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FLUID FLOW AND HEAT TRANSFER IN STRATIFIED SYSTEMS

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ABSTRACT

The flow of fluids in stratified porous systems and the analogous problem of heat conduction in stratified solids have been investigated both from an experimental and a purely mathematical standpoint. The individual layers of the stratified systems studied were assumed to be both homogeneous and isotropic at all interior points. The models used allowed for complete communication between layers, so that interlayer fluid flow occurred in the unsteady state.

A total of five heat transfer models were used to obtain experimental data. The primary purpose of obtaining the data was to verify the validity of the mathematical solutions obtained. The two two-layer models which were investigated were fabricated from a clad steel plate in which one quarter inch of type 316L stainless steel was uniformly bonded to one quarter inch of mild steel.

The investigation of the problems from a mathematical standpoint was completely independent of the experimental investigation. Solutions were obtained directly from the partial differential equations governing flow in the stratified systems, using the technique of separation of variables.

In the process of developing the mathematical theory of stratified systems, considerable work was done on single-layer homogeneous systems. Tables in the literature for dimensionless flux have been extended. Tables of dimensionless pressure (or temperature) distribution are also presented.

The mathematical solutions for multi-layer systems occur in the form of double infinite series. Tables of results for various values of the physical parameters have been computed and are presented in the Appendices. These tables permite the calculation of pressure (or temperature) distribution and various fluxes at the producing face in two-layer systems.

The IBM 704 digital computer was used to calculate eigenvalues and sum the series solutions obtained. Tables of eigenvalues are presented in the Appendices.

The majority of the mathematical work on stratified systems was concerned with two-layer systems, both linear and radial. The means of extending the two-layer solutions to the case of general multi-layer systems is presented. Tables of results were not computed for more than two layers.

The application of the solutions obtained to problems of fluid flow in petroleum reservoirs is discussed. Seven example problems are presented to illustrate the use of the tables of results in such problems. Approximation methods are discussed for representing the behavior of multi-layer systems by a single homogeneous layer having mean physical properties.

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NOMENCLATURE

Cp	Heat capacity, BTU/#°F
c	Fractional parameter denoting location of the interface in two-layer models. Also used to denote fluid plus formation compressibility in dimensionless time terms.
c_w	Water plus formation compressibility, vol/vol atm or vol/vol psi
a	Eigenvalues for the separated Y equation in multi-layer flow
b	Eigenvalues for the separated X equation in both single-layer and multi-layer flow
f,g	Fractional parameters denoting interface location in three-layer models
h	Thickness of a layer, centimeters or feet
H	Total thickness of a multi-layer system, centimeters or feet
J_o,J_1	Bessel functions, first kind, zeroth and first order respectively
L_a	Real length, centimeters or feet
L	Dimensionless length, La/H
k	Thermal conductivity, Btu/hr ft^2(°F/ft) for heat conduction Permeability, darcys, for fluid flow
K	Dimensionless ratio of permeabilities and porosities, or thermal properties, for multi-layer systems
N	Number of layers in a multi-layer system
P	Dimensionless pressure or temperature. Also real pressure, psia
p	Functional parameter for the finite Fourier cosine transformation
q	Instantaneous flux, vol/sec or Btu/sec
Q	Cumulative flux, volume or Btu
q_1,q_2	Dimensionless instantaneous individual fluxes for two-layer systems
Q(t)	Dimensionless cumulative total flux for both single-layer and two-layer systems

r_a	Real radius, centimeters or feet
r	Dimensionless radius
r_e	Exterior radius, centimeters or feet
r_b	Interior radius, centimeters or feet
R	Dimensionless radius ratio, r_e/r_b
s	Functional parameter for the LaPlace transformation
T_a	Real temperature, °F
T	Dimensionless temperature
t_a	Real time, as minutes or hours
u,v,w	Velocity components for derivation of the diffusivity equation
W_e	Cumulative flux, Btu or cubic feet
x_a, y_a	Real cartesian space coordinates, centimeters or feet
x,y	Dimensionless cartesian space coordinates
Y_o, Y_1	Bessel functions, second kind, zeroth and first order, respectively

Greek Letters

α	Separation constant; also eigenvalues for the constant terminal rate case
β	A function of K_2 plus the eigenvalues a and b
γ	A function of the eigenvalues a and b
ϵ	A dimensionless ratio used in defining $Q_{(t)}$
λ	Separation constant
ϕ	Porosity, fractional
μ	Fluid viscosity, centipoise
ρ	Density, $\#/ft^3$
Θ	Dimensionless time

NOMENCLATURE (CONT'D)

Subscripts

i	Denotes the layer of a multi-layer system
m,n	Running interfers for double series evaluation
a	Denotes an actual, rather than dimensionless value
$1,2$	Denote quantities for layers 1 or 2, respectively

I INTRODUCTION

The primary purpose of the research which is the basis of the dissertation has been to develop a more quantitative understanding of the flow properties of fluids in underground formations, particularly in aquifers connected with oil and gas reservoirs. Such formations frequently exhibit changes in the physical properties of the porous rock matrix as a function of depth. These inherent inhomogeneities can often be approximated by a "layer-cake" type of model, or other stratified system.

Since the flow of heat in solids is governed by the same equation as the flow of slightly compressible liquids (such as water) in porous media, the two problems may be investigated concurrently. Considerable use has been made of this fact, since the experimental models which were studied in this investigation were heat transfer analogies of the fluid flow problems. It should perhaps be explicitly stated that the research reported herein in no instance involves simultaneous heat transfer and fluid flow. The results obtained with respect to heat transfer are simply corollaries of the basic fluid flow problem, resulting from the identity of the governing flow equations.

The primary factor distinguishing the study of stratified systems from that of homogeneous, "single-layer" systems is the phenomenon of flow of fluid (or heat) between adjacent layers of the stratified system due to driving forces (pressure or temperature differences) induced in unsteady-state flow. The phenomenon herein referred to as "inter-layer flow" or "crossflow" is indicated schematically in Figure I-1. Thus, an extra dimension is necessary in solving for the flow properties of the system.

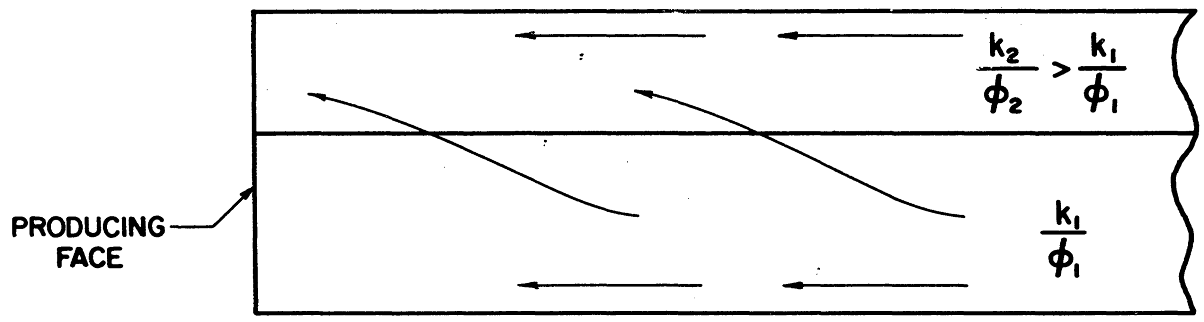


Figure I-1. Crossflow in a Two-Layer System.

For example, the temperature in a solid, homogeneous steel bar which is being heated at one end can be expressed in terms of only one length dimension plus time. However, if the bar is composed of two parallel pieces of metal, intimately bonded at the interface, the solution for temperature distribution must be a function of position with regard to the axis perpendicular to this interface, as well as distance from the end being heated, and time.

Problems of flow in stratified systems have been investigated both experimentally and analytically in this research. A total of five heat transfer models were used to obtain experimental data on temperature distribution in the unsteady state. These data were used to verify analytical solutions which were independently obtained from the governing partial differential equations.

The majority of the work which has been done is concerned with two-layer systems. The analytical solutions obtained for the two-layer case have been extended to the more general case of several layers. However, it will be shown that the complexity of the solutions mounts so rapidly with increasing number of layers that solutions of the type obtained are of only theoretical interest for more than three or four layers, since obtaining numerical answers then becomes extremely difficult.

All of the results presented in this dissertation are for bounded systems, i.e. for systems in which the length or radius is finite. It is possible, however, to make use of the solutions presented for solving problems associated with infinite systems. This can be done by considering the time range of the solution for which the exterior boundary of the system does not as yet have a significant effect on the results.

Considerable use has been made of the IBM 704 digital computer in this investigation. All of the analytical solutions which have been obtained are in the form of infinite Fourier and Fourier-Bessel series. The calculation of numerical results from these solutions by hand is, in nearly all cases, sufficiently tedious as to make this impractical. Using the IBM 704, however, it has proven possible to compute extensive tables for dimensionless pressure distribution (or analogously, temperature distribution) and flux for several important flow models.

II LITERATURE REVIEW

To the best of the author's knowledge, there does not now exist in the literature any material directly relating to the analytical treatment of flow in stratified systems where the effect of interlayer flow is considered. There does exist, however, a considerable volume of material relating to the solution of the diffusivity equation in homogeneous systems with various boundary conditions, as well as on experimental model studies relating to both homogeneous and heterogeneous flow systems.

The diffusivity equation, which governs both the flow of slightly compressible fluids in porous media and heat conduction in solids, is discussed in some detail in many mathematics textbooks, such as those by Churchill,[8,9] Wylie, [44] Snedden, [35] and Mickley, Sherwood, and Reed.[30] A very comprehensive treatment of solutions of this equation as applied to heat conduction has been given in a book by Carslaw and Jaeger.[5]

The paper by Van Everdingen and Hurst[40] in 1949 presented the first comprehensive treatment of the application of solutions of the diffusivity equation to unsteady state radial flow in reservoirs. Other authors, such as Muskat[32,33] had previously presented analytical solutions which could be applied to certain reservoir problems, but the paper by Van Everdingen and Hurst was certainly a milestone in the field. Since that time, noteworthy papers have also been published by Chatas,[7] Mortada,[31] Coats, [10,11,12] and others. The two basic types of boundary conditions considered in these papers are the "constant terminal pressure" (CTP) case, in which a step function change in pressure is applied at some boundary, and the "constant terminal rate" (CTR) case in which a step function change in flux is applied. Dimensionless cumulative flux, $Q_{(t)}$, for the CTP case and dimensionless pressure change at the boundary, $P_{(t)}$, for the CTR case have been tabulated both by Van Everdingen and Hurst and by Chatas. Mortada includes graphs showing pressure distribution for the infinite CTR case in his paper. Coats has shown the application of these results to actual reservoir problems, using field data.[11,12]

The work in the above papers is all principally concerned with single-phase isothermal horizontal radial flow of slightly compressible liquids in homogeneous and isotropic porous media. Considerable work has also been done in recent years on developing analytical solutions for other, more complex flow systems in reservoirs. Solutions are now available for a thick-sand model and a hemispherical flow model.[26] Some solutions to the diffusivity equation in elliptic coordinates have been obtained.[10] Stevens and Thodos have presented some point source solutions.[37] Interference effects between wells or between storage fields situated on a common aquifer have been studied.[31,37,12]

Material which has heretofore been published on heterogeneous or anisotropic flow systems in reservoirs has been to a considerable degree based on an approximate, statistical, or semi-empirical approach to the problems. Such is true of the works by Muskat[33] and Law[28]. Cardwell and Parsons have published some results for simple cases of heterogeneity,[4] but this author believes that their conclusions with regard to unsteady state behavior are incorrect. Some recent papers[19,21] have made use of "influence" - or "resistance-functions" to utilize known field data to predict future reservoir performance without the necessity of postulating a model.

A recent paper by Lefkovits, et al, has been directly concerned with the analytical treatment of stratified systems.[29] This paper is a very significant contribution to the knowledge of the behavior of such systems. However, it was assumed in the paper that the various layers of the porous matrix were connected only at the well bore (or analogously at the reservoir-aquifer interface). Thus, there was no consideration of cross-flow between adjacent layers of the system. This was also true in the earlier paper by Standing, et al.[36]

Much of the current knowledge of the behavior of stratified systems has been obtained from model studies. Several informative papers, such as those by Ferrell, et al,[16] Caudle and Witte,[6] Craig, et al,[13] Gaucher and Lindley[17] and Van Meurs[41] have recently been published dealing with the use of fluid flow models. In most cases the models used involved two phase flow.

In addition to fluid flow models, some work has recently been done on analog models. Potentiometric models, such as those described by Fatt[15] are of considerable use in studying steady-state flow, but are not usually applicable to unsteady-state investigations. A much more useful model for transient phenomena is the thermal or heat-transfer model, which has been used to some extent by Crawford and Landrum, et al.[14,27] Without wishing to deprecate these latter articles in any way, this author must take issue with the idea expressed therein that the use of a heat transfer model is something basically "new," since the identity of the controlling partial differential equations for fluid flow and heat transfer has been known for many years.

III. BASIC MATHEMATICAL THEORY

This section of the dissertation is devoted to:

1. Derivation of the diffusivity equation, which governs both heat conduction in solids and flow of slightly compressible fluids in porous media.
2. Discussion of the necessity of an unsteady-state flow analysis in reservoir systems.
3. A brief discussion of some of the methods available for the solution of partial differential equations such as the diffusivity equation.

A. Derivations of the Diffusivity Equation

The diffusivity equation for heat conduction in solids has been derived in several references. The following derivation is based largely on the work of Jakob[23] and that of Carslaw and Jaeger,[5] but uses the nomenclature of this dissertation where applicable.

The basic law of heat conduction is

$$q = -k \frac{A}{L} \Delta T \qquad \text{(III-1)}$$

It originates from Biot (1804, 1816) but is generally called Fourier's equation, because Fourier (1822) used it as a fundamental equation in his analytic theory of heat.[5] The basic equation of heat storage is*

$$q = \rho C_p V \frac{\Delta T}{\Delta t_a} \qquad \text{(III-2)}$$

* Due to the frequent use of dimensionless variables in the derivations contained in this dissertation, the subscript "a" has been applied to real spatial and time variables. Thus, for example, x_a denotes an actual spatial distance, as feet, inches, centimeters, etc., whereas $\underline{x}$ denotes the corresponding dimensionless variable.

where q is the heat energy stored in unit time in the volume V of a medium of density ρ and specific heat C_p, when the temperature increases by ΔT in a time interval Δt_a.

Equation (III-1) may also be written in the differential form

$$dQ = -kA \frac{\partial T}{\partial x_a} dt_a \tag{III-3}$$

where dQ is the heat conducted in the direction x_a during the time interval dt_a.

Consider the differential parallelepipedon shown in Figure III-1. Initially assume the conducting solid to be homogeneous and isotropic throughout this differential element. A heat balance will be set up which is valid in the differential of time dt_a. The heat entering from the left will be

$$dQ_{1,x} = -k(dy_a dz_a) \frac{\partial T}{\partial x_a} dt_a \tag{III-4}$$

and the heat leaving at the right side will be

$$\begin{aligned} dQ_{2,x} &= -k(dy_a dz_a) \frac{\partial}{\partial x_a} (T + \frac{\partial T}{\partial x_a} dx_a) \; dt_a \\ &= -k(dy_a dz_a)(\frac{\partial T}{\partial x_a} + \frac{\partial^2 T}{\partial x_a^2} dx) \; dt_a \quad . \end{aligned} \tag{III-5}$$

Similar equations exist for the heat quantities $dQ_{1,y}$, $dQ_{2,y}$, $dQ_{1,z}$ and $dQ_{2,z}$ which are conducted in the directions y_a and z_a. The heat energy stored in the differential body is

$$dQ_s = \rho C_p (dx_a dy_a dz_a) \frac{\partial T}{\partial t_a} dt_a \quad . \tag{III-6}$$

The total heat energy entering the differential element in time dt_a is

$$dQ_1 = dQ_{1,x} + dQ_{1,y} + dQ_{1,z} \tag{III-7}$$

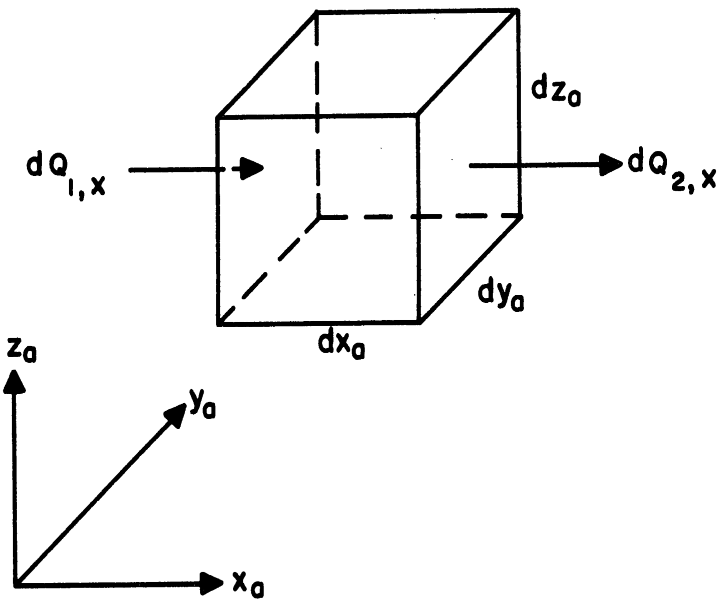


Figure III-1. Heat Conduction Through a Differential Volume Element.

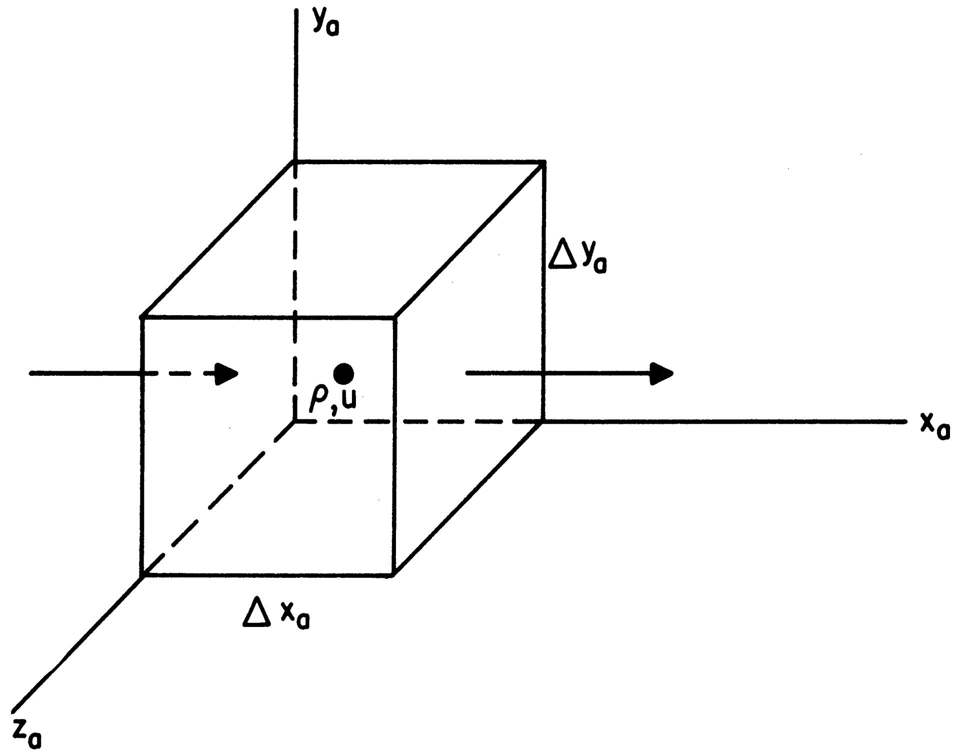


Figure III-2. Volume Element for Derivation of the Continuity Equation for Fluid Flow.

and that leaving is

$$dQ_2 = dQ_{2,x} + dQ_{2,y} + dQ_{2,z} \quad . \qquad \text{(III-8)}$$

Assuming no heat sources or sinks within the differential element, the law of conservation of energy may be written

$$dQ_1 = dQ_2 + dQ_s \quad . \qquad \text{(III-9)}$$

Substituting for dQ_1, dQ_2, and dQ_s and simplifying,

$$\frac{\rho C_p}{k} \frac{\partial T}{\partial t_a} = \frac{\partial^2 T}{\partial x_a{}^2} + \frac{\partial^2 T}{\partial y_a{}^2} + \frac{\partial^2 T}{\partial z_a{}^2} \quad . \qquad \text{(III-10)}$$

This is the form in which the diffusivity equation for heat conduction is most usually written. For the steady-state case, where temperature is not a function of time, this reduces to LaPlace's equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x_a{}^2} + \frac{\partial^2 T}{\partial y_a{}^2} + \frac{\partial^2 T}{\partial z_a{}^2} = 0 \quad . \qquad \text{(III-11)}$$

Now consider the case where thermal conductivity, k, is a function of position. Then Equation (III-5) must be written

$$dQ_{2,x} = -(dy_a dz_a)\left[k \frac{\partial T}{\partial x_a} + \frac{\partial}{\partial x_a}\left(k \frac{\partial T}{\partial x_a}\right) dx_a\right] dt_a \qquad \text{(III-12)}$$

Proceeding as before, the diffusivity equation in this case takes the form

$$\rho C_p \frac{\partial T}{\partial t_a} = \frac{\partial}{\partial x_a}\left(k \frac{\partial T}{\partial x_a}\right) + \frac{\partial}{\partial y_a}\left(k \frac{\partial T}{\partial y_a}\right) + \frac{\partial}{\partial z_a}\left(k \frac{\partial T}{\partial z_a}\right) . \qquad \text{(III-13)}$$

This form of the equation will be used in this dissertation in all cases involving flow in heterogeneous systems. The assumptions made in the derivation of Equation (III-13) were:

1. The conducting medium is isotropic at all interior points.
2. There are no heat sources or sinks within the conducting medium.
3. The density, ρ, and the heat capacity, C_p, of the conducting medium are independent of both position and temperature.

The diffusivity equation for flow of slightly compressible fluids has also been derived by many authors. The following derivation is largely excepted from the Handbook of Natural Gas Engineering.

Consider an element of volume such as that shown in Figure III-2, where the fluid density at the center of the element is ρ. The superficial fluid velocities u, v, and w in the x_a, y_a, and z_a directions respectively, at the center of the element will be considered. These superficial velocities are based on the entire cross-sectional flow area without regard to the fractional porosity of the porous medium.

The mass rate of flow through any face of this volume element is (velocity) (density) (area). Thus the mass rate of flow in the x_a direction into the left face is

$$\left[u - \left(\frac{\partial u}{\partial x_a}\right)\left(\frac{\Delta x_a}{2}\right)\right]\left[\rho - \left(\frac{\partial \rho}{\partial x_a}\right)\left(\frac{\Delta x_a}{2}\right)\right] \Delta y_a \Delta z_a \qquad \text{(III-14)}$$

and the mass rate of flow in the x_a direction out of the right face is

$$\left[u + \left(\frac{\partial u}{\partial x_a}\right)\left(\frac{\Delta x_a}{2}\right)\right]\left[\rho + \left(\frac{\partial \rho}{\partial x_a}\right)\left(\frac{\Delta x_a}{2}\right)\right] \Delta y_a \Delta z_a \quad . \qquad \text{(III-15)}$$

The net rate of flow into the volume element in the x_a direction is then

$$- \Delta x_a \Delta y_a \Delta z_a \left[\frac{\partial(\rho u)}{\partial x_a}\right] \qquad \text{(III-16)}$$

Similarly, the net mass flow rates into the volume element in the y_a and z_a directions, respectively, are

$$- \Delta x_a \Delta y_a \Delta z_a \left[\frac{\partial(\rho v)}{\partial y_a}\right] \qquad \text{(III-17)}$$

and

$$- \Delta x_a \Delta y_a \Delta z_a \left[\frac{\partial(\rho w)}{\partial z_a}\right] \qquad \text{(III-18)}$$

The rate of accumulation of fluid within the volume element, due to a change in density of the fluid within the element, is

$$\phi \Delta x_a \Delta y_a \Delta z_a \left(\frac{\partial \rho}{\partial t_a}\right) \qquad \text{(III-19)}$$

Setting the net input equal to the accumulation gives the equation of continuity,

$$\frac{\partial(\rho u)}{\partial x_a} + \frac{\partial(\rho v)}{\partial y_a} + \frac{\partial(\rho w)}{\partial z_a} = - \phi \left(\frac{\partial \rho}{\partial t_a}\right) \qquad \text{(III-20)}$$

Note that no assumptions with regard to homogeneity and/or isotropy are necessary in the derivation of this equation of continuity.

Darcy's law in its original form is written

$$\frac{q}{A} = - \frac{k}{\mu} \frac{dP}{dL} \qquad \text{(III-21)}$$

In terms of the velocity components, this is equivalent to the three equations

$$u = - \frac{k}{\mu} \frac{\partial P}{\partial x_a} ; \quad v = - \frac{k}{\mu} \frac{\partial P}{\partial y_a} ; \quad w = - \frac{k}{\mu} \frac{\partial P}{\partial z_a} \qquad \text{(III-22)}$$

The remaining equation necessary in the derivation is an equation of state relating fluid density to pressure. It is at this point that the concept of what constitutes a "slightly compressible" liquid becomes important. In this dissertation, a slightly compressible liquid is defined as any liquid which has a pressure versus density relationship expressible as

$$\rho = \rho_o [1 + c(P - P_o)] \qquad \text{(III-23)}$$

over the pressure range of interest in a given problem. This definition designates water and most crude oils as slightly compressible liquids in nearly all problems associated with flow in underground reservoirs.

Combining Equations (III-20), (III-22), and (III-23) gives the diffusivity equation for the flow of slightly compressible fluids in porous media.

$$\frac{\partial}{\partial x_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial x_a}\right) + \frac{\partial}{\partial y_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial y_a}\right) + \frac{\partial}{\partial z_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial z_a}\right) = \phi c \frac{\partial P}{\partial t_a} \qquad \text{(III-24)}$$

For permeability and fluid viscosity independent of position, this simplifies to

$$\frac{\partial^2 P}{\partial x_a^2} + \frac{\partial^2 P}{\partial y_a^2} + \frac{\partial^2 P}{\partial z_a^2} = \frac{\mu \phi c}{k}\frac{\partial P}{\partial t_a} \qquad \text{(III-25)}$$

The identity of the governing equations for heat conduction and flow of slightly compressible fluids in porous media is thus apparent, with of course the exception of different factors appearing in the coefficient of the time terms.

The assumptions made in deriving the diffusivity equation for fluid flow were:

1. Negligible capillary or gravity effects.
2. Single phase flow of isothermal slightly compressible liquid.
3. Isotropic porous matrix at any interior point.
4. Darcy's law applies.

The diffusion equation may be written in terms of cylindrical coordinates by a simple transformation of variables of the form

$$x_a = r_a \cos \sigma$$
$$z_a = r_a \sin \sigma \qquad \text{(III-26)}$$
$$ds^2 = dr_a^2 + r_a\, d\sigma^2 + dy_a^2$$

For radial flow with no gradient in the circumferential direction (i.e., $\partial P/\partial\sigma = 0$), and with homogeneity in the radial direction (i.e., $\partial k/\partial r_a = 0$), the diffusion equation thus becomes

$$k\left(\frac{\partial^2 P}{\partial r_a^2} + \frac{1}{r_a}\frac{\partial P}{\partial r_a}\right) + \frac{\partial}{\partial y_a}\left(k\frac{\partial P}{\partial y_a}\right) = \mu\phi c \frac{\partial P}{\partial t_a} \qquad \text{(III-27)}$$

for fluid flow, and

$$k\left(\frac{\partial^2 T}{\partial r_a^2} + \frac{1}{r_a}\frac{\partial T}{\partial r_a}\right) + \frac{\partial}{\partial y_a}\left(k\frac{\partial T}{\partial y_a}\right) = \rho C_p \frac{\partial T}{\partial t_a} \qquad \text{(III-28)}$$

for heat conduction.

The unsteady state flow equations for gas flow and flow of highly compressible liquids are derived in the Handbook of Natural Gas Engineering and will be shown here for comparison in the case of linear flow.

Gas Flow:

$$\frac{\partial^2 P^2}{\partial x_a^2} = \frac{\mu\phi}{kP}\frac{\partial P^2}{\partial t_a} \qquad \text{(III-29)}$$

Flow of highly compressible liquids:

$$\frac{\partial^2 P}{\partial x_a^2} + c\left(\frac{\partial P}{\partial x_a}\right)^2 = \frac{\mu\phi c}{k}\frac{\partial P}{\partial t_a} \qquad \text{(III-30)}$$

whereas for slightly compressible liquids:

$$\frac{\partial^2 P}{\partial x_a^2} = \frac{\mu\phi c}{k}\frac{\partial P}{\partial t_a}$$

B. <u>The Necessity of An Unsteady-State Flow Analysis in Reservoir Systems</u>

Chatas has published a detailed discussion of criteria for the necessity of using an unsteady-state flow model for reservoir systems.[7] A few of the pertinent points from his paper will be discussed here.

There are two ratios which serve to indicate the degree to which a system will behave in an unsteady-state manner upon the application of a pressure disturbance at its boundaries. The first ratio is that of the speed of propagation of disturbances in a reservoir to the maximum speed of fluid movement through the porous matrix. Since the former of these is the speed of sound in the fluid, this ratio may be expressed as

$$\frac{\mu r_b \ln\left(\frac{r_e}{r_b}\right)}{k(P_e - P_b)}\sqrt{\frac{B}{\rho}} \qquad \text{(III-31)}$$

where B is the bulk modulus of the reservoir fluid.

A second informative ratio is that of the rate of change in the fluid-mass content of a reservoir caused by pressure variations at its boundaries to the steady-state mass-flow ability of the reservoir. This ratio may be approximated by

$$\frac{r_e^2 \phi c \ \ln\left(\frac{r_e}{r_b}\right) \frac{dP_B}{dt_a}}{2\left(\frac{k}{\mu}\right)(P_e - P_b)} \qquad \text{(III-32)}$$

where P_B is the pressure at a boundary. This expression serves as a measure of the time required for the readjustment of the internal pressure distribution in a reservoir to a steady-state distribution when pressure variations occur at the boundaries.

For completely incompressible fluids, contained in a completely incompressible porous matrix, the ratio (III-31) becomes infinitely great, whereas that of (III-32) vanishes. In this case, steady-state flow always prevails, with the applicable flow equation for radial flow being

$$\frac{\partial^2 P}{\partial r_a^2} + \frac{1}{r_a} \frac{\partial P}{\partial r_a} = 0 \qquad \text{(III-33)}$$

This is true because in incompressible fluids the transmission of pressure variations at a boundary and the readjustment of internal pressure distributions occur instantaneously and without attenuation.

For actual reservoir fluids such as water or crude oil, the magnitude of the ratio (III-31) is very large. That is, boundary disturbances are transmitted many thousands of times faster than fluid can flow in the system. However, the large extent of these underground reservoirs usually causes the ratio (III-32) also to be quite large. Thus, steady-state flow no longer obtains, and an unsteady-state flow analysis is necessary.

The effect of various parameters on the ratios of (III-31) and (III-32) is of interest. Increasing the extent of the system (r_e), the porosity (ϕ) or the fluid compressibility (c) all tend to increase the time required for the system to reach steady-state. Increasing the permeability k, on the other hand, tends to make the system approach steady-state more quickly.

C. Methods of Solution of Partial Differential Equations

The author certainly does not plan to present here a comprehensive treatise on means of solving partial differential equations. However, some justification of the mathematical methods of attack used on the problems in this research seems to be in order.

There are two basic mathematical means of attack on partial differential equations. The first involves setting up a grid or network for the problem and using finite-difference techniques to obtain a solution. This method has been facilitated by the recent availability of comparatively high speed computing machinery such as the IBM 704. The justification for not using such methods in this investigation is largely based on the number of parameters involved in the problems. It is entirely possible that for a given set of physical parameters a finite-difference solution would be more advantageous. However, the general solution to the problem, quantitatively showing the functional relationship of the many different physical parametric groups, would require the solution of many different grid net-works. It was therefore decided at the beginning of the research that an analytic solution to the problems, if obtainable, would be preferred.

The second basic mathematical method of attack involves conversion of the partial differential equation into ordinary differential equation(s). This can be most easily accomplished using either some transformation technique (LaPlace transformations, Fourier transformations, etc.) or the classical method of separation of variables. As previously noted, Van Everdingen and Hurst[40] and other authors have principally used transformation methods of solution, in particular the LaPlace transformation.

The question of whether or not problems exist which can be solved by separation of variables but cannot be solved using transformation techniques, and vice versa, is of considerable interest. The author cannot cite any examples of such problems. It is certainly true, however, that many problems are more amenable to attack by one means than the other.

In the case of single layer flow in homogeneous, isotropic, bounded systems, there appears to be little basis to choose between the method of separation of variables and that using the LaPlace transformation. The method of separation of variables is somewhat more straightforward, but the LaPlace transformation solution has the advantage of presenting a clearer picture of the behavior of the system at very large and very small values of dimensionless time.

In the case of flow in stratified systems, this author believes the method of separation of variables to be superior. A solution of the problems was attempted using a LaPlace transformation with respect to time followed by finite Fourier trigonometric transformations with respect to the space variables. Sufficient difficulties were encountered, particularly with regard to introduction of the interface conditions, to cause the abandonment of this method at an early stage. The author does, however, believe that a solution based on some type of transformation method should be possible.

The method of separation of variables has been used in nearly all cases in this derivation (the single exception being the solution for single-layer linear flow of liquids in the constant terminal rate mode). For a detailed description of the method of separation of variables the reader is referred to the books by Churchill[8,9] and by Mickley, Sherwood, and Reed.[30] For a comparison of methods on the same problem, the reader may compare the solutions in this dissertation for radial flow in single-layer systems to those presented by Van Everdingen and Hurst.[40]

IV SINGLE-LAYER FLOW; MATHEMATICAL ANALYSIS

In the course of the investigation of flow in stratified systems, a considerable amount of work has been done on single layer homogeneous systems. The reasons for this work are three-fold. The primary reason was the necessity of having a complete set of tables for single-layer flow in order to adequately compare the results with those obtained in stratified systems. Although most of the problems presented in this section have been previously investigated by other authors, notably Van Everdingen and Hurst,[40] the tables which are available in the current literature are not sufficiently complete, especially with regard to pressure (or temperature) distribution, to enable an accurate comparison to be made.

The second reason for the work was to check the results obtained from the single-layer experimental models studied in this investigation. Again, the tables in the literature are not sufficiently complete for an accurate check.

A third reason concerns the means by which mathematical solutions may be obtained. The solutions obtained by Van Everdingen and Hurst[40] which were amplified and extended by Chatas,[7] Mortada,[31] and others, were all based on the use of the La Place Transformation. It was decided to use the classical method of separation of variables in attacking problems of flow in stratified systems in the research leading to this dissertation. Thus the work reported herein on single layer flow, which was also based largely on the technique of separation of variables, served as a mathematical foundation for the more complex work on stratified systems.

The work reported herein is in two categories. First the mathematical models used will be described and the solutions presented. Then the means of using some of the results will be explained using example calculations. The tables of results obtained are presented in appendices.

A. Single-layer Linear Flow

Figure IV-1 shows schematically the mathematical model used for both the constant terminal pressure and constant terminal rate cases. It is postulated in the following derivations that the model shown in Figure IV-1 is homogeneous and isotropic throughout. Considered as a liquid-flow system, the additional specifications are the assumptions used in deriving the diffusivity equation, namely:

1. Negligible capillary or gravity effects
2. Single phase flow of an isothermal slightly compressible liquid
3. Darcy's law applies.

Considered as a heat transfer model, it is specified that the thermal diffusivity of the conducting solid be independent of temperature.

The nomenclature used in the derivations will be that for liquid flow. The solutions will then also be written in terms of heat transfer nomenclature where any significant difference occurs.

1. The Constant Terminal Pressure Case

The equation governing the flow in this case is the diffusivity equation for k independent of position, as expressed in cartesian coordinates. This equation was derived in Section III of this dissertation and appears there as equation III-25.

$$\frac{\partial^2 P}{\partial x_a^2} + \frac{\partial^2 P}{\partial y_a^2} + \frac{\partial^2 P}{\partial x_a^2} = \frac{\mu \phi c}{k} \frac{\partial P}{\partial t_a} \qquad (IV - 1)$$

Since in this case there will be no net flow in the y_a or z_a directions, the equation reduces to:

$$\frac{\partial^2 P}{\partial x_a^2} = \frac{\mu \phi c}{k} \frac{\partial P}{\partial t_a} \qquad (IV - 2)$$

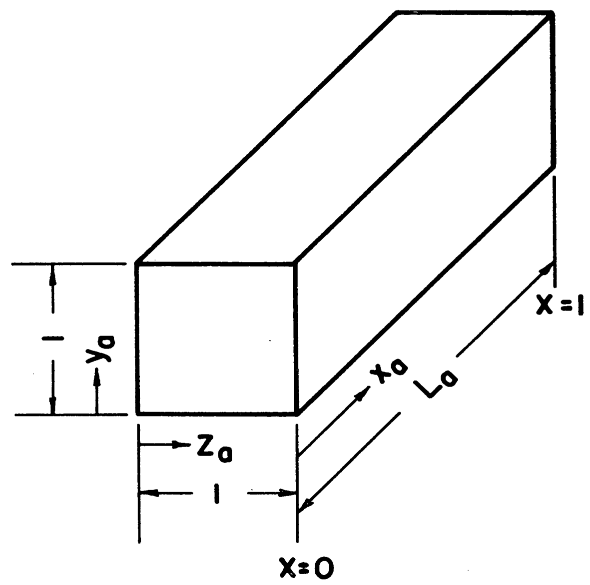


Figure IV-1. Mathematical Model for Single-Layer Linear Flow.

Defining dimensionless variables as follows:

$$x = \frac{x_a}{L_a} \tag{IV - 3}$$

$$\theta = \frac{k\ t_a}{\mu\phi c L_a{}^2} \tag{IV - 4}$$

the equation may be written

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial \theta} \tag{IV - 5}$$

The applicable boundary and initial conditions for the constant terminal pressure case are

Initial Condition: $P\ (x,o) = 1$ (IV - 6)

Boundary Conditions: $P\ (o,\theta) = 0$ (IV - 7)

$$\frac{\partial P}{\partial x}\ (1,\theta) = 0 \tag{IV - 8}$$

Equations IV-6 and IV-7 serve to introduce the step function in pressure at the $x=0$ (i.e. $x_a = 0$) end of the mathematical model. Equation IV-8 expresses the condition that the other end of the model have no flow across it. It is also known that

$$\lim_{\theta \to \infty} P = 0 \tag{IV - 9}$$

The method of separation of variables will be used as follows. It is first desired to find all solutions of the type

$$P = X(x)\ T(\theta) \tag{IV - 10}$$

which satisfy the equation (IV-5), the boundary conditions (IV-7) and (IV-8), and the limiting condition of (IV-9). Substituting (IV-10) into (IV-5),

$$X'' T = X T' \qquad \text{(IV - 11)}$$

Therefore

$$\frac{X''}{X} = \frac{T'}{T} = - a^2 \qquad \text{(IV - 12)}$$

The reader unfamiliar with the basis of the technique of separation of variables is again referred to the references cited(8,9,30) for an explanation of this equation. The resulting ordinary differential equations are thus

$$X'' + a^2 X = 0 \qquad \text{(IV - 13)}$$
$$T' + a^2 T = 0 \qquad \text{(IV - 14)}$$

For $a = 0$:

$$X = C_1 + C_2 x \qquad \text{(IV - 15)}$$

$$T = C_3 \qquad \text{(IV - 16)}$$

But from (IV-9), $C_3 = 0$. The case of $a = o$ thus contributes no terms to the solution.

For $a \neq o$:

$$X = C_4 \cos ax + C_5 \sin ax$$

$$T = C_6 e^{-a^2 \theta}$$

From (IV-7), $X(o) = 0$, so that $C_4 = 0$

From (IV-8),

$$X'(1) = 0 = C_5 a \cos a \qquad \text{(IV - 17)}$$

But $C_5 \neq 0$ or a trivial solution results. The case of $a = 0$ has already been considered.

Therefore

$$\cos a = 0 \tag{IV-18}$$

This equation is of the type to be herein referred to as a characteristic equation, and defines eigenvalues as

$$a = \left(\frac{2m - 1}{2}\right) \pi \; ; \; m = 1, 2, 3 \ldots\ldots\ldots \tag{IV-19}$$

Therefore

$$P = X(x)\ T(\theta) = C \sin ax\ e^{-a^2\theta} \tag{IV-20}$$

There is no single value of the constant C which will satisfy the initial condition (IV-6) for all values of x. However, since the original differential equation (IV-5) was linear, any sum of solutions of the form of (IV-20) is also a solution. The initial condition may thus be written

$$1 = \sum_{m=1}^{\infty} C_m \sin ax \tag{IV-21}$$

Multiplying both sides of this equation by sin ax, integrating both sides with respect to x over the range x = 0 to x = 1, and applying the applicable orthogonality condition that

$$\int_0^1 \sin a_u x \ \ \sin a_v x \, dx = 0 \quad \text{for } u \neq v \tag{IV-22}$$

The equation for C_m is found to be

$$C_m = \frac{\int_0^1 \sin ax \, dx}{\int_0^1 \sin^2 ax \, dx} = \frac{2 \left[\cos a - 1\right]}{\left[\sin a \cos a - a\right]} \tag{IV-23}$$

or since from (IV-18), $\cos a = 0$,

$$C_m = \frac{2}{a}$$

The solution is thus

$$P = \sum_{m=1}^{\infty} \frac{2}{a} \sin ax \, e^{-a^2\Theta} \qquad \text{(IV-24)}$$

where

$$a = \left(\frac{2m-1}{2} \right) \pi; \quad m = 1, 2, 3, \ldots\ldots \qquad \text{(IV-19)}$$

The orthogonality condition for well-known cases such as this will be stated as such without proof. In some of the more complicated cases to be encountered in multi-layer flow, the derivation of the applicable orthogonality condition will be given. In terms of heat transfer, the dimensionless time, Θ, is defined as

$$\Theta = \frac{k\, t_a}{\rho C_p L_a^2} \qquad \text{(IV-25)}$$

and the solution is again

$$T = \sum_{m=1}^{\infty} \frac{2}{a} \sin ax \, e^{-a^2\Theta} \qquad \text{(IV-26)}$$

A table of the dimensionless pressure (or analogously the dimensionless temperature) for this case, with parameters of x and Θ, is presented in Appendix F. These results are also shown graphically in Figure IV-2.

2. The Constant Terminal Rate Case

The governing equation for this case is again

$$\frac{\partial^2 P}{\partial x_a^2} = \frac{\mu \phi c}{k} \frac{\partial P}{\partial t_a} \qquad \text{(IV-2)}$$

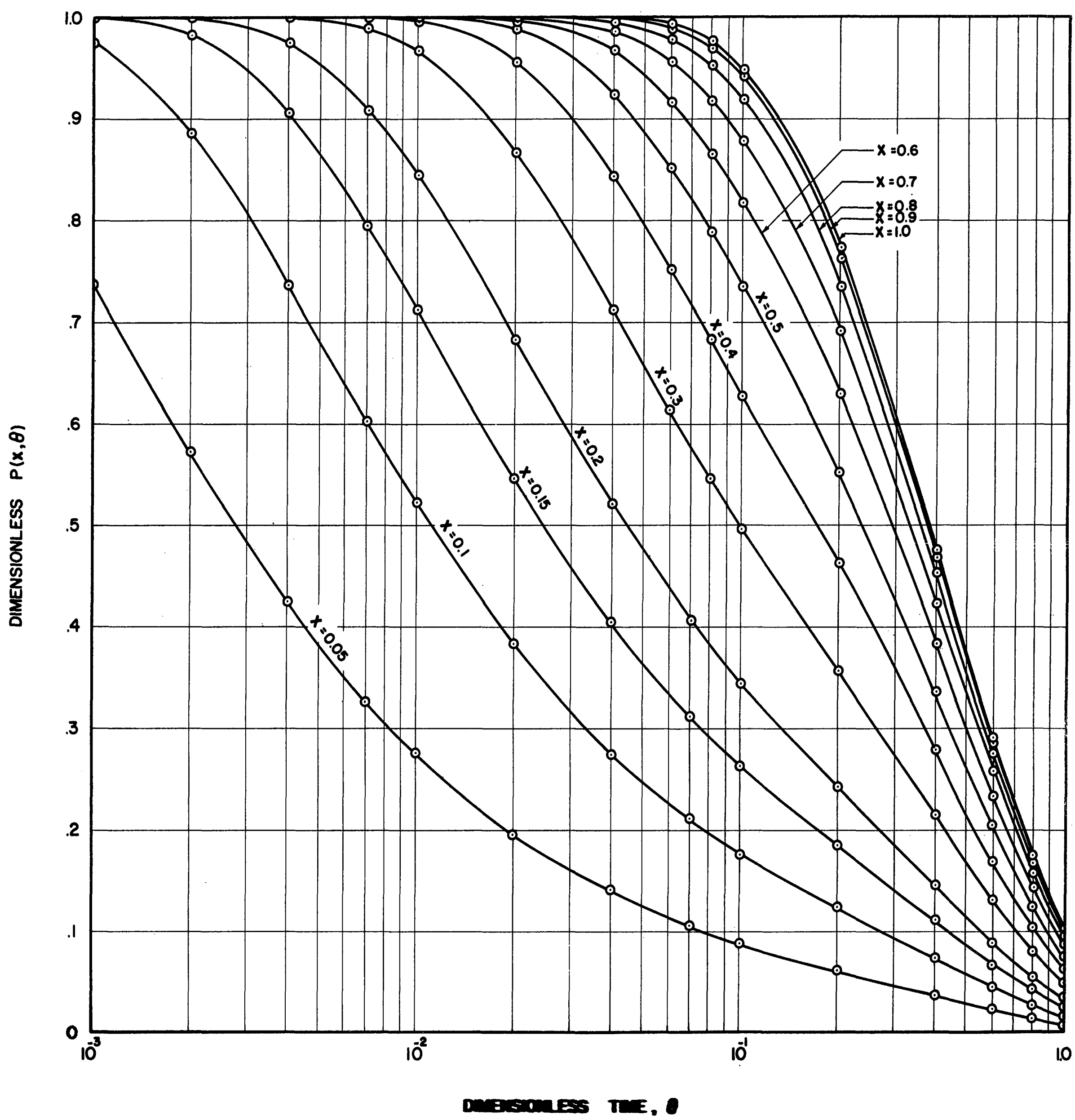


Figure IV-2. Dimensionless Pressure Distribution for Single-Layer Linear Flow, Constant Terminal Pressure.

This case will be used to illustrate a transformation method of solution. A finite Fourier cosine transformation will be employed on the spatial dimension in conjunction with a LaPlace transformation with respect to time. Defining dimensionless variables as:

$$\Theta = \frac{k t_a \pi^2}{\mu \phi c L_a^2} \tag{IV-27}$$

$$x = \pi \left(\frac{x_a}{L_a} \right) \tag{IV-28}$$

equation (IV-2) may be written as

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial \Theta} \tag{IV-5}$$

The applicable boundary and initial conditions for the constant terminal rate case are

$$P\,(x,o) = 0 \tag{IV-29}$$

$$\frac{\partial P}{\partial x}\,(0,\Theta) = -1 \tag{IV-30}$$

$$\frac{\partial P}{\partial x}\,(\pi,\Theta) = 0 \tag{IV-31}$$

Equations (IV-29) and (IV-30) introduce the step function in flux at x = 0, and equation (IV-31) expresses the condition of no flow across the other end of the model. The operational property of the finite Fourier cosine transformation, as given by Churchill,[8] is

$$C_n \left\{ F''(x) \right\} = -n^2 f_c(n) - F'(o) + (-1)^n F'(\pi) \tag{IV-32}$$

for n=0, 1, 2, ...

Let $\bar{P}$ represent the finite Fourier cosine transform of P (x,Θ) with respect to the variable x. Then

$$\frac{d\bar{P}}{dt} + n^2 \bar{P} - 1 = 0 \qquad \text{(IV-33)}$$

Note that the transformation has reduced the partial differential equation (IV-5) to a total differential equation, and has at the same time introduced the boundary conditions (IV-30) and (IV-31). This ordinary differential equation will be solved using the LaPlace transformation for purposes of illustration. Let $\bar{\bar{P}}$ represent the LaPlace transform of $\bar{P}$ with respect to dimensionless time Θ. Then

$$s\bar{\bar{P}} + n^2\bar{\bar{P}} - \frac{1}{s} = 0 \qquad \text{(IV-34)}$$

Solving algebraically

$$\bar{\bar{P}} = \frac{1}{s(s+n^2)} \qquad \text{(IV-35)}$$

This is of the form

$$\bar{\bar{P}} = \frac{1}{(s-a)(s-b)} \qquad \text{(IV-36)}$$

where $a = 0$
$b = -n^2$

From a table of inverse transformations,[8] the solution for $\bar{P}$ is thus

$$\bar{P} = \frac{1}{n^2}\left(1-e^{-n^2\Theta}\right) = \frac{1}{n^2} - \frac{e^{-n^2\Theta}}{n^2} \qquad \text{(IV-37)}$$

The necessary property for inverting the transform $\bar{P}$ given by Churchill[8] as

$$F(x) = \frac{1}{\pi} f_c\,(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} f_c(n)\cos nx \qquad \text{(IV-38)}$$

for $0 < x < \pi$

At n = 0

$$\bar{P} = L^{-1}\{\bar{\bar{P}}_{n=o}\} = L^{-1}\{\frac{1}{s^2}\} = \Theta \qquad \text{(IV-39)}$$

Therefore

$$\bar{P}(n,\Theta) = \frac{1}{n^2} - \frac{e^{-n^2\Theta}}{n^2} \qquad \text{(IV-37)}$$

$$\bar{P}(0,\Theta) = \Theta \qquad \text{(IV-40)}$$

The inverse transformation then yields

$$P(x,\Theta) = \frac{(\pi-x)^2}{2\pi} - \frac{\pi}{6} + \frac{\Theta}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2\Theta}}{n^2}\cos nx \qquad \text{(IV-41)}$$

For the heat transfer case, dimensionless variables are defined as

$$\Theta = \frac{k\, t_a \pi^2}{\rho C_p L_a^2} \qquad \text{IV-42}$$

$$x = \pi \frac{x_a}{L_a} \qquad \text{(IV-28)}$$

and the solution is again

$$T(x,\Theta) = \frac{(\pi-x)^2}{2\pi} - \frac{\pi}{6} + \frac{\Theta}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2\Theta}}{n^2}\cos nx \qquad \text{(IV-43)}$$

It is of interest to note that this form of the solution is somewhat easier to work with than the solution given by Carslaw and Jaeger, case III on p. 310 of their book.

It has been verified that this solution, IV-43, converges rapidly for values of Θ greater than 1.0. Tables of results have not, however, been computed, since many values of interest would unfortunately occur in the range of $\Theta < 1.0$.

B. Single-Layer Radial Flow

Figure IV-3 shows schematically the mathematical model used for both the constant terminal pressure and constant terminal rate cases. The same assumptions with regard to homogeneity, isotropy, etc. will be made in the radial case as in the case of single-layer linear flow.

The equation governing the flow in the case of single-layer radial flow is the diffusivity equation for k independent of position, as expressed in radial coordinates. The equation can be obtained directly from equation III-27 by assuming k constant.

$$\frac{\partial^2 P}{\partial r_a^2} + \frac{1}{r_a}\frac{\partial P}{\partial r_a} + \frac{\partial^2 P}{\partial y_a^2} = \frac{\mu \phi c}{k}\frac{\partial P}{\partial t_a} \qquad \text{(IV-44)}$$

Since in this case there will be no gradient in the y_a direction, the equation reduces to

$$\frac{\partial^2 P}{\partial r_a^2} + \frac{1}{r_a}\frac{\partial P}{\partial r_a} = \frac{\mu \phi c}{k}\frac{\partial P}{\partial t_a} \qquad \text{(IV-45)}$$

Defining dimensionless variables as

$$r = \frac{r_a}{r_b} \qquad \text{(IV-46)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(IV-47)}$$

$$\Theta = \frac{k t_a}{\mu \phi c r_b^2} \quad \text{for fluid flow} \qquad \text{(IV-48)}$$

$$\Theta = \frac{k t_a}{\rho C_p r_b^2} \quad \text{for heat conduction}$$

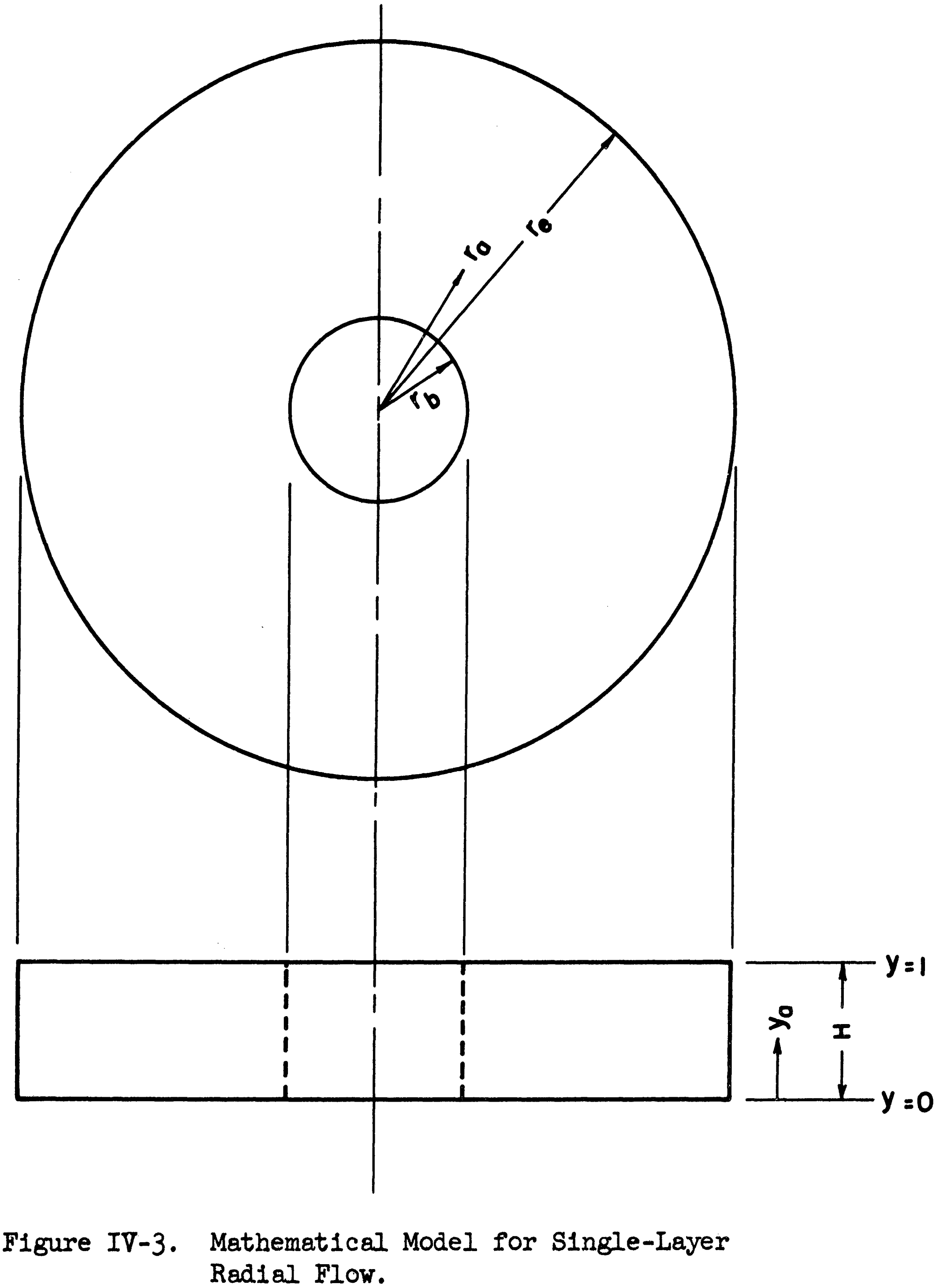


Figure IV-3. Mathematical Model for Single-Layer Radial Flow.

equation (IV-45) may be written

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \quad \frac{\partial P}{\partial r} = \frac{\partial P}{\partial \theta} \qquad \text{(IV-49)}$$

1. The Constant Terminal Pressure Case

The solution for $P(r,\theta)$ will first be obtained using the technique of separation of variables. The cumulative dimensionless flux across the producing face will then be obtained by appropriate differentiation and integration of this solution.

a. Pressure Distribution

The boundary and initial conditions for this constant terminal pressure case are

Initial Condition: $P(r,0) = 1$ (IV-50)

Boundary Conditions: $P(1,\theta) = 0$ (IV-51)

$$\frac{\partial P}{\partial r}(R,\theta) = 0 \qquad \text{(IV-52)}$$

It is also known that

$$\lim_{\theta \to \infty} P = 0 \qquad \text{(IV-53)}$$

It is first desired to find all solutions of the type

$$P = X(r) \quad Y(\theta) \qquad \text{(IV-54)}$$

which satisfy equations (IV-49), (IV-51), and (IV-52).

$$\frac{Y''}{Y} = \frac{X''}{X} + \frac{1}{r} \quad \frac{X}{X} = -b^2 \qquad \text{(IV-55)}$$

$$X' + b^2 \; Y = 0 \qquad \text{(IV-56)}$$

$$X'' + \frac{1}{r} X' + b^2 X = 0 \qquad \text{(IV-57)}$$

The equation (IV-57) is one form of Bessel's equation.

For $b = 0$: $Y = C_4$ (IV-58)

$$X = C_5 \ln r + C_6 \qquad \text{(IV-59)}$$

But from (IV-53), $C_4 = 0$

Therefore $b \neq 0$.

For $b \neq 0$:

$$Y = C_3 \ e^{-b^2\Theta} \qquad \text{(IV-60)}$$

$$X = C_1 \ J_o(br) + C_2 \ Y_o(br) \qquad \text{(IV-61)}$$

From (IV-51):

$$X\ (1) = 0 = C_1 \ J_o(b) + C_2 \ Y_o(b) \qquad \text{(IV-62)}$$

$$X'\ (R) = 0 = -b[C_1 J_1(bR) + C_2 Y_1(bR)] \qquad \text{(IV-63)}$$

Since $b \neq 0$

$$C_1 J_1(bR) + C_2 Y_1(bR) = 0 \qquad \text{(IV-64)}$$

In order that there be a non-trivial solution, the determinant of the coefficients in (IV-62) and (IV-64) must equal zero. This gives the characteristic equation

$$J_o(b)Y_1(bR) - Y_o(b)J_1(bR) = 0 \qquad \text{(IV-65)}$$

which defines an infinite number of real and positive eigenvalues b for each value of R. The means by which eigenvalues were determined from characteristic equations of the type of (IV-65) in this research are discussed in section (IV-C) of this dissertation.

From (IV-64)

$$C_2 = -C_1 \frac{J_1(bR)}{Y_1(bR)} \qquad \text{(IV-66)}$$

Then from (IV-54)

$$P = Ce^{-b^2\Theta} \ [J_o(br)Y_1(bR) - J_1(bR)Y_o(br)] \qquad \text{(IV-67)}$$

No single value of the constant C will satisfy the initial condition (IV-50). However, due to the linearity of (IV-49), any sum of solutions of the form of (IV-67) will also be a solution. The equation for the initial condition may thus be written

$$1 = \sum_{m=1}^{\infty} C_m \, [J_o(br)Y_1(bR) - J_1(bR)Y_o(br)] \qquad \text{(IV-68)}$$

Define

$$U(br) = J_o(br)Y_1(bR) - Y_o(br)J_1(bR) \qquad \text{(IV-69)}$$

Then multiplying both sides of equation (IV-68) by rU(br), integrating over the range r = 1 to r = R, and applying the applicable orthogonality condition that

$$\int_1^R rU_m(br)\, U_n(br)\, dr = 0 \quad \text{for } m \neq n \qquad \text{(IV-70)}$$

we obtain

$$C_m = \frac{\int_1^R rU\,(br)\,dr}{\int_1^R rU\,(br)\,dr} \qquad \text{(IV-71)}$$

From Muskat,[32] page 631:

$$\int_1^R rU\,(br)\,dr = -\frac{1}{b^2}\left[r\,\frac{\partial U}{\partial r}\right]_1^R \qquad \text{(IV-72)}$$

$$\int_1^R rU^2(br)\,dr = \frac{1}{2}\left\{r^2U^2 + \frac{1}{b^2}\left[r\,\frac{\partial U}{\partial r}\right]^2\right\}_1^R \qquad \text{(IV-73)}$$

$$U\,(bR) = \frac{-2}{\pi bR} \qquad \text{(IV-74)}$$

Also, by direct differentiation

$$\frac{\partial U}{\partial r} = -b[J_1(br)Y_1(bR) - Y_1(br)J_1(bR)] \qquad \text{(IV-75)}$$

$$\frac{\partial U}{\partial r}\,(bR) \equiv 0 \qquad \text{(IV-76)}$$

From (IV-65)

$$U\,(b) = 0 \qquad \text{(IV-77)}$$

Define

$$V(br) = J_1(br)Y_1(bR) - Y_1(br)J_1(bR) \qquad \text{(IV-78)}$$

Then

$$\frac{\partial U(br)}{\partial r} = -bV(br) \qquad \text{(IV-79)}$$

and

$$V(bR) \equiv 0 \qquad \text{(IV-80)}$$

Substituting into (IV-71)

$$C_m = \frac{\frac{-1}{b^2}\,[bV(b)]}{\frac{1}{2}\left\{R^2\left(\frac{-2}{\pi bR}\right)^2 - \frac{1}{b^2}[bV(b)]^2\right\}} = \frac{2bV(b)}{\left[b^2V^2(b) - \frac{4}{\pi^2}\right]} \qquad \text{(IV-81)}$$

The solution is thus

$$P(r,\theta) = \sum_{m=1}^{\infty} C_m\, U(br)e^{-b^2\theta} \qquad \text{(IV-82)}$$

where C_m is given by (IV-81) and (IV-78), and U(br) is defined by (IV-69).

Equation (IV-82) was used to compute the tables of $\bar{P}(r,\theta)$ appearing in Appendix G.

b. Dimensionless Cumulative Flux

A dimensionless cumulative flux across the producing face (at $r_a=r_b$, r=1) may be defined as

$$Q_{(t)} = \int_0^{\theta} \left(\frac{\partial P}{\partial r}\right)_{r=1} d\theta \qquad \text{(IV-83)}$$

This is the definition used by Van Everdingen and Hurst.[40]

$$\left(\frac{\partial P}{\partial r}\right)_{r=1} \left\{\sum_{m=1}^{\infty} C_m e^{-b^2\theta} \frac{\partial}{\partial r}\left[U(br)\right]\right\}_{r=1}$$

$$= -\sum_{m=1}^{\infty} b\ C_m\ V(b)e^{-b^2\theta} \qquad \text{(IV-84)}$$

$$Q_{(t)} = -\int_0^{\theta}\sum_{m=1}^{\infty} b\ C_m\ V(b)e^{-b^2\theta}\ d\theta$$

$$= -\sum_{m=1}^{\infty} b\ C_m\ V(b)\int_0^{\theta} e^{-b^2\theta}\ d\theta$$

$$= -\sum_{m=1}^{\infty} \frac{C_m}{b}\ V(b)\left[1-e^{-b^2\theta}\right] \qquad \text{(IV-85)}$$

Equation (IV-85) may be conveniently separated into two series, the first independent of time, the second time dependent.

$$Q_{(t)} = -\sum_{m=1}^{\infty} \frac{C_m}{b}\ V(b)$$

$$+\sum_{m=1}^{\infty} \frac{C_m}{b}\ V(b)e^{-b^2\theta} \qquad \text{(IV-86)}$$

This solution is perfectly valid, and was actually used to compute several values of $Q_{(t)}$. Unfortunately, however, the time independent series converges extremely slowly, requiring of the order of 500 terms to attain four-place accuracy.

In this particular instance, the technique of the LaPlace Transformation proves to be superior. Van Everdingen and Hurst have used this method to obtain the solution

$$Q(t) = \left(\frac{R^2-1}{2}\right) - 2\sum_{m=1}^{\infty} \frac{e^{-b^2\theta}J_1^{\,2}(bR)}{b^2[J_o^{\,2}(b) - J_1^{\,2}(bR)]} \qquad \text{(IV-87)}$$

It can easily be shown that the Equations (IV-85), (IV-86) and (IV-87) are mathematically identical. In both solutions the eigenvalues, b, are obtained from the characteristic equation

$$J_o(b)Y_1(bR) - Y_o(b)J_1(bR) = 0 \qquad \text{(IV-65)}$$

Rearranging

$$Y_1(bR) = \frac{J_1(bR)Y_o(b)}{J_o(b)} \qquad \text{(IV-88)}$$

$$\begin{aligned} V(b) &= J_1(b)Y_1(bR) - J_1(bR)Y_1(b) \\ &= \frac{J_1(bR)}{J_o(b)}\,[J_1(b)Y_o(b) - Y_1(b)J_o(b)] \\ &= \frac{J_1(bR)}{J_o(b)}\left(\frac{2}{\pi b}\right) \qquad \text{(IV-89)} \end{aligned}$$

Substituting into (IV-85)

$$Q_{(t)} = -2\sum_{m=1}^{\infty} \frac{[1-e^{-b^2\theta}]V^2(b)}{b^2V^2(b) - \frac{4}{\pi^2}}$$

$$= 2\sum_{m=1}^{\infty} \frac{[1-e^{-b^2\theta}]\,\frac{J_1^{\,2}(bR)}{J_o^{\,2}(b)}\left[\frac{4}{\pi^2b^2}\right]}{\left[\frac{J_1^{\,2}(bR)}{J_o^{\,2}(b)}\left[\frac{4}{\pi^2}\right] - \left[\frac{4}{\pi^2}\right]\right]}$$

$$= -2\sum_{m=1}^{\infty} \frac{[1-e^{-b^2\theta}]\,J_1^{\,2}(bR)}{b^2[J_1^{\,2}(bR) - J_o^{\,2}(b)]}$$

$$= 2\sum_{m=1}^{\infty} \frac{J_1^{\,2}(bR)}{b^2[J_o^{\,2}(b) - J_1^{\,2}(bR)]}$$

$$-2\sum_{m=1}^{\infty} \frac{e^{-b^2\theta}J_1^{\,2}(bR)}{b^2[J_o^{\,2}(b) - J_1^{\,2}(bR)]} \qquad \text{(IV-90)}$$

The solutions are thus equivalent if

$$\frac{R^2-1}{2} = 2 \sum_{m=1}^{\infty} \frac{J_1{}^2(bR)}{b^2[J_o{}^2(b) - J_1{}^2(bR)]} \qquad \text{(IV-91)}$$

From (IV-68)

$$1 = \sum_{m=1}^{\infty} C_m U(br)$$

$$= \sum_{m=1}^{\infty} \frac{2bV(b)}{b^2V^2(b) - (\frac{4}{\pi^2})} \; U(br) \qquad \text{(IV-92)}$$

Substituting for V(b) from (IV-89)

$$1 = \sum_{m=1}^{\infty} \frac{2b \frac{J_1(bR)}{J_o(b)} \frac{2}{\pi b}}{b^2 \frac{J_1{}^2(bR)}{J_o{}^2(b)} \frac{4}{\pi^2 b^2} - \frac{4}{\pi^2}} \; U(br)$$

$$= \pi \sum_{m=1}^{\infty} \frac{J_o(b) J_1(bR)}{[J_1{}^2(bR) - J_o{}^2(b)]} \; U(br) \qquad \text{(IV-93)}$$

Multiplying both sides by r and integrating with respect to r between 1 and R:

$$\int_1^R r\,dr = \sum_{m=1}^{\infty} C_m \int_1^R rU(br)\,dr$$

$$\frac{R^2-1}{2} = \pi \sum_{m=1}^{\infty} C_m \left[-\frac{1}{b^2} \left(r \frac{\partial U}{\partial r} \right)_1^R \right] \qquad \text{(IV-94)}$$

$$= \pi \sum_{m=1}^{\infty} C_m \left[- \frac{1}{b} V(b) \right]$$

$$= \pi \sum_{m=1}^{\infty} \frac{J_o(b) J_1(bR) V(b)}{b \left[J_o{}^2(b) - J_1{}^2(bR) \right]} \qquad \text{(IV-95)}$$

$$\left(\frac{R^2 - 1}{2} \right) = \pi \sum_{m=1}^{\infty} \frac{J_1{}^2(bR)}{b^2 \left[J_o{}^2(b) - J_1{}^2(bR) \right]} \quad \frac{J_1(bR)}{J_o(b)} \quad \frac{2}{\pi b}$$

$$= 2 \sum_{m=1}^{\infty} \frac{J_1{}^2(bR)}{b^2 \left[J_o{}^2(b) - J_1{}^2(bR) \right]} \qquad \text{(IV-96)}$$

This is the same as equation (IV-91). The two solutions are thus mathematically equivalent.

It is evident that the solution obtained using separation of variables utilizes a very slowly convergent series to represent the simple algebraic quantity $\left(\frac{R^2-1}{2}\right)$. The LaPlace Transformation solution represented by equation (IV-87) was therefore used to compute the table of results presented in Appendix I, using the IBM 704 computer. These tables are an extension of the tables presented by Van Everdingen and Hurst, who presented values of $Q_{(t)}$ up to R = 10. Appendix I presents tables of $Q_{(t)}$ from R = 10 through R = 1000.

Comparison with Van Everdingen and Hurst's tables is possible at R = 10. The maximum difference between tables is 0.2%. This slight error is due to two factors. Firstly, the first eigenvalue, b_1, used by Van Everdingen and Hurst is slightly in error (0.1104 should be 0.110269). Secondly, and more important, Van Everdingen and Hurst used only two terms in the series for all values of dimensionless time, whereas at low values of Θ (where the discrepancy between tables occurs) it was actually found necessary to use up to six terms to obtain four-place accuracy.

2. The Constant Terminal Rate Case

The governing flow equation is again.

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \theta} \qquad \text{(IV-49)}$$

The applicable boundary and initial conditions for the case of constant terminal rate are:

Initial condition: $P(r,o) = 0$ (IV-97)

Boundary conditions: $\frac{\partial P}{\partial r}(1,\theta) = -1$ (IV-98)

$$\frac{\partial P}{\partial r}(R,\theta) = 0 \qquad \text{(IV-99)}$$

A variation of the usual technique of separation of variables will be used for this case. Both a product-type and a sum-type of separated solution will be postulated. First, consider all solutions of the form

$$P = X(r)Y(\theta) \qquad \text{(IV-100)}$$

which satisfy (IV-49), (IV-98) and (IV-99). Then

$$\frac{Y'}{Y} = \frac{X''}{X} + \frac{1}{r}\frac{X'}{X} = -\alpha^2 \qquad \text{(IV-101)}$$

The resultant ordinary differential equations are

$$Y' + \alpha^2 Y = 0 \qquad \text{(IV-102)}$$

$$X'' + \frac{1}{r}X' + \alpha^2 X = 0 \qquad \text{(IV-103)}$$

For $\alpha \neq 0$:

$$Y = C_3 e^{-\alpha^2\theta} \qquad \text{(IV-104)}$$

$$X = C_1 J_o(\alpha r) + C_2 Y_o(\alpha r) \qquad \text{(IV-105)}$$

$$P = e^{-a^2\theta}\left[C_1 J_o(\alpha r) + C_2 Y_o(\alpha r)\right] \qquad \text{(IV-106)}$$

For $\alpha = 0$:

$$Y = C_4 \qquad \text{(IV-107)}$$

$$X = C_5 \ln r + C_6 \qquad \text{(IV-108)}$$

$$P = C_3 \ln r + C_4 \qquad \text{(IV-109)}$$

Now, consider all solutions of the form

$$P = X(r) + Y(\theta)$$

which satisfy (IV-49), (IV-98), and (IV-99). For this case,

$$Y' = X'' + \frac{1}{r} X' = C_7 \qquad \text{(IV-110)}$$

Solving

$$Y = C_7\theta \qquad \text{(IV-111)}$$

$$X = \frac{C_7 r^2}{4} \qquad \text{(IV-112)}$$

$$P = C_7(\frac{r^2}{4} + \theta) = C_5(r^2 + 4\theta) \qquad \text{(IV-113)}$$

Since a sum of these solutions will also be a solution, let

$$P = e^{-\alpha^2\theta}[C_1 J_o(\alpha r) + C_2 Y_o(\alpha r)] + C_3 \ln r + C_4 + C_5(r^2 + 4\theta) \qquad \text{(IV-114)}$$

Differentiating,

$$\frac{\partial P}{\partial r} = -\alpha[C_1 J_1(\alpha r) + C_2 Y_1(\alpha r)]e^{-\alpha^2\theta} + \frac{C_3}{r} + 2C_5 r \qquad \text{(IV-115)}$$

From (IV-98)

$$\left(\frac{\partial P}{\partial r}\right)_{r=1} = -1 = -\alpha[C_1 J_1(\alpha) + C_2 Y_1(\alpha)]e^{-\alpha^2\theta} + C_3 + 2C_5 \qquad \text{(IV-116)}$$

From (IV-99)

$$\left(\frac{\partial P}{\partial r}\right)_{r=R} = 0 = -\alpha[C_1J_1(\alpha R) + C_2Y_1(\alpha R)]e^{-\alpha^2\theta} + \frac{C_3}{R} + 2C_5R \qquad \text{(IV-117)}$$

Since equations (IV-116) and (IV-117) must be valid for all values of θ,

$$C_1J_1(\alpha) + C_2Y_1(\alpha) = 0 \qquad \text{(IV-118)}$$

$$C_1J_1(\alpha R) + C_2Y_1(\alpha R) = 0 \qquad \text{(IV-119)}$$

In order that there be a non-trivial solution, the determinant of the coefficients must equal zero. This defines the characteristic equation

$$J_1(\alpha)Y_1(\alpha R) - Y_1(\alpha)J_1(\alpha R) = 0 \qquad \text{(IV-120)}$$

from which an infinite number of eigenvalues α, all real and positive, may be determined for a given value of R.

The criteria of (IV-116) and (IV-117) being valid for all values of θ also necessitates

$$C_3 + 2C_5 = -1 \qquad \text{(IV-121)}$$

$$\frac{C_3}{R} + 2C_5R = 0 \qquad \text{(IV-122)}$$

Therefore

$$C_3 = \frac{-R^2}{R^2-1} \qquad \text{(IV-123)}$$

$$C_5 = \frac{1}{2(R^2-1)} \qquad \text{(IV-124)}$$

Since the initial pressure was set arbitrarily, the additive constant, C_4, may be set equal to zero.

From (IV-118)

$$C_2 = -C_1\frac{J_1(\alpha)}{Y_1(\alpha)} \qquad \text{(IV-125)}$$

Therefore

$$[C_1J_o(\alpha r) + C_2Y_o(\alpha r)] = [C_1J_o(\alpha r) - C_1 \frac{J_1(\alpha)}{Y_1(\alpha)} Y_o(\alpha r)]$$

$$= C[J_o(\alpha r)Y_1(\alpha) - J_1(\alpha)Y_o(\alpha r)] \qquad \text{(IV-126)}$$

Define

$$U(\alpha r) = J_o(\alpha r)Y_1(\alpha) - J_1(\alpha)Y_o(\alpha r) \qquad \text{(IV-127)}$$

Substituting into (IV-114),

$$P = CU(\alpha r)e^{-\alpha^2\theta} - \left(\frac{R^2}{R^2-1}\right) \ln r$$

$$+ \left[\frac{1}{2(R^2-1)}\right] r^2 \qquad \text{(IV-128)}$$

There is no single value of the constant C which will satisfy the initial condition (IV-97) for all values of r. However, the initial condition can be represented by

$$P(r,0) = 0 = C_3 \ln r + C_5r^2 + \sum_{n=1}^{\infty} C_nU(\alpha r) \qquad \text{(IV-129)}$$

Multiplying both sides of equation (IV-129) by $rU(\alpha r)$, integrating with respect to r from 1 to R, and applying the orthogonality condition (IV-70),

$$C_n \int_1^R rU^2(\alpha r)dr = -C_3\int_1^R rU(\alpha r) \ln r \, dr$$

$$-C_5\int_1^R r^3U(\alpha r)dr \qquad \text{(IV-130)}$$

It is evident that

$$\left(\frac{\partial U}{\partial r}\right)_{r=1} = 0 \qquad \text{(IV-131)}$$

From (IV-120)

$$\left(\frac{\partial U}{\partial r}\right)_{r=R} = 0 \qquad \text{(IV-132)}$$

From Muskat[32], p. 631:

$$U(\alpha) = \frac{-2}{\pi\alpha} \qquad \text{(IV-133)}$$

Evaluating the integrals:

$$\int_1^R r^2U^2(\alpha r)dr = \frac{1}{2}\left\{ r^2U^2 + \frac{1}{\alpha^2}\left[\frac{\partial U}{r\partial r}\right]^2\right\} = \frac{R^2U^2(\alpha R)}{2} - \frac{2}{\pi^2\alpha^2} \qquad \text{(IV-134)}$$

$$\int_1^R r\,\ell n\, r\; U(\alpha r)dr = \frac{1}{\alpha^2}\left[U - r\frac{\partial U}{\partial r}\,\ell nr\right]_1^R = \frac{U(\alpha R)}{\alpha^2} + \frac{2}{\pi\alpha^3} \qquad \text{(IV-135)}$$

$$\int_1^R r^3U(\alpha r)dr = \left[\frac{-r^3}{\alpha^2}\frac{\partial U}{\partial r} + \frac{2r^2U}{\alpha^2} + \frac{4r}{\alpha^4}\frac{\partial U}{\partial r}\right]_1^R \qquad \text{(IV-136)}$$

$$= \frac{2R^2U(\alpha R)}{\alpha^2} + \frac{4}{\pi\alpha^3}$$

$$C_n = \frac{\dfrac{R^2}{1-R^2}\left[\dfrac{U(\alpha R)}{\alpha^2} + \dfrac{2}{\pi\alpha^3}\right] - \dfrac{1}{2(R^2-1)}\left[\dfrac{2R^2U(\alpha R)}{\alpha^2} + \dfrac{4}{\pi\alpha^3}\right]}{\left[\dfrac{R^2\pi^2\alpha^2U^2(\alpha R) - 4}{2\pi^2\alpha^2}\right]}$$

$$= \frac{4\pi}{\alpha\left[R^2\pi^2\alpha^2U^2(\alpha R) - 4\right]} \qquad \text{(IV-137)}$$

From (IV-120), rearranging,

$$Y_1(\alpha) = \frac{J_1(\alpha)\;Y_1(\alpha R)}{J_1(\alpha R)} \qquad \text{(IV-138)}$$

Substituting into (IV-127)

$$U(\alpha R) = J_o(\alpha R)\ Y_1\ (\alpha) - J_1\ (\alpha)\ Y_o\ (\alpha R)$$

$$= \frac{J_1(\alpha)}{J_1(\alpha R)} \left[J_o(\alpha R)\ Y_1(\alpha R) - Y_o(\alpha R) J_1(\alpha R) \right]$$

$$= \frac{J_1(\alpha)}{J_1(\alpha R)} \left[\frac{-2}{\pi \alpha R} \right] \quad \text{(IV-139)}$$

Substituting (IV-139) into (IV-137),

$$C_n = \frac{4\pi}{\alpha \left[R^2 \pi^2 \alpha^2 \left(\frac{4}{\pi^2 R^2 \alpha^2} \right) \frac{J_1{}^2(\alpha)}{J_1{}^2(\alpha R)} - 4 \right]}$$

$$= \frac{\pi J_1{}^2(\alpha R)}{\alpha [J_1{}^2(\alpha) - J_1{}^2(\alpha R)]} \quad \text{(IV-140)}$$

Then it should be true that

$$P(r,\theta) = \frac{R^2 \ell n\ r}{(1-R^2)} + \frac{(r^2+4\theta)}{2(R^2-1)} + \pi \sum_{n=1}^{\infty} C_n U(\alpha r) e^{-\alpha^2 \theta} \quad \text{(IV-141)}$$

But this equation does not satisfy the initial condition (IV-97). It can be shown that letting $P(r,0) = B$, carrying out the previous analysis, then setting $B = 0$, results in the same solution, since

$$B \int_1^R rU(\alpha r) dr = 0 \quad \text{(IV-142)}$$

In order to fit the initial condition, a material balance will be employed.

$$q = \frac{-k}{\mu}\ 2\pi h r_b \left(\frac{\partial P}{\partial r}\right)_{r_b} \frac{ft^3}{sec}$$

$$= \frac{2\pi hk}{\mu}\ \frac{ft^3}{sec} \quad \text{(IV-143)}$$

$$Q = \int_0^\Theta q\, d\Theta = \frac{\mu\phi c r_b^2}{k} \int_0^\Theta q\, d\Theta$$

$$= 2\pi h\phi c r_b^2 \Theta \quad ft^3 \qquad \text{(IV-144)}$$

$$W = 2\pi h\phi c r_b^2 \rho\Theta \quad lbs. \qquad \text{(IV-145)}$$

At any time Θ it must also be true that

$$W = \int_1^R P\ 2\pi h\phi c r_b^2 \rho r dr$$

$$= 2\pi h\phi c r_b^2 \rho \int_1^R rPdr \qquad \text{(IV-146)}$$

Consider large Θ, where $e^{-\alpha^2\Theta} = 0$. Assume that one additional term, which must be independent of Θ, is necessary in the solution (IV-141), and let this term be represented by "b". Then

$$W = 2\pi h\phi c r_b^2 \rho \int_1^R \left[\frac{r^3}{2(R^2-1)} + \frac{2r\Theta}{(R^2-1)} - \frac{R^2 r \ln r}{(R^2-1)} + br\right] dr$$

$$= 2\pi h\phi c r_b^2 \rho\Theta \qquad \text{(IV-147)}$$

$$+2\pi h\Theta c r_b^2 \rho \int_1^R \left[br + \frac{r^3}{2(R^2-1)} - \frac{R^2 r}{(R^2-1)} \ln r \right] dr$$

Then, from (IV-145),

$$\int_1^R \left[br + \frac{r^3}{2(R^2-1)} - \frac{R^2 r \ln r}{(R^2-1)}\right] dr = 0 \qquad \text{(IV-148)}$$

$$\frac{b}{2}(R^2-1) + \frac{1}{8}\frac{(R^4-1)}{(R^2-1)} - \frac{R^2}{R^2-1}\left[\frac{R^2 \ln R}{2} - \frac{1}{4}(R^2-1)\right] = 0$$

$$\text{or} \quad b = \frac{4R^4 \ln R - 3R^4 + 2R^2 + 1}{4\,(R^2-1)} \qquad \text{(IV-149)}$$

The final solution is then

$$P(r,\theta) = \frac{r^2 + 4\theta}{2(R^2-1)} - \frac{R^2 \ln r}{R^2-1} + \frac{(1 + 2R^2 + 4R^4 \ln R - 3R^4)}{r(R^2-1)^2} + \pi \sum_{n=1}^{\infty} \frac{e^{-\alpha^2\theta} J_1{}^2(\alpha R) \left[J_1(\alpha)Y_o(\alpha r) - Y_1(\alpha)J_o(\alpha r)\right]}{\alpha \, [J_1{}^2(\alpha R) - J_1{}^2(\alpha)]} \tag{IV-150}$$

This is the same solution which Van Everdingen and Hurst derived using LaPlace Transforms. It should be noted that in their paper,[(40)] in equations (VII-17) and (VII-18), a misprint occurs in the numerator of the Fourier-Bessel series (the term $J_1(\beta_n R)$).

Using equation (IV-150), tables of dimensionless $P(r, \theta)$ for the constant terminal rate case have been computed on an IBM 704. These tables appear in Appendix H. Provision was made in the computer program to use up to 100 terms in the series. It was found that, in nearly all cases, less than 10 terms were actually required to obtain the desired accuracy.

Appendix H presents $P(r,\theta)$ for the constant terminal rate case for values of R from 1.5 to 100. It should be noted that $P(1, \theta)$ in Appendix H corresponds to the dimensionless $P_{(t)}$ defined by Van Everdingen and Hurst. Comparison with their tables indicates excellent agreement between $P(1,\theta)$ and $P_{(t)}$ in all cases where comparison is possible.

C. Calculation of Eigenvalues from Characteristic Equations

The simplest types of characteristic equations, such as

$$\cos a = 0 \tag{IV-18}$$

can be solved by inspection to obtain the eigenvalues

$$a = \left(\frac{2m - 1}{2}\right) \pi ; \quad m = 1,2,3,\ldots \qquad \text{(IV-19)}$$

In the more complicated cases, such as

$$J_o(b)\ Y_1\ (bR) - Y_o(b)\ J_1(bR) \qquad \text{(IV-65)}$$

the interrelation between terms of the equation is such as to make a solution by inspection impossible. It is this second type of characteristic equation with which this discussion is concerned.

The eigenvalues which are desired occur as roots of the characteristic equation, i.e. when the value of the function does indeed equal zero. This is shown graphically in Figure IV-4, in which the value of the function

$$F = J_o(b)\ Y_1\ (bR) - Y_o(b)\ J_1\ (bR) \qquad \text{(IV-151)}$$

is plotted versus the variable b, for R = 10.

The method used to obtain eigenvalues from all of the characteristic equations in this dissertation (except of course (IV-18) which can be solved by inspection) was the "half-interval" method. This is an interative type of method, in which successively better approximations to the eigenvalue are made until the desired accuracy is obtained. An IBM 704 digital computer was used to obtain final values in all cases. Some preliminary work was done on a Bendix G-15 computer.

In the half-interval method, the search for roots of the characteristic equation is initiated by specifying some initial point for the variable (b=0 in the case of (IV-65)) and some initial search increment. The increment is successively added to the current value of the variable until the function changes in sign. At this point, the interval

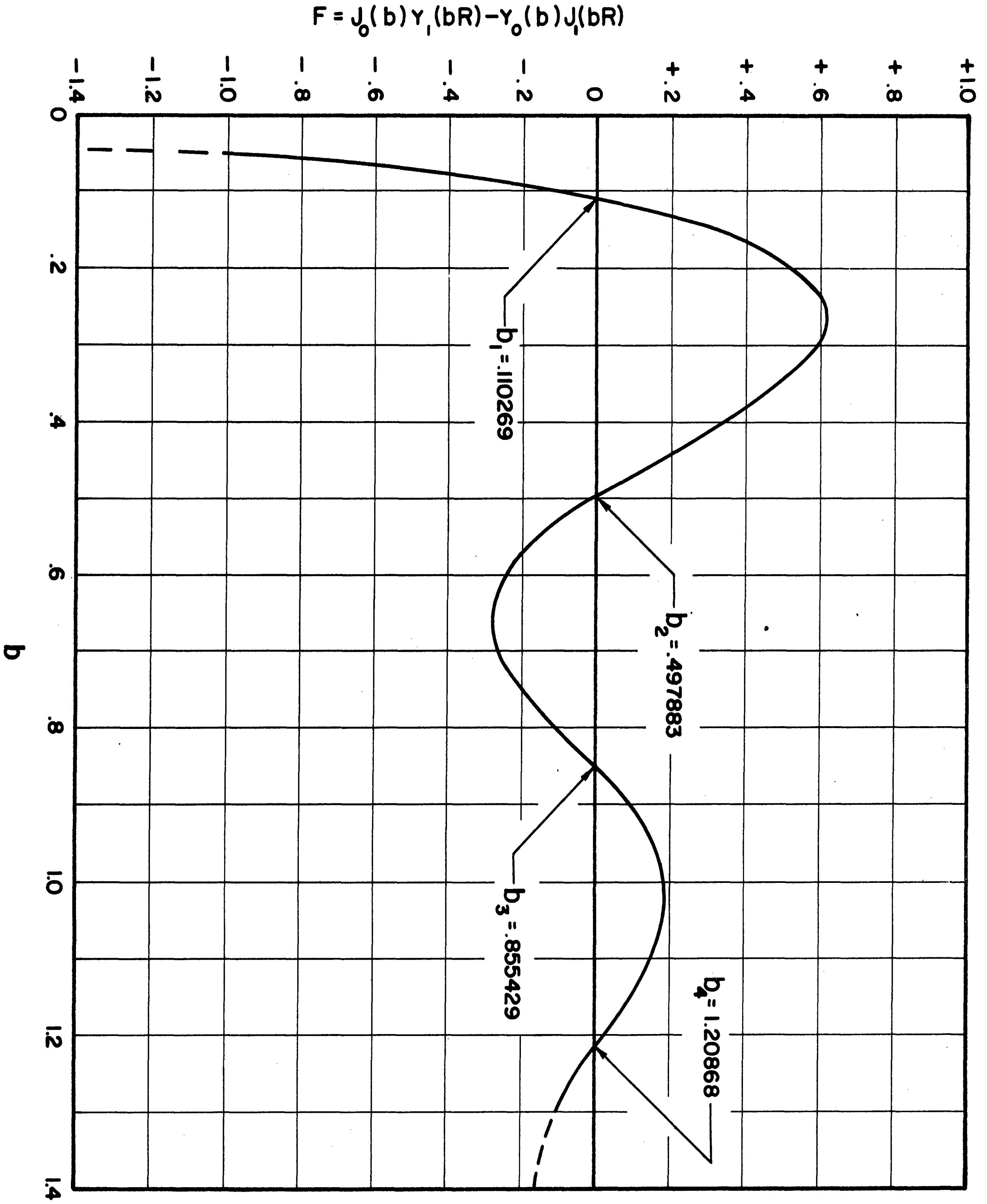


Figure IV-4. Functional Properties of a Characteristic Equation.

in which the change in sign occurs is successively "halved" or divided by two, always retaining that half in which the change in sign occurs. When the size of the interval has decreased to the limits on accuracy desired, the eigenvalue is taken as the midpoint of this interval. The procedure of adding the search increment is then repeated until the next change in sign occurs, etc. The search increment is frequently modified during the course of the search, once a few eigenvalues have been obtained.

