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THE PHENOMENON OF BLOCKING IN STRATIFIED FLOWS

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TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

Dimensional Form	Dimensionless Form	
x, z	ξ, η	Cartesian co-ordinates
p	Π	pressure
ρ		density
ρ_o		density at bottom of channel
ρ_1		density at top of channel
ρ_s		density on dividing streamline
ρ_B, ρ_2		density of stagnant zone
ρ_o'		another reference density
ψ		stream function
ψ'	Ψ, Ψ^*	stream function of the associated flow field
u, w		components of velocity
u', w'	U', W'	components of associated velocity
$q^2 = u^2 + w^2$	$q'^2 = U'^2 + W'^2$	
A, A_1		horizontal velocities of the associated flow far upstream
	$F, F_o, F_A, F_B, F_1, F_2$	Froude numbers
d, d_2		height of channel
d_1, d_o		depths far upstream of zone in which flow is concentrated
	$h_o = d_o / d$	a reference height far upstream
	b	a reference length
	r, s	reference lengths of obstacle

Dimensional Form	Dimensionless Form	
Q		discharge
$\nabla_{*}^{2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$	$\nabla^2 = \frac{\partial^2}{\partial \xi^2} \quad \frac{\partial^2}{\partial \eta^2}$	Laplacian operator
$\psi_x = \frac{\partial \psi}{\partial x}$		
$\psi_z = \frac{\partial \psi}{\partial z}$		

I. INTRODUCTION

In this study stagnant zones that occur in the flow of a stratified fluid are investigated. Numerous examples of such stagnant zones can be found in industrial processes and in nature. At sufficiently low rates of discharge, preferential withdrawal of heavier fluid from oil reservoirs, or the partial separation of fresh water from sea water which has intruded into a fresh water reservoir, or of cool water from warm water in thermoelectric plants, can be achieved due to part of the fluid becoming essentially stagnant. In nature, extensive regions of stagnant air are usually present in front of mountain ridges when the prevailing winds blow at a low speed, and when the stratification is sufficiently strong. If a city is situated in such a location, air pollution is the result, as is illustrated by the smog in the Los Angeles area.

The flows considered here are assumed to be two-dimensional and steady, and the fluid is taken to be inviscid, incompressible, and of variable density. This type of flow admits and sometimes demands solutions with velocity discontinuity along a streamline (contact discontinuity) creating a vortex sheet. A degenerate case is the classical problem of free streamlines in potential flows. In the problems under consideration, the fluid on one side of the discontinuity is required to be stagnant, so that the pressure on the discontinuity is known. The position of the dividing streamline is however not known *a priori*. Bernoulli's equation is satisfied along the dividing streamline and this provides a non-linear dynamic boundary condition to the flowing part. Since the motion is in general rotational, this is a case of a free streamline problem in rotational flow.

For the purpose of this investigation, the flow under consideration will be restricted between two horizontal planes and two cases are studied. In the first case, the symmetric flow into a line sink is considered, in which the fluid on the upper part is stagnant. In the second case, an obstruction is present on the lower boundary and a stagnant zone lies in front of, and is caused by the obstacle. An appropriate stratification profile is assumed far upstream to render the governing differential equation exactly linear. The parameter governing the flow is the Froude number. In the first case, a flow with stagnant zone is the actual mode of flow at Froude numbers below a number somewhat greater than $1/\pi$. In the second case, the height of the stagnant zone is related to the Froude number and the shape of the obstruction, depending on whether the obstacle has steep upstream face or gentle upstream face.

The fact that an appropriate stratification is chosen to render the equation exactly linear is no real restriction, since upstream profiles that lead to linearity are physically realistic in themselves, and furthermore, a perturbation from these assumed profiles will exhaust most of the physically realistic cases.

II. THE GOVERNING EQUATION

For steady two-dimensional flow of an inviscid, incompressible, density-stratified fluid in a gravitational field, with the gravity force acting in the negative z direction, the equations of motion are

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} , \tag{1}$$

$$u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g , \tag{2}$$

in which x and z are Cartesian co-ordinates, u and w are the corresponding velocity components, ρ is the density and p is the pressure, and g the gravitational acceleration. Since the equation of continuity is

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho w)}{\partial z} = 0 , \tag{3}$$

and the equation of incompressibility is

$$u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0 , \tag{4}$$

the equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 . \tag{5}$$

Since the density varies from streamline to streamline, irrotationality in general does not persist. The above system can however be rendered into a more convenient form by introducing an associated flow field (indicated by a prime) through the following transformation due to Yih:(1)

$$(u', w') = \sqrt{\rho/\rho_0}\,(u, w)\,. \tag{6}$$

It then follows from (5) that

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad , \tag{7}$$

whence it follows immediately that there exists a stream function for the associated flow ψ' such that

$$u' = -\frac{\partial \psi'}{\partial z} \quad , \qquad w' = \frac{\partial \psi}{\partial x} \quad . \tag{8}$$

From the equation of incompressibility for steady flow (4), it is obvious that

$$u \frac{\partial (f(\rho))}{\partial x} + w \frac{\partial (f(\rho))}{\partial z} = 0 \, , \tag{9}$$

where $f(\rho)$ is any function of density. It is also obvious from (4) that density is constant along a streamline in both the actual and the associated flow fields. Therefore, integration of (1) and (2) along a streamline shows that Bernoulli's equation is still true along a streamline.

By use of (9) the equations of motion (1) and (2) can now be written as

$$\rho_0 \left(u' \frac{\partial u'}{\partial x} + w' \frac{\partial u'}{\partial z}\right) = -\frac{\partial p}{\partial x} \, , \tag{10}$$

$$\rho_0 \left(u' \frac{\partial w'}{\partial x} + w' \frac{\partial w'}{\partial z}\right) = -\frac{\partial p}{\partial z} - g\rho \, . \tag{11}$$

If η' is the second component of the vorticity of the associated flow,

$$\eta' \equiv \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \equiv -\nabla_*^2 \psi', \tag{12}$$

and (10) and (11) become

$$\rho_0 \eta' \frac{\partial \psi'}{\partial x} = -\frac{\partial}{\partial x}\left(p + \frac{\rho_0 (u'^2 + w'^2)}{2}\right), \tag{13}$$

$$\rho_0 \eta' \frac{\partial \psi'}{\partial z} = -\frac{\partial}{\partial z}\left(p + \frac{\rho_0 (u'^2 + w'^2)}{2}\right) - g\rho. \tag{14}$$

Multiplication of (13) by dx and (14) by dz, and addition of the results yield

$$\begin{aligned} \rho_0 \eta' \, d\psi' &= -d\left(p + \frac{\rho_0 (u'^2 + w'^2)}{2}\right) - g\rho \, dz. \\ &= -d\left(p + \frac{\rho_0 (u'^2 + w'^2)}{2} + g\rho z\right) + gz \, d\rho \\ &= -dH(\psi') + gz \, d\rho(\psi'), \end{aligned} \tag{15}$$

or

$$\rho_0 \nabla_*^2 \psi' = \frac{dH}{d\psi'} - gz \frac{d\rho}{d\psi'}, \tag{16}$$

which can be written as

$$\rho_0 \nabla_*^2 \psi' + gz \frac{d\rho}{d\psi'} = F(\psi'). \tag{17}$$

This equation is originally obtained by Yih,[1] and possesses a form which is very much more suitable for further studies than the equation governing the stream function ψ of the actual flow field, first obtained by Long:[2]

$$\nabla_*^2\psi + \frac{1}{\rho}\frac{d\rho}{d\psi}\left(\frac{\psi_x^2+\psi_z^2}{2} + gz\right) = H(\psi) \quad . \tag{18}$$

It is easy to show that Long's equation can be simplified to (17) by Yih's transformation written in the form

$$\psi' = \int \sqrt{\rho/\rho_0}\, d\psi \quad .$$

In this study, the flow under consideration will be restricted between two rigid horizontal planes forming a channel. Two cases are investigated. In the first case, two-dimensional symmetric flow into a line sink is considered. In the second case, an obstacle is present on the lower boundary. In both cases, we seek a class of solutions exhibiting a discontinuity in the flow field, on one side of which the fluid is stagnant. In the second case, the stagnant zone lies in front and is caused by the obstacle.

It has been shown by Yih,[3] that if the fluid originates from a large reservoir, where the velocity is zero and flows into the channel horizontally, the associated flow is irrotational far upstream. If we now restrict our attention to a linear density stratification far upstream, equation (17) can be rendered exactly linear if u' = a positive constant A. For then

$$\psi' = -Az \quad . \tag{19}$$

If

$$\rho = \rho_0(1-\beta z) \,, \qquad \beta = \frac{\rho_0-\rho_1}{\rho_0 d} \,,$$

then (17) becomes

$$\rho_0 \nabla_*^2 \psi' + \frac{g\beta\rho_0}{A} z = F(\psi') ,$$

with

$$F(\psi') = -\frac{\rho_0 g\beta}{A^2} \psi' ,$$

by virtue of (19).

Therefore,

$$\nabla_*^2 \psi' + k\psi' = -Akz , \tag{20}$$

with

$$k = \frac{g\beta}{A^2} .$$

Equation (20) can be made dimensionless by

$$\Psi = \frac{\psi'}{Ad} , \qquad \xi = \frac{x}{d} , \qquad \eta = \frac{z}{d} . \tag{21}$$

The dimensionless form of (20) is then

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} + F^{-2}\Psi = -F^{-2}\eta , \tag{22}$$

where $F = \dfrac{A}{\sqrt{d^2 g \beta}}$ is a Froude number. Equation (22) is the equation to be solved, subject to the boundary conditions to be considered in the next section.

III. THE BOUNDARY CONDITIONS

Along the dividing streamline the pressure is fixed by the static pressure of the stagnant zone. The position of the dividing streamline is not fixed *a priori*. Physically, the conditions for stability of the total flow configuration are that the density of the stagnant fluid must never increase upwards, and that in the neighborhood of the streamline, the density must be greater than or less than the density along the streamline depending on whether the stagnant zone lies below or on top of the dividing streamline.

Bernoulli's equation,

$$\frac{q^2}{2} + gz + \frac{p}{\rho_s} = C \qquad (23)$$

holds along the dividing streamline, in which ρ_s is the density along the streamline and is a constant. Differentiation with respect to the distance s measured along the streamline yields,

$$\frac{d}{ds}\left(\frac{q^2}{2}\right) + g\frac{dz}{ds} + \frac{1}{\rho_s}\frac{dp}{ds} = 0 ,$$

or

$$\frac{d}{ds}\left(\frac{q^2}{2}\right) = -\frac{1}{\rho_s}\frac{dp}{ds} - g\frac{dz}{ds}$$

$$= -\left(g + \frac{1}{\rho_s}\frac{dp}{dz}\right)\frac{dz}{ds} .$$

But, since the pressure distribution in the stagnant zone is hydrostatic,

$$dp = -\rho g\, dz ,$$

so that

$$\frac{d}{ds}\left(\frac{q^2}{2}\right) = -g\left(1 - \frac{\rho}{\rho_s}\right)\frac{dz}{ds} \quad , \tag{24}$$

and

$$\frac{d^2}{dz^2}\left(\frac{q^2}{2}\right) = \frac{1}{\rho}\frac{d\rho}{dz} \quad . \tag{25}$$

Now $\frac{dz}{ds}$ can be either positive or negative depending on whether the dividing streamline is concave upward or downward as shown in Figure (1), for case (B) and case (A) respectively. In the geometry considered here, the stagnant zone is situated on the convex side of the dividing streamline.

