\frac{\pi}{q}}{\sin\frac{\pi}{p}})}$, the dihedral angle in terms of p and q.

<u>Evaluate</u>: $\{p,q\}$:

$$\{3,3\}:\quad 2\sin^{-1}(\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}) = 2\sin^{-1}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \cong 2\sin^{-1}(.5773) \cong 70°36'$$

$$\{4,3\}:\quad 2\sin^{-1}(\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{4}}) = 2\sin^{-1}\frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} \cong 2\ \sin^{-1}(.7071) \cong 90°$$

$$\{3,4\}: \quad 2\sin^{-1}\left(\frac{\cos\frac{\pi}{4}}{\sin\frac{\pi}{3}}\right) = 2\sin^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}}\right)$$

$$\cong 2\ \sin^{-1}\frac{\sqrt{2}}{\sqrt{3}} \cong 2\ \sin^{-1}(.8164) \cong 108°24'.$$

$$\{5,3\}: \quad 2\ \sin^{-1}\left(\frac{\cos\ \frac{\pi}{3}}{\sin\frac{\pi}{5}}\right) = 116°36'$$

$$\{3,5\}: \quad 2\ \sin^{-1}\left(\frac{\cos\ \frac{\pi}{5}}{\sin\frac{\pi}{3}}\right) = 138°12'$$

are the dihedral angles of the Platonic solids.

APPENDIX B: SOME LINEAR ALGEBRA[25]

ADDITION OF VECTORS

Given two vectors, $\underline{u}$ and $\underline{v}$, issuing from O(Figure B.1); the sum, $\underline{u} + \underline{v}$, of these vectors represents the displacement of O along $\underline{u}$, followed by $\underline{v}$; thus $\underline{u + v}$ is the vector that is the diagonal of the parallelogram determined by u and v.

SUBTRACTION OF VECTORS

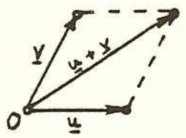


Figure B.1

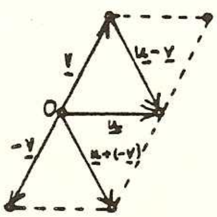


Figure B.2

Given two vectors, $\underline{u}$ and $\underline{v}$, issuing from O (Figure B.2); the difference, $\underline{u - v} = \underline{u + (-v)}$, is represented by the diagonal of the parallelogram that is not $\underline{u + v}$.

DOT PRODUCT OF TWO VECTORS

The following procedure for finding the 'product' of two vectors will produce an answer that is a scalar (real number). This product is referred to as the 'scalar product' or 'dot product' or 'inner product.'

Definition: The 'dot product,' $\underline{u \cdot v}$ of two vectors, $\underline{u} = (u_1, u_2, \ldots, u_n)$ and $\underline{v} = (v_1, v_2, \ldots, v_n)$, with all components real numbers, is the scalar

$$\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i .$$

Thus in three-dimensional space, $\underline{u} \cdot \underline{v} = \sum_{i=1}^{3} u_i v_i$.

---

[25]For more detailed treatment the reader is referred to texts on linear algebra; a classic algebra text is Garret Birkhoff and Saunders Mac Lane, A Survey of Modern Algebra, (New York: Macmillan, revised 1953).

Property: $(\underline{u}+\underline{v}) \cdot \underline{w} = \underline{u}\cdot\underline{w} + \underline{v}\cdot\underline{w}$; apply the distributive law to the definition.

LENGTH OF A VECTOR IN R^3

The dot product will be used to associate a single positive scalar, that represents vector length, with a non-trivial vector $\underline{u} \neq (0,0,0)$. From the definition of dot product, it follows that $\underline{u}\cdot\underline{u}=u_1^{\,2}+u_2^{\,2}+u_3^{\,2}>0$.