In all cases, the eigenvalues obtained in this investigation were tested to at least six significant figures. These eigenvalues are presented in Appendices B, C, D, and E.

Since the half-interval method is a relatively slow method for determining the eigenvalues, consideration was given to using a more rapid method, such as Newton's method. It was decided, however, that the relative simplicity of the half-interval method, which allowed the use of simple computer programs, minimized the possibility of error sufficiently to compensate for its deficiency in speed. In particular, the half interval method, properly used, makes it virtually impossible to "skip" an eigenvalue. This feature was of particular importance in the case of two-layer linear flow, in which the distance between eigenvalues was not monotone decreasing, but varied in a somewhat random manner.

D. Example Calculations for Single-Layer Flow

In this section, the use of the tables of results presented in the appendices to this dissertation will be illustrated, for the case of single layer flow. The tables to be used are those in Appendices F, G, H, and I.

A total of five example problems will be used here for illustration. The use of Appendix F will be illustrated by both a heat transfer problem and a reservoir flow problem in order to emphasize the dual usefulness of such tables. The use of Appendices G and I will be illustrated by a reservoir flow problem involving the constant terminal pressure case. An aquifer storage reservoir problem will illustrate the use of Appendix H. A final problem will then illustrate how the tables, in particular those of Appendix G, may be used to obtain solutions for problems involving "infinite" aquifers.

The example problems in this dissertation have in all cases been kept as simple as possible. The problems are intended solely to illustrate the use of the tables in the Appendices. For this reason, problems involving the use of the superposition principle have not been included. The use of the superposition principle is necessary for example, when a varying pressure at the gas-water interface in a gas storage reservoir-aquifer system is known as a function of time. The reader is referred to works by Coats[11, 12, 25] and others[26,40] for a more complete treatment of reservoir flow problems.

1. Example Problem No. 1

Consider a steel bar which is initially at a uniform temperature of 70°F. The pertinent dimensions and physical properties of the bar are known to be

Length, L_a = 20 inches

Width, w_a = 1 inch

Height, h_a = 1 inch

Thermal conductivity, k = 26 Btu/hr ft ^{2}(^{o}F/ft)

Density, = 490 #/ft^3

Heat Capacity, C_p = 0.12 Btu/#^{o}F

At time zero, one end of this bar ($x_a = 0$) is rapidly heated to 270^{o}F, and is maintained at this temperature thereafter. It is desired to calculate the temperature distribution in the bar after one hour, assuming that there are no heat losses from the bar.

This problem is a typical example for which the "constant terminal pressure" case tables may be used for heat transfer. Since heat flow will be in only one direction, the width, w_a and height, h_a, are irrelevant, and the proper table is that in Appendix F. The results in Appendix F are shown graphically in Figure IV-2.

Since Appendix F involves only dimensionless quantities, it is necessary to first dedimensionalize the known physical parameters. The applicable equations for this dedimensionalization are given on the first page of Appendix F as:

$$x = \frac{x_a}{L_a} \qquad \text{(IV-152)}$$

$$\Theta = \frac{kt_a}{\rho C_p L_a^2} \qquad \text{(IV-153)}$$

At a time of one hour, in this problem

$$\Theta = \frac{(26)\ (1)}{(490)(0.12)\left(\frac{20}{12}\right)^2} \quad \frac{\text{Btu ft}}{\text{hr ft}^2\ {}^o\text{F}} \quad \text{hr} \quad \frac{\text{ft}^3}{\#} \frac{\#^o\text{F}}{\text{Btu}}$$

$= 0.1592$ (dimensionless) (IV-154)

The length conversion for this problem is

$$x_a = 20x \text{ inches} \qquad \text{(IV-155)}$$

The conversion from dimensionless temperature to actual temperature in this problem is given by

$$T_a = 270 - (270-70)T$$
$$= 270 - 200T \ {}^{\circ}F \qquad \text{(IV-156)}$$

Reading values from Figure IV-2 at $\Theta = 0.1592$, or from Appendix F (interpolating with respect to Θ) the values in Table IV-1 may be obtained. This table shows the desired temperature distribution after one hour.

Table IV - 1

Temperature Distribution for Example Problem 1

x	x_a (inches)	T	T_a (°F)
.05	1	.069	256.2
.1	2	.137	242.6
.15	3	.203	229.4
.2	4	.268	216.4
.3	6	.391	191.8
.4	8	.520	166.0
.5	10	.617	146.6
.6	12	.700	130.0
.7	14	.765	117.0
.8	16	.815	107.0
.9	18	.836	102.8
1.0	20	.849	100.2

2. Example Problem 2

Consider a gas reservoir which is bounded on three sides by faulting planes, such as that shown in Figure IV-5. Such a reservoir-aquifer system would exhibit essentially linear flow in the aquifer.

The pertinent dimensions and physical properties of the aquifer are known to be

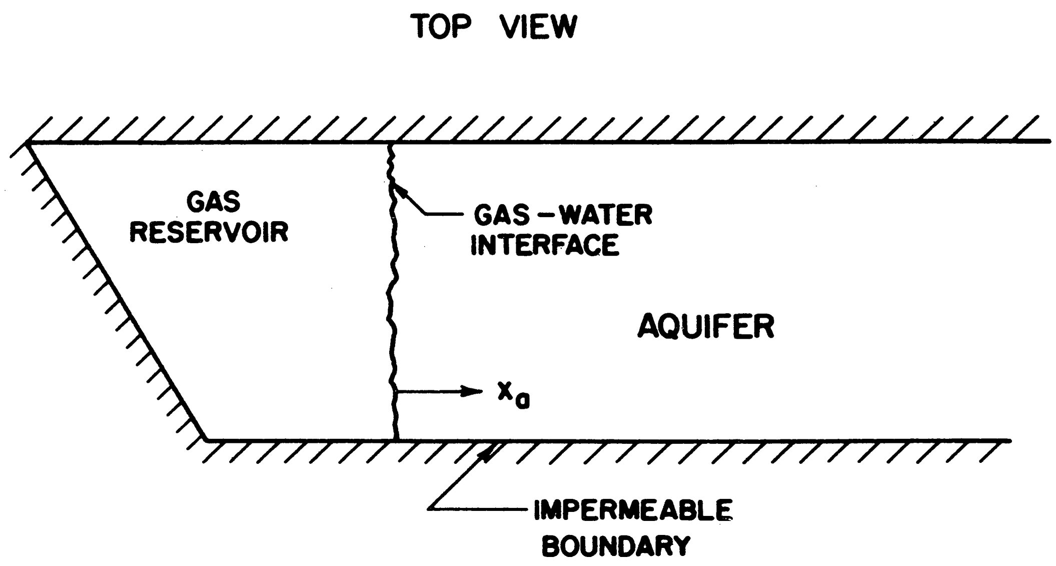


Figure IV-5. Hypothetical Reservoir for Example Problem 2.

Linear Extent of the aquifer, L_a . = 30 miles

Permeability, k = 2000 millidarcies

Porosity, = 20%

Water viscosity, = 1.0 centipoise

Water plus formation compressibility, c = 7×10^{-6} vol/vol psi

If the pressure throughout the gas reservoir is suddenly dropped by 100 psi, and maintained at the lower pressure, it is desired to find the time at which an observation well six miles from the gas-water interface (i.e. x_a = 6 miles) will indicate a pressure decline of 10 psi in the aquifer.

Use will again be made of Figure IV-2 and/or Appendix F, for single-layer linear flow. The applicable equations for dedimensionalization of the physical parameters are in this case.

$$x = \frac{x_a}{La} \qquad \text{(IV-157)}$$

$$\Theta = \frac{kt_a}{\mu \phi c L_a^2} \qquad \text{(IV-158)}$$

A compatible set of units to make Θ dimensionless are:

k = 2 darcys

μ = 1 centipoise

ϕ = 0.2 (dimensionless)

$c = 7 \times 10^{-6} \times 14.7$ vol/vol atm

$L_a = 30 \times 5280 \times 30.48$ cm

t_a = time, seconds

The dimensionless distance of the observation well is

$$x = \frac{x_a}{La} = \frac{6 \text{ miles}}{30 \text{ miles}} = 0.2 \qquad \text{(IV-159)}$$

To find the time desired, the dimensionless time at which $P(0.2,\theta) = 1-\left(\frac{10}{100}\right) = 0.9$ is simply converted back to real time using equation IV-158. From Figure IV-2 or Appendix F, this value of θ is 0.0074. The time in question is thus

$$t_a = \frac{\mu \phi c L_a^2 \theta}{k}$$

$$= \frac{(1)(0.2)(7\text{x}10^{-6}\text{x}14.7)(30\text{x}5280\text{x}30.48)^2\ (.0074)}{(2)}$$

$$= 1.772 \times 10^6 \quad \text{seconds}$$

$$= \frac{1.772\text{x}10^6}{(3600)(24)} = 20.5 \quad \text{days} \qquad \text{(IV-160)}$$

It should be noted that this analysis implicitly assumes a stationary gas-water interface, whereas in actuality the interface would shift somewhat due to water encroachment into the reservoir.

3. Example Problem 3

Consider a circular gas reservoir surrounded on all sides by a bounded uniform aquifer, with dimensions and physical properties as follow

h = 100 feet, thickness of aquifer formation

r_b = 2000 feet, radius at gas-water interface

μ = 0.8 centipoise, water viscosity

ϕ = 0.1 porosity

$c = 7 \times 10^{-6}$ vol/vol psi water plus formation compressibility

k = 1000 millidarcy permeability

R = 100 so that the exterior boundary radius, r_e, of the aquifer is 200,000 feet, or about 38 miles

If this gas reservoir is maintained at 100 psi above its initial pressure, it is desired to find the time at which the effect of the exterior aquifer boundary becomes significant, and to determine the pressure profile in the aquifer at and after this time. It is also desired to find the increase in the pore volume of the reservoir at and after the effect of the exterior aquifer boundary has become significant.

The system in this case should be approximated by the single-layer radial flow mathematical model. The pressure profile will therefore be obtained using the tables in Appendix G. It is necessary to assume a stationary gas-water interface in determining the pressure profile. The increase in the reservoir pore volume can be found using the tables in Appendix I.

The relation between actual and dimensionless time must first be calculated

$$\Theta = \frac{kt_a}{\mu\phi c r_b^2} \qquad \text{(IV-48)}$$

$$\frac{k}{\mu\phi c r_b^2} = \frac{1 \text{ darcy}}{(0.8\text{cp})(0.1)(7\times10^{-6} \times 14.7 \text{ vol/vol atm})(2000\times30.48\text{cm})^2}$$

$$= 3.26 \times 10^{-5} \text{ sec}^{-1} \qquad \text{(IV-161)}$$

Now $\quad 1 \text{ year} = (365)(24)(3600) = 3.155\times10^7 \text{ sec}$

Therefore

$$\frac{k}{\mu\phi c r_b^2} = (3.26\times10^{-5})\ (3.155\times10^7) = 1030 \text{ year}^{-1}$$

Then

$$\Theta = 1030 t_a \text{ for } t_a \text{ in years} \qquad \text{(IV-162)}$$

From the tables in Appendix G, at R = 100, the dimensionless pressure begins to deviate significantly from 1.0 at about $\Theta = 10^3$.

Thus, the time at which the effect of the exterior boundary becomes significant is approximately one year. Also, using the table at R =100, the values of $P(r, \theta)$ may be determined for succeeding values of time. These values are plotted in Figure IV-6.

The pressure profile in question is shown in Table IV-2. As an example from this table, the dimensionless pressure P after two years at $r = 20$ is $P = 0.69$. Thus, at a point in the aquifer (20)(2000)= 40,000 feet (7.58 miles) from the center of the gas reservoir, the pressure at this time is (1-0.69)(100) = 31 psi above the initial pressure.

A plot of the aquifer pressure profile for this problem is shown in Figure IV-7.

In order to calculate the increase in the pore volume of the reservoir, the total cumulative water efflux will be calculated, using the tables in Appendix I. A convenient formula for calculating the water efflux in field units is[26]

$$W_e = 6.283\ \phi c r_b^2 h(P_b - P_f) Q_{(t)} \qquad \text{(IV-163)}$$

where:

W_e = cumulative water influx or efflux, cu.ft.

ϕ = fractional porosity

c = formation plus water compressibility, vol/vol psi

h = formation thickness, ft.

r_b = inner boundary radius ft.

P_b = inner boundary pressure, psia

P_f = initial aquifer pressure, psia

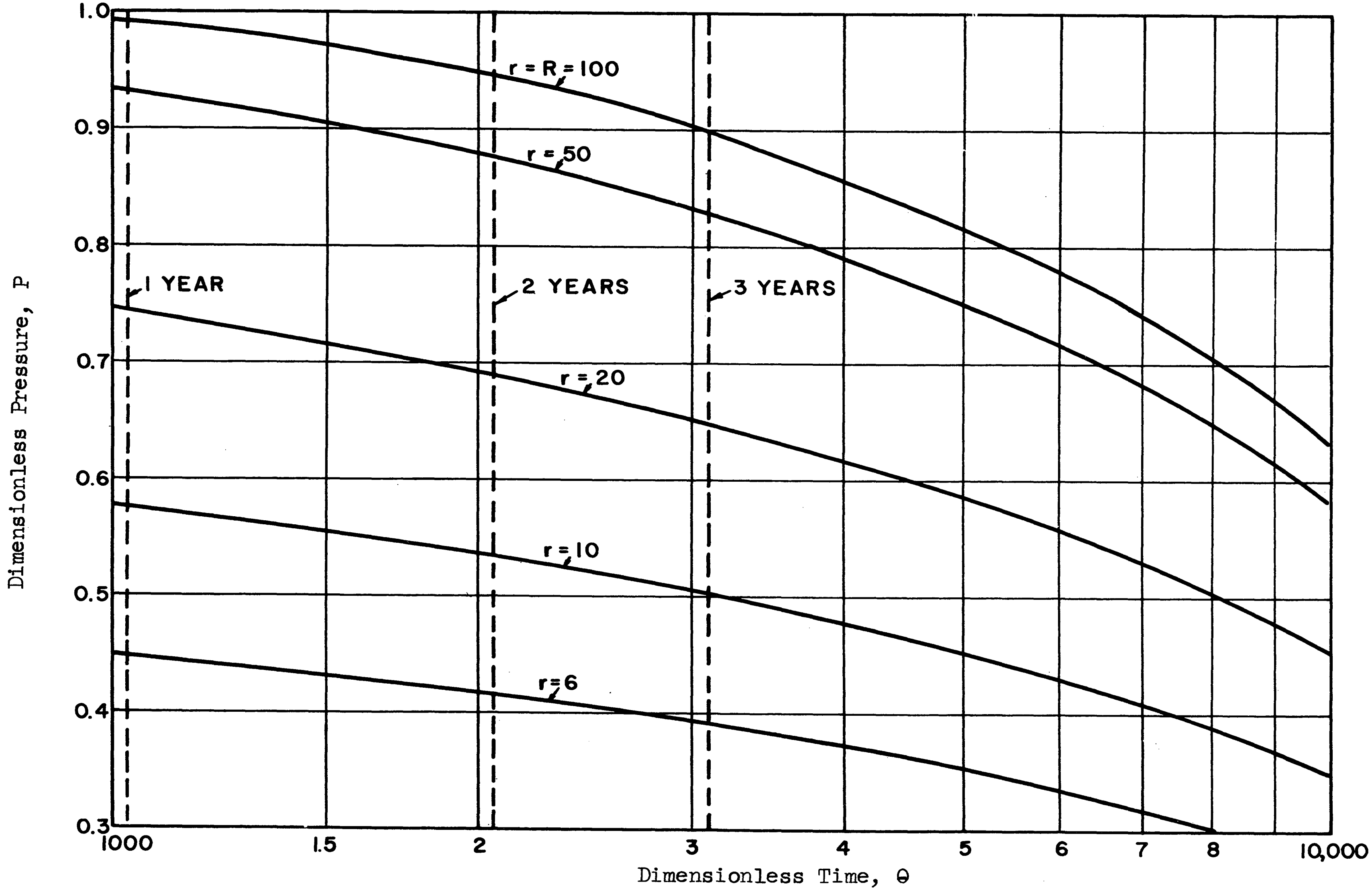


Figure IV-6. Dimensionless Pressure Distribution, $P(r,\Theta)$ for $R = 100$, for Example Problem 3.

Table IV-2

Pressure Distribution for Example Problem 3

r	r_a	r_a	At 1 year		At 2 years		At 3 years	
dimensionless	feet	miles	P dimensionless	$P-P_o$ psi	P dimensionless	$P-P_o$ psi	P dimensionless	$P-P_o$ psi
6	12,000	2.27	.448	55.2	.415	58.5	.391	60.9
10	20,000	3.79	.575	42.5	.535	46.5	.502	49.8
20	40,000	7.58	.744	25.6	.690	31.0	.648	35.2
50	100,000	18.95	.932	6.8	.876	12.4	.829	17.1
100	200,000	37.9	.991	0.9	.947	5.3	.900	10.0

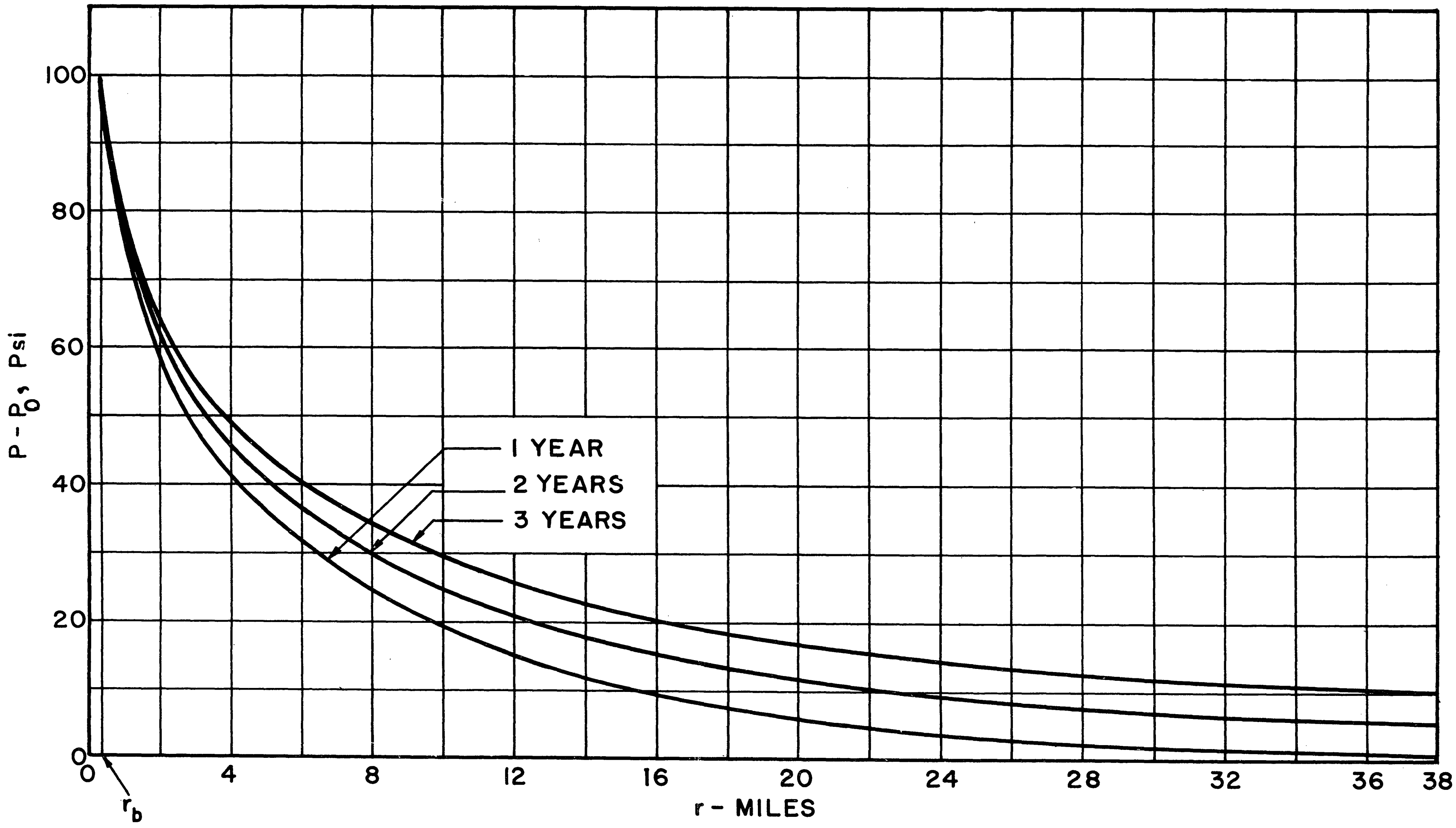


Figure IV-7. Aquifer Pressure Profile for Example Problem 3.

The relation between actual and dimensionless time is the same as that used in calculating the pressure profile. The water efflux, W_e, in this case denotes the increase in pore volume of the reservoir, in cubic feet. This change in volume is shown in Table IV-3, at times of one, two and three years. The relation between $Q_{(t)}$ and W_e used in this case was

$$W_e = 6.283(0.1)(7\text{x}10^{-6})(2000)^2(100)(100)Q_{(t)}$$

$$= 1.76 \times 10^5 \ Q_{(t)} \quad \text{cubic feet} \qquad \text{(IV-164)}$$

Table IV-3

Change In Reservoir Pore Volume for Example Problem 3

Θ dimensionless	t_a years	$Q_{(t)}$ dimensionless	$W_e = \Delta V$ cubic feet
1030	1	300.1	$5.28\text{x}10^7$
2060	2	546.0	$9.62\text{x}10^7$
3090	3	776.0	$13.66\text{x}10^7$

It is of interest to note that the original reservoir pore volume was

$$V_i = \pi \phi r_b^2 h = \pi(0.1)(2000)^2 (100)$$

$$= 12.6 \times 10^7 \text{ cubic feet} \qquad \text{(IV-165)}$$

After three years, the reservoir would have more than doubled in volume. The reservoir radius after three years would be

$$r_b = \sqrt{\frac{V}{\pi \phi h}} = \sqrt{\frac{(12.6 + 13.66) \times 10^7}{\pi(0.1)(100)}}$$

$$= 2890 \text{ feet} \qquad \text{(IV-165)}$$

This much change in the reservoir radius (from 2000 to 2890 feet) is somewhat incompatible with the assumption of a constant radius. The results obtained should, however, be reasonable approximations. The exact solution to the problem, considering the effect of the moving boundary, is considerably more complicated and is beyond the scope of this example problem.

4. Example Problem 4

Consider an aquifer storage reservoir with the following dimensions and physical properties

P_o = 700 psia

h = 80 feet

r_b = 1000 feet

k = 400 millidarcys

c = 7×10^{-6} vol/vol psi

ϕ = 0.17

μ = 1 centipoise

r_e = 100,000 feet

It is desired to grow this storage reservoir at a constant rate of 1.5 MMcf pore volume per month. The required reservoir pressure and the aquifer pressure profile are to be calculated for 1, 2, 3, 4, and 6 months after initiation of the growth.

The tables in Appendix H will be used for these calculations. The applicable relation between real and dimensionless time for these tables is

$$\Theta = \frac{kt_a}{\mu\phi c r_b^2} \qquad \text{(IV-166)}$$

For this case, at one month

$$\Theta = \frac{(0.4)(30)(24)(3600)}{(1)(0.17)(7 \times 10^{-6} \times 14.7)(1000 \times 30.48)^2}$$

$$= 64 \qquad \text{(IV-167)}$$

From Appendix H, the dimensionless time, Θ, at which the effect of the exterior boundary at R = 100 first becomes significant is approximately

Θ = 500. Moreover, values of $P(r,\Theta)$ are not listed in the table for R = 100 for values of Θ less than 100. This difficulty may be alleviated by using the value of $P(r,\Theta)$ from the table for R = 50 at the time Θ = 64. Since at this time the effect of the exterior boundary at R = 50 has no significant effect either, the result should be exact. This technique will be covered in more detail in Example Problem 5.

To facilitate the interpolation with respect to Θ from the tables in Appendix H, it is convenient to plot $P(r,\Theta)$ for the Θ range of interest. This plot is shown in Figure IV-8. The pertinent values of $P(r,\Theta)$ are then shown in Table IV-4.

Table IV-4

Dimensionless Pressure Distribution, $P(r,\Theta)$, for Example Problem 4

t_a months	Θ dimensionless	Dimensionless Pressure, $P(r,\Theta)$ r=1	r=3	r=6	r=20	r=50	r=100
1	64	2.51	1.41	0.77	0.05	0.00	0.00
2	128	2.85	1.75	1.07	0.16	0.00	0.00
3	192	3.05	1.95	1.26	0.27	0.00	0.00
4	256	3.19	2.09	1.40	0.37	0.01	0.00
6	384	3.39	2.28	1.60	0.51	0.04	0.00

By equating the rate of growth of the pore volume to the rate of efflux of water into the aquifer

$$q = \frac{1.5 \times 10^6 \ \ ft^3/month}{30 \ \ days/month} = 50{,}000 \ \ ft^3/day \qquad \text{(IV-168)}$$

One form of the equation for transforming the dimensionless $P(r,\Theta)$ to actual pressures is[26] [units as for Equation (IV-163]:

$$P = P_o + \frac{25.2\ q\ \mu}{kh} P(r,\Theta) \qquad \text{(IV-169)}$$

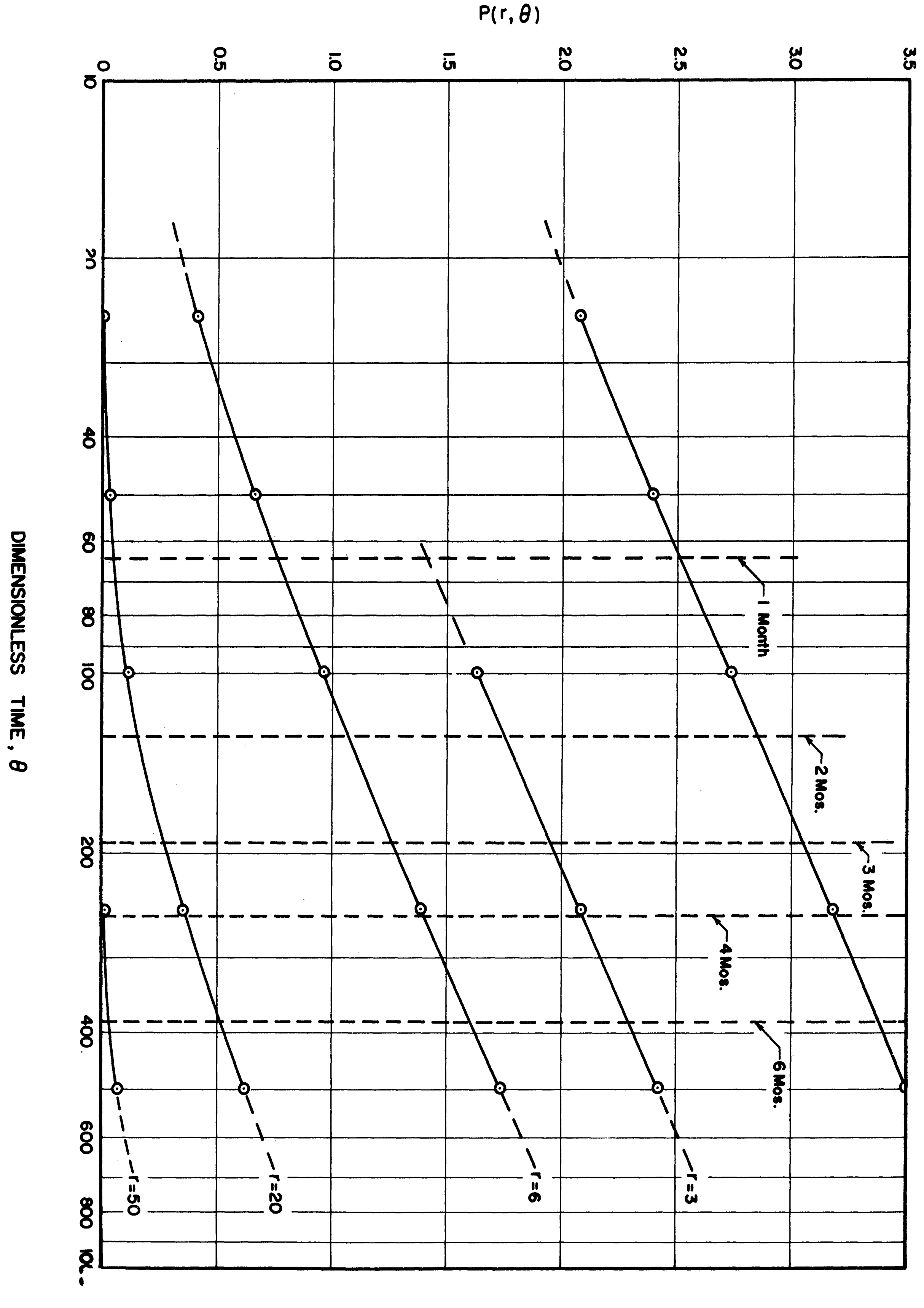


Figure IV-8. Dimensionless Pressure Distribution for Example Problem 4.

In this problem

$$P = 700 + \frac{(25.2)(50,000)(1)}{(400)(80)} \quad P(r,\theta)$$

$$= 700 + 39.4\ P(r,\theta) \qquad \text{(IV-170)}$$

Applying Equation (IV-170) to the dimensionless $P(r,\theta)$ in Table IV-4, the actual pressure distribution, $P(r_a,\ t_a)$ is that shown in Table IV-5. The pressure at r_a = 1000 feet is the required reservoir pressure.

Table IV-5

Actual Pressure Distribution, $P(r_a,\ t_a)$, for Example Problem 4

t_a months	Pressure, $P(r_a,\ t_a)$, psia: $r_a = r_b =$ 1,000 ft	$r_a =$ 3,000 ft	$r_a =$ 6,000 ft	$r_a =$ 20,000 ft	$r_a =$ 50,000 ft	$r_a =$ 100,000 ft
1	798.9	755.5	730.3	702.0	700.0	700.0
2	812.1	768.9	742.1	706.3	700.0	700.0
3	820.1	776.8	749.6	710.6	700.0	700.0
4	825.9	782.4	755.1	714.6	700.4	700.0
6	833.6	789.8	763.0	720.1	701.6	700.0

It should be noted that this analysis again assumed the gas-water interface to be essentially stationary throughout the time period of interest.

5. Example Problem 5

This problem is intended to illustrate the means by which the tables, such as those in Appendix G, for finite or bounded systems may be used in solving problems where the extent of an aquifer is assumed to be infinite. The technique to be illustrated is also applicable to determination of factors such as $P(r,\theta)$ and $Q_{(t)}$ for those low values of time at which the known exterior boundary has no significant effect, and for which values do not appear in the table with the particular R value desired.

An "infinite" aquifer is usually defined as one in which no significant effect of an exterior boundary can be observed. The behavior of any bounded system at time values sufficiently low that the effect of its exterior boundary is negligible thus is exactly the same as that for the "infinite" system. Thus, for example, all values in Appendix G at times, Θ, for which $P(R,\Theta)$ is approximately 1.0 can be used for the calculation of the behavior of infinite systems.

Consider a circular reservoir which is surrounded on all sides by an aquifer of essentially infinite extent. The known dimensions and physical parameters are:

P_o = 1100 psia

r_b = 1000 feet, radius of reservoir

μ = 1 centipoise water viscosity

ϕ = 0.1 porosity

c = 7×10^{-6} vol/vol psi water plus formation compressibility

k = 1000 millidarcy permeability

If this reservoir is maintained at a pressure of 1000 psia, it is desired to find the pressure profile in the aquifer as a function of time.

$$\Theta = \frac{kt_a}{\mu\phi c r_b^2}$$

$$= \frac{(1)t_a}{(1)(0.1)(7 \times 10^{-6} \times 14.7)(1000 \times 30.48)^2}$$

$$= 1.046 \times 10^{-4}\ t_a \qquad \text{for } t_a \text{ in seconds}$$

$$= 1.046 \times 10^{-4} \times 3600 \times 24\ t_a = 9.03\,t_a \text{ for } t_a \text{ in days} \qquad \text{(IV-171)}$$

Values of $P(r,\Theta)$ will be obtained from the tables in Appendix G. The conversion from dimensionless $P(r,\Theta)$ to real $P(r_a, t_a)$ is then

$$P(r_a, t_a) = 1000 + (1100 - 1000)\ P(r,\Theta)$$
$$= 1000 + 100\ P(r,\Theta)\ \text{psia}$$
$$x_a = 1000\ x,\ \text{feet} \qquad \text{(IV-172)}$$

The values of $P(r,\Theta)$ and corresponding $P(r_a, t_a)$ are shown in Table IV-6. The column "R Table" in Table IV-6 indicates the value of R in Appendix G from which the values of $P(r,\Theta)$ were obtained at that value of Θ or t_a. The only limit on this method is the range of R values for which the factors desired are available. In this case, the maximum value of Θ which may be used is $\Theta = 10^5$.

Table IV-6

Pressure Distribution for Example Problem 5

θ dimensionless	t_a days	r = 3 P(r,θ)	r_a = 3000 ft psia	r = 4 P(r,θ)	r_a = 4000 ft psia	r = 10 P(r,θ)	r_a = 10,000 ft psia
3	0.332	.7435	1074.4	.8814	1088.1	1.	1100.
10	1.109	-	-	.7163	1071.6	.9843	1098.4
30	3.32	-	-	.5851	1058.5	.9059	1090.6
100	11.09	-	-	.4780	1047.8	.7782	1077.8
300	33.2	-	-	.4063	1040.6	.6710	1067.1
1000	110.9	-	-	.3478	1034.8	.5769	1057.7
3000	332.	-	-	-	-	.5096	1051.0
10000	1109.	-	-	-	-	.4511	1045.1
30000	3320.	-	-	-	-	.4080	1040.8
100000	11090.	-	-	-	-	.3692	1036.9

θ dimensionless	r = 20 P(r,θ)	r_a = 20,000 ft psia	r = 50 P(r,θ)	r_a = 50,000 ft psia	r = 100 P(r,θ)	r_a = 100,000 ft psia	R Table
3	1.	1100.	1.	1100.	1.	1100.	10,12
10	1.	1100.	1.	1100.	1.	1100.	20
30	.9963	1099.6	1.	1100.	1.	1100.	75
100	.9468	1094.7	1.	1100.	1.	1100.	75
300	.8536	1085.4	.9908	1099.1	1.	1100.	100
1000	.7459	1074.6	.9343	1093.4	.9957	1099.6	100,200
3000	.6617	1066.2	.8524	1085.2	.9611	1096.1	200
10000	.5866	1058.7	.7631	1076.3	.8864	1088.6	500
30000	.5308	1053.1	.6923	1069.2	.8116	1081.2	1000
100000	.4804	1048.0	.6271	1062.7	.7374	1073.7	1000

V. MULTI-LAYER FLOW; MATHEMATICAL ANALYSIS

The principal purpose of the research which was the basis of this dissertation was the investigation of flow in stratified systems, i.e., multi-layer flow. This section presents the mathematical analysis which was done on this problem.

All of the mathematical analysis in this section is based on the case of constant terminal pressure. The majority of the section is devoted to presenting analytical solutions for the pressure distribution and the fluxes across the producing face in two-layer systems, both linear and radial. Tables of results for these quantities, in dimensionless form, are presented in Appendices J through O. Example problems demonstrating the use of these tables are included in this section. Part of the section will also cover the extension of the solutions derived for two-layer systems to three, four, and more layers.

A. Two-Layer Linear Flow; Constant Terminal Pressure

The mathematical model used for two-layer linear flow is shown in Figure V-1. It is postulated in the following derivations that each of the two layers of this model is in itself homogeneous and isotropic. A perfect bond, i.e. no resistance to flow, is postulated at the interface between layers. Considered as a liquid flow system, the additional requirements are the assumptions used in deriving the applicable diffusivity equation, namely:

1. Negligible capillary or gravity effects.
2. Single phase flow of an isothermal slightly compressible liquid.
3. Darcy's law applies.

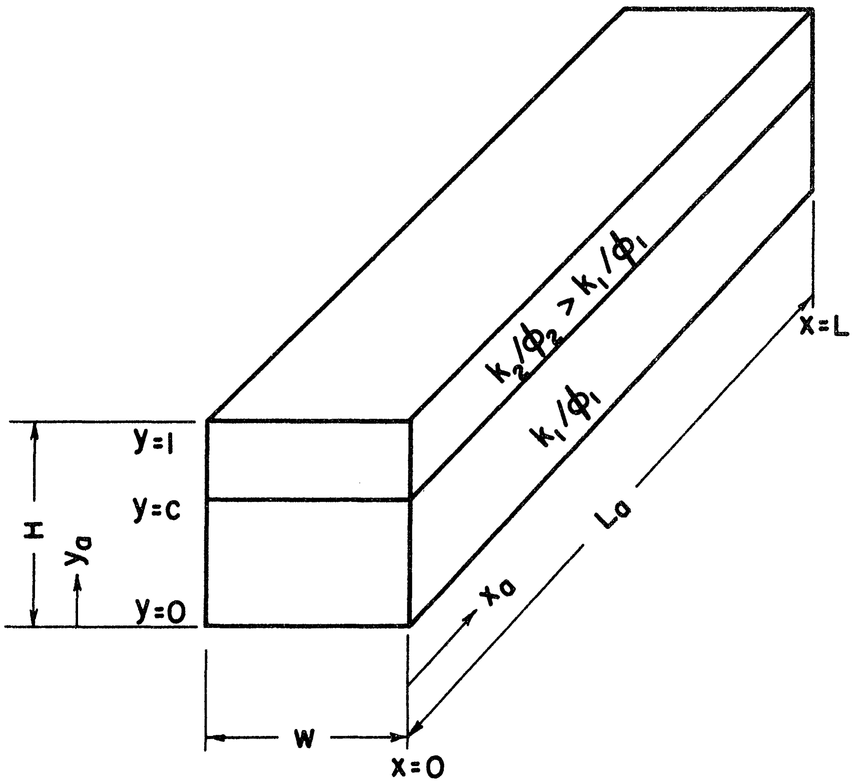


Figure V-1. Mathematical Model for Two-Layer Linear Flow.

Considered as a heat transfer model, it is postulated that the thermal diffusivity of the conducting solid is independent of temperature.

1. Pressure Distribution, $P(x,y,\theta)$

The equation governing the flow in this case is the diffusivity equation as expressed in cartesian coordinates, as derived with k/μ being a function of position within the flow model. This equation was derived in Chapter III of this dissertation, and appears there as Equation (III-24).

$$\frac{\partial}{\partial x_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial x_a}\right) + \frac{\partial}{\partial y_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial y_a}\right) + \frac{\partial}{\partial z_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial z_a}\right) = \phi c \frac{\partial P}{\partial t_a} \quad \text{(V-1)}$$

Referring to Figure V-1; there will be no flow in the z_a direction. Equation (V-1) thus reduces to

$$\frac{\partial}{\partial x_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial x_a}\right) + \frac{\partial}{\partial y_a}\left(\frac{k}{\mu}\frac{\partial P}{\partial y_a}\right) = \phi c \frac{\partial P}{\partial t_a} \quad \text{(V-2)}$$

Define dimensionless variables

$$x = \frac{x_a}{H} \quad \text{(V-3)}$$

$$y = \frac{y_a}{H} \quad \text{(V-4)}$$

$$L = \frac{L_a}{H} \quad \text{(V-5)}$$

Equation (V-2) may then be written

$$\frac{\partial}{\partial x}\left(\frac{k}{\mu}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k}{\mu}\frac{\partial P}{\partial y}\right) = \phi c H^2 \frac{\partial P}{\partial t_a} \quad \text{(V-6)}$$

or

$$\left(\frac{k}{\mu}\right)\frac{\partial^2 P}{\partial x^2} + \frac{\partial P}{\partial x}\frac{\partial}{\partial x}\left(\frac{k}{\mu}\right) + \left(\frac{k}{\mu}\right)\frac{\partial^2 P}{\partial y^2} + \frac{\partial P}{\partial y}\frac{\partial}{\partial y}\left(\frac{k}{\mu}\right) = \phi c H^2 \frac{\partial P}{\partial t_a} \quad \text{(V-7)}$$

Define

$$k = k_1 \qquad 0 \leq y < c$$
$$\;\; = k_2 \qquad 0 < y \leq 1 \qquad \text{(V-8)}$$

Then

$$\frac{\partial k}{\partial x} = 0 \qquad 0 \leq x \leq L \qquad \text{(V-9)}$$

and

$$\frac{\partial k}{\partial y} = 0 \qquad 0 \leq y < c \quad \text{and} \quad c < y \leq 1 \quad . \qquad \text{(V-10)}$$

Since the model is postulated to have the same fluid in both layers, viscosity μ and compressibility c are the same in both layers. Equation (V-7) may then be written

$$\frac{k}{\mu \phi c H^2} \frac{\partial^2 P}{\partial x^2} + \frac{k}{\mu \phi c H^2} \frac{\partial^2 P}{\partial y^2} = \frac{\partial P}{\partial t_a}$$

for all x, $\quad 0 \leq y < c \quad$ and $\quad c < y \leq 1.$ (V-11)

Define

$$\phi = \phi_1 \qquad 0 \leq y < c$$
$$\;\; = \phi_2 \qquad c < y \leq 1 \qquad \text{(V-12)}$$

$$K = 1 \qquad 0 \leq y < c$$
$$\;\; = K_2^* = \frac{k_2 \phi_1}{k_1 \phi_2} \qquad c < y \leq 1 \qquad \text{(V-13)}$$

$$K_2 = \frac{k_2}{k_1} \qquad \text{(V-14)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c H^2} \qquad \text{(V-15)}$$

Θ is thus the dimensionless time in this case. The governing flow equation now reduces to

$$K \frac{\partial^2 P}{\partial x^2} + K \frac{\partial^2 P}{\partial y^2} = \frac{\partial P}{\partial \Theta} \qquad \text{(V-16)}$$

The applicable initial and boundary conditions for the case of constant terminal pressure in this model are

Initial Condition: $P(x,y,0) = 1$ (V-17)

Boundary Conditions: $\frac{\partial P}{\partial y}(x,0,\Theta) = 0$ (V-18)

$$\frac{\partial P}{\partial y}(x,1,\Theta) = 0 \qquad \text{(V-19)}$$

$$P(0,y,\Theta) = 0 \qquad \text{(V-20)}$$

$$\frac{\partial P}{\partial x}(L,y,\Theta) = 0 \qquad \text{(V-21)}$$

Equations (V-17) and (V-20) introduce the step function pressure change at the face $x = 0$. Equations (V-18) and (V-19) characterize the fact of no flow across the $y = 0$ and $y = 1$ faces, and Equation (V-21) does likewise for the end face $x = L$.

Two equations apply at the interface $y = c$, denoting the continuity of both pressure and flux at this interface.

$$P(c-) = P(c+) \qquad \text{(V-22)}$$

$$\frac{\partial P}{\partial y}(c-) = K_2 \frac{\partial P}{\partial y}(c+) \qquad \text{(V-23)}$$

A solution will be obtained using the technique of separation of variables. It is first desired to find all solutions of the form

$$P = X(x)\, Y(y)\, T(\Theta) \qquad \text{(V-24)}$$

which satisfy Equation (V-16) and the boundary and interface conditions (V-18) through (V-23).

$$K\frac{X''}{X} + K\frac{Y''}{Y} = \frac{T'}{T} = \alpha \tag{V-25}$$

$$-\frac{X''}{X} = \frac{Y''}{Y} - \frac{\alpha}{K} = \lambda \tag{V-26}$$

It is of interest to note that in this case the order of separation of variables is important. It is impossible, for example, to separate variables by initially setting a function of y equal to a function of x and θ, due to the fact that K is a function of y.

The ordinary differential equations resulting from this separation of variables, together with their associated boundary conditions are

$$T' - \alpha T = 0 \tag{V-27}$$

$$\lim_{\theta \to \infty} T = 0 \tag{V-28}$$

$$X'' + \lambda X = 0 \tag{V-29}$$

$$X(o) = X'(L) = 0 \tag{V-30}$$

$$KY'' - [\alpha + \lambda K]Y = 0 \tag{V-31}$$

$$Y'(o) = Y'(1) = 0 \tag{V-32}$$

If $\alpha = 0$:

$$T = C_1 \text{ (constant)}$$

But from (V-28), this is impossible. Therefore, $\alpha \neq 0$.

If $\lambda = 0$:

$$X = C_2 + C_3 x$$

From (V-30) $\quad X(o) = 0 \quad$ Therefore $C_2 = 0$

$\quad X'(L) = 0 \quad$ Therefore $C_3 = 0$

Therefore, $\lambda \neq 0$.

For $\alpha \neq 0$:

$$T = C_1 e^{\alpha\theta}$$

In order to satisfy (V-28), α must be negative. Therefore, define

$$\alpha = -a^2 \qquad \text{(V-33)}$$

If λ is negative:

Let $\lambda = -g^2$

Then (V-29) becomes

$$X'' - g^2 X = 0$$

and

$$X = C_1 \sinh gx + C_2 \cosh gx$$

From (V-30)

$$X(o) = 0 \qquad \text{Therefore } C_2 = 0$$

$$X''(L) = 0 = C_1 g \cosh gL$$

But $g \neq 0$, or else $\lambda = 0$.

If $C_1 = 0$, the solution is trivial. Therefore it is required that

$$\cosh gL = 0 \quad .$$

But cosh gL is positive for all real g,L. There are therefore no real roots g to this equation. Therefore λ must be positive.

Define

$$\lambda = +b^2 \qquad \text{(V-34)}$$

Equations (IV-27) through (IV-32) then become

$$T' + a^2 T = 0 \qquad \text{(V-35)}$$

$$\lim_{\theta \to \infty} T = 0 \qquad \text{(V-36)}$$

$$X'' + b^2 X = 0 \qquad \text{(V-37)}$$

$$X(o) = X'(L) = 0 \qquad \text{(V-38)}$$

$$KY'' + [a^2 - b^2 K]\, Y = 0 \qquad \text{(V-39)}$$

$$Y'(o) = Y'(1) = 0 \qquad \text{(V-40)}$$

By inspection

$$T = C_5 e^{-a^2\theta} \qquad \text{(V-41a)}$$

and

$$X = C_6 \sin bx + C_7 \cos bx \qquad \text{(V-41b)}$$

From (V-38) X (o) = 0. Therefore $C_7 = 0$.

$$X'(L) = 0 = C_6 b \cos bL$$

Since neither C_6 nor b can be zero

$$\cos bL = 0 \qquad \text{(V-42)}$$

This characteristic equation, (V-42), defines an infinite number of real and positive eigenvalues b for each value of L.

$$b = \left(\frac{2m-1}{2}\right)\frac{\pi}{L} \; ; \quad m = 1,2,3,\ldots\ldots \qquad \text{(V-43)}$$

The solution to Equation (V-39) presents the most difficulty. Rewriting for convenience

$$KY'' + [a^2 - b^2K]\, Y = 0 \qquad \text{(IV-39)}$$

$$\begin{aligned} K &= 1 & 0 \leq y < c \\ &= K_2^* & c < y \leq 1 \end{aligned} \qquad \text{(IV-14)}$$

Since K is a constant <u>in</u> <u>each</u> <u>region</u>, it is possible to divide both sides of Equation (IV-39) by K, obtaining

$$Y'' + [a^2 - b^2]\, Y = 0 \qquad 0 \leq y < c \qquad \text{(V-44)}$$

$$Y'' + [\frac{a^2}{K_2^*} - b^2]\, Y = 0 \qquad c < y \leq 1 \qquad \text{(V-45)}$$

Define

$$\gamma = \sqrt{a^2 - b^2} \qquad \text{(V-46)}$$

$$\beta = \sqrt{\frac{a^2}{K_2^*} - b^2} \qquad \text{(V-47)}$$

Equations (IV-44) and (IV-45) are then

$$Y'' + \gamma^2 Y = 0 \qquad 0 \leq y < c \tag{V-48}$$

$$Y'' + \beta^2 Y = 0 \qquad c < y \leq 1 \tag{V-49}$$

Solving by inspection

$$Y = C_1 \sin \gamma y + C_2 \cos \gamma y \qquad 0 \leq y < c \tag{V-50}$$

$$Y = C_3 \sin \beta y + C_4 \cos \beta y \qquad c < y \leq 1 \tag{V-51}$$

From (V-40):

$$Y'(o) = 0 = C_1\gamma \cos \gamma y - C_2\gamma \sin \gamma y \Big|_{y = 0} \tag{V-52}$$

Therefore $C_1 = 0$

and

$$Y = C_2 \cos \gamma y \qquad 0 \leq y < c \tag{V-53}$$

Also from (V-40):

$$Y'(1) = 0 = C_3\beta \cos \beta - C_4\beta \sin \beta$$

or

$$C_3 = C_4 \frac{\sin \beta}{\cos \beta} \tag{V-54}$$

Substituting (V-54) into (V-51)

$$\begin{aligned} Y &= C_4 \left[\frac{\sin \beta}{\cos \beta} \sin \beta y + \cos \beta y\right] \\ &= \frac{C_4}{\cos \beta}\left[\sin \beta \sin \beta y + \cos \beta \cos \beta y\right] \\ &= \frac{C_4}{\cos \beta} \cos \beta(1-y) \qquad c < y \leq 1 \end{aligned} \tag{V-55}$$

The interface conditions (V-22) and (V-23) may be written

$$Y(c-) = Y(c+) \tag{V-56}$$

$$Y(c-) = K_2 Y'(c+) \tag{V-57}$$

Substituting

$$C_2 \cos \gamma c = \left[\frac{C_4}{\cos \beta}\right] \cos \beta\,(1-c) \tag{V-58}$$

$$-C_2\gamma \sin \gamma c = \left[\frac{C_4}{\cos \beta}\right] \beta K_2 \sin \beta(1-c) \qquad \text{(V-59)}$$

In order that there be a non-trivial solution, the determinant of the coefficients must equal zero, i.e.

$$\beta K_2 \sin \beta(1-c) \cos \gamma c + \gamma \sin \gamma c \cos \beta(1-c) = 0 \qquad \text{(V-60)}$$

This characteristic equation implicitly defines an infinite number of eigenvalues, a, for <u>each</u> value of b. This can perhaps be more clearly seen by substituting the definitions for γ and β from (V-46) and (V-47) into (V-60).

$$K_2 \sqrt{\frac{a^2}{K_2} - b^2} \sin\left[(1-c)\sqrt{\frac{a^2}{K_2} - b^2}\right] \cos\left[c\sqrt{a^2 - b^2}\right]$$
$$+ \sqrt{a^2 - b^2} \sin\left[c\sqrt{a^2 - b^2}\right] \cos\left[(1-c)\sqrt{\frac{a^2}{K_2} - b^2}\right] = 0 \qquad \text{(V-61)}$$

From (V-58)

$$\frac{C_4}{\cos \beta} = C_2 \frac{\cos \gamma c}{\cos \beta(1-c)}$$

Then the solution for Y may be written

$$Y = C_2 \cos \gamma y \qquad 0 \leq y < c \qquad \text{(V-53)}$$

$$Y = C_2 \frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y) \qquad c < y \leq 1 \qquad \text{(V-62)}$$

Substituting (V-41a), (V-41b), (V-53), and (V-62) into (V-24), and combining the values of the arbitrary constants C_2, C_5, and C_6,

$$P = C \sin bx\, e^{-a^2\Theta} \cos \gamma y \qquad 0 \leq y < c$$
$$= C \sin bx\, e^{-a^2\Theta} \left[\frac{\cos \gamma c}{\cos \beta(1-c)}\right] \cos \beta(1-y) \qquad c < y \leq 1 \qquad \text{(V-63)}$$

This solution, (V-63), satisfies the flow Equation (V-16), the boundary conditions (V-18) through (V-21), and the interface conditions (V-22) and

(V-23). There is, however, no single value of the constant C which will satisfy the initial condition

$$P(x,y,0) = 1 \tag{V-17}$$

for all values of x and y. Due to the linearity of (V-16), any sum of terms of the type (V-63) will also be a solution. Since two parameters, x and y, are involved, this leads to the representation of the initial condition by a double infinite series, analogous to the single infinite series obtained for single-layer flow.

The necessary orthogonality conditions for this case are

$$\int_0^L X(b_m,x)_{m=\ell}\; X(b_m,x)_{m=r}\, dx = 0 \quad \text{for} \quad \ell \neq r \tag{V-64}$$

and

$$\int_0^1 Y(m,a_{m,n},y)_{n=\ell}\; Y(m,a_{m,n},y)_{n=r}\, dy = 0 \quad \text{for } \ell \neq r, \text{ any } m. \tag{V-65}$$

The orthogonality condition, (V-64), is a well known case, and will thus be stated as such without proof. The condition (V-65), however, is not self evident, and is not a well known case. The derivation of (V-65) has therefore been included in this dissertation, as Appendix A. The reader who is interested in some general theory of differential equations with discontinuous coefficients, together with associated orthogonality properties, is referred to the work of Sangren.[34]

Proceding analogously to the previously shown single-layer case:

$$1 = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} CY\right]_m \sin bx \tag{V-66}$$

$$\left[\sum_{n=1}^{\infty} CY\right] = \frac{\int_0^L \sin bx\, dx}{\int_0^L \sin^2 bx\, dx} = \frac{-\frac{1}{b}[\cos bL - 1]}{\frac{1}{2b}[bL - \cos bL \sin bL]} \qquad \text{(V-67)}$$

From (V-42), $\cos bL = 0$.

Thus

$$\left[\sum_{n=1}^{\infty} CY\right] = \frac{2}{bL} \qquad \text{(V-68)}$$

Since (2/bL) is independent of n, Equation (V-66) may now be written as

$$1 = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} C_{mn}\, Y\right]_m \left(\frac{2}{bL}\right) \sin bx \qquad \text{(V-69)}$$

where:

$$\sum_{n=1}^{\infty} C_{mn}\, Y = 1 \qquad \text{(V-70)}$$

Then

$$C_{mn} = \frac{\int_0^1 Y dy}{\int_0^1 Y^2 dy} = \frac{\int_0^c \cos \gamma y\, dy + \int_c^1 \frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y)\, dy}{\int_0^c \cos^2 \gamma y\, dy + \int_c^1 \frac{\cos^2 \gamma c}{\cos^2 \beta(1-c)} \cos^2 \beta(1-y)\, dy} \qquad \text{(V-71)}$$

Evaluating the integrals

$$\int_0^c \cos \gamma y\, dy = \frac{1}{\gamma} \sin \gamma c \qquad \text{(V-72)}$$

$$\int_c^1 \frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y)\, dy = \frac{\cos \gamma c}{\cos \beta(1-c)} \left[-\frac{1}{\beta} \sin \beta(1-y)\right]_c^1$$

$$= \frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \qquad \text{(V-73)}$$

$$\int_0^c \cos^2 \gamma y \, dy = \frac{1}{4\gamma} [2\gamma c + \sin 2\gamma c] \qquad \text{(V-74)}$$

$$\int_c^1 \frac{\cos^2 \gamma c}{\cos^2 \beta(1-c)} \cos^2 \beta(1-y) \, dy$$

$$= \frac{\cos^2 \gamma c}{4\beta \cos^2 \beta(1-c)} [2\beta(1-c) + \sin 2\beta(1-c)] \qquad \text{(V-75)}$$

Substituting into (V-71)

$$C_{mn} = \frac{\left[\frac{\sin \gamma c}{\gamma}\right] + \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)}\right]}{\left[\frac{2\gamma c + \sin 2\gamma c}{4\gamma}\right] + \left\{\frac{\cos^2 \gamma c \, [2\beta(1-c) + \sin 2\beta(1-c)]}{4\beta \cos^2 \beta(1-c)}\right\}} \qquad \text{(V-76)}$$

Simplifying

$$C_{mn} = \frac{4 \cos \beta(1-c) \, [\beta \sin \gamma c \cos \beta(1-c) + \gamma \cos \gamma c \sin \beta(1-c)]}{\beta[2\gamma c + \sin 2\gamma c] \cos^2 \beta(1-c) + \gamma \cos^2 \gamma c [2\beta(1-c) + \sin 2\beta(1-c)]} \qquad \text{(V-77)}$$

Equation (V-77) for C_{mn} is perfectly valid, but is very highly sensitive to slight errors in the eigenvalues a and b. Working with a Bendix G-15 digital computer, using double precision arithmetic which amounted to working with about 12 decimal digits, the author found that in one case an error of one part in 10^7 in the eigenvalue a caused the value for C_{mn} to be incorrect by a factor of more than two. A much more stable equivalent equation for C_{mn} can be obtained for the case of $K_2 = K_2^*$ (i.e., $\phi_1 = \phi_2$).

Consider the term in (V-77)

$$F = \beta \sin \gamma c \cos \beta(1-c) + \gamma \cos \gamma c \sin \beta(1-c) \qquad \text{(V-78)}$$

From the characteristic equation, (V-60) or (V-61), defining the eigenvalues a:

$$0 = \gamma \sin \gamma c \cos \beta(1-c) + \beta K_2 \cos \gamma c \sin \beta(1-c) \qquad \text{(V-79)}$$

Rearranging

$$\frac{F}{\gamma} = \frac{\beta}{\gamma} \sin \gamma c \cos \beta(1-c) + \cos \gamma c \sin \beta(1-c) \qquad \text{(V-80)}$$

$$0 = \frac{\gamma}{\beta K_2} \sin \gamma c \cos \beta(1-c) + \cos \gamma c \sin \beta(1-c) \qquad \text{(V-81)}$$

Subtracting (V-81) from (V-80)

$$\frac{F}{\gamma} = \left[\frac{\beta}{\gamma} - \frac{\gamma}{\beta K_2}\right] \sin \gamma c \cos \beta(1-c) \qquad \text{(V-82)}$$

Now using the definitions for γ and β, from (V-46) and (V-47), letting $K_2 = K_2^*$

$$\left[\frac{\beta}{\gamma} - \frac{\gamma}{\beta K_2}\right] = \frac{\beta^2 K_2 - \gamma^2}{\gamma\beta K_2} = \frac{\left[\frac{a^2}{K_2} - b^2\right]K_2 - [a^2 - b^2]}{\gamma\beta K_2} = \frac{b^2(1-K_2)}{\gamma\beta K_2} \qquad \text{(V-83)}$$

Then, substituting back into (V-82),

$$F = \left[\frac{b^2(1-K_2)}{\beta K_2}\right] \sin \gamma c \cos \beta(1-c) \qquad \text{(V-84)}$$

Substituting (V-84) into (V-77)

$$C_{mn} = \frac{\left[\frac{4(1-K_2)}{K_2}\right]\left(\frac{b^2}{\beta}\right) \sin \gamma c \cos^2 \beta(1-c)}{\beta \cos^2 \beta(1-c)[2\gamma c + \sin 2\gamma c] + \gamma \cos^2 \gamma c[2\beta(1-c) + \sin 2\beta(1-c)]} \qquad \text{(V-85)}$$

Equation (V-85) is much less sensitive to slight errors in the eigenvalues a and b than is (V-77).

Summarizing, the solution is thus:

$$P(x,y,\theta) = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} C_{mn}\, e^{-a^2\theta}\, Y_{mn} \right]_m \left(\frac{2}{bL}\right) \sin bx \qquad \text{(V-86)}$$

where: C_{mn} is defined by (V-85)

$$Y_{mn} = \cos \gamma y \qquad 0 \leq y < c$$

$$= \frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y) \qquad c < y \leq 1 \qquad \text{(V-87)}$$

$$b = \left(\frac{2m-1}{2}\right) \frac{\pi}{L} ; \quad m = 1,2,3,\ldots\ldots \qquad \text{(V-43)}$$

$$\gamma \sin \gamma c \cos \beta(1-c) + \beta K_2 \sin \beta(1-c) \cos \gamma c = 0 \qquad \text{(V-60)}$$

$$\gamma = \sqrt{a^2 - b^2} \qquad \text{(V-46)}$$

$$\beta = \sqrt{\frac{a^2}{K_2^*} - b^2} \qquad \text{(V-47)}$$

One very important part of the derivation yet remains. It has been shown that all the eigenvalues b and a are real and positive. Moreover, by inspection it can be seen that the sign chosen for the square root in (V-46) and (V-47) defining γ and β is immaterial, since the cosine function is an even function. However, it has not yet been shown whether γ and β are real or imaginary. It can be seen that the characteristic equation, (V-60) or (V-61) has an infinite number of roots, a, for each value of b when both γ and β are real. The question thus becomes whether any eigenvalues are obtained as roots of (V-60) when γ, β, or both are imaginary.

If both γ and β are imaginary:

Define $$\gamma = i\,\epsilon_1 \qquad \text{(V-88)}$$

$$\beta = i\,\epsilon_2 \qquad \text{(V-89)}$$

The applicable relations between real and imaginary functions are:

$$\sin(ix) = i \sinh x \qquad \text{(V-90)}$$

$$\cos(ix) = \cosh x \qquad \text{(V-91)}$$

Substituting into (V-60)

$$\epsilon_1 \sinh \epsilon_1 c \cosh \epsilon_2(1-c) + \epsilon_2 K_2 \cosh \epsilon_1 c \sinh \epsilon_2(1-c) = 0 \qquad \text{(V-92)}$$

Now the terms $\cosh \epsilon_1 c$, $\cosh \epsilon_2(1-c)$, $\epsilon_2 \sinh \epsilon_2(1-c)$, and $\epsilon_1 \sinh \epsilon_1 c$ are always positive for all real values of ϵ_1, ϵ_2, and c. Moreover, K_2 is always positive. There are thus no real roots ϵ_1, ϵ_2 of Equation (V-92). It therefore follows that the case of both γ and β being imaginary contributes no eigenvalues a, and thus no terms to the series solution for $P(x,y,\Theta)$.

Now, for convenience, let the layers always be numbered such that

$$K_2^* = \frac{k_2 \phi_1}{k_1 \phi_2} > 1 \qquad \text{(V-93)}$$

Then in the range

$$b < a < b\sqrt{K_2} \qquad \text{(V-94)}$$

γ is real, but β is imaginary.

Define

$$\beta = i\omega \qquad \text{(V-95)}$$

Substituting into (V-60), using (V-90) and (V-91),

$$\gamma \sin \gamma c \cosh \omega(1-c) - \omega K_2 \sinh \omega(1-c) \cos \gamma c = 0 \qquad \text{(V-96)}$$

This characteristic equation, (V-96), does yield a finite number of roots, a, for each value of the eigenvalue b. The number of roots of (V-96) (which of course applies only over the range specified in (V-94)) is usually small for low values of m, and increases as m increases, due largely to the fact that the range specified by (V-94) increases as m increases for a given value of L. For example, it has been found that for the values

$$K_2 = K_2^* = 10$$
$$L = 10 \qquad \text{(V-97)}$$
$$c = 0.5$$

the number of eigenvalues which appear as roots of (V-96), in the range (V-94), is as follows for the first forty values of b_m.

m	Number of eigenvalues in the range (V-94)
1 - 7	1
8 - 13	2
14 - 20	3
21 - 27	4
28 - 33	5
34 - 40	6

It is, of course, necessary to make use of the relations (V-90) and (V-91) between imaginary and real functions when computing the values of C_{mn} and Y_{mn} from the eigenvalues found in the range (V-94). Equation (V-85) for C_{mn} then becomes

$$C_{mn} = \frac{\left[\frac{4(K_2-1)}{K_2}\right]\left(\frac{b^2}{\omega}\right) \sin \gamma c \cosh^2 \omega(1-c)}{\omega \cosh \omega(1-c)[2\gamma c + \sin 2\gamma c] + \gamma \cos^2 \gamma c[2\omega(1-c) + \sinh 2\omega(1-c)]} \qquad \text{(V-98)}$$

This may be written as

$$C_{mn} = \frac{\left[\frac{4(K_2-1)}{K_2}\right]\left(\frac{b^2}{\omega}\right) \sin \gamma c}{\omega[2\gamma c + \sin 2\gamma c] + 2\gamma \cos^2 \gamma c \left[\frac{\omega(1-c)}{\cosh^2 \omega(1-c)} + \tanh \omega(1-c)\right]} \qquad \text{(V-99)}$$

The form (V-99) is convenient for further simplification at high values of $\omega(1-c)$, at which point $\cosh^2\omega(1-c)$ becomes very large, and $\tanh\omega(1-c)$ approaches one.

The equations defining Y_{mn} in the range (V-94) become

$$Y_{mn} = \cos \gamma y \qquad 0 \leqq y < c$$

$$= \frac{\cos \gamma c}{\cosh \omega(1-c)} \cosh \omega(1-y) \qquad c < y \leqq 1 \qquad \text{(V-100)}$$

Values of dimensionless $P(x,y,\Theta)$ are presented in Appendix J. The use of these tables will be illustrated by an example problem later in this section.

2. Dimensionless Flux at the Producing Face

There are three types of dimensionless quantities useful in calculating the flux across the producing face ($x = x_a = 0$). Two of these, the individual instantaneous dimensionless fluxes, q_1 and q_2, and the individual cumulative dimensionless fluxes Q_1 and Q_2, may be derived directly from the solution for $P(x,y,\Theta)$ by appropriate differentiation and integration. The third, which is the total cumulative dimensionless flux from both layers taken as a whole, $Q_{(t)}$, is analogous to the dimensionless flux $Q_{(t)}$ for the case of single-layer flow. The expression for $Q_{(t)}$ may be most easily found by combining the solution for $P(x,y,\Theta)$ with a mass balance (or analogously, a heat balance) on the model.

a. Individual Instantaneous Dimensionless Flux

The dimensionless q_1 and q_2 will be derived considering the model to be a heat transfer model.

For either layer:

$$\frac{Btu}{hr} = kA\left[\left(\frac{\partial T_a}{\partial x_a}\right)_{x_a=0}\right]_{mean} = \frac{kA}{H}\left[\left(\frac{\partial T}{\partial x}\right)_{x=0}\right]_{mean} \Delta T \qquad \text{(V-101)}$$

$$\left[\left(\frac{\partial T}{\partial x}\right)_{x=0}\right]_{mean} = \frac{1}{c}\int_0^c \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \qquad 0 \leqq y < c$$

$$= \frac{1}{1-c}\int_c^1 \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \qquad c < y \leqq 1 \qquad \text{(V-102)}$$

Let the layer $0 < y < c$ be referred to as "layer 1", and the layer $c < y < 1$ be referred to as "layer 2". From the derivation of $P(x,y,\Theta)$, Equation (V-93), it is required that layer 2 have the higher thermal diffusivity.

Substituting (V-102) into (V-101)

$$\left(\frac{Btu}{hr}\right)_1 = \frac{k_1 cHw\Delta T}{cH}\int_0^c \left(\frac{\partial T}{\partial x}\right)_{x=0} dy = k_1 w\Delta T\int_0^c \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \qquad \text{(V-103)}$$

$$\left(\frac{Btu}{hr}\right)_2 = \frac{k_2(1-c)Hw\Delta T}{(1-c)H}\int_c^1 \left(\frac{\partial T}{\partial x}\right)_{x=0} dy = k_2 w\Delta T\int_c^1 \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \qquad \text{(V-104)}$$

The solution for $T(x,y,\Theta)$ is given by (V-86).

$$T(x,y,\Theta) = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} Y_{mn}\right]\left(\frac{2}{bL}\right) \sin bx \qquad \text{(V-86)}$$

The only changes in the liquid flow solution to convert it to a heat transfer solution are that for heat transfer

$$\Theta = \frac{k_1 t_a}{\rho_1 C_{p1} H^2} \qquad \text{(V-105)}$$

and

$$K_2^* = \frac{k_2 \rho_1 C_{p1}}{k_1 \rho_2 C_{p2}} \tag{V-106}$$

In both layers

$$\left(\frac{\partial T}{\partial x}\right)_{x=0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} Y_{mn} \left(\frac{2}{bL}\right) \left[\frac{\partial}{\partial x} (\sin bx)\Big|_{x=0}\right]$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} Y_{mn} \left(\frac{2}{bL}\right) \left[b \cos bx\Big|_{x=0}\right]$$

$$= \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} Y_{mn} \tag{V-107}$$

Define

$$q_1 = \int_0^c \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \tag{V-108}$$

$$q_2 = \int_c^1 \left(\frac{\partial T}{\partial x}\right)_{x=0} dy \tag{V-109}$$

Then

$$\left(\frac{Btu}{hr}\right)_1 = w\Delta T k_1 q_1 \tag{V-110}$$

and

$$\left(\frac{Btu}{hr}\right)_2 = w\Delta T k_2 q_2 \tag{V-111}$$

Now, evaluating integrals

$$\int_0^c Y_{mn} dy = \int_0^c \cos \gamma y \, dy = \frac{\sin \gamma c}{\gamma} \tag{V-112}$$

$$\int_c^1 Y_{mn} dy = \frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \tag{V-113}$$

The solutions are thus

$$q_1 = \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{\sin \gamma c}{\gamma} \right] \qquad \text{(V-114)}$$

$$q_2 = \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \right] \qquad \text{(V-115)}$$

It is, of course, necessary to use both real and imaginary values of β, as in computing $T(x,y,\Theta)$.

An analogous derivation of q_1 and q_2 for the flow of slightly compressible fluids leads to the equations

$$\left(\frac{ft^3}{hr}\right)_1 = 0.02635\left(\frac{k_1}{\mu}\right) w \Delta P q_1 \qquad \text{(V-110a)}$$

$$\left(\frac{ft^3}{hr}\right)_2 = 0.02635\left(\frac{k_2}{\mu}\right) w \Delta P q_2 \qquad \text{(V-110b)}$$

where:

k_1, k_2 = darcys

μ = centipoises

w = feet of width

ΔP = original pressure difference driving force, psia

The numerical constant in these equations results from the conversion of units to those designated above.

Tables of q_1 and q_2 are presented in Appendix L. An example calculation illustrating the use of these tables appears later in this section.

b. Individual Cumulative Dimensionless Flux

The individual cumulative dimensionless fluxes, Q_1 and Q_2, are obtained directly by an integration of q and q with respect to time.

$$\int_0^{\theta} e^{-a^2\theta} = \left[\frac{1-e^{-a^2\theta}}{a^2} \right] \tag{V-116}$$

Therefore

$$Q_1 = \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\frac{\sin \gamma c}{\gamma} \right] \left[\frac{1-e^{-a^2\theta}}{a^2} \right] \tag{V-117}$$

$$Q_2 = \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \right] \left[\frac{1-e^{-a^2\theta}}{a^2} \right] \tag{V-118}$$

Both Equations (V-117) for Q_1 and (V-118) for Q_2 may be separated into two series, one independent of time, the other time dependent. Unfortunately, the time independent series is, for both Q_1 and Q_2, very slow to converge. Unlike the case of $Q_{(t)}$ for single layer radial flow, where a simple algebraic quantity was found to be the mathematical equivalent of the time independent series, there is no great simplification possible in this ease. Since the preparation of tables of Q_1 and Q_2 would require the calculation of many more values of the eigenvalues, a, than are necessary for calculating $T(x,y,\theta)$ and $Q(t)$, it was not thought to be worthwhile to calculate such tables for this dissertation. It is believed that the availability of tables of q_1, q_2, and $Q_{(t)}$ should be sufficient for the great majority of practical problems. The expressions for Q_1 and Q_2 are included in this dissertation only for those readers who might wish to calculate these quantities for some particular values of the controlling parameters.

c. Total Cumulative Dimensionless Flux

The total cumulative dimensionless flux from both layers considered as a whole may be derived by combining a heat (or mass) balance with the expression previously derived for $T(x,y,\theta)$. This derivation will again treat the model as a heat transfer model.

The initial "heat-in-place" for the model, assuming unit width, is given by

$$Q_i = V_\rho C_p \Delta T_i \quad \text{Btu} \tag{V-119}$$

$$V_1 = H_c L_a = cH^2L \tag{V-120}$$

$$V_2 = H(1-c)L_a = (1-c)H^2L \tag{V-121}$$

Therefore, since a ΔT_i of 1.0 was used for this model

$$Q_i = H^2L[c\rho\, C_{p1} + (1-c)\rho_2 C_{p2}] \tag{V-122}$$

At any time, the heat-in-place, Q_{hip}, must be given by a volumetric integration of the temperature, i.e.

$$Q_{hip} = H^2L\{c\rho_1 C_{p1}[\Delta T_{mean}]_1 + (1-c)\rho_2 C_{p2}[\Delta T_{mean}]_2\}$$

$$= H^2L\{c\rho_1 C_{p1}\left[\frac{1}{Lc}\int_0^L\int_0^c T\,dy\,dx\right]$$

$$+ (1-c)\rho_2 C_{p2}\left[\frac{1}{L(1-c)}\int_c^L\int_c^1 T\,dy\,dx\right]\}$$

$$= H^2L\left[\frac{\rho_1 C_{p1}}{L}\int_0^L\int_0^c T\,dy\,dx + \frac{\rho_2 C_{p2}}{L}\int_0^L\int_c^1 T\,dy\,dx\right] \tag{V-123}$$

Define

$$\epsilon = \frac{\frac{\rho_1 C_{p1}}{L}\int_0^L\int_0^c T\,dy\,dx + \frac{\rho_2 C_{p2}}{L}\int_0^L\int_c^1 T\,dy\,dx}{c\rho_1 C_{p1} + (1-c)\rho_2 C_{p2}}$$

$$= \frac{\int_0^L\int_0^c T\,dy\,dx + \frac{\rho_2 C_{p2}}{\rho_1 C_{p1}}\int_0^L\int_c^1 T\,dy\,dx}{L\left[c + (1-c)\frac{\rho_2 C_{p2}}{\rho_1 C_{p1}}\right]} \tag{V-124}$$

Then ϵ represents the fraction of the heat energy still present at any time, and $(1-\epsilon)$ is the fraction of the total heat energy initially present which has been lost. Then, per unit width

$$Q = H^2L\,[c\rho_1C_{p1} + (1-c)\rho_2C_{p2}]\,(1-\epsilon)\Delta T \tag{V-125}$$

is the Btu of heat loss.

Define

$$Q_{(t)} = L(1-\epsilon) \tag{V-126}$$

Then

$$Q(Btu) = H^2w\Delta T\,[c\rho_1C_{p1} + (1-c)\rho_2C_{p2}]\,Q_{(t)} \tag{V-127}$$

where w is the width of the model, and ΔT is the original overall temperature difference driving force. Note that $0 < Q_{(t)} < L$.

Rewriting the solution for $T(x,y,\theta)$ for convenience

$$T = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} C_{mn}e^{-a^2\theta}Y_{mn}\right]_m \left(\frac{2}{bL}\right)\sin bx \tag{V-86}$$

where

$$Y_{mn} = \cos\gamma y \qquad 0 \leqq y < c$$

$$= \frac{\cos\gamma c}{\cos\beta(1-c)}\cos\beta(1-y) \quad c < y \leqq 1 \tag{V-87}$$

Evaluating integrals

$$\int_0^L \sin bx\,dx = -\frac{1}{b}\cos bx\,\Big|_0^L = \frac{1}{b} \tag{V-128}$$

$$\int_0^c Y_{mn}dy = \int_0^c \cos\gamma y\,dy = \frac{\sin\gamma c}{\gamma} \tag{V-129}$$

$$\int_c^1 Y_{mn}dy = \frac{\cos\gamma c}{\cos\beta(1-c)}\int_c^1 \cos\beta(1-y)dy = \frac{\cos\gamma c\,\sin\beta(1-c)}{\beta\cos\beta(1-c)} \tag{V-130}$$

Then

$$\frac{1}{L}\int_0^L\int_0^c T\,dy\,dx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-a^2\Theta}\,\frac{\sin\gamma c}{\gamma}\left(\frac{2}{b^2L^2}\right) \qquad \text{(V-131)}$$

and

$$\frac{\rho_2 C_{p2}}{L\rho_1 C_{p1}}\int_0^L\int_c^1 T\,dy\,dx = \frac{\rho_2 C_{p2}}{\rho_1 C_{p1}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-a^2\Theta}\left(\frac{2}{b^2L^2}\right)\frac{\cos\gamma c\,\sin\beta(1-c)}{\beta\cos\beta(1-c)} \qquad \text{(V-132)}$$

Therefore

$$\epsilon = \frac{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-a^2\Theta}\left(\frac{2}{b^2L^2}\right)\left[\frac{\sin\gamma c}{\gamma} + \frac{\rho_2 C_{p2}}{\rho_1 C_{p1}}\,\frac{\cos\gamma c\,\sin\beta(1-c)}{\beta\cos\beta(1-c)}\right]}{\left[c + (1-c)\,\frac{\rho_2 C_{p2}}{\rho_1 C_{p1}}\right]} \qquad \text{(V-133)}$$

and, as defined

$$Q_{(t)} = L(1-\epsilon) \qquad \text{(V-126)}$$

The corresponding derivation using a mass balance instead of a heat balance is similar in all respects and yields

$$\epsilon = \frac{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-a^2\Theta}\left(\frac{2}{b^2L^2}\right)\left[\frac{\sin\gamma c}{\gamma} + \frac{\phi_2}{\phi_1}\,\frac{\cos\gamma c\,\sin\beta(1-c)}{\beta\cos\beta(1-c)}\right]}{\left[c + (1-c)\,\frac{\phi_2}{\phi_1}\right]} \qquad \text{(V-134)}$$

and

$$Q_{(t)} = L(1-\epsilon) \qquad \text{(V-126)}$$

The use of the quantity $Q_{(t)}$ for computing the actual flux for the flow of slightly compressible fluids is given by

$$Q(ft^3) = H^2 w c_w \Delta P[c\phi_1 + (1-c)\,\phi_2]\,Q_{(t)} \qquad \text{(V-127a)}$$

where c_w is the formation plus water compressibility, and ΔP is the original overall pressure difference driving force.

Tables of $Q(t)$ are presented in Appendix N. All of these tables are based on the criterion that $\phi_1 = \phi_2$ or analogously $\rho_1 C_{p1} = \rho_2 C_{p2}$. The use of these tables will be illustrated by an example calculation later in this dissertation.

B. Two-Layer Radial Flow; Constant Terminal Pressure

The mathematical model used for two-layer radial flow is shown in Figure V-2. The same specifications with regard to homogeneity, isotropy, etc., apply to this model as applied to the two-layer linear model.

Since much of the mathematical analysis of this case is similar to that of either single layer radial flow or two-layer linear flow, reference will be made to these solutions rather than repeating the corresponding part of the derivations whenever possible.

1. Pressure Distribution, $P(r,y,\theta)$

The applicable flow equation in this case is

$$\frac{k}{\mu}\left[\frac{\partial^2 p}{\partial r_a^{\,2}} + \frac{1}{r_a}\frac{\partial p}{\partial r_a}\right] + \frac{\partial}{\partial y_a}\left[\frac{k}{\mu}\frac{\partial p}{\partial y_a}\right] \quad \phi c \frac{\partial p}{\partial t_a} \qquad \text{(III-}$$

Define dimensionless variables

$$r = \frac{r_a}{r_b} \qquad \text{(V-135)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(V-136)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-137)}$$

$$A = \frac{r_b}{H} \qquad \text{(V-138)}$$

$$K_2^* = \frac{k_2\phi_1}{k_1\phi_2}\ ; \qquad K_2 = \frac{k_1}{k_2} \qquad \text{(V-139)}$$

$$\Theta = \frac{k_1 t_a}{\mu\phi_1 c r_b^{\,2}} \qquad \text{(V-140)}$$

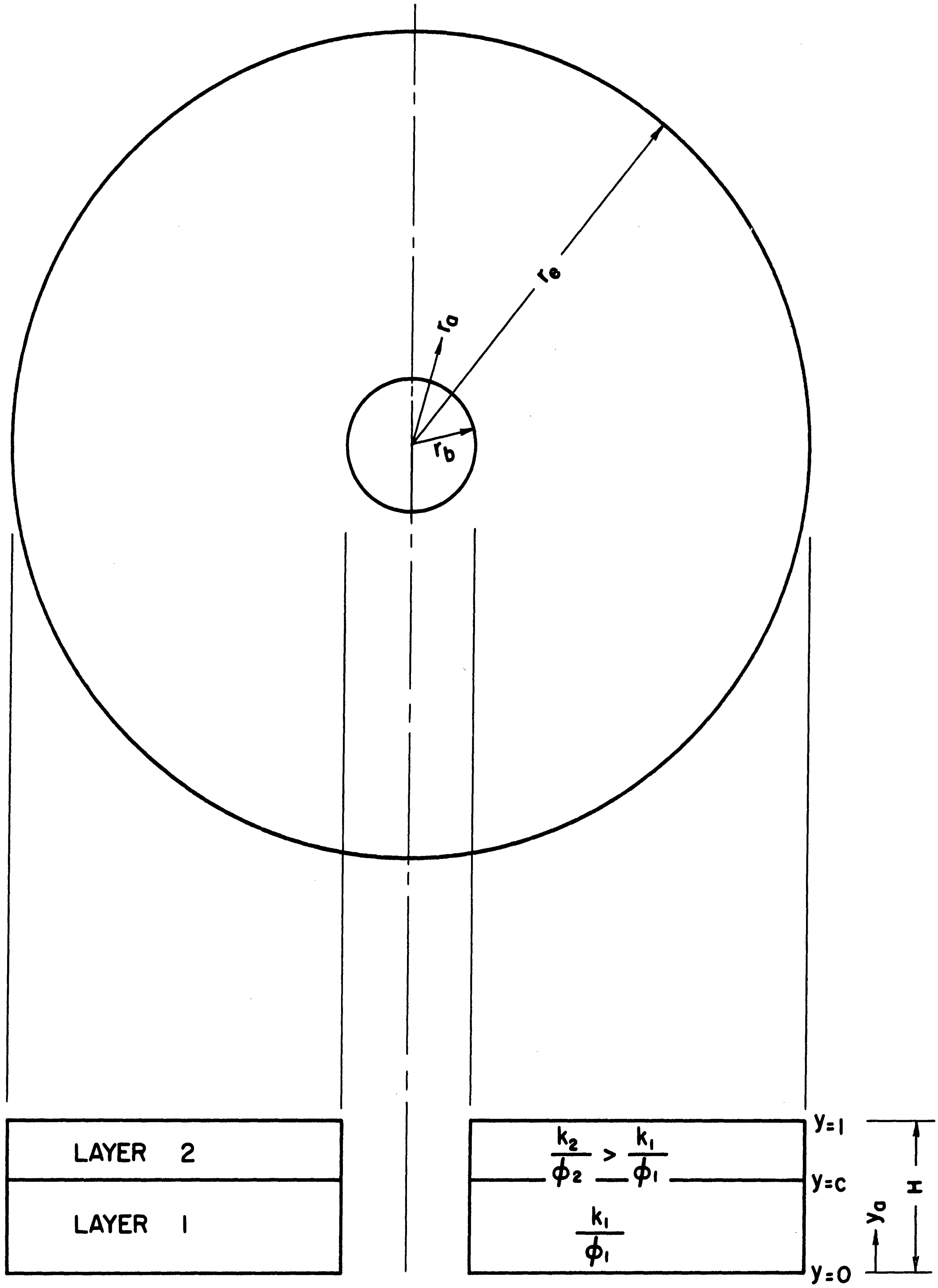


Figure V-2. Mathematical Model for Two-Layer Radial Flow.

The governing flow equation may then be written

$$K \left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right] + A^2 K \frac{\partial^2 p}{\partial y^2} = \frac{\partial p}{\partial \Theta} \qquad \text{(V-141)}$$

where:

$$K = 1 \qquad 0 \leqq y < c$$

$$= K_2^* \qquad c < y \leqq 1 \qquad \text{(V-14)}$$

The applicable initial, boundary, and interface conditions for the case of constant terminal pressure in this model are

Initial Condition: $P(r,y,0) = 1$ (V-142)

Boundary Conditions: $\frac{\partial p}{\partial y}(r,0,\Theta) = 0$ (V-143)

$\frac{\partial p}{\partial y}(r,1,\Theta) = 0$ (V-144)

$P(1,y,\Theta) = 0$ (V-145)

$\frac{\partial p}{\partial r}(R,y,\Theta) = 0$ (V-146)

Interface Conditions: $P(c-) = P(c+)$ (V-147)

$\frac{\partial p}{\partial y}(c-) = K_2 \frac{\partial p}{\partial y}(c+)$ (V-148)

The technique of separation of variables will be used to obtain a solution. It is first desired to find all solutions of the form

$$P = X(r)\ Y(y)\ T(\Theta) \qquad \text{(V-149)}$$

which satisfy Equation (V-141), and the boundary and interface conditions (V-143) through (V-148).

$$K \left[\frac{X''}{X} + \frac{1}{r}\frac{X'}{X}\right] + A^2 K \frac{Y''}{Y} = \frac{T'}{T} = \alpha \qquad \text{(V-150)}$$

$$-\left[\frac{X''}{X} + \frac{1}{r}\frac{X'}{X}\right] = A^2 \frac{Y''}{Y} - \frac{\alpha}{K} = \lambda \qquad \text{(V-151)}$$

The resultant ordinary differential equations, together with associated boundary conditions are thus

$$T' - \alpha T = 0 \quad \text{(V-152)}$$

$$\lim_{\theta \to \infty} T = 0 \quad \text{(V-153)}$$

$$X'' + \frac{1}{r} X' + \lambda X = 0 \quad \text{(V-154)}$$

$$X(1) = X'(R) = 0 \quad \text{(V-155)}$$

$$A^2 K Y'' - [\alpha + \lambda K] Y = 0 \quad \text{(V-156)}$$

$$Y'(0) = Y'(1) = 0 \quad \text{(V-157)}$$

As in the linear case, α must be negative.

Define

$$\alpha = -a^2 \quad \text{(V-33)}$$

If $\lambda = 0$:

$$X'' + \frac{1}{r} X' = 0$$

$$X = C_1 + C_2 \ln r$$

$$X' = \frac{C_2}{r}$$

$$X'(R) = \frac{C_2}{R} = 0$$

Therefore, $C_2 = 0$

$$X = C_1$$

$$X(1) = C_1 = 0$$

Therefore $\lambda \neq 0$

If λ is negative:

Let $\lambda = -g^2$

$$X'' + \frac{1}{r} X' - g^2 X = 0$$

$$r^2 X'' + r X - g^2 r^2 X = 0$$

This is one form of Bessels equation.

$$X = C_1 I_o(gr) + C_2 K_o(gr)$$

$$\frac{\partial X}{\partial r} = C_1 g I_1(gr) - C_2 g K_1(gr)$$

$$X(1) = 0 = C_1 I_o(g) + C_2 K_o(g) \qquad \text{(V-158)}$$

$$X'(R) = 0 = g[C_1 I_1(gR) - C_2 K_1(gR)] \qquad \text{(V-159)}$$

Since $\lambda \neq 0$, $g \neq 0$.