Thus, for a stable flow configuration, the density of the stagnant fluid bordering the dividing streamline at any point along it is greater or less than ρ_s , depending on whether $\frac{dz}{ds}$ is positive or negative. It therefore follows from equation (24) that $\frac{d}{ds}\left(\frac{q^2}{2}\right)$ is positive. Conversely, if $\frac{d}{ds}\left(\frac{q^2}{2}\right)$ is positive, then it follows from equation (24) that ρ must be greater or less than ρ_s depending on whether $\frac{dz}{ds}$ is positive or negative. Furthermore, from equation (25), for a stable stagnant stratification, $\frac{d^2}{dz^2}\left(\frac{q^2}{2}\right)$ along the dividing streamline must be negative or zero. Indeed, if the stratification is stable $\frac{d\rho}{dz}$ is negative or zero, and $\frac{d^2}{dz^2}\left(\frac{q^2}{2}\right)$ is therefore negative or zero. Conversely, if $\frac{d^2}{dz^2}\left(\frac{q^2}{2}\right)$ is negative or zero, then $\frac{d\rho}{dz}$ is negative or zero, and the stagnant stratification is therefore stable. Hence, the necessary and sufficient conditions for a stable configuration with a stable stagnant zone and a flowing zone to exist are that, along the dividing streamline, the square of the

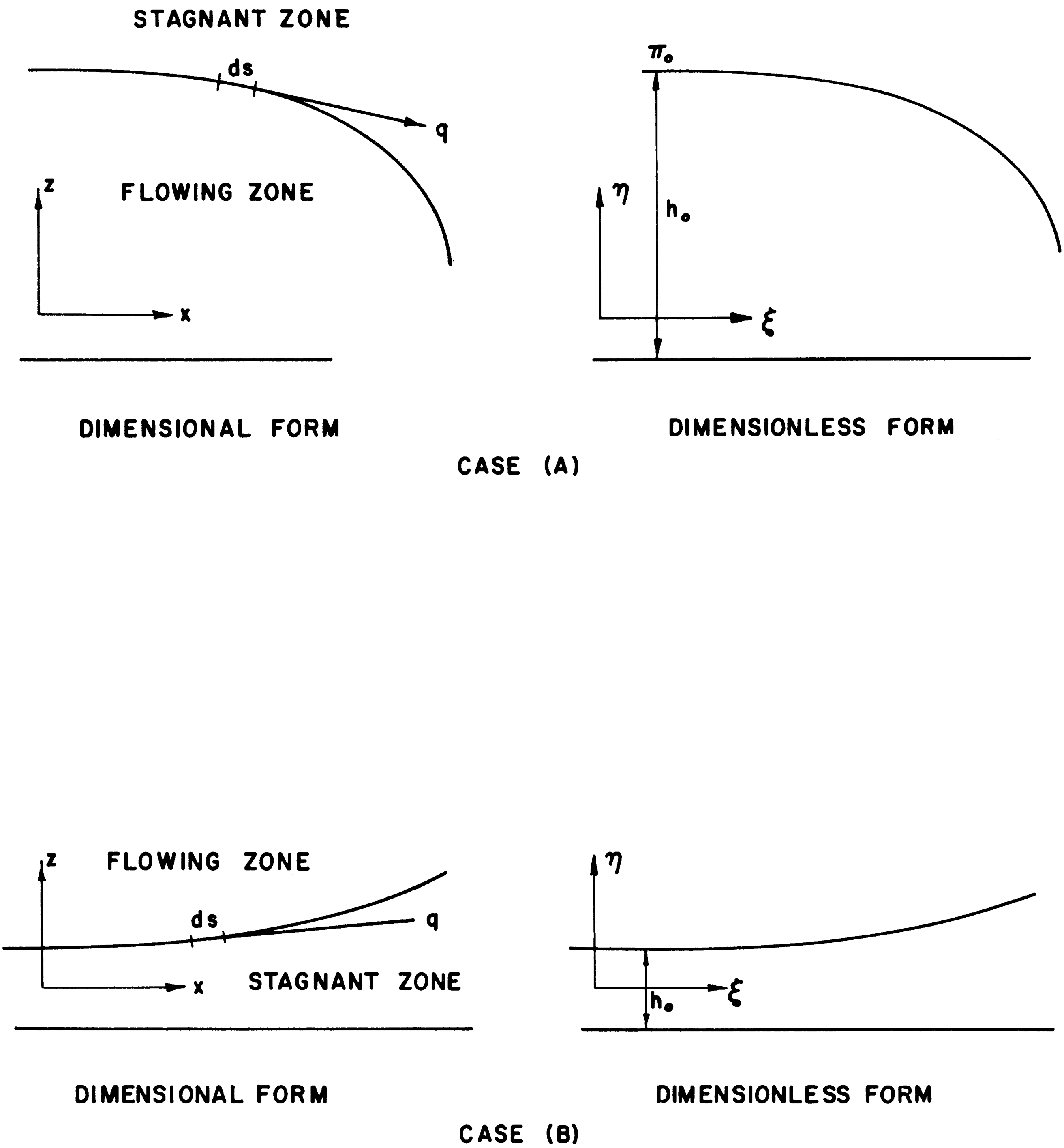


Figure 1. Configurations showing the dividing streamline and the location of the stagnant zone.

velocity be monotonically increasing, and $\frac{d^2}{dz^2}(\frac{q^2}{2})$ be negative or zero. For the inverse method of solution of this investigation, the above conditions are utilized to produce a posteriori a stable stagnant stratification, as will be seen in the next two sections.

The boundary condition along the dividing streamline is the satisfaction of Bernoulli's equation, which, in terms of the stream function ψ, can be written as

$$\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right\} + \frac{p}{\rho_s} + gz = C, \qquad (26)$$

Now

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial\psi'}\,\frac{\partial\psi'}{\partial\Psi}\,\frac{\partial\Psi}{\partial\xi}\,\frac{\partial\xi}{\partial x} = \sqrt{\frac{\rho_0}{\rho}}\,A d\,\frac{\partial\Psi}{\partial\xi}\,\frac{1}{d} = \sqrt{\frac{\rho_0}{\rho}}\,A\,\frac{\partial\Psi}{\partial\xi},$$

and, similarly,

$$\frac{\partial\psi}{\partial z} = \sqrt{\frac{\rho_0}{\rho}}\,A\,\frac{\partial\Psi}{\partial\eta} \quad .$$

Therefore (26) becomes

$$\frac{1}{2}\frac{\rho_0}{\rho_s}A^2\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2\right\} + \frac{p}{\rho_s} + g\eta d = C,$$

or

$$\frac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2\right\} + \frac{p}{\rho_0 A^2} + \frac{g\eta d}{A^2}\frac{\rho_s}{\rho_0} = C_1 \;,$$

or

$$\frac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2\right\} + \Pi + F_A^{-2}\eta = C_1 \;, \qquad (27)$$

with $\Pi = \frac{p}{\rho_0 A^2} =$ the non-dimensional pressure, and $F_A = \dfrac{A}{\sqrt{g(\rho_s/\rho_o)d}} =$ a Froude number.

If Π_o is the dimensionless pressure at a point $\eta = h_o$ on the dividing streamline far upstream, it then follows from (27) that

$$\tfrac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2\right\} + \Pi + F_A^{-2}\eta = \tfrac{1}{2}(1) + \Pi_o + F_A^{-2}h_o,$$

or

$$\tfrac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2 - 1\right\} = -F_A^{-2}(\eta - h_o) - (\Pi - \Pi_o), \quad (28)$$

which is the boundary condition along the dividing streamline and in which Π is the dimensionless hydrostatic pressure due to the stagnant fluid of some stable stratification.

It is useful at this stage to examine the variation of $(q')^2$ with η along the dividing streamline for some given stable stratifications. For a stagnant zone with stable linear stratification of the type

$$\rho_2 = \rho_o' - \rho_o\beta z,$$

where ρ_o' is a reference density at z = 0, and

$$p(z) = \int_z^d \rho_2 g\, dz .$$

Therefore

$$p(z) = \frac{\beta g\rho_o}{2}(z^2 - d^2) - \rho_o' g(z - d),$$

or

$$p(z) - p(h_0 d) = \frac{\beta g \rho_0}{2}(z^2 - h_0^2 d^2) - \rho_0' g (z - h_0 d).$$

Therefore

$$\Pi(\eta) - \Pi_0 = \tfrac{1}{2} F^{-2}(\eta^2 - h_0^2) - F_0^{-2}(\eta - h_0),$$

where

$$F_0 \equiv \frac{A}{\sqrt{g(\rho_0'/\rho_0)\, d}} .$$

Thus,

$$\tfrac{1}{2}\{(q')^2 - 1\} = -F_A^{-2}(\eta - h_0) - \tfrac{1}{2} F^{-2}(\eta^2 - h_0^2) + F_0^{-2}(\eta - h_0)$$

$$= -\tfrac{1}{2} F^{-2}(\eta^2 - h_0^2) - (F_A^{-2} - F_0^{-2})(\eta - h_0),$$

which is a quadratic in η .

For a stagnant zone of constant density ρ_B ,

$$\Pi(\eta) - \Pi_0 = -F_B^{-2}(\eta - h_0),$$

where

$$F_B \equiv \frac{A}{\sqrt{g(\rho_B/\rho_0)\, d}} .$$

Therefore

$$\tfrac{1}{2}\{(q')^2 - 1\} = -(F_A^{-2} - F_B^{-2})(\eta - h_0),$$

which is linear in η .

IV. FLOW INTO A SINK

A. Discussion and Statement of the Problem.

In this section the flow into a line sink is to be examined. The fluid is assumed to be confined between two parallel planes, one at $z = 0$, and one at $z = d$. The sink is situated in the lower plane at $x = 0$. Since everything is symmetric with respect to the z-axis, only one half of the flow field need be considered. The solution for flows with Froude's number greater than $1/\pi$ has been given by Yih(3), in which it was pointed out that no stagnant zone is possible at these rates of discharge. For lower rates of discharge, i.e.

$$Q \leq \frac{d_2^2}{\pi}\sqrt{g\beta}$$

experimental results of Debler(4) indicates the presence of a stagnant zone. The flow field is separated into two regions: a region where the fluid is essentially stagnant, and a region where the flow is concentrated. The line of discontinuity is a streamline and forms a vortex sheet. The Froude's numbers

$$F_1 \equiv \frac{Q}{d_1^2}\sqrt{\frac{1}{g\beta}}$$

and

$$F_2 \equiv \frac{Q}{d_2^2}\sqrt{\frac{1}{g\beta}} ,$$

based on the actual discharge rates are now the significant experimental parameters of the separated flow. In these definitions for F_1 and F_2, d_1 denotes the depth far upstream of the flowing part and d_2 denotes the total depth of the channel. It then follows immediately from these definitions that for a given discharge, the following relationship holds:

$$F_2 = F_1 \left(\frac{d_1}{d_2}\right)^2 . \tag{29}$$

The mathematical problem for flow with stagnant zone is then to find a solution to the equation

$$\nabla^2 \Psi^* + F_1^{-2} \Psi^* = -F_1^{-2} \eta^* ,$$

in which

$$F_1 \equiv \frac{A_1}{\sqrt{d_1^2 g \beta}} ,$$

and the asterisk indicates non-dimensionalizing by A_1 and d_1, with the boundary conditions

$$\Psi^* = 0 \quad \text{for} \quad \xi^* < 0 , \quad \eta^* = 0 ,$$

$$\Psi^* = -1 \quad \text{for} \quad \xi^* = 0 , \quad 0 < \eta^* < b^* ,$$

$$\Psi^* = -\eta^* \quad \text{for} \quad \xi^* = -\infty , \quad 0 \leq \eta^* \leq 1 ,$$

and along the dividing streamline,

$$\tfrac{1}{2} \left\{ \left(\frac{\partial \Psi^*}{\partial \xi^*}\right)^2 + \left(\frac{\partial \Psi^*}{\partial \eta^*}\right)^2 \right\} = C_1^* - F_A^{*-2} \eta^* - \Pi^* ,$$

in which Π^* is the dimensionless hydrostatic pressure due to the stagnant fluid. In order to solve this problem in the framework of the inverse method used in this study, an auxiliary problem is solved. the auxiliary problem is