Definition: The length of a vector $\underline{u}$, denoted $|\underline{u}|$ and sometimes called the 'absolute value' or the 'norm' of $\underline{u}$, is

$$|\underline{u}| = (\underline{u}\cdot\underline{u})^{1/2} = (u_1^{\,2} + u_2^{\,2} + u_3^{\,2})^{1/2}.$$

ANGLE BETWEEN TWO VECTORS IN R^3

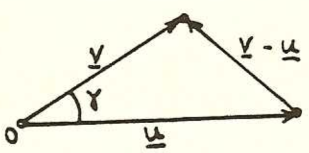


Figure B.3

Suppose $\underline{u}$ and $\underline{v}$ are two vectors in R^3 issuing from O. Use the law of cosines on the triangle formed by the vectors $\underline{u}$, $\underline{v}$, $\underline{v-u}$. (Figure B.3)

Thus $|\underline{v-u}|^2 = |\underline{v}|^2+|\underline{u}|^2 - 2|\underline{v}|\,|\underline{u}|\cos\gamma$. But, $|\underline{v-u}|^2$

$= (\underline{v-u}) \cdot(\underline{v-u})$ (by definition of length) and $(\underline{v-u})\cdot(\underline{v-u})$

$= \underline{v}\cdot(\underline{v-u})-\underline{u}\cdot(\underline{v-u})$ (by a property of dot products)

$= \underline{v}\cdot\underline{v}-\underline{v}\cdot\underline{u}-\underline{u}\cdot\underline{v}+\underline{u}\cdot\underline{u}$ (same property)

$= \underline{v}\cdot\underline{v}+\underline{u}\cdot\underline{u}-2(\underline{u}\cdot\underline{v})$ (multiplication in R^3 is commutative)

$= |\underline{v}|^2 + |\underline{u}|^2 - 2(\underline{u}\cdot\underline{v})$

Thus $|\underline{v-u}|^2=|\underline{v}|^2+|\underline{u}|^2-2(\underline{u}\cdot\underline{v})=|\underline{v}|^2+|\underline{u}|^2-2|\underline{v}|\,|\underline{u}|\cos\gamma$.

Therefore, $-2(\underline{u}\cdot\underline{v})=-2|\underline{u}|\,|\underline{v}|\cos\gamma$ or,

$$\cos\gamma = \frac{u\cdot v}{|\underline{u}|\,|\underline{v}|}.$$

LINEAR TRANSFORMATIONS IN R^3

Definition: A linear transformation $T{:}R^3 \to R^3$ of R^3 to R^3 is such that $(a\underline{u} + b\underline{v})T = a(\underline{u}T) + b(\underline{v}T)$, for $\underline{u}$, $\underline{v}$ vectors in R^3 and a, b, scalars in R^3.

<u>Theorem</u>: There is a one-to-one correspondence between the set of linear transformations $T:R \to R^3$ and the 3 x 3 matrices with real numbers as entries.

ROTATION MATRICES

Suppose T is a linear transformation, $T:R^3 \to R^3$, that rotates a vector in the (xy)-plane, through an angle of θ in the (xy)-plane, about the z-axis (Figure B.4). (T is linear since it is a rigid motion.)

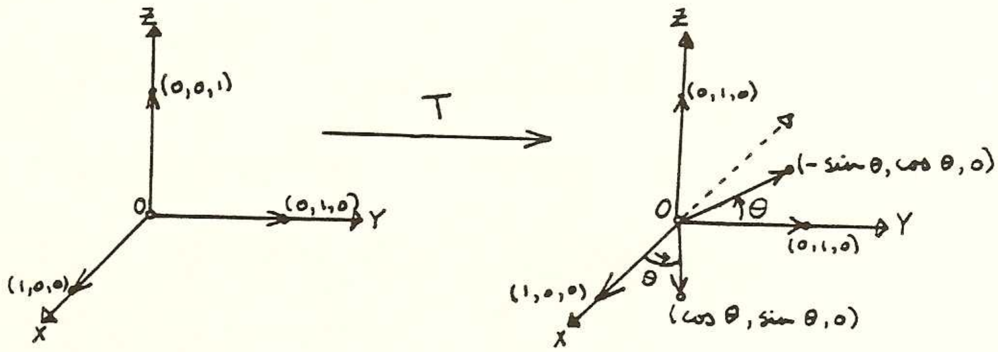


Figure B.4

Under this transformation, the unit vectors (1,0,0),(0,1,0), and (0,0,1) are carried to the vectors ($\cos\theta$, $\sin\theta$, 0), ($-\sin\theta$, $\cos\theta$, 0), and (0,0,1) respectively. Viewing these row vectors as matrix rows,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix on the right is a rotation matrix through angle θ about the z-axis, to be applied to row-vectors.