Therefore, for a non-trivial solution

$$I_o(g)K_1(gR) + K_o(g)I_1(gR) = 0 \qquad \text{(V-160)}$$

But for all real g, R, the values of all terms in this characteristic equation are always positive. There are thus no real roots, g, to Equation (V-160). Consequently, λ must be positive.

Define

$$\lambda = +b^2 \qquad \text{(V-161)}$$

Substituting (V-161) and (V-33) into (V-152) through (V-157), the ordinary differential equations become

$$T' + a^2 T = 0 \qquad \text{(V-162)}$$

$$\lim_{\Theta \to \infty} T = 0 \qquad \text{(V-163)}$$

$$X'' + \frac{1}{r} X' + b^2 X = 0 \qquad \text{(V-164)}$$

$$X(1) = X'(R) = 0 \qquad \text{(V-165)}$$

$$A^2 K Y'' + [a^2 - b^2 K]Y = 0 \qquad \text{(V-166)}$$

$$Y'(0) = Y'(1) = 0 \qquad \text{(V-167)}$$

By inspection

$$T = C_1 e^{-a^2\Theta} \qquad \text{(V-168)}$$

Equation (V-164) is one form of Bessels equation. It is identical to Equation (IV-57) for single layer radial flow, and has the same boundary conditions. From Equations (IV-57) through (IV-67),

$$X = C_4[J_o(br)Y_1(bR) - Y_o(br)J_1(bR)] \tag{V-169}$$

and

$$J_o(b)Y_1(bR) - Y_o(b)J_1(bR) = 0 \tag{V-170}$$

The characteristic Equation (V-170) is identical to Equation (IV-65). The eigenvalues, b, are thus the same for both single-layer and two-layer radial flow.

Since K is constant in both layer 1 and layer 2 individually, Equation (V-166) may be written

$$Y'' + \frac{1}{A^2}\left[\frac{a^2}{K} - b^2\right] Y = 0$$

$$0 \leqq y < c, \qquad c < y \leqq 1 \tag{V-171}$$

Define

$$\gamma = \frac{1}{A}\sqrt{a^2 - b^2} \tag{V-172}$$

$$\beta = \frac{1}{A}\sqrt{\frac{a^2}{K_2^*} - b^2} \tag{V-173}$$

Then

$$Y'' + \gamma^2 Y = 0 \qquad 0 \leqq y < c \tag{V-174}$$

$$Y'' + \beta^2 Y = 0 \qquad c < y \leqq 1 \tag{V-175}$$

Now Equations (V-174) and (V-175) are identical to Equations (V-48) and (V-49) for two-layer linear flow, and have the same boundary and interface conditions. Thus, from (V-48) through (V-62),

$$Y = C_5 \cos \gamma y \qquad 0 \leqq y < c$$

$$= C_5 \frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y) \qquad c < y \leqq 1 \tag{V-176}$$

and

$$\beta K_2 \sin \beta(1-c) \cos \gamma c + \gamma \sin \gamma c \cos \beta(1-c) = 0 \qquad \text{(V-177)}$$

Equations (V-177) and (V-60) are identical, yielding an identical set of eigenvalues, a, for a given b. However, since the values of b are not the same for two-layer linear and two-layer radial flow, the eigenvalues, a, are also different.

Now, as in the case of single-layer radial flow, define

$$U(br) = J_o(br)Y_1(bR) - Y_o(br)J_1(bR) \qquad \text{(V-178)}$$

Then, substituting into (V-149),

$$P = Ce^{-a^2\theta}U(br) \cos \gamma y \qquad 0 \leqq y < c$$

$$= Ce^{-a^2\theta}U(br) \left[\frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y) \right] \qquad c < y \leqq 1 \qquad \text{(V-179)}$$

Equation (V-179) satisfies Equations (V-141) and (V-143) through (V-148). Again, however, there is no single value of the constant C which satisfies the initial condition (V-142). As for two-layer linear flow, this initial condition may be represented by a double infinite series.

The necessary orthogonality conditions have been given previously as

$$\int_0^1 Y(m,a_{m,n},y)_{n=\ell} \; Y(m,a_{m,n},y)_{n=r} \, dy = 0$$

$$\text{for } \ell \neq r, \text{ any } m \qquad \text{(V-65)}$$

$$\int_1^R rU_m(br)U_n(br)dr = 0 \qquad \text{for } m \neq n \qquad \text{(IV-70)}$$

It should again be noted that Equation (V-65) is derived in Appendix A of this dissertation.

Proceding as before,

$$1 = \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} CY\right]_m U(br) \qquad (V\text{-}180)$$

$$\sum_{n=1}^{\infty} CY = \frac{\int_1^R rU(br)dr}{\int_1^R rU^2(br)dr} \qquad (V\text{-}181)$$

From (IV-71) through (IV-81)

$$\sum_{n=1}^{\infty} CY = \frac{2bV(b)}{b^2V^2(b) - \frac{4}{\pi^2}} \qquad (V\text{-}182)$$

where:

$$V(br) = J_1(br)Y_1(bR) - Y_1(br)J_1(bR) \qquad (IV\text{-}78)$$

Define

$$F(b) = \sum_{n=1}^{\infty} CY = \frac{2bV(b)}{b^2V^2(b) - \frac{4}{\pi^2}} \qquad (V\text{-}183)$$

Then

$$\frac{C}{F(b)} = C_{mn} = \frac{\int_0^1 Ydy}{\int_0^1 Y^2dy} \qquad (V\text{-}184)$$

From Equations (V-71) through (V-77)

$$\frac{C}{F(b)} = C_{mn} = \frac{4 \cos \beta(1-c)[\beta \sin \gamma c \cos \beta(1-c) + \gamma \cos \gamma c \sin \beta(1-c)]}{\beta[2\gamma c + \sin 2\gamma c]\cos^2\beta(1-c) + \gamma \cos^2\gamma c[2\beta(1-c) + \sin 2\beta(1-c)]}$$

(V-185)

As in the case of (V-77), this equation for C_{mn} is very highly sensitive to slight errors in the eigenvalues a and b. A modification is possible, analogous to that of Equations (V-78) through (V-85) for the case of single-layer linear flow. The only difference in these simplifications occurs due to the slightly different definitions of γ and β.

Equation (V-83) now becomes (for $\phi_1 = \phi_2$, $K_2 = K_2^*$)

$$\frac{\beta}{\gamma} - \frac{\gamma}{\beta K_2} = \frac{\beta^2 K_2 - \gamma^2}{\gamma \beta K_2}$$

$$= \frac{\frac{1}{A^2}\left[\frac{a^2}{K^2} - b^2\right] K_2 - \frac{1}{A^2}[a^2 - b^2]}{\gamma \, \beta \, K_2}$$

$$= \frac{b^2(1 - K_2)}{A^2 \, \gamma \, \beta \, K_2} \qquad \text{(V-186)}$$

The modified equation for C_{mn} then becomes

$$C_{mn} = \frac{\left[\frac{4(1-K_2)}{A^2 K_2}\right]\left(\frac{b^2}{\beta}\right)\sin \gamma c \, \cos^2\beta(1-c)}{\beta \cos^2\beta(1-c)[2\gamma c + \sin 2\gamma c] + \gamma \cos^2\gamma c[2\beta(1-c) + \sin 2\beta(1-c)]} \qquad \text{(V-187)}$$

Equation (V-187) is much less sensitive to slight errors in the eigenvalues a and b than is Equation (V-185), although both should give identical numerical answers for absolutely correct values of the eigenvalues. The solution may then be written

$$P(r,y,\theta) = \sum_{m=1}^{\infty} \cdot \left[\sum_{n=1}^{\infty} C_{mn} e^{-a^2\theta} Y_{mn} \right]_m F(b)U(br) \qquad \text{(V-188)}$$

where: C_{mn} is given by (V-187)

$$Y_{mn} = \cos \gamma y \qquad 0 \leq y < c$$

$$= \frac{\cos \gamma c}{\cos \beta(1-c)} \cos \beta(1-y) \qquad c < y \leq 1 \qquad \text{(V-176)}$$

$$\gamma = \frac{1}{A}\sqrt{a^2 - b^2} \qquad \text{(V-172)}$$

$$\beta = \frac{1}{A}\sqrt{\frac{a^2}{K_2^*} - b^2} \qquad \text{(V-173)}$$

$$F(b) = \frac{2bV(b)}{b^2V^2(b) - \left(\frac{4}{\pi^2}\right)} \qquad \text{(V-183)}$$

$$V(br) = J_1(br)Y_1(bR) - Y_1(br)J_1(bR) \qquad \text{(IV-78)}$$

$$U(br) = J_o(br)Y_1(bR) - Y_o(br)J_1(bR) \qquad \text{(V-178)}$$

and the characteristic equations are

$$U(b) = 0 \qquad \text{(IV-77)}$$

and

$$\gamma \sin \gamma c \cos \beta(1-c) + \beta K_2 \sin \beta(1-c) \cos \gamma c = 0 \qquad \text{(V-177)}$$

As in the case of two-layer linear flow, both real and imaginary values of β must be considered in order to obtain valid numerical results.

Values of dimensionless $P(r,y,\theta)$ are presented in Appendix K. The use of these tables will be illustrated by an example problem later in this section.

2. Dimensionless Flux at the Producing Face

The three types of dimensionless flux for the radial case are analogous to those for the linear case, namely

1. Individual instantaneous flux, q_1 and q_2 ,
2. Individual cumulative flux, Q_1 and Q_2, and
3. Total cumulative flux, $Q_{(t)}$.

The derivations of these quantities are sufficiently similar to the derivations for two-layer linear flow that the author does not feel that including these derivations would be worthwhile. The results will therefore be stated without proof.

The individual instantaneous fluxes for the radial case are:

$$q_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{2b^2 V^2(b)}{b^2 V^2(b) - \frac{4}{\pi^2}} \right] \frac{\sin \gamma c}{\gamma} \qquad \text{(V-189)}$$

$$q_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{2b^2 V^2(b)}{b^2 V^2(b) - \frac{4}{\pi^2}} \right] \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \right] \qquad \text{(V-190)}$$

The individual cumulative fluxes for the radial case are:

$$Q_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\frac{1 - e^{-a^2\Theta}}{a^2} \right] \left[\frac{2b^2 V^2(b)}{b^2 V^2(b) - \frac{4}{\pi^2}} \right] \frac{\sin \gamma c}{\gamma} \qquad \text{(V-191)}$$

$$Q_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[\frac{1 - e^{-a^2\Theta}}{a^2} \right] \left[\frac{2b^2 V^2(b)}{b^2 V^2(b) - \frac{4}{\pi^2}} \right] \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \right]$$

(V-192)

The total cumulative flux from both layers for the radial case is

$$Q(t) = \left(\frac{R^2 - 1}{2} \right) (1 - \epsilon) \qquad \text{(V-193)}$$

where:

$$\epsilon = \frac{\left(\frac{2}{R^2 - 1} \right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{\sin \gamma c}{\gamma} + \frac{\rho_2 C_{p2}}{\rho C} \frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)} \right] \left[\frac{2V^2(b)}{\frac{4}{\pi^2} - b^2 V^2(b)} \right]}{\left[c + \frac{\rho_2 C_{p2}}{\rho_1 C_{p1}} (1-c) \right]}$$

(V-194)

for heat transfer, or the same with $\left[\frac{\rho_2 C_{p2}}{\rho_1 C_{p1}} \right]$ replaced by $\left[\frac{\phi_2}{\phi_1} \right]$ for liquid flow.

The equations by which these dimensionless quantities may be used for heat transfer computations are given by

$$\left(\frac{Btu}{hr}\right)_1 = 2\pi H k_1 q_1 \Delta T \qquad \text{(V-195)}$$

$$\left(\frac{Btu}{hr}\right)_2 = 2\pi H k_2 q_2 \Delta T \qquad \text{(V-196)}$$

$$(Btu)_1 = 2\pi H k_1 Q_1 \Delta T \qquad \text{(V-197)}$$

$$(Btu)_2 = 2\pi H k_2 Q_2 \Delta T \qquad \text{(V-198)}$$

$$\text{Total Btu} = 2\pi r_b{}^2 H[c\rho_1 C_{p1} + (1-c)\rho_2 C_{p2}]Q_{(t)}\Delta T \qquad \text{(V-199)}$$

Corresponding equations for fluid flow are

$$\left(\frac{ft^3}{hr}\right)_1 = 0.1654\left(\frac{k_1}{\mu}\right) H \Delta P q_1 \qquad \text{(V-195a)}$$

$$\left(\frac{ft^3}{hr}\right)_2 = 0.1654\left(\frac{k_2}{\mu}\right) H \Delta P q_2 \qquad \text{(V-196a)}$$

where:

k_1, k_2 = darcys

H = feet

μ = centipoises

ΔP = original overall pressure difference driving force, psia

$$\text{Total } Q(ft^3) = 2\pi r_b{}^2 c_w H \Delta P[c\phi_1 + (1-c)\phi_2]Q_{(t)} \qquad \text{(V-199a)}$$

where:

c_w = water plus formation compressibility

Tables of q_1 and q_2 are presented in Appendix M of this dissertation. Tables of $Q_{(t)}$ for the case

$$\phi_1 = \phi_2$$

or analogously

$$\rho_1 C_{p1} = \rho_2 C_{p2}$$

appear in Appendix Q. Tables of Q_1 and Q_2 were not computed, due to the convergence difficulties discussed in the corresponding section for two-layer linear flow. The means of using the tables is illustrated by an example calculation in a later part of this dissertation.

C. Generalization of the Solutions for Two-Layer Systems to More Than Two Layers

Derivations have now been presented for two-layer flow in both linear and radial systems. This section will show how these solutions may be generalized to apply to a model of any number of layers.

Since the only essential difference in the derivation of the solutions for more than two layers occurs in the separated Y equation, which is the same for both linear and radial flow, it unnecessary to consider linear and radial flow separately. The derivations will be presented for linear flow models. The corresponding radial flow solutions will then be presented without a complete rederivation.

Consider the three-layer mathematical model shown schematically in Figure V-3. The specifications with regard to homogeneity, isotropy, etc. will be taken as the same used for two-layer linear flow.

Defining dimensionless variables

$$x = \frac{x_a}{H} \tag{V-200}$$

$$y = \frac{y_a}{H} \tag{V-201}$$

$$L = \frac{L_a}{H} \tag{V-202}$$

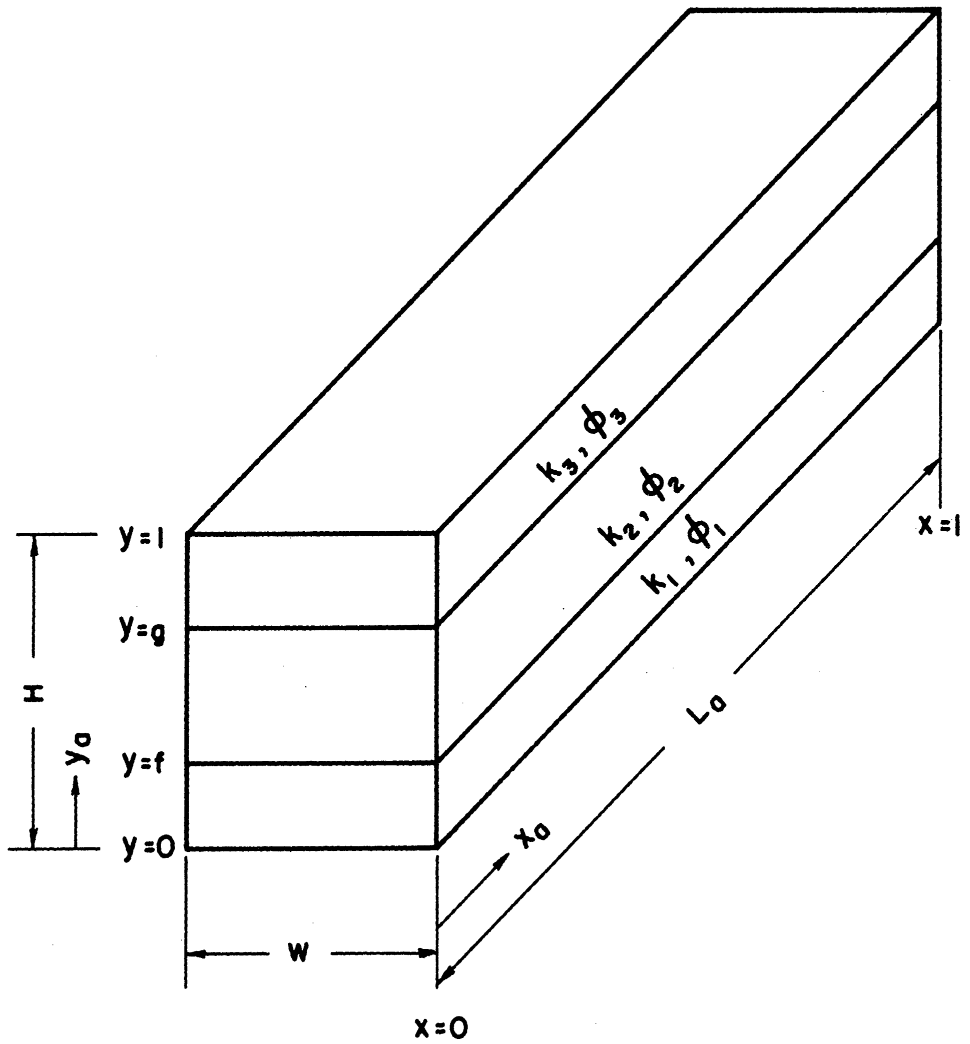


Figure V-3. Mathematical Model for Three-Layer Linear Flow.

the governing flow equation may be written

$$k \frac{\partial^2 P}{\partial x^2} + \frac{\partial}{\partial y}\left(k \frac{\partial P}{\partial y} \right) = \mu \phi c H^2 \frac{\partial P}{\partial t_a} \qquad \text{(V-203)}$$

For the constant terminal pressure case, the applicable initial and boundary conditions are

Initial Condition:

$$P(x,y,0) = 1 \qquad \text{(V-204)}$$

Boundary Conditions:

$$P(0,y,t_a) = 0 \qquad \text{(V-205)}$$

$$\frac{\partial P}{\partial x}(L,y,t_a) = 0 \qquad \text{(V-206)}$$

$$\frac{\partial P}{\partial y}(x,0,t_a) = 0 \qquad \text{(V-207)}$$

$$\frac{\partial P}{\partial y}(x,1,t_a) = 0 \qquad \text{(V-208)}$$

Separating variables in the usual manner, for

$$P = X(x)\ Y(y)\ T(t_a) \qquad \text{(V-209)}$$

the ordinary differential equations from the separation are

$$X'' + b^2 X = 0 \qquad \text{(V-210)}$$

$$T' + \frac{a^2}{\mu c H^2} T = 0 \qquad \text{(V-211)}$$

$$\left(\frac{k}{\phi}\right) Y'' + \left(\frac{k}{\phi}\right)' Y' + \left[a^2 - b^2 \left(\frac{k}{\phi}\right)\right] Y = 0 \qquad \text{(V-212)}$$

The solutions to (V-210) and (V-211) are, by inspection

$$X = C \sin bx \qquad \text{(V-213)}$$

$$T = C\, e^{\frac{-a^2 t_a}{\mu c H^2}} \qquad \text{(V-214)}$$

where:

$$b = \left(\frac{2m-1}{2}\right) \frac{\pi}{L} ; \quad m=1,2,3,\ldots \tag{V-215}$$

The question of whether a and b are real or imaginary, etc., is the same as for two-layer linear flow. The difference from the two-layer flow derivation occurs in the solution of the Y equation, (V-212).

Let the permeability, k, and porosity, ϕ, of the layers be that indicated in Figure V-3. The layers are assumed to be homogeneous and isotropic at all interior points. The applicable boundary and interface conditions become

$$Y'(o) = Y'(1) = 0 \tag{V-216}$$

$$Y(f-) = Y(f+) \tag{V-217}$$

$$Y(g-) = Y(g+) \tag{V-218}$$

$$Y'(f-) = \frac{k_2}{k_1} \quad Y'(f+) \tag{V-219}$$

$$Y'(g-) = \frac{k_3}{k_2} \quad Y'(g+) \tag{V-220}$$

Define

$$\frac{\phi_1}{k_1}\left[a^2 - b^2\left(\frac{k_1}{\phi_1}\right)\right] = \alpha^2 \tag{V-221}$$

$$\frac{\phi_2}{k_2}\left[a^2 - b^2\left(\frac{k_2}{\phi_2}\right)\right] = \lambda^2 \tag{V-222}$$

$$\frac{\phi_3}{k_3}\left[a^2 - b^2\left(\frac{k_3}{\phi_3}\right)\right] = \epsilon^2 \tag{V-223}$$

The equations and corresponding solutions are then

$$\begin{aligned} &Y'' + \alpha^2 Y = 0 \\ &Y = C_1 \sin \alpha y + C_2 \cos \alpha y \end{aligned} \qquad 0 < y < f \tag{V-224}$$

$$Y'' + \lambda^2 Y = 0$$
$$Y = C_3 \sin \lambda y + C_4 \cos \lambda y \qquad f < y < g \qquad \text{(V-225)}$$

$$Y'' + \epsilon^2 Y = 0$$
$$Y = C_5 \sin \epsilon y + C_6 \cos \epsilon y \qquad g < y < 1 \qquad \text{(V-226)}$$

From (V-216)

$$Y'(o) = 0 = [\alpha C_1 \cos \alpha y - \alpha C_2 \sin \alpha y]_{y=0} = \alpha C_1$$

Therefore

$$C_1 \neq 0$$

$$Y'(1) = 0 = [\epsilon\ C_5 \cos \epsilon - \epsilon\ C_6 \sin \epsilon]$$

Therefore

$$C_5 = C_6 \tan \epsilon$$

The solutions may thus be written

$$Y = C_2 \cos \alpha y \qquad 0 < y < f$$
$$= C_3 \sin \lambda y + C_4 \cos \lambda y \qquad f < y < g$$
$$= C_7 \cos \epsilon(1-y) \qquad g < y < 1 \qquad \text{(V-227)}$$

The interface conditions (V-217) through (V-220) may next be applied.

From (V-217)

$$C_2 \cos \alpha f = C_3 \sin \lambda f + C_4 \cos \lambda f \qquad \text{(V-228)}$$

From (V-218)

$$C_3 \sin \lambda g + C_4 \cos \lambda g = C_7 \cos \epsilon(1-g) \qquad \text{(V-229)}$$

From (V-219)

$$-C_2 \alpha \sin \alpha f = \frac{k_2}{k_1} [C_3\ \lambda \cos \lambda f - C_4\ \lambda \sin \lambda f] \qquad \text{(V-230)}$$

From (V-220)

$$\lambda[C_3 \cos \lambda g - C_4 \sin \lambda g] = \frac{k_3}{k_2} \epsilon\ C_7 \sin \epsilon(1-g) \qquad \text{(V-231)}$$

This set of four equations [(V-228) through (V-231)] can be considered as four equations in the four unknowns C_2, C_3, C_4, and C_7. In order that there be a non-trivial solution (i.e., $C_2 \neq C_3 \neq C_4 \neq C_7 \neq 0$) the fourth order determinant of the coefficients must be equal to zero.

Define

$$K_2 = \frac{k_2}{k_1} \tag{V-232}$$

$$K_3 = \frac{k_3}{k_2} \tag{V-233}$$

Rewriting the equations

$$C_2[\cos \alpha f] + C_3[-\sin \lambda f] + C_4[-\cos \lambda f] = 0 \tag{V-234}$$

$$C_3[\sin \lambda g] + C_4[\cos \lambda g] + C_7[-\cos \epsilon(1-g)] = 0 \tag{V-235}$$

$$C_2[-\alpha \sin \alpha f] + C_3[-K_2\lambda \cos \lambda f] + C_4[K_2\lambda \sin \lambda f] = 0 \tag{V-236}$$

$$C_3[\lambda \cos \lambda g] + C_4[-\lambda \sin \lambda g] + C_7[-K_3\epsilon \sin \epsilon(1-g)] = 0 \tag{V-237}$$

Define:

$$\begin{aligned}
F_1 &= \cos \alpha f & F_7 &= -\alpha \sin \alpha f \\
F_2 &= -\sin \lambda f & F_8 &= -K_2\lambda \cos \lambda f \\
F_3 &= -\cos \lambda f & F_9 &= K_2\lambda \sin \lambda f \\
F_4 &= \sin \lambda g & F_{10} &= \lambda \cos \lambda g \\
F_5 &= \cos \lambda g & F_{11} &= -\lambda \sin \lambda g \\
F_6 &= -\cos \epsilon(1-g) & F_{12} &= -K_3\epsilon \sin \epsilon(1-g)
\end{aligned} \tag{V-238}$$

The characteristic equation defining the eigenvalues, a, may then be written in the form of the determinant

$$D = \begin{vmatrix} F_1 & F_2 & F_3 & 0 \\ 0 & F_4 & F_5 & F_6 \\ F_7 & F_8 & F_9 & 0 \\ 0 & F_{10} & F_{11} & F_{12} \end{vmatrix} = 0 \tag{V-239}$$

Expanding by minors, the equation becomes

$$F_1F_4F_9F_{12} - F_1F_5F_8F_{12} + F_1F_6F_8F_{11}$$
$$- F_1F_6F_9F_{10} + F_2F_5F_7F_{12} - F_2F_6F_7F_{11}$$
$$- F_3F_4F_7F_{12} + F_3F_6F_7F_{10} = 0 \qquad \text{(V-240)}$$

Due to the zeros present, the 24 terms of the fourth order determinant simplify to only eight. The characteristic equation for the three-layer case thus contains eight terms, each of which is composed of four elements, F_i.

Now consider a four-layer linear case. The solutions to the separated Y equation would in this case contain eight constants, $C_1 - C_8$. The boundary conditions

$$Y'(o) = Y'(1) = 0 \qquad \text{(V-216)}$$

would eliminate two of these. The six interface conditions would then provide the necessary six equations for determining these six constants. The characteristic equation is thus of the form of a sixth order determinant equaling zero. Such a determinant, expanded by minors, would normally involve 6! = 720 terms, each of which would have six elements. It can be shown that the zeros present in the determinant reduce the number of non-zero factors in the characteristic equation to thirty-two terms, each of which involves six elements.

The extension to more than four layers is apparent. Let the number of layers in the model be N. The number of constants appearing in the solutions to the separated equation will be 2N. Of these, two may always be eliminated using the boundary conditions (V-216). Now for N layers, there will be (N-1) interfaces between layers. Since at each interface, both the pressure and flux must be equal in the adjoining layers, a total of 2(N-1) = 2N-2 equations will result from the interface conditions.

The characteristic equation for an N-layer system is thus in the form of a determinant of order (2N-2), set equal to zero. Expansion by minors of this determinant will yield $2^{(2N-3)}$ terms, each of which will contain (2N-2) elements. For example, a five-layer system would have a characteristic equation of the form of an eighth order determinant equal to zero. Expansion by minors would yield <u>128</u> terms, each of which would contain <u>eight</u> elements.

Let us now return to consideration of the three-layer system. The characteristic Equation (V-239) or (V-240) defines an infinite number of the desired eigenvalues, a, for each value of the parameter b. From inspection of (V-221) through (V-223), it is apparent that the factors α, λ, and ϵ might be either real or imaginary, depending on the relationship between a, b, and the ratio (k/ϕ). As in the two-layer flow case, it is necessary to consider the <u>complete</u> spectrum of real and positive values of the eigenvalue, a, for each value of b. The elements, F, appearing in the characteristic Equation (V-240) can always be written in terms of real quantities, even though α, λ, and/or ϵ may be imaginary by using the equations

$$\sin ix = i \sinh x \tag{V-241}$$

$$\cos ix = \cosh x \tag{V-242}$$

$$(i)^2 = -1 \tag{V-243}$$

relating real and imaginary numbers.

Having determined the eigenvalues a and b, the only remaining difficulty in obtaining the solution for three-layer flow is the establishment of the applicable orthogonality conditions. The orthogonality conditions for two separate and distinct layers, and for a continuous variation of (k/ϕ) with y are derived in Appendix A of this dissertation. In both cases it was found that

$$\int_0^1 Y_m Y_n dy = 0 \qquad \text{for} \quad m \neq n \tag{V-244}$$

The applicable condition for three or more separate and distinct layers may be easily derived using the same basic procedure as for two layers. The author has done so for a three-layer system, but does not believe the inclusion of the derivation in this dissertation is warranted, due to the similarity to the two-layer flow derivation. It was found that the condition (V-244) also applies to three-layer systems. Careful examination of the two- and three-layer derivations shows, moreover, that Equation (V-244) should be valid for <u>any</u> number of separate and distinct layers, as long as the criteria of continuity of pressure and flux obtain at all interfaces.

The three-layer solution continues by determining the relation between C_3 and C_4 in Equation (V-227), in the range $f < y < g$. This can be most easily done using (V-234) and (V-236), eliminating C_2. Substituting from (V-238) into these equations.

$$C_2F_1 + C_3F_2 + C_4F_3 = 0 \tag{V-245}$$

$$C_2F_7 + C_3F_8 + C_4F_9 = 0 \tag{V-246}$$

Eliminating C_2, it is found that

$$C_4 = \left[\frac{F_2F_7 - F_1F_8}{F_1F_9 - F_3F_7} \right] C_3 \tag{V-247}$$

Define

$$E = \frac{F_2F_7 - F_1F_8}{F_1F_9 - F_3F_7} \tag{V-248}$$

The expansion to fit the initial condition (V-204) then proceeds as follows:

$$1 = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} CY \right] \sin bx \tag{V-249}$$

$$\sum_{n=1}^{\infty} CY = \frac{\int_0^L \sin bx \, dx}{\int_0^L \sin^2 bx \, dx} = \frac{2}{bL} \tag{V-250}$$

Then

$$1 = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} C_{mn} Y \right] \left(\frac{2}{bL} \right) \sin bx \tag{V-251}$$

where

$$C_{mn} = \frac{\int_0^1 Y \, dy}{\int_0^1 Y^2 dy} \tag{V-252}$$

and

$$\begin{aligned} Y &= \cos \alpha y \qquad 0 < y < f \\ &= \sin \lambda y + E \cos \lambda y \qquad f < y < g \\ &= \cos \epsilon(1-y) \qquad g < y < g \end{aligned} \tag{V-253}$$

Expanding (V-252), substituting from (V-253)

$$C_{mn} = \frac{\int_0^f \cos \alpha y \, dy + \int_f^g [\sin \lambda y + E \cos \lambda y] dy + \int_g^1 \cos \epsilon(1-y) dy}{\int_0^f \cos^2 \alpha y \, dy + \int_f^g [\sin \lambda y + E \cos \lambda y]^2 dy \int_g^1 \cos^2 \epsilon(1-y) dy}$$

$$= \frac{I_1 + I_2 + I_3}{I_4 + I_5 + I_6} \tag{V-254}$$

Evaluating the integrals

$$I_1 = \int_0^f \cos \alpha y \, dy = \frac{\sin \alpha f}{\alpha} \qquad \text{(V-255)}$$

$$I_2 = \int_f^g [\sin \lambda y + E \cos \lambda y] dy$$

$$= \frac{1}{\lambda}[\cos \lambda f - \cos \lambda g] + \frac{E}{\lambda}[\sin \lambda g - \sin \lambda f] \qquad \text{(V-256)}$$

$$I_3 = \int_g^1 \cos \epsilon(1-y) dy = \frac{\sin \epsilon(1-g)}{\epsilon} \qquad \text{(V-257)}$$

$$I_4 = \int_0^f \cos^2 \alpha y \, dy = \frac{1}{2\alpha} [\sin \alpha f \cos \alpha f + \alpha f] \qquad \text{(V-258)}$$

$$I_5 = \int_f^g [\sin \lambda y + E \cos \lambda y]^2 \, dy$$

$$= \frac{1}{2\lambda} [g\lambda - \sin g\lambda \cos g\lambda - f\lambda + \sin f\lambda \cos f\lambda]$$

$$+ \frac{E}{2\lambda} [\cos 2\lambda f - \cos 2\lambda g]$$

$$+ \frac{E^2}{2\lambda} [g\lambda + \sin g\lambda \cos g\lambda - f\lambda - \sin f\lambda \cos f\lambda] \qquad \text{(V-259)}$$

$$I_6 = \int_g^1 \cos^2 \epsilon(1-y) dy = \frac{1}{4\epsilon} [2\epsilon(1-g) + \sin 2\epsilon(1-g)] \qquad \text{(V-260)}$$

The solution for three-layer linear flow in the constant terminal pressure case is then

$$P(x,y,t_a) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} Y\left(\frac{2}{bL}\right) \sin bx \; e^{\frac{-a^2 t_a}{\phi c H^2}} \qquad \text{(V-261)}$$

where Y is given by (V-248) and (V-253)

C_{mn} is given by (V-252) through (V-260)

b is defined by (V-215)

a is defined by (V-238) through (V-240)

The corresponding solution for three-layer radial flow, constant terminal pressure is

$$P(r,y,t_a) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} Y e^{\frac{-a^2 t_a}{\phi c H^2}} F(b)U(br) \qquad \text{(V-262)}$$

where C_{mn} and Y are as for linear flow

F(b) is given by (V-183)

U(br) is given by (V-178)

b is defined by (IV-77), and

a is defined by (V-238) through (V-240) with the change that

$$\alpha^2 = \frac{\phi_1}{A^2 k_1} \left[a^2 - b^2\left(\frac{k_1}{\phi_1}\right)\right] \qquad \text{(V-263)}$$

$$\lambda^2 = \frac{\phi_2}{A^2 k_2} \left[a^2 - b^2\left(\frac{k_2}{\phi_2}\right)\right] \qquad \text{(V-264)}$$

$$\epsilon^2 = \frac{\phi_3}{A^2 k_3} \left[a^2 - b^2\left(\frac{k_3}{\phi_3}\right)\right] \qquad \text{(V-265)}$$

rather than by (V-221) through (V-223).

D. Convergence of the Analytical Solutions For Two-Layer Flow

All of the solutions obtained for two-layer flow, and also for multi-layer flow, have been found in the form of double Fourier or Fourier-Bessel series. In the case of linear flow, the eigenvalues b are found by inspection, but the eigenvalues a are defined by an implicit expression. In the case of two-layer radial flow, the eigenvalues b were defined by the same characteristic equation as for single-layer radial flow, and the eigenvalues a are again defined by an implicit equation. The eigenvalues for the two-layer case were calculated on the IBM 704 using the half-interval method, as explained in Section IV-C of this dissertation. Since the calculation of the eigenvalues for the series involved a considerable amount of time on the IBM 704 (a total of approximately twelve hours for the eigenvalues in this dissertation) the rate of convergence of the analytical solutions was important.

The number of eigenvalues necessary to obtain the desired accuracy in the numerical tables was determined by trial and error. In the linear case, a twelve by eight matrix of eigenvalues was used for all values of L. That is, a total of eight eigenvalues, a, were computed for each of twelve values of the eigenvalue b for each value of L. This proved to be adequate for the computation of $P(r,y,\theta)$, q_1 and q_2, and $Q_{(t)}$. As noted in Section V-A-2-b, however, this proved to be entirely insufficient for calculation of the individual cumulative dimensionless fluxes Q_1 and Q_2 [Equations (V-117) and (V-118)]. A forty by eight matrix of eigenvalues was computed for L = 10 and used in an attempt to compute Q_1 and Q_2. This, too, was found to be insufficient. The calculation of tables of Q_1 and Q_2 was therefore abandoned.

In the radial case, unlike the linear case, it was found that the number of eigenvalues required was a function of a parameter in the problem, the ratio (r_b/H). The number of eigenvalues b required was essentially constant. The number of eigenvalues a required per b was strongly dependent on the value of (r_b/H).

It was originally intended that the solution for two-layer radial flow should have two principal applications. First, the solutions should apply to reservoir-aquifer systems (where r_b denotes the radius of the reservoir) and second, they should apply to the flow within an oil reservoir to a given well bore (where r_b denotes the radius of the well bore). In the former case, the value of (r_b/H) is normally quite large, of the order of ten to a hundred or more. In the latter case, however, the ratio is normally quite small, of the order of 0.01.

The number of eigenvalues a per eigenvalues b required for an accurate numerical result was found to be only one for values of (r_b/H) greater than 10.0. This may be contrasted with the linear case, where 3 to 8 such eigenvalues were usually necessary. It was found, moreover, that the solution for $P(r,y,\Theta)$ was essentially independent of the parameter (r_b/H) for values of $(r_b/H) = 10$ or more. Consequently, values of q_1, q_2, and $Q_{(t)}$ were also independent of (r_b/H) in this range.

For values of (r_b/H) less than about 1.0, on the other hand, it proved to be nearly impossible to obtain valid numerical results. The number of eigenvalues a per eigenvalue b required was found to be several hundred at $(r_b/H) = 0.05$. This would require a prohibitive amount of computer time for computation of eigenvalues. Moreover, the problem of round-off error with such a large number of terms in the series might very likely invalidate any numerical results thus obtained.

For the two-layer radial heat transfer model in this investigation, $(r_b/H) = 0.6$. A ten by forty matrix of eigenvalues (i.e. forty a's per each of ten b's) proved insufficient to give even an approximate check of the experimental data.

It thus proved to be impossible to compute values of $P(r,y,\Theta)$, etc., for values of (r_b/H) which would be useful for flow to a well in an oil reservoir. It seems probable that a point source solution to the problem would alleviate this difficulty.

It is of interest to consider the physical reason behind the failure of the series to converge for low values of (r_b/H). The author believes that this failure is largely due to the nature of the pressure gradient in the y direction in such cases. For high H, and thus low (r_b/H), points far removed from the interface between layers are largely unaffected by the presence of the other layer. It is only near the interface that a comparatively sharp change in pressure with respect to y at a given x occurs. The approximation of such a gradient by a series solution is tantamount to fitting a step-function curve. Thus, a very large number of series terms are required for an accurate approximation.

All values in the Appendices of this dissertation for two-layer radial flow were computed using a five by one matrix of eigenvalues. This proved to be sufficient for accurate calculation of all factors desired.

E. Example Calculations

This section is intended to illustrate the use of the tables in Appendices J, L, and N for two-layer linear flow, and Appendices K, M, and O for two-layer radial flow. Two example problems will be presented. In order to once again illustrate the dual usefuleness of the tables, the problem for two-layer linear flow will be considered to be a heat transfer problem, while that for two-layer radial flow will be concerned with a reservoir-aquifer system.

Both problems to be considered here will be for bounded systems. The use of the tables in Appendices J through O for infinite systems is analogous to that indicated in Example Problem 5 for single-layer flow.

1. <u>Example Problem 6</u>

Consider a two-layer bar such as that shown schematically in Figure V-4, with the following dimensions and physical properties.

L_a = 24 inches = 2.0 feet

H = 0.48 inches = 0.04 feet

w = 1.0 inches (width)

c = 0.5 (i.e. interface between layers is at y_a = 0.24 inches)

k_1 = 16 Btu/hr ft^2(°F/ft)

k_2 = 160 Btu/hr ft^2(°F/ft)

$\rho_1 = \rho_2$ = 500 lb/ft^3

$C_{p1} = C_{p2}$ = 0.1 Btu/lb °F

The bar is initially at 70°F throughout. At time zero (i.e. $t_a = \Theta = 0$) one end of the bar (at $x = x_a = 0$) is rapidly heated to 270°F and maintained at that temperature thereafter. It is assumed that there are no significant heat losses from the bar. It is desired to calculate

1. The temperature distribution, $T(x_a, y_a, t_a)$ in the bar at a time 30 minutes after the heating began (i.e. t_a = 30 min.).
2. The rate of heat transfer, Btu/hr, across the heating face into layers 1 and 2 (individually) at time t_a = 30 minutes.
3. The total heat input to the bar, Btu, at time t_a = 30 minutes.

For this bar

$$L = \frac{L_a}{H} = \frac{24 \text{ inches}}{0.48 \text{ inches}} = 50 \qquad \text{(V-266)}$$

$$x = \frac{x_a}{H} = \frac{x_a}{0.48} \text{ for } x_a \text{ in inches} \qquad \text{(V-267)}$$

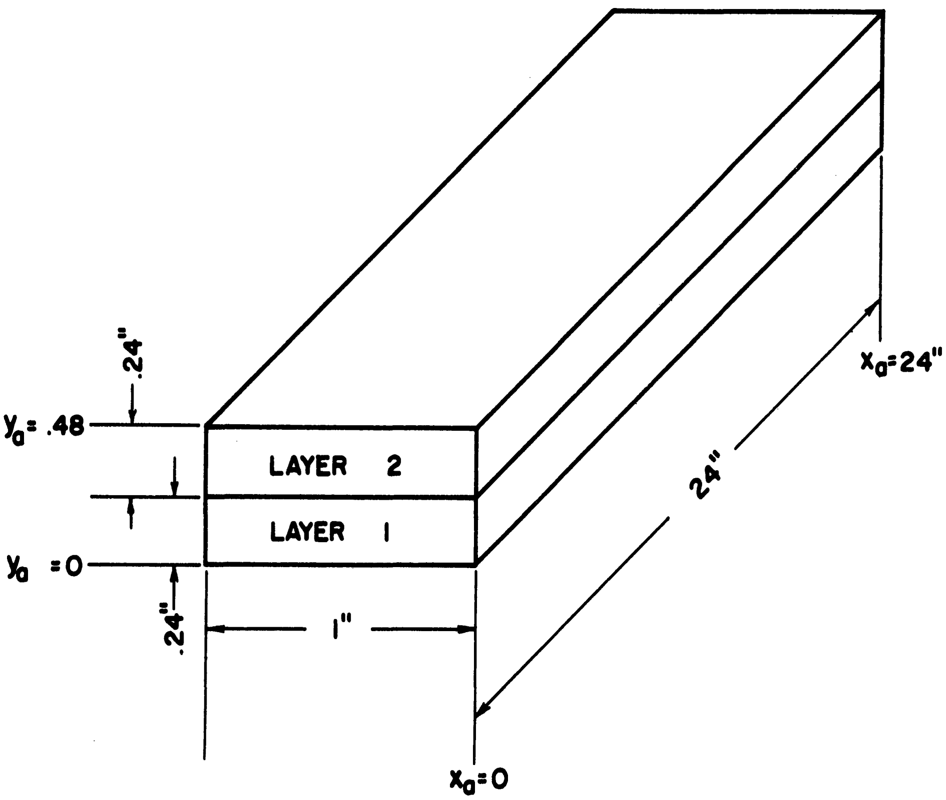


Figure V-4. Heat Transfer Model for Example Problem 6.

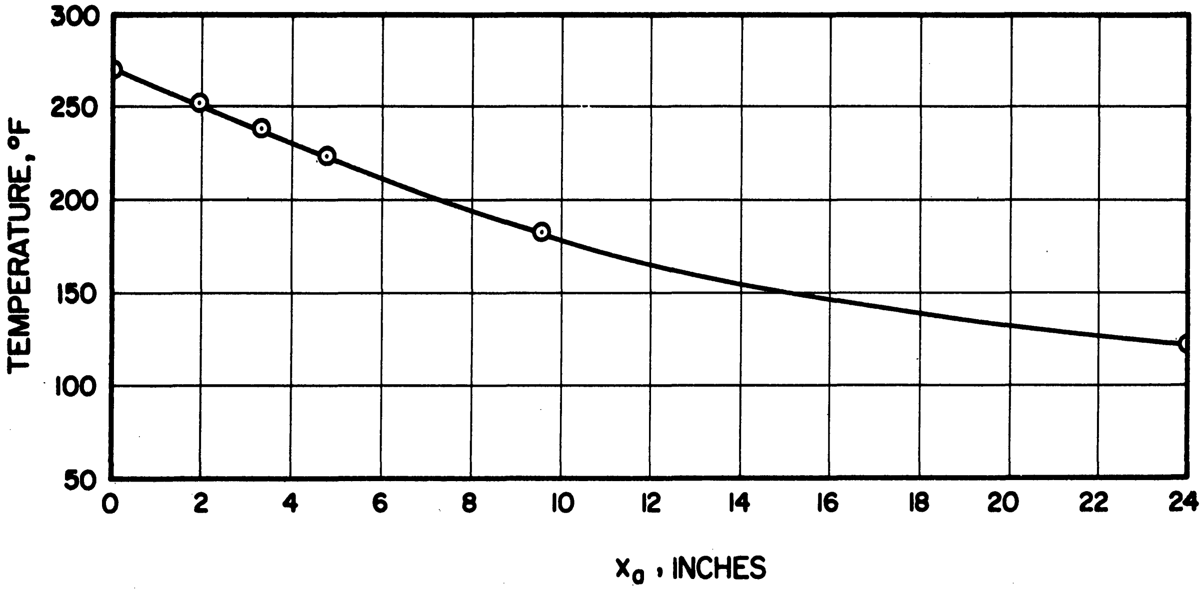


Figure V-5. Temperature Distribution for Example Problem 6.

$$y = \frac{y_a}{H} = \frac{y_a}{0.48} \quad \text{for } y_a \text{ in inches} \tag{V-268}$$

$$K_2^* = \frac{k_2 \rho_1 C_{p1}}{k_1 \rho_2 C_2} = \frac{(160)(500)(0.1)}{(16)(500)(0.1)} = K_2 = 10 \tag{V-269}$$

Layer 1 is defined as $0 < y_a < 0.24$ inches

Layer 2 is defined as $0.24 < y_a < 0.48$ inches

At $t_a = 30$ minutes

$$\Theta = \frac{k_1 t_a}{\rho_1 C_{p1} H^2} = \frac{(16)(0.5)}{(500)(0.1)(.04)^2}$$

$$= 100 \text{ (dimensionless)} \tag{V-270}$$

To determine $T(x_a, y_a, t_a)$ the table in Appendix J will be used, for $L = 50$, $K_2 = 10$, $c = 0.5$, and $\Theta = 100$. It can be seen from this table that there is no significant temperature variation in the y_a or y direction at this time. The dimensionless $P(x, y, \Theta)$ or analogously $T(x, y, \Theta)$ for all y at $\Theta = 100$ is given as

x	$T(x, y, \Theta)$
4	0.094
7	0.163
10	0.231
20	0.438
50	0.737

The corresponding actual temperatures as a function of actual position are found using

$$x_a = 0.48x \text{ (inches)} \tag{V-271}$$

$$T(x_a, y_a, t_a) = 270 - (270 - 70)\, T(x, y, \Theta)$$
$$= 270 - 200\, T(x, y, \Theta) \tag{V-272}$$

The results are shown in Figure V-5.

The individual instantaneous fluxes across the heating face into the two layers can be determined using the tables in Appendix L. From the table for L = 50, K_2 = 10, c = 0.5 at Θ = 100

$$q_1 = 0.01309$$

$$q_2 = 0.01308$$

To convert these dimensionless fluxes to real flux in Btu/hr, it is necessary to make use of the equation

$$\left(\frac{\text{Btu}}{\text{hr}}\right)_i = k_i q_i w \Delta T \qquad \text{(V-110)}$$

where: w is the width in feet

ΔT is the temperature difference driving force, °F.

Then for this model

$$\left(\frac{\text{Btu}}{\text{hr}}\right)_1 = (16)(0.01309)\left(\frac{1}{12}\right)(200) = 3.4907 \ \frac{\text{Btu}}{\text{hr}} \qquad \text{(V-273)}$$

$$\left(\frac{\text{Btu}}{\text{hr}}\right)_2 = (160)(0.01308)\left(\frac{1}{12}\right)(200) = 34.88 \ \frac{\text{Btu}}{\text{hr}} \qquad \text{(V-274)}$$

It is of interest to note that the fluxes are in this case nearly directly proportional to the respective thermal conductivities.

The total cumulative flux across the heating face is found from the tables in Appendix N. From these tables, for L = 50, K_2 = 10, and c = 0.5 at Θ = 100.

$$Q_{(t)} = 23.79$$

The applicable equation for converting this dimensionless flux to a real flux is

$$Q(\text{Btu}) = H^2 w \Delta T[c\rho_1 C_{p1} + (1-c)\rho_2 C_{p2}]Q_{(t)} \qquad \text{(V-127)}$$

so that for this problem

$$Q = (0.04)^2\left(\frac{1}{12}\right)(200)[(0.5)(500)(0.1) + (0.5)(500)(0.1)](23.79)$$

$$= 31.7 \text{ Btu total heat input} \qquad \text{(V-275)}$$

2. Example Problem 7

Consider a circular gas reservoir surrounded on all sides by a uniform aquifer. It is known that this aquifer is composed of two inter-communicating layers. The pertinent dimensions and physical properties are as follows

r_b = 2000 feet, radius of reservoir

r_e = 200,000 feet, exterior radius of the aquifer

h_1 = 50 feet, thickness of layer 1

h_2 = 50 feet, thickness of layer 2

k_1 = 5 millidarcys

k_2 = 100 millidarcys

ϕ = 10%

μ = 1.0 centipoise water viscosity

c = 7×10^{-6} vol/vol psi water plus formation compressibility

The initial reservoir and aquifer pressure is 1100 psia. If the reservoir pressure is maintained at 1000 psia, it is desired to calculate

1. The pressure distribution in the aquifer at a time of t_a = 88 days, assuming the gas-water interface to be stationary.
2. The influx of water, ft^3/hr, into the reservoir at t_a = 88 days from each of the two layers.

3. The total cumulative flux of water in cubic feet into the reservoir at 88 days.

4. The actual change in the radius of the reservoir, r_b, assuming the porous rock matrix of the reservoir to be the same as that of the aquifer, based on the foregoing calculations.

The pertinent dimensionless ratios for this problem are:

$$R = \frac{r_e}{r_b} = \frac{200{,}000 \text{ ft}}{2000 \text{ ft}} = 100 \qquad \text{(V-276)}$$

$$A = \frac{r_b}{H} = \frac{r_b}{h_1 + h_2} = \frac{2000}{50 + 50} = 20 \qquad \text{(V-277)}$$

$$K_2^* = \frac{k_2\, \phi_1}{k_1\, \phi_2} = \frac{(100)(0.1)}{(5)\ (0.1)} = K_2 = 20 \qquad \text{(V-278)}$$

$$c = \frac{h_1}{h_1 + h_2} = 0.5 \quad \text{(point of division on y coordinate)} \qquad \text{(V-279)}$$

$$r_a = r\, r_b = 2000\ r \quad \text{feet} \qquad \text{(V-280)}$$

$$y_a = Hy = (h_1 + h_2)y = 100\ y \quad \text{feet} \qquad \text{(V-281)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c r_b^2} = \frac{(.005)(88 \times 24 \times 3600)}{(1)(0.1)(7 \times 10^{-6} \times 14.7)(2000 \times 30.48)^2}$$

$$= 0.1 \qquad \text{(V-282)}$$

Since $A = r_b/H > 10$, the tables of Appendix K may be used to calculate the pressure distribution in the aquifer. The first step is to determine whether the effect of the exterior aquifer boundary is significant at t_a = 88 days, $\Theta = 0.1$. From the table for R = 100, K_2 = 20, c = 0.5 it can be seen that the exterior boundary will in this case have no significant effect until $\Theta = 50$. Since in this problem $\Theta = 0.1$, it is necessary to use a table for a smaller value of R. In this case the table for $R = 5.0$ should give a very close approximation to $P(r, y, \Theta)$ since the value of $P(5, y, \Theta)$ is 0.994 (nearly equal to 1.000) at $\Theta = 0.1$. As is true of all the tables in Appendix K, the variation in pressure with respect to y is too small to be noted with only three place accuracy. Reading from the table for $R = 5.0$, $K_2 = 20$, $c = 0.5$ at $\Theta = 0.1$ the dimensionless pressure distribution is found to be (for all y)

r	$P(r, y, \Theta)$
2	0.641
3	0.899
4	0.980
5	0.994

These dimensionless values may then be converted to real values using the equations

$$r_a = 2000\ r \quad \text{feet} \tag{V-283}$$

$$P(r_a, y_a, t_a) = 1000 + 100\ P(r, y, \Theta) \quad \text{psia} \tag{v-284}$$

The real values are thus

r_a (feet)	$P(r_a, y_a, t_a)$ (psia)
4,000	1064.1
6,000	1089.9
8,000	1098.0
10,000	1099.4

Next, the influx of water from each layer into the reservoir may be determined using the tables of Appendix M. From these tables, for $R = 5.0$, $K_2 = 20$, $c = 0.5$, at $\theta = 0.1$

$$q_1 = q_2 = 0.4845$$

The applicable equations for converting these dimensionless fluxes to real fluxes are

$$\left(\frac{ft^3}{hr}\right)_1 = 0.1654\left(\frac{k_1}{\mu}\right) H \Delta P q_1 \qquad \text{(V-195a)}$$

$$\left(\frac{ft^3}{hr}\right)_2 = 0.1654\left(\frac{k_2}{\mu}\right) H \Delta P q_2 \qquad \text{(V-196a)}$$

For this problem

$$\left(\frac{ft^3}{hr}\right)_1 = \frac{(0.1654)(0.005)(100)(100)(0.4845)}{(1.0)}$$

$$= 4.0 \quad ft^3/hr \quad \text{for layer 1} \qquad \text{(V-285)}$$

$$\left(\frac{ft^3}{hr}\right)_2 = \frac{(0.1654)(0.100)(100)(100)(0.4845)}{(1.0)}$$

$$= 80.0 \ ft^3/hr \quad \text{for layer 2} \qquad \text{(V-286)}$$

The total cumulative flux from both layers may be found using the tables in Appendix O. From these tables, for $R = 5.0$, $K_2 = 10$, $c = 0.5$ at $\theta = 0.1$

$$Q_{(t)} = 1.085$$

The applicable equation for converting to real flux is

$$\text{Total } Q(ft^3) = 2\pi r_b^2 c_w H \Delta P[c\phi_1 + (1-c)\phi_2]Q_{(t)} \qquad \text{(V-199a)}$$

where:

c_w = water plus formation compressibility

For this problem

$$\text{Total} \quad Q = 2\pi(2000)^2(7 \times 10^{-6})(100)(100)(0.1)(1.085)$$

$$= 1.91 \times 10^5 \quad ft^3 \quad \text{total water influx} \qquad (V\text{-}287)$$

Now the original volume of the reservoir was

$$V_i = \pi r_b^2 H\phi$$

$$= \pi(2000)^2(100)(0.1)$$

$$= 1.25674 \times 10^8 \quad ft^3 \qquad (V\text{-}288)$$

The total water influx was 0.00191×10^8 ft^3. The reservoir volume at 88 days is thus 1.25483×10^8 ft^3. The corresponding change in the reservoir radius is only a decrease of 1.52 feet. The assumption of a stationary gas-water interface for the problem is therefore entirely valid.

VI EXPERIMENTAL WORK, MODEL STUDIES

As previously noted, the majority of the information heretofore available on the flow properties of heterogeneous and/or anisotropic systems has been obtained experimentally using models of some sort. In many cases, particularly where highly irregular geometries are involved such as in simulating an actual reservoir, model studies offer the only means available for studying the overall flow patterns. This is also true in the case of many multi-phase flow problems, with which this dissertation is not concerned.

The purpose of obtaining experimental data in this investigation was primarily to verify the analytical work. Since it has proven possible to obtain solutions to the problems of flow in stratified systems herein considered by a purely mathematical approach, it was not necessary to use model studies in determining the required solutions. However, since there is nothing in the literature to directly check the validity of the analytical solutions obtained, it was deemed advisable to verify the solutions by comparison with experimental data. It should be emphasized that the two methods of attack on the problem were completely independent. No experimental results were utilized in obtaining the analytical solutions. The experimental data were used solely to verify the validity of the mathematical work.

Many different types of models have been used to study fluid flow problems. The most obvious type is simply a porous matrix of the desired size and shape, in which any desired liquid or gas may be used as the flowing liquid. Such models may be constructed by packing sand or glass beads, graded so as to obtain the desired permeability and porosity, into a restraining container. By varying the size distribution of the sand or glass beads, it is possible to construct several kinds of heterogeneous models. Flow models of this type have been extensively used to determine flooding patterns for water floods in both homogeneous and heterogeneous systems.[6,16,41]

The use of permeable fluid flow models with single phase flow is usually restricted to studies of steady state behavior. This is not true for multi-phase studies, such as in studying the flow patterns for water displacing oil. When the entire free pore volume of a porous model is initially filled with the same fluid throughout, unsteady state flow is dependent on the compressibility of fluid to provide the driving force. In an underground aquifer, which extends for many miles, the compressibility of the water, though small, is sufficient to provide a driving force for water to enter zones of low pressure (such as an oil or gas reservoir which is being depleted). In a laboratory model, however, the compressibility of liquids is clearly insufficient to instigate measureable phenomena of this type. Even with a gas as the flowing fluid, a very large model would be required to allow accurate measurement of the transient pressure distribution.

Unfortunately, steady state models are of little use in studying inter-layer fluid flow of the type with which this dissertation is concerned. Consider a model packed with sand such as that shown schematically in Figure VI-1. At first glance it might appear that by operating this model in the steady state, using either gas or liquid as the flowing fluid, some information with regard to cross-flow pressure distribution could be

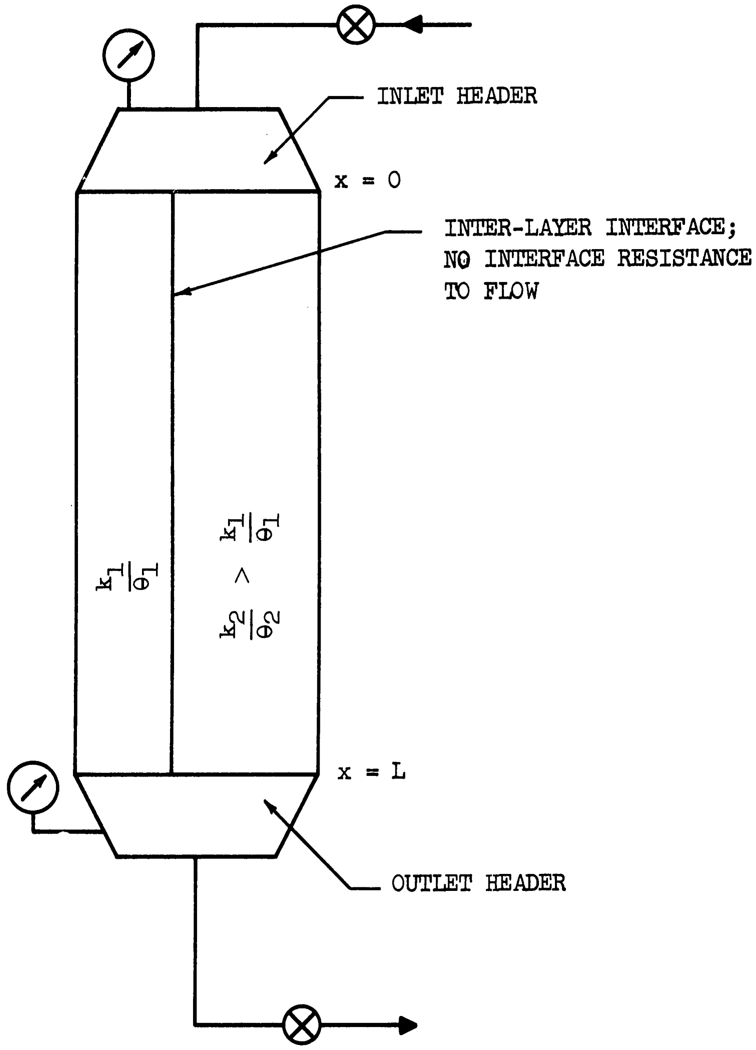


Figure VI-1. Schematic Drawing of a Heterogeneous Steady-State Flow Model.

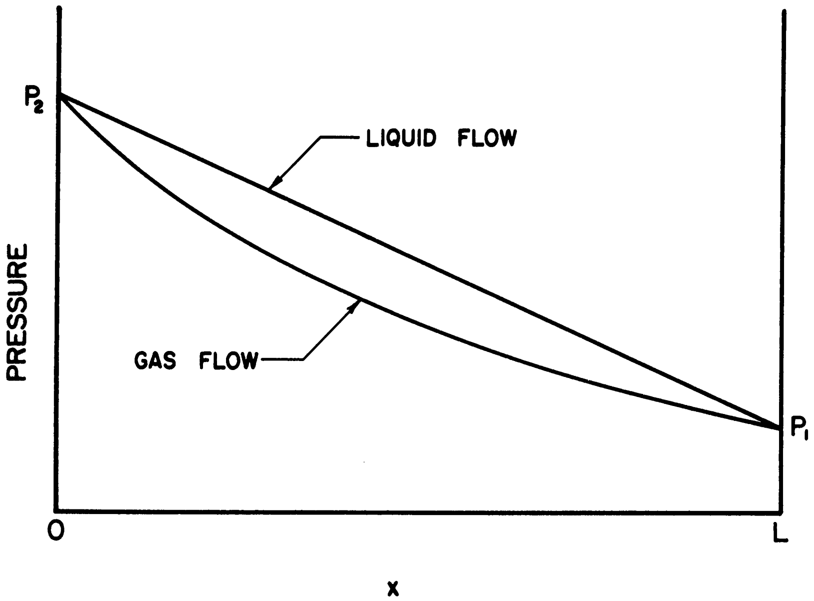


Figure VI-2. Pressure Distribution for Heterogeneous Steady-State Flow Model.

obtained. It can easily be shown, however, that in the steady state this model will involve no net inter-layer flow. Let the pressure at the inlet header be maintained at P_2, and that at the outlet header at P_1, with $P_2 > P_1$. Then the pressure distribution in either layer is that shown in Figure VI-2. This may be clearly seen by considering an impermeable barrier to be initially present at the interface, with the model permitted to reach steady state. Then the pressure distribution in each layer will become that of Figure VI-2. Upon removing this hypothetical barrier, there exists no pressure gradient tending to cause inter-layer flow. Such a steady-state model is thus useless for studying crossflow patterns.

The lack of crossflow in the steady-state also eliminates consideration of electric potential models, such as those used by Fatt, in this investigation. Such models are very useful in studying steady-state flow in systems of irregular geometry, and may also be used advantageously in studies of many heterogeneous systems in the steady state. They are not, however, well suited to the problems of this investigation.

The model best suited to studies of transient effects such as those herein considered is the heat transfer model. In this model the flowing "fluid" is heat, and pressure difference driving force is directly simulated by a temperature difference. As previously shown, a heat transfer model simulates the flow of a slightly compressible liquid in a porous media.

A. Design Considerations for Heat Transfer Models

One of the first decisions which must be made in designing any heat-transfer-analog model is that of selecting the material of construction.

The major factors to be considered in selecting the heat transfer medium are (1) the thermal diffusivity, $k/\rho C_p$, of the material, (2) the ease of fabrication of models from the material, and (3) the availability of the material in the desired size and shape at a reasonable cost.

The thermal diffusivity of the heat transfer medium is important in several respects. First of all, it is of major importance in determining the size of the model. As a general rule, the lower the diffusivity of the heat transfer medium, the smaller is the size of the model required for a given experiment. As model size decreases, problems of handling and generally problems of insulation also decrease. However, below a certain size, fabrication of the model, particularly with regard to placement of thermocouples, becomes very difficult.

Secondly, the thermal diffusivity of the heat transfer medium directly affects the type of insulation technique required. Transient response of materials of high thermal diffusity is usually sufficiently rapid that only superficial insulation is required in most models. However, with low thermal diffusivity materials, particularly those where the diffusity is of the same order of magnitude as common insulating materials, heat losses so serious as to invalidate the experiment may be all but impossible to prevent.

Thirdly, the variability of thermal diffusivity with temperature may be a factor in selecting the temperature level and temperature difference driving force at which a model may be successfully run.

A considerable range of materials could be used in building models. At the upper limit on thermal diffusivity are silver, with thermal conductivity, $k = 240$ Btu/hr ft^2 (°F/ft), and pure copper, with $k = 220$. At the lower limit in this respect are such material as hard rubber, k = 0.1 Btu/hr ft^2 (°F/ft) and pyroxylin plastics, with k = 0.075. Many materials with low diffusivities are, however, not suitable for the majority of models due to heterogeneity or anisotropy. For example, although various types of wood might seem attractive from the standpoint of ease of fabrication, the heterogeneity and anisotropy inherent in almost all types of wood eliminates the possibility of using it in any but highly specialized models.

The "ease of fabrication" involved with various heat transfer materials involves, to a large degree, the complexity of the model to be built. There are, however, certain important features which are involved in nearly all models.

One very important factor which must be considered in all heat-transfer-analog models is the means by which the model is to be heated (or cooled). This directly leads to the problem of bonding the heating (or cooling) medium to the model without introducing appreciable unwanted resistance at this interface.

The two most common methods of heating a model are electrical resistance heating and convection-conduction heating (by passing a hot fluid along some boundary). In electrical resistance heating, unless the entire model is heated internally by utilizing its own resistance, the medium in which heat is generated must be bonded in some manner to the medium to be heated. This is by no means a simple problem. It must be borne in mind that the models under discussion are all unsteady-state models, in which a given resistance has a variable temperature difference across it as a function of time. For example, if a heating tape, containing wires as heating elements, were attached to some part of the heat-transfer medium with epoxy resin, the temperature drop across the tape insulation plus resin would, in nearly all instances, be an unknown function of time. This could frequently prove troublesome, especially if it were desired to maintain the boundary or region in question at some fixed temperature.

Analogous problems are associated with convection-conduction type heating. With some materials, it might be possible to maintain the heating (or cooling) fluid in direct contact with the heat-transfer medium under investigation. This, however, is frequently impractical if only a small, isolated area is to be heated or cooled, with the remainder of the model being insulated. The alternative is to pass the fluid through some channel which is in turn connected to the model proper. Since contact resistances associated with pressure-type joints are nearly always variable with such factors as temperature difference, either a bonding agent such as solder must be used, or the two surfaces must be directly joined by welding.

Another "ease of fabrication" factor comes into play in comparatively large models, namely the ease of actually shaping the model. For example, the two-layer radial model which was studied in this investigation was fifty inches in diameter by one-half inch thick. The model was cut to circular shape on a Doall band-saw, starting with a rectangualr slab, 50" x 60", which weighed about 425 pounds. The weight of the slab, added to the fact that stainless steel is not easily cut, made this shaping a difficult task.

The third major factor in selecting material--that of cost and availability--also depends to a large extent on the type of model desired.

Little difficulty should be encountered in obtaining materials for small models, such as the single-layer linear model studied. However, surprisingly few satisfactory materials are readily available for building large two or three-dimensional models. This situation becomes extremely acute in the fabrication of multi-layer models, as will be explained later.

After selecting material and heating medium, the next major consideration is the means of temperature measurement. It seems apparent that the most satisfactory temperature sensing device in models of the type under discussion is the thermocouple. The type of thermocouple material which would be most satisfactory is usually apparent from consideration of the temperature level and temperature difference driving force at which the model is to be operated. Thus, the major problem in this area is the means of utilizing the thermocouples so as to obtain accurate temperature measurements.

The third major consideration is that of insulation. In simulating fluid flow by a heat-transfer-analog, a fluid flow boundary through which no fluid passes corresponds to a completely adiabatic boundary. In many models, particularly those fabricated of materials of high thermal diffusivity (e.g. copper), the short operating time required precludes serious heat losses. However, with models fabricated of steel, such as in this investigation, or other materials of comparably low thermal diffusivity, heat losses frequently prove to be the limiting factor in determining the length of time during which the model will yield useful data in a given run.

It is of interest to note the difference in the insulation criteria between steady-state and unsteady-state models. In steady-state models, the heat capacity of the insulation is unimportant; the purpose of the insulation is fulfilled if negligible heat losses to the surroundings occur, regardless of how much heat energy is present in the insulation itself. In unsteady-state models, however, the heat absorbed by the insulating medium itself may cause significant error, even though essentially no heat loss to the surroundings occurs.

The principal additional factor involved in the design of multilayer models is that of bonding at the interface between adjacent layers. This problem is closely associated with that of bonding the heat source (or sink) to the model in that any additional resistance introduced at the interface has a variable temperature difference associated with it. Thus, if a flux or bonding film (such as solder) is present at the interface in sufficient quantity to cause a measurable temperature difference across the bond, it is imperative to know the resistance of this bonding layer. A much superior alternative is, of course, to eliminate this resistance entirely, as was done in the two-layer models studied in this investigation.

The range of satisfactory materials which are readily available in a bonded condition is extremely limited. It was originally planned to fabricate the two-layer models in this investigation from copper and cupro-nickel. Depending on the type of cupro-nickel used, this combination of materials would have a thermal diffusivity ratio of five to eight. Moreover, the comparatively high thermal conductivity of both media would minimize insulation problems. Also, such materials would minimize physical problems of construction, since both copper and cupro-nickel are relatively easily cut and drilled. Unfortunately, however, it was not possible to obtain such materials in an already-bonded state without the presence of flux and other bonding materials at the interface.

In addition to the factor of interface-bonding, multi-layer models in general tend to make the other problems discussed even more critical. Consider, for example, the problem of insulation in a linear model. In the single-layer case, the only temperature gradient which should be present is in the direction of the length dimension of the model. Heat losses at the surface of the model naturally cause some gradients in the transverse direction, particularly near the surface of the model. By placing the thermocouples near the center of the model cross-section, it is possible to minimize the error caused by these heat losses. In multi-layer models, however, there should be temperature gradients in the transverse plane. Thus, the position of thermocouples with respect to the transverse axes of the model cannot be chosen arbitrarily. It is usually desired to place thermocouples near the surface of the model in order to measure cross-flow gradients. The heat-loss-induced surface temperature gradients, and consequently the insulation problems, are therefore more critical in the multi-layer case.

B. Description of Heat Transfer Models Studied in This Investigation

A total of five heat transfer models were studied in this investigation. These were

1. Single-layer linear model
2. Single-layer radial model
3. Single-layer radial "fault-plane" model
4. Two-layer linear model
5. Two-layer radial model

Schematic diagrams of these models (not drawn to scale) together with pertinent dimensions are shown in Figures VI-3 through VI-7. The steam fitting used at the heating end of the linear models is shown in Figure VI-8. Figure VI-9 shows two photographs of the "fault-plane" model.

All models were designed to simulate the "constant terminal pressure" case of liquid flow. For heat transfer models, this case corresponds to introducing a step function in temperature over some surface of an initially isothermal conducting medium.

1. Features Common to All Models

All models used in this investigation utilized steel as the conducting medium. For the single-layer homogeneous models, the entire conducting medium was mild steel. The two-layer models utilized both mild steel (k = 26) and 316L stainless steel (k = 9.3), bonded intimately at the interface. These two-layer models were cut from a 50" x 60" slab of clad steel in which 1/4" of 316L stainless was bonded to 1/4" of mild steel by hot rolling. This slab was generously donated by the Lukens Steel Company. The process of hot rolling produced a completely resistance-free interface for heat transfer.

All models were heated with steam at from 2-20 psig, which was drawn from a 60 psig steam line. With the models initially at room temperature, this provided a temperature difference driving force of approximately 170°F. Since the steam line which was used had a tendency to give considerable condensate in the steam, it was necessary to drain all possible condensate from this line before starting an experimental run. As an additional precaution, the line was continuously vented to the atmosphere at a point just ahead of the experimental models during all runs.

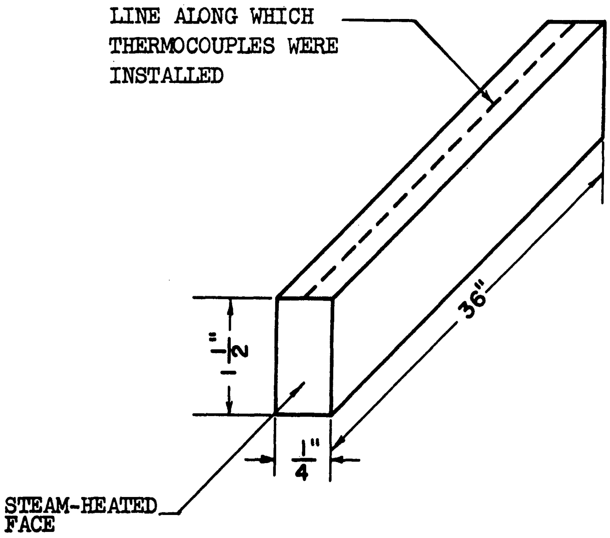


Figure VI-3. Single-Layer Linear Experimental Model.

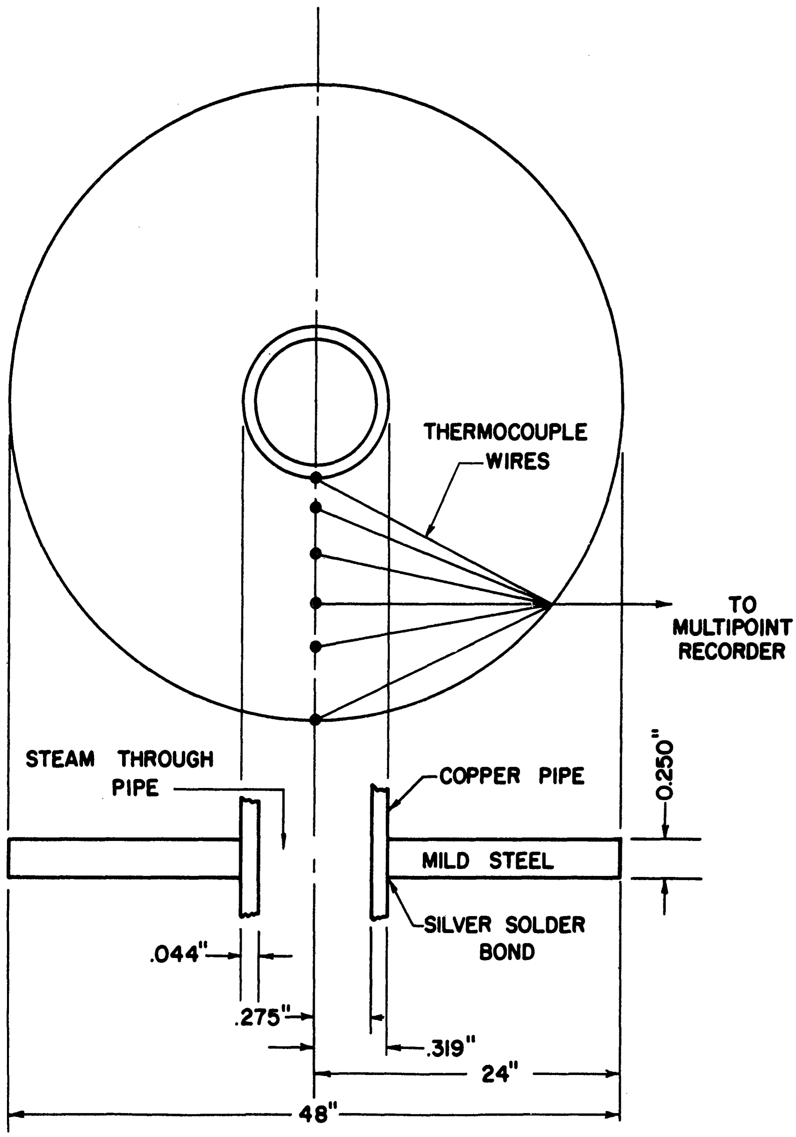


Figure VI-4. Single-Layer Radial Experimental Model.

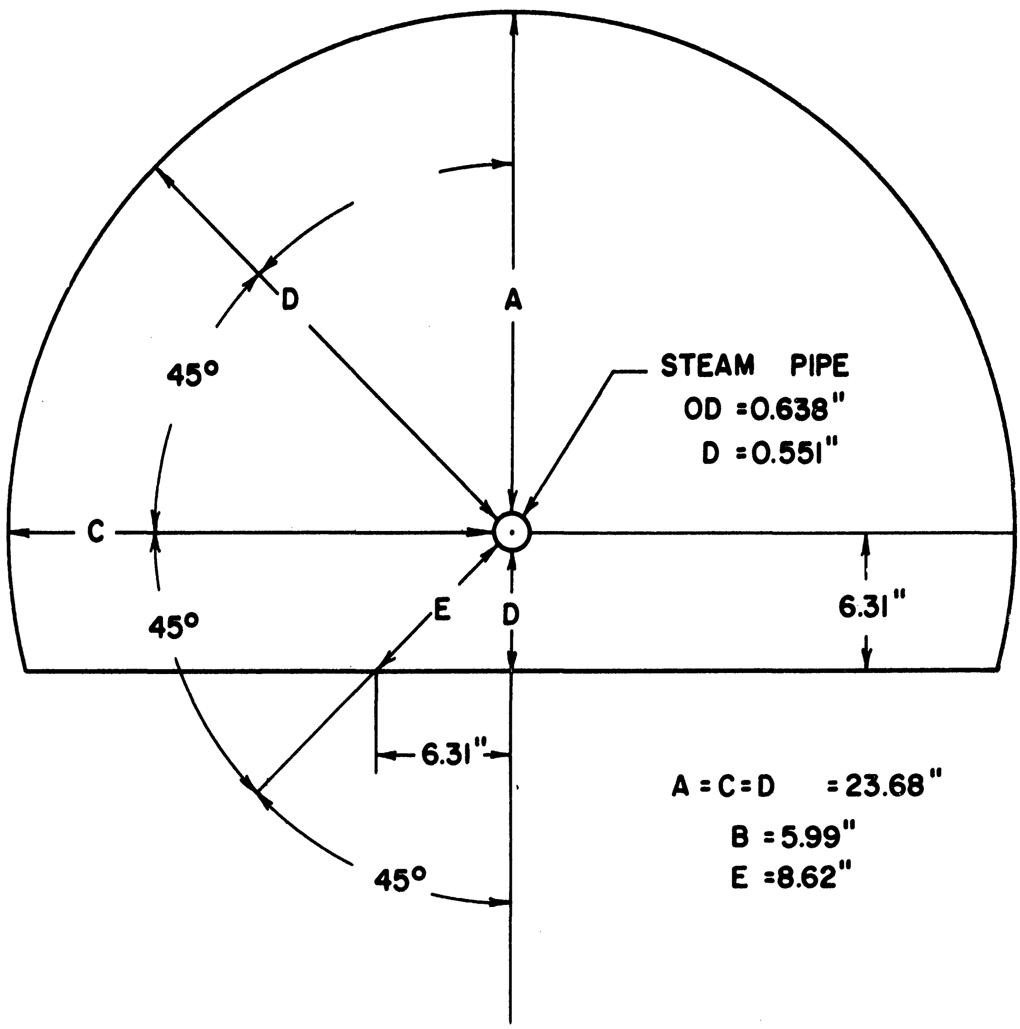


Figure VI-5. Single Layer Radial "Fault-Plane" Experimental Model.

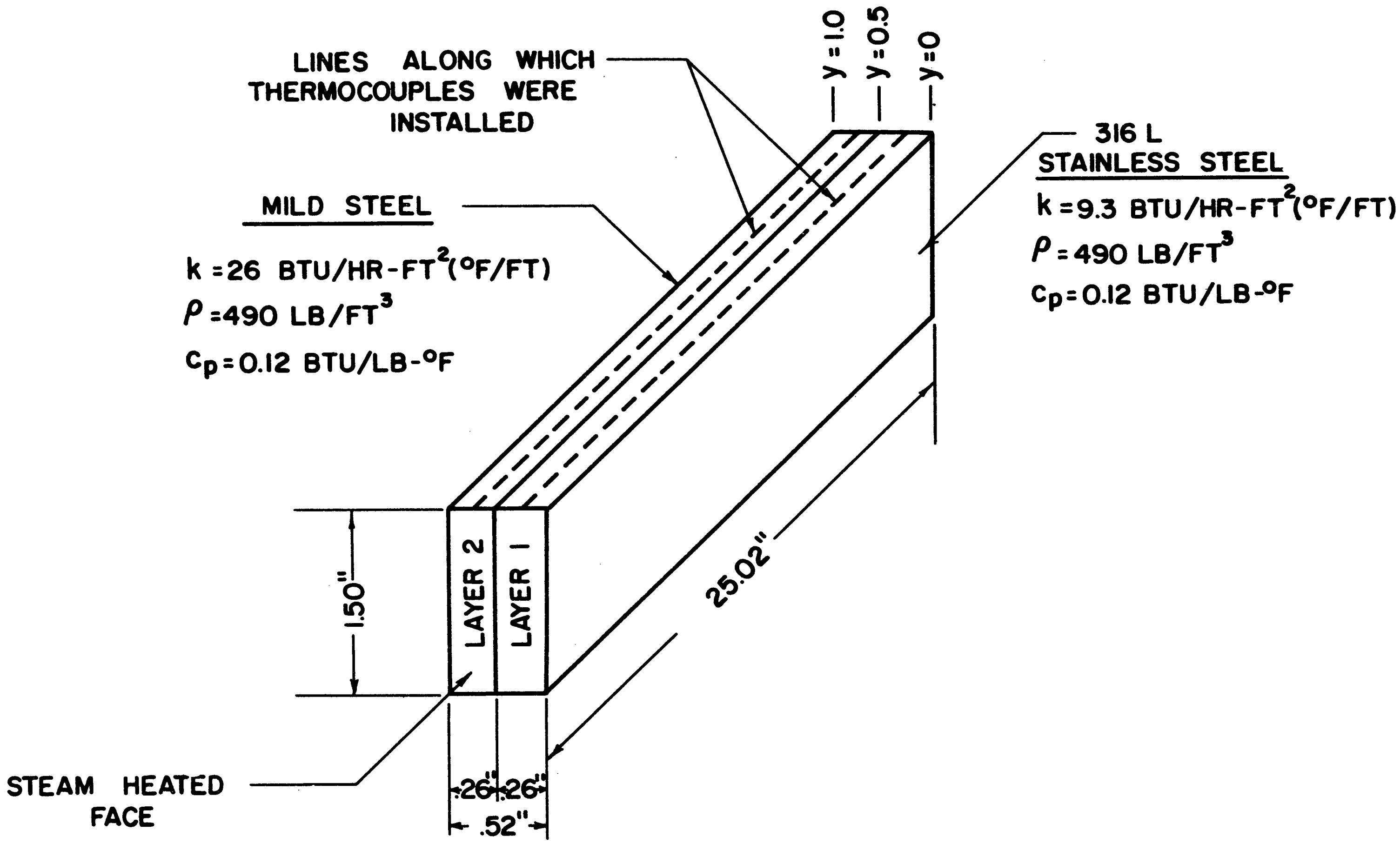


Figure VI-6. Two-Layer Linear Experimental Model.

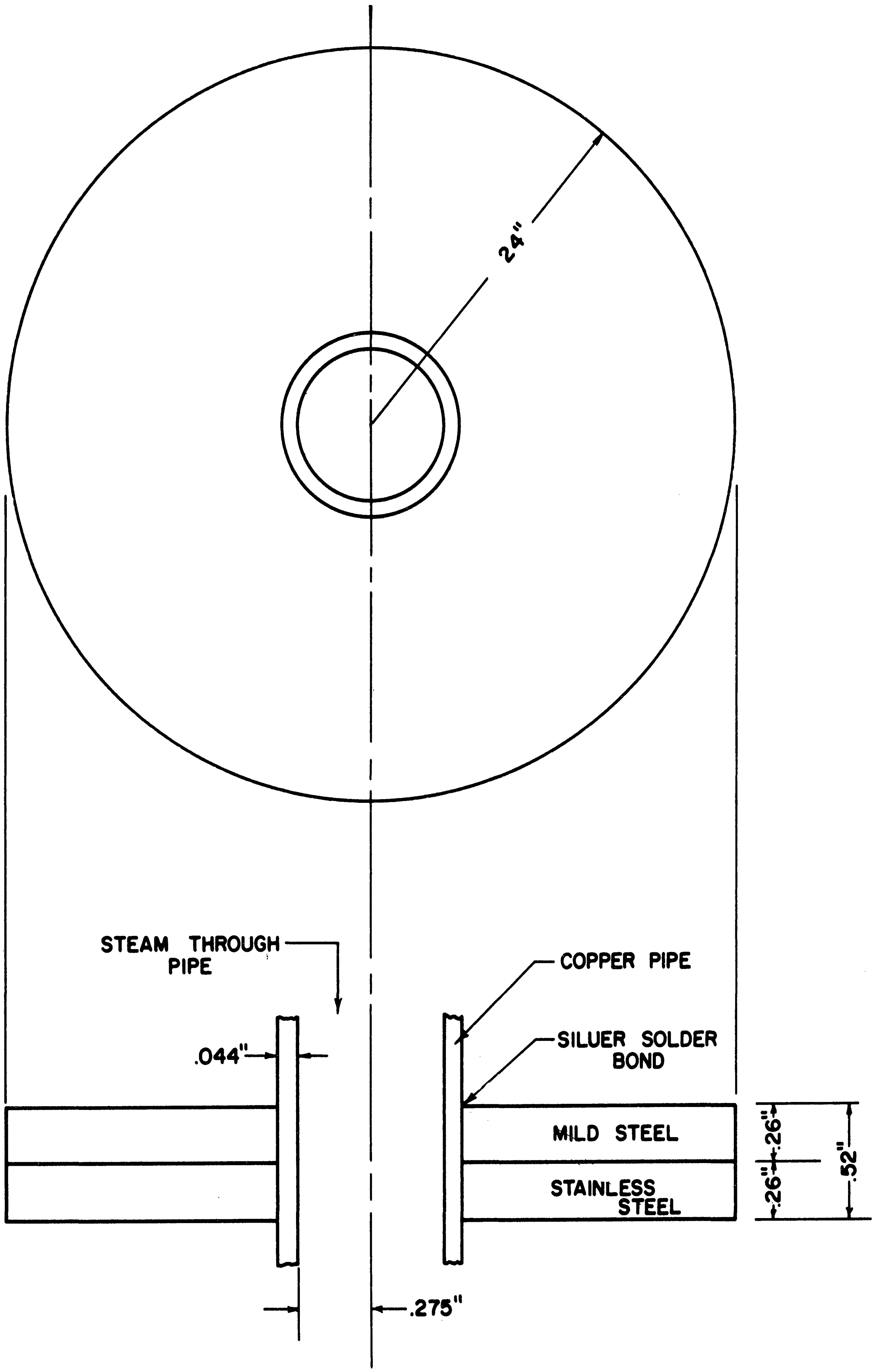


Figure VI-7. Two-Layer Radial Experimental Model.