$$\nabla^2 \Psi + F^{-2}\Psi = -F^{-2}\eta ,$$

with $F = F_1$ and subject to the boundary conditions

$$\Psi = 0 \qquad \text{for} \qquad \xi < 0 , \ \eta = 0,$$

$$\Psi = -h_0 \qquad \text{for} \qquad \xi = 0 , \ 0 < \eta < b,$$

$$\Psi = -\eta \qquad \text{for} \qquad \xi = -\infty , \ 0 \le \eta \le h_0 ,$$

and along the dividing streamline

$$\tfrac{1}{2}\left\{ \left(\frac{\partial \Psi}{\partial \xi}\right)^2 + \left(\frac{\partial \Psi}{\partial \eta}\right)^2 \right\} = c_1 - F_A^{-2}\eta - \Pi ,$$

in which Π is the hydrostatic pressure due to the stagnant fluid. Because of the conditions of stability established in Section II, the dividing streamline must meet the line $\xi = 0$ tangentially. The solution to this auxiliary problem can then be cast back into the solution to the actual problem by the similarity transformations

$$\Psi^* = \frac{\Psi}{h_0} , \qquad h_0 = \frac{d_0}{d} ,$$

and

$$A d_0 = A_1 d_1 ,$$

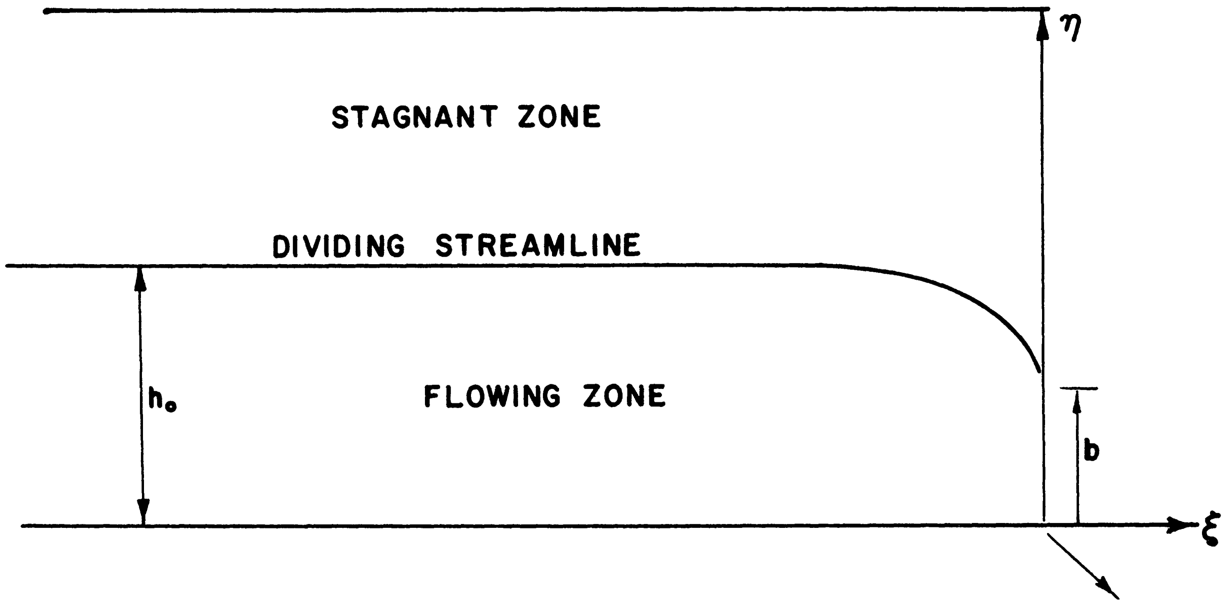


Figure 2. Flow into a sink.

and

$$dd_o = d_1^2 .$$

In the inverse method outlined below, various values of F and h_o are assumed and with the introduction of a suitable sink distribution on $\xi = o$, a solution satisfying the boundary value problem above is to be obtained.

B. Method of Solution.

A direct method of solution to the problem just stated is difficult, and in fact no advantage can be gained from attacking the problem directly. An inverse method is therefore employed. Inverse methods have yielded large classes of solutions to otherwise difficult problems in fluid mechanics.

The method is to introduce a distribution of sink on $\xi = o$. In this way the flow field is still continuous everywhere but there is one streamline which divides the flow into two regions; one part flowing completely into the original sink and the other into the sink distribution that has been introduced. This new problem can be stated as follows:

$$\nabla^2 \Psi + F^{-2} \Psi = - F^{-2} \eta , \tag{30}$$

in which Ψ is assumed to be of the form

$$\Psi = a \Psi_1 + (1-a) \Psi_2 ,$$

with Ψ_1 representing the flow into the original sink and satisfying the following boundary conditions:

$$\Psi_1 = -1 \quad \text{for} \quad \xi = 0 , \quad \eta = 1 ,$$

$$\Psi_1 = 0 \quad \text{for} \quad -\infty < \xi < 0 , \quad \eta = 0 ,$$

and

$$\Psi_1 = -\eta \quad \text{for} \quad \xi = -\infty , \; 0 \le \eta \le 1 ,$$

and Ψ_2 representing the flow into the sink distribution and satisfying the following boundary conditions:

$$\Psi_2 = -1 \quad \text{for} \quad -\infty < \xi < 0 , \quad \eta = 1 ,$$

$$\Psi_2 = 0 \quad \text{for} \quad -\infty < \xi < 0 , \quad \eta = 0 ,$$

$$\Psi_2 = -\eta \quad \text{for} \quad \xi = -\infty , \; 0 \le \eta \le 1 ,$$

$$\Psi_2 = g(\eta) \quad \text{for} \quad \xi = 0 , \; b < \eta \le 1 ,$$

$$\Psi_2 = 0 \quad \text{for} \quad \xi = 0 , \; 0 < \eta \le b .$$

Here, "a" represents the percentage of the total flow field that flows into the original sink. The solution Ψ obtained exhibits a dividing streamline, along which velocity can be calculated and therefore the pressure distribution can be computed. Now if the upper region, namely the part that flows into the fictitious sink distribution introduced on $\xi = 0$, is replaced by a stagnant layer of fluid of a stable stratification, and if the static pressure produced by the stagnant layer is equal to the pressure computed before, then this is a solution to the original boundary value problem.

The inverse method consists of the suitable choice of the sink distribution such that the velocity along the dividing streamline satisfies the conditions of stability; namely that q^2 be monotonically increasing and $\frac{d^2}{dz^2}\left(\frac{q^2}{2}\right) \leq 0$ along it. For, when these conditions are satisfied a unique stable stratification of the stagnant layer is determined.

Returning to the solution of Ψ_1 and Ψ_2 and with the assumption that

$$\Psi = -\eta + f, \quad \Psi_1 = -\eta + f_1, \quad \Psi_2 = -\eta + f_2,$$

then (30) becomes

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} + F^{-2} f = 0, \tag{31}$$

and the boundary conditions for f_1 are

$$f_1 = 0 \quad \text{for} \quad -\infty < \xi < 0, \quad \eta = 1, \tag{32}$$

$$f_1 = -1 + \eta \quad \text{for} \quad \xi = 0, \quad 0 < \eta \leq 1, \tag{33}$$

$$f_1 = 0 \quad \text{for} \quad -\infty < \xi < 0, \quad \eta = 0, \tag{34}$$

$$f_1 = 0 \quad \text{for} \quad \xi = -\infty, \quad 0 \leq \eta \leq 1. \tag{35}$$

By separation of variables and with conditions (32), (34), and (35), it is found that

$$f_1 = \sum_{n=1}^{\infty} B_n e^{a_n \xi} \sin n\pi\eta , \tag{36}$$

with

$$a_n^2 = n^2\pi^2 - F^{-2} .$$

From (33), the B's, which are the Fourier coefficients of $\eta - 1$ in terms of the complete orthogonal set $\{\sin n\pi\eta\}$, are given by

$$B_n = -\frac{2}{n\pi} . \tag{37}$$

Therefore

$$\Psi_1 = -\eta - \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{a_n \xi} \sin n\pi\eta . \tag{38}$$

For Ψ_2, it is necessary to assume the form of $g(\eta)$. A sink distribution of uniform strength from $\eta = b$ to $\eta = 1$ is assumed in this analysis. If the strength of the sink distribution is α, then

$$\Psi_2 = \int_b^{\eta} \alpha \, d\eta = \alpha (\eta - b) ,$$

and

$$-1 = \int_b^{1} \alpha \, d\eta = \alpha (1 - b) .$$

Therefore

$$\alpha = -\frac{1}{(1-b)} \,.$$

Hence

$$\left.\begin{aligned} \Psi_2 &= -\left(\frac{\eta - b}{1-b}\right) \quad \text{at} \quad \xi = 0\,, \quad b < \eta \le 1\,, \\ &\text{and} \\ \Psi_2 &= 0 \quad \text{at} \quad \xi = 0\,, \quad 0 \le \eta \le b\,. \end{aligned}\right\} \tag{39}$$

The boundary value problem for Ψ_2 can be restated in terms of f_2 as follows:

$$\frac{\partial^2 f_2}{\partial \xi^2} + \frac{\partial^2 f_2}{\partial \eta^2} + F^{-2} f_2 = 0\,, \tag{40}$$

$$f_2 = 0 \quad \text{for} \quad -\infty < \xi < 0\,, \quad \eta = 1\,, \tag{41}$$

$$f_2 = 0 \quad \text{for} \quad -\infty < \xi < 0\,, \quad \eta = 0\,, \tag{42}$$

$$f_2 = 0 \quad \text{for} \quad \xi = -\infty\,, \quad 0 \le \eta \le 1\,, \tag{43}$$

$$f_2 = \eta \quad \text{for} \quad \xi = 0\,, \quad 0 \le \eta \le b\,, \tag{44}$$

$$f_2 = \eta - \left(\frac{\eta - b}{1-b}\right) \quad \text{for} \quad \xi = 0\,, \quad b < \eta \le 1\,. \tag{45}$$

Therefore

$$f_2 = \sum_{n=1}^{\infty} C_n e^{a_n \xi} \sin n\pi\eta\,, \tag{46}$$

with

$$a_n^2 = n^2\pi^2 - F^{-2} .$$

At $\xi = 0$,

$$\sum_{n=1}^{\infty} C_n \sin n\pi\eta = \eta \qquad \text{for} \qquad 0 \le \eta \le b ,$$

$$= \eta - \left(\frac{\eta - b}{1 - b}\right) \quad \text{for} \quad b < \eta \le 1 .$$

Therefore

$$C_n = 2\int_0^1 \eta \sin n\pi\eta \, d\eta - 2\int_b^1 \left(\frac{\eta - b}{1-b}\right) \sin n\pi\eta \, d\eta$$

$$= \left(\frac{2}{1-b}\right)\left(\frac{1}{n\pi}\right)^2 \sin n\pi b. \qquad (47)$$

With the above value for C_n, Ψ_2 becomes

$$\Psi_2 = -\eta + \sum_{n=1}^{\infty} \left(\frac{2}{1-b}\right)\left(\frac{1}{n\pi}\right)^2 \sin n\pi b \; e^{a_n\xi} \sin n\pi\eta . \qquad (48)$$

Finally

$$\Psi = -\eta - \sum_{n=1}^{\infty} \frac{2a}{n\pi} e^{a_n\xi} \sin n\pi\eta +$$

$$+ (1-a) \sum_{n=1}^{\infty} \left(\frac{2}{1-b}\right)\left(\frac{1}{n\pi}\right)^2 \sin n\pi b \; e^{a_n\xi} \sin n\pi\eta . \qquad (49)$$

The series converges uniformly for all values of $\xi < 0$ and $0 \le \eta \le 1$. Now

$$U' = -\frac{\partial\Psi}{\partial\eta} , \qquad W' = \frac{\partial\Psi}{\partial\xi} .$$

Therefore

$$U' = 1 + \sum_{n=1}^{\infty} 2a\, e^{a_n \xi} \cos n\pi\eta - (1-a) \sum_{n=1}^{\infty} \left(\frac{2}{1-b}\right)$$

$$\left(\frac{1}{n\pi}\right) \sin n\pi b\, e^{a_n \xi} \sin n\pi\eta \,, \tag{50}$$

$$W' = -\sum_{n=1}^{\infty} \frac{2a}{n\pi} a_n e^{a_n \xi} \sin n\pi\eta + (1-a) \sum_{n=1}^{\infty} \left(\frac{2}{1-b}\right)$$

$$\left(\frac{1}{n\pi}\right)^2 a_n \sin n\pi b\, e^{a_n \xi} \sin n\pi\eta \,. \tag{51}$$

For any assumed value of F, "a" (which is equivalent to assuming h_o), and "b", the velocity along the dividing streamline can be calculated. From this a graph of q'^2 against η can be plotted to see whether the stablilty conditions are satisfied. The detailed calculations involve a process of trial and error.