Viewing the unit vectors as column vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So viewing these column vectors as matrix columns, the identity matrix is carried to a rotation matrix, to be applied to column vectors, as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{T}{\rightarrow} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similar procedure leads to rotation matrices through θ about the x-axis or the y-axis.

APPENDIX C

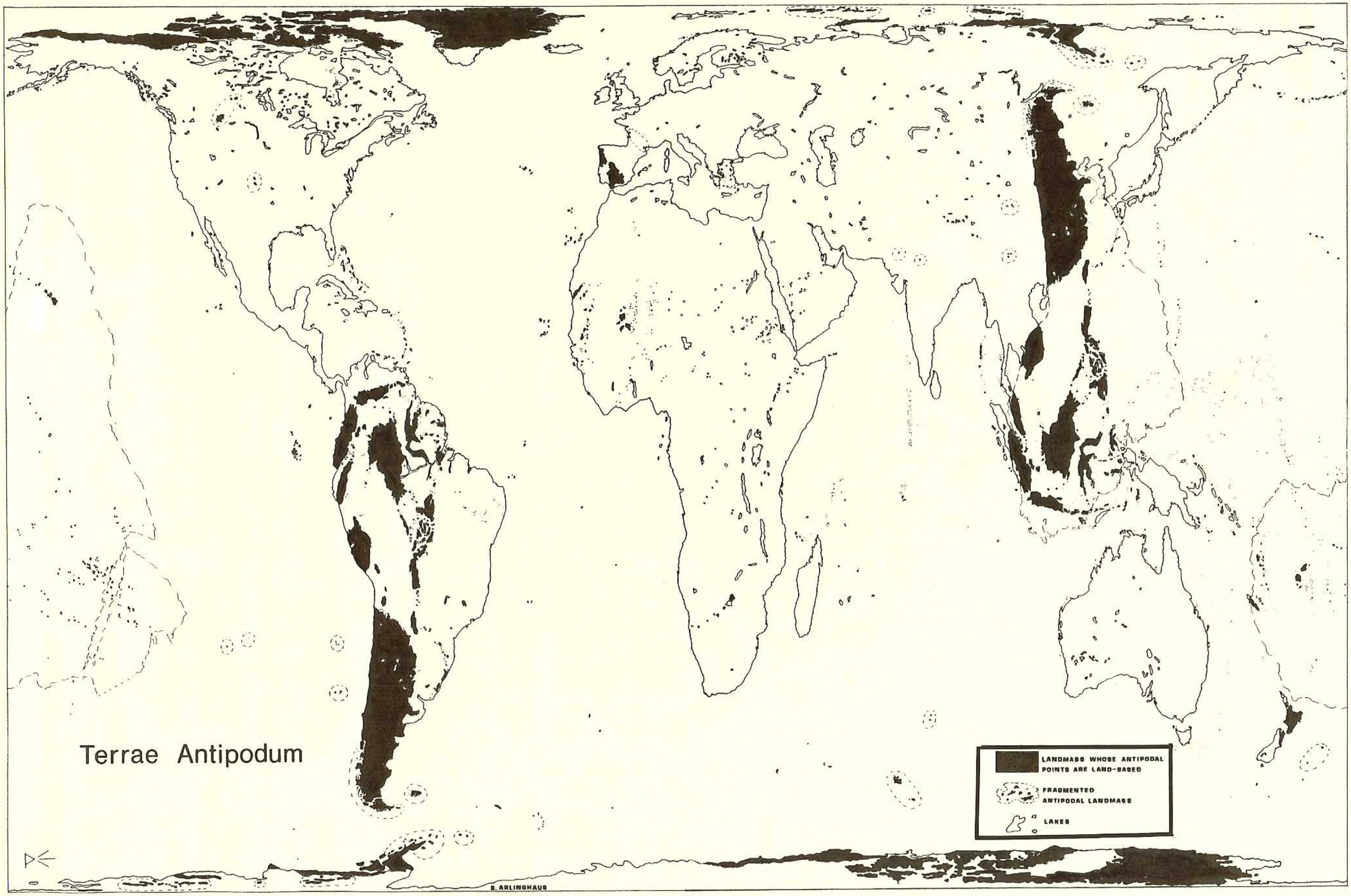
TERRAE ANTIPODUM: ANTIPODAL LANDMASS MAP[26]