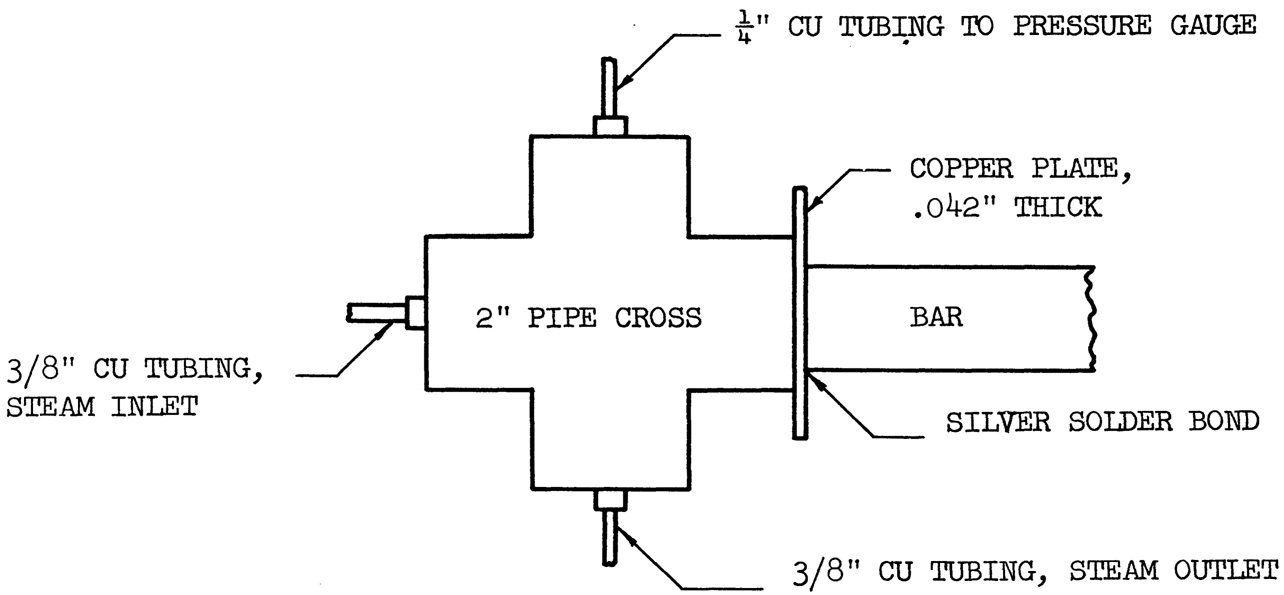
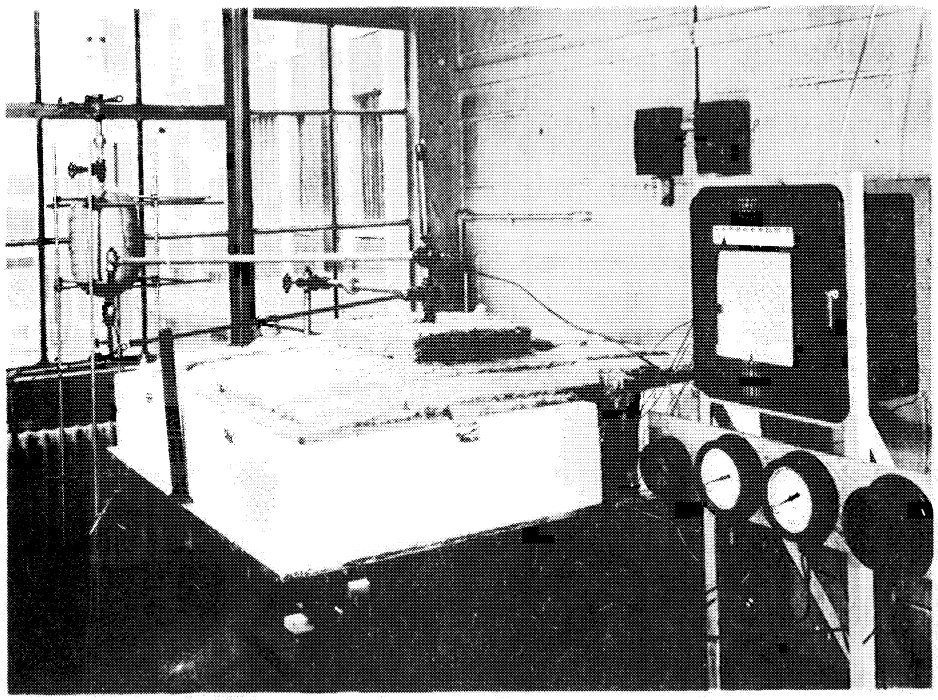


Figure VI-8. Steam Fitting for Linear Experimental Models.



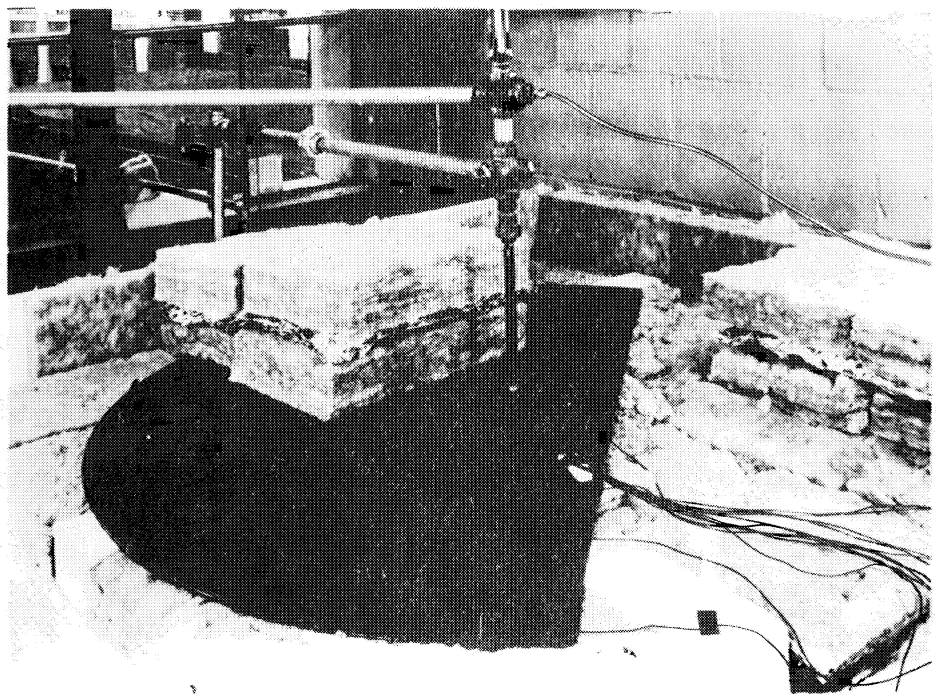


Figure VI-9 Photographs of the Single-Layer Radial Fault-Plane Experimental Model.

The steam pressure within the models was manually controlled by regulating the inlet and outlet valves on the model. A high steam velocity through the model was maintained in all cases. This high steam velocity was principally designed to prevent condensate build-up in the model. It had the additional effect of preventing the build-up of non-condensable gases within the model, and of increasing the steam condensing coefficient.

In all models, a thin layer of copper separated the steel conducting medium from the heating steam. In the linear models, a flat copper plate, .042 inches thick, was brazed across one opening of a two-inch pipe cross, and the steel bars were silver soldered to this plate (see Figure VI-8). In the radial models, a copper pipe with an outside diameter of 0.625" was used to contain the steam, with this pipe again being silver soldered to the steel conducting medium.

A twelve-point Leeds and Northrup Speedomax recording potentiometer was used in conjunction with copper-constantan thermocouples to record all temperature measurements. This recorder was a one second full-scale-balance type, and recorded a temperature every two seconds. A complete traverse of all twelve thermocouples was therefore completed every twenty-four seconds. The scale read directly in degrees Fahrenheit, with a scale range of 60-260°F. A chart speed of 60 inches per hour was used in order to separate the points recorded. A total of six different colored points were used to distinguish between thermocouples on the chart. The chart could be read within $\pm$ 0.5°F.

All thermocouples were calibrated at room temperature and in boiling water before installation in the models. The multi-point recorder was used as the sensing device in the calibration, so that the temperature measurement would reflect errors in either the thermocouple itself or in the recorder. No measureable difference between thermocouples was detected in any of these calibrations. The comparison between the thermocouple readings and that of a calibrated mercury thermometer was within 0.5°F at both room temperature and boiling water temperature. Since the recorder chart could only be read to $\pm$ 0.5°F, it was concluded that no correction due to thermocouple or recorder error was necessary. This does not, however, mean that the temperatures measured were exact, since this is a function not only of thermocouple and recorder error, but also of errors induced in the installation of the thermocouples in the model. This factor will be discussed in more detail at a later point in this dissertation.

The means of installing the thermocouples was essentially the same for all models. First, a very small thermocouple bead was formed, either by electric arc welding, or by careful soldering. A hole slightly larger in diameter than this bead was then drilled at the desired spot in the model. The depth of drilling was immaterial for all but the two-layer radial model, in which it was carefully controlled. The thermocouple wires near the bead were electrically insulated by coating them repeatedly with a solution of Duco cement thinned by acetone. The thermocouples were then placed in the hole, and held in place both by filling the void volume in the hole with additional Duco cement, and by using Scotch brand electrical tape to anchor the wires near the hole.

2. Individual Features of the Models

For the purposes of this section, the models will be discussed in the order in which they were actually fabricated and run.

a. Single-Layer Radial Model

As indicated, this was the first model to be fabricated and run. The diagram of this model appears in Figure VI-4.

This model was principally designed to check out the experimental technique to be used on the multi-layer models. The analytical solution for single-layer radial flow was unquestionably valid, so that there was no necessity to verify the solution using experimental data.

The thermocouples used with this model were 24 gauge copper-constantan. The thermocouples were installed with Duco cement as previously described. The thermocouple wire was then taped to the steel plate, leading in a direct line from the point of measurement to the multipoint recorder, as indicated on Figure VI-4.

Two quick-opening three-way valves were used with this model in an attempt to raise the heating face of the model to the desired temperature as rapidly as possible. The upper of these two valves can be seen in the photographs in Figure VI-9. A line diagram of the steam system for the model is shown in Figure VI-10. The steam was initially routed through the bypass line until the system became fairly well stabilized with respect to pressure variations. Both three-way valves were then turned simultaneously, to reroute the steam through the heating pipe of the model. The valve in the bypass line was used to equalize the pressure drops in the bypass line and in the heating pipe of the model. It was thus originally planned that the pressure equilibrium of the model should not be much disturbed when the steam was rerouted through the model itself. This did not, however, prove to be true, due to the considerable condensation of steam occurring in the heating pipe of the model in the first few seconds after steam was admitted to it.

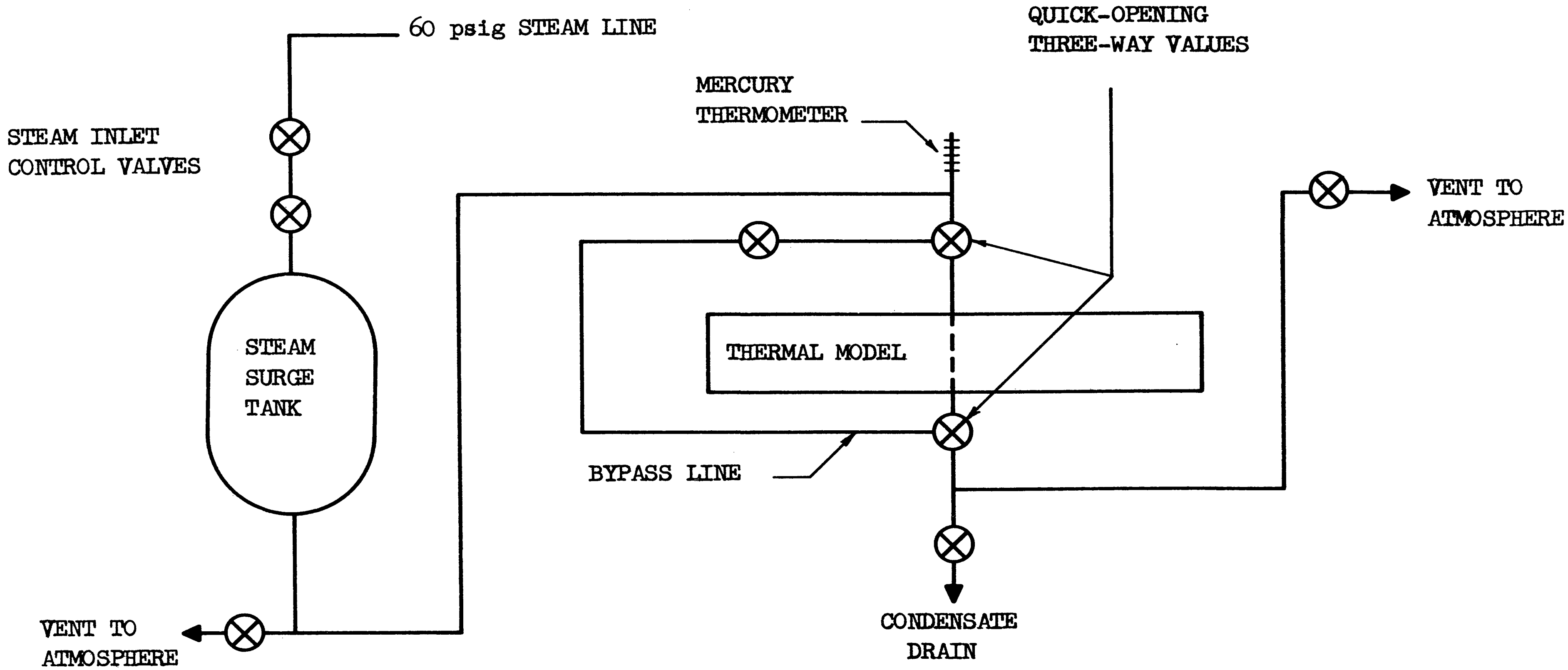


Figure VI-10. Line Diagram of Steam Apparatus for Single-Layer Radial and "Fault-Plane" Experimental Models.

This model was insulated by a combination of glass wool, styrofoam, and aluminum foil. The insulation actually touching the steel plate was glass wool, except for small styrofoam blocks used to support the plate. Outside one thickness (about four inches) of glass wool, a layer of aluminum foil was used to minimize radiation losses. Another layer of glass wool followed. The top and sides of the model were then covered with styrofoam, and the crevices between the styrofoam blocks were sealed with tape. The entire model rested on a plywood base. The insulation shown in the photographs in Figure VI-9 is nearly identical to that used for this model, except that in the fault-plane model the top was not covered with styrofoam.

Thermocouples were placed at various points in the insulation of the model in order to find points of maximum heat loss, and gain a qualitative idea of the efficiency of the insulation. The results thus obtained were entirely compatable with the deviation between analytical and experimental results which were found.

b. <u>Single-Layer Radial "Fault-Plane" Model</u>

This model is simply the single-layer radial model, with a slice cut from it in such a manner as to simulate a reservoir fault plane. The principal reason for running this model was to illustrate the application of heat transfer models to the study of irregular geometries.

The thermocouples used with this model were the same as for the single-layer radial model, but were installed in two different arrangements in order to gain a complete picture of the temperature distribution. The major difference in temperature measurement technique was the fact that the thermocouples were in this case taped to the model along an arc of approximately constant radius for at least six inches from the thermocouple bead. This technique was designed to decrease the error caused by conduction of heat by the thermocouple wire.

The insulation used for this model, along with the thermocouples used, are shown in Figure VI-9. As previously noted, the only difference from that used for the single-layer radial model is the absence of a styrofoam top layer.

The two three-way quick-opening valves, in conjunction with a steam bypass line, were also used with this model.

c. Single-Layer Linear Model

This was the third model to be fabricated and run. The primary purpose of the model was to refine the experimental techniques. In particular, the method of insulating linear models was thoroughly investigated. Also, this model made use of 30 gauge copper-constantan thermocouples, rather than the 24 gauge wires previously used.

The model was first constructed and run with no insulation in order to establish a limit on possible heat losses. For the next set of experimental runs, the bar was first wrapped with asbestos sheeting, then with aluminum foil (over the asbestos), and then covered on all sides with glass wool. This insulation was roughly comparable to that used in the two previous models. The results obtained checked moderately well with the analytical solution, but still indicated significant heat loss from the bar after a short period of time.

As previously mentioned, the problems of insulating unsteady-state models are often quite different from those for steady-state models.

Heat lost from the model due to heating of the insulation in an unsteady state model is frequently a major source of error, even though no heat is lost to the atmosphere beyond the insulation.

The ideal situation for unsteady-state insulation is to have the insulation immediately around a model heated by some external means to a temperature exactly equal to that of the model. This can be most accurately done by using a feedback loop, whereby thermocouples sense the temperature of the model, and the model temperature is used to actuate and regulate some electrical heating device. Such a system is usually extremely accurate, but is also quite expensive.

A much simpler, but very satisfactory, device was used in this investigation to heat the insulation surrounding the linear models. As shown in Figure VI-8, the linear models were heated at one end, at which they were silver soldered to a steam-heated copper plate. In order to heat the insulation from an "external" source (i.e. not the bar itself), a galvanized sheet steel "channel" was also soldered to this copper plate, as shown schematically in Figure VI-11. This sheet metal channel was then also heated by the copper plate, at approximately the same rate as the model. This is, at any time, the temperature at a given distance from the copper plate would be nearly the same in the bar and the sheet metal channel if there were no heat losses.

The bar was wrapped first with asbestos sheeting, then aluminum foil, and was supported inside the channel with styrofoam. The remaining void volume inside the channel was then filled with glass wool and/or powdered asbestos. The entire assembly was surrounded with approximately six inches of glass wool.

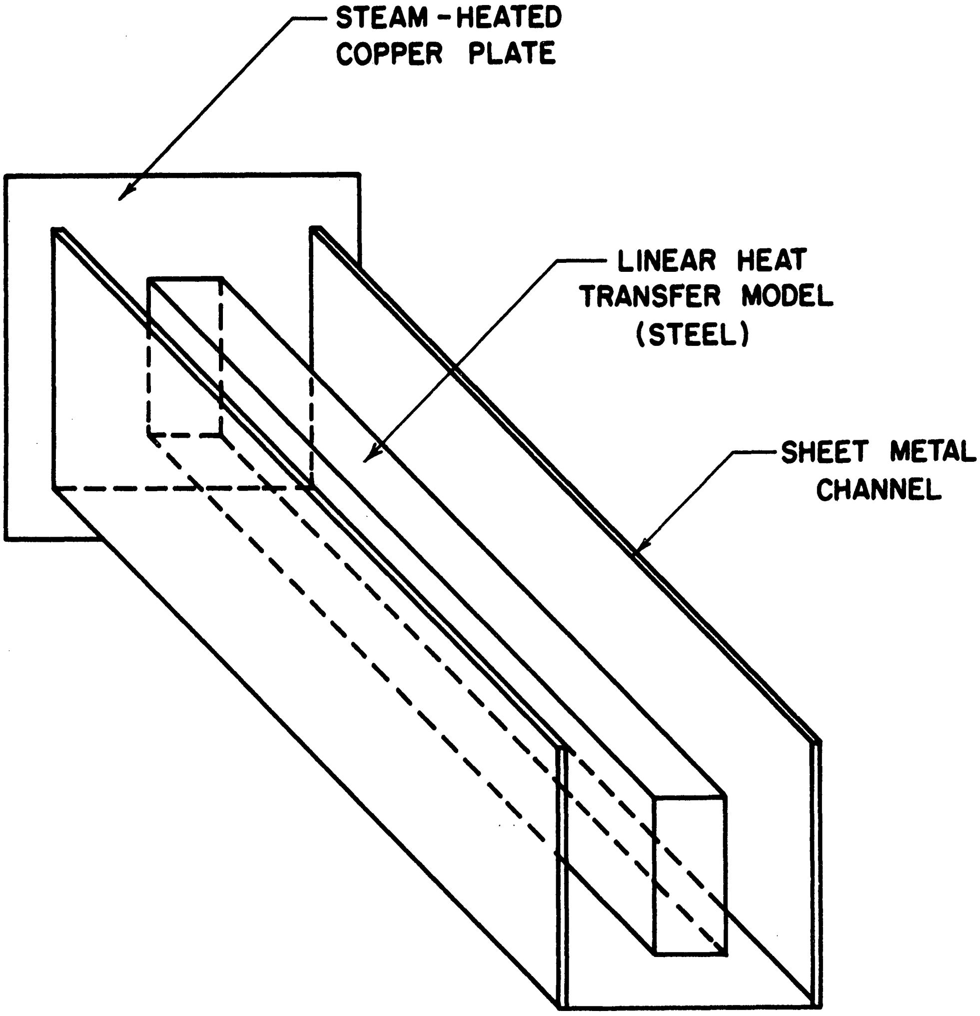


Figure VI-11. Sheet Metal "Channel" for Insulating Linear Experimental Models.

Since the sheet metal channel had a considerably greater surface area than the model proper (i.e. the steel bar), the insulation received the majority of the heat necessary to raise its temperature from the channel, rather than the bar. Heat losses from the bar were therefore minimized.

The thermocouple wires in this model were withdrawn vertically from the point of measurement. Since the wires were thus perpendicular to the direction of heat flow, they were thus approximately along an isotherm. This tended to minimize errors in temperature measurement due to heat conduction along the thermocouple wires.

Experience with the two single-layer radial models had indicated that the steam system incorporating quick-opening valves (see Figure VI-10) was not entirely satisfactory. The steam system was therefore modified to that shown in Figure VI-12 for the three remaining models. In this system, the steam was vented to the atmosphere immediately after the surge tank until the amount of condensate in the steam decreased to the minimum possible. To initiate the run, the steam pressure in the surge tank was allowed to build up to 20-35 psig, and the 3/8" globe valve was then rapidly opened. The resultant surge of steam through the model raised the heating face to the desired operating temperature in less than ten seconds. It was then necessary to simultaneously close the 3/8" globe valve and open the 1/4" needle valve, in such a manner as to maintain the temperature of the heating face constant. The temperature of the heating face was then controlled for the remainder of the run using the 1/4" needle valve.

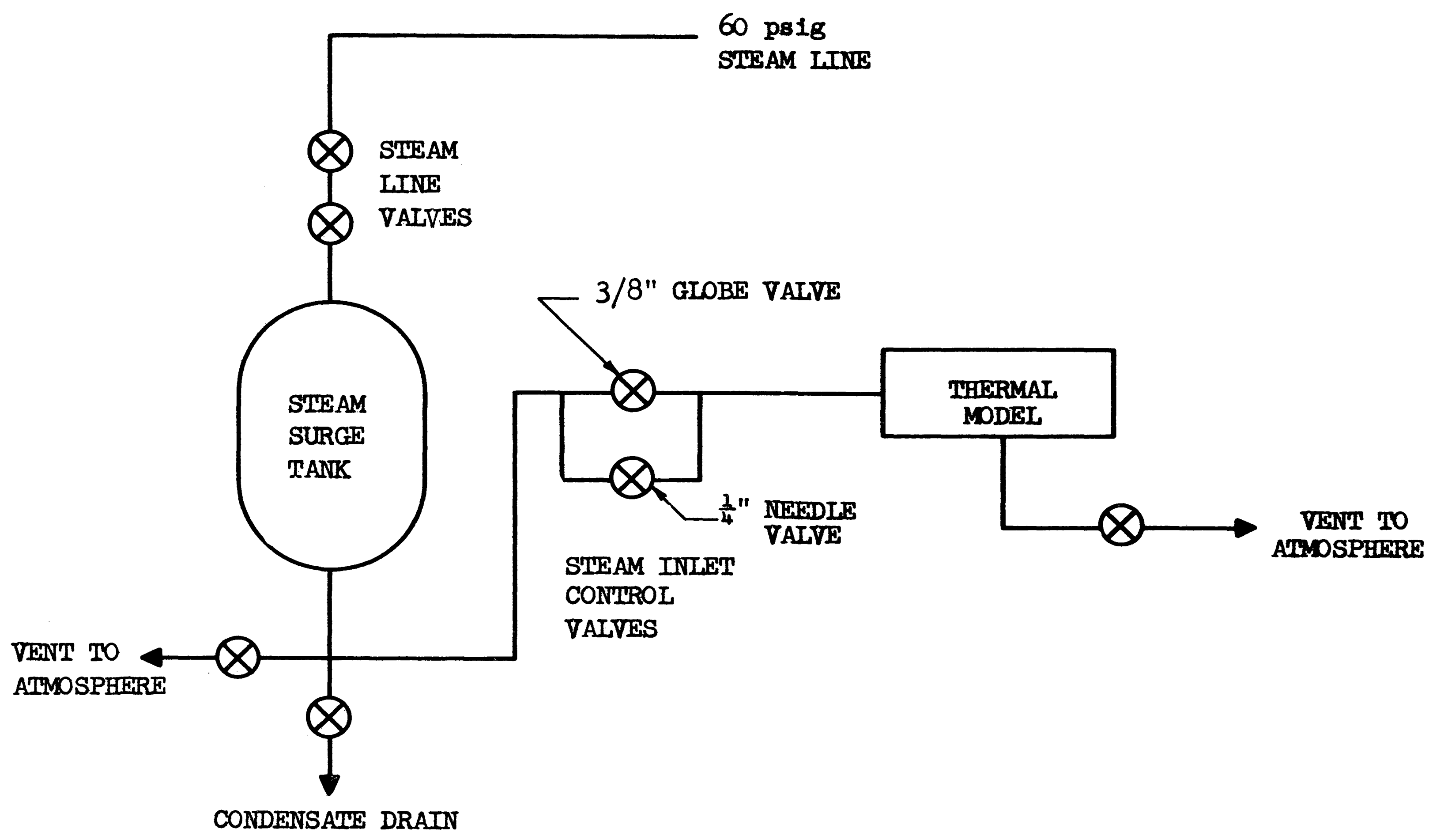


Figure VI-12. Line Diagram of Steam Apparatus for Both Linear and the Two-Layer Radial Experimental Models.

d. Two-Layer Linear Model

This was the first of the two two-layer models to be run. The model was nearly identical in design to the single-layer linear model, except for the bi-metal character of the bar. The insulation methods used, including the galvanized sheet steel channel, were identical.

As indicated in Figure VI-6, the thermocouples (30 guage copper-constantan) were inserted from the "edge" of the bar. In this way, it was unnecessary to carefully control the depth of the thermocouple bead, as was true for the two-layer radial model. The thermocouple wires were again withdrawn perpendicularly from the model in order to minimize the effect of heat conduction along these wires.

e. Two-Layer Radial Model

This model was the last to be run, and as such incorporated most of the improvements in experimental technique which were found from experience with the other models. This model was run first with the mild steel side up, with all thermocouples in the mild steel side, and was then inverted and rerun with thermocouples in both layers.

The fabrication of this model was the most difficult in several respects. First of all, the sheer weight of the metal plate (over 300 lbs.) made handling difficult. Second, and more important, this was the only model in which the depth to which the thermocouple bead was inserted had to be carefully controlled. This in turn required that the thermocouple bead be very small.

The mean of inserting thermocouples in this model is illustrated schematically in Figure VI-13. The ceramic insulation had a diameter of

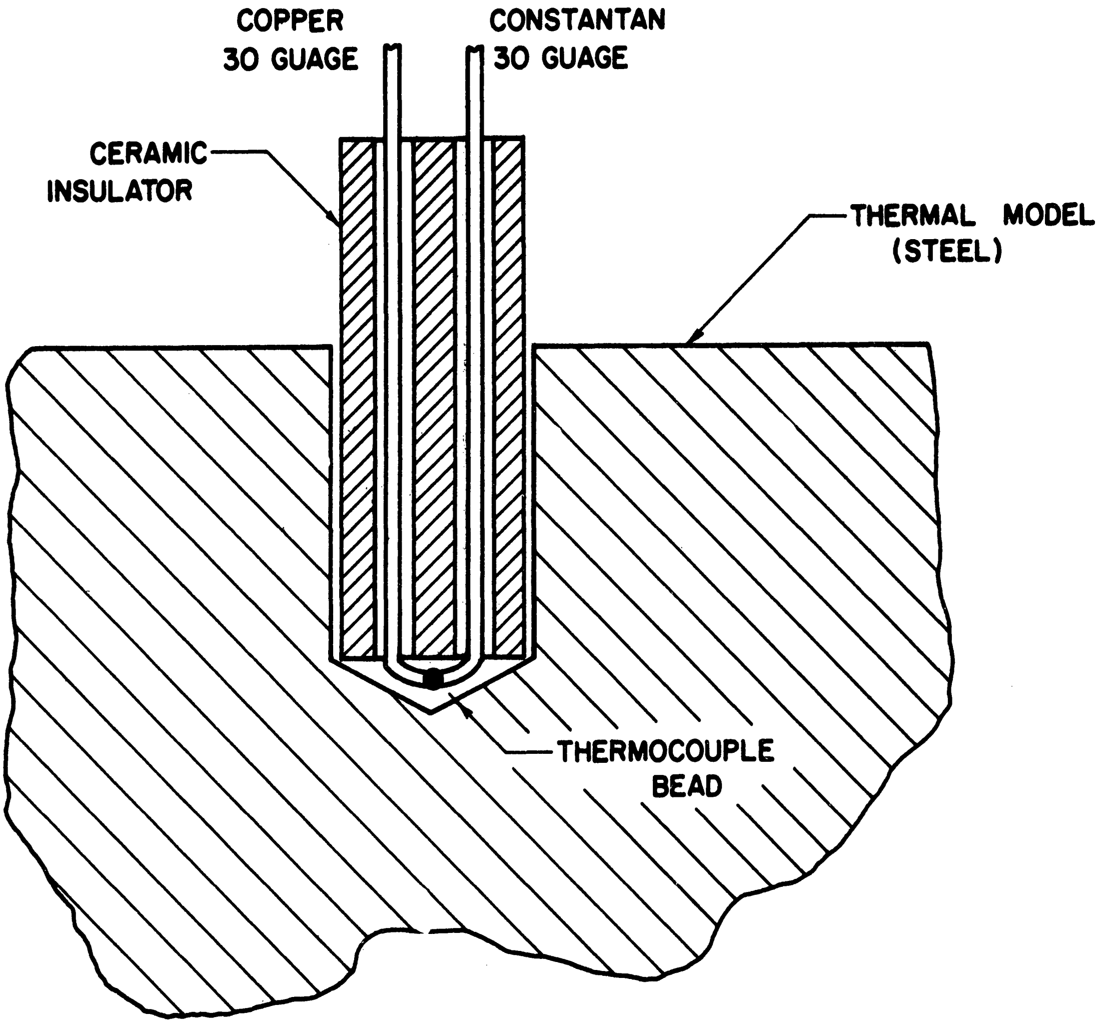


Figure VI-13. Thermocouple Installation in the Two-Layer Radial Experimental Model.

0.058 inches, and was predrilled to accept two 30 guage bare thermocouple wires (diameter 0.010 inches each). The thermocouple beads in this case were formed by electric arc welding under Xylene. The beads obtained were all of the order of 0.010 inches in diameter. By carefully controlling the depth of the hole drilled in the steel, and by inserting the ceramic insulation (together with the thermocouple bead) to the bottom of this hole, it was possible to position the thermocouple bead within ± 0.020 inches of depth. It should be noted that the thickness of each of the two layers of the model was 0.260 inches, so that the uncertainty in position of the thermocouple bead represented approximately 8% of the total thickness.

The ceramic elements were held in place in the model using Duco cement and Scotch brand electrical tape. The thermocouple wires were withdrawn perpendicular to the model surface in order to minimize errors in temperature measurement due to conduction along the thermocouple wire.

C. Experimental Data

The original data obtained from the experimental models in this investigation were obtained in the form of multi-point recorder charts. It should be noted that the pressure gauges associated with the experimental apparatus were used only to facilitate the manual control of the steam pressure in such a manner as to maintain a constant temperature at the heating face of the models.

The experimental data will be reported in two forms. First, the basic data, as obtained directly from the recorder charts will be presented. These data are in the form of actual temperatures as a function of actual time and actual position. Tables will also be presented of the dedimensionalized data, in the form of dimensionless temperatures as a function of dimensionless position and real time. This second set of tables is much more useful from the standpoint of interpretation, since it is difficult to readily note such factors as reproducibility using the tables of basic data.

As previously noted, the recorder chart speed used for the majority of the experimental runs was 60 inches per hour. In most cases the initiation of the temperature step function at time zero was accomplished sufficiently well to obtain a time scale of $\pm$ 0.25 minutes.

The conversion of actual temperatures, as recorded on the multipoint recorder chart, to dimensionless temperature was accomplished as follows:

1. Before each experimental run, the temperature at all points in the model was recorded using the multi-point recorder. The maximum gradient occurring as an initial condition in any of the runs was 2 °F over the maximum dimension of the model. In most cases, no detectable initial gradient was present. The values reported in the tables of basic data represent the average initial temperature of the model.
2. The time at which the temperature step-function was initiated was estimated from the recorder graph. This estimate was facilitated by always starting the steam flow through the model at a time 4 to 8 seconds before the first heating-face temperature measurement was due on the recorder. The time thus obtained was designated as $\Theta = t_a = 0$.

3. The actual temperature measured by all thermocouples was determined as a function of time, t_a, from the recorder chart. Near the beginning of the experimental run, during which time the temperatures near the heating face were changing fairly rapidly (i.e. up to 30°F during the 24 second recorder interval), it was necessary to draw a curve through the recorder points to facilitate interpolation. The temperatures were read from the chart within 0.5°F in most cases. These readings of actual temperatures are presented in Tables VI-1, VI-4, VI-6, VI-8, and VI-10 of this dissertation.

4. The actual temperatures, T_a, were converted to dimensionless temperatures, T, using the equation

$$T = 1 - \frac{T_a - T_i}{T_s - T_i} = \frac{T_s - T_a}{T_s - T_i} \qquad \text{(VI-1)}$$

in which:

T_a is the actual temperature, °F, obtained from the recorder chart

T_i is the corresponding initial temperature at that point, °F

T_s is the mean temperature of the heating face of the model over the course of the experimental run.

The heating face temperature was maintained within ± 2°F (after the first minute) in nearly all cases. Any extreme fluctuations in this temperature (as frequently occurred due to steam line surges, which in turn were due to the large amount of condensate) caused the run to be discontinued. The experimental runs reported herein constitute perhaps 30% of the runs which were actually begun on the various experimental models. The remainder of the runs were discontinued at a sufficiently early stage as to be totally worthless, or had some other distinct defect (such as a steam leak within the model insulation) which would negate their value.

The experimental data for the models in the form of dimensionless temperature as a function of real time, with parameters of dimensionless position, are presented in Tables VI-2, VI-5, VI-7, VI-9, and VI-11.

TABLE VI-1

BASIC EXPERIMENTAL DATA FOR THE SINGLE-LAYER RADIAL MODEL

Run Number	Heating-Face Temperature (°F)	Initial Condition (°F)
5	229.	84.5
7	230.	85.0
8	226.	82.5
9	233.	82.5
10	229.	74.0
11	242.	75.5

Actual Temperatures from Recorder Chart, °F

t_a min.	Run No. 5	Run No. 7	Run No. 8	Run No. 9	Run No. 10	Run No. 11
			r_a = 1.32 inches			
1	120.	120.	117.	119.	111.	117.
3	-	139.	136.	138.	130.5	137.
10	153.5	154.	151.	154.	147.	155.
60	169.	169.	167.	170.5	164.5	173.
90	172.	172.5	169.5	173.	167.5	176.
150						181.
225						185.
365						191.
			r_a = 2.57 inches			
1	95.	96.	92.	92.5	84.5	87.
3	111.	111.	108.	109.	101.	105.
10	127.5	128.	125.	127.5	120.	125.5
60	148.	148.5	145.5	148.5	142.	149.
90	152.	152.5	149.5	152.5	146.	153.
150						160.
225						165.
365						173.5
			r_a = 5.07 inches			
3		89.5	86.5	87.	79.	80.5
10	101.5	102.	99.	100.5	92.	95.
60	124.5	125.	122.	124.	117.	122.
90	129.	130.	127.	129.	122.	127.
150						133.5
225						143.
365						154.

t_a min.	Run No. 5	Run No. 7	Run No. 8	Run No. 9	Run No. 10	Run No. 11
			r_a = 10.07 inches			
10	87.	87.5	85.	85.5	77.	79.
60	103.5	104.5	102.	102.5	94.5	98.
90	109.	110.	107.	108.	100.5	104.
150						113.5
225						123.
365						136.
			r_a = 17.57 inches			
10	84.5	85.	82.5	82.5	-	-
60	91.	91.5	89.	89.5	81.	83.5
90	96.	96.5	94.	94.5	86.5	89.
150						99.
225						109.5'
365						123.5
			r_a = 24.00 inches			
10	84.	85.	82.5	82.5	-	80.
60	88.5	89.	86.5	86.5	78.	85.
90	93.	93.5	91.	91.	83.	95.5
150						106.
225						120.
365						

TABLE VI-2

DIMENSIONLESS TEMPERATURES FOR THE SINGLE-LAYER RADIAL MODEL

Dimensionless Temperatures

$$T = \frac{T_s - T_a}{T_s - T_i}$$

t_a min.	Run No. 5	Run No. 7	Run No. 8	Run No. 9	Run No. 10	Run No. 11	Weighted Average
1		.76	.760	.758	.761	.751	.758
3		.63	.627	.632	.635	.631	.631
10		.52	.523	.525	.529	.523	.525
60		.42	.412	.416	.416	.415	.415
90		.40	.394	.400	.396	.397	.397
225				.346		.342	.344
r = 8.04							
1	.93	.92	.934	.934	.932	.931	.933
3	.82	.82	.822	.824	.826	.823	.824
10	.70	.70	.704	.701	.703	.700	.702
60	.56	.56	.561	.562	.561	.559	.561
90	.53	.53	.533	.535	.535	.534	.534
225				.465		.463	.464
365						.411	.411
r = 15.85							
3		.97	.972	.970	.968	.970	.970
10	.89	.88	.885	.881	.884	.883	.883
60	.73	.72	.724	.725	.722	.721	.723
90	.69	.69	.690	.691	.690	.691	.690
225				.598		.595	.597
365						.529	.529

t_a min.	Run No. 5	Run No. 7	Run No. 8	Run No. 9	Run No. 10	Run No. 11	Weighted Average
10	.98	.98	.982	.980	.981	.979	.980
60	.87	.87	.864	.867	.868	.865	.867
90	.83	.83	.829	.831	.829	.829	.830
225				.718		.715	.716
365						.637	.637
r = 54.9							
10	1.00	1.00	1.000	1.000	1.000	1.000	1.000
60	.955	.955	.955	.954	.955	.952	.954
90	.92	.92	.920	.920	.919	.919	.920
225				.800		.796	.796
365						.712	.712
r = 75.2							
10	1.00	1.00	1.000	1.000	1.000	1.000	1.000
60	.97	.97	.972	.973	.974	.973	.973
90	.94	.94	.941	.944	.942	.943	.943
225							
365						.733	.733

TABLE VI-3

THERMOCOUPLE LOCATIONS FOR THE SINGLE-LAYER RADIAL FAULT-PLANE MODEL

Note: The letters A, B, C, D, E refer to the corresponding lines shown on Figure VI-5

Thermocouple No.	First Set of Thermocouples			Second Set of Thermocouples		
	Radius Vector	r_a inches	$r = \frac{r_a}{r_b}$	Radius Vector	r_a inches	$r = \frac{r_a}{r_b}$
2	A	1.32	4.11	D	1.28	4.01
3	B	1.32	4.11	E	1.32	4.11
4	A	3.18	9.98	A	3.18	9.98
5	B	3.26	10.22	B	3.26	10.22
6	C	3.32	10.41	D	3.84	12.05
7	A	6.23	19.52	E	3.84	12.05
8	B	6.23	19.52	D	8.34	26.2
9	C	6.23	19.52	E	6.32	19.83
10	A	15.32	48.1	D	15.32	48.1
11	B	15.32	48.1	E	8.76	27.5
12	C	24.00	75.2	A	24.00	75.2
	These locations were used for experimental runs 1, 2, 3, and 4.			These locations were used for experimental runs 5, 6, and 7.		

TABLE VI-4

BASIC EXPERIMENTAL DATA FOR THE SINGLE-LAYER RADIAL FAULT PLANE MODEL

Run Number	Heating-Face Temperature (°F)	Initial Condition (°F)
1	235.	85.
2	240.	101.
3	240.	74.
4	240.	75.
5	240.	70.
6	240.	78.
7	240.	74.5

Actual Temperatures from Recorder Chart, °F

First Set of Thermocouple Locations

t_a (min)	Thermocouple 2				Thermocouple 3				Thermocouple 4				Thermocouple 5				Thermocouple 6			
	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4
1	129	140	126	124	122	134	114	118	88	104	78	79	88	104	78	79	88	104	77	79
2	141	153	135	137	135	107	129	131	96	110	85	87	96	110	85	87	95	109	84	86
4	152	162	147	148	146	157	141	142	105	119	96	98	105	119	96	98	103	117	94	96
7	160	169	156	156	154	164	151	151	114	128	107	107	115	128	107	107	112	126	104	105
10	163	173	160	161	159	169	156	157	120	133	112	113	122	135	114	115	118	131	110	111
20	170	182	168	169	168	181	166	167	130	145	123	124	136	151	130	131	129	145	123	124
40	179	186	176	176	179	186	176	176	140	151	135	135	149	160	145	145	141	152	136	136
70	182		181	182	182		181	182	147		142	143	158		154	155	149		145	146
100			184				185				148				159				151	
200																				

t_a (min)	Thermocouple 7				Thermocouple 8				Thermocouple 9				Thermocouple 10				Thermocouple 11				Thermocouple 12		
	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 1	Run No. 3	Run No. 4
4	88	104	77	79	90	106	80	82	88	104	77	79											
7	93	109	84	84	99	114	90	90	93	108	83	84											
10	97	112	88	88	107	120	97	98	97	111	87	88											
20	106	122	98	98	122	139	115	116	106	123	98	99	86	102	76	77	86	102	76	77		75	
40	116	129	108	109	138	149	132	132	119	132	111	112	90	105	80	81	91	106	82	82	86	76	77
70	124		117	118	147		142	143	128		121	123	95		86	87	98		89	90	89	80	81
100			123				148				128				92				96			86	
200																							

Second Set of Thermocouple Locations

t_a (min)	Thermocouple 2			Thermocouple 3			Thermocouple 4			Thermocouple 5			Thermocouple 6			Thermocouple 7		
	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7
1	112	120	115	107	115	110	74	82	79	74	82	79	72	80	76	72	80	76
2	128	132	129	120	127	124	82	90	86	82	90	86	76	85	81	77	85	81
4	137	143	141	133	139	137	94	100	97	93	100	97	86	93	90	87	93	90
7	146	151	150	142	147	146	102	109	106	103	110	107	94	101	98	95	102	99
10	151	156	154	148	153	151	109	115	112	111	118	114	100	107	104	102	109	106
20	160	165	164	158	163	162	120	126	124	127	133	131	112	119	116	118	124	121
40	169	173	172	169	173	172	132	137	135	142	147	145	124	130	128	133	138	136
70		179	178		179	178		145	143		157	155		139	137		148	146
100																		
200																		

t_a (min)	Thermocouple 8			Thermocouple 9			Thermocouple 10			Thermocouple 11			Thermocouple 12		
	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7	Run No. 5	Run No. 6	Run No. 7
4	71			74	81	77				71					
7	73	81	77	80	87	83				75	82	79			
10	75	83	80	86	92	89				79	86	83			
20	83	90	87	101	107	104	72	80	76	92	99	96			
40	93	100	97	117	123	120	77	84	81	108	114	112	72	80	76
70		109	106		134	132		91	88		125	123		85	81
100															
200															

TABLE VI-5

DIMENSIONLESS TEMPERATURES FOR THE SINGLE-LAYER RADIAL FAULT PLANE MODEL

First Set of Thermocouple Locations

t_a (min)	Thermocouple 2 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 3 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 4 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 5 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 6 Run No. 1	Run No. 2	Run No. 3	Run No. 4
1	.706	.719	.722	.703	.753	.762	.759	.739	.980	.978	.975	.975	.980	.978	.975	.975	.980	.978	.981	.975
2	.626	.625	.632	.624	.666	-	.668	.660	.926	.935	.933	.927	.926	.935	.933	.927	.933	.942	.939	.933
4	.553	.561	.560	.557	.593	.597	.596	.593	.866	.870	.867	.860	.866	.870	.867	.860	.880	.884	.879	.872
7	.500	.510	.506	.509	.540	.546	.536	.539	.806	.805	.801	.806	.800	.805	.801	.806	.820	.820	.819	.818
10	.480	.482	.481	.478	.506	.510	.506	.503	.766	.769	.771	.769	.753	.755	.759	.757	.780	.784	.783	.781
20	.433	.417	.433	.430	.446	.424	.445	.442	.700	.683	.704	.703	.660	.640	.662	.660	.706	.683	.704	.703
40	.373	.388	.385	.387	.373	.388	.385	.387	.633	.640	.632	.636	.573	.575	.572	.575	.626	.633	.626	.630
70	.353		.355	.351	.353		.355	.351	.586		.590	.587	.513		.518	.515	.573		.572	.569
100			.337				.331				.554				.487				.536	

t_a (min)	Thermocouple 7 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 8 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 9 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 10 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 11 Run No. 1	Run No. 2	Run No. 3	Run No. 4	Thermocouple 12 Run No. 1	Run No. 2	Run No. 3	Run No. 4
4	.980	.978	.981	.975	.966	.964	.963	.957	.980	.978	.981	.975												
7	.946	.942	.939	.945	.906	.906	.903	.909	.946	.949	.945	.945												
10	.920	.920	.915	.921	.853	.863	.861	.860	.920	.928	.921	.921												
20	.860	.848	.855	.860	.753	.726	.753	.751	.860	.841	.855	.854	.993	.992	.987	.987	.993	.992	.987	.987	1.000	1.000	.993	1.000
40	.793	.798	.795	.793	.646	.654	.650	.654	.773	.776	.777	.775	.966	.971	.963	.963	.960	.964	.951	.957	.993	1.000	.987	.987
70	.740		.740	.739	.586		.590	.587	.713		.716	.709	.933		.927	.927	.913		.909	.909	.973		.963	.963
100			.704				.554				.674				.891				.867				.927	

Second Set of Thermocouple Locations

t_a (min)	Thermocouple 2 Run No. 5	Run No. 6	Run No. 7	Thermocouple 3 Run No. 5	Run No. 6	Run No. 7	Thermocouple 4 Run No. 5	Run No. 6	Run No. 7	Thermocouple 5 Run No. 5	Run No. 6	Run No. 7	Thermocouple 6 Run No. 5	Run No. 6	Run No. 7	Thermocouple 7 Run No. 5	Run No. 6	Run No. 7
1	.752	.740	.755	.782	.771	.785	.976	.975	.972	.976	.975	.972	.988	.987	.990	.988	.987	.990
2	.658	.666	.670	.705	.697	.700	.929	.925	.930	.929	.925	.930	.964	.956	.960	.958	.956	.960
4	.605	.598	.598	.629	.623	.622	.858	.864	.864	.864	.864	.864	.905	.907	.906	.900	.907	.906
7	.552	.549	.543	.576	.574	.567	.811	.808	.809	.805	.802	.803	.858	.858	.858	.852	.851	.851
10	.523	.518	.519	.541	.537	.537	.770	.771	.773	.758	.753	.761	.823	.820	.821	.811	.808	.809
20	.470	.462	.459	.482	.475	.471	.705	.703	.700	.664	.660	.658	.752	.746	.749	.717	.716	.719
40	.417	.413	.410	.417	.413	.410	.635	.635	.634	.576	.574	.574	.682	.679	.676	.629	.629	.628
70		.376	.374		.376	.374		.586	.586		.512	.513		.623	.622		.567	.567

t_a (min)	Thermocouple 8 Run No. 5	Run No. 6	Run No. 7	Thermocouple 9 Run No. 5	Run No. 6	Run No. 7	Thermocouple 10 Run No. 5	Run No. 6	Run No. 7	Thermocouple 11 Run No. 5	Run No. 6	Run No. 7	Thermocouple 12 Run No. 5	Run No. 6	Run No. 7
4	.994	1.000	1.000	.976	.981	.984				.994	1.000	1.000			
7	.982	.981	.984	.941	.944	.948				.970	.975	.972			
10	.970	.969	.966	.905	.913	.912				.947	.950	.948			
20	.923	.925	.924	.817	.820	.821	.988	.987	.990	.870	.870	.870			
40	.864	.864	.864	.723	.722	.725	.958	.962	.960	.776	.777	.773	.988	.987	.990
70		.808	.809		.654	.652		.919	.918		.709	.706		.956	.960

TABLE VI-6

BASIC EXPERIMENTAL DATA FOR THE SINGLE-LAYER LINEAR MODEL

Run Number	Heating-Face Temperature (°F)	Initial Condition (°F)
1	241.	73.5
2	241.	77.3
3	241.	71.
4	242.	72.8
7	240.	72.

Run No. 1: No Insulation on Bar

Runs No. 2 and 3: Asbestos, Aluminum Foil, and Glass Wool Insulation. No Sheet Steel Channel.

Runs No. 4 and 7: Asbestos, Aluminum Foil, Sheet Steel Channel, and Glass Wool Insulation.

Actual Temperatures from Recorder Chart, °F

x_a = 1.8 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
1	118.	112.	109.	110.	111.
2	141.5	138.	135.	137.	136.5
4	162.5	162.5	160.	162.	161.
7	175.	179.	176.5	180.	178.5
10	181.	188.	185.5	189.	188.
20	189.5	201.	199.5	204.	202.5
40	193.5	210.	209.5	214.	212.5
70	195.	215.5	215.	219.5	218.
82		217.			
100			217.	223.5	220.5
110					
150					
225			219.5		224.5

x_a = 3.6 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
1	79.	80.	74.5	76.	76.
2	93.	93.	87.5	90.	89.
4	115.	115.	110.5	112.	111.5
7	133.	136.	131.5	134.	132.5
10	143.	148.	144.5	146.5	146.
20	158.	170.	167.	171.	170.
40	165.5	186.	184.	189.5	188.
70	169.	195.	194.	199.5	198.
82		197.5			
100			198.5	206.	203.
110					
150					
225			204.	210.5	

x_a = 5.4 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
2	77.	79.	73.	75.	74.5
4	88.	89.	84.	85.5	85.
7	102.5	105.	99.5	102.	101.
10	112.	117.	112.	114.	114.
20	130.	141.	137.5	142.	140.5
40	140.	162.	159.5	165.5	164.
70	144.	175.	173.	180.	178.5
82		178.	-		
100			179.	189.	185.5
110					
150					
225			188.		196.5'

x_a = 7.2 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
4	78.	80.5	74.5	76.5	75.5
7	87.5	90.	84.	86.	85.
10	96.	99.	93.5	95.5	95.
20	114.	122.	117.5	121.	120.
40	125.	144.5	141.	146.5	146.
70	130.	160.	157.	164.	162.5
82		163.5			
100			165.	175.	172.
110					
150			172.		
225			177.		185.

x_a = 10.8 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
7	76.	79.	72.5	74.5	73.5
10	79.	82.	76.	77.5	76.5
20	90.5	95.	90.	92.5	91.5
40	102.	115.5	111.	116.	115.
70	108.	132.5	129.	135.	134.
82		137.			
100			139.	149.	145.
110					
150			148.		
225			155.		163.5

x_a = 14.4 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
10	74.7	83.	71.5	73.5	72.5
20	80.	96.5	77.	79.5	78.
40	89.	111.5	91.5	95.	94.
70	94.	116.	107.	112.5	111.5
82					
100			117.5	127.	123.
110					
150			127.5		
225			136.5		144.

x_a = 18.0 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
20	76.	79.5	73.	74.5	73.5
40	82.	87.	81.5	83.5	83.
70	87.	99.	94.	98.	97.
82		102.5			
100			103.5	111.5	108.
110					
150			113.5		
225			123.4		131.

x_a = 21.6 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
20	75.		72.		
40	78.5	81.5	76.	77.5	76.5
70	82.5	89.5	84.5	87.	86.
82		92.5			
100			92.5	98.5	95.
110					
150			102.5		
225			112.5		118.

x_a = 25.2 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
20	74.5				
40	76.5	79.3	73.5	75.	74.
70	80.	84.5	79.	81.	80.
82		86.5			
100			85.5	90.5	87.5
110					
150			94.5		
225			105.		110.

x_a = 28.8 inches

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
20	74.5				
40	75.5	78.	72.5	73.5	73.
70	78.5	81.	76.	77.5	76.5
82		82.5			
100			81.	85.	82.
110					
150			89.		
225			99.5		104.

TABLE VI-7

DIMENSIONLESS TEMPERATURES FOR THE SINGLE-LAYER LINEAR MODEL

Run No. 1: No Insulation or Bar

Runs No. 2 and 3: Asbestos, Aluminum Foil, and Glass Wool Insulation. No Sheet Steel Channel

Runs No. 4 and 7: Asbestos, Aluminum Foil, Sheet Steel Channel, and Glass Wool Insulation

t_a (min)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 7
			$x = \frac{x_a}{L_a} = 0.05$					x = .10					x = 1.5		
1	.734	.786	.774	.788	.768	.967	.982	.976	.982	.976	1.000	1.000	1.000	1.000	1.000
2	.594	.627	.621	.622	.616	.884	.902	.900	.900	.899	.979	.988	.985	.988	.985
4	.469	.477	.474	.475	.470	.752	.768	.764	.770	.765	.913	.927	.921	.926	.923
7	.394	.376	.376	.369	.366	.645	.639	.641	.640	.640	.827	.829	.829	.829	.827
10	.358	.322	.324	.315	.310	.585	.566	.565	.566	.560	.770	.755	.756	.758	.750
20	.308	.242	.241	.228	.223	.495	.431	.432	.421	.416	.663	.609	.606	.593	.593
40	.284	.187	.184	.168	.164	.451	.334	.332	.313	.310	.603	.480	.476	.454	.453
70	.275	.154	.150	.135	.131	.430	.279	.274	.254	.250	.579	.401	.397	.369	.366
82		.144					.264					.383			
100			.139		.116			.247		.220			.362		.325
110				.112					.215					.315	
150															
225			.124		.093			.215		.176			.309		.259
			x = .20					x = .30					x = .40		
4	.973	.982	.976	.976	.979	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
7	.916	.924	.921	.920	.923	.985	.991	.991	.988	.991	1.000	1.000	1.000	1.000	1.000
10	.866	.869	.865	.864	.863	.967	.972	.971	.970	.973	.993	1.000	.997	.994	.997
20	.758	.728	.724	.714	.714	.899	.893	.888	.882	.884	.961	.967	.965	.959	.964
40	.692	.591	.585	.563	.560	.830	.768	.764	.743	.744	.907	.884	.879	.867	.869
70	.663	.495	.491	.460	.461	.794	.664	.659	.631	.631	.878	.792	.794	.764	.765
82		.474					.636					.762			
100			.444		.405			.680		.565			.726		.696
110				.395					.548					.678	
150			.404					.547					.668		
225			.374		.328			.506		.455			.615		.571
			x = .50					x = .60					x = .70		
20	.985	.990	.988	.988	.991	.991	1.000	.997	1.000	1.000	.994	1.000	1.000	1.000	1.000
40	.949	.942	.938	.935	.935	.970	.976	.974	.970	.973	.982	.989	.988	.985	.988
70	.919	.869	.865	.849	.851	.946	.927	.924	.914	.917	.961	.957	.956	.950	.952
82		.847					.908					.945			
100			.808		.786			.876		.863			.918		.908
110				.770					.847					.894	
150			.750					.818					.865		
225			.691		.649			.759		.726			.803		.774
			x = .80				x = 1.0								
20	.994	1.000	1.000	1.000	1.000		1.000	1.000							
40	.988	.997	.997	.994	.994		.997	.997							
70	.970	.979	.976	.970	.973		.991	.991							
82		.969					.985								
100			.947		.940			.968							
110				.926											
150			.900					.926							
225			.838		.809			.868							

TABLE VI-8

BASIC EXPERIMENTAL DATA FOR THE
TWO-LAYER LINEAR MODEL

Run Number	Heating-face Temperature (°F)	Initial Condition (°F)
1	240.	71.
2	241.	71.3
3	243.	79.
4	242.	77.2
5	242.	81.5
6	242.	74.5

Actual Temperatures from Recorder Chart, °F

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 5	Run No. 6
	x_a=1.70inches; y_a=0.39inches					
1	97.	97.	107.	102.5	110.5	103.5
2	119.	119.	127.	123.	130.	125.5
4	142.	141.	149.	147.	151.5	147.5
7	161.	159.	166.	164.	168.	164.5
10	171.	170.	176.	175.	177.	174.
20	190.	190.	194.	193.5		191.
40	205.	205.	207.	207.		204.5
70	213.	213.	217.	215.		213.
100	217.	217.		218.5		217.
190		221.5		221.5		
	x_a=4.96inches; y_a=0.13, 0.39inches					
2		72.	80.		82.5	75.
4	77.5	78.	86.	84.	88.5	81.
7	90.	90.	97.	95.	100.	93.
10	101.	101.	107.	105.5	109.	104.
20	126.5	126.	131.	129.5		127.
40	152.5	153.	155.	154.		151.
70	169.	170.	171.	171.		162.
100	178.	179.		179.		176.
190		190.5		190.		
	x_a=20.00inches; y_a=0.39inches					
40	73.5	73.5	80.5	79.		
70	80.	80.	86.5	85.		81.5
100	89.	89.		92.5		89.
190		110.5		112.		

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 5	Run No. 6
	x_a=1.70inches; y_a=0.13inches					
1	97.	97.	107.	100.	107.5	100.
2	119.	119.	127.	120.	127.	122.
4	142.	141.	149.	143.5	148.	144.
7	161.	159.	166.	161.	164.5	161.
10	171.	170.	176.	172.	174.	171.
20	190.	190.	194.	191.		189.
40	205.	205.	207.	205.		203.
70	213.	213.	217.	214.		212.
100	217.	217.		217.5		216.
190		221.5		220.5		
	x_a=9.97inches; y_a=0.13, 0.39inches					
10	73.	73.	80.5	79.	83.	76.
20	82.	82.	89.	87.5		84.
40	101.5	101.	106.	105.		102.
70	121.	121.	124.	124.		120.
100	133.	133.		135.		131.
190		152.		152.5		
	x_a=14.91inches; y_a=0.13inches					
40	72.5	72.5		77.5		
70	76.5	76.5		81.5		78.
100	83.5	83.5		87.5		84.
190		104.5		105.5		

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 5	Run No. 6
	x_a=2.98inches; y_a=0.39inches					
1	74.	75.	83.	81.	85.5	79.
2	85.	86.	93.	90.5	96.	89.5
4	105.	105.	111.	110.	114.5	109.
7	125.	124.	130.	129.	132.	128.
10	138.	137.	142.	141.	144.	140.
20	162.	162.	165.	165.		163.
40	183.	183.	185.	184.		181.
70	195.5	196.	197.	197.		194.
100	201.5	202.		202.		200.
190		209.		208.5		
	x_a=4.96inches; y_a=0.13, 0.39inches					
20	72.5	73.	80.	78.5		75.5
40	80.5	80.	87.	85.		82.
70	93.5	93.	98.	97.5		93.
100	104.5	105.		107.		104.
190		126.		127.		

TABLE VI-9

DIMENSIONLESS TEMPERATURE DISTRIBUTION FOR THE TWO-LAYER LINEAR MODEL

t_a (min)	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 5	Run No. 6	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 5	Run No. 6	Run No. 1	Run No. 2	Run No. 3	Run No. 4	Run No. 5	Run No. 6
	x = 3.27; y = 0.75						x = 3.27; y = 0.25						x = 5.73; y = .75					
1	.840	.847	.829	.848	.819	.830	.840	.847	.829	.864	.838	.851	.976	.976	.976	.979	.975	.973
2	.711	.718	.707	.724	.692	.698	.711	.718	.707	.742	.716	.720	.912	.912	.915	.921	.910	.910
4	.575	.588	.573	.579	.564	.567	.575	.588	.573	.600	.585	.589	.794	.800	.805	.803	.794	.794
7	.464	.482	.469	.475	.461	.465	.464	.482	.469	.494	.483	.486	.675	.688	.689	.688	.685	.681
10	.404	.418	.408	.409	.405	.409	.404	.418	.408	.427	.424	.426	.599	.612	.616	.615	.611	.609
20	.292	.300	.299	.297		.308	.292	.300	.299	.313		.320	.457	.465	.475	.470		.472
40	.203	.212	.220	.215		.227	.203	.212	.220	.228		.236	.334	.341	.354	.355		.365
70	.166	.165	.159	.166		.176	.166	.165	.159	.173		.182	.260	.265	.281	.275		.287
100	.134	.142		.145		.152	.134	.142		.151		.159	.225	.230		.245		.250
190		.115		.127				.115		.134				.189		.206		
	x = 9.57; y = 0.25, 0.75						x = 19.17; y = 0.25, 0.75						x = 28.7; y = 0.25, 0.75					
1	1.000	1.000	1.000	1.000	1.000	1.000												
2	1.000	.994	.994	1.000	.994	.997												
4	.959	.959	.957	.961	.956	.961												
7	.885	.888	.890	.894	.885	.890												
10	.820	.823	.829	.830	.829	.824	.988	.988	.991	.988	.991	.988	1.000	1.000	1.000	1.000	1.000	1.000
20	.670	.676	.683	.685		.686	.935	.935	.939	.936		.940	.994	.991	.994	.991		.991
40	.516	.518	.536	.536		.543	.820	.823	.836	.830		.832	.947	.950	.951	.951		.952
70	.419	.418	.439	.433		.448	.705	.706	.725	.715		.723	.870	.874	.884			.887
100	.365	.365		.385		.394	.634	.635		.648		.660	.805	.803		.818		.821
190		.297		.318				.524		.542				.679		.697		
	x = 38.5; y = 0.75						x = 46.7; y = 0.25											
40	.991	.988	.991	.988		1.000	.997	.997		.997		1.000						
70	.953	.950	.956	.951		.955	.973	.973		.973		.976						
100	.900	.897		.906		.910	.932	.932		.936		.940						
190		.771		.788				.809		.827								

TABLE VI-10

BASIC EXPERIMENTAL DATA FOR THE TWO-LAYER RADIAL MODEL

Runs No. 1, 2, and 3 were made with the mild steel side up, all thermocouples in the mild side.

Runs No. 4, 5, 6, and 7 were made with the stainless steel side up, with thermocouples in both layers.

Run Number	Heating-face Temperature (°F)	Initial Condition (°F)
1	224.	73.5
2	224.5	81.
3	223.	82.
4	222.	79.
5	222.	78.
6	225.	80.
7	225.	78.9

Actual Temperatures from the Recorder Chart, °F

t_a (min.)	Run No. 1	Run No. 2	Run No. 3	Run No. 1	Run No. 2	Run No. 3	Run No. 1	Run No. 2	Run No. 3	Run No. 1	Run No. 2	Run No. 3
	r_a=1.26in.; y_a=0.39in.			r_a=2.20in.; y_a=0.39in.			r_a=3.13in.;y_a=0.39in.			r_a=4.39in.; y_a=0.39in.		
1	107.	113.5	114.	81.5	88.5	90.	75.	82.	83.5			
2	120.	125.	125.5	91.	98.	99.5	79.5	86.5	88.	75.	82.	83.
4	132.	134.5	135.	103.	108.	109.	88.	94.	95.	79.	85.5	87.
7	139.5	142.	142.5	112.	116.5	117.	96.	101.5	102.5	84.5	91.	92.
10	144.	146.	147.	117.5	121.5	122.	101.5	106.5	107.	89.	95.	96.
20	151.	153.5	154.	127.	131.	131.5	112.	116.5	117.	98.5	104.	105.
40	157.	161.	160.5	135.	140.	140.	120.4	126.	126.5	107.5	113.5	114.5
70	162.5	165.	165.	141.5	145.5	145.	128.	132.5	133.	116.	121.	121.5
100	164.	167.5	167.	144.	149.	148.5	131.	136.5	136.5	119.	125.	125.
200		172.5			155.			143.5			133.	
300		176.			159.5			149.			139.	
	r_a=6.26in.;y_a=0.39in.			r_a=12.51in.;y_a=0.39in.			r_a=18.76in.;y_a=0.39in.			r_a=25.00in.;y_a=0.39in.		
4	74.5	81.5	83.									
7	77.	83.5	85.									
10	80.	86.	87.									
20	87.	93.	94.	74.5	82.	83.						
40	95.	101.5	102.5	77.	85.	85.5		81.5	82.5			
70	103.5	108.5	109.5	83.	89.5	90.	76.5	83.	84.	75.	81.5	82.5
100	106.5	113.5	113.5	83.5	93.5	93.5	78.	85.5	86.	75.5	83.5	83.5
200		122.0			102.5			96.			91.	
300		128.			109.5			101.5			98.5	

t_a (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7	Run No. 4	Run No. 5	Run No. 6	Run No. 7	Run No. 4	Run No. 5	Run No. 6	Run No. 7	Run No. 4	Run No. 5	Run No. 6	Run No. 7	Run No. 4	Run No. 5	Run No. 6	Run No. 7
	r_a=1.26in.;y_a=0.13in.				r_a=1.26in.;y_a=0.39in.				r_a=2.20in.;y_a=0.13in.				r_a=3.13in.;y_a=0.13in.				r_a=3.13in.;y_a=0.39in.			
1	107.5	104.	106.	105.5	113.	110.	112.5	112.	86.	84.	85.5	85.	80.	78.5	81.	80.	80.5	79.	81.5	80.5
2	118.5	116.	119.	118.	123.5	121.5	124.	123.	94.5	92.5	95.	94.	81.5	82.	84.	83.	85.	83.	85.5	84.5
4	129.	128.	130.5	129.5	133.	132.	135.	134.	104.5	103.	105.	104.5	90.	88.5	91.	90.	92.5	90.5	93.	92.5
7	138.	136.5	139.5	138.5	140.	139.	142.	141.	113.5	112.	114.	113.	97.	96.	98.	97.	100.	98.	100.5	100.
10	142.5	141.5	144.5	143.5	144.5	143.5	146.5	145.5	118.5	117.	120.	119.	102.	101.	103.	102.	104.5	103.5	106.	105.
20	150.5	149.5	153.	152.	151.5	150.5	154.	153.	128.	127.	130.	128.5	112.	110.5	113.	112.5	114.5	113.	115.5	115.
40	157.5	156.	159.	158.5	158.5	157.	160.	159.5	137.	135.	138.	137.5	121.5	120.	123.	122.	123.	122.	124.5	124.
70		161.	163.	162.5		161.5	164.	163.5		141.	143.5	143.		127.	129.5	128.5		128.	130.5	130.
100		163.	166.	165.		163.5	166.5	166.		144.5	147.	147.		131.	133.5	133.		132.	134.5	134.
200				169.5				170.5				152.5				139.5				140.5
	r_a=18.76in.;y_a=0.13in.				r_a=6.26in.;y_a=0.13in.				r_a=25.00in.;y_a=0.13in.				r_a=12.51in.;y_a=0.13in.				r_a=12.51in.;y_a=0.39in.			
4					80.	78.5	81.	79.5												
7					81.5	80.5	83.	81.5												
10					84.	82.5	85.	84.												
20					90.	89.	91.5	90.					80.	78.5	81.	80.	80.	78.5	81.	80.
40	80.		80.5	79.5	98.5	97.	100.	99.					83.	81.5	83.5	82.5	83.	81.5	84.	83.
70		80.	82.	81.		104.	106.5	105.5		79.	80.5	79.5		85.5	87.5	86.		86.	88.	87.
100		82.	84.5	83.		108.5	111.	110.		80.5	82.5	81.		89.	91.	89.5		89.5	91.5	90.5
200				90.5				118.5				87.5				98.				99.

TABLE VI-11

DIMENSIONLESS TEMPERATURES FOR THE TWO-LAYER RADIAL MODEL

Runs No. 1, 2, and 3 were made with all thermocouples in the mild steel layer.

Runs No. 4, 5, 6, and 7 were made with thermocouples in both layers.

r=4.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
1	.777	.773	.773
2	.691	.693	.691
4	.611	.627	.624
7	.561	.575	.571
10	.532	.547	.539
20	.485	.495	.490
40	.445	.442	.444
70	.409	.414	.411
100	.399	.397	.397
200		.362	
300		.338	

r=7.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
1	.947	.948	.943
2	.884	.882	.876
4	.804	.812	.809
7	.744	.752	.752
10	.768	.718	.716
20	.644	.651	.649
40	.591	.589	.589
70	.548	.550	.553
100	.532	.526	.529
200		.484	
300		.453	

r=10.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
1	.990	.993	.989
2	.960	.962	.957
4	.904	.909	.908
7	.851	.857	.855
10	.814	.822	.823
20	.744	.752	.752
40	.688	.686	.684
70	.638	.641	.638
100	.618	.613	.613
200		.565	
300		.526	

r=14.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
2	.990	.993	.993
4	.964	.969	.965
7	.927	.930	.929
10	.897	.902	.901
20	.834	.840	.837
40	.774	.773	.770
70	.718	.721	.720
100	.698	.693	.695
200		.637	
300		.595	

r=20.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
2	1.000	1.000	1.000
4	.993	.997	.993
7	.977	.983	.979
10	.957	.965	.965
20	.910	.916	.915
40	.857	.857	.855
70	.801	.808	.805
100	.781	.773	.777
200		.714	
300		.672	

r=40.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
10	1.000	1.000	1.000
20	.993	.993	.993
40	.977	.972	.975
70	.937	.941	.943
100	.920	.913	.918
200		.850	
300		.801	

r=60.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
10	1.000	1.000	1.000
20	1.000	1.000	1.000
40	1.000	.997	.996
70	.980	.986	.986
100	.970	.969	.972
200		.895	
300		.857	

r=80.0; y=.75

ta (min.)	Run No. 1	Run No. 2	Run No. 3
40	1.000	1.000	1.000
70	.990	.997	.996
100	.987	.983	.989
200		.930	
300		.878	

r=4.0; y=.25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
1	.801	.819	.821	.818
2	.724	.736	.731	.732
4	.650	.653	.652	.654
7	.587	.594	.590	.592
10	.556	.559	.555	.559
20	.500	.504	.496	.500
40	.451	.459	.455	.455
70		.424	.427	.429
100		.410	.406	.411
200				.380

r=4.0; y=.75

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
1	.762	.778	.776	.774
2	.689	.698	.696	.698
4	.622	.625	.621	.624
7	.574	.576	.572	.575
10	.542	.545	.541	.545
20	.493	.496	.490	.494
40	.444	.451	.448	.449
70		.420	.420	.421
100		.406	.404	.404
200				.374

r=7.0; y=.25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
1	.951	.958	.962	.959
2	.892	.899	.897	.897
4	.822	.827	.828	.825
7	.758	.764	.766	.767
10	.724	.729	.724	.726
20	.657	.660	.655	.661
40	.595	.604	.600	.599
70		.563	.562	.562
100		.539	.538	.534
200				.496

r=10.0; y=.25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
1	.993	.997	.993	.993
2	.983	.972	.972	.973
4	.923	.927	.924	.925
7	.874	.875	.876	.877
10	.839	.840	.841	.843
20	.769	.774	.772	.770
40	.703	.708	.704	.706
70		.660	.658	.661
100		.632	.631	.630
200				.586

r=10.0; y=.75

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
1	.990	.993	.990	.990
2	.958	.965	.962	.962
4	.906	.913	.910	.908
7	.853	.861	.859	.856
10	.839	.840	.841	.843
20	.752	.757	.755	.753
40	.692	.694	.693	.692
70		.653	.651	.651
100		.632	.631	.630
200				.579

r=60.0; y= .25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
20	1.000	1.000	1.000	1.000
40	.993	1.000	.997	.997
70		.986	.986	.986
100		.625	.624	.623
200				.921

r=20.0; y=.25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
4	.993	.997	.993	.997
7	.983	.983	.979	.983
10	.965	.969	.965	.966
20	.923	.924	.921	.925
40	.864	.868	.862	.863
70		.819	.817	.819
100		.972	.969	.973
200				.729

r=80.0; y=.25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
70		.993	.997	.993
100		.788	.786	.788
200				.938

r=40.0; y=.25

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
20	.993	.997	.993	.993
40	.972	.976	.976	.976
70		.948	.948	.952
100		.924	.924	.928
200				.870

r=40.0; y=.75

ta (min.)	Run No. 4	Run No. 5	Run No. 6	Run No. 7
20	.993	.997	.993	.993
40	.972	.976	.972	.973
70		.944	.945	.945
100		.920	.921	.921
200				.863

VII DISCUSSION OF RESULTS

A. Comparison of Experimental Data and Analytical Results

The comparison between the experimental data and the results obtained from the mathematical analysis can be made most conveniently using graphical methods. Three of the thermal models which were studied can be compared directly with an analytical solution by plotting the dimensionless temperature distribution of the models as a function of time, with parameters of position, and superimposing the corresponding curves from the mathematical analysis.

For the single-layer radial and single-layer linear models, the comparison of results serves to indicate the accuracy of the experimental technique. In these cases, the analytical solutions are known to be valid. Any deviation from the analytical solution which is observed is thus due to experimental error. A knowledge of this experimental error is important in interpreting results for the two-layer linear model. The data obtained on the two-layer linear model serve as a verification of the corresponding analytical solution.

The data for the single-layer "fault-plane" model have no direct bearing on the subject of this dissertation. As previously noted, data were obtained on this model solely to illustrate the use of heat transfer models in studies of systems of irregular geometry. No analysis of the data, beyond the process of dedimensionalization, was therefore attempted.

The dimensions of the two-layer radial model, in particular the ratio of the radius at the heating surface to the thickness of the model

(r_b/H), unfortunately fall in a range where the analytical solution obtained for two-layer radial flow does not readily yield valid numerical results. As discussed in Section V of this dissertation, the convergence of the analytical solution for values of (r_b/H) less than 1.0 was extremely slow. For this model, (r_b/H) was 0.61. Comparison of results was therefore not possible. It should be explained that all of the experimental data which appear in this dissertation were obtained before valid mathematical analyses were completed. The model dimensions were chosen in an attempt to simulate an actual reservoir system, since at that time the range of convergence of the analytical solution was unknown.

For the single-layer linear model, the dimensionless temperature distribution from the analytical solution could be plotted directly from the values presented in Appendix F, without interpolation. For the single-layer radial model, the dimensionless temperatures from Appendix G were interpolated by plotting dimensionless time, θ, as a function of dimensionless radius, r. The analytical solution for the values of r used in the model was then read from these plots.

In the case of the two-layer linear model, values of dimensionless temperature at the values of x in the model were computed directly from the analytical solution, in order to increase the accuracy of comparison. These results are presented in Table VII-1.

The comparison of results for the single-layer radial model is shown in Figure VII-1. The average of the experimental data (dotted line) for this case appears to indicate a lower dimensionless temperature at a given time and position than is obtained from the analytical solution (solid line). This is true in all cases except for $r = 4.13$. This discrepancy is believed to be primarily due to the following two factors.

TABLE VII-1

DIMENSIONLESS TEMPERATURE DISTRIBUTION FOR THE TWO-LAYER LINEAR MODEL, AS CALCULATED FROM THE ANALYTICAL SOLUTION

	x=3.27		x=5.73		x=.957		x=19.17		x=28.7	x=38.5	x=46.7
	y=.25	y=.75	y=.25	y=.75	y=.25	y=.75	y=.25	y=.75	all y	all y	all y
.10											
.15	1.000	1.000									
.2	1.000	.999									
.3	.998	.996									
.4	.994	.989									
.5	.985	.980									
.6	.974	.967	1.000	1.000							
.7	.961	.953	1.000	.999							
.8	.947	.938	.999	.999							
1.0	.916	.907	.997	.996							
1.5	.840	.832	.986	.984							
1.5	.840	.832	.986	.984	1.000	1.000					
2.	.776	.769	.966	.963	1.000	.999					
3.	.677	.673	.916	.913	.996	.996					
4.	.608	.604	.866	.863	.987	.987					
5.	.555	.553	.819	.817	.974	.973					
6.	.514	.512	.778	.776	.958	.957					
7.	.481	.479	.742	.740	.941	.940	1.000	1.000			
8.	.454	.452	.709	.708	.922	.921	.999	.999			
10.	.411	.409	.655	.654	.886	.884	.998	.998			
15.	.340	.339	.559	.558	.802	.801	.990	.990	1.000		
20.	.297	.286	.495	.495	.735	.734	.974	.974	.999		
30.	.244	.244	.414	.414	.637	.637	.931	.931	.994	1.000	
40.	.212	.212	.363	.363	.569	.569	.885	.885	.982	.998	1.000
50.	.190	.190	.327	.327	.519	.519	.842	.841	.965	.995	.999
60.	.174	.174	.300	.300	.480	.480	.802	.802	.946	.990	.997
70.	.161	.161	.278	.278	.448	.448	767	.767	.926	.983	.994
80.	.151	.151	.261	.261	.422	.422	.735	.735	.905	.974	.990
100.	.135	.135	.234	.234	.381	.381	.681	.681	.864	.952	.975
150.	.110	.110	.192	.192	.315	.315	.584	.583	.773	.884	.918
200.	.095	.095	.166	.166	.274	.274	.515	.515	.696	.810	.847
300.	.076	.076	.132	.132	.219	.219	.415	.415	.570	.671	.704
400.	.062	.062	.108	.108	.179	.179	.340	.340	.468	.552	.580
500.	.051	.051	.089	.089	.147	.147	.279	.279	.384	.454	.477
600.	.042	.042	.073	.073	.120	.120	.229	.229	.316	.373	.392
700.	.034	.034	.060	.060	.099	.099	.189	.189	.259	.306	.322
800.	.028	.028	.049	.049	.081	.081	.155	.155	.213	.252	.265
1000.	.019	.019	.033	.033	.055	.055	.105	.105	.144	.170	.179

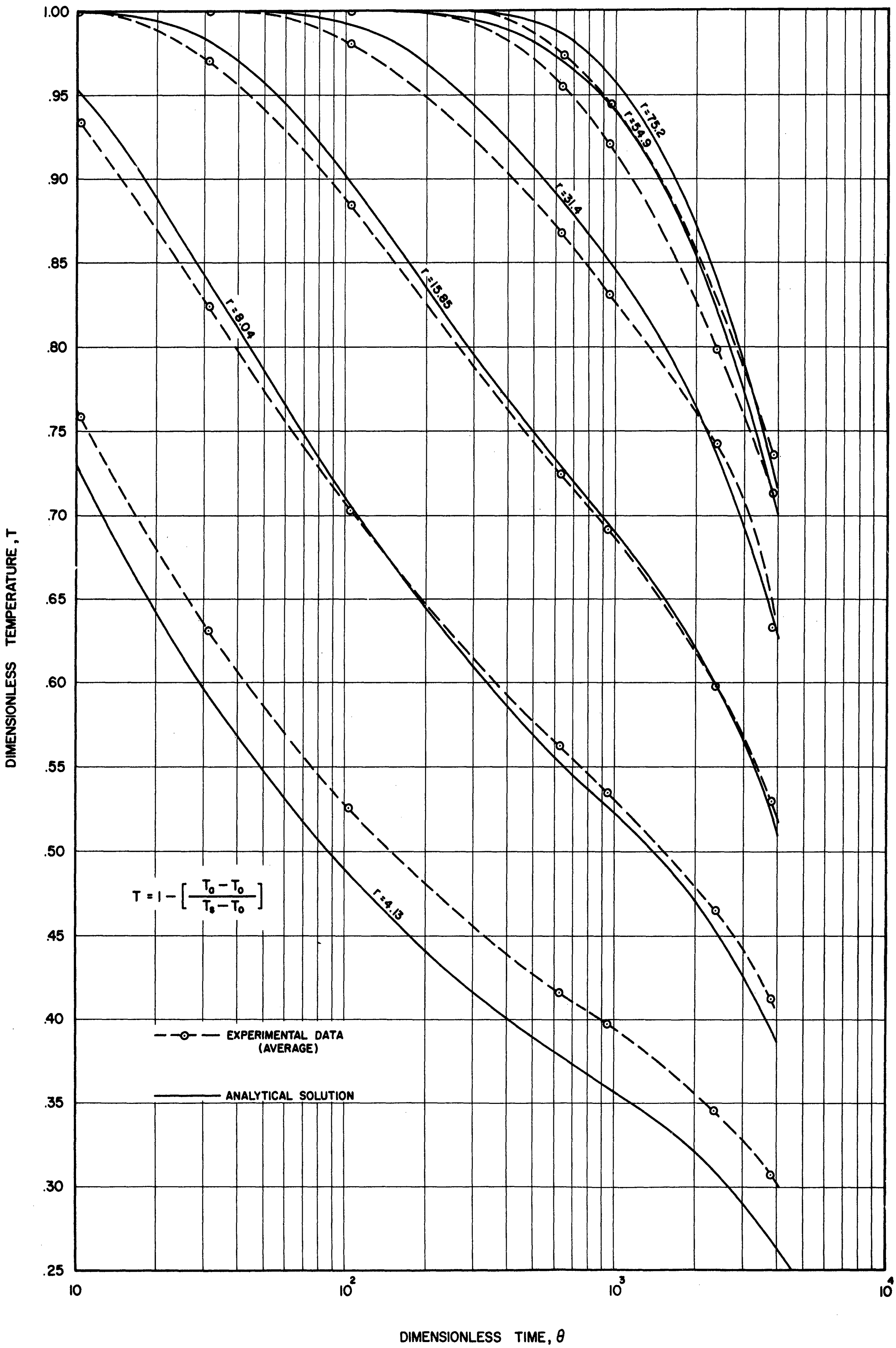


Figure VII-1. Comparison Between Experimental Data and the Analytical Solution for the Single-Layer Radial Model.

First, the introduction of the step function temperature change at time zero for this model was not precisely accomplished. The valving system used for this model was such that a period of at least 20 to 30 seconds was necessary to bring the heating face temperature to that desired. It was thus difficult to locate the $t_a = 0$ point on the recorder chart. This factor would be of importance only at low values of Θ in Figure VII-1.

Secondly, and more important, the 24 gauge thermocouple wires used in this model were evidently too large in diameter to obtain very accurate temperature readings, due to conduction of heat along the thermocouple wires. The thermocouple wires for large values of r passed over a hotter spot of the model at a short distance from the thermocouple bead. Conduction of heat along the thermocouple wires thus tended to increase the temperature reading, which in turn decreased the dimensionless temperature. This effect is particularly noticeable for $r = 31.4$ and above. It should again be noted that the single-layer radial model was the first model to be fabricated and run. On the basis of the comparison presented for this model, it was decided to use the smaller 30 gauge thermocouple wires in all subsequent models, in order to minimize this effect.

It may also be noted from Figure VII-1 that the experimental results tend to cross the line from the analytical solution, so that at high values of Θ the data predict too high a value of dimensionless temperature. This effect, which is more pronounced in the graphs for the other models, is largely due to heat loss from the model. Due to the definition of dimensionless temperature which was used, an error in actual temperature being too <u>low</u> is reflected by the dimensionless temperature being too high. Thus, heat losses, which tend to decrease the model temperature at a given time and position, cause the corresponding dimensionless temperature to be too high.

Three separate graphs are presented for the single-layer linear model, in order to more clearly show the effect of heat losses. Figure VII-2 shows the comparison of results for the data obtained with no insulation on the model (Run No. 1 on this model). The effect of heat losses are very apparent, since the experimental data deviate greatly from the analytical solution at high values of time.

Figure VII-3 shows the data for the model from Run Number 3, with asbestos, aluminum foil, and glass wool insulation on the single-layer linear model, but without the sheet steel channel. The improvement over the data in Figure VII-2 is very apparent, due to the reduction in heat loss.

Figure VII-4 shows the data obtained in experimental runs number 4 and 7 on the single-layer linear model. In these runs, the sheet steel channel cut the heat losses from the model to a very low value. Comparing with Figures (VII-2) and (VII-3), it can be seen that this sheet steel channel caused a significant reduction in the experimental error.

The comparison of results for the two-layer linear model is shown in Figure VII-5. In this figure, all of the experimental data points are indicated, and the dotted lines represent the weighted average of these data. The temperature gradient in the y or crossflow direction is so slight in this model that the lines of experimental data closely represent the temperature for a given x and θ at all values of y. It can be seen from Figure VII-5 that the experimental data fall very close to the analytical

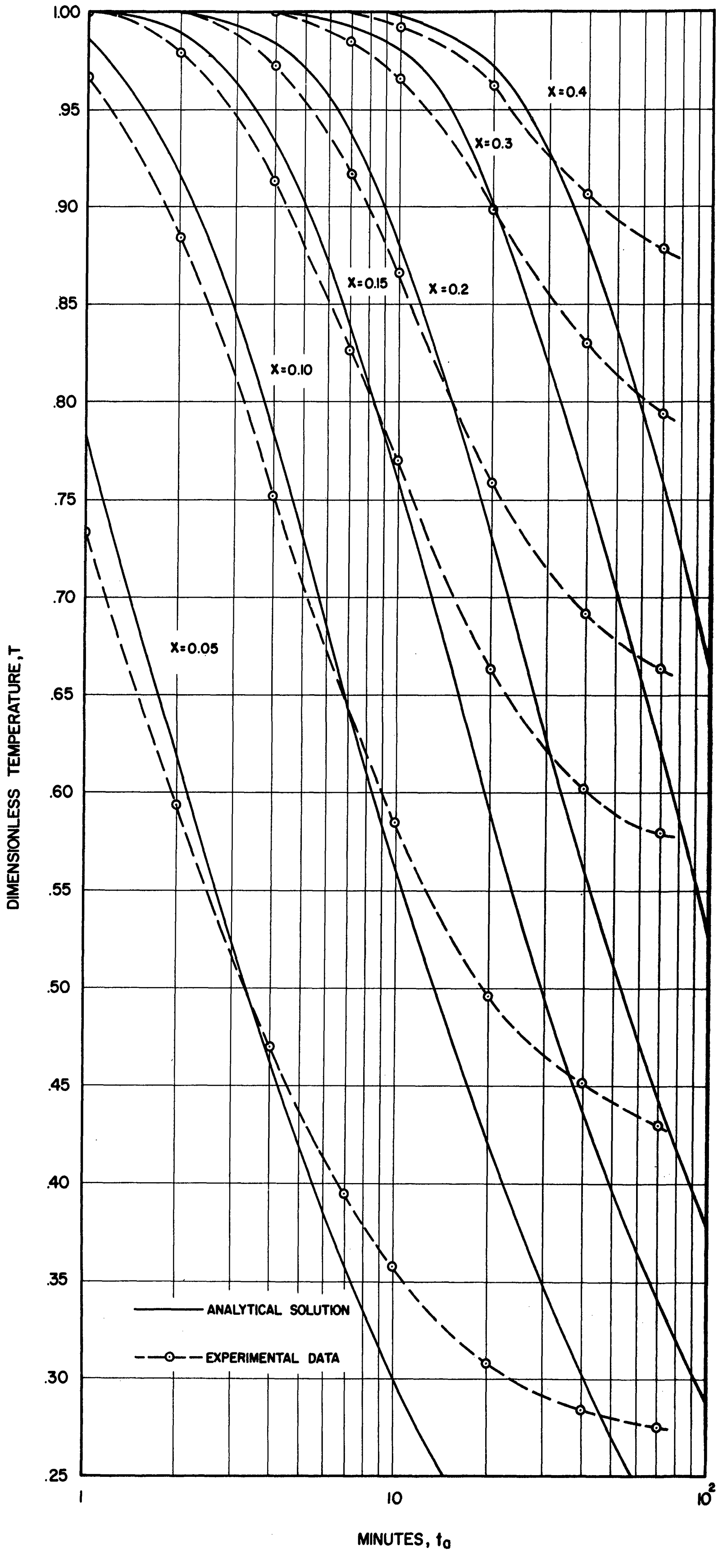


Figure VII-2 Comparison Between Experimental Data and the Analytical Solution for the Single-Layer Linear Model - No Insulation on Model.