C. Results and Discussion.

The solutions found indicate a Froude number of 0.345 for the flowing part for all separated flows. It has been shown analytically by Yih[5] that there exists an unique Froude number for the flowing part for such flows. That the number found here is a reasonable figure of this unique Froude number can be seen by the following consideration. For F slightly bigger than $1/\pi$, Yih's solution[3] shows a large eddy which is nearly horizontal, resulting in return flow to infinity. These eddies are moreover unstable. It is therefore indicated that a flow with discontinuity in the flow field, as given here, is called for, with the flowing part possessing a Froude number of a magnitude in the neighborhood of a number somewhat bigger than $1/\pi$, which is indeed the case found here. The results produced with discontinuity in the flow field are thus the desired solutions.

Experimental values of Debler[4] indicates that the Froude number for all separated flows lies in the neighborhood of 0.28. The discrepancy with the number obtained here is actually superficial rather than real. This is because of the fact that in the experimental measurements the effect of viscosity tends to make the depth of the stagnant zone smaller so that for the same discharge the measured d_1 is bigger in the case with viscosity than if the viscosity is completely absent. Also the presence of the boundary layer at the bottom of the channel in the experimental case increases the observed depth of the flowing zone. Furthermore the side-wall effect also tends to reduce the actual discharge compared with the theoretical discharge. Now, since the error in the depth of the flowing zone enters as a squared term, the experimental values when suitably corrected appear to be in agreement with the result obtained here.

Figures (3a) and (4a) show that q'^2 varies linearly with η. Thus the density of the stagnant zone is a constant. For Figure (3a), the equation for q'^2 against η is

$$\frac{(q')^2 - 1}{2} = 0.9\eta + 0.46 .$$

The density of the stagnant zone is therefore given by

$$\rho_B = (\rho_s - \frac{0.9 A^2}{gd}\rho_0) .$$

The Froude number F_2 based on the discharge and the total depth is determined to be 0.176. For Figure (4a) the equation for q'^2 against

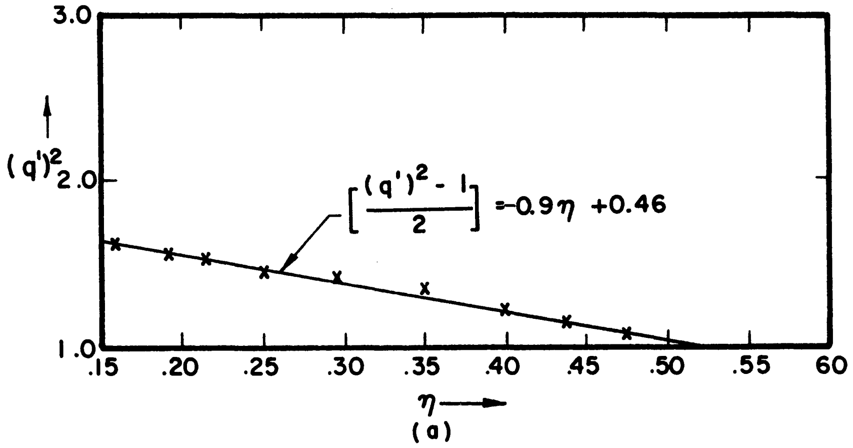


Figure 3(a). Graph of q'^2 against η .

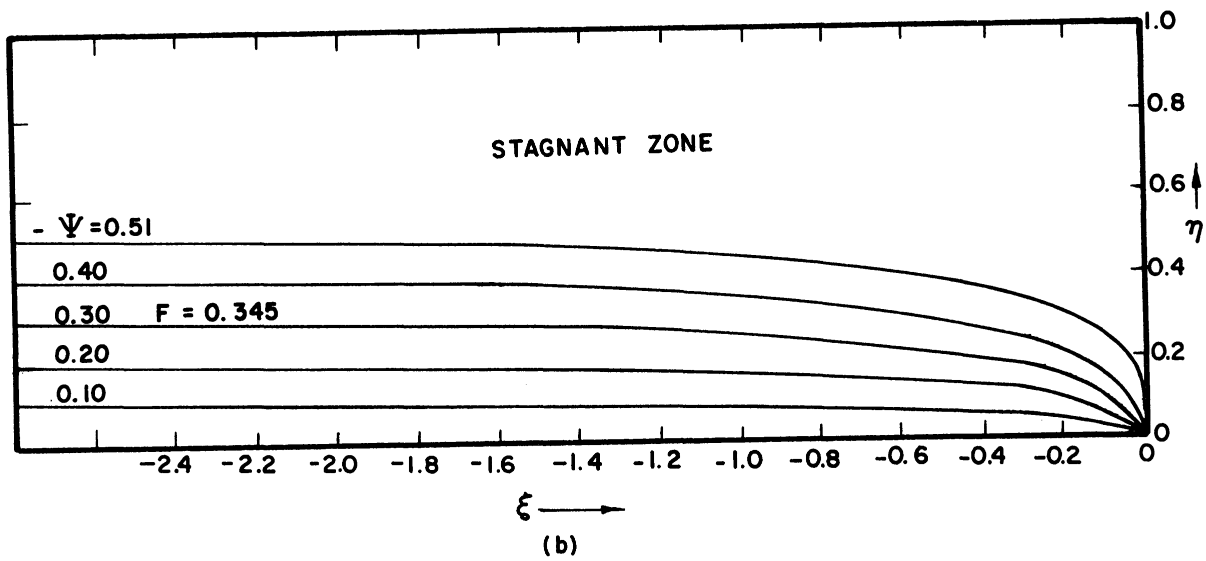


Figure 3(b). Flow pattern into a sink with stagnant zone at Froude number equal to 0.345.

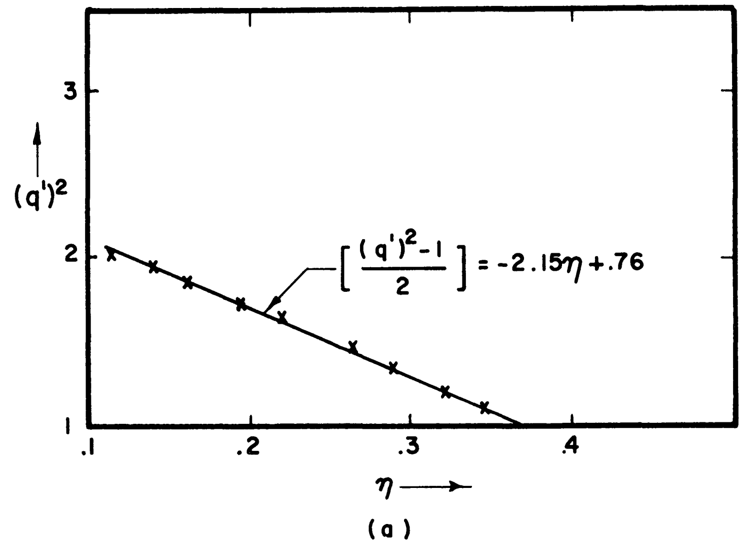


Figure 4(a). Graph of q'^2 against η .

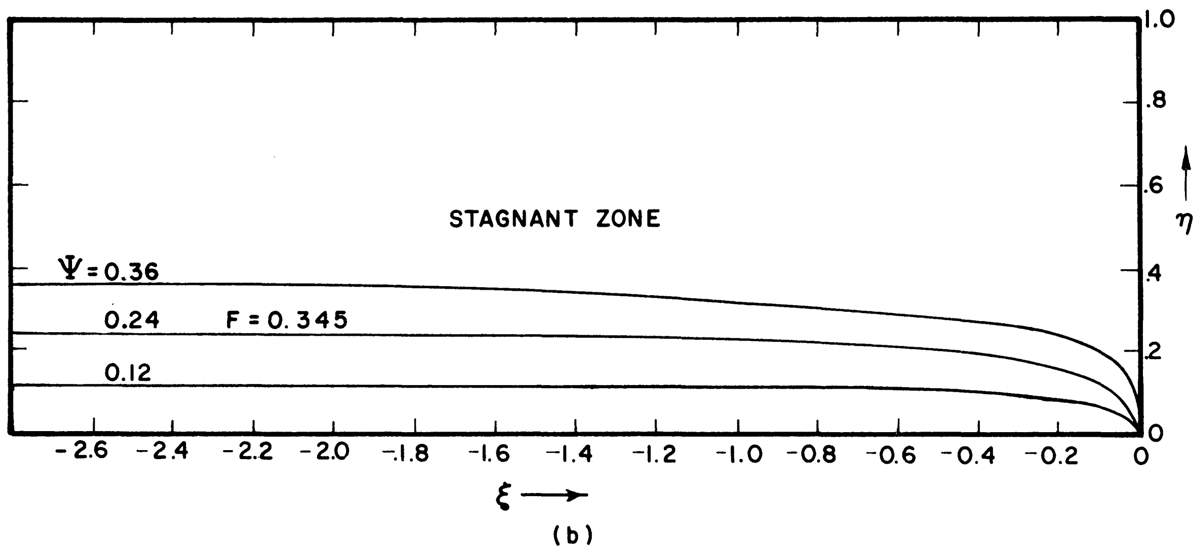


Figure 4(b). Flow pattern into a sink with stagnant zone at Froude number equal to 0.345.