The antipodal landmass map, Map C.1, is critical in determining whether or not a particular solid may be land-based.

[26]For map construction, see S. Arlinghaus, "Terrae Antipodum", Preprint, March 1984, Ann Arbor, Michigan. The base map is a Peters' projection, as indicated by the symbol in the lower left-hand corner of the map. This is an equal area projection constructed by Arno Peters, displayed as a "World Map," distributed by Christian Aid, P.O. Box 1, London SW9 8BH England.

<u>ADDENDUM</u>

[Tobler noted that Fisher and Miller (among others), in World Maps and Globes (1944), discuss some advantages and disadvantages associated with using the various Platonic solids as "globe-forming maps."]


Terrae Antipodum
LANDMASS WHOSE ANTIPODAL POINTS ARE LAND-BASED
FRAGMENTED ANTIPODAL LANDMASS
LAKES
S. ARLINGHAUS

INSTITUTE OF MATHEMATICAL GEOGRAPHY (IMaGe)
2790 BRIARCLIFF
ANN ARBOR, MI 48105-1429; U.S.A.
(313) 761-1231; IMaGe@UMICHUM

"Imagination is more important than knowledge"
A. Einstein

MONOGRAPH SERIES

Exclusive of shipping and handling; prices listed and payable in U.S. funds on a U.S. bank, only.

1. Sandra L. Arlinghaus and John D. Nystuen. *Mathematical Geography and Global Art: the Mathematics of David Barr's "Four Corners Project,"* 1986.

This monograph contains Nystuen's calculations, actually used by Barr to position his abstract tetrahedral sculpture within the earth. Placement of the sculpture vertices in Easter Island, South Africa, Greenland, and Indonesia was chronicled in film by The Archives of American Art for The Smithsonian Institution. In addition to the archival material, this monograph also contains Arlinghaus's solutions to broader theoretical questions–was Barr's choice of a tetrahedron unique within his initial constraints, and, within the set of Platonic solids?

2. Sandra L. Arlinghaus. *Down the Mail Tubes: the Pressured Postal Era, 1853-1984*, 1986.

The history of the pneumatic post, in Europe and in the United States, is examined for the lessons it might offer to the technological scenes of the late twentieth century. As Sylvia L. Thrupp, Alice Freeman Palmer Professor Emeritus of History, The University of Michigan, commented in her review of this work "Such brief comment does far less than justice to the intelligence and the stimulating quality of the author's writing, or to the breadth of her reading. The detail of her accounts of the interest of American private enterprise, in New York and other large cities on this continent, in pushing for construction of large tubes in systems to be leased to the government, brings out contrast between American and European views of how the new technology should be managed. This and many other sections of the monograph will set readers on new tracks of thought."

3. Sandra L. Arlinghaus. *Essays on Mathematical Geography*, 1986.

A collection of essays intended to show the range of power in applying pure mathematics to human systems. There are two types of essay: those which employ traditional mathematical proof, and those which do not. As mathematical proof may itself be regarded as art, the former style of essay might represent "traditional" art, and the latter, "surrealist" art. Essay titles are: "The well-tempered map projection," "Antipodal graphs," "Analogue clocks," "Steiner transformations," "Concavity and urban settlement patterns," "Measuring the vertical city," "Fad and permanence in human systems," "Topological exploration in geography," "A space for thought," and "Chaos in human systems–the Heine-Borel Theorem."