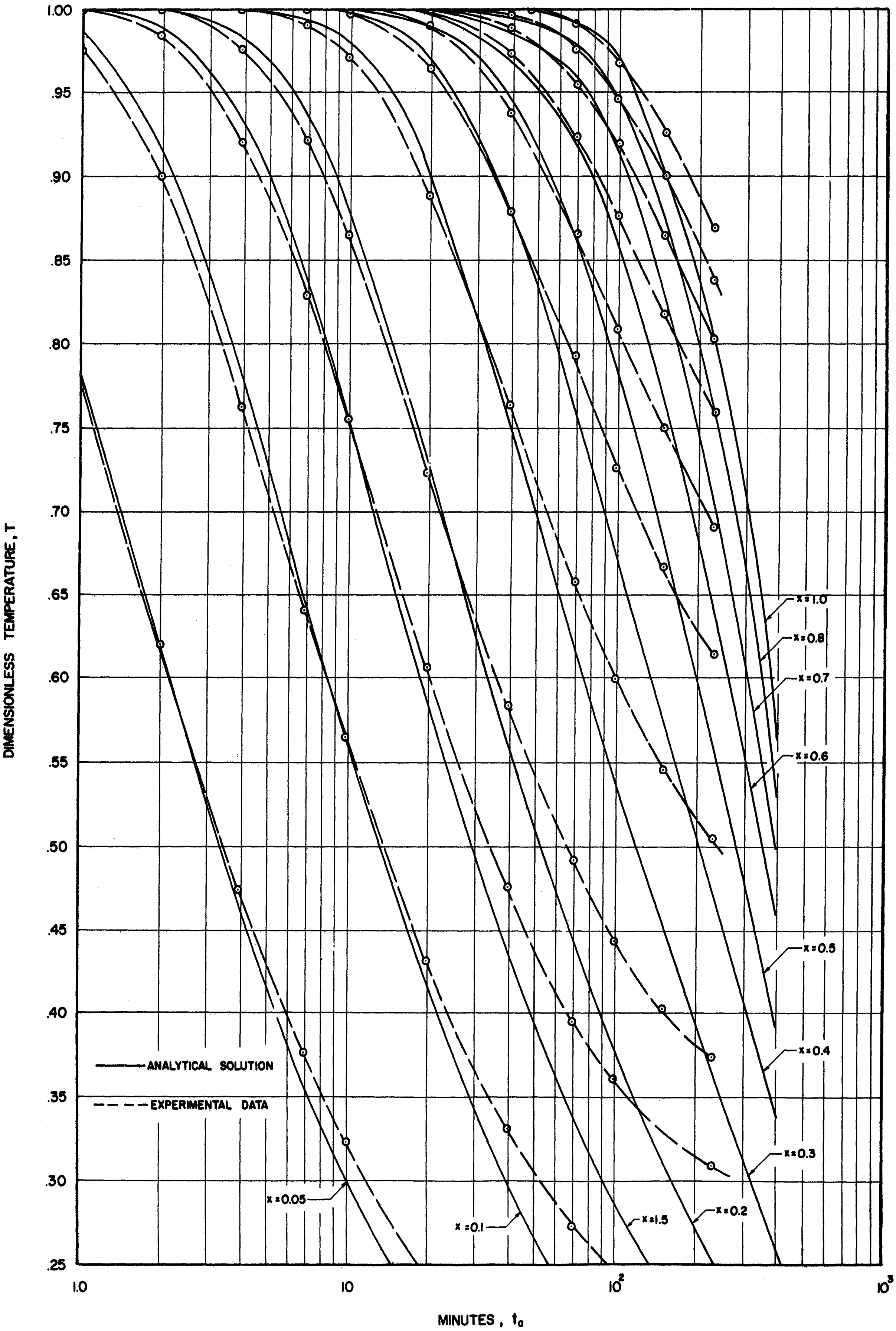


Figure VII-3. Comparison Between Experimental Data and the Analytical Solution for the Single-Layer Linear Model - Asbestos, Aluminum Foil, and Glass Wool Insulation, Without Sheet Steel Channel.

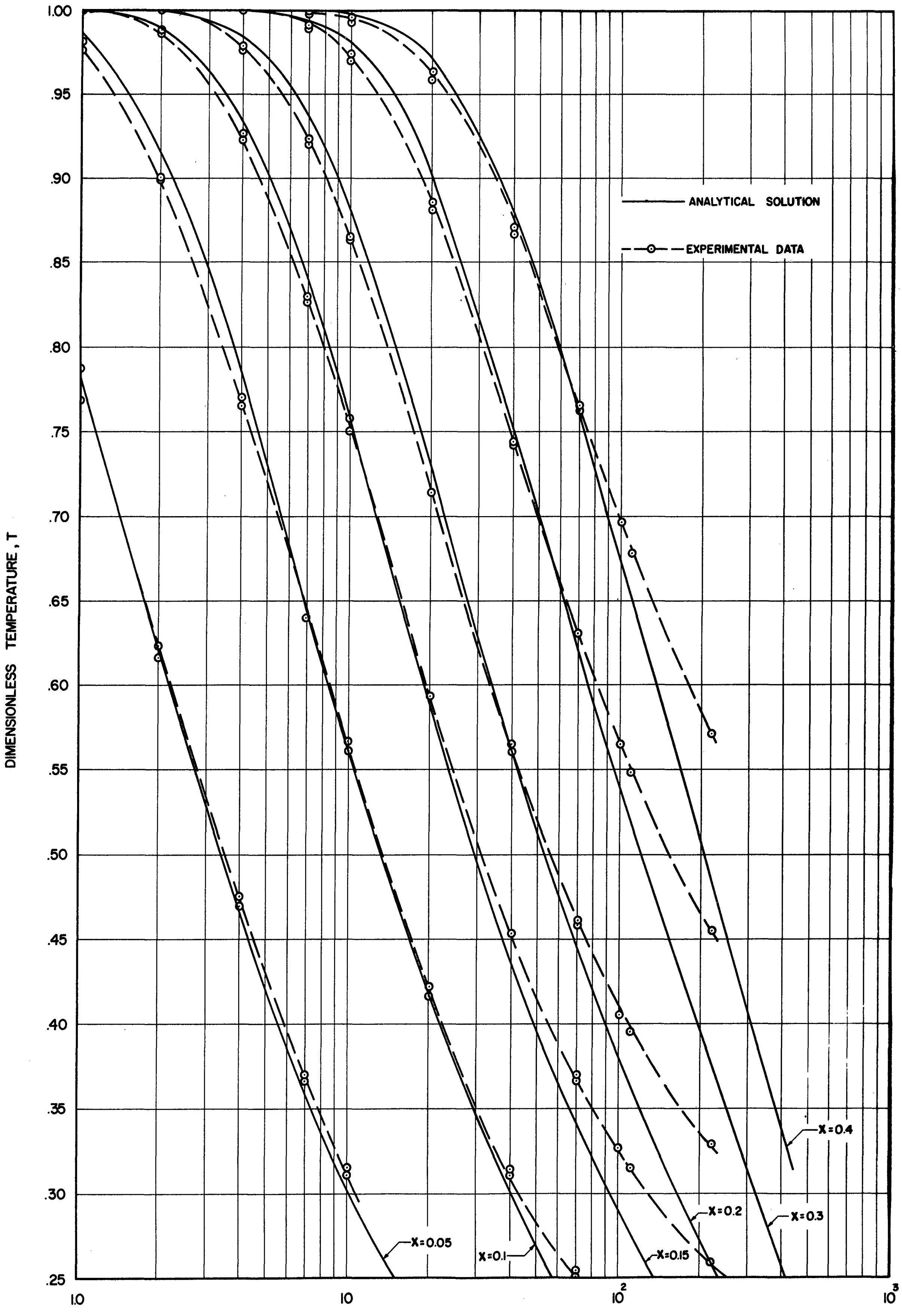


Figure VII-4. Comparison Between Experimental Data and the Analytical Solution for the Single-Layer Linear Model - Complete Insulation, Including Sheet Steel Channel.

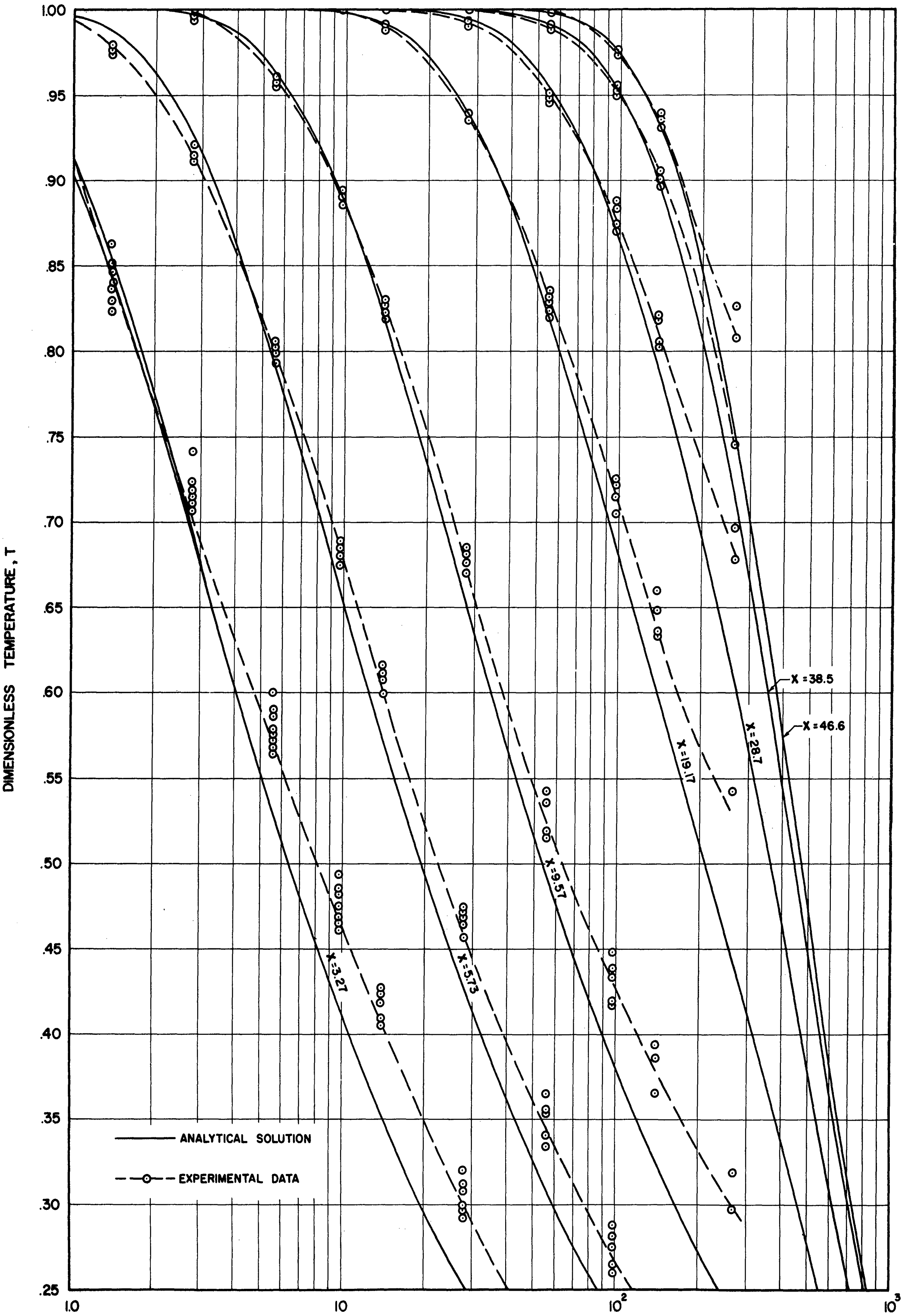


Figure VII-5. Comparison Between Experimental Data and the Analytical Solution for the Two-Layer Linear Model.

solution, with the normal deviation due to heat losses at higher values of Θ. On the basis of this comparison, the experimental data are thought by the author to constitute an adequate experimental verification of the corresponding analytical solution.

B. Comparison of Two-Layer Flow With Single-Layer Flow

In order to show some of the ramifications of the solutions for two-layer flow which have been obtained, this section will be devoted to comparisons between the flow as predicted by these solutions and the flow which might be predicted using only single-layer theory. The principal emphasis will be placed on flow of liquids in underground reservoirs. Analogous comparisons to those presented herein could also be made for heat conduction.

1. Comparison of the Form of the Analytical Solutions

The solutions for the two-layer flow systems which were obtained in this research are largely based on an extension of the technique of separation of variables which was used for single-layer flow. The form of the two-layer solutions would therefore be expected to be similar to that for single-layer flow. This has proven to be true. A comparison of the form of the solutions can be made for the pressure distribution in single-layer and two-layer radial flow of slightly compressible liquids, for the case of constant terminal pressure.

For single-layer flow:

$$P(r,\Theta) = \sum_{m=1}^{\infty} C_m U(br) e^{-b^2\Theta} \qquad \text{(IV-82)}$$

For two-layer flow

$$P(r,y,\Theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}e^{-a^2\Theta}Y_{mn}F(b)U(br) \qquad \text{(V-188)}$$

Now in both cases the eigenvalues b are roots of the characteristic equation

$$J_o(b)Y_1(bR) - Y_o(b)J_1(bR) = 0 \qquad \text{(IV-65)}$$

In both cases the function U(br) is defined as

$$U(br) = J_o(br)Y_1(bR) - Y_o(br)J_1(bR) \qquad \text{(IV-69)}$$

The expression for C_m for single-layer flow is given by

$$C_m = \frac{2bV(b)}{b^2V^2(b) - \frac{4}{\pi^2}} \qquad \text{(IV-81)}$$

This is the same function as that defined for two-layer flow as

$$F(b) = \frac{2bV(b)}{b^2V^2(b) - \frac{4}{\pi^2}} \qquad \text{(V-183)}$$

It can thus be seen that the solution for two-layer flow contains many of the same functions as for single-layer flow. The added complexity of the two-layer solution is due to the fact that the pressure distribution with respect to y must be represented by an infinite series in the two-layer case, whereas it is constant for single-layer flow.

The behavior of the two-layer solution as it approaches a condition of single-layer flow is worthy of mention. Such a circumstance would occur as K_2 approached one (i.e. k_1 approached k_2), or as the dividing point, c, on the y axis approached either zero or one. It has been determined that the solution for two-layer flow loses its validity at these limits. For the sake of brevity the complete mathematical analysis showing the breakdown of the two-layer solution to that for single-layer flow will not be presented here. The reader may however note that at $K_2 = 1.0$, $\gamma = \beta$. In the two-layer solution, it was assumed that neither γ nor β was equal to zero. It can be shown that for $K_2 = 1.0$ the only term in the "n" series must be for $\gamma = \beta = 0$. That is, for two-layer flow, the one term which is neglected $(n = 0)$ is precisely that one term which gives the solution for single-layer flow. The two-layer solution thus becomes invalid at this limit.

2. Comparison Between Separated and Intercommunicating Two-Layer Systems

The effect of crossflow or interlayer flow can be easily seen by comparing the fluxes obtained from a given two-layer system with and without interlayer fluid communication.

For single-layer unsteady-state flow of slightly compressible liquids:

$$W_e(ft^3) = 2\pi\phi c_w r_b^2 h\Delta P Q_{(t)}$$

where:

$$\Theta = \frac{kt_a}{\mu\phi c_w r_b^2}$$

The corresponding flux for a two-layer system is

$$W_e(ft^3) = 2\pi r_b^2 c_w H\Delta P[c\phi_1 + (1-c)\ \phi_2]Q_{(t)}$$

where $Q_{(t)}$ is a function of dimensionless time,

$$\Theta = \frac{k_1 t_a}{\mu\phi_1 c_w r_b^2}$$

For $\phi_1 = \phi_2$ this becomes

$$W_e(ft^3) = 2\pi\phi c_w r_b^2 H\Delta P Q_{(t)} \qquad \text{(VII-1)}$$

Then for $\phi_1 = \phi_2$ the ratio of the fluxes in cubic feet is equal to the ratio of the dimensionless fluxes, $Q_{(t)}$. Consider a two-layer radial system where R = 20.0 and c = 0.5. The dimensionless flux from this system will be computed both for separated and intercommunicating layers, for $K_2 = 5$ and $K_2 = 100$. Let Θ_{2L} represent the dimensionless time for the composite interconnected two-layer system, and let Θ_1 and Θ_2 represent the corresponding dimensionless times for layers 1 and 2 respectively. From the definitions of dimensionless time

$$\Theta_1 = \Theta_{2L} \tag{VII-2}$$

$$\Theta_2 = K_2\Theta_{2L} \tag{VII-3}$$

The total dimensionless flux assuming separated layers is given in this case (c = 0.5) by

$$Q(t)_{SL} = 0.5\ Q(t)_1 + 0.5\ Q(t)_2 \tag{VII-4}$$

Values of the dimensionless cumulative fluxes $Q(t)_1$ and $Q(t)_2$ for the separate individual layers may be obtained from the tables in Appendix I of this dissertation and the paper by Van Everdingen and Hurst.[40] Values of the total dimensionless cumulative flux, $Q(t)_{2L}$, assuming intercommunicating layers may be obtained from the tables of Appendix O. These dimensionless fluxes are shown in Table VII-2.

By comparing the values of $Q(t)_{2L}$ for the intercommunicating system with the values of $Q(t)_{SL}$ for the system of separated layers, it can immediately be seen that the separated system always exhibits less flux at a given time than the corresponding intercommunicating system. The increased flux from the intercommunicating system is due to crossflow. It can also be seen that the amount of increase in the flux is greater for the greater K_2.

TABLE VII-2

DIMENSIONLESS CUMULATIVE FLUXES FOR SEPARATED AND INTERCOMMUNICATING TWO-LAYER SYSTEMS

$K_2 = 5.0$

$\Theta_{2L} = \Theta_1$	Θ_2	$Q(t)_1$	$Q(t)_2$	$Q(t)_{2L}$	$Q(t)_{SL}$
5	25	4.499	14.57	9.958	9.535
10	50	7.417	24.84	16.75	16.13
20	100	12.32	42.91	28.66	27.62
50	250	24.84	86.22	58.97	55.53
100	500	42.91	133.52	97.91	88.17
200	1000	73.37	177.15	146.4	125.26
400	2000	117.67	196.93	185.0	157.30
800	4000	165.05	199.47	198.4	182.26
1000	5000	177.15	199.50	199.2	188.33

$K_2 = 100.0$

$\Theta_{2L} = \Theta_1$	Θ_2	$Q(t)_1$	$Q(t)_2$	$Q(t)_{2L}$	$Q(t)_{SL}$
0.2	20	0.606	12.32	7.463	6.46
0.5	50	1.020	24.84	14.69	12.93
1.	100	1.571	42.91	25.04	22.21
2.	200	2.442	73.37	43.25	37.90
5.	500	4.499	133.52	86.92	69.00
10.	1000	7.417	177.15	134.3	92.28
20.	2000	12.32	196.93	177.6	104.63
50.	5000	24.84	199.50	198.7	112.17
80.	8000	35.96	199.50	199.5	117.73

It can be stated unequivocally that crossflow between layers always increases the flux from the system over that which would be obtained from the separate layers, in systems of the "layer-cake" type which were studied for this dissertation. This is due to the fact that a part of the liquid present in the layer of lower permeability is carried to the producing face through the zone of higher permeability. Note for example in Table VII-1 the fluxes for $K_2 = 100$ at $\Theta_{2L} = 80$. In the system of separated layers, the layer 2, with higher permeability, has been depleted completely at this time, but only approximately 18% of the liquid in layer 1 has been produced. If the layers were intercommunicating, a good share of this liquid could be produced through the high permeability zone. Indeed, for complete communication, both layers would be completely depleted and depressurized by this time.

3. Approximation of Multi-Layer Systems by a Single-Layer Homogeneous System Having Mean Physical Properties

This section will be devoted to a study of the question of when and how the behavior of two-layer (and in general multi-layer) systems in the unsteady-state may be approximated by a single-layer homogeneous system having some mean physical properties. The concept of "millidarcy-feet" will be examined in conjunction with this question. Maximum and minimum limits on flux from a system where the degree of interface resistance is unknown will also be shown.

The concept of "millidarcy-feet" has frequently been applied to heterogeneous systems of a stratified type by reservoir engineers. With this method, for a stratified system of N layers, the layers of which each have some permeability k_i and thickness h_i, a mean permeability k_m is defined as

$$k_m = \frac{\sum_{i=1}^{N} (k_i h_i)}{\sum_{i=1}^{N} h_i} \qquad \text{(VII-5)}$$

That is, the mean permeability by which the system is treated as a homogeneous single-layer system is the total millidarcy-feet of the system divided by its total thickness.

Systems of separated (i.e. non-intercommunicating) layers may first be considered. Several papers have recently been published on such systems, such as that by Lefkovits, et.al.[29] It can easily be shown that there is no single constant mean permeability by which a system of two or more separated layers may be represented by a single-layer system. Consider the equation for cumulative dimensionless flux in the radial flow of slightly compressible liquids in a single layer.

$$Q_{(t)} = \left(\frac{R^2-1}{2}\right) - 2 \sum_{m=1}^{\infty} \frac{e^{-b^2\Theta} J_1^{\,2}(bR)}{b^2[J_o^2(b) - J_1^{\,2}(bR)]} \qquad \text{(IV-87)}$$

where

$$\Theta = \frac{k t_a}{\mu \phi c r_b^{\,2}} \qquad \text{(IV-48)}$$

Now consider two layers with the same physical dimensions r_b, r_e, and thus R, which have different values of (k/ϕ). Then the only difference in the solutions for these layers appears in the dimensionless times, Θ. Define

$$\Theta_m = \left(\frac{k}{\phi}\right)_m \frac{t_a}{\mu c r_b^{\,2}} \qquad \text{(VII-6)}$$

$$\theta_1 = \left(\frac{k}{\phi} \right)_1 \frac{t_a}{\mu c r_b^2} \qquad \text{(VII-7)}$$

$$\theta_2 = \left(\frac{k}{\phi} \right)_2 \frac{t_a}{\mu c r_b^2} \qquad \text{(VII-8)}$$

where $\left(\frac{k}{\phi} \right)_m$ is some mean property by which it is hoped to approximate the total cumulative dimensionless flux from both layers using a single-layer system. In order to accomplish this, it must be true that

$$e^{-b^2\theta_m} = C\ [e^{-b^2\theta_1} + e^{-b^2\theta_2}] \qquad \text{(VII-9)}$$

where C is some constant. But this equation cannot be valid for all values of real time t_a. It would be necessary to either allow the constant C or the value $\left(\frac{k}{\phi} \right)_m$ to be a function of real time, t_a, in order to make this equation valid for all values of t_a. It is thus impossible to represent a system of separated layers of different physical properties by a single-layer system with some constant mean value of the physical properties. This in turn means that the concept of millidarcy-feet, Equation (VII-5), is not valid for systems of separate and distinct layers in unsteady-state flow.

In considering the question of whether intercommunicating layers may be represented by a single layer with some mean physical properties, it is necessary to define what is meant by "complete crossflow". Since the crossflow is dependent upon the pressure gradient in the y direction, it would seem at first glance that complete crossflow might well be defined in terms of this gradient. The interlayer flux or crossflow is, however, also a direct function of the flow resistance in the y direction, which is in turn a function of the interface area, the thickness of the layers, and the permeability of the layers. The author thus proposes a definition in terms of the flux at the producing face. In this dissertation, the term "complete crossflow" will be used to represent that condition for which the instantaneous flux in cubic feet per second per square foot of the producing face from each layer of a multi-layer system in unsteady-state flow is directly proportional to the permeability of that layer for all values of time. That is, regardless of the initial distribution of fluid in the system, the flux at the producing face is dependent only on the permeability. This necessarily implies that a part of the fluid being produced from a given layer initially may have originated in another layer of the system.

This definition of complete crossflow leads to consideration of Appendices L and M of this dissertation, which present values of the instantaneous dimensionless fluxes, q_1 and q_2, for linear and radial unsteady-state flow, respectively. The application of the definition of complete crossflow to these tables indicates that for complete crossflow the values of q_1 and q_2 should be directly proportional to the relative thickness of the layers. That is, for $c = 0.5$, the crossflow between layers is complete if $q_1 = q_2$. Inspection of the tables of Appendix M thus indicates that (except for the single case of $R = 5$ and $K_2 = 100$) the crossflow in the two-layer radial systems for which values are tabulated is complete. The values tabulated in Appendix L for linear flow, however, indicate that the crossflow in most of these systems is <u>not</u> complete, although this limit is approached for large values of $L = L_a/H$.

The reader may well at this point question why crossflow is complete for the radial systems, but not for the linear systems. It must be remembered that all of the radial systems for which values have been tabulated in the Appendices are comparatively thin, so that (r_b/H) is greater than 10.0. This means that the distance from a given point in the lower permeability layer to the interface with the high permeability layer is in most cases much less than the distance to the producing face. There is thus very little resistance to crossflow. Indeed, although the crossflow in the radial systems is complete, it has been previously noted that the pressure gradient in the y direction is so slight that the tables of $P(r,y,\Theta)$ in Appendix K do not include parameters of y. In the linear case, however, the systems for all but very high values of $L = L_a/H$ are quite thick in comparison with the radial systems. The resistance to crossflow is thus higher in these systems. The tables of $P(x,y,\Theta)$ for two-layer linear flow in Appendix J show a significant pressure gradient in the y direction although crossflow in these cases is not complete. This factor of the pressure gradient in the y direction being smaller for the case of complete crossflow than for only partial crossflow again serves to show the merits of defining crossflow in terms of fluxes rather than pressure gradients.

Now consider the implication of the definition of complete crossflow with regard to the concept of millidarcy-feet. According to the definition for complete crossflow the flux from a given layer is independent both of the position of the layer in the system, and of the thickness of the layer. Then in this case the concept of millidarcy-feet, and the use of Equation (VII-5) to define a mean permeability for the system should be entirely valid! This in turn implies that when a multi-layer system exhibits complete crossflow, it should be possible to accurately predict the behavior of the system using a single-layer homogeneous system with mean physical properties, where k_m is defined by Equation (VII-5), and the mean porosity, ϕ_m, is given by the analogous equation

$$\phi_m = \frac{\sum_{i=1}^{N} (\phi_i h_i)}{\sum_{i=1}^{N} h_i} \qquad \text{(VII-10)}$$

The foregoing discussion indicates that it should be possible to predict the behavior of the two-layer radial systems for which values appear in Appendices K, M, and O by using a mean permeability as defined by Equation (VII-5) in conjunction with the tables for single-layer flow. The validity of this assertion will be examined by comparing the fluxes from a two-layer intercommunicating radial system as predicted using two-layer theory to the fluxes predicted using a mean permeability in conjunction with single-layer tables. In order to use Appendix O, it is necessary to consider $\phi_1 = \phi_2$. The preceding section of this dissertation has shown that in this case the ratio of the real cumulative fluxes, in cubic feet, is equal to the ratio of the corresponding cumulative dimensionless fluxes, $Q_{(t)}$. In terms of the nomenclature used for two-layer flow, Equation (VII-5) in this case, indicates the use of a mean permeability

$$k_m = c + (1-c)\, K_2 \qquad \text{(VII-11)}$$

The definition of dimensionless times, Θ, is such that in approximating the behavior of the two-layer system by using a single layer of mean permeability,

$$\Theta_{SL} = k_m \Theta_{2L} \qquad \text{(VII-12)}$$

where:

Θ_{SL} = dimensionless time for the single-layer approximation

Θ_{2L} = dimensionless time for the two-layer system.

Values of cumulative dimensionless flux for the two-layer system, $Q(t)_{2L}$, may be obtained from Appendix O. Values of $Q(t)_{SL}$ for the single-layer approximation may be obtained from Appendix I plus the tables of Van Everdingen and Hurst.[40] For this example, the values in the tables were interpolated linearly where necessary. Table VII-3 shows the values of flux obtained for a system of R = 20, K_2 = 5 and 100, c = 0.1, 0.5, and 0.9, for $r_b/H > 10$.

Examination of Table VII-3 clearly shows that the fluxes predicted using a mean permeability and single-layer tables, $Q(t)_{SL}$, are the same as those from two-layer theory, $Q(t)_{2L}$, within the limits of accuracy of linear interpolation of the tables. Thus the use of a mean k in conjunction with single-layer theory yields a perfectly valid representation of the total cumulative flux from this two-layer system, as predicted due to the complete crossflow.

As additional verification, an example showing the comparison between the pressure distribution $P(r,y,\Theta)$ for a radial two-layer system as computed from two-layer theory and that computed using a mean permeability and single-layer tables will be shown. Values of $P(r,y,\Theta)$ from two-layer theory will be obtained directly from Appendix K for R = 20, c = 0.5, K_2 = 5, and $r_b/H > 10$. Corresponding values from single-layer flow may be obtained from Appendix G. In this case, the values for single-layer flow were interpolated graphically. The comparison is shown in Table VII-4.

TABLE VII-3

DIMENSIONLESS CUMULATIVE FLUXES FOR THE APPROXIMATION OF A TWO-LAYER RADIAL SYSTEM BY A HOMOGENEOUS SINGLE-LAYER SYSTEM

θ_{2L}	θ_{SL}	$Q(t)_{2L}$	$Q(t)_{SL}$	θ_{2L}	θ_{SL}	$Q(t)_{2L}$	$Q(t)_{SL}$
$R = 20,\ c = 0.5,\ K_2 = 5$				$R = 20,\ c = 0.5,\ K_2 = 100$			
5	15	9.958	9.95	0.2	10.1	7.463	7.46
10	30	16.75	16.74	0.5	25.25	14.69	14.68
20	60	28.66	28.65	1.	50.5	25.04	25.03
50	150	58.97	58.93	2.	101.	43.25	43.24
100	300	97.91	97.81	5.	252.5	86.92	86.82
200	600	146.4	146.40	10.	505.	134.3	134.23
400	1200	185.0	185.00	20.	1010.	177.6	177.54
800	2400	198.4	198.42	50.	2525.	198.7	198.67
1000	3000	199.2	199.20	80.	4040.	199.5	199.47
$R = 20,\ c = 0.1,\ K_2 = 5$				$R = 20,\ c = 0.9,\ K_2 = 100$			
5	7	5.740	5.749	0.2	2.18	2.625	2.559
10	14	9.459	9.439	0.5	5.45	4.816	4.700
20	28	15.88	15.88	1.	10.9	7.878	7.872
50	70	32.36	32.35	2.	21.8	13.14	13.14
100	140	55.90	55.89	5.	54.5	26.57	26.57
200	280	93.41	93.41	10.	109.	45.93	45.89
400	560	141.6	141.61	20.	218.	78.19	78.08
800	1120	182.3	181.86	50.	545.	139.7	139.64
1000	1400	190.1	190.09	100.	1090.	181.1	180.68
2000	2800	199.0	199.04	200.	2180.	197.8	197.74
$R = 20,\ c = 0.1,\ K_2 = 5$				$R = 20,\ c = 0.1,\ K_2 = 100$			
5	23	13.70	13.68	0.2	18.02	11.41	11.38
10	46	23.29	23.26	0.5	45.05	22.91	22.90
20	92	40.18	40.16	1.	90.1	39.51	39.50
50	230	81.31	81.21	2.	180.2	67.87	67.86
100	460	127.6	127.56	5.	450.5	126.1	126.08
200	920	172.9	172.31	10.	901.	171.8	171.16
400	1840	195.9	195.82	20.	1802.	195.6	195.6
800	3680	199.4	199.42	50.	4505.	199.5	199.5

TABLE VII-4

DIMENSIONLESS PRESSURE DISTRIBUTION FOR
THE APPROXIMATION OF A TWO-LAYER RADIAL SYSTEM
BY A HOMOGENEOUS SINGLE-LAYER SYSTEM

$R = 20$, $c = 0.5$, $K_2 = 5$

Dimensionless Time		Dimensionless Pressure Distribution for all Values of y					
		From Two-Layer Theory			From Single-Layer Tables		
θ_{2L}	θ_{SL}	r = 4	r = 10	r = 20	r = 4	r = 10	r = 20
5	15	.664	.963	1.000	.663	.962	1.000
10	30	.585	.906	.992	.585	.908	.990
20	60	.518	.829	.948	.515	.828	.948
50	150	.420	.678	.784	.416	.671	.785
100	300	.304	.490	.567	.305	.490	.567
200	600	.159	.256	.296	.153	.245	.288
500	1500	.023	.037	.042	.022	.042	.052

The pressure distribution from the two methods checks within the accuracy with which the single-layer tables could be graphically interpolated.

It may thus be concluded that the Equations (VII-5) and (VII-10) in conjunction with tables for single-layer flow may be used as an alternative to the use of Appendices K and O in determining dimensionless pressure distribution, $P(r,y,\Theta)$ and dimensionless cumulative flux, $Q_{(t)}$ for two-layer radial systems of intercommunicating layers, for unsteady-state flow, and for $r_b/ > 10.0$. Moreover, Equations (VII-5) and (VII-10) and single-layer tables may be conveniently used for values of K_2 between 2 and 100 for which values do not appear in Appendices K and O, and for cases in which $\phi_1 \neq \phi_2$.

Most importantly, it can be seen by direct extension of the foregoing discussion that it should be valid to use the method of single-layer approximation for <u>any</u> multi-layer radial system in the unsteady state, under the restrictions that

1. $r_b/H > 10$
2. The ratio of (k/ϕ) of any two layers not be greater than 100
3. $R > 10$

These restrictions are estimations on the basis of the results for two layers.

These approximations by a single layer which are valid for radial-flow would not be expected to apply to the cases of two-layer linear flow for which values appear in Appendices J, L, and N, since for these cases the crossflow is not complete. The author has verified this by attempting to compute dimensionless pressure distribution, $P(x,y,\Theta)$, for the two-layer case of $L = 10$, $c = 0.5$, $K_2 = 2.0$, by using a mean permeability and tables for single-layer linear flow. The results obtained were very poor approximations.

The validity of the application of the concept of millidarcy-feet, i.e. the use of Equations (VII-5) and (VII-10) and single-layer tables to represent the behavior of multi-layer systems, may be summarized as follows:

1. Millidarcy-feet may be used for true steady-state flow, with or without impermeable barriers between layers.
2. Millidarcy-feet may be used for multi-layer systems in which there is <u>complete</u> crossflow. It is usually necessary to use multi-layer theory to determine whether or not the crossflow for a given system is truly complete.
3. Millidarcy-feet is <u>not</u> a valid concept for use with systems of separate and distinct layers, i.e. where the layers are separated by impermeable barriers.

For stratified systems of more than three or four layers, or in systems where there is some finite but unknown interface resistance, the use of the multi-layer theory as presented in this dissertation to determine the completeness of crossflow is very difficult. This is also true in anisotropic stratified systems in which the vertical permeability is not equal to the horizontal permeability. If the crossflow is not known to be complete, the validity of approximation by a single layer is questionable. It is, however, possible to establish maximum and minimum limits on the total cumulative flux at the producing face for multi-layer systems of the type studied in this dissertation. The minimum flux is that which is predicted treating each layer of the system individually, assuming no crossflow (see Table VII-1). The maximum flux is that assuming complete crossflow, for which the flux may be calculated using single-layer theory and Equations (VII-5) and (VII-10). For partial crossflow, the flux should fall between these limits.

C. Steady-State and Unsteady-State Permeabilities of Stratified Systems

The term "permeability" as applied to heterogeneous systems must be carefully defined. By choosing a sufficiently small volume element, nearly all heterogeneous systems can be resolved into a combination of homogeneous elements. In each of these homogeneous volume elements, the original concept of permeability as defined by Darcy applies. That is, the permeability of these homogeneous elements is a function solely of the physical properties of the porous matrix, and is independent of whether the system is in steady-state or unsteady-state flow.

On the macroscopic scale, however, the term "permeability" is frequently used to indicate the average physical properties of a heterogeneous system, as in "in-situ permeability". In such cases, the permeability which is measured in the steady-state is not necessarily that which would accurately predict the unsteady-state behavior of the system.

Consider, for example, a stratified system of separate and distinct non-intercommunicating layers. A macroscopic mean permeability of this system could be determined by measuring the flux from the system in steady-state flow. This permeability would not, however, be valid for predicting the performance of the system in unsteady-state flow. As shown in the preceding section, in order to use a single mean permeability for representing the behavior of such a system in the unsteady-state, the mean permeability would necessarily be time dependent.

The difference between steady-state and unsteady-state macroscopic mean permeabilities must also be considered in predicting the performance of a heterogeneous reservoir on the basis of core analyses. It is for this reason that measurements of in-situ permeability by unsteady-state draw down and/or buildup tests usually indicate a much more valid macroscopic mean permeability (and porosity) than can be obtained from core analyses, even though a very large number of cores are tested. In stratified systems of the type considered in this dissertation, it has been shown that the use of a macroscopic mean permeability as obtained by using the concept of millidarcy-feet as applied to core analyses would indicate too high a flux for a system that did not exhibit complete crossflow.

D. The Flow of Gas in Stratified Systems

The non-linearity of the equation governing the flow of gas in porous media (Equation III-29) makes it impossible to obtain a complete analytical solution for gas flow in stratified systems, using mathematics of the type used in this dissertation. It is possible, however, to investigate certain aspects of the flow of gas in stratified systems on the basis of the known solutions for the flow of slightly compressible liquids.

For low pressure differences, the density of a gas is relatively constant, and may be approximately expressed by Equation (III-23) which was used for slightly compressible liquids. Under these conditions, the solutions in this dissertation for the flow of slightly compressible liquids should provide accurate approximations to the gas flow problem. With high pressure differences, on the other hand, the gas density may no longer be validly approximated by Equation (III-23). In this case, the predictive accuracy of the liquid-flow solutions may become completely void.

Another problem encountered to a great extent in gas flow, but generally of little significance for liquid flow, is the phenomenon of "turbulent" flow. However, with a sufficiently high pressure gradient to induce turbulent flow, the density variation of the gas in the system is usually too large to permit use of the liquid flow solutions even without this additional complication.

For gas flow with high pressure drawdowns, it is possible to make only qualitative statements regarding the flow in stratified systems. First of all, in porous systems which would exhibit crossflow for liquid flow, the systems will also exhibit crossflow in gas flow. Secondly, this crossflow must tend to increase the flux at the producing face above that which would be obtained for separate and distinct (i.e. non-intercommunicating) layers.

One factor with regard to gas flow which is worthy of mention here is the behavior of a stratified gas reservoir in an isochronal back-pressure test. In these tests a gas well is opened to flow for a given length of time for several different flow rates. The instantaneous flux at the end of this time period is plotted versus the difference in the squares of the pressures at the well bottom and far out in the reservoir at this time. The term "isochronal" refers to the fact that the same time period is used for all test points. Since the pressure in the reservoir is allowed to return approximately to equilibrium after each point is obtained, the same "radius of drainage" is obtained for all points.

It was thought at the outset of the research leading to this dissertation that the slope of the isochronal back pressure curve might be a function of the degree of stratification of the system. The results, however, indicate that for stratified systems of the types studied herein this would not be true. In all systems studied, the solutions obtained for the constant terminal pressure case are independent of the pressure difference applied to the system. That is, doubling the initial pressure difference driving force in all cases exactly doubles the flux obtained from the system at a given time. The behavior of the stratified systems in this regard is exactly the same as for a homogeneous system. It is therefore concluded that the slope of the isochronal back pressure curve is not a function of the degree of stratification of the system. It seems very probable that this slope is entirely independent of all types of reservoir heterogeneity (since the same radius of drainage is applicable for all points), except insofar as the heterogeneity affects the turbulence level of the gas.

Attempts to show that turbulence is the cause of the variation in the slope of the back pressure curve have heretofore been unsuccessful in that values of the reciprocal slope below 0.9 cannot be predicted, but do occur in practice. In all such attempts, an average permeability as from Equation (VII-5) was used for the system. The use of an average permeability inherently implies the assumption of a mean gas velocity. Since turbulence level is not a linear function of velocity, the linear average velocity which is thereby calculated would give inadequate predictions of the actual turbulence effects for that portion of the zone actually delivering the bulk of the gas.

VIII RECOMMENDATIONS FOR FUTURE WORK

The investigation of fluid flow in heterogeneous and/or anisotropic systems is certainly of great importance in understanding the complex flow problems associated with underground formations. To date, the surface has only been scratched. Opportunities for both analytical and experimental work are abundant. This author will not pretend to indicate which problems are of the most importance in this field, but will indicate some problems of particular interest to himself.

In the field of mathematical work:

1. As indicated, the solutions presented in this dissertation for multi-layer radial flow are very slowly convergent for small values of r_b/H, so that accurate numerical values are very difficult to obtain. The author believes that a "point-source" solution to the multi-layer radial flow problem would yield valuable results in this regard. It is also possible that resolving the problem using some transformation technique, possibly a combination of a LaPlace transformation and finite Fourier trigonometric transformations, might yield pertinent approximations not apparent from the solution using separation of variables.

2. Although solutions are presented in this dissertation for more than two layers, no numerical values have been obtained for these cases. The computation of a complete set of tables of pressure distribution, etc. for three- and four-layered systems would require a prohibitive amount of time, due both

to the complexity of the solutions and the multiplicity of parameters. It should, however, be feasible to compute values for certain particular cases. It would be of considerable interest to determine how the arrangement of layers in three- and four-layered systems affects the flow patterns and flux across the producing face. For example, given three layers, of 1, 10 and 100 millidarcy permeability respectively, three separate and distinct arrangements of these layers may be made (i.e. 1, 10, 100; 1, 100, 10; and 10, 1, 100). The author believes that the arrangement of layers should in many cases have a distinct effect on the flux obtained.

3. All of the work in this dissertation has been done on systems which are completely isotropic within a given "layer" of the system. Actual underground formations frequently are not. A multi-layer solution for vertical permeability (i.e. parallel to the y axis in the models) being some fraction of horizontal permeability would therefore be of great value.

4. The "layer-cake" types of models used in this research are poor approximations to many formations which involve continuous permeability distributions as a function of depth. Investigation of the flow patterns in systems with random permeability distribution would be very informative. Such an investigation would probably be based on some statistical distribution, and would most likely require finite difference techniques to obtain a useable solution.

In the field of experimental work:

1. The use of heat transfer models is very convenient in simulating irregular geometries. This is true of both homogeneous and heterogeneous systems. The linear and radial cases have been considered in the research leading to this dissertation. The next logical step would seem to be an elliptical model.

2. Types of heterogeneity other than the "layer cake" type of model could be simulated without undue difficulty using thermal models. The first step in this direction might be the introduction of an additional resistance at the heating surface. This could be easily done by using a material other than copper to contain the heating steam.

   It should also be possible to investigate systems of "random" permeability using thermal models. Such models could be built up of several blocks of different materials. It would probably be necessary to use a solder bond between blocks, which would make the results less quantitative than for the models used in the research leading to this dissertation. By using high conductivity materials such as copper and various cupro-nickels, the effect of bond resistance could probably be minimized sufficiently to obtain results of adequate accuracy.

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APPENDIX A

DERIVATION OF THE ORTHOGONALITY CONDITIONS FOR THE "Y" EQUATIONS IN TWO-LAYER FLOW

The orthogonality condition (V-65) for the equations

$$Y'' + \gamma^2 Y = 0 \qquad 0 \leqq y < c \tag{A-1}$$

$$Y'' + \beta^2 Y = 0 \qquad c < y \leqq 1 \tag{A-2}$$

$$\gamma = \sqrt{a^2 - b^2}\,; \qquad \beta = \sqrt{\frac{a^2}{K_2} - b^2} \tag{A-3}$$

$$Y'(0) = Y'(1) = 0 \tag{A-4}$$

with the interface conditions

$$Y(c-) = Y(c+) \tag{A-5}$$

$$Y'(c-) = K_2 Y'(c+) \tag{A-6}$$

will first be derived. The more general case of K being a continuous function of y will then be considered.

Consider Equation (A-1) through (A-4) for some given value of the parameter b_m. At $a = a_1$, (i.e., $n = 1$), $\gamma = \gamma_1$, and $\beta = \beta_1$. There will then be some solution Y_1 satisfying the equations

$$Y_1'' + \gamma_1^2 Y_1 = 0 \qquad 0 \leqq y < c \tag{A-7}$$

$$Y_1'' + \beta_1^2 Y_1 = 0 \qquad c < y \leqq 1 \tag{A-8}$$

Similarly, for $a = a_2$

$$Y_2'' + \gamma_2^2 Y_2 = 0 \qquad 0 \leqq y < c \tag{A-9}$$

$$Y_2'' + \beta_2^2 Y_2 = 0 \qquad c < y \leqq 1 \tag{A-10}$$

Multiplying both sides of (A-7) and (A-8) by Y_2, both sides of (A-9) and (A-10) by $(-Y_1)$, and adding

$$Y_1''Y_2 - Y_1Y_2'' = (\gamma_2^2 - \gamma_1^2)Y_1Y_2 \qquad 0 \leqq y < c \tag{A-11}$$

$$Y_1''Y_2 - Y_1Y_2'' = (\beta_2^2 - \beta_1^2)Y_1Y_2 \qquad c < y \leqq 1 \tag{A-12}$$

Integrating both sides of (A-11) and (A-12) with respect to y over the intervals on y in which they are valid

$$\int_0^c [Y_1''Y_2 - Y_1Y_2'']dy = \int_0^c (\gamma_2^2 - \gamma_1^2)Y_1Y_2\, dy \tag{A-13}$$

$$\int_c^1 [Y_1''Y_2 - Y_1Y_2'']dy = \int_c^1 (\beta_2^2 - \beta_1^2)Y_1Y_2 dy \tag{A-14}$$

Now

$$Y_1''Y_2 - Y_1Y_2'' = \frac{d}{dy}[Y_1'Y_2 - Y_2'Y_1] \qquad \text{(A-15)}$$

Therefore

$$[Y_1'Y_2 - Y_2'Y_1]_o^c = \int_o^c (\gamma_2^2 - \gamma_1^2)Y_1Y_2\, dy \qquad \text{(A-16)}$$

$$[Y_1'Y_2 - Y_2'Y_1]_c^1 = \int_c^1 (\beta_2^2 - \beta_1^2)Y_1Y_2\, dy \qquad \text{(A-17)}$$

From Equation (A-4), the quantity

$$[Y_1'Y_2 - Y_2'Y_1]$$

vanishes at $y = 0$ and $y = 1$, but not at $y = c$.

Define

$$\phi(c) = Y_1'(c)Y_2(c) - Y_2'(c)Y_1(c) \qquad \text{(A-18)}$$

Then

$$(\gamma_2^2 - \gamma_1^2)\int_o^c Y_1Y_2dy = \phi(c-) \qquad \text{(A-19)}$$

$$(\beta_2^2 - \beta_1^2)\int_c^1 Y_1Y_2dy = -\phi(c+) \qquad \text{(A-20)}$$

Multiplying (A-19) by $\phi(c+)$, (A-20) by $\phi(c-)$ and adding

$$(\gamma_2^2 - \gamma_1^2)\phi(c+)\int_o^c Y_1Y_2dy + (\beta_2^2 - \beta_1^2)\phi(c-)\int_c^1 Y_1Y_2dy = 0 \qquad \text{(A-21)}$$

Rewriting Equation (A-18) in terms of the limits approaching $y \mp c$ from either side

$$\phi(c-) = Y_1'(c-)Y_2(c-) - Y_2'(c-)Y_1(c-) \qquad \text{(A-22)}$$

$$\phi(c+) = Y_1'(c+)Y_2(c+) - Y_2'(c+)Y_1(c+) \qquad \text{(A-23)}$$

But the interface conditions (A-5) and (A-6) are valid for all solutions, Y. Therefore, substituting

$$\phi(c-) = K_2Y_1'(c+)Y_2(c+) - K_2Y_2'(c+)Y_1(c+)$$

$$= K_2\phi(c+) \qquad \text{(A-24)}$$

Substituting into (A-21) and cancelling

$$(\gamma_2^2 - \gamma_1^2) \int_0^c Y_1 Y_2 dy + (\beta_2^2 - \beta_1^2) K_2 \int_c^1 Y_1 Y_2 dy = 0 \tag{A-25}$$

But from the definitions (A-3),

$$(\gamma_2^2 - \gamma_1^2) = K_2(\beta_2^2 - \beta_1^2) \tag{A-26}$$

Substituting (A-26) into (A-25) and cancelling

$$\int_0^c Y_1 Y_2 dy + \int_c^1 Y_1 Y_2 dy = \int_0^1 Y_1 Y_2 dy = 0 \tag{A-27}$$

Since the entire foregoing analysis holds equally as well for $a_{n=\ell}$, $a_{n=r}$ as for a_1, a_2, it can be written in general that

$$\int_0^1 Y_{n=\ell}\, Y_{n=r}\, dy = 0 \tag{A-28}$$

for $\ell \neq r$.

This is the Equation (IV-65) which was to be derived.

Now consider the more general case where K is a continuous function of y, so that $\partial k/\partial y$ is not zero. In this case, the separated equation becomes

$$k(y)Y'' + k'(y)Y' + [a^2 - b^2 k(y)]Y = 0 \tag{A-29}$$

where

$$b = (\frac{2m-1}{2})\frac{\pi}{L}\ , \ m = 1,2,3,\ldots \tag{A-30}$$

This may be written

$$\frac{d}{dy}(kY') + a^2 Y = b^2 kY \tag{A-31}$$

Consider the eigenvalues a_m, a_n with corresponding eigenfunctions Y_m, Y_n.

$$\frac{d}{dy}(kY_m') + a_m^2 Y_m = b^2 kY_m \tag{A-32}$$

$$\frac{d}{dy}(kY_n') + a_n^2 Y_n = b^2 kY_n \tag{A-33}$$

Then multiplying (A-32) by Y_n and (A-33) by $(-Y_m)$ and adding

$$Y_n \frac{d}{dy}(kY_m') - Y_m \frac{d}{dy}(kY_n') + (a_m^2 - a_n^2)Y_mY_n = 0 \qquad \text{(A-34)}$$

Rearranging

$$(a_m^2 - a_n^2)Y_mY_n = Y_m \frac{d}{dy}(kY_n') - Y_n \frac{d}{dy}(kY_m')$$

$$= \frac{d}{dy}[(kY_n')Y_m - (kY_m')Y_n] \qquad \text{(A-35)}$$

Integrating both sides with respect to y over the range $0 < y < 1$

$$(a_m^2 - a_n^2) \int_0^1 Y_mY_n dy = [k(Y_mY_n' - Y_m'Y_n)]_0^1 \qquad \text{(A-36)}$$

But from (A-4)

$$Y_m'(0) = Y_n'(0) = Y_m'(1) = Y_n'(1) = 0 \qquad \text{(A-37)}$$

Therefore

$$(a_m^2 - a_n^2) \int_0^1 Y_mY_n dy = 0 \qquad \text{(A-38)}$$

or for $m \neq n$

$$\int_0^1 Y_mY_n dy = 0 \qquad \text{(A-39)}$$

This is the same condition as was found to apply for two separate and distinct layers.

It should be noted that it is very important that the relationship of the eigenvalues a and b in Equations (A-1) - (A-4) and (A-29) be clearly specified. In all problems in this dissertation, the value b is merely a parameter for the problem in Y, leading to a set of Sturm-Liouville systems with eigenvalues a (one such system for each value of the parameter b). The reverse situation for the same equations, treating a's as the known parameters, leads to an orthogonality condition

$$\int_0^1 kY_mY_n dy = 0 \qquad \text{(A-40)}$$

in which the k(y) appears as a weight function. The Equation (A-40) is not valid for any problems in this dissertation.

APPENDIX B

EIGENVALUES, b, FOR SINGLE-LAYER AND MULTI-LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Characteristic Equation:

$$J_o(b)Y_1(bR) - Y_o(b)J_1(bR) = 0 \qquad \text{(IV-65)}$$

Solved by: Half-interval method on IBM 704

Estimated Accuracy: $\pm$ 2 parts in 10^6

Note: The values shown are photographed from the IBM 704 computer output. The eigenvalues are listed by rows rather than columns (i.e., row number one contains b_1 through b_5, row two contains b_6 through b_{10}, etc.). The format used is one of the "floating point" output formats used by the Fortran System on 704, and can be easily interpreted as follows:

6.321872E00 = 6.321872

6.321872E-01 = 0.6321872

6.321872E01 = 63.21872

6.321872E02 = 632.1872

That is, the number following the "E" in each case represents the power of ten by which the number preceding the "E" should be multiplied to obtain a normal, fixed point number.

b's FOR R=1.5

2.889893E 00	9.344790E 00	1.566013E 01	2.195701E 01	2.824779E 01
3.453581E 01	4.082234E 01	4.710797E 01	5.339303E 01	5.967769E 01
6.596208E 01	7.224625E 01	7.853026E 01	8.481416E 01	9.109795E 01
9.738167E 01	1.036653E 02	1.099489E 02	1.162325E 02	1.225160E 02
1.287995E 02	1.350829E 02	1.413664E 02	1.476498E 02	1.539332E 02
1.602165E 02	1.664999E 02	1.727833E 02	1.790666E 02	1.853499E 02
1.916332E 02	1.979166E 02	2.041998E 02	2.104831E 02	2.167664E 02
2.230497E 02	2.293330E 02	2.356163E 02	2.418995E 02	2.481828E 02
2.544661E 02	2.607493E 02	2.670326E 02	2.733158E 02	2.795991E 02
2.858823E 02	2.921656E 02	2.984488E 02	3.047320E 02	3.110153E 02
3.172985E 02	3.235817E 02	3.298649E 02	3.361481E 02	3.424313E 02
3.487145E 02	3.549978E 02	3.612810E 02	3.675642E 02	3.738474E 02
3.801306E 02	3.864138E 02	3.926970E 02	3.989802E 02	4.052634E 02
4.115466E 02	4.178298E 02	4.241131E 02	4.303963E 02	4.366795E 02
4.429627E 02	4.492459E 02	4.555291E 02	4.618123E 02	4.680955E 02
4.743787E 02	4.806619E 02	4.869451E 02	4.932284E 02	4.995116E 02
5.057948E 02	5.120780E 02	5.183612E 02	5.246444E 02	5.309276E 02
5.372108E 02	5.434940E 02	5.497772E 02	5.560604E 02	5.623436E 02
5.686269E 02	5.749101E 02	5.811933E 02	5.874765E 02	5.937597E 02
6.000429E 02	6.063261E 02	6.126093E 02	6.188925E 02	6.251757E 02

b's FOR R=2.0

1.360777E 00	4.645895E 00	7.814162E 00	1.096714E 01	1.411506E 01
1.726067E 01	2.040505E 01	2.354868E 01	2.669183E 01	2.983466E 01
3.297725E 01	3.611967E 01	3.926195E 01	4.240413E 01	4.554623E 01
4.868826E 01	5.183025E 01	5.497218E 01	5.811409E 01	6.125595E 01
6.439779E 01	6.753961E 01	7.068142E 01	7.382319E 01	7.696496E 01
8.010671E 01	8.324845E 01	8.639018E 01	8.953190E 01	9.267361E 01
9.581531E 01	9.895701E 01	1.020987E 02	1.052404E 02	1.083821E 02
1.115237E 02	1.146654E 02	1.178071E 02	1.209487E 02	1.240904E 02
1.272320E 02	1.303737E 02	1.335153E 02	1.366570E 02	1.397986E 02
1.429403E 02	1.460819E 02	1.492236E 02	1.523652E 02	1.555068E 02
1.586484E 02	1.617900E 02	1.649316E 02	1.680733E 02	1.712149E 02
1.743565E 02	1.774981E 02	1.806397E 02	1.837813E 02	1.869229E 02
1.900645E 02	1.932062E 02	1.963478E 02	1.994894E 02	2.026310E 02
2.057726E 02	2.089142E 02	2.120558E 02	2.151975E 02	2.183391E 02
2.214807E 02	2.246223E 02	2.277639E 02	2.309055E 02	2.340471E 02
2.371887E 02	2.403304E 02	2.434720E 02	2.466136E 02	2.497552E 02
2.528968E 02	2.560384E 02	2.591800E 02	2.623217E 02	2.654633E 02
2.686049E 02	2.717465E 02	2.748881E 02	2.780297E 02	2.811713E 02
2.843130E 02	2.874546E 02	2.905962E 02	2.937378E 02	2.968794E 02
3.000210E 02	3.031626E 02	3.063043E 02	3.094459E 02	3.125875E 02

b's FOR R=2.5

8.660581E-01	3.083539E 00	5.201066E 00	7.305411E 00	9.405344E 00
1.150327E 01	1.360011E 01	1.569629E 01	1.779206E 01	1.988754E 01
2.198281E 01	2.407794E 01	2.617294E 01	2.826785E 01	3.036269E 01
3.245748E 01	3.455222E 01	3.664691E 01	3.874158E 01	4.083621E 01
4.293083E 01	4.502542E 01	4.712000E 01	4.921456E 01	5.130911E 01
5.340365E 01	5.549817E 01	5.759268E 01	5.968719E 01	6.178169E 01
6.387618E 01	6.597067E 01	6.806515E 01	7.015962E 01	7.225409E 01
7.434856E 01	7.644302E 01	7.853748E 01	8.063194E 01	8.272639E 01
8.482084E 01	8.691529E 01	8.900974E 01	9.110417E 01	9.319861E 01
9.529305E 01	9.738749E 01	9.948192E 01	1.015764E 02	1.036708E 02
1.057652E 02	1.078596E 02	1.099540E 02	1.120484E 02	1.141429E 02
1.162373E 02	1.183317E 02	1.204261E 02	1.225205E 02	1.246149E 02
1.267093E 02	1.288037E 02	1.308982E 02	1.329926E 02	1.350870E 02
1.371814E 02	1.392758E 02	1.413702E 02	1.434646E 02	1.455590E 02
1.476535E 02	1.497479E 02	1.518423E 02	1.539367E 02	1.560311E 02
1.581255E 02	1.602199E 02	1.623143E 02	1.644088E 02	1.665032E 02
1.685976E 02	1.706920E 02	1.727864E 02	1.748808E 02	1.769752E 02
1.790696E 02	1.811641E 02	1.832585E 02	1.853529E 02	1.874473E 02
1.895417E 02	1.916361E 02	1.937305E 02	1.958249E 02	1.979194E 02
2.000138E 02	2.021082E 02	2.042026E 02	2.062970E 02	2.083914E 02

b's FOR R=3.0

6.255976E-01	2.304045E 00	3.895421E 00	5.475156E 00	7.050951E 00
8.624941E 00	1.019795E 01	1.177037E 01	1.334241E 01	1.491419E 01
1.648579E 01	1.805724E 01	1.962859E 01	2.119986E 01	2.277106E 01
2.434221E 01	2.591332E 01	2.748439E 01	2.905543E 01	3.062645E 01
3.219744E 01	3.376842E 01	3.533938E 01	3.691033E 01	3.848126E 01
4.005218E 01	4.162310E 01	4.319401E 01	4.476490E 01	4.633579E 01
4.790668E 01	4.947756E 01	5.104843E 01	5.261930E 01	5.419017E 01
5.576103E 01	5.733189E 01	5.890274E 01	6.047359E 01	6.204444E 01
6.361528E 01	6.518613E 01	6.675697E 01	6.832781E 01	6.989865E 01
7.146949E 01	7.304032E 01	7.461115E 01	7.618198E 01	7.775281E 01
7.932362E 01	8.089443E 01	8.246525E 01	8.403606E 01	8.560687E 01
8.717768E 01	8.874850E 01	9.031931E 01	9.189012E 01	9.346093E 01
9.503175E 01	9.660256E 01	9.817337E 01	9.974419E 01	1.013150E 02
1.028858E 02	1.044566E 02	1.060274E 02	1.075982E 02	1.091691E 02
1.107399E 02	1.123107E 02	1.138815E 02	1.154523E 02	1.170231E 02
1.185939E 02	1.201647E 02	1.217356E 02	1.233064E 02	1.248772E 02
1.264480E 02	1.280188E 02	1.295896E 02	1.311604E 02	1.327312E 02
1.343021E 02	1.358729E 02	1.374437E 02	1.390145E 02	1.405853E 02
1.421561E 02	1.437269E 02	1.452977E 02	1.468686E 02	1.484394E 02
1.500102E 02	1.515810E 02	1.531518E 02	1.547226E 02	1.562934E 02

b's FOR R=4.0

3.934562E-01	1.526607E 00	2.590821E 00	3.645581E 00	4.697059E 00
5.747008E 00	6.796123E 00	7.844732E 00	8.893010E 00	9.941064E 00
1.098895E 01	1.203672E 01	1.308440E 01	1.413201E 01	1.517956E 01
1.622708E 01	1.727454E 01	1.832198E 01	1.936939E 01	2.041678E 01
2.146416E 01	2.251151E 01	2.355885E 01	2.460618E 01	2.565349E 01
2.670081E 01	2.774811E 01	2.879540E 01	2.984269E 01	3.088997E 01
3.193724E 01	3.298452E 01	3.403177E 01	3.507904E 01	3.612630E 01
3.717355E 01	3.822080E 01	3.926805E 01	4.031530E 01	4.136254E 01
4.240975E 01	4.345697E 01	4.450419E 01	4.555140E 01	4.659862E 01
4.764583E 01	4.869305E 01	4.974026E 01	5.078748E 01	5.183469E 01
5.288191E 01	5.392912E 01	5.497634E 01	5.602355E 01	5.707077E 01
5.811798E 01	5.916520E 01	6.021241E 01	6.125963E 01	6.230684E 01
6.335406E 01	6.440127E 01	6.544849E 01	6.649570E 01	6.754292E 01
6.859013E 01	6.963735E 01	7.068456E 01	7.173178E 01	7.277900E 01
7.382621E 01	7.487343E 01	7.592064E 01	7.696786E 01	7.801507E 01
7.906229E 01	8.010950E 01	8.115672E 01	8.220393E 01	8.325115E 01
8.429836E 01	8.534558E 01	8.639279E 01	8.744001E 01	8.848722E 01
8.953444E 01	9.058165E 01	9.162887E 01	9.267608E 01	9.372330E 01
9.477051E 01	9.581773E 01	9.686494E 01	9.791216E 01	9.895938E 01
1.000066E 02	1.010538E 02	1.021010E 02	1.031482E 02	1.041954E 02

b's FOR R=5.0

2.823583E-01	1.139214E 00	1.939182E 00	2.731207E 00	3.520405E 00
4.308266E 00	5.095388E 00	5.882061E 00	6.668439E 00	7.454613E 00
8.240639E 00	9.026561E 00	9.812398E 00	1.059817E 01	1.138389E 01
1.216957E 01	1.295522E 01	1.374083E 01	1.452643E 01	1.531200E 01
1.609756E 01	1.688311E 01	1.766863E 01	1.845415E 01	1.923966E 01
2.002516E 01	2.081065E 01	2.159613E 01	2.238161E 01	2.316709E 01
2.395255E 01	2.473802E 01	2.552348E 01	2.630894E 01	2.709439E 01
2.787984E 01	2.866529E 01	2.945073E 01	3.023617E 01	3.102162E 01
3.180703E 01	3.259245E 01	3.337786E 01	3.416328E 01	3.494869E 01
3.573411E 01	3.651952E 01	3.730493E 01	3.809035E 01	3.887576E 01
3.966118E 01	4.044659E 01	4.123201E 01	4.201742E 01	4.280283E 01
4.358825E 01	4.437366E 01	4.515908E 01	4.594449E 01	4.672991E 01
4.751532E 01	4.830073E 01	4.908615E 01	4.987156E 01	5.065698E 01
5.144239E 01	5.222781E 01	5.301322E 01	5.379864E 01	5.458405E 01
5.536946E 01	5.615488E 01	5.694029E 01	5.772571E 01	5.851112E 01
5.929654E 01	6.008195E 01	6.086736E 01	6.165278E 01	6.243819E 01
6.322361E 01	6.400902E 01	6.479444E 01	6.557985E 01	6.636527E 01
6.715068E 01	6.793609E 01	6.872151E 01	6.950692E 01	7.029234E 01
7.107775E 01	7.186317E 01	7.264858E 01	7.343400E 01	7.421941E 01
7.500482E 01	7.579024E 01	7.657565E 01	7.736107E 01	7.814648E 01

b's FOR R=6.0

2.180720E-01	9.074934E-01	1.548563E 00	2.182803E 00	2.814564E 00
3.445132E 00	4.075036E 00	4.704531E 00	5.333756E 00	5.962794E 00
6.591701E 00	7.220503E 00	7.849228E 00	8.477897E 00	9.106516E 00
9.735099E 00	1.036365E 01	1.099217E 01	1.162067E 01	1.224916E 01
1.287762E 01	1.350608E 01	1.413452E 01	1.476295E 01	1.539137E 01
1.601978E 01	1.664819E 01	1.727660E 01	1.790499E 01	1.853338E 01
1.916176E 01	1.979014E 01	2.041852E 01	2.104689E 01	2.167526E 01
2.230363E 01	2.293199E 01	2.356035E 01	2.418871E 01	2.481707E 01
2.544540E 01	2.607374E 01	2.670207E 01	2.733040E 01	2.795874E 01
2.858707E 01	2.921541E 01	2.984374E 01	3.047207E 01	3.110041E 01
3.172874E 01	3.235707E 01	3.298541E 01	3.361374E 01	3.424207E 01
3.487041E 01	3.549874E 01	3.612708E 01	3.675541E 01	3.738374E 01
3.801208E 01	3.864041E 01	3.926874E 01	3.989708E 01	4.052541E 01
4.115375E 01	4.178208E 01	4.241041E 01	4.303875E 01	4.366708E 01
4.429541E 01	4.492375E 01	4.555208E 01	4.618042E 01	4.680875E 01
4.743708E 01	4.806542E 01	4.869375E 01	4.932208E 01	4.995042E 01
5.057875E 01	5.120708E 01	5.183542E 01	5.246375E 01	5.309208E 01
5.372042E 01	5.434875E 01	5.497709E 01	5.560542E 01	5.623375E 01
5.686209E 01	5.749042E 01	5.811875E 01	5.874709E 01	5.937542E 01
6.000376E 01	6.063209E 01	6.126042E 01	6.188876E 01	6.251709E 01

b's FOR R=7.0

1.765096E-01	7.534490E-01	1.288383E 00	1.817344E 00	2.344099E 00
2.869780E 00	3.394854E 00	3.919552E 00	4.444002E 00	4.968277E 00
5.492429E 00	6.016489E 00	6.540472E 00	7.064402E 00	7.588288E 00
8.112130E 00	8.635947E 00	9.159746E 00	9.683518E 00	1.020727E 01
1.073101E 01	1.125474E 01	1.177845E 01	1.230216E 01	1.282586E 01
1.334954E 01	1.387323E 01	1.439690E 01	1.492058E 01	1.544424E 01
1.596790E 01	1.649156E 01	1.701521E 01	1.753887E 01	1.806252E 01
1.858616E 01	1.910980E 01	1.963344E 01	2.015707E 01	2.068072E 01
2.120433E 01	2.172795E 01	2.225156E 01	2.277517E 01	2.329878E 01
2.382240E 01	2.434601E 01	2.486962E 01	2.539324E 01	2.591685E 01
2.644046E 01	2.696408E 01	2.748769E 01	2.801130E 01	2.853492E 01
2.905853E 01	2.958214E 01	3.010575E 01	3.062937E 01	3.115298E 01
3.167659E 01	3.220021E 01	3.272382E 01	3.324743E 01	3.377105E 01
3.429466E 01	3.481827E 01	3.534188E 01	3.586550E 01	3.638911E 01
3.691272E 01	3.743634E 01	3.795995E 01	3.848356E 01	3.900718E 01
3.953079E 01	4.005440E 01	4.057802E 01	4.110163E 01	4.162524E 01
4.214885E 01	4.267247E 01	4.319608E 01	4.371969E 01	4.424331E 01
4.476692E 01	4.529053E 01	4.581415E 01	4.633776E 01	4.686137E 01
4.738498E 01	4.790860E 01	4.843221E 01	4.895582E 01	4.947944E 01
5.000305E 01	5.052666E 01	5.105028E 01	5.157389E 01	5.209750E 01

b's FOR R=8.0

1.475980E-01	6.437055E-01	1.102702E 00	1.556404E 00	2.008123E 00
2.458863E 00	2.909047E 00	3.358883E 00	3.808487E 00	4.257929E 00
4.707253E 00	5.156491E 00	5.605653E 00	6.054769E 00	6.503839E 00
6.952878E 00	7.401887E 00	7.850873E 00	8.299836E 00	8.748784E 00
9.197717E 00	9.646642E 00	1.009555E 01	1.054445E 01	1.099335E 01
1.144223E 01	1.189111E 01	1.233998E 01	1.278885E 01	1.323772E 01
1.368658E 01	1.413543E 01	1.458429E 01	1.503313E 01	1.548198E 01
1.593083E 01	1.637967E 01	1.682851E 01	1.727734E 01	1.772617E 01
1.817498E 01	1.862380E 01	1.907261E 01	1.952142E 01	1.997023E 01
2.041905E 01	2.086786E 01	2.131667E 01	2.176549E 01	2.221430E 01
2.266311E 01	2.311192E 01	2.356074E 01	2.400955E 01	2.445836E 01
2.490718E 01	2.535599E 01	2.580480E 01	2.625361E 01	2.670243E 01
2.715124E 01	2.760005E 01	2.804887E 01	2.849768E 01	2.894649E 01
2.939531E 01	2.984412E 01	3.029293E 01	3.074174E 01	3.119056E 01
3.163937E 01	3.208818E 01	3.253700E 01	3.298581E 01	3.343462E 01
3.388343E 01	3.433225E 01	3.478106E 01	3.522987E 01	3.567869E 01
3.612750E 01	3.657631E 01	3.702512E 01	3.747394E 01	3.792275E 01
3.837156E 01	3.882038E 01	3.926919E 01	3.971800E 01	4.016681E 01
4.061563E 01	4.106444E 01	4.151325E 01	4.196207E 01	4.241088E 01
4.285969E 01	4.330851E 01	4.375732E 01	4.420613E 01	4.465494E 01

b's FOR R= 10.0

1.102686E-01	4.978831E-01	8.554285E-01	1.208680E 00	1.560290E 00
1.911073E 00	2.261379E 00	2.611381E 00	2.961181E 00	3.310834E 00
3.660381E 00	4.009851E 00	4.359258E 00	4.708615E 00	5.057936E 00
5.407223E 00	5.756486E 00	6.105724E 00	6.454947E 00	6.804152E 00
7.153345E 00	7.502525E 00	7.851696E 00	8.200858E 00	8.550014E 00
8.899161E 00	9.248301E 00	9.597437E 00	9.946569E 00	1.029569E 01
1.064482E 01	1.099394E 01	1.134305E 01	1.169217E 01	1.204128E 01
1.239038E 01	1.273949E 01	1.308859E 01	1.343770E 01	1.378679E 01
1.413589E 01	1.448499E 01	1.483408E 01	1.518318E 01	1.553227E 01
1.588136E 01	1.623045E 01	1.657954E 01	1.692863E 01	1.727772E 01
1.762679E 01	1.797587E 01	1.832495E 01	1.867402E 01	1.902310E 01
1.937218E 01	1.972125E 01	2.007033E 01	2.041940E 01	2.076848E 01
2.111756E 01	2.146663E 01	2.181571E 01	2.216479E 01	2.251386E 01
2.286294E 01	2.321201E 01	2.356109E 01	2.391017E 01	2.425924E 01
2.460832E 01	2.495740E 01	2.530647E 01	2.565555E 01	2.600463E 01
2.635370E 01	2.670278E 01	2.705185E 01	2.740093E 01	2.775001E 01
2.809908E 01	2.844816E 01	2.879724E 01	2.914631E 01	2.949539E 01
2.984446E 01	3.019354E 01	3.054262E 01	3.089169E 01	3.124077E 01
3.158985E 01	3.193892E 01	3.228800E 01	3.263707E 01	3.298615E 01
3.333523E 01	3.368430E 01	3.403338E 01	3.438246E 01	3.473153E 01
3.508060E 01	3.542967E 01	3.577874E 01	3.612781E 01	3.647688E 01
3.682595E 01	3.717502E 01	3.752408E 01	3.787315E 01	3.822222E 01
3.857129E 01	3.892036E 01	3.926943E 01	3.961850E 01	3.996757E 01
4.031664E 01	4.066571E 01	4.101478E 01	4.136384E 01	4.171291E 01
4.206198E 01	4.241105E 01	4.276012E 01	4.310919E 01	4.345826E 01
4.380733E 01	4.415640E 01	4.450547E 01	4.485454E 01	4.520360E 01
4.555267E 01	4.590174E 01	4.625081E 01	4.659988E 01	4.694895E 01
4.729802E 01	4.764709E 01	4.799616E 01	4.834523E 01	4.869430E 01
4.904336E 01	4.939243E 01	4.974150E 01	5.009057E 01	5.043964E 01
5.078871E 01	5.113778E 01	5.148685E 01	5.183592E 01	5.218499E 01
5.253405E 01	5.288312E 01	5.323219E 01	5.358126E 01	5.393033E 01
5.427940E 01	5.462846E 01	5.497753E 01	5.532660E 01	5.567567E 01
5.602474E 01	5.637381E 01	5.672288E 01	5.707194E 01	5.742101E 01
5.777008E 01	5.811915E 01	5.846822E 01	5.881729E 01	5.916635E 01
5.951542E 01	5.986449E 01	6.021356E 01	6.056263E 01	6.091170E 01
6.126077E 01	6.160983E 01	6.195890E 01	6.230797E 01	6.265704E 01
6.300611E 01	6.335518E 01	6.370424E 01	6.405331E 01	6.440238E 01
6.475145E 01	6.510052E 01	6.544959E 01	6.579865E 01	6.614772E 01
6.649679E 01	6.684586E 01	6.719493E 01	6.754400E 01	6.789306E 01
6.824213E 01	6.859120E 01	6.894027E 01	6.928934E 01	6.963841E 01

b's FOR R= 12.0

8.739113E-02	4.054898E-01	6.983276E-01	9.875770E-01	1.275430E 00
1.562574E 00	1.849298E 00	2.135755E 00	2.422029E 00	2.708173E 00
2.994223E 00	3.280197E 00	3.566117E 00	3.851992E 00	4.137831E 00
4.423639E 00	4.709422E 00	4.995187E 00	5.280935E 00	5.566667E 00
5.852386E 00	6.138095E 00	6.423794E 00	6.709485E 00	6.995169E 00
7.280845E 00	7.566519E 00	7.852183E 00	8.137848E 00	8.423502E 00
8.709159E 00	8.994809E 00	9.280456E 00	9.566097E 00	9.851740E 00
1.013738E 01	1.042302E 01	1.070865E 01	1.099429E 01	1.127992E 01
1.156555E 01	1.185118E 01	1.213681E 01	1.242243E 01	1.270806E 01
1.299368E 01	1.327930E 01	1.356492E 01	1.385054E 01	1.413616E 01
1.442178E 01	1.470740E 01	1.499302E 01	1.527864E 01	1.556425E 01
1.584987E 01	1.613548E 01	1.642110E 01	1.670671E 01	1.699233E 01
1.727794E 01	1.756355E 01	1.784916E 01	1.813478E 01	1.842039E 01
1.870600E 01	1.899161E 01	1.927722E 01	1.956283E 01	1.984844E 01
2.013405E 01	2.041966E 01	2.070526E 01	2.099087E 01	2.127648E 01
2.156209E 01	2.184770E 01	2.213331E 01	2.241891E 01	2.270452E 01
2.299013E 01	2.327574E 01	2.356134E 01	2.384695E 01	2.413255E 01
2.441816E 01	2.470377E 01	2.498937E 01	2.527498E 01	2.556058E 01
2.584619E 01	2.613179E 01	2.641740E 01	2.670300E 01	2.698862E 01
2.727421E 01	2.755983E 01	2.784542E 01	2.813103E 01	2.841664E 01
2.870225E 01	2.898785E 01	2.927345E 01	2.955905E 01	2.984466E 01
3.013026E 01	3.041586E 01	3.070147E 01	3.098707E 01	3.127267E 01
3.155827E 01	3.184388E 01	3.212948E 01	3.241508E 01	3.270069E 01
3.298629E 01	3.327189E 01	3.355749E 01	3.384310E 01	3.412870E 01
3.441430E 01	3.469990E 01	3.498551E 01	3.527111E 01	3.555671E 01
3.584232E 01	3.612792E 01	3.641352E 01	3.669912E 01	3.698472E 01
3.727033E 01	3.755593E 01	3.784153E 01	3.812713E 01	3.841273E 01
3.869834E 01	3.898394E 01	3.926954E 01	3.955514E 01	3.984075E 01
4.012635E 01	4.041195E 01	4.069755E 01	4.098316E 01	4.126876E 01
4.155436E 01	4.183996E 01	4.212556E 01	4.241117E 01	4.269677E 01
4.298237E 01	4.326797E 01	4.355357E 01	4.383917E 01	4.412478E 01
4.441038E 01	4.469598E 01	4.498158E 01	4.526718E 01	4.555278E 01
4.583839E 01	4.612399E 01	4.640959E 01	4.669519E 01	4.698079E 01
4.726639E 01	4.755200E 01	4.783760E 01	4.812320E 01	4.840880E 01
4.869440E 01	4.898000E 01	4.926561E 01	4.955121E 01	4.983681E 01
5.012241E 01	5.040801E 01	5.069361E 01	5.097921E 01	5.126481E 01
5.155041E 01	5.183601E 01	5.212161E 01	5.240721E 01	5.269281E 01
5.297841E 01	5.326401E 01	5.354962E 01	5.383522E 01	5.412082E 01
5.440642E 01	5.469202E 01	5.497762E 01	5.526322E 01	5.554882E 01
5.583442E 01	5.612002E 01	5.640562E 01	5.669122E 01	5.697682E 01

b's FOR R= 14.0

7.203469E-02	3.417686E-01	5.897348E-01	8.346231E-01	1.078308E 00
1.321370E 00	1.564061E 00	1.806516E 00	2.048806E 00	2.290974E 00
2.533059E 00	2.775074E 00	3.017037E 00	3.258956E 00	3.500845E 00
3.742708E 00	3.984544E 00	4.226363E 00	4.468165E 00	4.709953E 00
4.951728E 00	5.193495E 00	5.435253E 00	5.677002E 00	5.918742E 00
6.160479E 00	6.402209E 00	6.643934E 00	6.885658E 00	7.127372E 00
7.369089E 00	7.610796E 00	7.852505E 00	8.094208E 00	8.335909E 00
8.577612E 00	8.819310E 00	9.061006E 00	9.302699E 00	9.544391E 00
9.786081E 00	1.002777E 01	1.026946E 01	1.051115E 01	1.075283E 01
1.099452E 01	1.123620E 01	1.147788E 01	1.171956E 01	1.196125E 01
1.220292E 01	1.244460E 01	1.268628E 01	1.292796E 01	1.316964E 01
1.341131E 01	1.365299E 01	1.389467E 01	1.413634E 01	1.437802E 01
1.461969E 01	1.486137E 01	1.510304E 01	1.534471E 01	1.558638E 01
1.582806E 01	1.606973E 01	1.631140E 01	1.655307E 01	1.679474E 01
1.703641E 01	1.727809E 01	1.751975E 01	1.776142E 01	1.800309E 01
1.824477E 01	1.848643E 01	1.872810E 01	1.896977E 01	1.921144E 01
1.945311E 01	1.969478E 01	1.993644E 01	2.017811E 01	2.041978E 01
2.066145E 01	2.090312E 01	2.114478E 01	2.138645E 01	2.162812E 01
2.186978E 01	2.211145E 01	2.235312E 01	2.259478E 01	2.283645E 01
2.307811E 01	2.331979E 01	2.356145E 01	2.380311E 01	2.404477E 01
2.428644E 01	2.452810E 01	2.476977E 01	2.501143E 01	2.525309E 01
2.549476E 01	2.573643E 01	2.597809E 01	2.621975E 01	2.646142E 01
2.670309E 01	2.694475E 01	2.718641E 01	2.742808E 01	2.766974E 01
2.791141E 01	2.815307E 01	2.839474E 01	2.863640E 01	2.887807E 01
2.911973E 01	2.936140E 01	2.960306E 01	2.984473E 01	3.008639E 01
3.032806E 01	3.056972E 01	3.081139E 01	3.105305E 01	3.129472E 01
3.153638E 01	3.177804E 01	3.201971E 01	3.226137E 01	3.250304E 01
3.274470E 01	3.298637E 01	3.322803E 01	3.346969E 01	3.371136E 01
3.395302E 01	3.419469E 01	3.443635E 01	3.467802E 01	3.491968E 01
3.516135E 01	3.540301E 01	3.564467E 01	3.588634E 01	3.612800E 01
3.636966E 01	3.661133E 01	3.685299E 01	3.709465E 01	3.733631E 01
3.757797E 01	3.781964E 01	3.806130E 01	3.830296E 01	3.854462E 01
3.878628E 01	3.902795E 01	3.926961E 01	3.951127E 01	3.975293E 01
3.999459E 01	4.023626E 01	4.047792E 01	4.071958E 01	4.096124E 01
4.120290E 01	4.144456E 01	4.168623E 01	4.192789E 01	4.216955E 01
4.241121E 01	4.265288E 01	4.289454E 01	4.313620E 01	4.337786E 01
4.361953E 01	4.386119E 01	4.410285E 01	4.434452E 01	4.458618E 01
4.482784E 01	4.506950E 01	4.531117E 01	4.555283E 01	4.579449E 01
4.603616E 01	4.627782E 01	4.651948E 01	4.676114E 01	4.700281E 01
4.724447E 01	4.748613E 01	4.772779E 01	4.796946E 01	4.821112E 01

b's FOR R= 16.0

6.106301E-02	2.952034E-01	5.102131E-01	7.225449E-01	9.338200E-01
1.144538E 00	1.354932E 00	1.565105E 00	1.775132E 00	1.985049E 00
2.194886E 00	2.404662E 00	2.614389E 00	2.824077E 00	3.033734E 00
3.243365E 00	3.452974E 00	3.662563E 00	3.872140E 00	4.081702E 00
4.291253E 00	4.500793E 00	4.710324E 00	4.919849E 00	5.129369E 00
5.338882E 00	5.548387E 00	5.757890E 00	5.967386E 00	6.176880E 00
6.386372E 00	6.595860E 00	6.805343E 00	7.014824E 00	7.224304E 00
7.433782E 00	7.643255E 00	7.852730E 00	8.062200E 00	8.271672E 00
8.481142E 00	8.690607E 00	8.900073E 00	9.109538E 00	9.319001E 00
9.528465E 00	9.737924E 00	9.947385E 00	1.015684E 01	1.036630E 01
1.057576E 01	1.078522E 01	1.099468E 01	1.120413E 01	1.141359E 01
1.162304E 01	1.183250E 01	1.204195E 01	1.225141E 01	1.246086E 01
1.267031E 01	1.287976E 01	1.308921E 01	1.329867E 01	1.350812E 01
1.371757E 01	1.392702E 01	1.413647E 01	1.434592E 01	1.455537E 01
1.476481E 01	1.497427E 01	1.518371E 01	1.539316E 01	1.560261E 01
1.581206E 01	1.602150E 01	1.623095E 01	1.644040E 01	1.664985E 01
1.685930E 01	1.706874E 01	1.727819E 01	1.748763E 01	1.769708E 01
1.790652E 01	1.811597E 01	1.832542E 01	1.853486E 01	1.874431E 01
1.895375E 01	1.916320E 01	1.937264E 01	1.958209E 01	1.979153E 01
2.000098E 01	2.021043E 01	2.041988E 01	2.062932E 01	2.083875E 01
2.104819E 01	2.125763E 01	2.146708E 01	2.167652E 01	2.188597E 01
2.209541E 01	2.230485E 01	2.251430E 01	2.272374E 01	2.293318E 01
2.314263E 01	2.335207E 01	2.356151E 01	2.377096E 01	2.398040E 01
2.418985E 01	2.439929E 01	2.460873E 01	2.481818E 01	2.502762E 01
2.523706E 01	2.544651E 01	2.565595E 01	2.586539E 01	2.607484E 01
2.628428E 01	2.649372E 01	2.670317E 01	2.691261E 01	2.712205E 01
2.733149E 01	2.754093E 01	2.775038E 01	2.795982E 01	2.816926E 01
2.837870E 01	2.858815E 01	2.879759E 01	2.900703E 01	2.921647E 01
2.942592E 01	2.963536E 01	2.984480E 01	3.005424E 01	3.026369E 01
3.047313E 01	3.068257E 01	3.089201E 01	3.110146E 01	3.131090E 01
3.152034E 01	3.172978E 01	3.193922E 01	3.214866E 01	3.235810E 01
3.256754E 01	3.277699E 01	3.298643E 01	3.319587E 01	3.340531E 01
3.361475E 01	3.382419E 01	3.403363E 01	3.424307E 01	3.445252E 01
3.466196E 01	3.487140E 01	3.508084E 01	3.529028E 01	3.549972E 01
3.570916E 01	3.591860E 01	3.612804E 01	3.633749E 01	3.654693E 01
3.675637E 01	3.696581E 01	3.717525E 01	3.738469E 01	3.759413E 01
3.780357E 01	3.801301E 01	3.822245E 01	3.843189E 01	3.864133E 01
3.885077E 01	3.906021E 01	3.926965E 01	3.947909E 01	3.968853E 01
3.989797E 01	4.010741E 01	4.031685E 01	4.052629E 01	4.073573E 01
4.094517E 01	4.115461E 01	4.136405E 01	4.157349E 01	4.178293E 01

b's FOR R= 18.0

5.285988E-02	2.596995E-01	4.494867E-01	6.369002E-01	8.233744E-01
1.009357E 00	1.195038E 00	1.380525E 00	1.565874E 00	1.751127E 00
1.936303E 00	2.121421E 00	2.306494E 00	2.491531E 00	2.676540E 00
2.861522E 00	3.046484E 00	3.231429E 00	3.416360E 00	3.601278E 00
3.786186E 00	3.971083E 00	4.155971E 00	4.340854E 00	4.525731E 00
4.710602E 00	4.895465E 00	5.080327E 00	5.265182E 00	5.450035E 00
5.634885E 00	5.819730E 00	6.004574E 00	6.189414E 00	6.374252E 00
6.559089E 00	6.743924E 00	6.928755E 00	7.113588E 00	7.298416E 00
7.483246E 00	7.668071E 00	7.852897E 00	8.037720E 00	8.222543E 00
8.407365E 00	8.592188E 00	8.777007E 00	8.961827E 00	9.146646E 00
9.331463E 00	9.516281E 00	9.701098E 00	9.885913E 00	1.007073E 01
1.025554E 01	1.044036E 01	1.062517E 01	1.080998E 01	1.099480E 01
1.117961E 01	1.136442E 01	1.154923E 01	1.173404E 01	1.191885E 01
1.210366E 01	1.228847E 01	1.247328E 01	1.265809E 01	1.284290E 01
1.302771E 01	1.321252E 01	1.339733E 01	1.358214E 01	1.376695E 01
1.395175E 01	1.413656E 01	1.432137E 01	1.450617E 01	1.469098E 01
1.487579E 01	1.506059E 01	1.524540E 01	1.543021E 01	1.561501E 01
1.579982E 01	1.598463E 01	1.616943E 01	1.635424E 01	1.653904E 01
1.672385E 01	1.690865E 01	1.709346E 01	1.727826E 01	1.746306E 01
1.764787E 01	1.783267E 01	1.801748E 01	1.820228E 01	1.838709E 01
1.857189E 01	1.875670E 01	1.894150E 01	1.912630E 01	1.931111E 01
1.949591E 01	1.968071E 01	1.986552E 01	2.005032E 01	2.023512E 01
2.041993E 01	2.060473E 01	2.078954E 01	2.097434E 01	2.115914E 01
2.134395E 01	2.152875E 01	2.171355E 01	2.189836E 01	2.208316E 01
2.226797E 01	2.245277E 01	2.263757E 01	2.282238E 01	2.300718E 01
2.319198E 01	2.337678E 01	2.356158E 01	2.374639E 01	2.393119E 01
2.411599E 01	2.430079E 01	2.448559E 01	2.467040E 01	2.485520E 01
2.504000E 01	2.522480E 01	2.540960E 01	2.559440E 01	2.577921E 01
2.596401E 01	2.614881E 01	2.633361E 01	2.651841E 01	2.670322E 01
2.688802E 01	2.707282E 01	2.725762E 01	2.744242E 01	2.762722E 01
2.781203E 01	2.799683E 01	2.818163E 01	2.836643E 01	2.855123E 01
2.873603E 01	2.892083E 01	2.910563E 01	2.929044E 01	2.947524E 01
2.966004E 01	2.984484E 01	3.002964E 01	3.021444E 01	3.039925E 01
3.058405E 01	3.076885E 01	3.095365E 01	3.113845E 01	3.132325E 01
3.150805E 01	3.169285E 01	3.187766E 01	3.206246E 01	3.224726E 01
3.243206E 01	3.261686E 01	3.280166E 01	3.298646E 01	3.317126E 01
3.335606E 01	3.354086E 01	3.372566E 01	3.391046E 01	3.409526E 01
3.428006E 01	3.446487E 01	3.464966E 01	3.483447E 01	3.501927E 01
3.520407E 01	3.538887E 01	3.557367E 01	3.575847E 01	3.594327E 01
3.612807E 01	3.631287E 01	3.649767E 01	3.668247E 01	3.686727E 01