η is given by

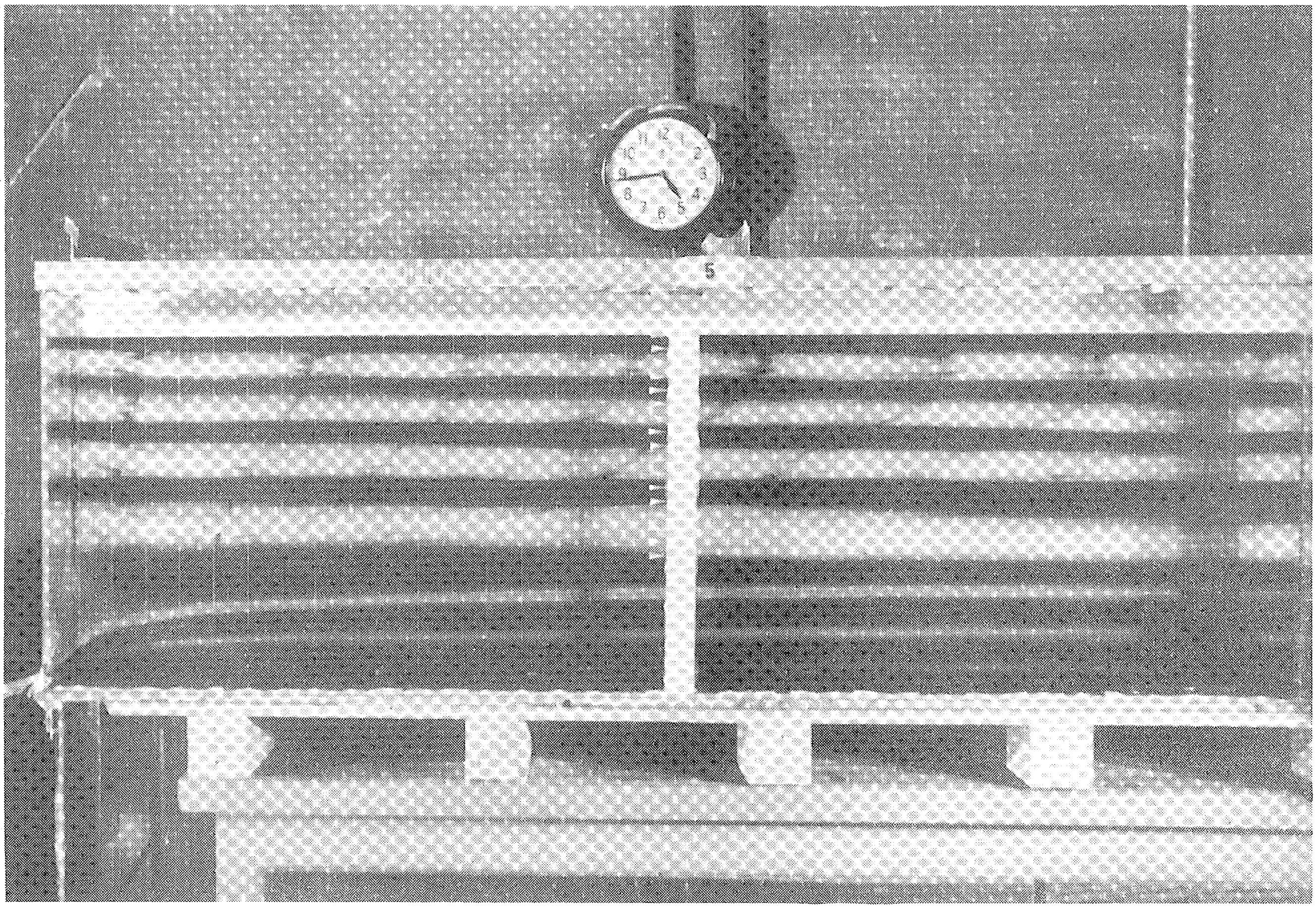
$$\frac{(q')^2 - 1}{2} \qquad -2.15\,\eta + 0.76\,.$$

The density of the stagnant zone is therefore given by

$$\rho_B \qquad (\rho_s - \frac{2.15\,A^2}{g\,d}\,\rho_o)\,.$$

The Froude number F_2 based on the discharge and the total depth is determined to be 0.124. Figures (3b) and (4b) show the flow pattern of the separated flow and compare rather well with the photographs taken by Debler(4) in his experiments.

It is to be noted that the stagnant fluid in the region between the dividing streamline and the horizontal tangent to the dividing streamline far upstream (we call this region the "wedge" region for brevity) is of constant density. The density of the stagnant fluid above this region may be of any stratification so long as it is statically stable. In the actual physical realization of these flows, the constancy of the density of the stagnant fluid in the wedge region is probably achieved. This is because at the initiation of this flow the layer of fluid above the dividing streamline is required to expand and spread out to fill up the wedge region. There is also some accompanied mixing due to viscosity at this initial stage before the flow conditions become steady.



Flow pattern into a sink for $F_1 = 0.245$.
Photograph taken by Debler.(4)

V. FLOW OVER AN OBSTACLE

A. Discussion and Statement of the Problem.

In this problem the fluid is still confined between two parallel planes as in the last problem. An obstacle is now introduced on the lower boundary. Physically, this corresponds to the study of atmospheric flows past mountain ridges. The interface between the troposphere and the stratosphere is approximated by the upper rigid plane. The investigation of large amplitude lee-waves downstream from the obstacle has recently been done by Long(2), Yih(6), and Claus(7); however, no theoretical work has been done regarding the phenomenom of blocking in front of the obstacle.

At low speeds of flow, it has been observed that regions of essentially stagnant fluid extend in front of the obstacle to far upstream. A well-known example that occurs in nature is the stagnant zone in front of mountain ridges when the prevailing winds blow at a low speed. If there is no viscosity, then there is a streamline across which the flow is discontinuous. The problem is therefore one which is similar to the one considered in the last section. Here, a streamline is to be found which extends from the obstacle to far upstream, dividing the flow into two regions. Below this streamline, the fluid is stagnant. Above it there is flow. The position of the dividing streamline is as before not known *a priori*. The pressure condition for a stable stagnant zone is to be satisfied also.

It is assumed that the obstacle is given by $\eta = h(\xi)$ where

$$h(\xi) \neq 0 \qquad \text{for} \qquad -r \leq \xi \leq s \, ,$$

and

$$h(\xi) = 0 \qquad \text{otherwise.}$$

The streamline which forms the line of discontinuity must meet the obstacle tangentially, otherwise there will be a stagnation point rendering the flow unstable, The line of discontinuity is assumed to meet the obstacle at $\xi = -e$. The boundary value problem is then

$$\nabla^2 \Psi + F^{-2}\Psi = -F^{-2}\eta \ , \tag{52}$$

with the boundary conditions

$$\Psi = -1 \qquad \text{for} \qquad -\infty < \xi < +\infty \ , \quad \eta = 1 \ , \tag{53}$$

$$\Psi = -\eta \qquad \text{for} \qquad \xi = -\infty \ , \quad 0 \le \eta \le 1 \ , \tag{54}$$

$$\Psi = -m_0 \qquad \text{for} \qquad s < \xi < +\infty \ , \quad \eta = 0 \ , \tag{55}$$

$$\Psi = 0 \qquad \text{for} \qquad -\infty < \xi < -r \ , \quad \eta = 0 \ , \tag{56}$$

$$\Psi = -m_0 \qquad \text{for} \qquad -e < \xi < s \ , \quad \eta = h(\xi) \ , \tag{57}$$

and, along the dividing streamline $\Psi = -m_0$,

$$\tfrac{1}{2}\left\{\left(\frac{\partial \Psi}{\partial \xi}\right)^2 + \left(\frac{\partial \Psi}{\partial \eta}\right)^2\right\} = -\Pi - F_A^{-2}\eta + C_2 \ , \tag{58}$$

where Π is the static pressure due to the stable stagnant zone.

In this problem, it is desired to study the nature of the stagnant zone with the variation of Froude number, and also to study

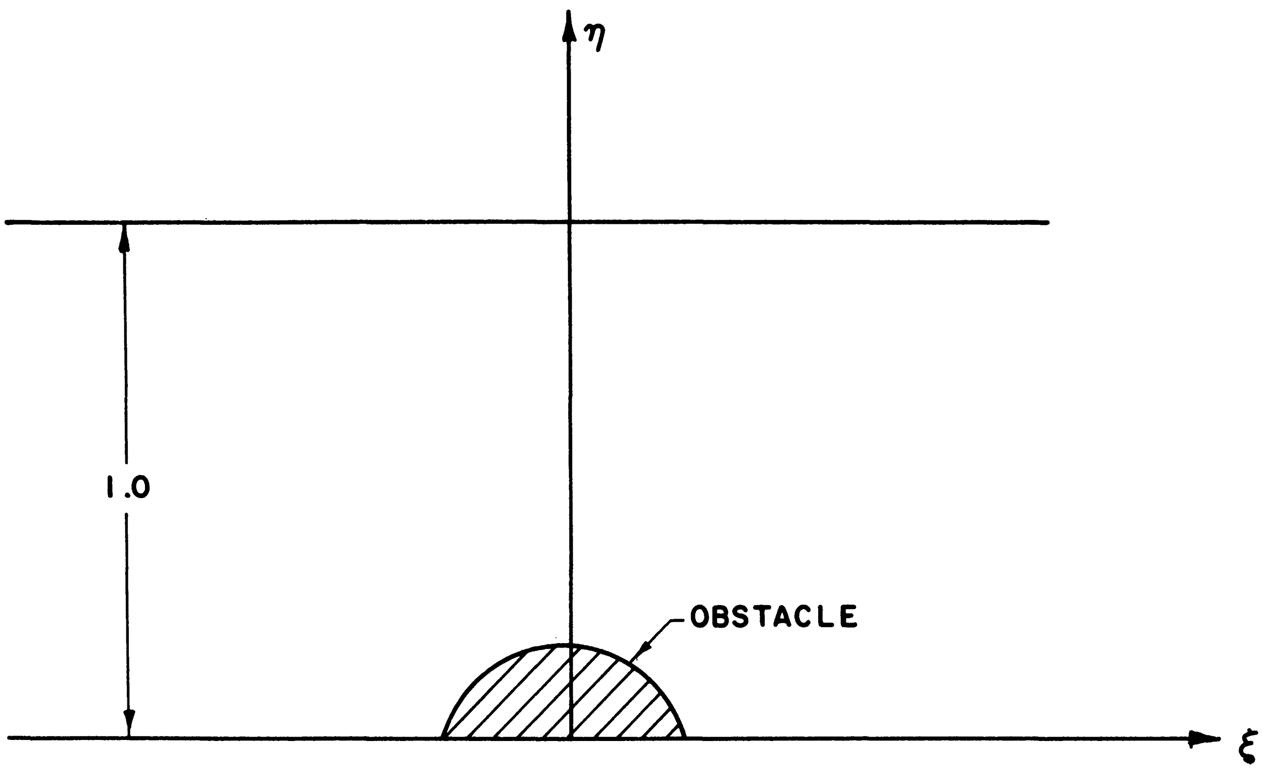


Figure 5. Obstacle in channel.

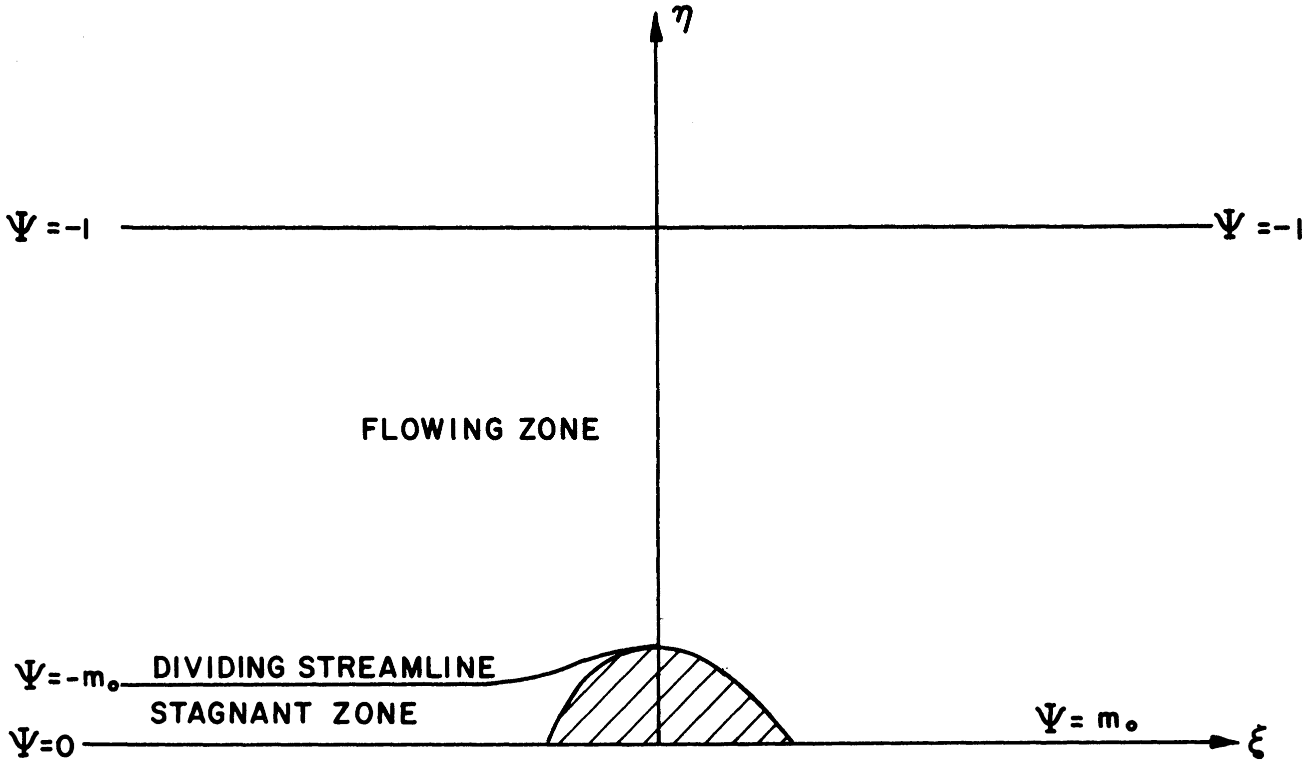


Figure 6. Flow over an obstacle.

the effect of the shape of the obstacle on the formation of the stagnant zone. It is reasonable to conjecture that in front of obstacles whose upstream face has a steep rise, the height of the stagnant zone should be as high as the obstacle itself. On the other hand, if the upstream face has a gentle slope then lower stagnant zones should also be possible.