4. Robert F. Austin, *A Historical Gazetteer of Southeast Asia*, 1986.

Dr. Austin's Gazetteer draws geographic coordinates of Southeast Asian place-names together with references to these place-names as they have appeared in historical and literary documents. This book is of obvious use to historians and to historical geographers specializing in Southeast Asia. At a deeper level, it might serve as a valuable source in establishing place-name linkages which have remained previously unnoticed, in documents describing trade or other communications connections, because of variation in place-name nomenclature.

5. Sandra L. Arlinghaus, *Essays on Mathematical Geography–II*, 1987.

Written in the same format as IMaGe Monograph #3, that seeks to use "pure" mathematics in real-world settings, this volume contains the following material: "Frontispiece–the Atlantic Drainage Tree," "Getting a Handel on Water-Graphs," "Terror in Transit: A Graph Theoretic Approach to the Passive Defense of Urban Networks," "Terrae Antipodum," "Urban Inversion," "Fractals: Constructions, Speculations, and Concepts," "Solar Woks," "A Pneumatic Postal Plan: The Chambered Interchange and ZIPPR Code," "Endpiece."

6. Pierre Hanjoul, Hubert Beguin, and Jean-Claude Thill, *Theoretical Market Areas Under Euclidean Distance*, 1988. (English language text; Abstracts written in French and in English.)

Though already initiated by Rau in 1841, the economic theory of the shape of two-dimensional market areas has long remained concerned with a representation of transportation costs as linear in distance. In the general gravity model, to which the theory also applies, this corresponds to a decreasing exponential function of distance deterrence. Other transportation cost and distance deterrence functions also appear in the literature, however. They have not always been considered from the viewpoint of the shape of the market areas they generate, and their disparity asks the question whether other types of functions would not be worth being investigated. There is thus a need for a general theory of market areas: the present work aims at filling this gap, in the case of a duopoly competing inside the Euclidean plane endowed with Euclidean distance.

(Bien qu'ébauchée par Rau dès 1841, la théorie économique de la forme des aires de marché planaires s'est longtemps contentée de l'hypothèse de coûts de transport proportionnels à la distance. Dans le modèle gravitaire généralisé, auquel on peut étendre cette théorie, ceci correspond au choix d'une exponentielle décroissante comme fonction de dissuasion de la distance. D'autres fonctions de coût de transport ou de dissuasion de la distance apparaissent cependant dans la littérature. La forme des aires de marché qu'elles engendrent n'a pas toujours été étudiée ; par ailleurs, leur variété amène à se demander si d'autres fonctions encore ne mériteraient pas d'être examinées. Il paraît donc utile de disposer d'une théorie générale des aires de marché : ce à quoi s'attache ce travail en cas de duopole, dans le cadre du plan euclidien muni d'une distance euclidienne.)

7. Keith J. Tinkler, Editor, *Nystuen—Dacey Nodal Analysis*, 1988.

Professor Tinkler's volume displays the use of this graph theoretical tool in geography, from the original Nystuen—Dacey article, to a bibliography of uses, to original uses by Tinkler. Some reprinted material is included, but by far the larger part is of previously unpublished material. (Unless otherwise noted, all items listed below are previously unpublished.) Contents: " 'Foreward' " by Nystuen, 1988; "Preface" by Tinkler, 1988; "Statistics for Nystuen—Dacey Nodal Analysis," by Tinkler, 1979; Review of Nodal Analysis literature by Tinkler (pre–1979, reprinted with permission; post—1979, new as of 1988); FORTRAN program listing for Nodal Analysis by Tinkler; "A graph theory interpretation of nodal regions" by John D. Nystuen and Michael F. Dacey, reprinted with permission, 1961; Nystuen—Dacey data concerning telephone flows in Washington and Missouri, 1958, 1959 with comment by Nystuen, 1988; "The expected distribution of nodality in random (p, q) graphs and multigraphs," by Tinkler, 1976.