b's FOR R= 20.0

4.650735E-02	2.317506E-01	4.016038E-01	5.693360E-01	7.362229E-01
9.026642E-01	1.068834E 00	1.234827E 00	1.400691E 00	1.566464E 00
1.732170E 00	1.897821E 00	2.063429E 00	2.229003E 00	2.394551E 00
2.560073E 00	2.725578E 00	2.891064E 00	3.056538E 00	3.221999E 00
3.387452E 00	3.552895E 00	3.718329E 00	3.883757E 00	4.049176E 00
4.214592E 00	4.380005E 00	4.545411E 00	4.710815E 00	4.876211E 00
5.041604E 00	5.207000E 00	5.372388E 00	5.537774E 00	5.703163E 00
5.868544E 00	6.033927E 00	6.199303E 00	6.364681E 00	6.530056E 00
6.695434E 00	6.860805E 00	7.026178E 00	7.191548E 00	7.356921E 00
7.522291E 00	7.687659E 00	7.853024E 00	8.018392E 00	8.183756E 00
8.349124E 00	8.514488E 00	8.679850E 00	8.845214E 00	9.010575E 00
9.175939E 00	9.341300E 00	9.506658E 00	9.672019E 00	9.837382E 00
1.000274E 01	1.016810E 01	1.033345E 01	1.049882E 01	1.066417E 01
1.082953E 01	1.099489E 01	1.116025E 01	1.132560E 01	1.149096E 01
1.165631E 01	1.182167E 01	1.198703E 01	1.215238E 01	1.231774E 01
1.248309E 01	1.264845E 01	1.281380E 01	1.297916E 01	1.314451E 01
1.330986E 01	1.347522E 01	1.364057E 01	1.380593E 01	1.397128E 01
1.413663E 01	1.430199E 01	1.446734E 01	1.463269E 01	1.479805E 01
1.496340E 01	1.512875E 01	1.529410E 01	1.545946E 01	1.562480E 01
1.579016E 01	1.595551E 01	1.612087E 01	1.628622E 01	1.645156E 01
1.661691E 01	1.678226E 01	1.694761E 01	1.711296E 01	1.727831E 01
1.744366E 01	1.760901E 01	1.777436E 01	1.793971E 01	1.810506E 01
1.827042E 01	1.843577E 01	1.860112E 01	1.876647E 01	1.893182E 01
1.909717E 01	1.926252E 01	1.942787E 01	1.959322E 01	1.975857E 01
1.992392E 01	2.008927E 01	2.025462E 01	2.041997E 01	2.058532E 01
2.075067E 01	2.091602E 01	2.108137E 01	2.124672E 01	2.141207E 01
2.157742E 01	2.174277E 01	2.190812E 01	2.207347E 01	2.223882E 01
2.240417E 01	2.256952E 01	2.273487E 01	2.290022E 01	2.306557E 01
2.323092E 01	2.339627E 01	2.356162E 01	2.372697E 01	2.389232E 01
2.405767E 01	2.422302E 01	2.438837E 01	2.455372E 01	2.471907E 01
2.488442E 01	2.504976E 01	2.521511E 01	2.538046E 01	2.554581E 01
2.571116E 01	2.587651E 01	2.604186E 01	2.620720E 01	2.637255E 01
2.653790E 01	2.670325E 01	2.686860E 01	2.703395E 01	2.719930E 01
2.736465E 01	2.752999E 01	2.769534E 01	2.786069E 01	2.802604E 01
2.819139E 01	2.835674E 01	2.852209E 01	2.868744E 01	2.885278E 01
2.901813E 01	2.918348E 01	2.934883E 01	2.951418E 01	2.967952E 01
2.984487E 01	3.001022E 01	3.017557E 01	3.034092E 01	3.050626E 01
3.067161E 01	3.083696E 01	3.100231E 01	3.116766E 01	3.133300E 01
3.149835E 01	3.166370E 01	3.182905E 01	3.199439E 01	3.215974E 01
3.232509E 01	3.249044E 01	3.265579E 01	3.282114E 01	3.298648E 01

b's FOR R= 50.0

1.578811E-02	8.792922E-02	1.538057E-01	2.189009E-01	2.836836E-01
3.483003E-01	4.128096E-01	4.772456E-01	5.416328E-01	6.059760E-01
6.702900E-01	7.345795E-01	7.988495E-01	8.631000E-01	9.273407E-01
9.915667E-01	1.055783E 00	1.119989E 00	1.184186E 00	1.248383E 00
1.312570E 00	1.376747E 00	1.440924E 00	1.505097E 00	1.569264E 00
1.633432E 00	1.697593E 00	1.761749E 00	1.825907E 00	1.890060E 00
1.954210E 00	2.018362E 00	2.082509E 00	2.146654E 00	2.210800E 00
2.274941E 00	2.339083E 00	2.403224E 00	2.467363E 00	2.531500E 00
2.595638E 00	2.659771E 00	2.723906E 00	2.788043E 00	2.852174E 00
2.916306E 00	2.980437E 00	3.044570E 00	3.108701E 00	3.172330E 00
3.236960E 00	3.301089E 00	3.365216E 00	3.429345E 00	3.493471E 00
3.557600E 00	3.621727E 00	3.685851E 00	3.749978E 00	3.814102E 00
3.878229E 00	3.942353E 00	4.006476E 00	4.070500E 00	4.134723E 00
4.198847E 00	4.262969E 00	4.327093E 00	4.391215E 00	4.455339E 00
4.519461E 00	4.583584E 00	4.647706E 00	4.711826E 00	4.775947E 00
4.840071E 00	4.904192E 00	4.968311E 00	5.032432E 00	5.096551E 00
5.160672E 00	5.224791E 00	5.288912E 00	5.353035E 00	5.417151E 00
5.481274E 00	5.545390E 00	5.609512E 00	5.673633E 00	5.737751E 00
5.801871E 00	5.865989E 00	5.930108E 00	5.994226E 00	6.058344E 00
6.122464E 00	6.186581E 00	6.250700E 00	6.314822E 00	6.378940E 00
6.443058E 00	6.507175E 00	6.571292E 00	6.635410E 00	6.699527E 00
6.763645E 00	6.827762E 00	6.891880E 00	6.955997E 00	7.020114E 00
7.084232E 00	7.148349E 00	7.212467E 00	7.276584E 00	7.340702E 00
7.404819E 00	7.468936E 00	7.533054E 00	7.597171E 00	7.661289E 00
7.725406E 00	7.789524E 00	7.853641E 00	7.917759E 00	7.981876E 00
8.045992E 00	8.110108E 00	8.174225E 00	8.238341E 00	8.302457E 00
8.366574E 00	8.430690E 00	8.494806E 00	8.558922E 00	8.623039E 00
8.687155E 00	8.751271E 00	8.815388E 00	8.879504E 00	8.943620E 00
9.007737E 00	9.071853E 00	9.135969E 00	9.200085E 00	9.264202E 00
9.328318E 00	9.392434E 00	9.456551E 00	9.520667E 00	9.584783E 00
9.648899E 00	9.713014E 00	9.777130E 00	9.841246E 00	9.905361E 00
9.969477E 00	1.003359E 01	1.009771E 01	1.016182E 01	1.022594E 01
1.029005E 01	1.035417E 01	1.041829E 01	1.048240E 01	1.054652E 01
1.061063E 01	1.067475E 01	1.073886E 01	1.080298E 01	1.086710E 01
1.093121E 01	1.099533E 01	1.105944E 01	1.112356E 01	1.118767E 01
1.125179E 01	1.131590E 01	1.138002E 01	1.144414E 01	1.150825E 01
1.157237E 01	1.163648E 01	1.170060E 01	1.176471E 01	1.182883E 01
1.189294E 01	1.195706E 01	1.202117E 01	1.208529E 01	1.214940E 01
1.221352E 01	1.227763E 01	1.234175E 01	1.240587E 01	1.246998E 01
1.253410E 01	1.259821E 01	1.266233E 01	1.272644E 01	1.279056E 01

b's FOR R= 75.0

9.924828E-03	5.771942E-02	1.013148E-01	1.444056E-01	1.873011E-01
2.300892E-01	2.728122E-01	3.154863E-01	3.581312E-01	4.007467E-01
4.433460E-01	4.859257E-01	5.284925E-01	5.710494E-01	6.135966E-01
6.561340E-01	6.986649E-01	7.411926E-01	7.837105E-01	8.262251E-01
8.687365E-01	9.112446E-01	9.537462E-01	9.962479E-01	1.038746E 00
1.081241E 00	1.123734E 00	1.166224E 00	1.208711E 00	1.251201E 00
1.293684E 00	1.336169E 00	1.378651E 00	1.421131E 00	1.463609E 00
1.506089E 00	1.548566E 00	1.591041E 00	1.633518E 00	1.675993E 00
1.718466E 00	1.760940E 00	1.803413E 00	1.845882E 00	1.888355E 00
1.930824E 00	1.973292E 00	2.015761E 00	2.058228E 00	2.100698E 00
2.143165E 00	2.185634E 00	2.228101E 00	2.270566E 00	2.313033E 00
2.355497E 00	2.397964E 00	2.440428E 00	2.482890E 00	2.525354E 00
2.567820E 00	2.610284E 00	2.652746E 00	2.695210E 00	2.737671E 00
2.780135E 00	2.822596E 00	2.865055E 00	2.907521E 00	2.949980E 00
2.992441E 00	3.034905E 00	3.077366E 00	3.119824E 00	3.162285E 00
3.204749E 00	3.247209E 00	3.289668E 00	3.332129E 00	3.374587E 00
3.417048E 00	3.459506E 00	3.501966E 00	3.544424E 00	3.586885E 00
3.629343E 00	3.671803E 00	3.714261E 00	3.756721E 00	3.799179E 00
3.841639E 00	3.884097E 00	3.926557E 00	3.969015E 00	4.011475E 00
4.053931E 00	4.096390E 00	4.138847E 00	4.181306E 00	4.223762E 00
4.266220E 00	4.308677E 00	4.351136E 00	4.393594E 00	4.436051E 00
4.478510E 00	4.520965E 00	4.563423E 00	4.605883E 00	4.648339E 00
4.690794E 00	4.733251E 00	4.775711E 00	4.818166E 00	4.860625E 00
4.903079E 00	4.945536E 00	4.987995E 00	5.030450E 00	5.072908E 00
5.115364E 00	5.157823E 00	5.200278E 00	5.242735E 00	5.285191E 00
5.327646E 00	5.370103E 00	5.412559E 00	5.455017E 00	5.497473E 00
5.539929E 00	5.582387E 00	5.624841E 00	5.667297E 00	5.709752E 00
5.752210E 00	5.794666E 00	5.837121E 00	5.879578E 00	5.922032E 00
5.964491E 00	6.006945E 00	6.049404E 00	6.091856E 00	6.134315E 00
6.176771E 00	6.219224E 00	6.261680E 00	6.304137E 00	6.346592E 00
6.389047E 00	6.431503E 00	6.473958E 00	6.516414E 00	6.558870E 00
6.601325E 00	6.643781E 00	6.686236E 00	6.728692E 00	6.771147E 00
6.813603E 00	6.856059E 00	6.898514E 00	6.940970E 00	6.983425E 00
7.025881E 00	7.068336E 00	7.110792E 00	7.153248E 00	7.195703E 00
7.238159E 00	7.280614E 00	7.323070E 00	7.365525E 00	7.407981E 00
7.450436E 00	7.492891E 00	7.535346E 00	7.577801E 00	7.620256E 00
7.662711E 00	7.705166E 00	7.747621E 00	7.790076E 00	7.832531E 00
7.874986E 00	7.917441E 00	7.959896E 00	8.002352E 00	8.044807E 00
8.087262E 00	8.129717E 00	8.172172E 00	8.214627E 00	8.257082E 00
8.299537E 00	8.341992E 00	8.384447E 00	8.426902E 00	8.469357E 00

b's FOR R= 100.0

7.167015E-03	4.290188E-02	7.546291E-02	1.076626E-01	1.397207E-01
1.717007E-01	2.036318E-01	2.355337E-01	2.674111E-01	2.992690E-01
3.311123E-01	3.629409E-01	3.947646E-01	4.265786E-01	4.583828E-01
4.901869E-01	5.219814E-01	5.537709E-01	5.855556E-01	6.173402E-01
6.491200E-01	6.808949E-01	7.126697E-01	7.444397E-01	7.762097E-01
8.079797E-01	8.397441E-01	8.715104E-01	9.032710E-01	9.350336E-01
9.667942E-01	9.985492E-01	1.030310E 00	1.062065E 00	1.093822E 00
1.125573E 00	1.157326E 00	1.189031E 00	1.220834E 00	1.252585E 00
1.284334E 00	1.316035E 00	1.347835E 00	1.379532E 00	1.411331E 00
1.443078E 00	1.474824E 00	1.506571E 00	1.538316E 00	1.570064E 00
1.601809E 00	1.633552E 00	1.665298E 00	1.697041E 00	1.728786E 00
1.760530E 00	1.792275E 00	1.824014E 00	1.855760E 00	1.887503E 00
1.919244E 00	1.950984E 00	1.982729E 00	2.014468E 00	2.046210E 00
2.077953E 00	2.109694E 00	2.141433E 00	2.173174E 00	2.204913E 00
2.236654E 00	2.268393E 00	2.300134E 00	2.331873E 00	2.363614E 00
2.395353E 00	2.427090E 00	2.458829E 00	2.490570E 00	2.522308E 00
2.554049E 00	2.585787E 00	2.617524E 00	2.649263E 00	2.680999E 00
2.712738E 00	2.744478E 00	2.776216E 00	2.807953E 00	2.839691E 00
2.871428E 00	2.903166E 00	2.934902E 00	2.966640E 00	2.998378E 00
3.030116E 00	3.061853E 00	3.093589E 00	3.125327E 00	3.157063E 00
3.188799E 00	3.220536E 00	3.252272E 00	3.284008E 00	3.315745E 00
3.347481E 00	3.379217E 00	3.410953E 00	3.442690E 00	3.474426E 00
3.506162E 00	3.537899E 00	3.569635E 00	3.601371E 00	3.633107E 00
3.664844E 00	3.696580E 00	3.728316E 00	3.760052E 00	3.791789E 00
3.823525E 00	3.855261E 00	3.886998E 00	3.918734E 00	3.950470E 00
3.982206E 00	4.013941E 00	4.045676E 00	4.077412E 00	4.109147E 00
4.140882E 00	4.172618E 00	4.204353E 00	4.236088E 00	4.267824E 00
4.299559E 00	4.331294E 00	4.363030E 00	4.394765E 00	4.426500E 00
4.458236E 00	4.489971E 00	4.521706E 00	4.553442E 00	4.585177E 00
4.616913E 00	4.648648E 00	4.680383E 00	4.712119E 00	4.743854E 00
4.775589E 00	4.807324E 00	4.839058E 00	4.870793E 00	4.902528E 00
4.934263E 00	4.965997E 00	4.997732E 00	5.029467E 00	5.061202E 00
5.092937E 00	5.124671E 00	5.156406E 00	5.188141E 00	5.219876E 00
5.251611E 00	5.283345E 00	5.315080E 00	5.346815E 00	5.378550E 00
5.410284E 00	5.442019E 00	5.473754E 00	5.505489E 00	5.537224E 00
5.568958E 00	5.600692E 00	5.632427E 00	5.664161E 00	5.695896E 00
5.727630E 00	5.759364E 00	5.791099E 00	5.822833E 00	5.854568E 00
5.886302E 00	5.918036E 00	5.949771E 00	5.981505E 00	6.013240E 00
6.044974E 00	6.076708E 00	6.108443E 00	6.140177E 00	6.171912E 00
6.203646E 00	6.235380E 00	6.267115E 00	6.298849E 00	6.330584E 00

b's FOR R= 200.0

3.303748E-03	2.108172E-02	3.725399E-02	5.325485E-02	6.919089E-02
8.509287E-02	1.009737E-01	1.168401E-01	1.326959E-01	1.485436E-01
1.643849E-01	1.802209E-01	1.960527E-01	2.118807E-01	2.277056E-01
2.435278E-01	2.593476E-01	2.751653E-01	2.909811E-01	3.067952E-01
3.226078E-01	3.384190E-01	3.542290E-01	3.700378E-01	3.858455E-01
4.016524E-01	4.174582E-01	4.332633E-01	4.490676E-01	4.648712E-01
4.806741E-01	4.964763E-01	5.122780E-01	5.280792E-01	5.438798E-01
5.596799E-01	5.754795E-01	5.912789E-01	6.070777E-01	6.228760E-01
6.386741E-01	6.544719E-01	6.702692E-01	6.860663E-01	7.018630E-01
7.176595E-01	7.334556E-01	7.492515E-01	7.650472E-01	7.808426E-01
7.966378E-01	8.124328E-01	8.282276E-01	8.440221E-01	8.598165E-01
8.756106E-01	8.914046E-01	9.071984E-01	9.229921E-01	9.387855E-01
9.545790E-01	9.703721E-01	9.861652E-01	1.001958E 00	1.017751E 00
1.033543E 00	1.049336E 00	1.065128E 00	1.080920E 00	1.096713E 00
1.112505E 00	1.128297E 00	1.144088E 00	1.159880E 00	1.175672E 00
1.191463E 00	1.207255E 00	1.223046E 00	1.238837E 00	1.254629E 00
1.270420E 00	1.286211E 00	1.302002E 00	1.317792E 00	1.333583E 00
1.349374E 00	1.365165E 00	1.380955E 00	1.396746E 00	1.412536E 00
1.428327E 00	1.444117E 00	1.459907E 00	1.475697E 00	1.491488E 00
1.507278E 00	1.523068E 00	1.538858E 00	1.554648E 00	1.570438E 00
1.586227E 00	1.602016E 00	1.617806E 00	1.633595E 00	1.649384E 00
1.665174E 00	1.680963E 00	1.696753E 00	1.712542E 00	1.728331E 00
1.744121E 00	1.759910E 00	1.775700E 00	1.791489E 00	1.807278E 00
1.823068E 00	1.838857E 00	1.854647E 00	1.870436E 00	1.886225E 00
1.902015E 00	1.917804E 00	1.933593E 00	1.949383E 00	1.965172E 00
1.980961E 00	1.996749E 00	2.012538E 00	2.028327E 00	2.044115E 00
2.059904E 00	2.075693E 00	2.091481E 00	2.107270E 00	2.123059E 00
2.138847E 00	2.154636E 00	2.170425E 00	2.186213E 00	2.202002E 00
2.217791E 00	2.233579E 00	2.249368E 00	2.265157E 00	2.280945E 00
2.296734E 00	2.312523E 00	2.328311E 00	2.344100E 00	2.359888E 00
2.375677E 00	2.391465E 00	2.407253E 00	2.423041E 00	2.438830E 00
2.454618E 00	2.470406E 00	2.486194E 00	2.501983E 00	2.517771E 00
2.533559E 00	2.549347E 00	2.565135E 00	2.580924E 00	2.596712E 00
2.612500E 00	2.628288E 00	2.644077E 00	2.659865E 00	2.675653E 00
2.691441E 00	2.707229E 00	2.723018E 00	2.738806E 00	2.754594E 00
2.770382E 00	2.786170E 00	2.801958E 00	2.817746E 00	2.833534E 00
2.849322E 00	2.865110E 00	2.880897E 00	2.896685E 00	2.912473E 00
2.928261E 00	2.944049E 00	2.959837E 00	2.975625E 00	2.991413E 00
3.007201E 00	3.022989E 00	3.038777E 00	3.054565E 00	3.070352E 00
3.086140E 00	3.101928E 00	3.117716E 00	3.133504E 00	3.149292E 00

b's FOR R= 500.0

1.206954E-03	8.296839E-03	1.473161E-02	2.110235E-02	2.744982E-02
3.378538E-02	4.011371E-02	4.643717E-02	5.275710E-02	5.907435E-02
6.538949E-02	7.170290E-02	7.801492E-02	8.432572E-02	9.063550E-02
9.694439E-02	1.032525E-01	1.095599E-01	1.158667E-01	1.221729E-01
1.284787E-01	1.347840E-01	1.410888E-01	1.473933E-01	1.536974E-01
1.600012E-01	1.663048E-01	1.726080E-01	1.789109E-01	1.852136E-01
1.915161E-01	1.978184E-01	2.041204E-01	2.104223E-01	2.167240E-01
2.230255E-01	2.293269E-01	2.356281E-01	2.419291E-01	2.482300E-01
2.545308E-01	2.608315E-01	2.671320E-01	2.734324E-01	2.797327E-01
2.860329E-01	2.923330E-01	2.986330E-01	3.049329E-01	3.112327E-01
3.175324E-01	3.238320E-01	3.301316E-01	3.364311E-01	3.427305E-01
3.490298E-01	3.553291E-01	3.616282E-01	3.679274E-01	3.742264E-01
3.805254E-01	3.868243E-01	3.931232E-01	3.994221E-01	4.057208E-01
4.120195E-01	4.183182E-01	4.246168E-01	4.309154E-01	4.372139E-01
4.435124E-01	4.498108E-01	4.561092E-01	4.624075E-01	4.687058E-01
4.750041E-01	4.813024E-01	4.876005E-01	4.938987E-01	5.001968E-01
5.064949E-01	5.127929E-01	5.190909E-01	5.253889E-01	5.316868E-01
5.379847E-01	5.442826E-01	5.505805E-01	5.568783E-01	5.631761E-01
5.694739E-01	5.757716E-01	5.820694E-01	5.883670E-01	5.946647E-01
6.009623E-01	6.072599E-01	6.135575E-01	6.198551E-01	6.261526E-01
6.324499E-01	6.387472E-01	6.450445E-01	6.513418E-01	6.576390E-01
6.639363E-01	6.702336E-01	6.765309E-01	6.828282E-01	6.891255E-01
6.954228E-01	7.017201E-01	7.080174E-01	7.143147E-01	7.206120E-01
7.269093E-01	7.332065E-01	7.395038E-01	7.458011E-01	7.520984E-01
7.583957E-01	7.646930E-01	7.709903E-01	7.772876E-01	7.835849E-01
7.898818E-01	7.961787E-01	8.024757E-01	8.087726E-01	8.150695E-01
8.213664E-01	8.276634E-01	8.339603E-01	8.402572E-01	8.465541E-01
8.528511E-01	8.591480E-01	8.654449E-01	8.717418E-01	8.780388E-01
8.843357E-01	8.906326E-01	8.969295E-01	9.032265E-01	9.095234E-01
9.158203E-01	9.221172E-01	9.284142E-01	9.347111E-01	9.410080E-01
9.473047E-01	9.536014E-01	9.598981E-01	9.661948E-01	9.724915E-01
9.787881E-01	9.850848E-01	9.913815E-01	9.976782E-01	1.003975E 00
1.010272E 00	1.016568E 00	1.022865E 00	1.029162E 00	1.035458E 00
1.041755E 00	1.048052E 00	1.054348E 00	1.060645E 00	1.066942E 00
1.073238E 00	1.079535E 00	1.085832E 00	1.092128E 00	1.098425E 00
1.104722E 00	1.111018E 00	1.117315E 00	1.123611E 00	1.129908E 00
1.136204E 00	1.142501E 00	1.148797E 00	1.155094E 00	1.161390E 00
1.167687E 00	1.173983E 00	1.180280E 00	1.186576E 00	1.192873E 00
1.199169E 00	1.205466E 00	1.211762E 00	1.218059E 00	1.224355E 00
1.230652E 00	1.236948E 00	1.243245E 00	1.249541E 00	1.255838E 00

b's FOR R= 1000.0

5.687979E-04	4.110895E-03	7.320277E-03	1.049891E-02	1.366666E-02
1.682897E-02	1.998800E-02	2.314487E-02	2.630019E-02	2.945434E-02
3.260757E-02	3.576007E-02	3.891196E-02	4.206333E-02	4.521427E-02
4.836483E-02	5.151507E-02	5.466500E-02	5.781468E-02	6.096412E-02
6.411336E-02	6.726240E-02	7.041127E-02	7.355996E-02	7.670852E-02
7.985694E-02	8.300524E-02	8.615341E-02	8.930147E-02	9.244944E-02
9.559730E-02	9.874507E-02	1.018927E-01	1.050404E-01	1.081879E-01
1.113353E-01	1.144827E-01	1.176301E-01	1.207773E-01	1.239245E-01
1.270716E-01	1.302187E-01	1.333658E-01	1.365128E-01	1.396597E-01
1.428066E-01	1.459534E-01	1.491002E-01	1.522470E-01	1.553937E-01
1.585404E-01	1.616871E-01	1.648337E-01	1.679803E-01	1.711268E-01
1.742733E-01	1.774198E-01	1.805663E-01	1.837127E-01	1.868591E-01
1.900055E-01	1.931519E-01	1.962982E-01	1.994445E-01	2.025908E-01
2.057370E-01	2.088832E-01	2.120294E-01	2.151756E-01	2.183218E-01
2.214679E-01	2.246140E-01	2.277602E-01	2.309062E-01	2.340523E-01
2.371984E-01	2.403444E-01	2.434904E-01	2.466364E-01	2.497824E-01
2.529283E-01	2.560743E-01	2.592202E-01	2.623662E-01	2.655121E-01
2.686580E-01	2.718038E-01	2.749497E-01	2.780955E-01	2.812414E-01
2.843872E-01	2.875330E-01	2.906788E-01	2.938246E-01	2.969704E-01
3.001162E-01	3.032619E-01	3.064076E-01	3.095534E-01	3.126991E-01
3.158447E-01	3.189903E-01	3.221359E-01	3.252815E-01	3.284271E-01
3.315727E-01	3.347183E-01	3.378639E-01	3.410095E-01	3.441551E-01
3.473007E-01	3.504463E-01	3.535919E-01	3.567375E-01	3.598831E-01
3.630287E-01	3.661743E-01	3.693199E-01	3.724655E-01	3.756111E-01
3.787567E-01	3.819023E-01	3.850479E-01	3.881935E-01	3.913391E-01
3.944845E-01	3.976299E-01	4.007754E-01	4.039208E-01	4.070662E-01
4.102116E-01	4.133571E-01	4.165025E-01	4.196479E-01	4.227933E-01
4.259388E-01	4.290842E-01	4.322296E-01	4.353750E-01	4.385205E-01
4.416659E-01	4.448113E-01	4.479567E-01	4.511022E-01	4.542476E-01
4.573930E-01	4.605384E-01	4.636839E-01	4.668293E-01	4.699747E-01
4.731200E-01	4.762653E-01	4.794106E-01	4.825559E-01	4.857012E-01
4.888465E-01	4.919918E-01	4.951371E-01	4.982824E-01	5.014277E-01
5.045730E-01	5.077183E-01	5.108636E-01	5.140089E-01	5.171542E-01
5.202995E-01	5.234448E-01	5.265901E-01	5.297354E-01	5.328807E-01
5.360260E-01	5.391713E-01	5.423166E-01	5.454619E-01	5.486072E-01
5.517524E-01	5.548976E-01	5.580428E-01	5.611880E-01	5.643332E-01
5.674784E-01	5.706237E-01	5.737689E-01	5.769141E-01	5.800593E-01
5.832045E-01	5.863497E-01	5.894949E-01	5.926401E-01	5.957853E-01
5.989305E-01	6.020758E-01	6.052210E-01	6.083662E-01	6.115114E-01
6.146566E-01	6.178018E-01	6.209470E-01	6.240922E-01	6.272374E-01

APPENDIX C

EIGENVALUES, α, FOR SINGLE-LAYER RADIAL FLOW, CONSTANT TERMINAL RATE

Characteristic Equation:

$$J_1(\alpha)Y_1(\alpha R) - Y_1(\alpha)J_1(\alpha R) = 0 \qquad \text{(IV-120)}$$

Solved by: Half-interval method on IBM 704

Estimated Accuracy: $\pm$ 1 part in 10^5

Note: See Appendix B for explanation of format.

ALPHAS FOR R= 1.5

6.321872E 00	1.258612E 01	1.886278E 01	2.514267E 01	3.142387E 01
3.770574E 01	4.398798E 01	5.027045E 01	5.655309E 01	6.283583E 01
6.911865E 01	7.540154E 01	8.168447E 01	8.796743E 01	9.425043E 01
1.005334E 02	1.068165E 02	1.130995E 02	1.193826E 02	1.256657E 02
1.319488E 02	1.382319E 02	1.445150E 02	1.507981E 02	1.570812E 02
1.633643E 02	1.696475E 02	1.759306E 02	1.822137E 02	1.884969E 02
1.947800E 02	2.010632E 02	2.073463E 02	2.136295E 02	2.199126E 02
2.261958E 02	2.324789E 02	2.387621E 02	2.450452E 02	2.513284E 02
2.576116E 02	2.638947E 02	2.701779E 02	2.764611E 02	2.827442E 02
2.890274E 02	2.953106E 02	3.015937E 02	3.078769E 02	3.141601E 02
3.204432E 02	3.267264E 02	3.330096E 02	3.392927E 02	3.455759E 02
3.518591E 02	3.581423E 02	3.644254E 02	3.707086E 02	3.769918E 02
3.832750E 02	3.895581E 02	3.958413E 02	4.021245E 02	4.084077E 02
4.146908E 02	4.209740E 02	4.272572E 02	4.335404E 02	4.398235E 02
4.461067E 02	4.523899E 02	4.586731E 02	4.649562E 02	4.712394E 02
4.775226E 02	4.838058E 02	4.900890E 02	4.963721E 02	5.026553E 02
5.089385E 02	5.152217E 02	5.215049E 02	5.277880E 02	5.340712E 02
5.403544E 02	5.466376E 02	5.529208E 02	5.592039E 02	5.654871E 02
5.717703E 02	5.780535E 02	5.843367E 02	5.906198E 02	5.969030E 02
6.031862E 02	6.094694E 02	6.157526E 02	6.220358E 02	6.283189E 02

ALPHAS FOR R= 2.0

3.196578E 00	6.312349E 00	9.444464E 00	1.258120E 01	1.571985E 01
1.885948E 01	2.199966E 01	2.514019E 01	2.828096E 01	3.142189E 01
3.456294E 01	3.770408E 01	4.084529E 01	4.398656E 01	4.712787E 01
5.026921E 01	5.341059E 01	5.655198E 01	5.969340E 01	6.283483E 01
6.597629E 01	6.911775E 01	7.225922E 01	7.540071E 01	7.854220E 01
8.168370E 01	8.482521E 01	8.796673E 01	9.110824E 01	9.424976E 01
9.739129E 01	1.005328E 02	1.036744E 02	1.068159E 02	1.099574E 02
1.130990E 02	1.162405E 02	1.193821E 02	1.225236E 02	1.256652E 02
1.288068E 02	1.319483E 02	1.350899E 02	1.382314E 02	1.413730E 02
1.445146E 02	1.476561E 02	1.507977E 02	1.539393E 02	1.570808E 02
1.602224E 02	1.633640E 02	1.665055E 02	1.696471E 02	1.727887E 02
1.759302E 02	1.790718E 02	1.822134E 02	1.853550E 02	1.884965E 02
1.916381E 02	1.947797E 02	1.979213E 02	2.010629E 02	2.042044E 02
2.073460E 02	2.104876E 02	2.136292E 02	2.167708E 02	2.199123E 02
2.230539E 02	2.261955E 02	2.293371E 02	2.324787E 02	2.356202E 02
2.387618E 02	2.419034E 02	2.450450E 02	2.481866E 02	2.513282E 02
2.544697E 02	2.576113E 02	2.607529E 02	2.638945E 02	2.670361E 02
2.701777E 02	2.733192E 02	2.764608E 02	2.796024E 02	2.827440E 02
2.858856E 02	2.890272E 02	2.921688E 02	2.953103E 02	2.984519E 02
3.015935E 02	3.047351E 02	3.078767E 02	3.110183E 02	3.141599E 02

ALPHAS FOR R= 2.5

2.156472E 00	4.223089E 00	6.306583E 00	8.395280E 00	1.048619E 01
1.257824E 01	1.467096E 01	1.676409E 01	1.885749E 01	2.095110E 01
2.304485E 01	2.513870E 01	2.723264E 01	2.932664E 01	3.142070E 01
3.351479E 01	3.560893E 01	3.770309E 01	3.979727E 01	4.189148E 01
4.398571E 01	4.607995E 01	4.817420E 01	5.026847E 01	5.236274E 01
5.445703E 01	5.655132E 01	5.864562E 01	6.073993E 01	6.283424E 01
6.492856E 01	6.702288E 01	6.911721E 01	7.121154E 01	7.330588E 01
7.540021E 01	7.749455E 01	7.958890E 01	8.168325E 01	8.377759E 01
8.587195E 01	8.796630E 01	9.006066E 01	9.215501E 01	9.424937E 01
9.634374E 01	9.843809E 01	1.005325E 02	1.026268E 02	1.047212E 02
1.068155E 02	1.089099E 02	1.110043E 02	1.130987E 02	1.151930E 02
1.172874E 02	1.193818E 02	1.214762E 02	1.235705E 02	1.256649E 02
1.277593E 02	1.298536E 02	1.319480E 02	1.340424E 02	1.361368E 02
1.382312E 02	1.403255E 02	1.424199E 02	1.445143E 02	1.466087E 02
1.487031E 02	1.507974E 02	1.528918E 02	1.549862E 02	1.570806E 02
1.591750E 02	1.612693E 02	1.633637E 02	1.654581E 02	1.675525E 02
1.696469E 02	1.717413E 02	1.738357E 02	1.759300E 02	1.780244E 02
1.801188E 02	1.822132E 02	1.843076E 02	1.864020E 02	1.884964E 02
1.905907E 02	1.926851E 02	1.947795E 02	1.968739E 02	1.989683E 02
2.010627E 02	2.031571E 02	2.052514E 02	2.073458E 02	2.094402E 02

ALPHAS FOR R= 3.0

1.635617E 00	3.178841E 00	4.738090E 00	6.302718E 00	7.869709E 00
9.437931E 00	1.100687E 01	1.257627E 01	1.414598E 01	1.571590E 01
1.728598E 01	1.885617E 01	2.042646E 01	2.199682E 01	2.356724E 01
2.513771E 01	2.670821E 01	2.827875E 01	2.984931E 01	3.141990E 01
3.299051E 01	3.456113E 01	3.613177E 01	3.770243E 01	3.927309E 01
4.084376E 01	4.241445E 01	4.398514E 01	4.555584E 01	4.712654E 01
4.869725E 01	5.026797E 01	5.183869E 01	5.340941E 01	5.498014E 01
5.655088E 01	5.812161E 01	5.969236E 01	6.126309E 01	6.283384E 01
6.440459E 01	6.597534E 01	6.754609E 01	6.911685E 01	7.068761E 01
7.225836E 01	7.382912E 01	7.539988E 01	7.697064E 01	7.854141E 01
8.011217E 01	8.168294E 01	8.325371E 01	8.482447E 01	8.639525E 01
8.796602E 01	8.953679E 01	9.110756E 01	9.267833E 01	9.424911E 01
9.581988E 01	9.739066E 01	9.896143E 01	1.005322E 02	1.021030E 02
1.036738E 02	1.052445E 02	1.068153E 02	1.083861E 02	1.099569E 02
1.115277E 02	1.130984E 02	1.146692E 02	1.162400E 02	1.178108E 02
1.193816E 02	1.209523E 02	1.225231E 02	1.240939E 02	1.256647E 02
1.272355E 02	1.288063E 02	1.303770E 02	1.319478E 02	1.335186E 02
1.350894E 02	1.366602E 02	1.382310E 02	1.398018E 02	1.413725E 02
1.429433E 02	1.445141E 02	1.460849E 02	1.476557E 02	1.492265E 02
1.507973E 02	1.523681E 02	1.539388E 02	1.555096E 02	1.570804E 02

ALPHAS FOR R= 4.0

1.111876E 00	2.134230E 00	3.169739E 00	4.210408E 00	5.253488E 00
6.297867E 00	7.343019E 00	8.388667E 00	9.434652E 00	1.048087E 01
1.152727E 01	1.257380E 01	1.362043E 01	1.466714E 01	1.571392E 01
1.676074E 01	1.780761E 01	1.885452E 01	1.990146E 01	2.094842E 01
2.199541E 01	2.304241E 01	2.408943E 01	2.513647E 01	2.618352E 01
2.723058E 01	2.827765E 01	2.932473E 01	3.037181E 01	3.141891E 01
3.246601E 01	3.351312E 01	3.456023E 01	3.560735E 01	3.665447E 01
3.770160E 01	3.874873E 01	3.979586E 01	4.084300E 01	4.189014E 01
4.293728E 01	4.398443E 01	4.503158E 01	4.607873E 01	4.712588E 01
4.817303E 01	4.922019E 01	5.026735E 01	5.131450E 01	5.236167E 01
5.340883E 01	5.445599E 01	5.550316E 01	5.655032E 01	5.759749E 01
5.864466E 01	5.969183E 01	6.073900E 01	6.178617E 01	6.283334E 01
6.388052E 01	6.492769E 01	6.597487E 01	6.702204E 01	6.806922E 01
6.911639E 01	7.016357E 01	7.121075E 01	7.225793E 01	7.330511E 01
7.435229E 01	7.539947E 01	7.644665E 01	7.749383E 01	7.854101E 01
7.958819E 01	8.063537E 01	8.168256E 01	8.272974E 01	8.377692E 01
8.482410E 01	8.587129E 01	8.691848E 01	8.796566E 01	8.901284E 01
9.006003E 01	9.110722E 01	9.215440E 01	9.320159E 01	9.424877E 01
9.529596E 01	9.634315E 01	9.739034E 01	9.843752E 01	9.948471E 01
1.005319E 02	1.015791E 02	1.026263E 02	1.036735E 02	1.047206E 02

ALPHAS FOR R= 5.0

8.471493E-01	1.611072E 00	2.385316E 00	3.164208E 00	3.945411E 00
4.727899E 00	5.511168E 00	6.294945E 00	7.079068E 00	7.863439E 00
8.647991E 00	9.432681E 00	1.021748E 01	1.100236E 01	1.178731E 01
1.257232E 01	1.335737E 01	1.414246E 01	1.492758E 01	1.571272E 01
1.649790E 01	1.728309E 01	1.806830E 01	1.885353E 01	1.963877E 01
2.042402E 01	2.120928E 01	2.199455E 01	2.277983E 01	2.356512E 01
2.435042E 01	2.513572E 01	2.592103E 01	2.670634E 01	2.749166E 01
2.827698E 01	2.906231E 01	2.984764E 01	3.063297E 01	3.141831E 01
3.220365E 01	3.298899E 01	3.377434E 01	3.455969E 01	3.534504E 01
3.613039E 01	3.691574E 01	3.770110E 01	3.848646E 01	3.927182E 01
4.005718E 01	4.084254E 01	4.162790E 01	4.241327E 01	4.319863E 01
4.398400E 01	4.476937E 01	4.555474E 01	4.634011E 01	4.712548E 01
4.791085E 01	4.869622E 01	4.948160E 01	5.026697E 01	5.105235E 01
5.183773E 01	5.262310E 01	5.340848E 01	5.419386E 01	5.497923E 01
5.576461E 01	5.654999E 01	5.733537E 01	5.812076E 01	5.890613E 01
5.969152E 01	6.047690E 01	6.126228E 01	6.204766E 01	6.283305E 01
6.361843E 01	6.440381E 01	6.518920E 01	6.597458E 01	6.675997E 01
6.754535E 01	6.833074E 01	6.911612E 01	6.990151E 01	7.068690E 01
7.147228E 01	7.225767E 01	7.304306E 01	7.382845E 01	7.461383E 01
7.539922E 01	7.618461E 01	7.696999E 01	7.775538E 01	7.854077E 01

ALPHAS FOR R= 6.0

6.864318E-01	1.296378E 00	1.914316E 00	2.536331E 00	3.160492E 00
3.785889E 00	4.412051E 00	5.038716E 00	5.665729E 00	6.292992E 00
6.920441E 00	7.548029E 00	8.175728E 00	8.803512E 00	9.431367E 00
1.005928E 01	1.068724E 01	1.131523E 01	1.194327E 01	1.257133E 01
1.319941E 01	1.382751E 01	1.445564E 01	1.508378E 01	1.571193E 01
1.634010E 01	1.696828E 01	1.759646E 01	1.822466E 01	1.885287E 01
1.948108E 01	2.010930E 01	2.073752E 01	2.136575E 01	2.199399E 01
2.262223E 01	2.325047E 01	2.387872E 01	2.450697E 01	2.513523E 01
2.576348E 01	2.639174E 01	2.702001E 01	2.764827E 01	2.827654E 01
2.890481E 01	2.953308E 01	3.016136E 01	3.078964E 01	3.141791E 01
3.204619E 01	3.267447E 01	3.330276E 01	3.393104E 01	3.455933E 01
3.518761E 01	3.581590E 01	3.644419E 01	3.707248E 01	3.770077E 01
3.832906E 01	3.895735E 01	3.958565E 01	4.021394E 01	4.084224E 01
4.147053E 01	4.209883E 01	4.272712E 01	4.335542E 01	4.398372E 01
4.461202E 01	4.524031E 01	4.586862E 01	4.649691E 01	4.712522E 01
4.775352E 01	4.838182E 01	4.901012E 01	4.963842E 01	5.026672E 01
5.089503E 01	5.152333E 01	5.215164E 01	5.277994E 01	5.340824E 01
5.403655E 01	5.466485E 01	5.529316E 01	5.592147E 01	5.654977E 01
5.717808E 01	5.780639E 01	5.843469E 01	5.906300E 01	5.969131E 01
6.031961E 01	6.094792E 01	6.157623E 01	6.220454E 01	6.283285E 01

ALPHAS FOR R= 7.0

5.780570E-01	1.085941E 00	1.599990E 00	2.117571E 00	2.637115E 00
3.157824E 00	3.679271E 00	4.201211E 00	4.723493E 00	5.246024E 00
5.768739E 00	6.291596E 00	6.814563E 00	7.337617E 00	7.860743E 00
8.383925E 00	8.907156E 00	9.430427E 00	9.953732E 00	1.047707E 01
1.100042E 01	1.152380E 01	1.204720E 01	1.257062E 01	1.309405E 01
1.361749E 01	1.414095E 01	1.466441E 01	1.518788E 01	1.571137E 01
1.623486E 01	1.675835E 01	1.728185E 01	1.780536E 01	1.832887E 01
1.885239E 01	1.937592E 01	1.989944E 01	2.042297E 01	2.094650E 01
2.147004E 01	2.199358E 01	2.251712E 01	2.304067E 01	2.356422E 01
2.408777E 01	2.461132E 01	2.513487E 01	2.565843E 01	2.618198E 01
2.670554E 01	2.722910E 01	2.775266E 01	2.827623E 01	2.879979E 01
2.932336E 01	2.984692E 01	3.037049E 01	3.089406E 01	3.141763E 01
3.194120E 01	3.246477E 01	3.298835E 01	3.351192E 01	3.403549E 01
3.455907E 01	3.508264E 01	3.560622E 01	3.612980E 01	3.665338E 01
3.717695E 01	3.770053E 01	3.822411E 01	3.874769E 01	3.927127E 01
3.979485E 01	4.031843E 01	4.084201E 01	4.136560E 01	4.188918E 01
4.241276E 01	4.293635E 01	4.345993E 01	4.398352E 01	4.450710E 01
4.503068E 01	4.555427E 01	4.607786E 01	4.660144E 01	4.712503E 01
4.764861E 01	4.817220E 01	4.869579E 01	4.921937E 01	4.974296E 01
5.026655E 01	5.079014E 01	5.131372E 01	5.183731E 01	5.236090E 01

ALPHAS FOR R= 8.0

4.998220E-01	9.351292E-01	1.375190E 00	1.818292E 00	2.263173E 00
2.709143E 00	3.155816E 00	3.602964E 00	4.050449E 00	4.498175E 00
4.946087E 00	5.394138E 00	5.842299E 00	6.290548E 00	6.738867E 00
7.187245E 00	7.635671E 00	8.084137E 00	8.532637E 00	8.981167E 00
9.429722E 00	9.878298E 00	1.032689E 01	1.077551E 01	1.122413E 01
1.167277E 01	1.212142E 01	1.257009E 01	1.301876E 01	1.346744E 01
1.391613E 01	1.436482E 01	1.481352E 01	1.526223E 01	1.571094E 01
1.615966E 01	1.660838E 01	1.705710E 01	1.750583E 01	1.795456E 01
1.840330E 01	1.885204E 01	1.930078E 01	1.974952E 01	2.019827E 01
2.064702E 01	2.109577E 01	2.154452E 01	2.199328E 01	2.244203E 01
2.289079E 01	2.333955E 01	2.378831E 01	2.423708E 01	2.468584E 01
2.513460E 01	2.558337E 01	2.603214E 01	2.648091E 01	2.692968E 01
2.737845E 01	2.782722E 01	2.827599E 01	2.872476E 01	2.917354E 01
2.962231E 01	3.007109E 01	3.051986E 01	3.096864E 01	3.141742E 01
3.186620E 01	3.231497E 01	3.276375E 01	3.321253E 01	3.366131E 01
3.411009E 01	3.455887E 01	3.500766E 01	3.545644E 01	3.590522E 01
3.635400E 01	3.680279E 01	3.725157E 01	3.770035E 01	3.814914E 01
3.859792E 01	3.904671E 01	3.949549E 01	3.994428E 01	4.039306E 01
4.084185E 01	4.129064E 01	4.173942E 01	4.218821E 01	4.263700E 01
4.308579E 01	4.353457E 01	4.398336E 01	4.443215E 01	4.488094E 01

ALPHAS FOR R= 10.0

3.940941E-01	7.330569E-01	1.074838E 00	1.418864E 00	1.764331E 00
2.110731E 00	2.457757E 00	2.805219E 00	3.152994E 00	3.501001E 00
3.849184E 00	4.197502E 00	4.545928E 00	4.894440E 00	5.243023E 00
5.591662E 00	5.940350E 00	6.289078E 00	6.637841E 00	6.986633E 00
7.335451E 00	7.684290E 00	8.033149E 00	8.382024E 00	8.730915E 00
9.079818E 00	9.428734E 00	9.777660E 00	1.012660E 01	1.047554E 01
1.082449E 01	1.117345E 01	1.152242E 01	1.187139E 01	1.222036E 01
1.256934E 01	1.291833E 01	1.326732E 01	1.361631E 01	1.396531E 01
1.431431E 01	1.466332E 01	1.501232E 01	1.536133E 01	1.571035E 01
1.605936E 01	1.640838E 01	1.675739E 01	1.710642E 01	1.745544E 01
1.780446E 01	1.815349E 01	1.850251E 01	1.885154E 01	1.920057E 01
1.954960E 01	1.989864E 01	2.024767E 01	2.059670E 01	2.094574E 01
2.129478E 01	2.164381E 01	2.199285E 01	2.234189E 01	2.269093E 01
2.303997E 01	2.338901E 01	2.373806E 01	2.408710E 01	2.443614E 01
2.478519E 01	2.513423E 01	2.548328E 01	2.583232E 01	2.618137E 01
2.653042E 01	2.687946E 01	2.722851E 01	2.757756E 01	2.792661E 01
2.827566E 01	2.862471E 01	2.897376E 01	2.932281E 01	2.967186E 01
3.002091E 01	3.036996E 01	3.071901E 01	3.106807E 01	3.141712E 01
3.176617E 01	3.211522E 01	3.246428E 01	3.281333E 01	3.316239E 01
3.351144E 01	3.386049E 01	3.420955E 01	3.455860E 01	3.490766E 01

ALPHAS FOR R= 12.0

3.257301E-01	6.035982E-01	8.831074E-01	1.164276E 00	1.446605E 00
1.729725E 00	2.013394E 00	2.297457E 00	2.581807E 00	2.866374E 00
3.151106E 00	3.435969E 00	3.720935E 00	4.005985E 00	4.291102E 00
4.576276E 00	4.861497E 00	5.146758E 00	5.432053E 00	5.717377E 00
6.002727E 00	6.288098E 00	6.573488E 00	6.858896E 00	7.144318E 00
7.429754E 00	7.715201E 00	8.000659E 00	8.286127E 00	8.571603E 00
8.857086E 00	9.142578E 00	9.428075E 00	9.713578E 00	9.999087E 00
1.028460E 01	1.057012E 01	1.085564E 01	1.114117E 01	1.142670E 01
1.171223E 01	1.199777E 01	1.228331E 01	1.256885E 01	1.285439E 01
1.313994E 01	1.342549E 01	1.371104E 01	1.399659E 01	1.428215E 01
1.456771E 01	1.485326E 01	1.513882E 01	1.542439E 01	1.570995E 01
1.599551E 01	1.628108E 01	1.656664E 01	1.685221E 01	1.713778E 01
1.742335E 01	1.770892E 01	1.799449E 01	1.828006E 01	1.856564E 01
1.885121E 01	1.913679E 01	1.942236E 01	1.970794E 01	1.999351E 01
2.027909E 01	2.056467E 01	2.085025E 01	2.113583E 01	2.142141E 01
2.170699E 01	2.199257E 01	2.227815E 01	2.256373E 01	2.284931E 01
2.313489E 01	2.342048E 01	2.370606E 01	2.399164E 01	2.427723E 01
2.456281E 01	2.484840E 01	2.513398E 01	2.541957E 01	2.570515E 01
2.599074E 01	2.627633E 01	2.656191E 01	2.684750E 01	2.713309E 01
2.741867E 01	2.770426E 01	2.798985E 01	2.827544E 01	2.856103E 01

ALPHAS FOR R= 14.0

2.777831E-01	5.134111E-01	7.499456E-01	9.877137E-01	1.226415E 00
1.465784E 00	1.705633E 00	1.945833E 00	2.186295E 00	2.426958E 00
2.667775E 00	2.908716E 00	3.149755E 00	3.390874E 00	3.632059E 00
3.873298E 00	4.114584E 00	4.355908E 00	4.597266E 00	4.838653E 00
5.080064E 00	5.321497E 00	5.562949E 00	5.804417E 00	6.045901E 00
6.287397E 00	6.528906E 00	6.770424E 00	7.011953E 00	7.253490E 00
7.495035E 00	7.736587E 00	7.978145E 00	8.219710E 00	8.461279E 00
8.702854E 00	8.944433E 00	9.186017E 00	9.427604E 00	9.669195E 00
9.910790E 00	1.015239E 01	1.039399E 01	1.063559E 01	1.087720E 01
1.111880E 01	1.136041E 01	1.160203E 01	1.184364E 01	1.208526E 01
1.232688E 01	1.256850E 01	1.281012E 01	1.305174E 01	1.329336E 01
1.353499E 01	1.377661E 01	1.401824E 01	1.425987E 01	1.450150E 01
1.474313E 01	1.498476E 01	1.522640E 01	1.546803E 01	1.570967E 01
1.595130E 01	1.619294E 01	1.643457E 01	1.667621E 01	1.691785E 01
1.715949E 01	1.740113E 01	1.764277E 01	1.788441E 01	1.812605E 01
1.836769E 01	1.860933E 01	1.885097E 01	1.909262E 01	1.933426E 01
1.957590E 01	1.981755E 01	2.005919E 01	2.030084E 01	2.054248E 01
2.078413E 01	2.102578E 01	2.126742E 01	2.150907E 01	2.175072E 01
2.199237E 01	2.223401E 01	2.247566E 01	2.271731E 01	2.295896E 01
2.320061E 01	2.344226E 01	2.368390E 01	2.392556E 01	2.416721E 01

ALPHAS FOR R= 16.0

2.422446E-01	4.468973E-01	6.519865E-01	8.579976E-01	1.064762E 00
1.272089E 00	1.479834E 00	1.687891E 00	1.896186E 00	2.104665E 00
2.313289E 00	2.522028E 00	2.730860E 00	2.939768E 00	3.148739E 00
3.357764E 00	3.566832E 00	3.775939E 00	3.985078E 00	4.194245E 00
4.403436E 00	4.612648E 00	4.821879E 00	5.031127E 00	5.240390E 00
5.449665E 00	5.658952E 00	5.868249E 00	6.077556E 00	6.286871E 00
6.496195E 00	6.705525E 00	6.914862E 00	7.124204E 00	7.333552E 00
7.542905E 00	7.752263E 00	7.961624E 00	8.170990E 00	8.380359E 00
8.589732E 00	8.799107E 00	9.008486E 00	9.217868E 00	9.427251E 00
9.636638E 00	9.846026E 00	1.005542E 01	1.026481E 01	1.047420E 01
1.068360E 01	1.089300E 01	1.110240E 01	1.131180E 01	1.152120E 01
1.173060E 01	1.194001E 01	1.214941E 01	1.235882E 01	1.256823E 01
1.277764E 01	1.298705E 01	1.319646E 01	1.340587E 01	1.361528E 01
1.382470E 01	1.403411E 01	1.424353E 01	1.445294E 01	1.466236E 01
1.487178E 01	1.508120E 01	1.529061E 01	1.550003E 01	1.570945E 01
1.591887E 01	1.612829E 01	1.633771E 01	1.654713E 01	1.675656E 01
1.696598E 01	1.717540E 01	1.738482E 01	1.759425E 01	1.780367E 01
1.801310E 01	1.822252E 01	1.843195E 01	1.864137E 01	1.885080E 01
1.906022E 01	1.926965E 01	1.947908E 01	1.968850E 01	1.989793E 01
2.010736E 01	2.031678E 01	2.052621E 01	2.073564E 01	2.094507E 01

ALPHAS FOR R= 18.0

2.148256E-01	3.957735E-01	5.768512E-01	7.586205E-01	9.410001E-01
1.123855E 00	1.307073E 00	1.490569E 00	1.674279E 00	1.858158E 00
2.042171E 00	2.226292E 00	2.410501E 00	2.594782E 00	2.779123E 00
2.963514E 00	3.147949E 00	3.332420E 00	3.516923E 00	3.701453E 00
3.886006E 00	4.070580E 00	4.255173E 00	4.439782E 00	4.624405E 00
4.809041E 00	4.993688E 00	5.178346E 00	5.363014E 00	5.547690E 00
5.732373E 00	5.917063E 00	6.101760E 00	6.286462E 00	6.471170E 00
6.655883E 00	6.840601E 00	7.025322E 00	7.210048E 00	7.394777E 00
7.579510E 00	7.764245E 00	7.948984E 00	8.133725E 00	8.318469E 00
8.503215E 00	8.687964E 00	8.872714E 00	9.057466E 00	9.242220E 00
9.426977E 00	9.611734E 00	9.796494E 00	9.981254E 00	1.016602E 01
1.035078E 01	1.053554E 01	1.072031E 01	1.090508E 01	1.108985E 01
1.127461E 01	1.145938E 01	1.164416E 01	1.182893E 01	1.201370E 01
1.219847E 01	1.238325E 01	1.256802E 01	1.275280E 01	1.293758E 01
1.312235E 01	1.330713E 01	1.349191E 01	1.367669E 01	1.386147E 01
1.404625E 01	1.423103E 01	1.441581E 01	1.460059E 01	1.478537E 01
1.497015E 01	1.515494E 01	1.533972E 01	1.552450E 01	1.570929E 01
1.589407E 01	1.607886E 01	1.626364E 01	1.644843E 01	1.663321E 01
1.681800E 01	1.700278E 01	1.718757E 01	1.737236E 01	1.755714E 01
1.774193E 01	1.792672E 01	1.811151E 01	1.829629E 01	1.848108E 01

ALPHAS FOR R= 20.0

1.930163E-01	3.552274E-01	5.173652E-01	6.800228E-01	8.431771E-01
1.006734E 00	1.170606E 00	1.334725E 00	1.499038E 00	1.663505E 00
1.828095E 00	1.992786E 00	2.157560E 00	2.322402E 00	2.487301E 00
2.652249E 00	2.817237E 00	2.982261E 00	3.147316E 00	3.312397E 00
3.477500E 00	3.642624E 00	3.807766E 00	3.972924E 00	4.138096E 00
4.303281E 00	4.468477E 00	4.633683E 00	4.798898E 00	4.964121E 00
5.129353E 00	5.294591E 00	5.459835E 00	5.625086E 00	5.790341E 00
5.955601E 00	6.120866E 00	6.286135E 00	6.451408E 00	6.616684E 00
6.781965E 00	6.947247E 00	7.112533E 00	7.277822E 00	7.443113E 00
7.608407E 00	7.773702E 00	7.939000E 00	8.104300E 00	8.269602E 00
8.434905E 00	8.600210E 00	8.765516E 00	8.930824E 00	9.096134E 00
9.261445E 00	9.426757E 00	9.592070E 00	9.757384E 00	9.922700E 00
1.008802E 01	1.025333E 01	1.041865E 01	1.058397E 01	1.074929E 01
1.091461E 01	1.107993E 01	1.124526E 01	1.141058E 01	1.157590E 01
1.174123E 01	1.190655E 01	1.207188E 01	1.223720E 01	1.240253E 01
1.256786E 01	1.273319E 01	1.289851E 01	1.306384E 01	1.322917E 01
1.339450E 01	1.355983E 01	1.372516E 01	1.389049E 01	1.405582E 01
1.422116E 01	1.438649E 01	1.455182E 01	1.471715E 01	1.488249E 01
1.504782E 01	1.521315E 01	1.537849E 01	1.554382E 01	1.570915E 01
1.587449E 01	1.603982E 01	1.620516E 01	1.637049E 01	1.653583E 01

ALPHAS FOR R= 50.0

7.672789E-02	1.406161E-01	2.040939E-01	2.675179E-01	3.309652E-01
3.944572E-01	4.579989E-01	5.215893E-01	5.852253E-01	6.489029E-01
7.126180E-01	7.763669E-01	8.401462E-01	9.039525E-01	9.677832E-01
1.031636E 00	1.095508E 00	1.159398E 00	1.223303E 00	1.287224E 00
1.351157E 00	1.415102E 00	1.479058E 00	1.543024E 00	1.606999E 00
1.670982E 00	1.734972E 00	1.798970E 00	1.862974E 00	1.926984E 00
1.991000E 00	2.055020E 00	2.119045E 00	2.183075E 00	2.247109E 00
2.311146E 00	2.375187E 00	2.439231E 00	2.503279E 00	2.567329E 00
2.631382E 00	2.695438E 00	2.759496E 00	2.823556E 00	2.887619E 00
2.951683E 00	3.015749E 00	3.079817E 00	3.143887E 00	3.207958E 00
3.272031E 00	3.336106E 00	3.400181E 00	3.464258E 00	3.528337E 00
3.592416E 00	3.656497E 00	3.720578E 00	3.784661E 00	3.848745E 00
3.912829E 00	3.976915E 00	4.041001E 00	4.105088E 00	4.169176E 00
4.233265E 00	4.297354E 00	4.361444E 00	4.425535E 00	4.489626E 00
4.553718E 00	4.617811E 00	4.681904E 00	4.745998E 00	4.810092E 00
4.874186E 00	4.938282E 00	5.002377E 00	5.066473E 00	5.130570E 00
5.194666E 00	5.258764E 00	5.322861E 00	5.386960E 00	5.451058E 00
5.515157E 00	5.579256E 00	5.643355E 00	5.707455E 00	5.771555E 00
5.835656E 00	5.899756E 00	5.963857E 00	6.027959E 00	6.092060E 00
6.156162E 00	6.220264E 00	6.284366E 00	6.348468E 00	6.412571E 00

ALPHAS FOR R= 100.0

3.832884E-02	7.019452E-02	1.018151E-01	1.333734E-01	1.649127E-01
1.964480E-01	2.279858E-01	2.595290E-01	2.910790E-01	3.226365E-01
3.542018E-01	3.857746E-01	4.173548E-01	4.489423E-01	4.805365E-01
5.121374E-01	5.437444E-01	5.753574E-01	6.069759E-01	6.385997E-01
6.702284E-01	7.018618E-01	7.334997E-01	7.651417E-01	7.967876E-01
8.284373E-01	8.600905E-01	8.917471E-01	9.234067E-01	9.550694E-01
9.867348E-01	1.018403E 00	1.050073E 00	1.081746E 00	1.113422E 00
1.145099E 00	1.176779E 00	1.208460E 00	1.240143E 00	1.271828E 00
1.303515E 00	1.335203E 00	1.366893E 00	1.398584E 00	1.430276E 00
1.461970E 00	1.493665E 00	1.525362E 00	1.557059E 00	1.588758E 00
1.620457E 00	1.652158E 00	1.683859E 00	1.715562E 00	1.747265E 00
1.778969E 00	1.810675E 00	1.842380E 00	1.874087E 00	1.905794E 00
1.937502E 00	1.969211E 00	2.000920E 00	2.032630E 00	2.064340E 00
2.096052E 00	2.127763E 00	2.159475E 00	2.191188E 00	2.222901E 00
2.254615E 00	2.286329E 00	2.318044E 00	2.349759E 00	2.381474E 00
2.413190E 00	2.444906E 00	2.476623E 00	2.508340E 00	2.540057E 00
2.571774E 00	2.603492E 00	2.635211E 00	2.666929E 00	2.698648E 00
2.730368E 00	2.762087E 00	2.793807E 00	2.825527E 00	2.857247E 00
2.888968E 00	2.920689E 00	2.952410E 00	2.984131E 00	3.015852E 00
3.047574E 00	3.079296E 00	3.111018E 00	3.142741E 00	3.174463E 00

ALPHAS FOR R= 200.0

1.916000E-02	3.508279E-02	5.087749E-02	6.663579E-02	8.237953E-02
9.811656E-02	1.138504E-01	1.295828E-01	1.453147E-01	1.610468E-01
1.767795E-01	1.925129E-01	2.082472E-01	2.239826E-01	2.397189E-01
2.554564E-01	2.711951E-01	2.869348E-01	3.026757E-01	3.184176E-01
3.341607E-01	3.499049E-01	3.656501E-01	3.813963E-01	3.971436E-01
4.128918E-01	4.286410E-01	4.443911E-01	4.601422E-01	4.758941E-01
4.916469E-01	5.074005E-01	5.231549E-01	5.389102E-01	5.546662E-01
5.704229E-01	5.861804E-01	6.019385E-01	6.176973E-01	6.334568E-01
6.492170E-01	6.649777E-01	6.807391E-01	6.965010E-01	7.122634E-01
7.280265E-01	7.437901E-01	7.595542E-01	7.753188E-01	7.910839E-01
8.068495E-01	8.226155E-01	8.383820E-01	8.541488E-01	8.699162E-01
8.856840E-01	9.014521E-01	9.172206E-01	9.329895E-01	9.487587E-01
9.645284E-01	9.802984E-01	9.960686E-01	1.011839E 00	1.027610E 00
1.043381E 00	1.059153E 00	1.074925E 00	1.090697E 00	1.106469E 00
1.122242E 00	1.138015E 00	1.153788E 00	1.169562E 00	1.185336E 00
1.201109E 00	1.216884E 00	1.232658E 00	1.248433E 00	1.264208E 00
1.279983E 00	1.295758E 00	1.311533E 00	1.327309E 00	1.343085E 00
1.358861E 00	1.374637E 00	1.390413E 00	1.406190E 00	1.421967E 00
1.437744E 00	1.453521E 00	1.469298E 00	1.485075E 00	1.500853E 00
1.516631E 00	1.532408E 00	1.548186E 00	1.563964E 00	1.579743E 00

ALPHAS FOR R= 500.0

7.663507E-03	1.403148E-02	2.034759E-02	2.664850E-02	3.294296E-02
3.923413E-02	4.552341E-02	5.181154E-02	5.809893E-02	6.438583E-02
7.067239E-02	7.695874E-02	8.324495E-02	8.953106E-02	9.581714E-02
1.021032E-01	1.083892E-01	1.146753E-01	1.209614E-01	1.272476E-01
1.335338E-01	1.398200E-01	1.461063E-01	1.523927E-01	1.586792E-01
1.649658E-01	1.712524E-01	1.775391E-01	1.838259E-01	1.901127E-01
1.963997E-01	2.026867E-01	2.089738E-01	2.152611E-01	2.215483E-01
2.278357E-01	2.341232E-01	2.404107E-01	2.466983E-01	2.529860E-01
2.592738E-01	2.655617E-01	2.718496E-01	2.781377E-01	2.844258E-01
2.907140E-01	2.970022E-01	3.032906E-01	3.095790E-01	3.158675E-01
3.221561E-01	3.284447E-01	3.347335E-01	3.410223E-01	3.473111E-01
3.536001E-01	3.598891E-01	3.661781E-01	3.724673E-01	3.787565E-01
3.850458E-01	3.913352E-01	3.976246E-01	4.039141E-01	4.102037E-01
4.164933E-01	4.227830E-01	4.290727E-01	4.353625E-01	4.416524E-01
4.479423E-01	4.542323E-01	4.605223E-01	4.668124E-01	4.731026E-01
4.793928E-01	4.856831E-01	4.919734E-01	4.982638E-01	5.045543E-01
5.108447E-01	5.171353E-01	5.234259E-01	5.297165E-01	5.360072E-01
5.422979E-01	5.485887E-01	5.548796E-01	5.611705E-01	5.674614E-01
5.737524E-01	5.800435E-01	5.863345E-01	5.926257E-01	5.989168E-01
6.052081E-01	6.114993E-01	6.177906E-01	6.240820E-01	6.303733E-01

ALPHAS FOR R= 1000.0

3.831718E-03	7.015626E-03	1.017355E-02	1.332383E-02	1.647084E-02
1.961616E-02	2.276049E-02	2.590420E-02	2.904749E-02	3.219049E-02
3.533329E-02	3.847593E-02	4.161845E-02	4.476089E-02	4.790325E-02
5.104557E-02	5.418785E-02	5.733009E-02	6.047231E-02	6.361451E-02
6.675670E-02	6.989887E-02	7.304104E-02	7.618321E-02	7.932538E-02
8.246754E-02	8.560971E-02	8.875187E-02	9.189404E-02	9.503622E-02
9.817841E-02	1.013206E-01	1.044628E-01	1.076050E-01	1.107472E-01
1.138894E-01	1.170317E-01	1.201739E-01	1.233161E-01	1.264584E-01
1.296007E-01	1.327430E-01	1.358852E-01	1.390276E-01	1.421699E-01
1.453122E-01	1.484545E-01	1.515969E-01	1.547392E-01	1.578816E-01
1.610240E-01	1.641664E-01	1.673088E-01	1.704512E-01	1.735936E-01
1.767360E-01	1.798785E-01	1.830209E-01	1.861634E-01	1.893059E-01
1.924484E-01	1.955909E-01	1.987334E-01	2.018759E-01	2.050185E-01
2.081610E-01	2.113036E-01	2.144462E-01	2.175887E-01	2.207313E-01
2.238740E-01	2.270166E-01	2.301592E-01	2.333018E-01	2.364445E-01
2.395871E-01	2.427298E-01	2.458725E-01	2.490152E-01	2.521579E-01
2.553006E-01	2.584434E-01	2.615861E-01	2.647288E-01	2.678716E-01
2.710144E-01	2.741572E-01	2.772999E-01	2.804427E-01	2.835855E-01
2.867284E-01	2.898712E-01	2.930140E-01	2.961569E-01	2.992998E-01
3.024426E-01	3.055855E-01	3.087284E-01	3.118713E-01	3.150142E-01

APPENDIX D

EIGENVALUES, a, FOR TWO LAYER LINEAR FLOW, CONSTANT TERMINAL PRESSURE

Characteristic Equation:

$$\beta K_2 \sin \beta(1-c) \cos \gamma c + \gamma \sin \gamma c \cos \beta(1-c) = 0 \qquad \text{(V-60)}$$

where:

$$\gamma = \sqrt{a^2 - b^2} \qquad \text{(V-46)}$$

$$\beta = \sqrt{\frac{a^2}{K_2^*} - b^2} \qquad \text{(V-47)}$$

$$K_2^* = \frac{k_2 \phi_1}{k_1 \phi_2} \quad \text{(liquid flow)} \qquad \text{(V-14)}$$

$$= \frac{k_2 \rho_1 C_{p1}}{k_1 \rho_2 C_{p2}} \quad \text{(heat conduction)}$$

$$K_2 = \frac{k_2}{k_1}$$

All values shown are for $K_2 = K_2^*$

$$b = \left(\frac{2m-1}{2}\right)\frac{\pi}{L} \quad ; \quad m = 1,2,3,\ldots \qquad \text{(V-43)}$$

Solved by: Half-interval method on IBM 704

Estimated Accuracy: $\pm$ 2 parts in 10^6

Note: See Appendix B for explanation of format.

EIGENVALUES A FOR K2= 2., L= 5., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.8456692E-01	3.5994206E 00	7.5505087E 00	1.0846518E 01	1.4892212E 01	1.8317832E 01	2.2068585E 01	2.5888784E 01
2	1.1489271E 00	3.7535744E 00	7.6224542E 00	1.0899899E 01	1.4927587E 01	1.8350241E 01	2.2092462E 01	2.5911187E 01
3	1.8988574E 00	4.0508796E 00	7.7641065E 00	1.1006111E 01	1.4998090E 01	1.8414899E 01	2.2140218E 01	2.5955858E 01
4	2.6250828E 00	4.4742908E 00	7.9713442E 00	1.1164074E 01	1.5103273E 01	1.8511487E 01	2.2211708E 01	2.6022716E 01
5	3.3209836E 00	5.0047188E 00	8.2387798E 00	1.1372204E 01	1.5242486E 01	1.8639510E 01	2.2306829E 01	2.6111481E 01
6	3.9857689E 00	5.6212448E 00	8.5605057E 00	1.1628470E 01	1.5414912E 01	1.8798311E 01	2.2425417E 01	2.6221889E 01
7	4.6249645E 00	6.3010898E 00	8.9308971E 00	1.1930404E 01	1.5619650E 01	1.8987106E 01	2.2567277E 01	2.6353518E 01
8	5.2469847E 00	7.0221007E 00	9.3453293E 00	1.2275161E 01	1.5855758E 01	1.9204945E 01	2.2732188E 01	2.6505984E 01
9	5.8592951E 00	7.7650841E 00	9.8009865E 00	1.2659530E 01	1.6122282E 01	1.9450735E 01	2.2919940E 01	2.6678777E 01
10	6.4670110E 00	8.5138273E 00	1.0297507E 01	1.3079983E 01	1.6418337E 01	1.9723277E 01	2.3130315E 01	2.6871328E 01
11	7.0731463E 00	9.2537366E 00	1.0837498E 01	1.3532709E 01	1.6743165E 01	2.0021250E 01	2.3363029E 01	2.7083071E 01
12	7.6793252E 00	9.9712521E 00	1.1425845E 01	1.4013709E 01	1.7096154E 01	2.0343266E 01	2.3617894E 01	2.7313371E 01

EIGENVALUES A FOR K2= 5., L= 5., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.4234225E-01	3.9009398E 00	9.0153379E 00	1.3304970E 01	1.6832639E 01	2.1841123E 01	2.6519540E 01	2.9900527E 01
2	1.5826507E 00	4.1592616E 00	9.0860996E 00	1.3387004E 01	1.6887900E 01	2.1871067E 01	2.6556950E 01	2.9935122E 01
3	2.4891825E 00	4.7031395E 00	9.2281753E 00	1.3547517E 01	1.6999767E 01	2.1931052E 01	2.6631053E 01	3.0004717E 01
4	3.2208528E 00	5.5216595E 00	9.4445635E 00	1.3779552E 01	1.7170898E 01	2.2021292E 01	2.6740351E 01	3:0110038E 01
5	3.8326070E 00	6.5092852E 00	9.7453477E 00	1.4073240E 01	1.7404843E 01	2.2142198E 01	2.6882739E 01	3.0252111E 01
6	4.3966089E 00	7.5348403E 00	1.0156606E 01	1.4416892E 01	1.7705462E 01	2.2294673E 01	2.7055522E 01	3.0432064E 01
7	4.9501706E 00	8.4831645E 00	1.0729259E 01	1.4799161E 01	1.8075470E 01	2.2480297E 01	2.7255637E 01	3.0651027E 01
8	5.5069703E 00	9.2678523E 00	1.1512490E 01	1.5212283E 01	1.8514655E 01	2.2701643E 01	2.7479940E 01	3.0909716E 01
9	6.0709769E 00	9.8889004E 00	1.2477974E 01	1.5656672E 01	1.9018074E 01	2.2962684E 01	2.7725479E 01	3.1208169E 01
10	6.6426968E 00	1.0411896E 01	1.3527139E 01	1.6147870E 01	1.9575190E 01	2.3269272E 01	2.7989946E 01	3.1545361E 01
11	7.2214854E 00	1.0891477E 01	1.4552892E 01	1.6727387E 01	2.0170497E 01	2.3629145E 01	2.8272027E 01	3.1918843E 01
12	7.8064187E 00	1.1355925E 01	1.5450790E 01	1.7463499E 01	2.0786095E 01	2.4051470E 01	2.8571742E 01	3.2324677E 01

EIGENVALUES A FOR K2= 10., L= 5., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	7.3056946E-01	4.0166155E 00	9.4809210E 00	1.5072073E 01	1.9367822E 01	2.3045773E 01	2.8418523E 01	3.4066220E 01
2	2.0350081E 00	4.4767958E 00	9.5608612E 00	1.5127433E 01	1.9473777E 01	2.3106045E 01	2.8444915E 01	3.4090368E 01
3	2.9294873E 00	5.4996082E 00	9.7348640E 00	1.5235727E 01	1.9679566E 01	2.3231035E 01	2.8498378E 01	3.4138184E 01
4	3.5238309E 00	6.8544838E 00	1.0049303E 01	1.5394057E 01	1.9971962E 01	2.3429886E 01	2.8580310E 01	3.4208916E 01
5	4.0327567E 00	8.1432474E 00	1.0633105E 01	1.5602797E 01	2.0329927E 01	2.3715813E 01	2.8693326E 01	3.4301605E 01
6	4.5361994E 00	9.0588602E 00	1.1661575E 01	1.5872788E 01	2.0726611E 01	2.4103392E 01	2.8841899E 01	3.4415554E 01
7	5.0533631E 00	9.6320740E 00	1.3013450E 01	1.6243117E 01	2.1136058E 01	2.4601664E 01	2.9033477E 01	3.4550425E 01
8	5.5867811E 00	1.0064538E 01	1.4342335E 01	1.6828132E 01	2.1543442E 01	2.5204620E 01	2.9280259E 01	3.4706860E 01
9	6.1348099E 00	1.0456660E 01	1.5327743E 01	1.7818724E 01	2.1954589E 01	2.5884984E 01	2.9601694E 01	3.4886807E 01
10	6.6950442E 00	1.0843628E 01	1.5938228E 01	1.9163328E 01	2.2407725E 01	2.6596025E 01	3.0025515E 01	3.5094354E 01
11	7.2652549E 00	1.1238126E 01	1.6371709E 01	2.0539414E 01	2.3009428E 01	2.7284321E 01	3.0581379E 01	3.5336825E 01
12	7.8435972E 00	1.1644850E 01	1.6743067E 01	2.1612658E 01	2.3964076E 01	2.7916299E 01	3.1280620E 01	3.5627097E 01

EIGENVALUES A FOR K2= 20., L= 5., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	9.9867679E-01	4.1056892E 00	9.6757212E 00	1.5634059E 01	2.1513508E 01	2.6509827E 01	2.9936826E 01	3.5012724E 01
2	2.5167027E 00	5.0337943E 00	9.7923553E 00	1.5681415E 01	2.1557308E 01	2.6623265E 01	3.0048096E 01	3.5046091E 01
3	3.2199478E 00	6.8407714E 00	1.0112743E 01	1.5781751E 01	2.1642091E 01	2.6832618E 01	3.0281913E 01	3.5116112E 01
4	3.6826406E 00	8.4671510E 00	1.0995659E 01	1.5954560E 01	2.1764597E 01	2.7105712E 01	3.0655732E 01	3.5230508E 01
5	4.1313935E 00	9.2689140E 00	1.2723105E 01	1.6270477E 01	2.1925364E 01	2.7404958E 01	3.1179761E 01	3.5404396E 01
6	4.6041593E 00	9.6825579E 00	1.4480081E 01	1.7011964E 01	2.2136883E 01	2.7702262E 01	3.1836525E 01	3.5665973E 01
7	5.1035175E 00	1.0011794E 01	1.5447112E 01	1.8604407E 01	2.2450191E 01	2.7989428E 01	3.2564530E 01	3.6063555E 01
8	5.6255578E 00	1.0331722E 01	1.5897440E 01	2.0456349E 01	2.3074723E 01	2.8280424E 01	3.3265410E 01	3.6660016E 01
9	6.1657901E 00	1.0662919E 01	1.6212267E 01	2.1626174E 01	2.4493579E 01	2.8621884E 01	3.3856103E 01	3.7483650E 01
10	6.7204118E 00	1.1011682E 01	1.6495161E 01	2.2148380E 01	2.6389891E 01	2.9172772E 01	3.4331580E 01	3.8461662E 01
11	7.2864222E 00	1.1379626E 01	1.6774993E 01	2.2475418E 01	2.7777301E 01	3.0401730E 01	3.4759631E 01	3.9428163E 01
12	7.8615287E 00	1.1766584E 01	1.7061885E 01	2.2749994E 01	2.8405023E 01	3.2288030E 01	3.5289121E 01	4.0222691E 01

EIGENVALUES A FOR K2= 50., L= 5., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.5039474E 00	4.2715099E 00	9.7891868E 00	1.5852664E 01	2.1999954E 01	2.8151718E 01	3.4242004E 01	4.0023961E 01
2	2.9643785E 00	6.5608179E 00	1.0082216E 01	1.5925079E 01	2.2036734E 01	2.8178965E 01	3.4272128E 01	4.0094531E 01
3	3.3992415E 00	8.7967975E 00	1.1658906E 01	1.6137357E 01	2.2119309E 01	2.8234843E 01	3.4329855E 01	4.0216446E 01
4	3.7752596E 00	9.3922133E 00	1.4421987E 01	1.7008226E 01	2.2285335E 01	2.8324866E 01	3.4412268E 01	4.0363885E 01
5	4.1890175E 00	9.6708181E 00	1.5498238E 01	1.9672584E 01	2.2757271E 01	2.8469804E 01	3.4519780E 01	4.0518699E 01
6	4.6440906E 00	9.9182644E 00	1.5820857E 01	2.1532786E 01	2.4749727E 01	2.8769303E 01	3.4663104E 01	4.0675771E 01
7	5.1331012E 00	1.0178243E 01	1.6041780E 01	2.2001092E 01	2.7341784E 01	2.9951017E 01	3.4891560E 01	4.0843198E 01
8	5.6484644E 00	1.0460936E 01	1.6250319E 01	2.2234832E 01	2.8164179E 01	3.2754219E 01	3.5526997E 01	4.1051813E 01
9	6.1840976E 00	1.0768556E 01	1.6466541E 01	2.2428653E 01	2.8445650E 01	3.4262796E 01	3.7867635E 01	4.1443968E 01
10	6.7353920E 00	1.1100779E 01	1.6696773E 01	2.2618019E 01	2.8642633E 01	3.4653684E 01	4.0197191E 01	4.2974237E 01
11	7.2989148E 00	1.1456317E 01	1.6943263E 01	2.2813436E 01	2.8821666E 01	3.4869595E 01	4.0841129E 01	4.5785701E 01
12	7.8720981E 00	1.1833544E 01	1.7206683E 01	2.3018891E 01	2.8999682E 01	3.5048302E 01	4.1098192E 01	4.6978828E 01

EIGENVALUES A FOR K2= 100., L= 5., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.9891734E 00	4.5429146E 00	9.8427944E 00	1.5916455E 01	2.2097738E 01	2.8309382E 01	3.4528900E 01	4.0742550E 01
2	3.1267010E 00	8.1831503E 00	1.0776374E 01	1.6057227E 01	2.2152016E 01	2.8340614E 01	3.4551350E 01	4.0761836E 01
3	3.4567758E 00	9.3011714E 00	1.4459710E 01	1.7055579E 01	2.2321345E 01	2.8415325E 01	3.4599788E 01	4.0801136E 01
4	3.8052864E 00	9.5573876E 00	1.5561824E 01	2.0746632E 01	2.3342445E 01	2.8598288E 01	3.4687593E 01	4.0863805E 01
5	4.2078904E 00	9.7621585E 00	1.5802555E 01	2.1840801E 01	2.7036966E 01	2.9631853E 01	3.4880050E 01	4.0962958E 01
6	4.6572427E 00	9.9812180E 00	1.5974588E 01	2.2075164E 01	2.8127045E 01	3.3328903E 01	3.5922352E 01	4.1163869E 01
7	5.1428705E 00	1.0226425E 01	1.6145804E 01	2.2233515E 01	2.8358447E 01	3.4417029E 01	3.9621752E 01	4.2213356E 01
8	5.6560411E 00	1.0499960E 01	1.6330227E 01	2.2383994E 01	2.8509726E 01	3.4647224E 01	4.0709292E 01	4.5915195E 01
9	6.1901563E 00	1.0801248E 01	1.6532027E 01	2.2541148E 01	2.8649030E 01	3.4794570E 01	4.0939338E 01	4.7003151E 01
10	6.7403480E 00	1.1128769E 01	1.6752464E 01	2.2709660E 01	2.8791106E 01	3.4927140E 01	4.1084461E 01	4.7233715E 01
11	7.3030457E 00	1.1480658E 01	1.6991738E 01	2.2891331E 01	2.8940834E 01	3.5059851E 01	4.1212754E 01	4.7377653E 01
12	7.8755963E 00	1.1854933E 01	1.7249560E 01	2.3086853E 01	2.9100263E 01	3.5197711E 01	4.1339200E 01	4.7503119E 01

EIGENVALUES A FOR K2= 2., L= 10., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.9235779E-01	3.5847567E 00	7.5437245E 00	1.0841504E 01	1.4888888E 01	1.8314779E 01	2.2066350E 01	2.5886694E 01
2	5.7647897E-01	3.6237785E 00	7.5618020E 00	1.0854869E 01	1.4897745E 01	1.8322892E 01	2.2072311E 01	2.5892293E 01
3	9.5881025E-01	3.7010365E 00	7.5978109E 00	1.0881572E 01	1.4915437E 01	1.8339106E 01	2.2084253E 01	2.5903495E 01
4	1.3381419E 00	3.8150863E 00	7.6514624E 00	1.0921533E 01	1.4941934E 01	1.8363391E 01	2.2102171E 01	2.5920258E 01
5	1.7132554E 00	3.9640130E 00	7.7223545E 00	1.0974655E 01	1.4977190E 01	1.8395729E 01	2.2126044E 01	2.5942614E 01
6	2.0829655E 00	4.1456345E 00	7.8099605E 00	1.1040805E 01	1.5021161E 01	1.8436076E 01	2.2155868E 01	2.5970502E 01
7	2.4461873E 00	4.3576360E 00	7.9136628E 00	1.1119810E 01	1.5073763E 01	1.8484376E 01	2.2191617E 01	2.6003948E 01
8	2.8020503E 00	4.5976424E 00	8.0327871E 00	1.1211475E 01	1.5134919E 01	1.8540578E 01	2.2233284E 01	2.6042863E 01
9	3.1500146E 00	4.8631837E 00	8.1665952E 00	1.1315566E 01	1.5204532E 01	1.8604593E 01	2.2280833E 01	2.6087245E 01
10	3.4899729E 00	5.1516508E 00	8.3143541E 00	1.1431842E 01	1.5282506E 01	1.8676355E 01	2.2334280E 01	2.6137065E 01
11	3.8222708E 00	5.4602721E 00	8.4753016E 00	1.1560015E 01	1.5368739E 01	1.8755773E 01	2.2393578E 01	2.6192278E 01
12	4.1476495E 00	5.7861698E 00	8.6487443E 00	1.1699773E 01	1.5463106E 01	1.8842737E 01	2.2458709E 01	2.6252818E 01

EIGENVALUES A FOR K2= 5., L= 10., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.7184587E-01	3.8773634E 00	9.0087074E 00	1.3297214E 01	1.6827477E 01	2.1838308E 01	2.6516012E 01	2.9897309E 01
2	8.1011308E-01	3.9404873E 00	9.0263900E 00	1.3317864E 01	1.6841245E 01	2.1845789E 01	2.6525400E 01	2.9905934E 01
3	1.3317469E 00	4.0691119E 00	9.0617675E 00	1.3358939E 01	1.6868848E 01	2.1860772E 01	2.6544119E 01	2.9923235E 01
4	1.8249972E 00	4.2674140E 00	9.1148814E 00	1.3419992E 01	1.6910478E 01	2.1883255E 01	2.6572087E 01	2.9949238E 01
5	2.2790771E 00	4.5393404E 00	9.1858802E 00	1.3500360E 01	1.6966368E 01	2.1913242E 01	2.6609155E 01	2.9984011E 01
6	2.6877274E 00	4.8847290E 00	9.2750814E 00	1.3599171E 01	1.7036873E 01	2.1950776E 01	2.6655146E 01	3.0027666E 01
7	3.0524675E 00	5.2958577E 00	9.3831296E 00	1.3715346E 01	1.7122371E 01	2.1995875E 01	2.6709830E 01	3.0080302E 01
8	3.3817245E 00	5.7580850E 00	9.5111862E 00	1.3847633E 01	1.7223339E 01	2.2048603E 01	2.6772944E 01	3.0142078E 01
9	3.6864002E 00	6.2539931E 00	9.6611959E 00	1.3994662E 01	1.7340250E 01	2.2109050E 01	2.6844178E 01	3.0213086E 01
10	3.9760404E 00	6.7668403E 00	9.8362639E 00	1.4154939E 01	1.7473608E 01	2.2177318E 01	2.6923192E 01	3.0293488E 01
11	4.2574949E 00	7.2814029E 00	1.0041074E 01	1.4326948E 01	1.7623877E 01	2.2253531E 01	2.7009626E 01	3.0383465E 01
12	4.5351807E 00	7.7833803E 00	1.0282171E 01	1.4509231E 01	1.7791399E 01	2.2337905E 01	2.7103110E 01	3.0483123E 01

EIGENVALUES A FOR K2= 10., L= 10., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.6761530E-01	3.9776413E 00	9.4736172E 00	1.5066834E 01	1.9357785E 01	2.3040206E 01	2.8416047E 01	3.4063943E 01
2	1.0839857E 00	4.0833172E 00	9.4931576E 00	1.5080784E 01	1.9384499E 01	2.3055098E 01	2.8422622E 01	3.4070002E 01
3	1.7413089E 00	4.3104314E 00	9.5329348E 00	1.5108503E 01	1.9437575E 01	2.3085159E 01	2.8435811E 01	3.4082060E 01
4	2.3003429E 00	4.6809466E 00	9.5945337E 00	1.5149661E 01	1.9516245E 01	2.3130942E 01	2.8455701E 01	3.4100107E 01
5	2.7447399E 00	5.1969399E 00	9.6808997E 00	1.5203875E 01	1.9619346E 01	2.3193298E 01	2.8482399E 01	3.4124036E 01
6	3.0954110E 00	5.8234000E 00	9.7972169E 00	1.5270703E 01	1.9745246E 01	2.3273376E 01	2.8516097E 01	3.4153762E 01
7	3.3889873E 00	6.5071670E 00	9.9525305E 00	1.5349868E 01	1.9891772E 01	2.3372582E 01	2.8557029E 01	3.4189115E 01
8	3.6541395E 00	7.1974981E 00	1.0162265E 01	1.5441349E 01	2.0056286E 01	2.3492622E 01	2.8605501E 01	3.4230054E 01
9	3.9075478E 00	7.8474355E 00	1.0450441E 01	1.5545576E 01	2.0235709E 01	2.3635346E 01	2.8661959E 01	3.4276414E 01
10	4.1578671E 00	8.4135281E 00	1.0845610E 01	1.5663733E 01	2.0426577E 01	2.3802666E 01	2.8726915E 01	3.4328138E 01
11	4.4093502E 00	8.8702279E 00	1.1362211E 01	1.5798187E 01	2.0625332E 01	2.3996297E 01	2.8801100E 01	3.4385084E 01
12	4.6639711E 00	9.2253870E 00	1.1982099E 01	1.5953313E 01	2.0828633E 01	2.4217483E 01	2.8885391E 01	3.4447291E 01