B. Method of Solution.

In this problem, an inverse method is again employed. An infinite channel is considered and singularities which perturb the original parallel flow are introduced. The singularities are in the form of sinks and vortex distributions located near $\xi = 0$ and extending from $\eta = 0$ to near the height of the obstacle to be realized in the flow field. This type of singularities introduces a streamline which is open upstream and closed downstream, so that the flow field is divided into two parts: one part flowing out downstream and the other into the sinks that have been introduced. As in the last problem, the pressure distribution along the dividing streamline can be calculated. If the lower region is considered partly as the obstacle and partly as a stagnant zone, and if the pressure calculated is equal to the static pressure on the dividing streamline produced by an equivalent stagnant fluid of stable stratification, then this is a solution to the boundary value problem posed.

The inverse method consists in choosing suitable sink and vortex distributions so that the desired obstacle shape is obtained and such that the velocity satisfies the conditions of stability along the dividing streamline, so that a stable stratification is determined.

The flow field Ψ_s with sink singularities only is now considered first, and the flow field Ψ_v for vortex distributions is superimposed later and suitably adjusted to give the flow field the desired features. Thus, with "a" representing the percentage of the total flow field that flows into the sinks,

$$\Psi = a\Psi_s + (1-a)\Psi_v \ , \qquad (59)$$

in which Ψ_s satisfies equation (52) with the following boundary conditions:

$$\Psi_s = 0 \quad \text{for} \quad -\infty < \xi < 0 \,,\ \eta = 0 \,, \qquad (60)$$

$$\Psi_s = -m \quad \text{for} \quad 0 < \xi < +\infty \,,\ \eta = 0 \,, \qquad (61)$$

$$(m_0 = am)$$

$$\Psi_s = -1 \quad \text{for} \quad -\infty < \xi < +\infty \,,\ \eta = 1 \,, \qquad (62)$$

$$\Psi_s = -\eta \quad \text{for} \quad \xi = -\infty \,, 0 \le \eta \le 1 \,, \qquad (63)$$

and at $\xi = 0$ the solution Ψ_{s-} for $-\infty < \xi < 0$ and the solution Ψ_{s+} for $0 < \xi < +\infty$ are matched by the requirements

$$\Psi_{s-} - \Psi_{s+} = f_1(\eta) \,, \qquad (64)$$

$$\frac{\partial \Psi_{s-}}{\partial \xi} - \frac{\partial \Psi_{s+}}{\partial \xi} = 0 \ . \qquad (65)$$

These requirements represent a singularity of the sink type. The part Ψ_V satisfies equation (52) with the boundary conditions

$$\Psi_v = 0 \qquad \text{for} \qquad -\infty < \xi < 0\ , \quad \eta = 0\ , \tag{66}$$

$$\Psi_v = 0 \qquad \text{for} \qquad 0 < \xi < +\infty\ , \quad \eta = 0\ , \tag{67}$$

$$\Psi_v = -1 \qquad \text{for} \qquad -\infty < \xi < +\infty\ , \quad \eta = 1\ , \tag{68}$$

$$\Psi_v = -\eta \qquad \text{for} \qquad \xi = -\infty\ , \quad 0 \leq \eta \leq 1\ , \tag{69}$$

and at $\xi = 0$, the solution Ψ_{V-} for $-\infty < \xi < 0$ and the solution Ψ_{V+} for $0 < \xi < +\infty$ are matched by the requirements

$$\Psi_{v-} - \Psi_{v+} = 0\ , \tag{70}$$

$$\frac{\partial \Psi_{v-}}{\partial \xi} - \frac{\partial \Psi_{v+}}{\partial \xi} = f_2(\eta)\ . \tag{71}$$

These requirements represent a singularity of the vortex type. The representation of singularities in the above manner is first introduced in fluid problems by Yih[6].

The above problem for Ψ_S is solved by assuming

$$\Psi_s(\xi,\eta) = \Psi_1(\eta) + \Psi_2(\xi,\eta)\ , \tag{72}$$

where Ψ_1 is the parallel part and satisfies the equation

$$\frac{d^2\Psi_1}{d\eta^2} + F^{-2}\Psi_1 = -F^{-2}\eta\ , \tag{73}$$

and the boundary conditions

$$\Psi_{1-}(0) = 0 , \qquad \Psi_{1-}(1) = -1 , \tag{74}$$

$$\Psi_{1+}(0) = -m , \qquad \Psi_{1+}(1) = -1 , \tag{75}$$

and Ψ_2 satisfies the homogeneous equation

$$\nabla^2 \Psi_2 + F^{-2} \Psi_2 = 0 , \tag{76}$$

with the boundary conditions

$$\Psi_{2-}(\xi, 0) = 0 , \quad \Psi_{2-}(\xi, 1) = 0 , \quad \Psi_{2-}(-\infty, \eta) = 0 , \tag{77}$$

$$\Psi_{2+}(\xi, 0) = 0 , \quad \Psi_{2+}(\xi, 1) = 0 . \tag{78}$$

Equation (73) with (74) yields the solution

$$\Psi_{1-} = -\eta . \tag{79}$$

Equation (73) with (75) yields the solution

$$\Psi_{1+} = -\eta + m \cot F^{-1} \sin F^{-1}\eta - m \cos F^{-1}\eta . \tag{80}$$

Equation (76) with (77) and (78) admits a pair of solutions

$$\Psi_{2-} = \sum_{n=1}^{\infty} A_n e^{\sqrt{a_n}\,\xi} \sin n\pi\eta , \tag{81}$$

$$\Psi_{2+} = \sum_{n=1}^{\infty} D_n e^{-\sqrt{a_n}\,\xi} \sin n\pi\eta , \tag{82}$$

where

$$a_n^2 = n^2\pi^2 - F^{-2}.$$

For $F < 1/\pi$ wave components are present. However, because of the last condition in (77), no oscillatory terms are allowed for $\xi < 0$, i.e. no waves are present upstream, Thus,

$$\Psi_{2-} = \sum_{n=N+1}^{\infty} A_n e^{\sqrt{a_n}\xi} \sin n\pi\eta \,, \tag{83}$$

$$\Psi_{2+} = \sum_{n=1}^{N} (B_n \cos\sqrt{-a_n}\xi + C_n \sin\sqrt{-a_n}\xi) \sin n\pi\eta$$

$$+ \sum_{n=N+1}^{\infty} D_n e^{-\sqrt{a_n}\xi} \sin n\pi\eta \,, \tag{84}$$

in which $N + 1$ is the smallest integer that makes a_n^2 positive. Thus equation (52) with boundary conditions (60) and (63) has the following solutions:

$$\Psi_{s-} = -\eta + \sum_{n=N+1}^{\infty} A_n e^{\sqrt{a_n}\xi} \sin n\pi\eta \,, \tag{85}$$

$$\Psi_{s+} = -\eta + m \cot F^{-1} \sin F^{-1}\eta - m \cos F^{-1}\eta +$$

$$+ \sum_{n=1}^{N} (B_n \cos\sqrt{-a_n}\xi + C_n \sin\sqrt{-a_n}\xi) \sin n\pi\eta +$$

$$+ \sum_{n=N+1}^{\infty} D_n e^{-\sqrt{a_n}\xi} \sin n\pi\eta \,. \tag{86}$$

This solution corresponds to waves downstream but not upstream, and a sink at $\xi = 0$. The constants are determined by conditions (64) and

(65). Substitution of (85) and (86) into (64) and (65) yields

$$\sum_{n=N+1}^{\infty} A_n \sin n\pi\eta - m \cot F^{-1} \sin F^{-1}\eta + m \cos F^{-1}\eta -$$

$$- \sum_{n=1}^{N} B_n \sin n\pi\eta - \sum_{n=N+1}^{\infty} D_n \sin n\pi\eta = f_1(\eta) , \qquad (87)$$

and

$$\sum_{n=N+1}^{\infty} A_n \sqrt{a_n} \sin n\pi\eta = \sum_{n=1}^{\infty} C_n \sqrt{-a_n} \sin n\pi\eta +$$

$$+ \sum_{n=N+1}^{\infty} (-\sqrt{a_n}) D_n \sin n\pi\eta . \qquad (88)$$

Therefore

$$C_n = 0 ,$$

and

$$D_n = -A_n .$$

Therefore (87) becomes

$$\sum_{n=1}^{N} B_n \sin n\pi\eta + \sum_{n=N+1}^{\infty} 2 A_n \sin n\pi\eta = f_1(\eta) +$$

$$+ m \cot F^{-1} \sin F^{-1}\eta - m \cos F^{-1}\eta . \qquad (89)$$

The B_n's and A_n's are then simply the Fourier coefficients.

Therefore

$$B_n = -2 \int_0^1 \left(f_1(\eta) + m \cot F^{-1} \sin F^{-1}\eta - m \cos F^{-1}\eta \right) \sin n\pi\eta \, d\eta , \qquad (90)$$

and

$$A_n = \int_0^1 (f_1(\eta) + m \cot F^{-1} \sin F^{-1}\eta - m \cos F^{-1}\eta) \sin n\pi\eta \, d\eta . \quad (91)$$

The solution of (52) with (66) to (69) yields a pair of solutions for Ψ_V. The solutions are

$$\Psi_{V-} = -\eta + \sum_{n=N+1}^{\infty} A_n' e^{\sqrt{a_n}\xi} \sin n\pi\eta , \quad (92)$$

$$\Psi_{V+} = -\eta + \sum_{n=1}^{N} (B_n' \cos\sqrt{-a_n}\xi \sin n\pi\eta + C_n' \sin\sqrt{-a_n}\xi \cdot$$

$$\sin n\pi\eta) + \sum_{n=N+1}^{\infty} D_n' e^{-\sqrt{a_n}\xi} \sin n\pi\eta . \quad (93)$$

The constants are determined by conditions (70) and (71). Use of (70) yields

$$A_n' = D_n' , \qquad B_n' = 0 . \quad (94)$$

Use of (71) and (94) yields

$$\sum_{n=N+1}^{\infty} \sqrt{a_n} A_n' \sin n\pi\eta - \sum_{n=1}^{N} C_n' \sqrt{-a_n} \sin n\pi\eta +$$

$$+ \sum_{n=1}^{N} \sqrt{a_n} A_n' \sin n\pi\eta = f_2(\eta) .$$

Therefore

$$-\sum_{n=1}^{N} C_n' \sqrt{-a_n} \sin n\pi\eta + \sum_{n=N+1}^{\infty} 2\sqrt{a_n} A_n' \sin n\pi\eta = f_2(\eta) ,$$

in which

$$C_n' = -\frac{2}{\sqrt{-a_n}} \int_0^1 f_2(\eta) \sin n\pi\eta \, d\eta , \qquad (95)$$

and

$$A_n' = \frac{1}{\sqrt{a_n}} \int_0^1 f_2(\eta) \sin n\pi\eta \, d\eta . \qquad (96)$$

For suitably chosen F, m, and $f_1(\eta)$ and $f_2(\eta)$, the velocity along the dividing streamline can be calculated. From this q'^2 against η can be plotted to see whether the stability conditions are satisfied. The detailed calculations involve a process of trial and error.