8. James W. Fonseca, *The Urban Rank–size Hierarchy: A Mathematical Interpretation*, 1989.

The urban rank–size hierarchy can be characterized as an equiangular spiral of the form $r = ae^{\theta \cot \alpha}$. An equiangular spiral can also be constructed from a Fibonacci sequence. The urban rank–size hierarchy is thus shown to mirror the properties derived from Fibonacci characteristics such as rank–additive properties. A new method of structuring the urban rank–size hierarchy is explored which essentially parallels that of the traditional rank–size hierarchy below rank 11. Above rank 11 this method may help explain the frequently noted concavity of the rank–size distribution at the upper levels. The research suggests that the simple rank–size rule with the exponent equal to 1 is not merely a special case, but rather a theoretically justified norm against which deviant cases may be measured. The spiral distribution model allows conceptualization of a new view of the urban rank–size hierarchy in which the three largest cities share functions in a Fibonacci hierarchy.

9. Sandra L. Arlinghaus, *An Atlas of Steiner Networks*, 1989.

A Steiner network is a tree of minimum total length joining a prescribed, finite, number of locations; often new locations are introduced into the prescribed set to determine the minimum tree. This Atlas explains the mathematical detail behind the Steiner construction for prescribed sets of n locations and displays the steps, visually, in a series of Figures. The proof of the Steiner construction is by mathematical induction, and enough steps in the early part of the induction are displayed completely that the reader who is well–trained in Euclidean geometry, and familiar with concepts from graph theory and elementary number theory, should be able to replicate the constructions for full as well as for degenerate Steiner trees.

10. Daniel A. Griffith, *Simulating $K = 3$ Christaller Central Place Structures: An Algorithm Using A Constant Elasticity of Substitution Consumption Function*, 1989.

An algorithm is presented that uses BASICA or GWBASIC on IBM compatible machines. This algorithm simulates Christaller $K = 3$ central place structures, for a four–level hierarchy. It is based upon earlier published work by the author. A description of the spatial theory, mathematics, and sample output runs appears in the monograph. A digital version is available from the author, free of charge, upon request; this request must be accompanied by a 5.5–inch formatted diskette. This algorithm has been developed for use in Social Science classroom laboratory situations, and is designed to (a) cultivate a deeper understanding of central place theory, (b) allow parameters of a central place system to be altered and then graphic and tabular results attributable to these changes viewed, without experiencing the tedium of massive calculations, and (c) help promote a better comprehension of the complex role distance plays in the space–economy. The algorithm also should facilitate intensive numerical research on central place structures; it is expected that even the sample simulation results will reveal interesting insights into abstract central place theory.

The background spatial theory concerns demand and competition in the space–economy; both linear and non–linear spatial demand functions are discussed. The mathematics is concerned with (a) integration of non–linear spatial demand cones on a continuous demand surface, using a constant elasticity of substitution consumption function, (b) solving for roots of polynomials, (c) numerical approximations to integration and root extraction, and (d) multinomial discriminant function classification of commodities into central place hierarchy levels. Sample output is presented for contrived data sets, constructed from artificial and empirical information, with the wide range of all possible central place structures being generated. These examples should facilitate implementation testing. Students are able to vary single or multiple parameters of the problem, permitting a study of how certain changes manifest themselves within the context of a theoretical central place structure. Hierarchical classification criteria may be changed, demand elasticities may or may not vary and can take on a wide range of non–negative values, the uniform transport cost may be set at any positive level, assorted fixed costs and variable costs may be introduced, again within a rich range of non–negative possibilities, and the number of commodities can be altered. Directions for algorithm execution are summarized. An ASCII version of the algorithm, written directly from GWBASIC, is included in an appendix; hence, it is free of typing errors.