EIGENVALUES A FOR K2= 20., L= 10., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.0662081E-01	4.0344732E 00	9.6657646E 00	1.5629698E 01	2.1509354E 01	2.6498887E 01	2.9926593E 01	3.5009645E 01
2	1.4603440E 00	4.2324398E 00	9.6926430E 00	1.5641365E 01	2.1520437E 01	2.6527954E 01	2.9953939E 01	3.5017878E 01
3	2.2285947E 00	4.6920831E 00	9.7498926E 00	1.5664965E 01	2.1542373E 01	2.6584937E 01	3.0009387E 01	3.5034499E 01
4	2.7461915E 00	5.4387310E 00	9.8463627E 00	1.5701129E 01	2.1574804E 01	2.6667656E 01	3.0094410E 01	3.5059950E 01
5	3.0851841E 00	6.3616779E 00	1.0001498E 01	1.5751036E 01	2.1617264E 01	2.6772868E 01	3.0211023E 01	3.5094850E 01
6	3.3429024E 00	7.3076314E 00	1.0256894E 01	1.5816807E 01	2.1669281E 01	2.6896399E 01	3.0361574E 01	3.5140131E 01
7	3.5716117E 00	8.1351831E 00	1.0688477E 01	1.5902321E 01	2.1730526E 01	2.7033563E 01	3.0548287E 01	3.5197090E 01
8	3.7934448E 00	8.7376744E 00	1.1365665E 01	1.6014908E 01	2.1801011E 01	2.7179439E 01	3.0772690E 01	3.5267574E 01
9	4.0174060E 00	9.1266730E 00	1.2246238E 01	1.6169185E 01	2.1881239E 01	2.7329523E 01	3.1034954E 01	3.5354107E 01
10	4.2470049E 00	9.3897840E 00	1.3201654E 01	1.6395636E 01	2.1972519E 01	2.7480175E 01	3.1333088E 01	3.5460092E 01
11	4.4833767E 00	9.5926424E 00	1.4096650E 01	1.6755064E 01	2.2077624E 01	2.7628943E 01	3.1662448E 01	3.5590104E 01
12	4.7266228E 00	9.7680961E 00	1.4804482E 01	1.7331397E 01	2.2201911E 01	2.7774841E 01	3.2015278E 01	3.5750234E 01

EIGENVALUES A FOR K2= 50., L= 10., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	7.8327194E-01	4.0931728E 00	9.7700638E 00	1.5846547E 01	2.1996641E 01	2.8149183E 01	3.4239114E 01	4.0016946E 01
2	2.0938926E 00	4.6149684E 00	9.8234205E 00	1.5863089E 01	2.2005558E 01	2.8155946E 01	3.4246764E 01	4.0035474E 01
3	2.7834509E 00	5.8200468E 00	9.9583738E 00	1.5898548E 01	2.2023816E 01	2.8169560E 01	3.4261893E 01	4.0071079E 01
4	3.0977702E 00	7.2919688E 00	1.0275777E 01	1.5959214E 01	2.2052404E 01	2.8190138E 01	3.4284120E 01	4.0121214E 01
5	3.3057770E 00	8.4475993E 00	1.1043308E 01	1.6060507E 01	2.2093231E 01	2.8217995E 01	3.4313029E 01	4.0182602E 01
6	3.4913618E 00	9.0267123E 00	1.2374306E 01	1.6244394E 01	2.2150096E 01	2.8253751E 01	3.4348230E 01	4.0251824E 01
7	3.6785246E 00	9.2998078E 00	1.3824589E 01	1.6640100E 01	2.2230914E 01	2.8298510E 01	3.4389426E 01	4.0325816E 01
8	3.8745955E 00	9.4709519E 00	1.4865016E 01	1.7535717E 01	2.2354347E 01	2.8354434E 01	3.4436632E 01	4.0402316E 01
9	4.0814925E 00	9.6075511E 00	1.5360162E 01	1.8934045E 01	2.2571807E 01	2.8425804E 01	3.4490257E 01	4.0479810E 01
10	4.2991579E 00	9.7327712E 00	1.5602098E 01	2.0346055E 01	2.3040799E 01	2.8521772E 01	3.4551425E 01	4.0557649E 01
11	4.5268199E 00	9.8559760E 00	1.5757171E 01	2.1276618E 01	2.4051626E 01	2.8664336E 01	3.4622498E 01	4.0636091E 01
12	4.7634783E 00	9.9814470E 00	1.5879674E 01	2.1705169E 01	2.5494595E 01	2.8916097E 01	3.4708310E 01	4.0716016E 01

EIGENVALUES A FOR K2= 100., L= 10., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.0872912E 00	4.1564109E 00	9.8067318E 00	1.5906643E 01	2.2093199E 01	2.8306609E 01	3.4526852E 01	4.0740733E 01
2	2.5569776E 00	5.2929746E 00	9.9138932E 00	1.5933771E 01	2.2105475E 01	2.8314066E 01	3.4532363E 01	4.0745552E 01
3	3.0156147E 00	7.3259079E 00	1.0294316E 01	1.5999465E 01	2.2131967E 01	2.8329528E 01	3.4543531E 01	4.0755196E 01
4	3.2162777E 00	8.7299964E 00	1.1575779E 01	1.6145949E 01	2.2178083E 01	2.8354254E 01	3.4560776E 01	4.0769734E 01
5	3.3767986E 00	9.1927948E 00	1.3614102E 01	1.6557378E 01	2.2257843E 01	2.8390930E 01	3.4584816E 01	4.0789343E 01
6	3.5390478E 00	9.3811080E 00	1.4997126E 01	1.7866191E 01	2.2415284E 01	2.8445345E 01	3.4617054E 01	4.0814357E 01
7	3.7130725E 00	9.5039541E 00	1.5454838E 01	1.9905823E 01	2.2837560E 01	2.8531852E 01	3.4660259E 01	4.0845538E 01
8	3.9009868E 00	9.6088445E 00	1.5639544E 01	2.1279676E 01	2.4158134E 01	2.8694960E 01	3.4720279E 01	4.0884180E 01
9	4.1024368E 00	9.7105476E 00	1.5754676E 01	2.1734512E 01	2.6198630E 01	2.9123179E 01	3.4811541E 01	4.0933075E 01
10	4.3162694E 00	9.8148124E 00	1.5847378E 01	2.1917582E 01	2.7567668E 01	3.0450584E 01	3.4978689E 01	4.0998031E 01
11	4.5411142E 00	9.9242008E 00	1.5932671E 01	2.2029549E 01	2.8021032E 01	3.2491904E 01	3.5411079E 01	4.1093507E 01
12	4.7756283E 00	1.0039903E 01	1.6016609E 01	2.2117243E 01	2.8203418E 01	3.3858268E 01	3.6743242E 01	4.1264229E 01

EIGENVALUES A FOR K2= 2., L= 20., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	9.6188140E-02	3.5810853E 00	7.5420303E 00	1.0840249E 01	1.4888060E 01	1.8314028E 01	2.2065770E 01	2.5886172E 01
2	2.8849018E-01	3.5908718E 00	7.5465550E 00	1.0843591E 01	1.4890273E 01	1.8316051E 01	2.2067270E 01	2.5887576E 01
3	4.8056943E-01	3.6103955E 00	7.5555947E 00	1.0850280E 01	1.4894699E 01	1.8320106E 01	2.2070269E 01	2.5890369E 01
4	6.7227671E-01	3.6395546E 00	7.5691310E 00	1.0860297E 01	1.4901341E 01	1.8326195E 01	2.2074749E 01	2.5894558E 01
5	8.6346120E-01	3.6782064E 00	7.5871404E 00	1.0873651E 01	1.4910181E 01	1.8334288E 01	2.2080710E 01	2.5900155E 01
6	1.0539722E 00	3.7261682E 00	7.6095831E 00	1.0890323E 01	1.4921238E 01	1.8344421E 01	2.2088180E 01	2.5907148E 01
7	1.2436566E 00	3.7832237E 00	7.6364177E 00	1.0910305E 01	1.4934481E 01	1.8356570E 01	2.2097129E 01	2.5915551E 01
8	1.4323631E 00	3.8491300E 00	7.6675916E 00	1.0933585E 01	1.4949930E 01	1.8370730E 01	2.2107583E 01	2.5925321E 01
9	1.6199387E 00	3.9236289E 00	7.7030417E 00	1.0960146E 01	1.4967558E 01	1.8386899E 01	2.2119528E 01	2.5936507E 01
10	1.8062346E 00	4.0064416E 00	7.7427140E 00	1.0989977E 01	1.4987370E 01	1.8405068E 01	2.2132942E 01	2.5949062E 01
11	1.9911068E 00	4.0972871E 00	7.7865245E 00	1.1023053E 01	1.5009356E 01	1.8425235E 01	2.2147851E 01	2.5963020E 01
12	2.1744177E 00	4.1958755E 00	7.8344005E 00	1.1059356E 01	1.5033503E 01	1.8447406E 01	2.2164243E 01	2.5978343E 01

EIGENVALUES A FOR K2= 5., L= 20., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.3600691E-01	3.8714875E 00	9.0070479E 00	1.3295278E 01	1.6826190E 01	2.1837597E 01	2.6515135E 01	2.9896485E 01
2	4.0734779E-01	3.8871711E 00	9.0114675E 00	1.3300449E 01	1.6829637E 01	2.1839480E 01	2.6517486E 01	2.9898631E 01
3	6.7665568E-01	3.9186957E 00	9.0203099E 00	1.3310780E 01	1.6836505E 01	2.1843221E 01	2.6522171E 01	2.9902970E 01
4	9.4253410E-01	3.9663620E 00	9.0335739E 00	1.3326233E 01	1.6846839E 01	2.1848842E 01	2.6529206E 01	2.9909427E 01
5	1.2035365E 00	4.0306049E 00	9.0512614E 00	1.3346778E 01	1.6860644E 01	2.1856331E 01	2.6538575E 01	2.9918097E 01
6	1.4581795E 00	4.1119736E 00	9.0733770E 00	1.3372349E 01	1.6877930E 01	2.1865689E 01	2.6550257E 01	2.9928912E 01
7	1.7049801E 00	4.2110554E 00	9.0999328E 00	1.3402887E 01	1.6898742E 01	2.1876929E 01	2.6564235E 01	2.9941917E 01
8	1.9425449E 00	4.3283923E 00	9.1309495E 00	1.3438305E 01	1.6923103E 01	2.1890035E 01	2.6580507E 01	2.9957092E 01
9	2.1696884E 00	4.4643509E 00	9.1664444E 00	1.3478502E 01	1.6951038E 01	2.1905028E 01	2.6599047E 01	2.9974490E 01
10	2.3855828E 00	4.6189597E 00	9.2064574E 00	1.3523368E 01	1.6982619E 01	2.1921918E 01	2.6619821E 01	2.9994100E 01
11	2.5898829E 00	4.7917753E 00	9.2510489E 00	1.3572789E 01	1.7017855E 01	2.1940687E 01	2.6642817E 01	3.0015915E 01
12	2.7827885E 00	4.9818278E 00	9.3002924E 00	1.3626630E 01	1.7056824E 01	2.1961335E 01	2.6668014E 01	3.0039979E 01

EIGENVALUES A FOR K2= 10., L= 20., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.8409626E-01	3.9680231E 00	9.4717988E 00	1.5065529E 01	1.9355269E 01	2.3038810E 01	2.8415433E 01	3.4063361E 01
2	5.4997285E-01	3.9937833E 00	9.4766574E 00	1.5069024E 01	1.9361967E 01	2.3042522E 01	2.8417075E 01	3.4064897E 01
3	9.0878708E-01	4.0463626E 00	9.4864178E 00	1.5075999E 01	1.9375335E 01	2.3049967E 01	2.8420358E 01	3.4067903E 01
4	1.2555061E 00	4.1278157E 00	9.5011601E 00	1.5086438E 01	1.9395315E 01	2.3061178E 01	2.8425309E 01	3.4072458E 01
5	1.5848260E 00	4.2409803E 00	9.5210215E 00	1.5100300E 01	1.9421863E 01	2.3076191E 01	2.8431897E 01	3.4078479E 01
6	1.8915363E 00	4.3889461E 00	9.5462000E 00	1.5117550E 01	1.9454872E 01	2.3095101E 01	2.8440163E 01	3.4086027E 01
7	2.1713486E 00	4.5741286E 00	9.5769553E 00	1.5138137E 01	1.9494239E 01	2.3117979E 01	2.8450108E 01	3.4095039E 01
8	2.4219542E 00	4.7970769E 00	9.6136447E 00	1.5162008E 01	1.9539784E 01	2.3144935E 01	2.8461748E 01	3.4105552E 01
9	2.6437721E 00	5.0556677E 00	9.6567515E 00	1.5189122E 01	1.9591372E 01	2.3176105E 01	2.8475079E 01	3.4117512E 01
10	2.8397760E 00	5.3451962E 00	9.7068977E 00	1.5219408E 01	1.9648752E 01	2.3211603E 01	2.8490170E 01	3.4130942E 01
11	3.0144947E 00	5.6592970E 00	9.7649189E 00	1.5252823E 01	1.9711735E 01	2.3251628E 01	2.8507007E 01	3.4145805E 01
12	3.1728130E 00	5.9910575E 00	9.8319191E 00	1.5289339E 01	1.9780028E 01	2.3296318E 01	2.8525651E 01	3.4162071E 01

EIGENVALUES A FOR K2= 20., L= 20., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.5420243E-01	4.0172539E 00	9.6633000E 00	1.5628611E 01	2.1508311E 01	2.6496142E 01	2.9924048E 01	3.5008883E 01
2	7.5542042E-01	4.0636942E 00	9.6698937E 00	1.5631514E 01	2.1511089E 01	2.6503460E 01	2.9930862E 01	3.5010929E 01
3	1.2343571E 00	4.1615158E 00	9.6832776E 00	1.5637340E 01	2.1516638E 01	2.6517997E 01	2.9944516E 01	3.5015036E 01
4	1.6745185E 00	4.3198386E 00	9.7038767E 00	1.5646129E 01	2.1524919E 01	2.6539642E 01	2.9965127E 01	3.5021237E 01
5	2.0599388E 00	4.5489559E 00	9.7323716E 00	1.5657917E 01	2.1535891E 01	2.6568180E 01	2.9992823E 01	3.5029545E 01
6	2.3806567E 00	4.8540694E 00	9.7698103E 00	1.5672787E 01	2.1549523E 01	2.6603328E 01	3.0027809E 01	3.5040026E 01
7	2.6379810E 00	5.2294437E 00	9.8177656E 00	1.5690860E 01	2.1565736E 01	2.6644728E 01	3.0070280E 01	3.5052746E 01
8	2.8432273E 00	5.6591306E 00	9.8785224E 00	1.5712254E 01	2.1584501E 01	2.6692007E 01	3.0120518E 01	3.5067756E 01
9	3.0111091E 00	6.1230528E 00	9.9554718E 00	1.5737169E 01	2.1605736E 01	2.6744662E 01	3.0178785E 01	3.5085201E 01
10	3.1544467E 00	6.6016573E 00	1.0053603E 01	1.5765879E 01	2.1629389E 01	2.6802198E 01	3.0245388E 01	3.5105160E 01
11	3.2825382E 00	7.0768332E 00	1.0180072E 01	1.5798701E 01	2.1655401E 01	2.6864059E 01	3.0320619E 01	3.5127765E 01
12	3.4015938E 00	7.5309685E 00	1.0344662E 01	1.5836125E 01	2.1683732E 01	2.6929612E 01	3.0404790E 01	3.5153217E 01

EIGENVALUES A FOR K2= 50., L= 20., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.9537795E-01	4.0524575E 00	9.7654355E 00	1.5845032E 01	2.1995807E 01	2.8148544E 01	3.4238403E 01	4.0015163E 01
2	1.1556935E 00	4.1645329E 00	9.7779065E 00	1.5849083E 01	2.1998011E 01	2.8150231E 01	3.4240329E 01	4.0019879E 01
3	1.8191617E 00	4.4200045E 00	9.8042083E 00	1.5857316E 01	2.2002481E 01	2.8153616E 01	3.4244144E 01	4.0029157E 01
4	2.3244183E 00	4.8579063E 00	9.8474394E 00	1.5870014E 01	2.2009212E 01	2.8158703E 01	3.4249854E 01	4.0042861E 01
5	2.6625974E 00	5.4693754E 00	9.9135727E 00	1.5887665E 01	2.2018336E 01	2.8165515E 01	3.4257433E 01	4.0060720E 01
6	2.8820153E 00	6.1870843E 00	1.0013632E 01	1.5910968E 01	2.2029947E 01	2.8174048E 01	3.4266797E 01	4.0082371E 01
7	3.0351678E 00	6.9321117E 00	1.0167953E 01	1.5941081E 01	2.2044205E 01	2.8184326E 01	3.4277908E 01	4.0107512E 01
8	3.1546158E 00	7.6313003E 00	1.0411673E 01	1.5979780E 01	2.2061357E 01	2.8196412E 01	3.4290733E 01	4.0135664E 01
9	3.2575350E 00	8.2147503E 00	1.0791767E 01	1.6029847E 01	2.2081714E 01	2.8210326E 01	3.4305201E 01	4.0166404E 01
10	3.3528493E 00	8.6402287E 00	1.1334378E 01	1.6095971E 01	2.2105731E 01	2.8226162E 01	3.4321254E 01	4.0199289E 01
11	3.4453207E 00	8.9234910E 00	1.2008549E 01	1.6186166E 01	2.2134047E 01	2.8244027E 01	3.4338853E 01	4.0233967E 01
12	3.5375666E 00	9.1120453E 00	1.2747364E 01	1.6314758E 01	2.2167587E 01	2.8264036E 01	3.4357959E 01	4.0269980E 01

EIGENVALUES A FOR K2= 100., L= 20., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.5458716E-01	4.0743161E 00	9.7983631E 00	1.5904255E 01	2.2092086E 01	2.8305909E 01	3.4526337E 01	4.0740290E 01
2	1.5738414E 00	4.3070880E 00	9.8212379E 00	1.5910685E 01	2.2095083E 01	2.8307761E 01	3.4527706E 01	4.0741515E 01
3	2.3163973E 00	4.8740618E 00	9.8729093E 00	1.5924087E 01	2.2101180E 01	2.8311496E 01	3.4530461E 01	4.0743899E 01
4	2.7282935E 00	5.7758337E 00	9.9693578E 00	1.5945750E 01	2.2110626E 01	2.8317163E 01	3.4534600E 01	4.0747495E 01
5	2.9434654E 00	6.8189614E 00	1.0148953E 01	1.5978058E 01	2.2123796E 01	2.8324869E 01	3.4540199E 01	4.0752320E 01
6	3.0751741E 00	7.7885543E 00	1.0497753E 01	1.6025452E 01	2.2141337E 01	2.8334762E 01	3.4547246E 01	4.0758343E 01
7	3.1731017E 00	8.4959980E 00	1.1138413E 01	1.6096541E 01	2.2164185E 01	2.8347087E 01	3.4555864E 01	4.0765601E 01
8	3.2574986E 00	8.9005175E 00	1.2067208E 01	1.6209298E 01	2.2193958E 01	2.8362155E 01	3.4566105E 01	4.0774122E 01
9	3.3372493E 00	9.1195720E 00	1.3111075E 01	1.6404131E 01	2.2233272E 01	2.8380400E 01	3.4578092E 01	4.0783933E 01
10	3.4165715E 00	9.2518687E 00	1.4071247E 01	1.6769005E 01	2.2286784E 01	2.8402520E 01	3.4592021E 01	4.0795062E 01
11	3.4975653E 00	9.3435946E 00	1.4767085E 01	1.7424285E 01	2.2363302E 01	2.8429522E 01	3.4608103E 01	4.0807552E 01
12	3.5812987E 00	9.4150645E 00	1.5165257E 01	1.8360071E 01	2.2481144E 01	2.8463050E 01	3.4626675E 01	4.0821536E 01

EIGENVALUES A FOR K2= 2., L= 50., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.8476298E-02	3.5800553E 00	7.5415528E 00	1.0839900E 01	1.4887828E 01	1.8313810E 01	2.2065631E 01	2.5886021E 01
2	1.1542416E-01	3.5816229E 00	7.5422755E 00	1.0840432E 01	1.4888187E 01	1.8314134E 01	2.2065872E 01	2.5886249E 01
3	1.9235773E-01	3.5847586E 00	7.5437273E 00	1.0841502E 01	1.4888889E 01	1.8314785E 01	2.2066341E 01	2.5886691E 01
4	2.6926751E-01	3.5894555E 00	7.5458954E 00	1.0843109E 01	1.4889960E 01	1.8315759E 01	2.2067064E 01	2.5887365E 01
5	3.4614418E-01	3.5957111E 00	7.5487898E 00	1.0845248E 01	1.4891370E 01	1.8317057E 01	2.2068010E 01	2.5888249E 01
6	4.2297807E-01	3.6035219E 00	7.5524068E 00	1.0847922E 01	1.4893139E 01	1.8318677E 01	2.2069214E 01	2.5889374E 01
7	4.9975923E-01	3.6128780E 00	7.5567434E 00	1.0851128E 01	1.4895267E 01	1.8320622E 01	2.2070648E 01	2.5890721E 01
8	5.7647884E-01	3.6237770E 00	7.5618032E 00	1.0854872E 01	1.4897750E 01	1.8322898E 01	2.2072320E 01	2.5892282E 01
9	6.5312697E-01	3.6362089E 00	7.5675762E 00	1.0859144E 01	1.4900581E 01	1.8325484E 01	2.2074219E 01	2.5894076E 01
10	7.2969369E-01	3.6501606E 00	7.5740682E 00	1.0863953E 01	1.4903764E 01	1.8328412E 01	2.2076386E 01	2.5896087E 01
11	8.0616966E-01	3.6656256E 00	7.5812668E 00	1.0869293E 01	1.4907304E 01	1.8331645E 01	2.2078761E 01	2.5898341E 01
12	8.8254496E-01	3.6825882E 00	7.5891877E 00	1.0875166E 01	1.4911193E 01	1.8335219E 01	2.2081385E 01	2.5900795E 01

EIGENVALUES A FOR K2= 5., L= 50., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.4412171E-02	3.8698431E 00	9.0065858E 00	1.3294738E 01	1.6825834E 01	2.1837415E 01	2.6514886E 01	2.9896273E 01
2	1.6319361E-01	3.8723478E 00	9.0072894E 00	1.3295560E 01	1.6826386E 01	2.1837714E 01	2.6515270E 01	2.9896605E 01
3	2.7184585E-01	3.8773637E 00	9.0087073E 00	1.3297214E 01	1.6827477E 01	2.1838308E 01	2.6516012E 01	2.9897309E 01
4	3.8028263E-01	3.8848945E 00	9.0108297E 00	1.3299698E 01	1.6829132E 01	2.1839198E 01	2.6517153E 01	2.9898319E 01
5	4.8841627E-01	3.8949545E 00	9.0136559E 00	1.3303007E 01	1.6831335E 01	2.1840408E 01	2.6518643E 01	2.9899725E 01
6	5.9615929E-01	3.9075603E 00	9.0171936E 00	1.3307142E 01	1.6834087E 01	2.1841890E 01	2.6520526E 01	2.9901443E 01
7	7.0342160E-01	3.9227289E 00	9.0214360E 00	1.3312090E 01	1.6837391E 01	2.1843698E 01	2.6522771E 01	2.9903517E 01
8	8.1011270E-01	3.9404856E 00	9.0263891E 00	1.3317864E 01	1.6841245E 01	2.1845794E 01	2.6525407E 01	2.9905942E 01
9	9.1614138E-01	3.9608577E 00	9.0320474E 00	1.3324454E 01	1.6845655E 01	2.1848179E 01	2.6528402E 01	2.9908696E 01
10	1.0214140E 00	3.9838701E 00	9.0384170E 00	1.3331862E 01	1.6850623E 01	2.1850885E 01	2.6531769E 01	2.9911796E 01
11	1.1258356E 00	4.0095608E 00	9.0454925E 00	1.3340084E 01	1.6856142E 01	2.1853871E 01	2.6535512E 01	2.9915255E 01
12	1.2293114E 00	4.0379623E 00	9.0532738E 00	1.3349107E 01	1.6862210E 01	2.1857177E 01	2.6539630E 01	2.9919075E 01

EIGENVALUES A FOR K2= 10., L= 50., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	7.3670700E-02	3.9653389E 00	9.4712901E 00	1.5065157E 01	1.9354574E 01	2.3038429E 01	2.8415260E 01	3.4063211E 01
2	2.2086476E-01	3.9694319E 00	9.4720662E 00	1.5065721E 01	1.9355641E 01	2.3039022E 01	2.8415523E 01	3.4063458E 01
3	3.6761534E-01	3.9776420E 00	9.4736178E 00	1.5066835E 01	1.9357786E 01	2.3040207E 01	2.8416048E 01	3.4063943E 01
4	5.1362265E-01	3.9900263E 00	9.4759518E 00	1.5068516E 01	1.9360998E 01	2.3041994E 01	2.8416833E 01	3.4064659E 01
5	6.5858115E-01	4.0066664E 00	9.4790672E 00	1.5070745E 01	1.9365279E 01	2.3044364E 01	2.8417896E 01	3.4065627E 01
6	8.0217725E-01	4.0276701E 00	9.4829725E 00	1.5073537E 01	1.9370625E 01	2.3047337E 01	2.8419196E 01	3.4066840E 01
7	9.4408900E-01	4.0531672E 00	9.4876646E 00	1.5076884E 01	1.9377032E 01	2.3050915E 01	2.8420795E 01	3.4068292E 01
8	1.0839854E 00	4.0833178E 00	9.4931539E 00	1.5080784E 01	1.9384500E 01	2.3055102E 01	2.8422625E 01	3.4070005E 01
9	1.2215282E 00	4.1182960E 00	9.4994576E 00	1.5085232E 01	1.9393026E 01	2.3059887E 01	2.8424741E 01	3.4071931E 01
10	1.3563729E 00	4.1582931E 00	9.5065751E 00	1.5090235E 01	1.9402600E 01	2.3065274E 01	2.8427107E 01	3.4074094E 01
11	1.4881754E 00	4.2035122E 00	9.5145208E 00	1.5095782E 01	1.9413228E 01	2.3071281E 01	2.8429744E 01	3.4076515E 01
12	1.6165957E 00	4.2541596E 00	9.5233003E 00	1.5101874E 01	1.9424884E 01	2.3077916E 01	2.8432658E 01	3.4079171E 01

EIGENVALUES A FOR K2= 20., L= 50., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.0178027E-01	4.0124755E 00	9.6626093E 00	1.5628298E 01	2.1508027E 01	2.6495378E 01	2.9923335E 01	3.5008661E 01
2	3.0488659E-01	4.0197644E 00	9.6636602E 00	1.5628765E 01	2.1508464E 01	2.6496544E 01	2.9924425E 01	3.5008989E 01
3	5.0662097E-01	4.0344729E 00	9.6657655E 00	1.5629701E 01	2.1509356E 01	2.6498889E 01	2.9926595E 01	3.5009647E 01
4	7.0604087E-01	4.0568569E 00	9.6689352E 00	1.5631095E 01	2.1510694E 01	2.6502404E 01	2.9929868E 01	3.5010635E 01
5	9.0216706E-01	4.0872984E 00	9.6731852E 00	1.5632953E 01	2.1512467E 01	2.6507065E 01	2.9934226E 01	3.5011947E 01
6	1.0939750E 00	4.1262913E 00	9.6785301E 00	1.5635286E 01	2.1514682E 01	2.6512876E 01	2.9939699E 01	3.5013593E 01
7	1.2803978E 00	4.1744462E 00	9.6850076E 00	1.5638082E 01	2.1517333E 01	2.6519854E 01	2.9946269E 01	3.5015568E 01
8	1.4603433E 00	4.2324386E 00	9.6926408E 00	1.5641361E 01	2.1520433E 01	2.6527950E 01	2.9953951E 01	3.5017870E 01
9	1.6327266E 00	4.3009745E 00	9.7014767E 00	1.5645114E 01	2.1523969E 01	2.6537168E 01	2.9962743E 01	3.5020512E 01
10	1.7965226E 00	4.3807394E 00	9.7115552E 00	1.5649347E 01	2.1527928E 01	2.6547495E 01	2.9972687E 01	3.5023493E 01
11	1.9508417E 00	4.4722994E 00	9.7229426E 00	1.5654058E 01	2.1532311E 01	2.6558916E 01	2.9983756E 01	3.5026813E 01
12	2.0949978E 00	4.5760463E 00	9.7356915E 00	1.5659264E 01	2.1537127E 01	2.6571400E 01	2.9995997E 01	3.5030496E 01

EIGENVALUES A FOR K2= 50., L= 50., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.5856425E-01	4.0413434E 00	9.7641449E 00	1.5844607E 01	2.1995571E 01	2.8148364E 01	3.4238205E 01	4.0014687E 01
2	4.7380105E-01	4.0583286E 00	9.7661071E 00	1.5845248E 01	2.1995933E 01	2.8148635E 01	3.4238499E 01	4.0015450E 01
3	7.8327215E-01	4.0931737E 00	9.7700626E 00	1.5846544E 01	2.1996634E 01	2.8149174E 01	3.4239122E 01	4.0016930E 01
4	1.0828553E 00	4.1475919E 00	9.7760712E 00	1.5848495E 01	2.1997704E 01	2.8149996E 01	3.4240035E 01	4.0019199E 01
5	1.3681236E 00	4.2240708E 00	9.7842457E 00	1.5851105E 01	2.1999122E 01	2.8151069E 01	3.4241273E 01	4.0022187E 01
6	1.6344710E 00	4.3255998E 00	9.7947235E 00	1.5854391E 01	2.2000891E 01	2.8152425E 01	3.4242803E 01	4.0025887E 01
7	1.8775322E 00	4.4551660E 00	9.8076980E 00	1.5858383E 01	2.2003046E 01	2.8154055E 01	3.4244628E 01	4.0030316E 01
8	2.0938930E 00	4.6149683E 00	9.8234183E 00	1.5863084E 01	2.2005551E 01	2.8155954E 01	3.4246772E 01	4.0035477E 01
9	2.2818583E 00	4.8055732E 00	9.8422176E 00	1.5868536E 01	2.2008445E 01	2.8158124E 01	3.4249200E 01	4.0041277E 01
10	2.4418818E 00	5.0254467E 00	9.8645004E 00	1.5874766E 01	2.2011706E 01	2.8160565E 01	3.4251936E 01	4.0047786E 01
11	2.5763590E 00	5.2710848E 00	9.8908021E 00	1.5881806E 01	2.2015353E 01	2.8163284E 01	3.4254973E 01	4.0054920E 01
12	2.6888911E 00	5.5377136E 00	9.9218031E 00	1.5889725E 01	2.2019381E 01	2.8166293E 01	3.4258283E 01	4.0062723E 01

EIGENVALUES A FOR K2= 100., L= 50., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.2302670E-01	4.0524936E 00	9.7960634E 00	1.5903596E 01	2.2091766E 01	2.8305721E 01	3.4526187E 01	4.0740178E 01
2	6.6361271E-01	4.0859500E 00	9.7995782E 00	1.5904608E 01	2.2092239E 01	2.8306012E 01	3.4526409E 01	4.0740366E 01
3	1.0872908E 00	4.1564100E 00	9.8067299E 00	1.5906648E 01	2.2093192E 01	2.8306610E 01	3.4526843E 01	4.0740752E 01
4	1.4814657E 00	4.2706653E 00	9.8178039E 00	1.5909740E 01	2.2094640E 01	2.8307502E 01	3.4527498E 01	4.0741320E 01
5	1.8330011E 00	4.4374066E 00	9.8332494E 00	1.5913934E 01	2.2096575E 01	2.8308677E 01	3.4528387E 01	4.0742080E 01
6	2.1310183E 00	4.6638966E 00	9.8537049E 00	1.5919284E 01	2.2099008E 01	2.8310179E 01	3.4529487E 01	4.0743040E 01
7	2.3710485E 00	4.9514139E 00	9.8801371E 00	1.5925860E 01	2.2101979E 01	2.8311975E 01	3.4530807E 01	4.0744188E 01
8	2.5569786E 00	5.2929744E 00	9.9138966E 00	1.5933779E 01	2.2105470E 01	2.8314084E 01	3.4532348E 01	4.0745542E 01
9	2.6985209E 00	5.6755447E 00	9.9568783E 00	1.5943158E 01	2.2109522E 01	2.8316504E 01	3.4534138E 01	4.0747082E 01
10	2.8067718E 00	6.0841325E 00	1.0011787E 01	1.5954179E 01	2.2114176E 01	2.8319243E 01	3.4536131E 01	4.0748825E 01
11	2.8912745E 00	6.5043237E 00	1.0082457E 01	1.5967034E 01	2.2119430E 01	2.8322323E 01	3.4538358E 01	4.0750731E 01
12	2.9591994E 00	6.9226490E 00	1.0174241E 01	1.5982021E 01	2.2125340E 01	2.8325752E 01	3.4540837E 01	4.0752863E 01

EIGENVALUES A FOR K2= 2., L= 100., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.9238211E-02	3.5799102E 00	7.5414842E 00	1.0839852E 01	1.4887796E 01	1.8313777E 01	2.2065595E 01	2.5886006E 01
2	5.7714069E-02	3.5803005E 00	7.5416647E 00	1.0839985E 01	1.4887889E 01	1.8313864E 01	2.2065652E 01	2.5886057E 01
3	9.6188146E-02	3.5810861E 00	7.5420309E 00	1.0840250E 01	1.4888060E 01	1.8314029E 01	2.2065770E 01	2.5886172E 01
4	1.3465927E-01	3.5822595E 00	7.5425689E 00	1.0840651E 01	1.4888325E 01	1.8314265E 01	2.2065955E 01	2.5886326E 01
5	1.7312620E-01	3.5838258E 00	7.5432965E 00	1.0841188E 01	1.4888687E 01	1.8314590E 01	2.2066195E 01	2.5886551E 01
6	2.1158771E-01	3.5857838E 00	7.5441996E 00	1.0841852E 01	1.4889129E 01	1.8315004E 01	2.2066494E 01	2.5886847E 01
7	2.5004283E-01	3.5881326E 00	7.5452840E 00	1.0842656E 01	1.4889660E 01	1.8315489E 01	2.2066846E 01	2.5887179E 01
8	2.8849020E-01	3.5908706E 00	7.5465544E 00	1.0843591E 01	1.4890274E 01	1.8316054E 01	2.2067273E 01	2.5887578E 01
9	3.2692865E-01	3.5940002E 00	7.5480001E 00	1.0844665E 01	1.4890982E 01	1.8316700E 01	2.2067746E 01	2.5888012E 01
10	3.6535695E-01	3.5975167E 00	7.5496294E 00	1.0845867E 01	1.4891779E 01	1.8317425E 01	2.2068291E 01	2.5888512E 01
11	4.0377424E-01	3.6014223E 00	7.5514384E 00	1.0847205E 01	1.4892670E 01	1.8318245E 01	2.2068886E 01	2.5889084E 01
12	4.4217871E-01	3.6057156E 00	7.5534252E 00	1.0848676E 01	1.4893640E 01	1.8319137E 01	2.2069545E 01	2.5889689E 01

EIGENVALUES A FOR K2= 5., L= 100., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.7206734E-02	3.8696089E 00	9.0065178E 00	1.3294659E 01	1.6825785E 01	2.1837389E 01	2.6514850E 01	2.9896230E 01
2	8.1614876E-02	3.8702341E 00	9.0066925E 00	1.3294865E 01	1.6825914E 01	2.1837452E 01	2.6514944E 01	2.9896318E 01
3	1.3600691E-01	3.8714878E 00	9.0070456E 00	1.3295282E 01	1.6826199E 01	2.1837612E 01	2.6515142E 01	2.9896487E 01
4	1.9037211E-01	3.8733664E 00	9.0075779E 00	1.3295896E 01	1.6826606E 01	2.1837837E 01	2.6515424E 01	2.9896753E 01
5	2.4469965E-01	3.8758734E 00	9.0082835E 00	1.3296726E 01	1.6827159E 01	2.1838123E 01	2.6515798E 01	2.9897104E 01
6	2.9897881E-01	3.8790099E 00	9.0091684E 00	1.3297765E 01	1.6827839E 01	2.1838511E 01	2.6516266E 01	2.9897516E 01
7	3.5319835E-01	3.8827740E 00	9.0102307E 00	1.3299002E 01	1.6828671E 01	2.1838962E 01	2.6516835E 01	2.9898027E 01
8	4.0734768E-01	3.8871720E 00	9.0114667E 00	1.3300448E 01	1.6829635E 01	2.1839479E 01	2.6517484E 01	2.9898657E 01
9	4.6141613E-01	3.8922024E 00	9.0128852E 00	1.3302101E 01	1.6830736E 01	2.1840074E 01	2.6518234E 01	2.9899348E 01
10	5.1539212E-01	3.8978680E 00	9.0144742E 00	1.3303960E 01	1.6831966E 01	2.1840755E 01	2.6519075E 01	2.9900105E 01
11	5.6926504E-01	3.9041700E 00	9.0162444E 00	1.3306026E 01	1.6833345E 01	2.1841504E 01	2.6520016E 01	2.9900985E 01
12	6.2302346E-01	3.9111111E 00	9.0181893E 00	1.3308299E 01	1.6834853E 01	2.1842316E 01	2.6521049E 01	2.9901920E 01

EIGENVALUES A FOR K2= 10., L= 100., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.6837617E-02	3.9649564E 00	9.4712156E 00	1.5065110E 01	1.9354464E 01	2.3038368E 01	2.8415227E 01	3.4063204E 01
2	1.1049457E-01	3.9659776E 00	9.4714075E 00	1.5065249E 01	1.9354733E 01	2.3038514E 01	2.8415304E 01	3.4063245E 01
3	1.8409621E-01	3.9680234E 00	9.4717951E 00	1.5065531E 01	1.9355274E 01	2.3038818E 01	2.8415444E 01	3.4063377E 01
4	2.5760576E-01	3.9710966E 00	9.4723806E 00	1.5065946E 01	1.9356072E 01	2.3039264E 01	2.8415623E 01	3.4063567E 01
5	3.3098556E-01	3.9752010E 00	9.4731599E 00	1.5066507E 01	1.9357150E 01	2.3039854E 01	2.8415885E 01	3.4063815E 01
6	4.0419874E-01	3.9803453E 00	9.4741330E 00	1.5067202E 01	1.9358484E 01	2.3040598E 01	2.8416214E 01	3.4064102E 01
7	4.7720717E-01	3.9865349E 00	9.4752977E 00	1.5068041E 01	1.9360096E 01	2.3041487E 01	2.8416610E 01	3.4064474E 01
8	5.4997296E-01	3.9937837E 00	9.4766589E 00	1.5069018E 01	1.9361956E 01	2.3042531E 01	2.8417070E 01	3.4064878E 01
9	6.2245668E-01	4.0021021E 00	9.4782192E 00	1.5070134E 01	1.9364098E 01	2.3043717E 01	2.8417602E 01	3.4065378E 01
10	6.9462011E-01	4.0115037E 00	9.4799670E 00	1.5071394E 01	1.9366517E 01	2.3045048E 01	2.8418195E 01	3.4065914E 01
11	7.6642333E-01	4.0220024E 00	9.4819167E 00	1.5072792E 01	1.9369181E 01	2.3046541E 01	2.8418853E 01	3.4066518E 01
12	8.3782597E-01	4.0336180E 00	9.4840708E 00	1.5074323E 01	1.9372126E 01	2.3048178E 01	2.8419565E 01	3.4067205E 01

EIGENVALUES A FOR K2= 20., L= 100., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.0897234E-02	4.0117947E 00	9.6625079E 00	1.5628257E 01	2.1507972E 01	2.6495276E 01	2.9923233E 01	3.5008641E 01
2	1.5263502E-01	4.0136132E 00	9.6627750E 00	1.5628378E 01	2.1508095E 01	2.6495561E 01	2.9923506E 01	3.5008711E 01
3	2.5420242E-01	4.0172552E 00	9.6632948E 00	1.5628608E 01	2.1508311E 01	2.6496144E 01	2.9924052E 01	3.5008886E 01
4	3.5548505E-01	4.0227396E 00	9.6640854E 00	1.5628957E 01	2.1508645E 01	2.6497020E 01	2.9924873E 01	3.5009122E 01
5	4.5636687E-01	4.0300901E 00	9.6651378E 00	1.5629421E 01	2.1509092E 01	2.6498190E 01	2.9925961E 01	3.5009441E 01
6	5.5673029E-01	4.0393358E 00	9.6664540E 00	1.5630004E 01	2.1509644E 01	2.6499650E 01	2.9927315E 01	3.5009864E 01
7	6.5645583E-01	4.0505233E 00	9.6680412E 00	1.5630700E 01	2.1510317E 01	2.6501415E 01	2.9928948E 01	3.5010359E 01
8	7.5542073E-01	4.0636933E 00	9.6698929E 00	1.5631513E 01	2.1511088E 01	2.6503459E 01	2.9930861E 01	3.5010929E 01
9	8.5349993E-01	4.0789068E 00	9.6720173E 00	1.5632444E 01	2.1511970E 01	2.6505791E 01	2.9933043E 01	3.5011590E 01
10	9.5056528E-01	4.0962225E 00	9.6744131E 00	1.5633494E 01	2.1512984E 01	2.6508404E 01	2.9935490E 01	3.5012330E 01
11	1.0464855E 00	4.1157120E 00	9.6770917E 00	1.5634660E 01	2.1514076E 01	2.6511317E 01	2.9938219E 01	3.5013151E 01
12	1.1411282E 00	4.1374493E 00	9.6800408E 00	1.5635944E 01	2.1515299E 01	2.6514518E 01	2.9941235E 01	3.5014053E 01

EIGENVALUES A FOR K2= 50., L= 100., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	7.9311458E-02	4.0397660E 00	9.7639643E 00	1.5844548E 01	2.1995542E 01	2.8148345E 01	3.4238158E 01	4.0014613E 01
2	2.3769921E-01	4.0439785E 00	9.7644517E 00	1.5844709E 01	2.1995633E 01	2.8148404E 01	3.4238242E 01	4.0014794E 01
3	3.9537795E-01	4.0524558E 00	9.7654320E 00	1.5845026E 01	2.1995798E 01	2.8148549E 01	3.4238403E 01	4.0015197E 01
4	5.5186540E-01	4.0653121E 00	9.7669090E 00	1.5845513E 01	2.1996063E 01	2.8148741E 01	3.4238620E 01	4.0015733E 01
5	7.0666488E-01	4.0827086E 00	9.7688812E 00	1.5846155E 01	2.1996417E 01	2.8149013E 01	3.4238940E 01	4.0016506E 01
6	8.5926290E-01	4.1048560E 00	9.7713666E 00	1.5846968E 01	2.1996871E 01	2.8149353E 01	3.4239320E 01	4.0017443E 01
7	1.0091236E 00	4.1320280E 00	9.7743715E 00	1.5847944E 01	2.1997406E 01	2.8149766E 01	3.4239779E 01	4.0018542E 01
8	1.1556935E 00	4.1645352E 00	9.7779100E 00	1.5849079E 01	2.1998018E 01	2.8150229E 01	3.4240321E 01	4.0019863E 01
9	1.2984000E 00	4.2027248E 00	9.7819911E 00	1.5850391E 01	2.1998727E 01	2.8150769E 01	3.4240935E 01	4.0021366E 01
10	1.4366633E 00	4.2469832E 00	9.7866405E 00	1.5851865E 01	2.1999539E 01	2.8151385E 01	3.4241622E 01	4.0023050E 01
11	1.5699067E 00	4.2976993E 00	9.7918806E 00	1.5853508E 01	2.2000420E 01	2.8152069E 01	3.4242392E 01	4.0024906E 01
12	1.6975758E 00	4.3552569E 00	9.7977223E 00	1.5855325E 01	2.2001392E 01	2.8152812E 01	3.4243228E 01	4.0026944E 01

EIGENVALUES A FOR K2= 100., L= 100., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.1159781E-01	4.0494184E 00	9.7957391E 00	1.5903497E 01	2.2091728E 01	2.8305689E 01	3.4526178E 01	4.0740148E 01
2	3.3411667E-01	4.0576427E 00	9.7966117E 00	1.5903754E 01	2.2091847E 01	2.8305763E 01	3.4526229E 01	4.0740219E 01
3	5.5458707E-01	4.0743152E 00	9.7983655E 00	1.5904261E 01	2.2092079E 01	2.8305925E 01	3.4526336E 01	4.0740312E 01
4	7.7158833E-01	4.0998815E 00	9.8010141E 00	1.5905020E 01	2.2092430E 01	2.8306131E 01	3.4526496E 01	4.0740457E 01
5	9.8362490E-01	4.1350029E 00	9.8045892E 00	1.5906041E 01	2.2092912E 01	2.8306431E 01	3.4526725E 01	4.0740620E 01
6	1.1891179E 00	4.1805443E 00	9.8091186E 00	1.5907317E 01	2.2093511E 01	2.8306805E 01	3.4526990E 01	4.0740888E 01
7	1.3864151E 00	4.2375291E 00	9.8146509E 00	1.5908869E 01	2.2094226E 01	2.8307243E 01	3.4527321E 01	4.0741149E 01
8	1.5738422E 00	4.3070859E 00	9.8212347E 00	1.5910689E 01	2.2095076E 01	2.8307761E 01	3.4527713E 01	4.0741494E 01
9	1.7497879E 00	4.3903382E 00	9.8289435E 00	1.5912780E 01	2.2096048E 01	2.8308365E 01	3.4528140E 01	4.0741894E 01
10	1.9128370E 00	4.4882454E 00	9.8378546E 00	1.5915161E 01	2.2097134E 01	2.8309017E 01	3.4528642E 01	4.0742319E 01
11	2.0619194E 00	4.6014628E 00	9.8480722E 00	1.5917829E 01	2.2098363E 01	2.8309780E 01	3.4529185E 01	4.0742795E 01
12	2.1964492E 00	4.7301894E 00	9.8597080E 00	1.5920806E 01	2.2099700E 01	2.8310600E 01	3.4529791E 01	4.0743339E 01

EIGENVALUES A FOR K2= 2., L= 1000., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.9238140E-03	3.5798622E 00	7.5414631E 00	1.0839835E 01	1.4887784E 01	1.8313770E 01	2.2065593E 01	2.5885986E 01
2	5.7714673E-03	3.5798663E 00	7.5414672E 00	1.0839833E 01	1.4887783E 01	1.8313771E 01	2.2065596E 01	2.5885992E 01
3	9.6191162E-03	3.5798739E 00	7.5414675E 00	1.0839839E 01	1.4887789E 01	1.8313774E 01	2.2065597E 01	2.5885990E 01
4	1.3466753E-02	3.5798851E 00	7.5414717E 00	1.0839844E 01	1.4887793E 01	1.8313779E 01	2.2065601E 01	2.5885995E 01
5	1.7314400E-02	3.5798997E 00	7.5414793E 00	1.0839845E 01	1.4887797E 01	1.8313775E 01	2.2065590E 01	2.5885998E 01
6	2.1162037E-02	3.5799216E 00	7.5414869E 00	1.0839853E 01	1.4887795E 01	1.8313787E 01	2.2065596E 01	2.5885998E 01
7	2.5009660E-02	3.5799433E 00	7.5415015E 00	1.0839862E 01	1.4887806E 01	1.8313789E 01	2.2065606E 01	2.5885994E 01
8	2.8857290E-02	3.5799722E 00	7.5415162E 00	1.0839871E 01	1.4887810E 01	1.8313789E 01	2.2065603E 01	2.5886011E 01
9	3.2704898E-02	3.5800011E 00	7.5415310E 00	1.0839880E 01	1.4887822E 01	1.8313802E 01	2.2065619E 01	2.5886007E 01
10	3.6552480E-02	3.5800370E 00	7.5415456E 00	1.0839895E 01	1.4887827E 01	1.8313811E 01	2.2065610E 01	2.5886025E 01
11	4.0400088E-02	3.5800765E 00	7.5415637E 00	1.0839908E 01	1.4887832E 01	1.8313810E 01	2.2065625E 01	2.5886033E 01
12	4.4247670E-02	3.5801195E 00	7.5415856E 00	1.0839924E 01	1.4887841E 01	1.8313825E 01	2.2065627E 01	2.5886021E 01

EIGENVALUES A FOR K2= 5., L= 1000., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.7206770E-03	3.8695312E 00	9.0064943E 00	1.3294633E 01	1.6825771E 01	2.1837369E 01	2.6514842E 01	2.9896228E 01
2	8.1620791E-03	3.8695361E 00	9.0064991E 00	1.3294638E 01	1.6825762E 01	2.1837363E 01	2.6514852E 01	2.9896224E 01
3	1.3603446E-02	3.8695484E 00	9.0065038E 00	1.3294637E 01	1.6825770E 01	2.1837362E 01	2.6514845E 01	2.9896212E 01
4	1.9044809E-02	3.8695681E 00	9.0065088E 00	1.3294646E 01	1.6825770E 01	2.1837370E 01	2.6514844E 01	2.9896232E 01
5	2.4486111E-02	3.8695914E 00	9.0065099E 00	1.3294651E 01	1.6825778E 01	2.1837381E 01	2.6514858E 01	2.9896219E 01
6	2.9927377E-02	3.8696258E 00	9.0065221E 00	1.3294660E 01	1.6825785E 01	2.1837385E 01	2.6514859E 01	2.9896219E 01
7	3.5368562E-02	3.8696600E 00	9.0065341E 00	1.3294678E 01	1.6825794E 01	2.1837384E 01	2.6514865E 01	2.9896230E 01
8	4.0809718E-02	3.8697055E 00	9.0065426E 00	1.3294687E 01	1.6825803E 01	2.1837393E 01	2.6514874E 01	2.9896239E 01
9	4.6250761E-02	3.8697545E 00	9.0065620E 00	1.3294705E 01	1.6825820E 01	2.1837407E 01	2.6514883E 01	2.9896244E 01
10	5.1691743E-02	3.8698108E 00	9.0065738E 00	1.3294727E 01	1.6825823E 01	2.1837408E 01	2.6514882E 01	2.9896270E 01
11	5.7132612E-02	3.8698745E 00	9.0065930E 00	1.3294747E 01	1.6825847E 01	2.1837413E 01	2.6514901E 01	2.9896272E 01
12	6.2573348E-02	3.8699454E 00	9.0066122E 00	1.3294771E 01	1.6825860E 01	2.1837413E 01	2.6514903E 01	2.9896276E 01

EIGENVALUES A FOR K2= 10., L= 1000., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.6838069E-03	3.9648288E 00	9.4711947E 00	1.5065088E 01	1.9354440E 01	2.3038355E 01	2.8415228E 01	3.4063188E 01
2	1.1051476E-02	3.9648404E 00	9.4711936E 00	1.5065090E 01	1.9354428E 01	2.3038346E 01	2.8415226E 01	3.4063195E 01
3	1.8419115E-02	3.9648616E 00	9.4712015E 00	1.5065093E 01	1.9354445E 01	2.3038360E 01	2.8415233E 01	3.4063194E 01
4	2.5786619E-02	3.9648923E 00	9.4712064E 00	1.5065098E 01	1.9354450E 01	2.3038365E 01	2.8415238E 01	3.4063197E 01
5	3.3154013E-02	3.9649329E 00	9.4712145E 00	1.5065106E 01	1.9354458E 01	2.3038360E 01	2.8415238E 01	3.4063207E 01
6	4.0521207E-02	3.9649828E 00	9.4712193E 00	1.5065107E 01	1.9354474E 01	2.3038375E 01	2.8415230E 01	3.4063207E 01
7	4.7888249E-02	3.9650455E 00	9.4712301E 00	1.5065116E 01	1.9354497E 01	2.3038384E 01	2.8415241E 01	3.4063185E 01
8	5.5255051E-02	3.9651145E 00	9.4712473E 00	1.5065131E 01	1.9354510E 01	2.3038398E 01	2.8415255E 01	3.4063201E 01
9	6.2621561E-02	3.9651962E 00	9.4712641E 00	1.5065140E 01	1.9354531E 01	2.3038401E 01	2.8415252E 01	3.4063193E 01
10	6.9987765E-02	3.9652905E 00	9.4712807E 00	1.5065154E 01	1.9354559E 01	2.3038417E 01	2.8415252E 01	3.4063204E 01
11	7.7353620E-02	3.9653909E 00	9.4712968E 00	1.5065168E 01	1.9354589E 01	2.3038433E 01	2.8415269E 01	3.4063225E 01
12	8.4719015E-02	3.9655039E 00	9.4713192E 00	1.5065182E 01	1.9354615E 01	2.3038455E 01	2.8415265E 01	3.4063226E 01

EIGENVALUES A FOR K2= 20., L= 1000., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.0899287E-03	4.0115684E 00	9.6624765E 00	1.5628242E 01	2.1507955E 01	2.6495240E 01	2.9923197E 01	3.5008626E 01
2	1.5269795E-02	4.0115883E 00	9.6624825E 00	1.5628242E 01	2.1507966E 01	2.6495241E 01	2.9923207E 01	3.5008629E 01
3	2.5449500E-02	4.0116253E 00	9.6624849E 00	1.5628251E 01	2.1507964E 01	2.6495248E 01	2.9923206E 01	3.5008614E 01
4	3.5628887E-02	4.0116794E 00	9.6624905E 00	1.5628248E 01	2.1507972E 01	2.6495247E 01	2.9923213E 01	3.5008634E 01
5	4.5807915E-02	4.0117505E 00	9.6625061E 00	1.5628257E 01	2.1507974E 01	2.6495261E 01	2.9923235E 01	3.5008628E 01
6	5.5986415E-02	4.0118418E 00	9.6625141E 00	1.5628263E 01	2.1507978E 01	2.6495282E 01	2.9923239E 01	3.5008626E 01
7	6.6164304E-02	4.0119499E 00	9.6625319E 00	1.5628272E 01	2.1507977E 01	2.6495291E 01	2.9923254E 01	3.5008628E 01
8	7.6341382E-02	4.0120779E 00	9.6625489E 00	1.5628275E 01	2.1507987E 01	2.6495305E 01	2.9923271E 01	3.5008650E 01
9	8.6517704E-02	4.0122226E 00	9.6625688E 00	1.5628285E 01	2.1508003E 01	2.6495346E 01	2.9923298E 01	3.5008657E 01
10	9.6693044E-02	4.0123871E 00	9.6625948E 00	1.5628293E 01	2.1508013E 01	2.6495357E 01	2.9923325E 01	3.5008664E 01
11	1.0686731E-01	4.0125678E 00	9.6626228E 00	1.5628307E 01	2.1508031E 01	2.6495395E 01	2.9923347E 01	3.5008664E 01
12	1.1704034E-01	4.0127680E 00	9.6626499E 00	1.5628324E 01	2.1508036E 01	2.6495428E 01	2.9923383E 01	3.5008664E 01

EIGENVALUES A FOR K2= 50., L= 1000., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	7.9320521E-03	4.0392447E 00	9.7639011E 00	1.5844525E 01	2.1995524E 01	2.8148331E 01	3.4238148E 01	4.0014608E 01
2	2.3796083E-02	4.0392877E 00	9.7639078E 00	1.5844529E 01	2.1995524E 01	2.8148329E 01	3.4238163E 01	4.0014598E 01
3	3.9659391E-02	4.0393701E 00	9.7639172E 00	1.5844531E 01	2.1995538E 01	2.8148336E 01	3.4238163E 01	4.0014594E 01
4	5.5521500E-02	4.0394984E 00	9.7639292E 00	1.5844533E 01	2.1995530E 01	2.8148337E 01	3.4238155E 01	4.0014614E 01
5	7.1382031E-02	4.0396653E 00	9.7639506E 00	1.5844540E 01	2.1995538E 01	2.8148327E 01	3.4238163E 01	4.0014603E 01
6	8.7240416E-02	4.0398737E 00	9.7639772E 00	1.5844547E 01	2.1995545E 01	2.8148334E 01	3.4238169E 01	4.0014610E 01
7	1.0309613E-01	4.0401267E 00	9.7640046E 00	1.5844562E 01	2.1995548E 01	2.8148344E 01	3.4238168E 01	4.0014632E 01
8	1.1894886E-01	4.0404203E 00	9.7640362E 00	1.5844569E 01	2.1995550E 01	2.8148341E 01	3.4238179E 01	4.0014656E 01
9	1.3479815E-01	4.0407577E 00	9.7640754E 00	1.5844579E 01	2.1995550E 01	2.8148347E 01	3.4238192E 01	4.0014641E 01
10	1.5064324E-01	4.0411383E 00	9.7641212E 00	1.5844602E 01	2.1995567E 01	2.8148359E 01	3.4238200E 01	4.0014682E 01
11	1.6648400E-01	4.0415581E 00	9.7641703E 00	1.5844614E 01	2.1995579E 01	2.8148372E 01	3.4238195E 01	4.0014696E 01
12	1.8231976E-01	4.0420204E 00	9.7642249E 00	1.5844630E 01	2.1995591E 01	2.8148382E 01	3.4238200E 01	4.0014734E 01

EIGENVALUES A FOR K2= 100., L= 1000., C= 0.5

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.1162487E-02	4.0484048E 00	9.7956282E 00	1.5903468E 01	2.2091708E 01	2.8305694E 01	3.4526173E 01	4.0740135E 01
2	3.3486955E-02	4.0484890E 00	9.7956402E 00	1.5903465E 01	2.2091708E 01	2.8305680E 01	3.4526162E 01	4.0740130E 01
3	5.5809466E-02	4.0486503E 00	9.7956580E 00	1.5903470E 01	2.2091702E 01	2.8305682E 01	3.4526170E 01	4.0740145E 01
4	7.8128506E-02	4.0488979E 00	9.7956824E 00	1.5903477E 01	2.2091708E 01	2.8305688E 01	3.4526177E 01	4.0740153E 01
5	1.0044287E-01	4.0492237E 00	9.7957133E 00	1.5903492E 01	2.2091709E 01	2.8305693E 01	3.4526168E 01	4.0740167E 01
6	1.2275103E-01	4.0496328E 00	9.7957560E 00	1.5903505E 01	2.2091728E 01	2.8305697E 01	3.4526177E 01	4.0740144E 01
7	1.4505191E-01	4.0501249E 00	9.7958094E 00	1.5903521E 01	2.2091727E 01	2.8305699E 01	3.4526182E 01	4.0740147E 01
8	1.6734392E-01	4.0506980E 00	9.7958720E 00	1.5903533E 01	2.2091740E 01	2.8305715E 01	3.4526181E 01	4.0740172E 01
9	1.8962593E-01	4.0513545E 00	9.7959459E 00	1.5903557E 01	2.2091751E 01	2.8305711E 01	3.4526181E 01	4.0740175E 01
10	2.1189637E-01	4.0520928E 00	9.7960228E 00	1.5903579E 01	2.2091755E 01	2.8305717E 01	3.4526190E 01	4.0740185E 01
11	2.3415371E-01	4.0529151E 00	9.7961117E 00	1.5903607E 01	2.2091778E 01	2.8305735E 01	3.4526204E 01	4.0740156E 01
12	2.5639719E-01	4.0538207E 00	9.7962046E 00	1.5903633E 01	2.2091793E 01	2.8305738E 01	3.4526193E 01	4.0740175E 01

EIGENVALUES A FOR K2= 2., L= 20., C= 0.1

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.0825942E-01	4.4293886E 00	8.7671119E 00	1.2941781E 01	1.6994037E 01	2.1109541E 01	2.5393856E 01	2.9799801E 01
2	3.2477199E-01	4.4393817E 00	8.7721407E 00	1.2945200E 01	1.6996706E 01	2.1111731E 01	2.5395723E 01	2.9801364E 01
3	5.4126570E-01	4.4592959E 00	8.7821903E 00	1.2952042E 01	1.7002024E 01	2.1116135E 01	2.5399424E 01	2.9804525E 01
4	7.5772796E-01	4.4890034E 00	8.7972442E 00	1.2962289E 01	1.7010007E 01	2.1122731E 01	2.5404955E 01	2.9809248E 01
5	9.7414558E-01	4.5283085E 00	8.8172730E 00	1.2975940E 01	1.7020658E 01	2.1131516E 01	2.5412374E 01	2.9815549E 01
6	1.1905058E 00	4.5769678E 00	8.8422558E 00	1.2992993E 01	1.7033949E 01	2.1142504E 01	2.5421614E 01	2.9823437E 01
7	1.4067954E 00	4.6346825E 00	8.8721403E 00	1.3013431E 01	1.7049880E 01	2.1155685E 01	2.5432710E 01	2.9832884E 01
8	1.6230006E 00	4.7011170E 00	8.9068844E 00	1.3037226E 01	1.7068461E 01	2.1171041E 01	2.5445632E 01	2.9843901E 01
9	1.8391064E 00	4.7759108E 00	8.9464376E 00	1.3064380E 01	1.7089671E 01	2.1188586E 01	2.5460409E 01	2.9856509E 01
10	2.0550986E 00	4.8586759E 00	8.9907297E 00	1.3094871E 01	1.7113510E 01	2.1208300E 01	2.5477000E 01	2.9870666E 01
11	2.2709600E 00	4.9490111E 00	9.0397005E 00	1.3128685E 01	1.7139959E 01	2.1230192E 01	2.5495441E 01	2.9886387E 01
12	2.4866785E 00	5.0465143E 00	9.0932822E 00	1.3165779E 01	1.7169007E 01	2.1254251E 01	2.5515717E 01	2.9903680E 01

EIGENVALUES A FOR K2= 5., L= 20., C= 0.1

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.6844779E-01	6.9248484E 00	1.3062406E 01	1.8220014E 01	2.4326689E 01	3.1241300E 01	3.8172106E 01	4.4341869E 01
2	5.0531045E-01	6.9392014E 00	1.3069115E 01	1.8225332E 01	2.4331290E 01	3.1244943E 01	3.8174935E 01	4.4343930E 01
3	8.4207394E-01	6.9678062E 00	1.3082515E 01	1.8235974E 01	2.4340489E 01	3.1252210E 01	3.8180578E 01	4.4348089E 01
4	1.1786716E 00	7.0104840E 00	1.3102606E 01	1.8251938E 01	2.4354298E 01	3.1263114E 01	3.8189056E 01	4.4354320E 01
5	1.5150335E 00	7.0669627E 00	1.3129351E 01	1.8273228E 01	2.4372703E 01	3.1277637E 01	3.8200364E 01	4.4362636E 01
6	1.8510874E 00	7.1368866E 00	1.3162728E 01	1.8299822E 01	2.4395690E 01	3.1295779E 01	3.8214467E 01	4.4372981E 01
7	2.1867549E 00	7.2198389E 00	1.3202698E 01	1.8331737E 01	2.4423261E 01	3.1317540E 01	3.8231395E 01	4.4385436E 01
8	2.5219586E 00	7.3153573E 00	1.3249214E 01	1.8368965E 01	2.4455379E 01	3.1342907E 01	3.8251126E 01	4.4399939E 01
9	2.8566070E 00	7.4229174E 00	1.3302237E 01	1.8411484E 01	2.4492036E 01	3.1371874E 01	3.8273660E 01	4.4416537E 01
10	3.1906042E 00	7.5419727E 00	1.3361719E 01	1.8459316E 01	2.4533235E 01	3.1404420E 01	3.8298991E 01	4.4435152E 01
11	3.5238526E 00	7.6719704E 00	1.3427605E 01	1.8512430E 01	2.4578956E 01	3.1440536E 01	3.8327104E 01	4.4455845E 01
12	3.8562300E 00	7.8123258E 00	1.3499831E 01	1.8570827E 01	2.4629146E 01	3.1480224E 01	3.8357989E 01	4.4478561E 01

EIGENVALUES A FOR K2= 10., L= 20., C= 0.1

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.3692150E-01	9.5614415E 00	1.6112511E 01	2.3224531E 01	3.2945117E 01	4.2260461E 01	4.8359113E 01	5.6066415E 01
2	7.1068422E-01	9.5796754E 00	1.6120805E 01	2.3233832E 01	3.2951885E 01	4.2264745E 01	4.8361998E 01	5.6070287E 01
3	1.1842044E 00	9.6160086E 00	1.6137413E 01	2.3252435E 01	3.2965420E 01	4.2273275E 01	4.8367789E 01	5.6078050E 01
4	1.6573142E 00	9.6701636E 00	1.6162369E 01	2.3280305E 01	3.2985711E 01	4.2286039E 01	4.8376473E 01	5.6089674E 01
5	2.1298380E 00	9.7417425E 00	1.6195731E 01	2.3317443E 01	3.3012764E 01	4.2303070E 01	4.8388052E 01	5.6105151E 01
6	2.6015862E 00	9.8302287E 00	1.6237573E 01	2.3363817E 01	3.3046518E 01	4.2324317E 01	4.8402548E 01	5.6124485E 01
7	3.0723530E 00	9.9350176E 00	1.6287985E 01	2.3419397E 01	3.3086962E 01	4.2349756E 01	4.8419938E 01	5.6147728E 01
8	3.5419098E 00	1.0055424E 01	1.6347092E 01	2.3484134E 01	3.3134086E 01	4.2379416E 01	4.8440294E 01	5.6174812E 01
9	4.0099971E 00	1.0190693E 01	1.6415028E 01	2.3557996E 01	3.3187842E 01	4.2413206E 01	4.8463559E 01	5.6205732E 01
10	4.4763275E 00	1.0340053E 01	1.6491945E 01	2.3640917E 01	3.3248187E 01	4.2451100E 01	4.8489821E 01	5.6240546E 01
11	4.9405617E 00	1.0502712E 01	1.6578012E 01	2.3732831E 01	3.3315117E 01	4.2493105E 01	4.8519027E 01	5.6279255E 01
12	5.4023119E 00	1.0677902E 01	1.6673441E 01	2.3833697E 01	3.3388523E 01	4.2539158E 01	4.8551257E 01	5.6321791E 01

EIGENVALUES A FOR K2= 20., L= 20., C= 0.1

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.3413254E-01	1.2596154E 01	1.8717079E 01	3.1242638E 01	4.3901314E 01	5.0026541E 01	6.2482591E 01	7.5205657E 01
2	1.0022034E 00	1.2614946E 01	1.8733790E 01	3.1256944E 01	4.3907607E 01	5.0032277E 01	6.2489774E 01	7.5209453E 01
3	1.6696847E 00	1.2652306E 01	1.8767338E 01	3.1285507E 01	4.3920171E 01	5.0043806E 01	6.2504058E 01	7.5217073E 01
4	2.3361608E 00	1.2707842E 01	1.8817864E 01	3.1328333E 01	4.3938960E 01	5.0061105E 01	6.2525494E 01	7.5228505E 01
5	3.0011737E 00	1.2780957E 01	1.8885636E 01	3.1385315E 01	4.3963971E 01	5.0084221E 01	6.2554095E 01	7.5243706E 01
6	3.6642101E 00	1.2870937E 01	1.8971007E 01	3.1456385E 01	4.3995045E 01	5.0113246E 01	6.2589771E 01	7.5262663E 01
7	4.3246801E 00	1.2976974E 01	1.9074314E 01	3.1541433E 01	4.4032156E 01	5.0148237E 01	6.2632604E 01	7.5285342E 01
8	4.9818765E 00	1.3098248E 01	1.9196021E 01	3.1640337E 01	4.4075139E 01	5.0189259E 01	6.2682512E 01	7.5311699E 01
9	5.6349491E 00	1.3233989E 01	1.9336539E 01	3.1752945E 01	4.4123936E 01	5.0236433E 01	6.2739494E 01	7.5341694E 01
10	6.2828496E 00	1.3383583E 01	1.9496302E 01	3.1879118E 01	4.4178321E 01	5.0289841E 01	6.2803510E 01	7.5375278E 01
11	6.9242837E 00	1.3546667E 01	1.9675665E 01	3.2018636E 01	4.4238222E 01	5.0349634E 01	6.2874606E 01	7.5412394E 01
12	7.5576079E 00	1.3723262E 01	1.9874979E 01	3.2171320E 01	4.4303389E 01	5.0415905E 01	6.2952656E 01	7.5452996E 01

EIGENVALUES A FOR K2= 50., L= 20., C= 0.1

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	5.2741866E-01	1.5176675E 01	2.5442459E 01	4.4929222E 01	5.1612378E 01	7.2471306E 01	8.0018116E 01	9.8261311E 01
2	1.5815868E 00	1.5189573E 01	2.5485536E 01	4.4939030E 01	5.1628678E 01	7.2483936E 01	8.0022738E 01	9.8272429E 01
3	2.6336913E 00	1.5215600E 01	2.5571529E 01	4.4958534E 01	5.1661344E 01	7.2509124E 01	8.0031927E 01	9.8294508E 01
4	3.6821601E 00	1.5255218E 01	2.5700130E 01	4.4987526E 01	5.1710553E 01	7.2546853E 01	8.0045862E 01	9.8327761E 01
5	4.7250856E 00	1.5309303E 01	2.5870844E 01	4.5025702E 01	5.1776475E 01	7.2596989E 01	8.0064586E 01	9.8371997E 01
6	5.7600334E 00	1.5379260E 01	2.6083040E 01	4.5072660E 01	5.1859455E 01	7.2659357E 01	8.0088201E 01	9.8427278E 01
7	6.7837182E 00	1.5467435E 01	2.6335908E 01	4.5127953E 01	5.1959810E 01	7.2733773E 01	8.0116836E 01	9.8493508E 01
8	7.7915550E 00	1.5577448E 01	2.6628483E 01	4.5190986E 01	5.2077926E 01	7.2819955E 01	8.0150778E 01	9.8570576E 01
9	8.7769586E 00	1.5715057E 01	2.6959612E 01	4.5261233E 01	5.2214217E 01	7.2917648E 01	8.0190101E 01	9.8658614E 01
10	9.7302984E 00	1.5889039E 01	2.7328044E 01	4.5338032E 01	5.2369130E 01	7.3026497E 01	8.0235230E 01	9.8757358E 01
11	1.0637634E 01	1.6112632E 01	2.7732366E 01	4.5420735E 01	5.2543082E 01	7.3146129E 01	8.0286360E 01	9.8866940E 01
12	1.1479612E 01	1.6404553E 01	2.8171052E 01	4.5508767E 01	5.2736486E 01	7.3276032E 01	8.0343863E 01	9.8987014E 01

EIGENVALUES A FOR K2= 100., L= 20., C= 0.1

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	7.4543436E-01	1.5854964E 01	3.4554869E 01	4.7582924E 01	6.9026589E 01	7.9395201E 01	1.0331162E 02	1.1140329E 02
2	2.2345001E 00	1.5871035E 01	3.4617814E 01	4.7588961E 01	6.9056925E 01	7.9399942E 01	1.0332998E 02	1.1140852E 02
3	3.7178605E 00	1.5904470E 01	3.4743230E 01	4.7601092E 01	6.9117464E 01	7.9409473E 01	1.0336662E 02	1.1141892E 02
4	5.1906244E 00	1.5958221E 01	3.4930208E 01	4.7619529E 01	6.9208036E 01	7.9423867E 01	1.0342136E 02	1.1143471E 02
5	6.6458791E 00	1.6037659E 01	3.5177387E 01	4.7644593E 01	6.9328288E 01	7.9443384E 01	1.0349403E 02	1.1145600E 02
6	8.0729432E 00	1.6152243E 01	3.5482899E 01	4.7676746E 01	6.9477868E 01	7.9468171E 01	1.0358443E 02	1.1148304E 02
7	9.4542273E 00	1.6318517E 01	3.5844483E 01	4.7716582E 01	6.9656206E 01	7.9498670E 01	1.0369202E 02	1.1151607E 02
8	1.0759819E 01	1.6565512E 01	3.6259477E 01	4.7764903E 01	6.9862595E 01	7.9535151E 01	1.0381655E 02	1.1155554E 02
9	1.1940601E 01	1.6941576E 01	3.6724696E 01	4.7822769E 01	7.0096297E 01	7.9578142E 01	1.0395733E 02	1.1160182E 02
10	1.2929901E 01	1.7512712E 01	3.7236510E 01	4.7891545E 01	7.0356300E 01	7.9628322E 01	1.0411378E 02	1.1165553E 02
11	1.3676123E 01	1.8329860E 01	3.7790809E 01	4.7973017E 01	7.0641496E 01	7.9686434E 01	1.0428510E 02	1.1171721E 02
12	1.4190256E 01	1.9381493E 01	3.8382793E 01	4.8069535E 01	7.0950468E 01	7.9753386E 01	1.0447054E 02	1.1178767E 02

EIGENVALUES A FOR K2= 2., L= 20., C= 0.3

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.0240161E-01	4.1082399E 00	7.6879869E 00	1.2011053E 01	1.5794139E 01	1.9622646E 01	2.3927740E 01	2.7490202E 01
2	3.0716211E-01	4.1169650E 00	7.6934674E 00	1.2014427E 01	1.5796553E 01	1.9624794E 01	2.3929378E 01	2.7491627E 01
3	5.1179428E-01	4.1343698E 00	7.7044229E 00	1.2021165E 01	1.5801363E 01	1.9629070E 01	2.3932680E 01	2.7494490E 01
4	7.1621132E-01	4.1603678E 00	7.7208264E 00	1.2031270E 01	1.5808584E 01	1.9635493E 01	2.3937601E 01	2.7498772E 01
5	9.2032411E-01	4.1948459E 00	7.7426523E 00	1.2044729E 01	1.5818202E 01	1.9644041E 01	2.3944151E 01	2.7504482E 01
6	1.1240415E 00	4.2376451E 00	7.7698604E 00	1.2061525E 01	1.5830228E 01	1.9654738E 01	2.3952372E 01	2.7511599E 01
7	1.3272677E 00	4.2885789E 00	7.8023996E 00	1.2081656E 01	1.5844646E 01	1.9667563E 01	2.3962218E 01	2.7520155E 01
8	1.5299042E 00	4.3474492E 00	7.8402105E 00	1.2105086E 01	1.5861458E 01	1.9682511E 01	2.3973671E 01	2.7530156E 01
9	1.7318477E 00	4.4140255E 00	7.8832258E 00	1.2131803E 01	1.5880665E 01	1.9699582E 01	2.3986781E 01	2.7541578E 01
10	1.9329880E 00	4.4880790E 00	7.9313748E 00	1.2161773E 01	1.5902237E 01	1.9718775E 01	2.4001513E 01	2.7554414E 01
11	2.1332123E 00	4.5693766E 00	7.9845701E 00	1.2194991E 01	1.5926195E 01	1.9740071E 01	2.4017888E 01	2.7568665E 01
12	2.3323987E 00	4.6576785E 00	8.0427223E 00	1.2231408E 01	1.5952516E 01	1.9763475E 01	2.4035858E 01	2.7584342E 01

EIGENVALUES A FOR K2= 5., L= 20., C= 0.3

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.5308944E-01	5.1678500E 00	1.0167774E 01	1.5499334E 01	2.0335038E 01	2.5829548E 01	3.0505170E 01	3.6156566E 01
2	4.5895848E-01	5.1785487E 00	1.0177062E 01	1.5502909E 01	2.0339670E 01	2.5831719E 01	3.0508231E 01	3.6158133E 01
3	7.6388777E-01	5.2000161E 00	1.0195614E 01	1.5510076E 01	2.0348921E 01	2.5836056E 01	3.0514345E 01	3.6161282E 01
4	1.0672177E 00	5.2323869E 00	1.0223369E 01	1.5520827E 01	2.0362782E 01	2.5842585E 01	3.0523534E 01	3.6166027E 01
5	1.3682398E 00	5.2758825E 00	1.0260234E 01	1.5535169E 01	2.0381266E 01	2.5851262E 01	3.0535771E 01	3.6172316E 01
6	1.6661744E 00	5.3308042E 00	1.0306096E 01	1.5553102E 01	2.0404330E 01	2.5862131E 01	3.0551068E 01	3.6180209E 01
7	1.9601542E 00	5.3975714E 00	1.0360818E 01	1.5574641E 01	2.0431978E 01	2.5875144E 01	3.0569416E 01	3.6189662E 01
8	2.2492115E 00	5.4767092E 00	1.0424224E 01	1.5599793E 01	2.0464201E 01	2.5890349E 01	3.0590809E 01	3.6200671E 01
9	2.5322679E 00	5.5688546E 00	1.0496116E 01	1.5628576E 01	2.0500957E 01	2.5907718E 01	3.0615258E 01	3.6213259E 01
10	2.8081489E 00	5.6747705E 00	1.0576286E 01	1.5661003E 01	2.0542230E 01	2.5927243E 01	3.0642733E 01	3.6227397E 01
11	3.0756107E 00	5.7952805E 00	1.0664479E 01	1.5697097E 01	2.0587993E 01	2.5948939E 01	3.0673263E 01	3.6243125E 01
12	3.3334060E 00	5.9312347E 00	1.0760448E 01	1.5736882E 01	2.0638208E 01	2.5972801E 01	3.0706803E 01	3.6260398E 01

EIGENVALUES A FOR K2= 10., L= 20., C= 0.3

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.1216335E-01	5.5994506E 00	1.2930591E 01	1.7060750E 01	2.5060523E 01	2.9373743E 01	3.6428171E 01	4.2369758E 01
2	6.3553409E-01	5.6141077E 00	1.2941527E 01	1.7067464E 01	2.5064053E 01	2.9379435E 01	3.6430000E 01	4.2373972E 01
3	1.0559722E 00	5.6438624E 00	1.2963321E 01	1.7080946E 01	2.5071116E 01	2.9390813E 01	3.6433665E 01	4.2382376E 01
4	1.4713039E 00	5.6896350E 00	1.2995795E 01	1.7101257E 01	2.5081658E 01	2.9407886E 01	3.6439176E 01	4.2395010E 01
5	1.8790226E 00	5.7528946E 00	1.3038697E 01	1.7128517E 01	2.5095661E 01	2.9430701E 01	3.6446545E 01	4.2411842E 01
6	2.2761622E 00	5.8357119E 00	1.3091694E 01	1.7162891E 01	2.5113095E 01	2.9459247E 01	3.6455730E 01	4.2432830E 01
7	2.6591954E 00	5.9408355E 00	1.3154382E 01	1.7204554E 01	2.5133869E 01	2.9493551E 01	3.6466762E 01	4.2458022E 01
8	3.0240522E 00	6.0716444E 00	1.3226274E 01	1.7253750E 01	2.5157974E 01	2.9533650E 01	3.6479616E 01	4.2487361E 01
9	3.3663440E 00	6.2318957E 00	1.3306827E 01	1.7310727E 01	2.5185302E 01	2.9579555E 01	3.6494344E 01	4.2520831E 01
10	3.6819243E 00	6.4251450E 00	1.3395459E 01	1.7375773E 01	2.5215814E 01	2.9631303E 01	3.6510895E 01	4.2558398E 01
11	3.9677511E 00	6.6538849E 00	1.3491503E 01	1.7449221E 01	2.5249395E 01	2.9688919E 01	3.6529274E 01	4.2600065E 01
12	4.2227619E 00	6.9186392E 00	1.3594320E 01	1.7531403E 01	2.5285982E 01	2.9752424E 01	3.6549526E 01	4.2645776E 01

EIGENVALUES A FOR K2= 20., L= 20., C= 0.3

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.9688691E-01	5.8132274E 00	1.4989593E 01	2.0336943E 01	2.6628899E 01	3.5762666E 01	4.0682473E 01	4.7460551E 01
2	8.8788024E-01	5.8368940E 00	1.4995641E 01	2.0354345E 01	2.6632464E 01	3.5765897E 01	4.0690932E 01	4.7462323E 01
3	1.4701460E 00	5.8861870E 00	1.5007684E 01	2.0389104E 01	2.6639622E 01	3.5772352E 01	4.0707850E 01	4.7465918E 01
4	2.0365559E 00	5.9652920E 00	1.5025638E 01	2.0441128E 01	2.6650439E 01	3.5781985E 01	4.0733255E 01	4.7471303E 01
5	2.5779893E 00	6.0810791E 00	1.5049374E 01	2.0510329E 01	2.6665008E 01	3.5794742E 01	4.0767180E 01	4.7478523E 01
6	3.0828274E 00	6.2434018E 00	1.5078750E 01	2.0596506E 01	2.6683423E 01	3.5810550E 01	4.0809563E 01	4.7487581E 01
7	3.5376431E 00	6.4642271E 00	1.5113633E 01	2.0699418E 01	2.6705863E 01	3.5829328E 01	4.0860500E 01	4.7498496E 01
8	3.9303546E 00	6.7543761E 00	1.5153892E 01	2.0818765E 01	2.6732519E 01	3.5850992E 01	4.0919982E 01	4.7511352E 01
9	4.2554495E 00	7.1182138E 00	1.5199473E 01	2.0954132E 01	2.6763625E 01	3.5875399E 01	4.0988020E 01	4.7526127E 01
10	4.5171928E 00	7.5503871E 00	1.5250391E 01	2.1105011E 01	2.6799497E 01	3.5902487E 01	4.1064601E 01	4.7542930E 01
11	4.7272140E 00	8.0381311E 00	1.5306772E 01	2.1270787E 01	2.6840473E 01	3.5932086E 01	4.1149759E 01	4.7561802E 01
12	4.8991029E 00	8.5666686E 00	1.5368950E 01	2.1450718E 01	2.6887012E 01	3.5964115E 01	4.1243434E 01	4.7582817E 01

EIGENVALUES A FOR K2= 50., L= 20., C= 0.3

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	4.6618090E-01	5.9399706E 00	1.5715697E 01	2.5522129E 01	3.1642514E 01	3.7392193E 01	4.7145430E 01	5.6974373E 01
2	1.3872191E 00	5.9928462E 00	1.5720771E 01	2.5527655E 01	3.1669901E 01	3.7397803E 01	4.7147138E 01	5.6976964E 01
3	2.2703226E 00	6.1117929E 00	1.5731049E 01	2.5538582E 01	3.1724545E 01	3.7409151E 01	4.7150505E 01	5.6982098E 01
4	3.0769680E 00	6.3268434E 00	1.5746861E 01	2.5554717E 01	3.1806193E 01	3.7426479E 01	4.7155548E 01	5.6989793E 01
5	3.7547398E 00	6.6845634E 00	1.5768749E 01	2.5575768E 01	3.1914369E 01	3.7450185E 01	4.7162335E 01	5.6999899E 01
6	4.2610931E 00	7.2230133E 00	1.5797580E 01	2.5601394E 01	3.2048444E 01	3.7480792E 01	4.7170894E 01	5.7012386E 01
7	4.6037758E 00	7.9304910E 00	1.5834701E 01	2.5631225E 01	3.2207537E 01	3.7519047E 01	4.7181184E 01	5.7027118E 01
8	4.8315644E 00	8.7541374E 00	1.5882205E 01	2.5664882E 01	3.2390428E 01	3.7565909E 01	4.7193348E 01	5.7044004E 01
9	4.9916888E 00	9.6417595E 00	1.5943453E 01	2.5702026E 01	3.2595550E 01	3.7622583E 01	4.7207370E 01	5.7062881E 01
10	5.1141652E 00	1.0556421E 01	1.6024057E 01	2.5742415E 01	3.2820890E 01	3.7690631E 01	4.7223341E 01	5.7083657E 01
11	5.2158003E 00	1.1470498E 01	1.6133630E 01	2.5785840E 01	3.3063892E 01	3.7772025E 01	4.7241378E 01	5.7106156E 01
12	5.3059143E 00	1.2356581E 01	1.6289116E 01	2.5832313E 01	3.3321470E 01	3.7869183E 01	4.7261562E 01	5.7130306E 01

EIGENVALUES A FOR K2= 100., L= 20., C= 0.3

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	6.5722890E-01	5.9859025E 00	1.5873787E 01	2.6093185E 01	3.6177930E 01	4.3902092E 01	4.8390794E 01	5.7855659E 01
2	1.9382600E 00	6.0921274E 00	1.5881354E 01	2.6096102E 01	3.6182179E 01	4.3935924E 01	4.8405141E 01	5.7857782E 01
3	3.0963475E 00	6.3618848E 00	1.5897149E 01	2.6101993E 01	3.6190562E 01	4.4002905E 01	4.8434370E 01	5.7862072E 01
4	3.9924352E 00	6.9163431E 00	1.5922682E 01	2.6110943E 01	3.6202925E 01	4.4101679E 01	4.8479713E 01	5.7868621E 01
5	4.5415463E 00	7.8282239E 00	1.5960805E 01	2.6123092E 01	3.6219022E 01	4.4230155E 01	4.8542873E 01	5.7877463E 01
6	4.8381160E 00	8.9919015E 00	1.6016792E 01	2.6138683E 01	3.6238500E 01	4.4385250E 01	4.8626429E 01	5.7888753E 01
7	5.0099342E 00	1.0265262E 01	1.6100846E 01	2.6158082E 01	3.6261030E 01	4.4563079E 01	4.8733673E 01	5.7902682E 01
8	5.1251624E 00	1.1555886E 01	1.6234010E 01	2.6181839E 01	3.6286328E 01	4.4758654E 01	4.8868756E 01	5.7919474E 01
9	5.2138304E 00	1.2784635E 01	1.6462517E 01	2.6210842E 01	3.6314119E 01	4.4966140E 01	4.9036466E 01	5.7939482E 01
10	5.2896795E 00	1.3838756E 01	1.6882432E 01	2.6246443E 01	3.6344171E 01	4.5178959E 01	4.9242113E 01	5.7963036E 01
11	5.3595235E 00	1.4584708E 01	1.7616887E 01	2.6290894E 01	3.6376367E 01	4.5390334E 01	4.9490891E 01	5.7990760E 01
12	5.4270015E 00	1.5016389E 01	1.8663294E 01	2.6347955E 01	3.6410677E 01	4.5593964E 01	4.9787300E 01	5.8023286E 01

EIGENVALUES A FOR K2= 2., L= 20., C= 0.7

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	8.9546518E-02	3.2565477E 00	6.8514774E 00	1.0506035E 01	1.3864256E 01	1.7052622E 01	2.0558412E 01	2.4237146E 01
2	2.6857598E-01	3.2671416E 00	6.8557691E 00	1.0508766E 01	1.3866523E 01	1.7054490E 01	2.0559862E 01	2.4238349E 01
3	4.4741470E-01	3.2882453E 00	6.8643476E 00	1.0514225E 01	1.3871053E 01	1.7058219E 01	2.0562729E 01	2.4240743E 01
4	6.2593751E-01	3.3196978E 00	6.8771988E 00	1.0522413E 01	1.3877832E 01	1.7063825E 01	2.0567045E 01	2.4244325E 01
5	8.0402094E-01	3.3612645E 00	6.8943068E 00	1.0533321E 01	1.3886876E 01	1.7071293E 01	2.0572799E 01	2.4249113E 01
6	9.8154541E-01	3.4126510E 00	6.9156590E 00	1.0546942E 01	1.3898171E 01	1.7080626E 01	2.0579994E 01	2.4255109E 01
7	1.1583955E 00	3.4735074E 00	6.9412193E 00	1.0563259E 01	1.3911717E 01	1.7091814E 01	2.0588629E 01	2.4262279E 01
8	1.3344615E 00	3.5434445E 00	6.9709676E 00	1.0582276E 01	1.3927492E 01	1.7104867E 01	2.0598688E 01	2.4270642E 01
9	1.5096439E 00	3.6220368E 00	7.0048638E 00	1.0603963E 01	1.3945502E 01	1.7119765E 01	2.0610203E 01	2.4280220E 01
10	1.6838485E 00	3.7088466E 00	7.0428753E 00	1.0628314E 01	1.3965716E 01	1.7136526E 01	2.0623130E 01	2.4290967E 01
11	1.8569957E 00	3.8034172E 00	7.0849622E 00	1.0655309E 01	1.3988142E 01	1.7155132E 01	2.0637494E 01	2.4302910E 01
12	2.0290175E 00	3.9052887E 00	7.1310819E 00	1.0684922E 01	1.4012769E 01	1.7175578E 01	2.0653287E 01	2.4316055E 01