C. Results and Discussion.

The values chosen for $f_1(\eta)$ and $f_2(\eta)$ have been determined to be as follows: $f_1(\eta) = T(\eta)$, $T(\eta)$ being a step function which is equal to one at $\eta = 0$ and equal to zero for $0 < \eta \leq 1$. (90) and (91) then gives

$$B_n = -m \cot F^{-1} \left\{ \frac{\sin(n\pi - F^{-1})}{(n\pi - F^{-1})} - \frac{\sin(n\pi + F^{-1})}{(n\pi + F^{-1})} \right\} -$$

$$-m \left\{ \frac{\cos(n\pi + F^{-1})}{(n\pi + F^{-1})} + \frac{\cos(n\pi - F^{-1})}{(n\pi - F^{-1})} - \frac{2n\pi}{(n^2\pi^2 - F^{-2})} \right\} ,$$

$$A_n = \frac{m \cot F^{-1}}{2} \left\{ \frac{\sin(n\pi - F^{-1})}{(n\pi - F^{-1})} - \frac{\sin(n\pi + F^{-1})}{(n\pi + F^{-1})} \right\} +$$

$$+ \frac{m}{2} \left\{ \frac{\cos(n\pi + F^{-1})}{(n\pi + F^{-1})} + \frac{\cos(n\pi - F^{-1})}{(n\pi - F^{-1})} - \frac{2n\pi}{(n^2\pi^2 - F^{-2})} \right\} .$$

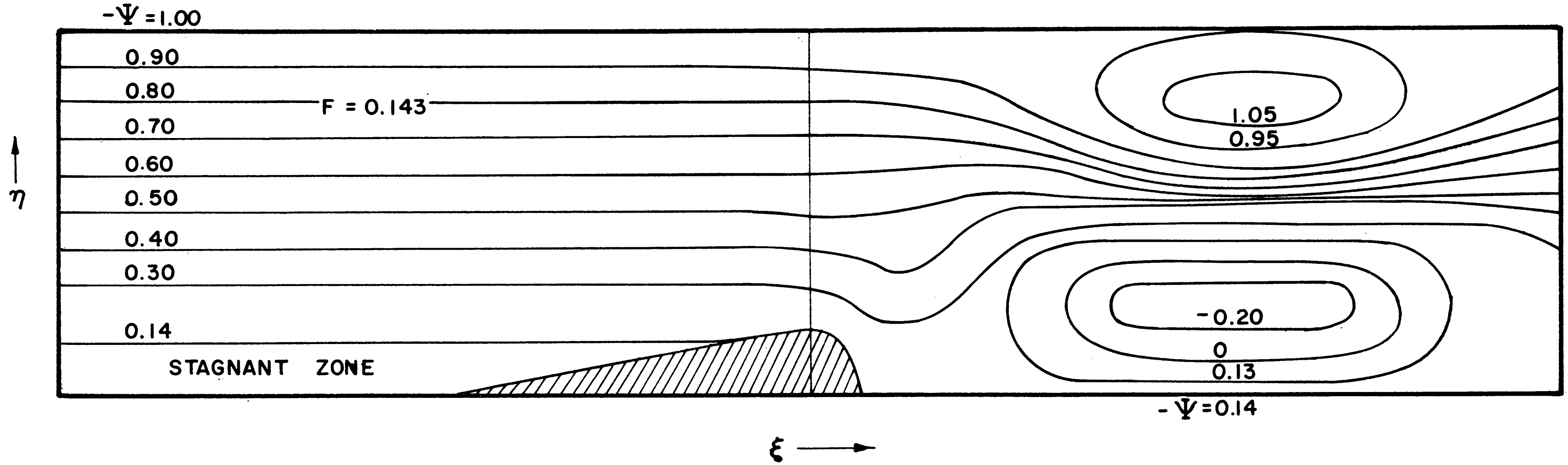


Figure 7. Flow pattern over an obstacle with stagnant zone in front of wedge at Froude number equal to 0.143.

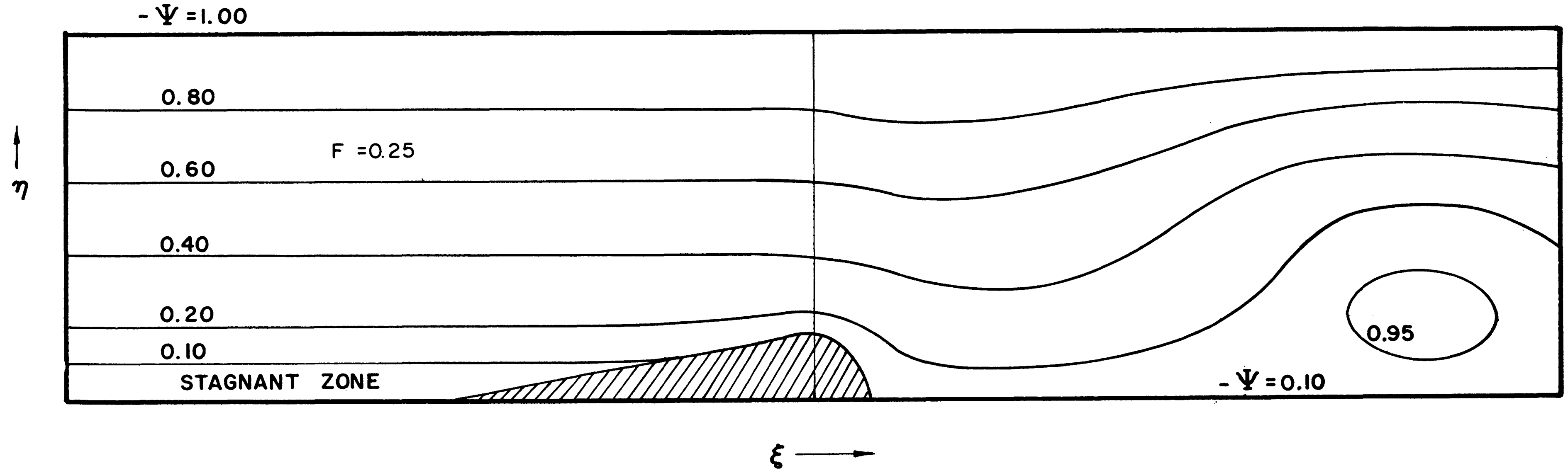


Figure 8. Flow pattern over an obstacle with stagnant zone in front of wedge at Froude number equal to 0.25.

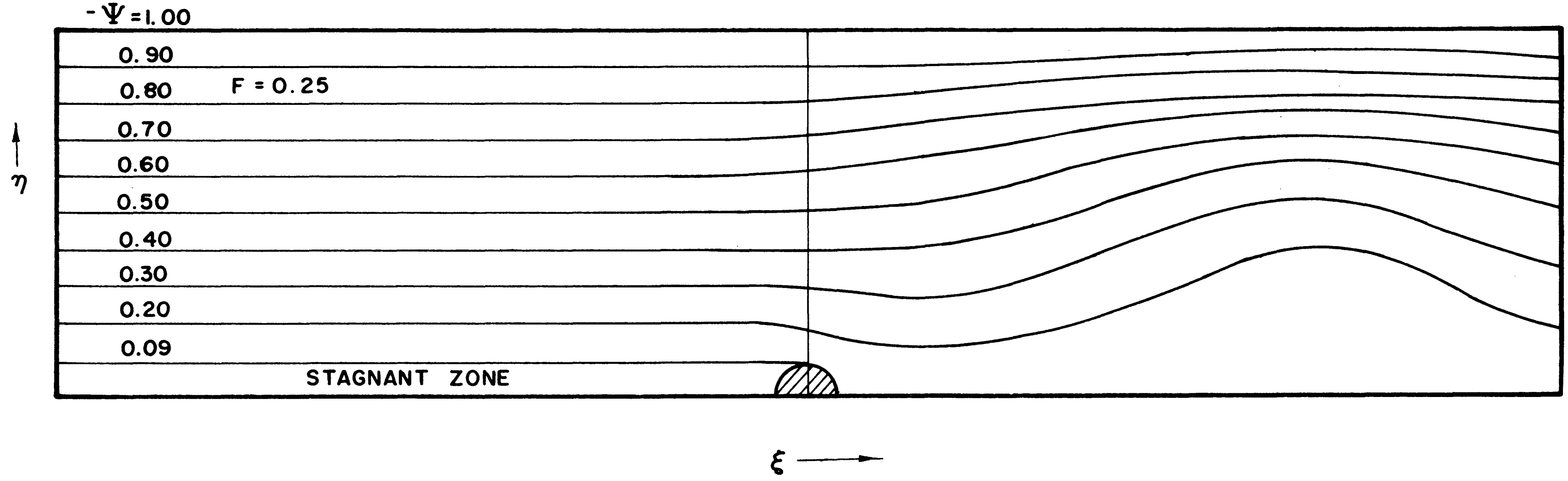


Figure 9. Flow pattern over an obstacle with stagnant zone of the same height as the obstacle at Froude number equal to 0.25.

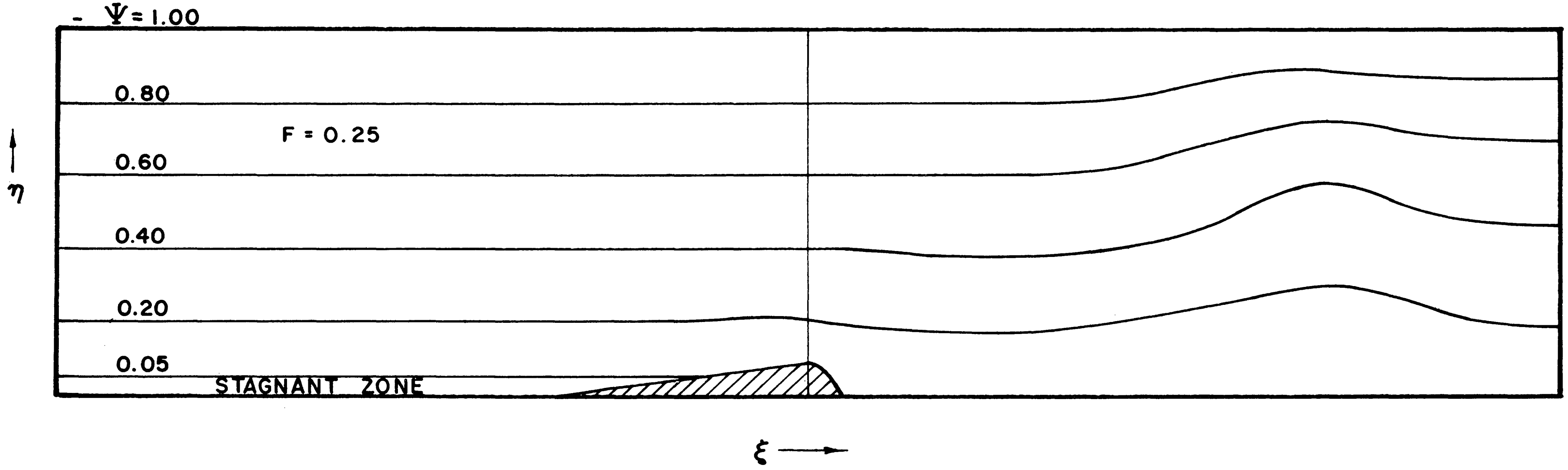


Figure 10. Flow pattern over an obstacle with stagnant zone in front of wedge at Froude number equal to 0.25.