11. Sandra L. Arlinghaus and John D. Nystuen, *Environmental Effects on Bus Durability*, 1990.

This monograph draws on the authors' previous publications on "Climatic" and "Terrain" effects on bus durability. Material on these two topics is selected, and reprinted, from three published papers that appeared in the *Transportation Research Record* and in the *Geographical Review*. New material concerning "congestion" effects is examined at the national level, to determine "dense," "intermediate," and "sparse" classes of congestion, and at the local level of congestion in Ann Arbor (as suggestive of how one might use local data). This material is drawn together in a single volume, along with a summary of the consequences of all three effects simultaneously, in order to suggest direction for more highly automated studies that should follow naturally with the release of the 1990 U. S. Census data.

12. Daniel A. Griffith, Editor. *Spatial Statistics: Past, Present, and Future*, 1990.

Proceedings of a Symposium of the same name held at Syracuse University in Summer, 1989. Content includes a Preface by Griffith and the following papers:

Brian Ripley, "Gibbsian interaction models";
J. Keith Ord, "Statistical methods for point pattern data";
Luc Anselin, "What is special about spatial data";
Robert P. Haining, "Models in human geography:
problems in specifying, estimating, and validating models for spatial data";
R. J. Martin, "The role of spatial statistics in geographic modelling";
Daniel Wartenberg, "Exploratory spatial analyses: outliers, leverage points, and influence functions";
J. H. P. Paelinck, "Some new estimators in spatial econometrics";
Daniel A. Griffith, "A numerical simplification for estimating parameters of spatial autoregressive models";
Kanti V. Mardia "Maximum likelihood estimation for spatial models";
Ashish Sen, "Distribution of spatial correlation statistics";

Sylvia Richardson, "Some remarks on the testing of association between spatial processes";
Graham J. G. Upton, "Information from regional data";
Patrick Doreian, "Network autocorrelation models: problems and prospects."

Each chapter is preceded by an "Editor's Preface" and followed by a Discussion and, in some cases, by an author's Rejoinder to the Discussion.

INSTITUTE OF MATHEMATICAL GEOGRAPHY
2790 Briarcliff
Ann Arbor, MI 48105
U.S.A.

MICMG DISCUSSION PAPERS, JOHN D. NYSTUEN, EDITOR

The collected work of the Michigan Interuniversity Community of Mathematical Geographers is reprinted in a single essay format (as they were originally reproduced). Royalties from sales will be deposited in the "MICMG" fund for the development of projects in mathematical geography, administered by IMaGe, and disposed of by Nystuen, in collaboration with IMaGe.

Consider ordering one number as reading supplementary to texts in an upper division course. The dates of original release and titles of the individual numbers are listed below. Prices are current as of 1988.

1. Arthur Getis, "Temporal land use pattern analysis with the use of nearest neighbor and quadrat methods." [Re-typed and maps re-drawn.] July, 1963. $1.50

2. Marc Anderson, "A working bibliography of mathematical geography." September, 1963. $3.50

3. William Bunge, "Patterns of location." February, 1964. $2.50

4. Michael F. Dacey, "Imperfections in the uniform plane." June, 1964. $2.50

5. Robert S. Yuill, "A simulation study of barrier effects in spatial diffusion problems." April, 1965. $3.00

6. William Warntz, "A note on surfaces and paths and applications to geographical problems." May, 1965. $2.00

7. Stig Nordbeck, "The law of allometric growth." June, 1965. $2.50

8. Waldo R. Tobler, "Numerical map generalization;" and "Notes on the analysis of geographical distributions." $3.00

9. Peter R. Gould, "On mental maps." September, 1966. $3.50

10. John D. Nystuen, "Effects of boundary shape and the concept of local convexity;" Julian Perkal, "On the length of empirical curves;" and, Julian Perkal, "An attempt at objective generalization." December, 1966. $4.50

11. E. Casetti and R. K. Semple, "A method for the stepwise separation of spatial trends." April, 1968. $2.50

12. W. Bunge, R. Guyot, A. Karlin, R. Martin, W. Pattison, W. Tobler, S. Toulmin, and W. Warntz, "The philosophy of maps." June, 1968. $4.50