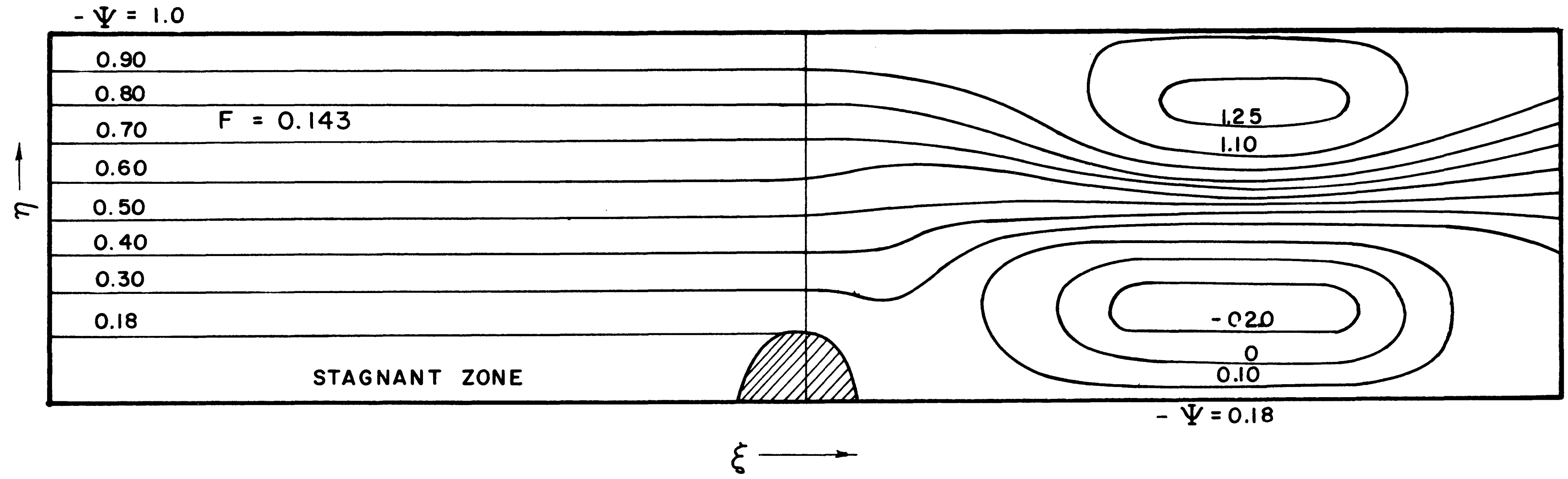


Figure 11. Flow pattern over an obstacle with stagnant zone of the same height as the obstacle at Froude number equal to 0.143.

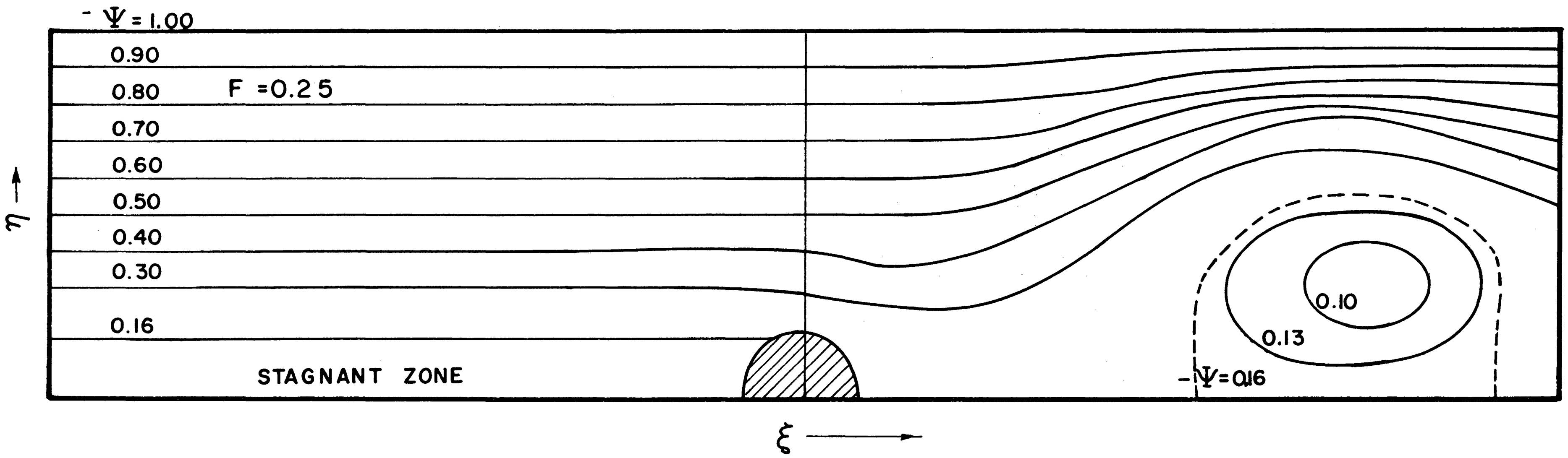


Figure 12. Flow pattern over an obstacle with stagnant zone of the same height as the obstacle at Froude number equal to 0.25.

$f_2(\eta) = C$ for $0 \leq \eta \leq \ell$. and $f_2(\eta) = 0$ for $\ell < \eta$. Then from (95) and (96), it is seen that

$$C_n' = \frac{2C}{\sqrt{-a_n}} \frac{1}{n\pi} (\cos n\pi\ell - 1) ,$$

$$A_n' = \frac{C}{\sqrt{a_n}} \frac{1}{n\pi} (-\cos n\pi\ell + 1) .$$

Two general types of obstacles have been generated. One is somewhat circular and the other is wedge-shaped. The downstream flow pattern is mainly dependent on the height of the obstacle and the magnitude of the Froude number F, and appears to be quite independent of the actual shape of the obstacle. For higher obstacles and lower F, larger downstream eddies are formed. However, it is not the purpose of this study to investigate the significance of these eddies. Suffice it to mention that these eddies have actually been realized in the laboratory. The region of actual interest is of course the stagnant zones extending upstream from the obstacle. It is seen from Figures (9), (11), and (12), that for the near-circular obstacles, or actually for obstacles with steep upstream face of arbitrary shape, the tendancy is for the stagnant zone to reach the same height as the obstacle, the flowing part skimming over the top of the obstacle. These are the only stable configurations that could be obtained in the trial and error calculations that have been carried out. This is a result that is to be expected, since this is an example of a weak flow in stratified fluids under the action of gravity, and it has been shown by Yih[8] that in such motions vertical movements are inhibited. In order to obtain a whole series of lower and lower stagnant zones, the obstacle has to be wedge-shaped. This is shown in Figures (7), (8), and (10). In fact, as shown in Figures (7) and (8), the higher the Froude number, the lower is the depth of the stagnant zone. This is also seen in Figures (11) and (12), though not so marked as in the other case.

In all cases considered the density of the fluid in the stagnant zone is determined to be constant. The Froude number for the various examples is either 0.143 or 0.25.

VI. CONCLUSION

A class of solutions exhibiting a contact discontinuity has been studied for two-dimensional, steady flow of an inviscid, incompressible, stratified fluid between two parallel plates. For certain upstream conditions, the governing equation is rendered exactly linear.

It has been found that along the dividing streamline, which separates the flow field into the flowing zone and the stagnant zone, the flow must satisfy Bernoulli's relation with the pressure given by the hydorstatic pressure imposed by the stagnant zone. In order that the whole flow configuration be stable, the density of the stagnant fluid must satisfy certain physical requirements. These physical requirements have been translated into a mathematical form in terms of a set of necessary and sufficient conditions which the velocity along the dividing streamline must obey. In the inverse method of solution used in this study, these conditions have been employed to produce *a posteriori* stable flow configurations.

In the case of the flow into a sink, stable solutions have been found exhibiting a stable stagnant zone which lies on top of the flowing zone. The line of discontinuity in the flow field is the dividing streamline. The flow patterns for these flows are seen to be in good agreement with experimental results. It has been found here that for all separated flows the Froude's number of the flowing zone has a constant value equal to 0.345. This is in agreement with the fact established by Yih[5] that in the case under consideration there is a unique Froude number for the flowing zone for all separated flows. It is also of the same order of magnitude as the number 0.28 established experimentally by Debler[4]. The Froude number F_2, based on the discharge and the total depth, for the cases considered are found to be 0.176 and 0.124. The depth of the stagnant zone increases with decrease in F_2.

For flows over an obstacle, it has been found that stagnant regions exist in front of the obstacle. Downstream eddies are also present, as well as the expected large amplitude lee-waves. It has been shown here that the depth of the stagnant zone does depend on the Froude number, the depth increasing with decrease in Froude number. It has also been shown that the height of the stagnant zone is nearly equal to the height of the obstacle with steep upstream face of arbitrary shape. On the other hand, a series of lower and lower stagnant zones has been obtained when the obstacle is wedge-shaped.

Finally it may be noted that by the inverse method utilized here, it is possible indeed to construct solutions to free-streamline problems in stratified flows. Furthermore, since the flow is rotational, the solutions obtained here illustrate a class of solutions to free-streamline problems in rotational flow.

APPENDIX

In the case of flow into a sink, detailed calculations have been carried out for the following:

For $F = .345$, $a = .51$, and $b = .14$, $F_2 = .176$, the values computed for q'^2 along the dividing streamline are tabulated below:

η	q'^2
.160	1.625
.190	1.575
.215	1.525
.250	1.460
.295	1.425
.350	1.350
.400	1.225
.440	1.125
.475	1.075

For $F = .345$, $a = .36$, $b = .10$, $F_2 = .124$, the values computed for q'^2 along the dividing streamline are tabulated below:

η	q'^2
.115	2.025
.140	1.950
.160	1.850
.190	1.725
.220	1.650

.265	1.475
.290	1.350
.320	1.175
.345	1.100

In the case of flow over an obstacle, detailed calculations have been carried out for the following:

$F = 0.143$, $m = 0.20$, $C = 10$, $l = 0.15$, and $a = 0.70$.

The sink is situated at $\xi = .1$, and the vortex sheet is situated at $\xi = 0$. The values computed for q'^2 along the dividing streamline are

η	q'^2
.140	1.000
.145	1.054
.150	1.117

Therefore

$$\frac{(q')^2 - 1}{2} = 5.85\,(\eta - .14) .$$

The density of the stagnant zone is therefore a constant and equal to the following:

$$\rho_B = \left(\rho_s + \frac{5.85\,A^2}{gd}\,\rho_o\right) .$$

$F = 0.25$, $m = 0.20$, $C = 10$, $l = 0.15$, and $a = 0.8$.

The singularities are situated as in the previous case. The values computed for q'^2 along the dividing streamline are

η	q'^2
.160	1.000
.162	1.030
.165	1.140

Therefore

$$\frac{(q')^2 - 1}{2} = 14(\eta - .16) .$$

The density of the stagnant zone is therefore a constant and equal to the following:

$$\rho_B = (\rho_s + \frac{14 A^2}{gd}\rho_0) .$$

F = 0.143, m = 0.20, C = 18, l = 0.14, and a = 0.90.

The singularities are situated as in the previous case. The velocity is a constant along the whole streamline. The density of the stagnant zone is therefore a constant and is in fact able to assume any value greater than or equal to ρ_s .

The case F = 0.25, m = 0.20, C = 10, l = 0.15, and a = 0.50, has the same constancy of velocity along the dividing streamline, and thus has the same result.

F = 0.25, m = 0.10, C = 18, l = 0.05, and a = 0.90.

The sink is situated at $\xi = .08$, and the vortex distribution is situated at $\xi = 0$. The velocity is again a constant along the streamline and the result is thus the same as the previous case. The case F = 0.25, m = 0.10, C = 18, l = 0.05, and a = 0.50, is the same as the previous case.

The above results are obtained through trial-and-error computations on the IBM 7090. Detailed tabulations are not attached here, but are available from the author.

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