EIGENVALUES A FOR K2= 5., L= 20., C= 0.7

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.1646417E-01	3.3217674E 00	7.1651044E 00	1.1258902E 01	1.5372457E 01	1.9299459E 01	2.2730796E 01	2.5967201E 01
2	3.4869326E-01	3.3401929E 00	7.1706899E 00	1.1261817E 01	1.5374549E 01	1.9301455E 01	2.2733115E 01	2.5969093E 01
3	5.7883520E-01	3.3770210E 00	7.1818730E 00	1.1267644E 01	1.5378732E 01	1.9305449E 01	2.2737759E 01	2.5972885E 01
4	8.0553982E-01	3.4322050E 00	7.1987085E 00	1.1276398E 01	1.5384992E 01	1.9311421E 01	2.2744701E 01	2.5978583E 01
5	1.0275313E 00	3.5056219E 00	7.2212506E 00	1.1288079E 01	1.5393349E 01	1.9319382E 01	2.2753958E 01	2.5986187E 01
6	1.2436664E 00	3.5970318E 00	7.2495972E 00	1.1302703E 01	1.5403781E 01	1.9329321E 01	2.2765526E 01	2.5995681E 01
7	1.4530008E 00	3.7059971E 00	7.2838678E 00	1.1320277E 01	1.5416313E 01	1.9341213E 01	2.2779375E 01	2.6007088E 01
8	1.6548629E 00	3.8318281E 00	7.3242140E 00	1.1340835E 01	1.5430909E 01	1.9355065E 01	2.2795497E 01	2.6020427E 01
9	1.8489166E 00	3.9735018E 00	7.3708219E 00	1.1364377E 01	1.5447601E 01	1.9370843E 01	2.2813886E 01	2.6035673E 01
10	2.0351873E 00	4.1296569E 00	7.4239238E 00	1.1390962E 01	1.5466357E 01	1.9388556E 01	2.2834513E 01	2.6052867E 01
11	2.2140365E 00	4.2985899E 00	7.4837894E 00	1.1420597E 01	1.5487199E 01	1.9408171E 01	2.2857381E 01	2.6072013E 01
12	2.3861057E 00	4.4783325E 00	7.5507435E 00	1.1453341E 01	1.5510128E 01	1.9429671E 01	2.2882456E 01	2.6093089E 01

EIGENVALUES A FOR K2= 10., L= 20., C= 0.7

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.5096486E-01	3.3439900E 00	7.2572441E 00	1.1455778E 01	1.5743213E 01	2.0041762E 01	2.4287645E 01	2.8347937E 01
2	4.5027466E-01	3.3754296E 00	7.2651079E 00	1.1459286E 01	1.5745396E 01	2.0043413E 01	2.4289139E 01	2.8349721E 01
3	7.4179516E-01	3.4387944E 00	7.2809737E 00	1.1466322E 01	1.5749757E 01	2.0046725E 01	2.4292138E 01	2.8353277E 01
4	1.0206410E 00	3.5347991E 00	7.3051054E 00	1.1476932E 01	1.5756320E 01	2.0051672E 01	2.4296615E 01	2.8358600E 01
5	1.2826192E 00	3.6638632E 00	7.3379382E 00	1.1491168E 01	1.5765091E 01	2.0058297E 01	2.4302597E 01	2.8365675E 01
6	1.5247995E 00	3.8254090E 00	7.3800707E 00	1.1509137E 01	1.5776075E 01	2.0066561E 01	2.4310046E 01	2.8374495E 01
7	1.7460454E 00	4.0171941E 00	7.4323288E 00	1.1530932E 01	1.5789317E 01	2.0076493E 01	2.4318992E 01	2.8385035E 01
8	1.9471671E 00	4.2350452E 00	7.4957734E 00	1.1556721E 01	1.5804823E 01	2.0088100E 01	2.4329409E 01	2.8397286E 01
9	2.1305473E 00	4.4731412E 00	7.5717461E 00	1.1586697E 01	1.5822656E 01	2.0101368E 01	2.4341292E 01	2.8411207E 01
10	2.2994162E 00	4.7246629E 00	7.6618906E 00	1.1621092E 01	1.5842841E 01	2.0116334E 01	2.4354647E 01	2.8426766E 01
11	2.4571332E 00	4.9825008E 00	7.7681051E 00	1.1660227E 01	1.5865441E 01	2.0132978E 01	2.4369445E 01	2.8443928E 01
12	2.6067129E 00	5.2398022E 00	7.8924498E 00	1.1704475E 01	1.5890518E 01	2.0151336E 01	2.4385697E 01	2.8462676E 01

EIGENVALUES A FOR K2= 20., L= 20., C= 0.7

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.0294021E-01	3.3573602E 00	7.3010425E 00	1.1541724E 01	1.5884106E 01	2.0264684E 01	2.4657458E 01	2.9045687E 01
2	6.0033624E-01	3.4151157E 00	7.3135753E 00	1.1546554E 01	1.5886772E 01	2.0266479E 01	2.4658836E 01	2.9046860E 01
3	9.7274996E-01	3.5333204E 00	7.3392624E 00	1.1556306E 01	1.5892129E 01	2.0270099E 01	2.4661607E 01	2.9049215E 01
4	1.3060149E 00	3.7148637E 00	7.3794036E 00	1.1571173E 01	1.5900195E 01	2.0275526E 01	2.4665765E 01	2.9052719E 01
5	1.5917574E 00	3.9583461E 00	7.4361408E 00	1.1591467E 01	1.5911067E 01	2.0282795E 01	2.4671311E 01	2.9057402E 01
6	1.8301394E 00	4.2543477E 00	7.5126773E 00	1.1617640E 01	1.5924839E 01	2.0291930E 01	2.4678271E 01	2.9063257E 01
7	2.0286592E 00	4.5858184E 00	7.6134906E 00	1.1650319E 01	1.5941624E 01	2.0302970E 01	2.4686642E 01	2.9070273E 01
8	2.1977971E 00	4.9319518E 00	7.7444044E 00	1.1690398E 01	1.5961587E 01	2.0315967E 01	2.4696451E 01	2.9078496E 01
9	2.3472370E 00	5.2721024E 00	7.9120989E 00	1.1739089E 01	1.5984966E 01	2.0330968E 01	2.4707706E 01	2.9087900E 01
10	2.4843642E 00	5.5885932E 00	8.1224860E 00	1.1798079E 01	1.6012038E 01	2.0348043E 01	2.4720430E 01	2.9098485E 01
11	2.6142741E 00	5.8691927E 00	8.3779212E 00	1.1869688E 01	1.6043167E 01	2.0367298E 01	2.4734666E 01	2.9110281E 01
12	2.7403203E 00	6.1089489E 00	8.6745009E 00	1.1957108E 01	1.6078854E 01	2.0388830E 01	2.4750435E 01	2.9123285E 01

EIGENVALUES A FOR K2= 50., L= 20., C= 0.7

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	3.0967765E-01	3.3731043E 00	7.3278300E 00	1.1589903E 01	1.5957325E 01	2.0367922E 01	2.4799225E 01	2.9241279E 01
2	8.9290265E-01	3.5120985E 00	7.3548438E 00	1.1598846E 01	1.5961608E 01	2.0370494E 01	2.4800993E 01	2.9242617E 01
3	1.3751487E 00	3.8033579E 00	7.4131924E 00	1.1617362E 01	1.5970316E 01	2.0375666E 01	2.4804547E 01	2.9245307E 01
4	1.7263731E 00	4.2373180E 00	7.5128049E 00	1.1646851E 01	1.5983745E 01	2.0383525E 01	2.4809909E 01	2.9249325E 01
5	1.9674440E 00	4.7550651E 00	7.6714523E 00	1.1689809E 01	1.6002410E 01	2.0394243E 01	2.4817137E 01	2.9254726E 01
6	2.1397661E 00	5.2728922E 00	7.9151382E 00	1.1750446E 01	1.6027095E 01	2.0408031E 01	2.4826325E 01	2.9261568E 01
7	2.2754261E 00	5.7185554E 00	8.2678396E 00	1.1835790E 01	1.6058993E 01	2.0425181E 01	2.4837571E 01	2.9269849E 01
8	2.3930587E 00	6.0574239E 00	8.7260221E 00	1.1957547E 01	1.6099894E 01	2.0446128E 01	2.4851005E 01	2.9279634E 01
9	2.5026353E 00	6.2985451E 00	9.2441512E 00	1.2134513E 01	1.6152593E 01	2.0471483E 01	2.4866810E 01	2.9291014E 01
10	2.6094216E 00	6.4709510E 00	9.7531166E 00	1.2392657E 01	1.6221511E 01	2.0502099E 01	2.4885267E 01	2.9304053E 01
11	2.7162206E 00	6.6006976E 00	1.0189760E 01	1.2754802E 01	1.6313805E 01	2.0539214E 01	2.4906664E 01	2.9318909E 01
12	2.8245058E 00	6.7050174E 00	1.0523220E 01	1.3216720E 01	1.6441262E 01	2.0584659E 01	2.4931487E 01	2.9335685E 01

EIGENVALUES A FOR K2= 100., L= 20., C= 0.7

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	4.3074731E-01	3.3914051E 00	7.3389394E 00	1.1606038E 01	1.5980327E 01	2.0398632E 01	2.4838503E 01	2.9290281E 01
2	1.1893717E 00	3.6714687E 00	7.3915737E 00	1.1622073E 01	1.5987390E 01	2.0402536E 01	2.4841011E 01	2.9292068E 01
3	1.6972478E 00	4.2452374E 00	7.5160275E 00	1.1656862E 01	1.6002129E 01	2.0410534E 01	2.4846092E 01	2.9295657E 01
4	1.9811037E 00	4.9754989E 00	7.7600003E 00	1.1717169E 01	1.6025884E 01	2.0423029E 01	2.4853889E 01	2.9301093E 01
5	2.1491633E 00	5.6344944E 00	8.1971142E 00	1.1817154E 01	1.6061226E 01	2.0440701E 01	2.4864659E 01	2.9308528E 01
6	2.2687096E 00	6.0861409E 00	8.8444096E 00	1.1986152E 01	1.6112741E 01	2.0464666E 01	2.4878778E 01	2.9318091E 01
7	2.3692313E 00	6.3545651E 00	9.5707663E 00	1.2280347E 01	1.6188967E 01	2.0496742E 01	2.4896782E 01	2.9329973E 01
8	2.4633703E 00	6.5186771E 00	1.0186081E 01	1.2770437E 01	1.6306566E 01	2.0539922E 01	2.4919516E 01	2.9344537E 01
9	2.5568029E 00	6.6305318E 00	1.0595639E 01	1.3447914E 01	1.6498895E 01	2.0599441E 01	2.4948253E 01	2.9362136E 01
10	2.6521554E 00	6.7160369E 00	1.0840888E 01	1.4165585E 01	1.6826837E 01	2.0684864E 01	2.4984981E 01	2.9383478E 01
11	2.7506216E 00	6.7878938E 00	1.0993819E 01	1.4746144E 01	1.7355439E 01	2.0814846E 01	2.5033034E 01	2.9409391E 01
12	2.8526828E 00	6.8526900E 00	1.1098997E 01	1.5122679E 01	1.8052563E 01	2.1026570E 01	2.5098123E 01	2.9441279E 01

EIGENVALUES A FOR K2= 2., L= 20., C= 0.9

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	8.2372631E-02	3.1477523E 00	6.3201208E 00	9.5321167E 00	1.2782031E 01	1.6061020E 01	1.9358544E 01	2.2664428E 01
2	2.4710377E-01	3.1570558E 00	6.3246506E 00	9.5350370E 00	1.2784147E 01	1.6062679E 01	1.9359896E 01	2.2665572E 01
3	4.1179233E-01	3.1755795E 00	6.3336992E 00	9.5408717E 00	1.2788393E 01	1.6065985E 01	1.9362605E 01	2.2667864E 01
4	5.7641043E-01	3.2031664E 00	6.3472478E 00	9.5496202E 00	1.2794752E 01	1.6070946E 01	1.9366660E 01	2.2671293E 01
5	7.4093115E-01	3.2395874E 00	6.3652741E 00	9.5612665E 00	1.2803221E 01	1.6077561E 01	1.9372068E 01	2.2675873E 01
6	9.0532725E-01	3.2845490E 00	6.3877389E 00	9.5758140E 00	1.2813814E 01	1.6085821E 01	1.9378828E 01	2.2681592E 01
7	1.0695741E 00	3.3377127E 00	6.4146051E 00	9.5932505E 00	1.2826513E 01	1.6095726E 01	1.9386932E 01	2.2688451E 01
8	1.2336461E 00	3.3986916E 00	6.4458197E 00	9.6135517E 00	1.2841311E 01	1.6107285E 01	1.9396394E 01	2.2696453E 01
9	1.3975225E 00	3.4670759E 00	6.4813263E 00	9.6367174E 00	1.2858208E 01	1.6120483E 01	1.9407193E 01	2.2705605E 01
10	1.5611815E 00	3.5424407E 00	6.5210614E 00	9.6627158E 00	1.2877191E 01	1.6135324E 01	1.9419340E 01	2.2715893E 01
11	1.7246031E 00	3.6243450E 00	6.5649571E 00	9.6915331E 00	1.2898261E 01	1.6151800E 01	1.9432826E 01	2.2727319E 01
12	1.8877707E 00	3.7123571E 00	6.6129314E 00	9.7231569E 00	1.2921412E 01	1.6169906E 01	1.9447660E 01	2.2739885E 01

EIGENVALUES A FOR K2= 5., L= 20., C= 0.9

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	9.2921363E-02	3.1512232E 00	6.3404939E 00	9.5876235E 00	1.2887286E 01	1.6226991E 01	1.9595002E 01	2.2982343E 01
2	2.7856961E-01	3.1648384E 00	6.3467498E 00	9.5913923E 00	1.2889865E 01	1.6228908E 01	1.9596507E 01	2.2983580E 01
3	4.6364021E-01	3.1919101E 00	6.3592570E 00	9.5989313E 00	1.2895022E 01	1.6232751E 01	1.9599532E 01	2.2986071E 01
4	6.4776672E-01	3.2321253E 00	6.3780107E 00	9.6102393E 00	1.2902760E 01	1.6238501E 01	1.9604062E 01	2.2989804E 01
5	8.3061132E-01	3.2850332E 00	6.4029866E 00	9.6253352E 00	1.2913092E 01	1.6246174E 01	1.9610108E 01	2.2994769E 01
6	1.0118759E 00	3.3500488E 00	6.4341804E 00	9.6442115E 00	1.2926004E 01	1.6255764E 01	1.9617669E 01	2.3000982E 01
7	1.1913095E 00	3.4264855E 00	6.4715514E 00	9.6668764E 00	1.2941512E 01	1.6267290E 01	1.9626747E 01	2.3008440E 01
8	1.3687175E 00	3.5135558E 00	6.5150766E 00	9.6933514E 00	1.2959632E 01	1.6280740E 01	1.9637334E 01	2.3017149E 01
9	1.5439667E 00	3.6104038E 00	6.5647174E 00	9.7236401E 00	1.2980358E 01	1.6296115E 01	1.9649436E 01	2.3027103E 01
10	1.7169855E 00	3.7161168E 00	6.6204147E 00	9.7577654E 00	1.3003703E 01	1.6313439E 01	1.9663064E 01	2.3038294E 01
11	1.8877660E 00	3.8297476E 00	6.6821218E 00	9.7957288E 00	1.3029681E 01	1.6332703E 01	1.9678225E 01	2.3050729E 01
12	2.0563495E 00	3.9503331E 00	6.7497468E 00	9.8375629E 00	1.3058306E 01	1.6353911E 01	1.9694898E 01	2.3064421E 01

EIGENVALUES A FOR K2= 10., L= 20., C= 0.9

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.0822466E-01	3.1530885E 00	6.3473079E 00	9.6049297E 00	1.2918755E 01	1.6274816E 01	1.9660665E 01	2.3067249E 01
2	3.2384107E-01	3.1738608E 00	6.3564244E 00	9.6101036E 00	1.2922103E 01	1.6277174E 01	1.9662452E 01	2.3068669E 01
3	5.3700863E-01	3.2150836E 00	6.3746934E 00	9.6204711E 00	1.2928785E 01	1.6281898E 01	1.9666021E 01	2.3071528E 01
4	7.4626276E-01	3.2760880E 00	6.4021713E 00	9.6360799E 00	1.2938850E 01	1.6289006E 01	1.9671392E 01	2.3075793E 01
5	9.5038459E-01	3.3558445E 00	6.4389428E 00	9.6569966E 00	1.2952314E 01	1.6298491E 01	1.9678564E 01	2.3081484E 01
6	1.1484796E 00	3.4529428E 00	6.4851009E 00	9.6833061E 00	1.2969220E 01	1.6310401E 01	1.9687538E 01	2.3088613E 01
7	1.3400332E 00	3.5655877E 00	6.5407470E 00	9.7151139E 00	1.2989622E 01	1.6324733E 01	1.9698339E 01	2.3097197E 01
8	1.5249203E 00	3.6916518E 00	6.6059462E 00	9.7525661E 00	1.3013595E 01	1.6341536E 01	1.9710978E 01	2.3107225E 01
9	1.7033654E 00	3.8287368E 00	6.6807359E 00	9.7957966E 00	1.3041213E 01	1.6360845E 01	1.9725471E 01	2.3118692E 01
10	1.8758751E 00	3.9743225E 00	6.7650576E 00	9.8449789E 00	1.3072565E 01	1.6382696E 01	1.9741833E 01	2.3131631E 01
11	2.0431372E 00	4.1258864E 00	6.8587522E 00	9.9003109E 00	1.3107762E 01	1.6407140E 01	1.9760098E 01	2.3146054E 01
12	2.2059307E 00	4.2810535E 00	6.9614969E 00	9.9619635E 00	1.3146922E 01	1.6434243E 01	1.9780296E 01	2.3161975E 01

EIGENVALUES A FOR K2= 20., L= 20., C= 0.9

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.3362315E-01	3.1553648E 00	6.3512002E 00	9.6136051E 00	1.2933988E 01	1.6297491E 01	1.9691268E 01	2.3106121E 01
2	3.9790164E-01	3.1904216E 00	6.3660589E 00	9.6215974E 00	1.2938856E 01	1.6300744E 01	1.9693607E 01	2.3107914E 01
3	6.5362591E-01	3.2596895E 00	6.3959796E 00	9.6376979E 00	1.2948641E 01	1.6307280E 01	1.9698311E 01	2.3111497E 01
4	8.9624132E-01	3.3612192E 00	6.4413515E 00	9.6621279E 00	1.2963448E 01	1.6317130E 01	1.9705378E 01	2.3116880E 01
5	1.1228144E 00	3.4915961E 00	6.5026793E 00	9.6952324E 00	1.2983425E 01	1.6330386E 01	1.9714867E 01	2.3124085E 01
6	1.3323373E 00	3.6457744E 00	6.5804766E 00	9.7374613E 00	1.3008799E 01	1.6347130E 01	1.9726806E 01	2.3133142E 01
7	1.5256048E 00	3.8173565E 00	6.6751078E 00	9.7893870E 00	1.3039853E 01	1.6367497E 01	1.9741270E 01	2.3144065E 01
8	1.7046966E 00	3.9992881E 00	6.7865506E 00	9.8516667E 00	1.3076917E 01	1.6391652E 01	1.9758334E 01	2.3156916E 01
9	1.8722986E 00	4.1847776E 00	6.9141375E 00	9.9250164E 00	1.3120431E 01	1.6419782E 01	1.9778081E 01	2.3171726E 01
10	2.0311321E 00	4.3680907E 00	7.0563350E 00	1.0010158E 01	1.3170891E 01	1.6452133E 01	1.9800629E 01	2.3188559E 01
11	2.1836059E 00	4.5450312E 00	7.2106433E 00	1.0107701E 01	1.3228894E 01	1.6489005E 01	1.9826130E 01	2.3207482E 01
12	2.3316726E 00	4.7130920E 00	7.3736942E 00	1.0218019E 01	1.3295098E 01	1.6530729E 01	1.9854722E 01	2.3228571E 01

EIGENVALUES A FOR K2= 50., L= 20., C= 0.9

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	1.9019134E-01	3.1610384E 00	6.3552365E 00	9.6195775E 00	1.2943385E 01	1.6310966E 01	1.9709103E 01	2.3128466E 01
2	5.5705479E-01	3.2386652E 00	6.3874812E 00	9.6361047E 00	1.2952870E 01	1.6316920E 01	1.9713126E 01	2.3131356E 01
3	8.8715339E-01	3.3893176E 00	6.4534699E 00	9.6699642E 00	1.2972195E 01	1.6328981E 01	1.9721255E 01	2.3137181E 01
4	1.1678047E 00	3.6000237E 00	6.5556915E 00	9.7227368E 00	1.3002075E 01	1.6347478E 01	1.9733629E 01	2.3146015E 01
5	1.3994557E 00	3.8479497E 00	6.6963035E 00	9.7968472E 00	1.3043660E 01	1.6372944E 01	1.9750524E 01	2.3157983E 01
6	1.5916788E 00	4.1057922E 00	6.8748446E 00	9.8953272E 00	1.3098593E 01	1.6406100E 01	1.9772266E 01	2.3173286E 01
7	1.7564798E 00	4.3504549E 00	7.0856389E 00	1.0021382E 01	1.3169041E 01	1.6447997E 01	1.9799343E 01	2.3192119E 01
8	1.9040333E 00	4.5685677E 00	7.3167624E 00	1.0177318E 01	1.3257762E 01	1.6499996E 01	1.9832397E 01	2.3214842E 01
9	2.0415648E 00	4.7565489E 00	7.5523196E 00	1.0362802E 01	1.3367908E 01	1.6563890E 01	1.9872279E 01	2.3241819E 01
10	2.1737706E 00	4.9171809E 00	7.7771524E 00	1.0573055E 01	1.3502605E 01	1.6641942E 01	1.9920050E 01	2.3273598E 01
11	2.3035664E 00	5.0558603E 00	7.9810147E 00	1.0798345E 01	1.3663948E 01	1.6736908E 01	1.9977161E 01	2.3310828E 01
12	2.4327146E 00	5.1781694E 00	8.1598487E 00	1.1025732E 01	1.3851528E 01	1.6851903E 01	2.0045395E 01	2.3354382E 01

EIGENVALUES A FOR K2= 100., L= 20., C= 0.9

M	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
1	2.5756229E-01	3.1700993E 00	6.3594404E 00	9.6229639E 00	1.2947245E 01	1.6315833E 01	1.9715214E 01	2.3135912E 01
2	7.3347080E-01	3.3175017E 00	6.4212251E 00	9.6540055E 00	1.2964550E 01	1.6326343E 01	1.9722075E 01	2.3140677E 01
3	1.1129442E 00	3.5907485E 00	6.5505569E 00	9.7193838E 00	1.3000634E 01	1.6348008E 01	1.9736094E 01	2.3150328E 01
4	1.3881975E 00	3.9318891E 00	6.7538820E 00	9.8257310E 00	1.3058619E 01	1.6382269E 01	1.9757955E 01	2.3165256E 01
5	1.5874765E 00	4.2672930E 00	7.0264100E 00	9.9820993E 00	1.3143683E 01	1.6431471E 01	1.9788785E 01	2.3185984E 01
6	1.7425971E 00	4.5493517E 00	7.3403839E 00	1.0196144E 01	1.3263258E 01	1.6499277E 01	1.9830250E 01	2.3213337E 01
7	1.8750447E 00	4.7684362E 00	7.6516277E 00	1.0465603E 01	1.3426471E 01	1.6591012E 01	1.9884892E 01	2.3248562E 01
8	1.9969855E 00	4.9363165E 00	7.9237258E 00	1.0769654E 01	1.3641006E 01	1.6713983E 01	1.9956314E 01	2.3293347E 01
9	2.1149194E 00	5.0687652E 00	8.1433398E 00	1.1072775E 01	1.3906021E 01	1.6877041E 01	2.0049661E 01	2.3350132E 01
10	2.2322480E 00	5.1784225E 00	8.3153552E 00	1.1342688E 01	1.4204457E 01	1.7088012E 01	2.0171965E 01	2.3422414E 01
11	2.3507104E 00	5.2739516E 00	8.4511855E 00	1.1564758E 01	1.4504962E 01	1.7347448E 01	2.0331769E 01	2.3515057E 01
12	2.4711479E 00	5.3609908E 00	8.5616567E 00	1.1740907E 01	1.4776654E 01	1.7641329E 01	2.0537097E 01	2.3634750E 01

APPENDIX E

EIGENVALUES, a, FOR TWO LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Characteristic Equation:

$$\beta K_2 \sin \beta(1-c) \cos \gamma c + \gamma \sin \gamma c \cos \beta(1-c) = 0 \qquad \text{(V-177)}$$

where:

$$\gamma = \frac{H}{r_b}\sqrt{a^2 - b^2} \qquad \text{(V-172)}$$

$$\beta = \frac{H}{r_b}\sqrt{\frac{a^2}{K_2^*} - b^2} \qquad \text{(V-173)}$$

$$K_2^* = \frac{k_2 \phi_1}{k_1 \phi_2} \qquad \text{(liquid flow)} \qquad \text{(V-174)}$$

$$= \frac{k_2 \rho_1 C_{p1}}{k_1 \rho_2 C_{p2}} \qquad \text{(heat conduction)}$$

$$K_2 = \frac{k_2}{k_1}$$

All values shown are for $K_2 = K_2^*$

$$J_o(b)Y_1(bR) - Y_o(b)J_1(bR) = 0 \qquad \text{(V-170)}$$

Solved by: Half-interval method on IBM 704

Estimated Accuracy: $\pm$ 5 parts in 10^6

Note: See Appendix B for explanation of Format.

All values are for $r_b/H = 50$

EIGENVALUES A FOR RB/H=50.   FIVE VALUES OF M, ALL FOR N=1

K2	L	C	EIGENVALUES A(5,1)				
2.	5.	0.5	3.4581655E-01	1.3952424E 00	2.3749845E 00	3.3449794E 00	4.3114859E 00
5.	5.	0.5	4.8905784E-01	1.9731411E 00	3.3585930E 00	4.7301175E 00	6.0965112E 00
10.	5.	0.5	6.6218597E-01	2.6715756E 00	4.5472065E 00	6.4036345E 00	8.2526255E 00
20.	5.	0.5	9.1493963E-01	3.6911114E 00	6.2818885E 00	8.8451537E 00	1.1396769E 01
50.	5.	0.5	1.4258116E 00	5.7512524E 00	9.7850111E 00	1.3771293E 01	1.7732947E 01
100.	5.	0.5	2.0064624E 00	8.0913409E 00	1.3759266E 01	1.9349512E 01	2.4889738E 01
2.	10.	0.5	1.3505090E-01	6.0977948E-01	1.0476796E 00	1.4803195E 00	1.9109470E 00
5.	10.	0.5	1.9099004E-01	8.6235479E-01	1.4816304E 00	2.0934533E 00	2.7024136E 00
10.	10.	0.5	2.5860127E-01	1.1676283E 00	2.0061078E 00	2.8344658E 00	3.6589026E 00
20.	10.	0.5	3.5730805E-01	1.6132934E 00	2.7717507E 00	3.9161377E 00	5.0549902E 00
50.	10.	0.5	5.5682470E-01	2.5140595E 00	4.3190669E 00	6.1017475E 00	7.8752345E 00
100.	10.	0.5	7.8359998E-01	3.5377591E 00	6.0771419E 00	8.5841516E 00	1.1076900E 01
2.	20.	0.5	5.6959215E-02	2.8383506E-01	4.9186186E-01	6.9729070E-01	9.0168405E-01
5.	20.	0.5	8.0552146E-02	4.0140266E-01	6.9559605E-01	9.8611420E-01	1.2751654E 00
10.	20.	0.5	1.0906807E-01	5.4350106E-01	9.4183798E-01	1.3351952E 00	1.7265629E 00
20.	20.	0.5	1.5069965E-01	7.5095361E-01	1.3013273E 00	1.8448139E 00	2.3855368E 00
50.	20.	0.5	2.3484740E-01	1.1702666E 00	2.0279317E 00	2.8748150E 00	3.7173360E 00
100.	20.	0.5	3.3049113E-01	1.6468605E 00	2.8537378E 00	4.0453544E 00	5.2306874E 00
2.	50.	0.5	1.9335933E-02	1.0769070E-01	1.8837228E-01	2.6809733E-01	3.4743954E-01
5.	50.	0.5	2.7344669E-02	1.5229752E-01	2.6639841E-01	3.7914608E-01	4.9135280E-01
10.	50.	0.5	3.7024230E-02	2.0621155E-01	3.6070485E-01	5.1336550E-01	6.6529424E-01
20.	50.	0.5	5.1158258E-02	2.8492106E-01	4.9838479E-01	7.0931463E-01	9.1923141E-01
50.	50.	0.5	7.9722924E-02	4.4401878E-01	7.7667290E-01	1.1053808E 00	1.4325076E 00
100.	50.	0.5	1.1219238E-01	6.2484805E-01	1.0929805E 00	1.5555485E 00	2.0158792E 00
2.	100.	0.5	8.7774193E-03	5.2543460E-02	9.2422442E-02	1.3185877E-01	1.7112190E-01
5.	100.	0.5	1.2413423E-02	7.4307096E-02	1.3070513E-01	1.8647612E-01	2.4200309E-01
10.	100.	0.5	1.6807505E-02	1.0061197E-01	1.7697560E-01	2.5248954E-01	3.2767174E-01
20.	100.	0.5	2.3221967E-02	1.3901673E-01	2.4452596E-01	3.4886459E-01	4.5274369E-01
50.	100.	0.5	3.6188926E-02	2.1664051E-01	3.8106502E-01	5.4366741E-01	7.0554960E-01
100.	100.	0.5	5.0930113E-02	3.0487365E-01	5.3626174E-01	7.6507459E-01	9.9289268E-01
2.	500.	0.5	1.4778102E-03	1.0161459E-02	1.8042014E-02	2.5844578E-02	3.3618577E-02
5.	500.	0.5	2.0902587E-03	1.4369223E-02	2.5515133E-02	3.6549390E-02	4.7543913E-02
10.	500.	0.5	2.8286424E-03	1.9456290E-02	3.4548157E-02	4.9488778E-02	6.4373606E-02
20.	500.	0.5	3.9104802E-03	2.6881829E-02	4.7733364E-02	6.8378004E-02	8.8947445E-02
50.	500.	0.5	6.0914373E-03	4.1892231E-02	7.4388912E-02	1.0655962E-01	1.3861172E-01
100.	500.	0.5	8.5750514E-03	5.8953357E-02	1.0468140E-01	1.4995527E-01	1.9506383E-01
2.	1000.	0.5	6.9628404E-04	5.0342341E-03	8.9653583E-03	1.2858141E-02	1.6737891E-02
5.	1000.	0.5	9.8389797E-04	7.1195512E-03	1.2678209E-02	1.8184562E-02	2.3670587E-02
10.	1000.	0.5	1.3338620E-03	9.6399906E-03	1.7165522E-02	2.4621557E-02	3.2050154E-02
20.	1000.	0.5	1.8408328E-03	1.3317959E-02	2.3718791E-02	3.4019806E-02	4.4282298E-02
50.	1000.	0.5	2.8681485E-03	2.0755066E-02	3.6962873E-02	5.3016788E-02	6.9012116E-02
100.	1000.	0.5	4.0393139E-03	2.9208624E-02	5.2018917E-02	7.4603417E-02	9.7114933E-02
2.	20.	0.1	6.4105953E-02	3.1944607E-01	5.5357248E-01	7.8477541E-01	1.0148130E 00
5.	20.	0.1	9.9746898E-02	4.9704939E-01	8.6134362E-01	1.2210889E 00	1.5790216E 00
10.	20.	0.1	1.4029479E-01	6.9910333E-01	1.2114856E 00	1.7174694E 00	2.2209035E 00
20.	20.	0.1	1.9786075E-01	9.8596064E-01	1.7085862E 00	2.4221863E 00	3.1321902E 00
50.	20.	0.1	3.1232554E-01	1.5563550E 00	2.6970294E 00	3.8234574E 00	4.9442077E 00
100.	20.	0.1	4.4145123E-01	2.1997979E 00	3.8120582E 00	5.4041816E 00	6.9882765E 00
2.	20.	0.3	6.0637855E-02	3.0216568E-01	5.2362706E-01	7.4232299E-01	9.5991627E-01
5.	20.	0.3	9.0659344E-02	4.5176426E-01	7.8286867E-01	1.1098379E 00	1.4351575E 00
10.	20.	0.3	1.2565485E-01	6.2615408E-01	1.0850704E 00	1.5382543E 00	1.9891512E 00
20.	20.	0.3	1.7586810E-01	8.7637038E-01	1.5186716E 00	2.1529442E 00	2.7840133E 00
50.	20.	0.3	2.7631758E-01	1.3769121E 00	2.3860548E 00	3.3825711E 00	4.3740312E 00
100.	20.	0.3	3.8993874E-01	1.9430993E 00	3.3671788E 00	4.7734042E 00	6.1724428E 00
2.	20.	0.7	5.3025903E-02	2.6423611E-01	4.5789807E-01	6.4914188E-01	8.3942226E-01
5.	20.	0.7	6.8979610E-02	3.4374034E-01	5.9567231E-01	8.4445677E-01	1.0919847E 00
20.	20.	0.7	1.2037912E-01	5.9986675E-01	1.0395041E 00	1.4736341E 00	1.9055466E 00
50.	20.	0.7	1.8427305E-01	9.1825273E-01	1.5911987E 00	2.2556557E 00	2.9166473E 00
100.	20.	0.7	2.5768033E-01	1.2840240E 00	2.2249521E 00	3.1538702E 00	4.0777627E 00
2.	20.	0.9	4.8776652E-02	2.4306113E-01	4.2120494E-01	5.9712393E-01	7.7215639E-01
5.	20.	0.9	5.5027382E-02	2.7420880E-01	4.7518191E-01	6.7364540E-01	8.7110661E-01
10.	20.	0.9	6.4101093E-02	3.1944249E-01	5.5356905E-01	7.8476608E-01	1.0147996E 00
20.	20.	0.9	7.9191297E-02	3.9464864E-01	6.8389547E-01	9.6952398E-01	1.2537000E 00
50.	20.	0.9	1.1296198E-01	5.6291011E-01	9.7545700E-01	1.3828091E 00	1.7880829E 00
100.	20.	0.9	1.5354060E-01	7.6509605E-01	1.3258003E 00	1.8793906E 00	2.4300639E 00

APPENDIX F

DIMENSIONLESS PRESSURE DISTRIBUTION, $P(x, \Theta)$, FOR SINGLE-LAYER LINEAR FLOW, CONSTANT TERMINAL PRESSURE

Calculated from

$$P(x, \Theta) = \sum_{m=1}^{\infty} \frac{2}{a} \sin a\, x\, e^{-a^2\Theta} \qquad \text{(IV-24)}$$

Eigenvalues, a, from:

$$a = \left(\frac{2m-1}{2}\right)\pi; \quad m = 1,2,3,\ldots \qquad \text{(IV-19)}$$

Truncation Test: $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$

Estimated Accuracy: $\pm$ 0.00002

Definition of Dimensionless Variables:

$$x = \frac{x_a}{L_a} \qquad \text{(IV-3)}$$

$$\Theta = \frac{k t_a}{\mu \phi c L_a^2} \quad \text{(liquid flow)} \qquad \text{(IV-4)}$$

$$= \frac{k t_a}{\rho C_p L_a^2} \quad \text{(heat conduction)} \qquad \text{(IV-25)}$$

$P(x,\theta)$

θ	x=.05	x=.10	x=.15	x=.2
.0001	.99959	-	-	-
.0002	.98758	1.00000	-	-
.0004	.92290	.99959	1.00000	-
.0007	.81855	.99247	.99994	1.00000
.001	.73645	.97465	.99920	.99999
.002	.57080	.88615	.98229	.99843
.004	.42385	.73645	.90647	.97465
.007	.32740	.60198	.79511	.90903
.01	.27633	.52050	.71116	.84270
.02	.19741	.38292	.54675	.68269
.04	.14032	.27633	.40412	.52050
.07	.10631	.21073	.31150	.40702
.1	.08901	.17692	.26265	.34522
.2	.06215	.12387	.18471	.24425
.4	.03725	.07426	.11082	.14669
.6	.02273	.04532	.06763	.08953
.8	.01388	.02767	.04129	.05465
1.	.00847	.01689	.02521	.03337
1.2	.00517	.01031	.01539	.02037
1.5	.00247	.00492	.00734	.00972
2.0	.00072	.00143	.00214	.00283
2.5	.00021	.00042	.00062	.00082
3.0	.00006	.00012	.00018	.00024

$P(x,\theta)$

θ	x=.3	x=.4	x=.5	x=.6	x=.7	x=.8	x=.9	x=1.0
.002	1.00000	1.00000	-	-	-	-	-	-
.004	.99920	.99999	1.00000	-	-	-	-	-
.007	.98877	.99928	.99998	1.00000	-	-	-	-
.01	.96611	.99532	.99959	.99998	1.00000	1.00000	1.00000	-
.02	.86639	.95450	.98758	.99730	.99953	.99994	.99995	1.00000
.04	.71116	.84270	.92290	.96610	.98667	.99530	.99844	.99919
.06	.61352	.75178	.85107	.91668	.95652	.97855	.98913	.99222
.08	.54672	.68263	.78852	.86592	.91873	.95180	.96959	.97516
.1	.49752	.62856	.37565	.81854	.87882	.91907	.94192	.94931
.2	.35783	.46165	.55318	.63040	.69181	.73633	.76329	.77231
.4	.21550	.27899	.33560	.38393	.42281	.45129	.46865	.47449
.6	.13153	.17029	.20486	.23438	.25813	.27553	.28614	.28971
.8	.08030	.10396	.12506	.14309	.15759	.16821	.7469	.17687
1.	.04902	.06347	.07635	.08736	.09621	.10269	.10665	.10798
1.2	.02993	-	.04661	.05333	.05873	-	.06511	.06592
1.5	.01428	.01848	.02223	.02544	.02802	.02991	.03106	.03144
2.0	.00416	.00538	.00647	.00741	.00816	.00871	.00904	.00916
2.5	.00121	.00157	.00189	.00216	.00238	.00254	.00263	.00267
3.0	.00035	.00046	.00055	.00063	.00069	.00074	.00077	.00078

APPENDIX G

DIMENSIONLESS PRESSURE DISTRIBUTION, $P(r, \Theta)$, FOR SINGLE-LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Calculated from:

$$P(r,\Theta) = \sum_{m=1}^{\infty} C_m \, U(br) e^{-b^2\Theta} \qquad \text{(IV-82)}$$

Eigenvalues, b, from: Appendix B

Truncation Test: $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$

Estimated Accuracy: $\pm$ 0.0002

Definition of Dimensionless Variables:

$$r = \frac{r_a}{r_b} \qquad \text{(IV-46)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(IV-47)}$$

$$\Theta = \frac{k t_a}{\mu \phi c r_b^2} \quad \text{for liquid flow}$$

$$= \frac{k t_a}{\rho C_p r_b^2} \quad \text{for heat conduction} \qquad \text{(IV-48)}$$

R = 1.5

θ	r=1.2	r=1.5
.005	.9584	
.006	.9380	1.0000
.008	.8960	.9999
.010	.8562	.9993
.015	.7730	.9935
.02	.7097	.9792
.03	.6198	.9303
.04	.5564	.8686
.05	.5061	.8044
.06	.4631	.7422
.08	.3905	.6293
.10	.3302	.5327
.15	.2175	.3509
.2	.1432	.2311
.3	.0621	.1003
.4	.0270	.0435
.5	.0117	.0189
.6	.0051	.0082
.8	.0010	.0015
1.0	.0002	.0003

R = 2.0

θ	r=1.5	r=2.0
.010	.9997	
.015	.9968	
.020	.9898	1.0000
.03	.9662	.9999
.04	.9368	.9994
.05	.9066	.9977
.06	.8777	.9943
.08	.8261	.9817
.10	.7824	.9624
.15	.6964	.8973
.2	.6290	.8249
.3	.5202	.6884
.4	.4320	.5724
.5	.3589	.4757
.6	.2983	.3953
.8	.2059	.2729
1.0	.1422	.1885
1.5	.0563	.0747
2	.0223	.0296
3	.0035	.0046
4	.0005	.0007
5	.0001	.0001

R = 2.5

θ	r=1.5	r=2.0	r=2.5
.05	.9066	.9989	
.06	.8777	.9972	1.0000
.08	.8263	.9911	.9998
.10	.7831	.9819	.9990
.15	.7019	.9512	.9918
.2	.6452	.9171	.9763
.3	.5690	.8505	.9277
.4	.5166	.7887	.8693
.5	.4749	.7315	.8099
.6	.4389	.6786	.7527
.8	.3769	.5841	.6485
1.0	.3243	.5027	.5582
1.5	.2229	.3455	.3837
2.0	.1532	.2374	.2637
3	.0723	.1122	.1245
4	.0342	.0530	.0588
5	.0161	.0250	.0278
6	.0076	.0118	.0131
8	.0017	.0026	.0029
10	.0004	.0006	.0007

R = 3.0

θ	r=1.5	r=2.0	r=3.0
.10	.7831	.9819	1.0000
.15	.7019	.9514	.9997
.2	.6453	.9183	.9981
.3	.5702	.8582	.9880
.4	.5213	.8088	.9684
.5	.4860	.7678	.9425
.6	.4585	.7323	.9131
.8	.4159	.6717	.8505
1.0	.3818	.6192	.7886
1.5	.3127	.5083	.6494
2.0	.2571	.4179	.5340
3	.1738	.2826	.3611
4	.1175	.1910	.2441
5	.0795	.1292	.1651
6	.0537	.0873	.1116
8	.0246	.0399	.0510
10	.0112	.0183	.0233
15	.0016	.0026	.0033
20	.0002	.0004	.0005
30	.0000	.0000	.0000

R = 4.0

θ	r=1.5	r=2.5	r=4.0
.10	.7831	.9995	
.15	.7019	.9961	
.20	.6453	.9886	1.0000
.3	.5702	.9660	.9999
.4	.5214	.9395	.9992
.5	.4864	.9132	.9971
.6	.4597	.8887	.9934
.8	.4210	.8454	.9807
1.0	.3937	.8089	.9624
1.5	.3486	.7364	.9042
2	.3177	.6776	.8410
3	.2703	.5789	.7220
4	.2314	.4958	.6186
5	.1982	.4246	.5299
6	.1697	.3637	.4539
8	.1245	.2669	.3331
10	.0914	.1958	.2444
15	.0421	.0903	.1127
20	.0194	.0416	.0520
30	.0041	.0089	.0111
40	.0009	.0019	.0024
50	.0002	.0004	.0005

R = 5.0

θ	r=1.2	r=1.5	r=2.0	r=2.5	r=3.0	r=4.0	r=5.0
.010	.8591	.9968					
.015	.7735	.9960					
.02	.7098	.9896					
.03	.6209	.9662	1.0000				
.04	.5609	.9368	.9997				
.05	.5171	.9066	.9989				
.06	.4834	.8777	.9972	1.0000			
.08	.4341	.8263	.9912	.9999			
.10	.3993	.7831	.9819	.9995	1.0000		
.15	.3433	.7019	.9514	.9961	.9998		
.2	.3088	.6453	.9183	.9886	.9991	1.0000	
.3	.2670	.5702	.8582	.9660	.9942	.9999	
.4	.2415	.5214	.8095	.9395	.9850	.9996	1.0000
.5	.2239	.4864	.7699	.9132	.9730	.9986	.9999
.6	.2107	.4598	.7373	.8887	.9596	.9968	.9998
.8	.1920	.4211	.6866	.8457	.9319	.9908	.9985
1.0	.1790	.3938	.6486	.8100	.9055	.9822	.9955
1.5	.1584	.3499	.5840	.7434	.8490	.9540	.9792
2	.1458	.3226	.5418	.6963	.8040	.9217	.9535
3	.1298	.2876	.4856	.6289	.7331	.8549	.8904
4	.1185	.2628	.4444	.5769	.6743	.7904	.8250
5	.1091	.2419	.4093	.5317	.6220	.7301	.7625
6	.1006	.2231	.3776	.4907	.5741	.6742	.7043
8	.0858	.1902	3218	.4183	.4894	.5749	.6005
10	.0731	.1621	2744	.3566	.4173	.4902	.5120
15	.0491	.1088	.1842	.2394	.2801	.3290	.3437
20	.0329	.0731	.1236	.1607	.1880	.2208	.2307
30	.0148	.0329	.0557	.0724	.0847	.0995	.1039
40	.0067	.0148	.0251	.0326	.0382	.0448	.0468
50	.0030	.0067	.0113	.0147	.0172	.0202	.0211
60	.0014	.0030	.0051	.0066	.0077	.0091	.0095
80	.0003	.0006	.0010	.0013	.0016	.0018	.0019
100	.0001	.0001	.0002	.0003	.0003	.0004	.0004

θ	R = 6.0 r=2.0	r=4.0	r=6.0
.5	.7699	.9986	
.6	.7373	.9968	1.0000
.8	.6866	.9910	.9999
1.0	.6486	.9824	.9996
1.5	.5840	.9562	.9965
2	.5422	.9289	.9887
3	.4890	.8788	.9611
4	.4545	.8346	.9247
5	.4282	.7943	.8855
6	.4061	.7568	.8459
8	.3678	.6877	.7702
10	.3342	.6252	.7005
15	.2634	.4929	.5523
20	.2077	.3886	.4354
30	.1291	.2415	.2706
40	.0802	.1501	.1682
50	.0499	.0933	.1045
60	.0310	.0580	.0650
80	.0120	.0224	.0251
100	.0046	.0087	.0097
150	.0004	.0008	.0009
200	.0000	.0001	.0001

θ	R = 7.0 r=2.0	r=4.0	r=7.0
.5	.7699	.9986	
.6	.7373	.9968	
.8	.6866	.9909	
1.0	.6486	.9824	1.0000
1.5	.5840	.9563	.9996
2	.5422	.9292	.9978
3	.4892	.8812	.9878
4	.4556	.8421	.9702
5	.4314	.8093	.9478
6	.4125	.7804	.9230
8	.3827	.7298	.8710
10	.3580	.6846	.8195
15	.3057	.5853	.7018
20	.2615	.5009	.6006
30	.1915	.3668	.4398
40	.1403	.2686	.3221
50	.1027	.1967	.2359
60	.0752	.1440	.1727
80	.0403	.0772	.0926
100	.0216	.0414	.0497
150	.0046	.0087	.0105
200	.0010	.0018	.0022

θ	R = 8.0 r=2.0	r=5.0	r=8.0
1.0	.6486	.9978	
1.5	.5840	.9902	1.0000
2	.5422	.9785	.9996
3	.4892	.9508	.9966
4	.4557	.9234	.9893
5	.4318	.8981	.9778
6	.4136	.8750	.9635
8	.3865	.8335	.9298
10	.3659	.7961	.8934
15	.3255	.7128	.8033
20	.2916	.6391	.7207
30	.2344	.5139	.5796
40	.1885	.4133	.4662
50	.1516	.3324	.3749
60	.1220	.2673	.3015
80	.0789	.1729	.1950
100	.0510	.1118	.1261
150	.0172	.0376	.0424
200	.0058	.0127	.0143
300	.0007	.0014	.0016
400	.0001	.0002	.0002
500	.0000	.0000	.0000

θ	R = 10.0 r=2.0	r=4.0	r=7.0	r=10.0
1.0	.6486	.9824	1.0000	
1.5	.5840	.9563	.9998	
2	.5422	.9292	.9989	
3	.4892	.8814	.9942	.9998
4	.4557	.8430	.9862	.9990
5	.4319	.8120	.9761	.9968
6	.4137	.7865	.9651	.9932
8	.3874	.7464	.9422	.9816
10	.3686	.7157	.9195	.9658
15	.3369	.6599	.8650	.9174
20	.3141	.6170	.8139	.8658
30	.2772	.5450	.7207	.7676
40	.2453	.4825	.6382	.6798
50	.2172	.4272	.5651	.6020
60	.1924	.3783	.5004	.5330
80	.1508	.2967	.3924	.4180
100	.1183	.2326	.3077	.3277
150	.0644	.1266	.1675	.1784
200	.0351	.0690	.0912	.0972
300	.0104	.0204	.0270	.0288
400	.0031	.0061	.0080	.0085
500	.0009	.0018	.0024	.0025
600	.0003	.0005	.0007	.0008
800	.0000	.0000	.0001	.0001
1000	.0000	.0000	.0000	.0000

θ	R = 12.0 r=3.0	r=6.0	r=12.0
1.0	.9055	.9998	
1.5	.8493	.9983	
2	.8059	.9947	
3	.7435	.9820	1.0000
4	.7004	.9659	.9999
5	.6682	.9491	.9997
6	.6430	.9329	.9990
8	.6053	.9033	.9960
10	.5779	.8778	.9905
15	.5322	.8276	.9678
20	.5021	.7889	.9381
30	.4589	.7261	.8731
40	.4240	.6718	.8097
50	.3926	.6223	.7503
60	.3636	.5765	.6952
80	.3121	.4948	.5967
100	.2679	.4247	.5122
150	.1829	.2899	.3496
200	.1248	.1979	.2386
300	.0582	.0922	.1112
400	.0271	.0430	.0518
500	.0126	.0200	.0241
600	.0059	.0093	.0112
800	.0013	.0020	.0024
1000	.0003	.0004	.0005

θ	R = 14.0 r=3.0	r=7.0	r=14.0
2	.8059	.9989	
3	.7435	.9942	
4	.7004	.9862	
5	.6682	.9762	1.0000
6	.6430	.9653	.9999
8	.6053	.9435	.9993
10	.5779	.9230	.9977
15	.5324	.8799	.9885
20	.5031	.8459	.9730
30	.4647	.7933	.9324
40	.4372	.7501	.8880
50	.4139	.7112	.8439
60	.3926	.6749	.8015
80	.3537	.6083	.7226
100	.3188	.5483	.6514
150	.2460	.4230	.5025
200	.1898	.3263	.3877
300	.1129	.1942	.2307
400	.0672	.1156	.1373
500	.0400	.0688	.0819
600	.0238	.0409	.0486
800	.0084	.0145	.0172
1000	.0030	.0051	.0061
1500	.0002	.0004	.0005
2000	.0000	.0000	.0000

R = 16.0

θ	r=4.0	r=8.0	r=16.0
5	.8120	.9897	
6	.7865	.9832	1.0000
8	.7465	.9684	.9999
10	.7162	.9530	.9995
15	.6641	.9175	.9963
20	.6296	.8877	.9891
30	.5847	.8411	.9653
40	.5544	.8043	.9353
50	.5304	.7724	.9033
60	.5095	.7431	.8712
80	.4720	.6890	.8091
100	.4379	.6394	.7511
150	.3634	.5306	.6233
200	.3016	.4404	.5173
300	.2077	.3033	.3563
400	.1431	.2089	.2454
500	.0985	.1439	.1690
600	.0679	.0991	.1164
800	.0322	.0470	.0552
1000	.0153	.0223	.0262
1500	.0024	.0035	.0406
2000	.0004	.0005	.0006
3000	.0000	.0000	.0000

R = 18.0

θ	r=4.0	r=8.0	r=18.0
5	.8120	.9897	
6	.7864	.9832	
8	.7464	.9684	1.0000
10	.7162	.9530	.9999
15	.6641	.9175	.9989
20	.6296	.8878	.9959
30	.5850	.8423	.9830
40	.5558	.8083	.9636
50	.5339	.7805	.9411
60	.5159	.7562	.9172
80	.4853	.7130	.8689
100	.4583	.6737	.8221
150	.3983	.5856	.7150
200	.3464	.5093	.6218
300	.2619	.3851	.4702
400	.1981	.2912	.3556
500	.1498	.2202	.2689
600	.1133	.1666	.2034
800	.0648	.0952	.1163
1000	.0370	.0545	.0665
1500	.0092	.0135	.0164
2000	.0022	.0033	.0041
3000	.0001	.0002	.0002
4000	.0000	.0000	.0000
5000	.0000	.0000	.0000

R = 20.0

θ	r=4.0	r=10.0	r=20.0
5	.8121	.9985	
6	.7865	.9968	
8	.7465	.9915	
10	.7163	.9843	1.0000
15	.6641	.9634	.9997
20	.6297	.9424	.9986
30	.5851	.9058	.9920
40	.5561	.8761	.9802
50	.5350	.8511	.9649
60	.5182	.8293	.9476
80	.4914	.7909	.9105
100	.4689	.7563	.8730
150	.4201	.6782	.7840
200	.3770	.6087	.7037
300	.3036	.4093	.5668
400	.2446	.3949	.4566
500	.1970	.3181	.3678
600	.1587	.2562	.2962
800	.1030	.1663	.1992
1000	.0668	.1079	.1247
1500	.0227	.0366	.0423
2000	.0077	.0124	.0143
3000	.0009	.0014	.0016
4000	.0001	.0002	.0002
5000	.0000	.0000	.0000

R = 50.0

θ	r=4.0	r=10.0	r=20.0	r=50.0
10	.7163	.9844	1.0000	
15	.6641	.9635	.9999	
20	.6297	.9424	.9993	
30	.5851	.9059	.9963	
40	.5562	.8766	.9909	
50	.5353	.8528	.9841	
60	.5191	.8331	.9767	
80	.4952	.8020	.9614	1.0000
100	.4779	.7782	.9468	.9998
150	.4491	.7364	.9154	.9982
200	.4304	.7083	.8905	.9942
300	.4064	.6709	.8531	.9798
400	.3904	.6455	.8250	.9605
500	.3781	.6255	.8014	.9391
600	.3675	.6082	.7801	.9170
800	.3488	.5774	.7412	.8731
1000	.3317	.5490	.7049	.8308
1500	.2928	.4846	.6223	.7335
2000	.2584	.4278	.5493	.6475
3000	.2014	.3334	.4281	.5047
4000	.1570	.2599	.3337	.3933
5000	.1224	.2025	.2601	.3065
6000	.0954	.1579	.2027	.2389
8000	.0579	.0959	.1231	.1451
10000	.0352	.0582	.0748	.0881
15000	.0101	.0167	.0215	.0253
20000	.0029	.0048	.0062	.0073
30000	.0002	.0004	.0005	.0006
40000	.0000	.0000	.0000	.0000

R = 75.0

θ	r=4.0	r=10.0	r=20.0	r=7
20	.6297	.9424	.9993	
30	.5851	.9059	.9963	
40	.5562	.8766	.9909	
50	.5353	.8528	.9841	
60	.5192	.8331	.9767	
80	.4952	.8020	.9614	
100	.4780	.7782	.9468	
150	.4491	.7364	.9154	1.0000
200	.4305	.7083	.8905	.9999
300	.4064	.6710	.8536	.9992
400	.3909	.6463	.8270	.9969
500	.3795	.6282	.8066	.9929
600	.3707	.6139	.7901	.9875
800	.3574	.5923	.7644	.9736
1000	.3474	.5759	.7442	.9575
1500	.3281	.5442	.7041	.9138
2000	.3119	.5172	.6694	.8703
3000	.2825	.4686	.6064	.7888
4000	.2560	.4246	.5495	.7148
5000	.2320	.3848	.4980	.6477
6000	.2102	.3487	.4513	.5870
8000	.1726	.2863	.3706	.4820
10000	.1418	.2351	.3043	.3958
15000	.0866	.1437	.1860	.2419
20000	.0529	.0878	.1136	.1478
30000	.0198	.0328	.0424	.0552
40000	.0074	.0122	.0158	.0206
50000	.0028	.0046	.0059	.0077
60000	.0010	.0017	.0022	.0029
80000	.0001	.0002	.0003	.0004
100000	.0000	.0000	.0000	.0001
150000	.0000	.0000	.0000	.0000
200000	.0000	.0000	.0000	.0000

R = 100.0

θ	r=4.0	r=7.0	r=10.0	r=20.0	r=50.0	r=100.0
$1.0x10^2$	.4778	.6658	.7782	.9468	.9999	
1.5	.4490	.6273	.7364	.9154	.9992	
2	.4303	.6020	.7082	.8905	.9973	
3	.4063	.5691	.6710	.8536	.9908	1.0000
4	.3907	.5476	.6463	.8270	.9825	.9999
5	.3794	.5319	.6281	.8066	.9736	.9995
6	.3706	.5197	.6139	.7901	.9649	.9987
8	.3574	.5014	.5925	.7648	.9486	.9956
$1x10^3$	.3478	.4880	.5768	.7458	.9341	.9907
1.5	.3314	.4650	.5499	.7125	.9037	.9729
2	.3200	.4491	.5312	.6888	.8780	.9511
3	.3024	.4244	.5019	.6512	.8326	.9051
4	.2870	.4028	.4764	.6181	.7907	.8601
5	.2726	.3825	.4525	.5871	.7510	.8170
6	.2589	.3634	.4398	.5577	.7134	.7761
8	.2337	.3279	.3878	.5032	.6438	.7004
$1x10^4$	.2108	.2959	.3500	.4541	.5809	.6320
1.5	.1631	.2289	.2707	.3513	.4493	.4888
2	.1261	.1770	.2094	.2717	.3476	.3781
3	.0755	.1059	.1253	.1626	.2079	.2262
4	.0452	.0634	.0750	.0973	.1244	.1354
5	.0270	.0379	.0448	.0582	.0744	.0810
6	.0162	.0227	.0268	.0348	.0445	.0485
8	.0058	.0081	.0096	.0125	.0159	.0173
$1x10^5$	.0021	.0029	.0034	.0045	.0057	.0062
1.5	.0002	.0002	.0003	.0003	.0004	.0005
2	.0000	.0000	.0000	.0000	.0000	.0000
3	.0000	.0000	.0000	.0000	.0000	.0000

R = 200

θ	r=10	r=20	r=50	r=100	r=200
$1x10^3$	.5769	.7459	.9343	.9957	1.0000
1.5	.5502	.7130	.9055	.9879	.9999
2	.5326	.6910	.8837	.9788	.9996
3	.5096	.6617	.8524	.9611	.9973
4	.4943	.6422	.8303	.9454	.9925
5	.4830	.6278	.8132	.9316	.9857
6	.4741	.6162	.7993	.9192	.9776
8	.4600	.5980	.7768	.8969	.9593
$1x10^4$	.4485	.5831	.7578	.8765	.9397
1.5	.4237	.5509	.7163	.8294	.8905
2	.4011	.5215	.6781	.7852	.8432
3	.3596	.4676	.6080	.7040	.7561
4	.3224	.4192	.5451	.6312	.6779
5	.2891	.3759	.4887	.5660	.6078
6	.2592	.3370	.4382	.5074	.5449
8	.2084	.2709	.3523	.4079	.4381
$1x10^5$	.1675	.2178	.2832	.3279	.3522
1.5	.0971	.1262	.1641	.1900	.2040
2	.0562	.0731	.0951	.1101	.1182
3	.0189	.0245	.0319	.0370	0397
4	.0063	.0082	.0107	.0124	.0133
5	.0021	.0028	.0036	.0042	.0047
6	.0007	.0009	.0012	.0014	.0015
8	.0001	.0001	.0001	.0002	.0002
$1x10^6$	.0000	.0000	.0000	.0000	.0000

R = 500

θ	r=10	r=20	r=50	r=100	r=200	r=500
$5x10^3$	.4831	.6278	.8134	.9322	.9935	
6 "	.4742	.6165	.7998	.9206	.9900	
8 "	.4609	.5993	.7789	.9016	.9823	1.0000
$1x10^4$	.4511	.5866	.7631	.8864	.9746	.9999
1.5 "	.4342	.5502	.7171	.8387	.9419	.9972
2 "	.4230	.5502	.7171	.8387	.9419	.9972
3 "	.4080	.5308	.6923	.8115	.9184	.9897
4 "	.3979	.5176	.6753	.7924	.9001	.9790
5 "	.3900	.5074	.6621	.7773	.8846	.9666
6 "	.3833	.4987	.6508	.7643	.8706	.9536
8 "	.3716	.4834	.6308	.7409	.8446	.9269
$1x10^5$	.3607	.4692	.6124	.7193	.8201	.9004
1.5 "	.3353	.4362	.5693	.6687	.7624	.8372
2 "	.3117	.4055	.5293	.6217	.7089	.7784
3 "	.2695	.3506	.4575	.5374	.6128	.6729
4	.2329	.3030	.3955	.4646	.5297	.5817
5	.2014	.2620	.3419	.4016	.4579	.5028
6	.1741	.2265	.2955	.3472	.3958	.4347
8	.1301	.1692	.2208	.2594	.2958	.3248
$1x10^6$	.0972	.1264	.1650	.1938	.2210	.2427
1.5	.0469	.0610	.0797	.0936	.1067	.1172
2	.0226	.0295	.0385	.0452	.0515	.0565
3	.0053	.0069	.0090	.0105	.0120	.0132
4	.0012	.0016	.0021	.0025	.0028	.0031
5	.0003	.0004	.0005	.0006	.0007	.0007

R=1000

θ	r=10	r=20	r=50	r=100	r=200	r=500	r=1000
$1x10^4$	.4511	.5866	.7631	.8864	.9746	1.0000	
1.5	.4342	.5647	.7357	.8584	.9568	.9996	
2	.4230	.5502	.7171	.8387	.9419	.9987	
3	.4080	.5308	.6923	.8116	.9186	.9953	1.0000
4	.3981	.5178	.6757	.7929	.9011	.9906	.9999
5	.3906	.5082	.6632	.7788	.8873	.9855	.9997
6	.3848	.5006	.6533	.7675	.8760	.9803	.9993
8	.3759	.4890	.6383	.7503	.8581	.9702	.9976
$1x10^5$	.3692	.4804	.6271	.7374	.8443	.9611	.9948
1.5	.3577	.4654	.6076	.7147	.8196	.9413	.9841
2	.3496	.4548	.5938	.6986	.8018	.9242	.9706
3	.3370	.4385	.5725	.6736	.7734	.8936	.9411
4	.3260	.4242	.5538	.6517	.7482	.8650	.9115
5	.3156	.4106	.5361	.6308	.7243	.8374	.8825
6	.3055	.3975	.5190	.6107	.7012	.8107	.8544
8	.2864	.3726	.4865	.5724	.6573	.7600	.8009
$1x10^6$	.2685	.3493	.4560	.5366	.6161	.7123	.7507
1.5	.2284	.2971	.3879	.4564	.5241	.6059	.6386
2	.1943	.2527	.3300	.3883	.4458	.5154	.5432
3	.1406	.1829	.2388	.2809	.3226	.3730	.3930
4	.1017	.1323	.1728	.2033	.2334	.2699	.2844
5	.0736	.0957	.1250	.1471	.1689	.1953	.2058
6	.0533	.0693	.0905	.1064	.1222	.1413	.1489
8	.0279	.0363	.0474	.0557	.0640	.0740	.0780
$1x10^7$	.0146	.0190	.0248	.0292	.0335	.0387	.0408
1.5	.0029	.0038	.0049	.0058	.0066	.0077	.0081
2	.0006	.0007	.0010	.0011	.0013	.0015	.0016
3	.0000	.0000	.0000	.0000	.0001	.0001	.0001

APPENDIX H

DIMENSIONLESS PRESSURE DISTRIBUTION, $P(r,\theta)$, FOR SINGLE-LAYER RADIAL FLOW, CONSTANT TERMINAL RATE

Calculated from:

$$P(r,\theta) = \frac{r^2 + 4\theta}{2(R^2-1)} - \frac{R^2 \ln r}{(R^2-1)}$$

$$+ \frac{(1 + 2R^2 + 4R^4 \ln R - 3R^4)}{r(R^2 - 1)^2}$$

$$+ \pi \sum_{n=1}^{\infty} \frac{e^{-\alpha^2\theta} J_1^{\,2}(\alpha R)[J_1(\alpha)Y_o(\alpha r) - Y_1(\alpha)J_o(\alpha r)]}{\alpha[J_1^{\,2}(\alpha R) - J_1^{\,2}(\alpha)]} \qquad \text{(IV-150)}$$

Eigenvalues, α, from: Appendix C

Truncation Test: $|\text{Term}| < |\frac{\text{Sum}}{10^6}|$

Estimated Accuracy: Four significant figures

Definition of Dimensionless Variables:

$$r = \frac{r_a}{r_b} \qquad \text{(IV-46)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(IV-47)}$$

$$\theta = \frac{kt_a}{\mu \phi c r_b^{\,2}} \quad \text{for liquid flow}$$

$$= \frac{kt_a}{\rho C_p r_b^{\,2}} \quad \text{for heat conduction} \qquad \text{(IV-48)}$$

R = 1.5

Θ	r = 1.0	r = 1.2	r = 1.5
.025	.1669	.0395	.0032
.050	.2303	.0870	.0248
.100	.3219	.1709	.0954
.250	.5637	.4115	.3339
.500	.9637	.8115	.7339
1.000	1.764	1.612	1.534
2.500	4.164	4.012	3.934
5.000	8.164	8.012	7.934
10.000	16.164	16.012	15.934

R = 2.0

Θ	r = 1.0	r = 1.5	r = 2.0
.025	.1669	.0016	.0000
.050	.2301	.0119	.0002
.100	.3142	.0446	.0055
.250	.4678	.1486	.0684
.500	.6478	.3165	.2255
1.000	.9823	.6500	.5581
2.500	1.982	1.650	1.558
5.000	3.649	3.317	3.225
10.000	6.982	6.650	6.558

R = 2.5

Θ	r = 1.0	r = 1.5	r = 2.0	r = 2.5
.05	.2301	.0119	.0001	.0000
.10	.3142	.0445	.0026	.0001
.25	.4659	.1414	.0325	.0105
.50	.6182	.2665	.1116	.0696
1.00	.8318	.4693	.2956	.2451
2.50	1.406	1.042	.8663	.8149
5.00	2.358	1.994	1.819	1.767
10.00	4.263	3.899	3.723	3.672
25.00	9.977	9.614	9.438	9.386
50.00	19.501	19.137	18.962	18.910

R = 3.0

Θ	r = 1.0	r = 1.5	r = 2.0	r = 3.0
.10	.3142	.0445	.0026	.0000
.25	.4659	.1413	.0318	.0011
.50	.6169	.2632	.0999	.0183
1.00	.8055	.4337	.2327	.1056
2.50	1.203	.8246	.6106	.4672
5.00	1.828	1.450	1.236	1.092
10.00	3.078	2.700	2.481	2.342
25.00	6.828	6.450	6.236	6.092
50.00	13.078	12.700	12.486	12.342

R = 4.0

Θ	r = 1.0	r = 1.5	r = 2.5	r = 4.0
.25	.4659	.1413	.0050	.0000
.50	.6169	.2631	.0321	.0007
1.00	.8022	.4283	.1069	.0156
2.50	1.108	.7197	.3220	.1592
5.00	1.460	1.069	.6581	.4824
10.00	2.127	1.736	1.325	1.149
25.00	4.127	3.736	3.325	3.149
50.00	7.461	7.070	6.658	6.482

R = 5.0

Θ	r = 1.0	r = 1.2	r = 1.5	r = 2.0	r = 2.5	r = 3.0	r = 4.0	r = 5.0
.5	.6169	.4436	.2631	.0994	.0321	.0087	.0003	.0000
1.0	.8022	.6253	.4283	.2204	.1066	.0477	.0074	.0016
2.5	1.101	.9209	.7107	.4639	.3006	.1923	.0794	.0498
5.0	1.378	1.198	.9831	.7226	.5407	.4121	.2651	.2228
10.0	1.808	1.628	1.412	1.149	.9636	.8311	.6774	.5325
25.0	3.059	2.878	2.663	2.399	2.214	2.081	1.927	1.882
50.0	5.142	4.961	4.746	4.483	4.297	4.164	4.011	3.966
100.0	9.309	9.128	8.913	8.649	8.464	8.331	8.177	8.132

R = 8.0

Θ	r = 1	r = 2	r = 5	r = 8
1.0	.8021	.2204	.0008	.0000
2.5	1.100	.4632	.0229	.0007
5	1.362	.7011	.0967	.0168
10	1.654	.9783	.2590	.1171
25	2.180	1.500	.7371	.5698
50	2.975	2.295	1.531	1.363
100	4.653	3.882	3.118	2.950

R = 12.0

Θ	r = 1	r = 3	r = 6	r = 12
5	1.362	.3737	.0448	.0002
10	1.651	.6133	.1501	.0087
25	2.066	.9948	.4182	.1356
50	2.462	1.384	.7826	.4648
100	3.165	2.087	1.483	1.163
250	5.263	4.184	3.581	3.261
500	8.760	7.681	7.077	6.757
1000	15.752	14.674	14.070	13.750

R = 10.0

Θ	r = 1	r = 2	r = 4	r = 7	r = 10
1	.8021	.2204	.0073	.0000	.0000
2.5	1.100	.4632	.0703	.0015	.0000
5	1.362	.7010	.1942	.0195	.0021
10	1.651	.9752	.3894	.0928	.0343
25	2.085	1.401	.7668	.3781	.2797
50	2.604	1.919	1.280	.8813	.7787
100	3.614	2.929	2.290	1.891	1.789
250	6.645	5.960	5.321	4.922	4.819
500	11.695	11.010	10.371	9.972	9.869

R = 14.0

Θ	r = 1	r = 3	r = 7	r = 14
5	1.362	.3736	.0195	.0000
10	1.651	.6133	.0897	.0019
25	2.062	.9906	.3048	.0633
50	2.411	1.329	.5889	.2805
100	2.939	1.855	1.106	.7868
250	4.478	3.394	2.645	2.325
500	7.042	5.958	5.209	4.889
1000	12.170	11.337	10.337	10.017

R = 16

θ	r = 1	r = 4	r = 8	r = 16
5	1.362	.1942	.0079	.0000
10	1.651	.3891	.0521	.0004
25	2.062	.7310	.2235	.0281
50	2.394	1.039	.4585	.1681
100	2.825	1.463	.8620	.5442
250	4.003	2.641	2.039	1.720
500	5.964	4.602	4.000	3.681
1000	0.886	8.523	7.921	7.602

R = 18.0

θ	r = 1	r = 4	r = 9	r = 18
5	1.362	.1942	.0029	.0000
10	1.651	.3891	.0292	.0001
25	2.062	.7310	.1633	.0118
50	2.390	1.033	.3638	.0991
100	2.768	1.402	.6935	.3798
250	3.705	2.337	1.625	1.305
500	5.253	3.885	3.173	2.853
1000	8.349	6.981	6.289	5.949

R = 20

θ	r = 1	r = 4	r = 10	r = 20
10	1.651	.3890	.0158	.0000
25	2.062	.7309	.1182	.0047
50	2.389	1.031	.2911	.0572
100	2.742	1.373	.5710	.2654
250	3.513	2.142	1.328	1.009
500	4.766	3.395	2.581	2.263
1000	7.272	5.901	5.088	4.769
2500	14.791	13.420	12.607	12.288

R = 50

θ	r = 1	r = 6	r = 20	r = 50
25	2.062	.4063	.0022	.0000
50	2.388	.6691	.0260	.0000
100	2.723	.9687	.1124	.0003
500	3.522	1.738	.6263	.1692
1000	3.963	2.178	1.047	.5526
2500	5.166	3.380	2.249	1.752
5000	7.167	5.381	4.249	3.753

R = 100

Θ	r = 1	r = 3	r = 6	r = 10	r = 20	r = 50	r = 100
100	2.723	1.632	.9689	.5294	.1126	.0002	.0000
250	3.173	2.077	1.396	.9157	.3539	.0128	.0000
500	3.516	2.419	1.732	1.237	.6133	.0738	.0012
1000	3.861	2.763	2.073	1.570	.9128	.2173	.0271
2500	4.335	3.237	2.545	2.038	1.362	.5628	.2590
5000	4.856	3.757	3.065	2.558	1.880	1.069	.7507
10000	5.856	4.758	4.066	3.558	2.880	2.069	1.751
25000	8.857	7.758	7.066	6.559	5.880	5.069	4.751

APPENDIX I

DIMENSIONLESS CUMULATIVE FLUX, $Q_{(t)}$, FOR SINGLE-LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Calculated from:

$$Q_{(t)} = \left(\frac{R^2-1}{2}\right) - 2 \sum_{m=1}^{\infty} \frac{e^{-b^2\theta} J_1^{\,2}(bR)}{b^2[J_o^2(b) - J_1^{\,2}(bR)]} \qquad \text{(IV-87)}$$

Eigenvalues, b, from: Appendix B

Truncation Test: $|\text{Term}| < \left|\frac{\text{Sum}}{10^6}\right|$

Estimated Accuracy: All figures shown are significant.

Definition of Dimensionless Variables:

$$r = \frac{r_a}{r_b} \qquad \text{(IV-46)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(IV-47)}$$

$$\theta = \frac{kt_a}{\mu\phi c r_b^{\,2}} \quad \text{for liquid flow}$$

$$= \frac{kt_a}{\rho C_p r_b^{\,2}} \quad \text{for heat conduction} \qquad \text{(IV-48)}$$

R = 10

θ	Q(t)
15	9.94
20	12.30
22	13.19
24	14.07
26	14.92
28	15.75
30	16.56
32	17.35
34	18.13
36	18.88
38	19.61
40	20.33
42	21.03
44	21.72
46	22.38
48	23.04
50	23.67
52	24.29
54	24.90
56	25.49
58	26.07
60	26.63
65	27.98
70	29.25
75	30.44
80	31.57
85	32.62
90	33.62
95	34.56
100	35.44
120	38.47
140	40.85
160	42.72
180	44.18

R = 10 (cont'd)

θ	Q(t)
200	45.33
240	46.94
280	47.92
320	48.53
360	48.90
400	49.13
440	49.27
480	49.36
520	49.41
560	49.45
600	49.47

R = 12

θ	Q(t)
15	9.95
20	12.32
22	13.23
24	14.12
26	15.00
28	15.86
30	16.71
32	17.54
34	18.36
36	19.17
38	19.96
40	20.74
42	21.51
44	22.27
46	23.02
48	23.75
50	24.48
52	25.19
54	25.89
56	26.58
58	27.26
60	27.94
65	29.57
70	31.14
75	32.65
80	34.11
85	35.51
90	36.86
95	38.15
100	39.40
120	43.95
140	47.85
160	51.20
180	54.08

R = 12 (cont'd)

θ	Q(t)
200	56.55
240	60.48
280	63.38
320	65.52
360	67.09
400	68.25
440	69.11
480	69.74
520	70.20
560	70.54
600	70.80
700	71.17
800	71.35
1000	71.47
1200	71.49
1400	71.50

R = 14

θ	Q(t)
15	9.95
20	12.32
22	13.23
24	14.13
26	15.01
28	15.88
30	16.74
32	17.58
34	18.41
36	19.24
38	20.05
40	20.85
42	21.65
44	22.43
46	23.21
48	23.98
50	24.74
52	25.49
54	26.24
56	26.97
58	27.70
60	28.42
65	30.19
70	31.92
75	33.60
80	35.23
85	36.83
90	38.38
95	39.90
100	41.37
120	46.90
140	51.89
160	56.39
180	60.44

R = 14 (cont'd)	
Θ	Q(t)
240	70.36
280	75.44
320	79.58
360	82.94
400	85.67
440	87.88
480	89.69
520	91.15
560	92.34
600	93.31
700	95.01
800	96.02
1000	96.97
1200	97.31
1400	97.43
1600	97.48
1800	97.49
2000	97.50

R = 16	
Θ	Q(t)
15	9.95
20	12.32
22	13.23
24	14.13
26	15.01
28	15.88
30	16.74
32	17.59
34	18.42
36	19.25
38	20.07
40	20.88
42	21.68
44	22.47
46	23.26
48	24.04
50	24.82
52	25.58
54	26.34
56	27.10
58	27.85
60	28.59
65	30.42
70	32.22
75	33.98
80	35.71
85	37.40
90	39.07
95	40.70
100	42.30
120	48.43

R = 16 (cont'd)	
Θ	Q(t)
140	54.11
160	59.38
180	64.28
200	68.82
240	76.95
280	83.96
320	89.99
360	95.19
400	99.66
440	103.52
480	106.84
520	109.71
560	112.17
600	114.29
700	118.40
800	121.24
1000	124.53
1200	126.09
1400	126.83
1600	127.18
1800	127.35
2000	127.43
2200	127.47
2400	127.48

R = 18	
Θ	Q(t)
30	16.74
32	17.59
34	18.43
36	19.25
38	20.07
40	20.88
42	21.69
44	22.48
46	23.27
48	24.06
50	24.84
52	25.61
54	26.37
56	27.14
58	27.89
60	28.64
65	30.50
70	32.32
75	34.12
80	35.89
85	37.64
90	39.36
95	41.06
100	42.73
120	49.18
140	55.29
160	61.06
180	66.52
200	71.68
240	81.18
280	89.67

R = 18 (cont'd)	
Θ	
320	97.27
360	104.06
400	110.14
440	115.57
480	120.42
520	124.77
560	128.65
600	132.13
700	139.29
800	144.70
1000	151.89
1200	156.01
1400	158.36
1600	159.70
1800	160.47
2000	160.91
2200	161.16
2400	161.31
2600	161.39
2800	161.44
3000	161.46
3500	161.49
4000	161.50

R = 20	
θ	Q(t)
30	16.74
32	17.59
34	18.42
36	19.25
38	20.07
40	20.88
42	21.69
44	22.48
46	23.27
48	24.06
50	24.84
52	25.61
54	26.38
56	27.14
58	27.90
60	28.65
65	30.52
70	32.35
75	34.17
80	35.96
85	37.73
90	39.47
95	41.20
100	42.91
120	49.54
140	55.89
160	61.98
180	67.80
200	73.37
240	83.83
280	93.41

R = 20 (cont'd)	
θ	Q(t)
320	102.21
360	110.27
400	117.67
440	124.45
480	130.67
520	136.37
560	141.61
600	146.40
700	156.73
800	165.05
1000	177.15
1200	185.00
1400	190.09
1600	193.39
1800	195.54
2000	196.93
2200	197.83
2400	198.42
2600	198.80
2800	199.04
3000	199.20
3500	199.40
4000	199.47
4500	199.49
5000	199.50

R = 50	
θ	Q(t)
70	32.36
75	34.18
80	35.98
90	39.43
100	43.02
110	46.45
120	49.83
140	56.46
160	62.95
180	69.31
200	75.58
225	83.28
250	90.86
275	98.33
300	105.7
350	120.2
400	134.4
450	148.4
500	162.1
550	175.7
600	189.0
700	215.2
800	240.7
900	265.5
1000	289.8
1200	336.4
1400	380.8
1600	423.1
1800	463.3
2000	501.5
2500	589.2

R = 50 (cont'd)	
θ	Q(t)
3000	666.5
3500	734.8
4000	795.1
4500	848.4
5000	895.4
5500	936.9
6000	973.5
6500	1005.9
7000	1034.4
7500	1059.6
8000	1081.9
8500	1101.5
9000	1118.9
9500	1134.2
10^4	1147.7
1.1×10^4	1170.1
1.2×10^4	1187.7
1.3×10^4	1201.3
1.4×10^4	1211.9
1.5×10^4	1220.2
1.6×10^4	1226.7
1.7×10^4	1231.7
1.8×10^4	1235.6
1.9×10^4	1238.7
2.0×10^4	1241.1
2.2×10^4	1244.4
2.4×10^4	1246.4
2.6×10^4	1247.6
2.8×10^4	1248.4
3.0×10^4	1248.8
3.5×10^4	1249.3
4.0×10^4	1249.4
4.5×10^4	1249.5

R = 75	
θ	Q(t)
4×10^2	134.4
4.5×10^2	148.4
5×10^2	162.2
5.5×10^2	175.8
6×10^2	189.3
7×10^2	215.7
8×10^2	241.7
9×10^2	267.3
10^3	292.5
1.2×10^3	342.0
1.4×10^3	390.4
1.6×10^3	437.7
1.8×10^3	484.1
2.0×10^3	529.5
2.5×10^3	639.2
3.0×10^3	743.6
3.5×10^3	843.0
4.0×10^3	937.7
4.5×10^3	1027.7
5.0×10^3	1113.5
5.5×10^3	1195.1
6.0×10^3	1272.8
6.5×10^3	1346.8
7.0×10^3	1417.2
7.5×10^3	1484.2
8.0×10^3	1548.0
8.5×10^3	1608.8
9.0×10^3	1666.6
9.5×10^3	1721.7
10^4	1774.1
1.1×10^4	1871.4
1.2×10^4	1959.7
1.3×10^4	2039.6

R = 75 (cont'd)

Θ	Q(t)
1.4×10^4	2112.1
1.5×10^4	2177.7
1.6×10^4	2237.2
1.7×10^4	2291.1
1.8×10^4	2340.0
1.9×10^4	2384.3
2.0×10^4	2424.4
2.2×10^4	2493.7
2.4×10^4	2550.6
2.6×10^4	2597.4
2.8×10^4	2635.7
3.0×10^4	2667.3
3.5×10^4	2723.6
4.0×10^4	2758.0
4.5×10^4	2779.0
5.0×10^4	2791.8
5.5×10^4	2799.7
6.0×10^4	2804.5
6.5×10^4	2897.4
7.0×10^4	2809.2
7.5×10^4	2810.3
8.0×10^4	2811.0
9.0×10^4	2811.6
10^5	2811.9

R = 100

Θ	Q(t)
10^3	292.6
1.2×10^3	342.3
1.4×10^3	390.9
1.6×10^3	438.7
1.8×10^3	485.8
2.0×10^3	532.3
2.5×10^3	646.1
3.0×10^3	756.6
3.5×10^3	864.3
4.0×10^3	969.2
4.5×10^3	1071.4
5.0×10^3	1171.0
5.5×10^3	1268.1
6.0×10^3	1362.7
6.5×10^3	1454.9
7.0×10^3	1544.8
7.5×10^3	1632.4
8.0×10^3	1717.7
8.5×10^3	1801.0
9.0×10^3	1882.1
9.5×10^3	1961.1
10^4	2038.1
1.1×10^4	2186.4
1.2×10^4	2327.3
1.3×10^4	2461.1
1.4×10^4	2588.2
1.5×10^4	2708.9
1.6×10^4	2823.6
1.7×10^4	2932.5
1.8×10^4	3036.0
1.9×10^4	3134.3
2.0×10^4	3227.7

R = 100 (cont'd)

Θ	Q(t)
2.2×10^4	3400.7
2.4×10^4	3556.8
2.6×10^4	3697.6
2.8×10^4	3824.7
3.0×10^4	3939.4
3.5×10^4	4179.5
4.0×10^4	4365.3
4.5×10^4	4508.9
5.0×10^4	4620.0
5.5×10^4	4706.0
6.0×10^4	4772.5
6.5×10^4	4823.9
7.0×10^4	4863.7
7.5×10^4	4894.4
8.0×10^4	4918.2
9.0×10^4	4950.9
10^5	4970.4
1.2×10^5	4989.1
1.4×10^5	4995.8
1.6×10^5	4998.2
1.8×10^5	4999.0
2.0×10^5	4999.3
2.5×10^5	4999.5

R = 200

Θ	Q(t)
2×10^3	532.5
2.5×10^3	646.8
3.0×10^3	758.5
3.5×10^3	868.3
4.0×10^3	976.4
4.5×10^3	1083.1
5.0×10^3	1188.6
5.5×10^3	1293.0
6.0×10^3	1396.4
6.5×10^3	1499.0
7.0×10^3	1600.7
7.5×10^3	1701.7
8.0×10^3	1801.9
8.5×10^3	1901.5
9.0×10^3	2000.4
9.5×10^3	2098.7
10^4	2196.4
1.1×10^4	2390.0
1.2×10^4	2581.5
1.3×10^4	2770.7
1.4×10^4	2957.8
1.5×10^4	3142.9
1.6×10^4	3325.9
1.7×10^4	3506.9
1.8×10^4	3686.0
1.9×10^4	3863.1
2.0×10^4	4038.3
2.2×10^4	4382.9
2.4×10^4	4720.1
2.6×10^4	5050.1
2.8×10^4	5372.9
3.0×10^4	5688.7

R = 200 (cont'd)

Θ	Q(t)
3.5×10^{4}	6448.8
4.0×10^{4}	7168.5
4.5×10^{4}	7849.9
5.0×10^{4}	8495.2
5.5×10^{4}	9106.2
6.0×10^{4}	9684.8
6.5×10^{4}	10233
7.0×10^{4}	10751
7.5×10^{4}	11242
8.0×10^{4}	11708
9.0×10^{4}	12565
10^{5}	13334
1.2×10^{5}	14641
1.4×10^{5}	15692
1.6×10^{5}	16537
1.8×10^{5}	17216
2.0×10^{5}	17762
2.5×10^{5}	18703
3.0×10^{5}	19248
3.5×10^{5}	19564
4.0×10^{5}	19747
4.5×10^{5}	19853
5.0×10^{5}	19915
6.0×10^{5}	19971
7.0×10^{5}	19990
8.0×10^{5}	19996
9.0×10^{5}	19998
10^{6}	19999

R = 500

Θ	Q(t)
10^{4}	2198.4
1.1×10^{4}	2393.4
1.2×10^{4}	2586.8
1.3×10^{4}	2778.6
1.4×10^{4}	2969.1
1.5×10^{4}	3158.2
1.6×10^{4}	3346.3
1.7×10^{4}	3533.2
1.8×10^{4}	3719.2
1.9×10^{4}	3904.2
2.0×10^{4}	4088.3
2.2×10^{4}	4454.1
2.4×10^{4}	4816.9
2.6×10^{4}	5177.1
2.8×10^{4}	5534.8
3.0×10^{4}	5890.3
3.5×10^{4}	6770.2
4.0×10^{4}	7639.2
4.5×10^{4}	8498.7
5.0×10^{4}	9349.7
5.5×10^{4}	10193
6.0×10^{4}	11029
6.5×10^{4}	11858
7.0×10^{4}	12680
7.5×10^{4}	13496
8.0×10^{4}	14306
9.0×10^{4}	15908
10^{5}	17486
1.2×10^{5}	20574
1.4×10^{5}	23572
1.6×10^{5}	26485
1.8×10^{5}	29314
2.0×10^{5}	32061
2.5×10^{5}	38590
3.0×10^{5}	44660
3.5×10^{5}	50304

R = 500 (cont'd)

Θ	Q(t)
4.0×10^{5}	55551
4.5×10^{5}	60429
5.0×10^{5}	64965
6.0×10^{5}	73103
7.0×10^{5}	80139
8.0×10^{5}	86220
9.0×10^{5}	91477
10^{6}	96021
1.2×10^{6}	1.033×10^{5}
1.4×10^{6}	1.088×10^{5}
1.6×10^{6}	1.129×10^{5}
1.8×10^{6}	1.160×10^{5}
2.0×10^{6}	1.182×10^{5}
3.0×10^{6}	1.234×10^{5}
4.0×10^{6}	1.246×10^{5}
5.0×10^{6}	1.249×10^{5}
6.0×10^{6}	1.250×10^{5}

R = 1000

Θ	
6×10^{4}	1.104×10^{4}
7×10^{4}	1.270×10^{4}
8×10^{4}	1.434×10^{4}
10^{5}	1.757×10^{4}
1.5×10^{5}	2.545×10^{4}
2.0×10^{5}	3.313×10^{4}
2.5×10^{5}	4.065×10^{4}
3.0×10^{5}	4.803×10^{4}
4×10^{5}	6.243×10^{4}
5×10^{5}	7.636×10^{4}
6×10^{5}	8.984×10^{4}
7×10^{5}	1.029×10^{5}
8×10^{5}	1.155×10^{5}
9×10^{5}	1.278×10^{5}
10^{6}	1.396×10^{5}
1.2×10^{6}	1.622×10^{5}
1.4×10^{6}	1.834×10^{5}
1.6×10^{6}	2.032×10^{5}
1.8×10^{6}	2.218×10^{5}
2.0×10^{6}	2.392×10^{5}
2.5×10^{6}	2.782×10^{5}
3.0×10^{6}	3.113×10^{5}
3.5×10^{6}	3.395×10^{5}
4×10^{6}	3.635×10^{5}
5×10^{6}	4.012×10^{5}
6×10^{6}	4.285×10^{5}
7×10^{6}	4.483×10^{5}
8×10^{6}	4.626 x 10-
10^{7}	4.804×10^{5}
1.2×10^{7}	4.897×10^{5}
1.4×10^{7}	4.956×10^{5}
1.6×10^{7}	4.971×10^{5}
1.8×10^{7}	4.985×10^{5}
2.0×10^{7}	4.992×10^{5}
2.5×10^{7}	4.998×10^{5}
3.0×10^{7}	5.000×10^{5}

APPENDIX J

DIMENSIONLESS PRESSURE DISTRIBUTION, $P(x,y,\Theta)$, FOR TWO-LAYER LINEAR FLOW, CONSTANT TERMINAL PRESSURE

Calculated from:

$$P(x,y,\Theta) = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} Y_{mn}\right] \left(\frac{2}{bL}\right) \sin bx \qquad \text{(V-86)}$$

Eigenvalues b from:

$$b = \left(\frac{2m-1}{2}\right)\frac{\pi}{L} ; \quad m = 1,2,3,\ldots \qquad \text{(V-43)}$$

Eigenvalues a from: Appendix D

Truncation Test (Both Series): $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$

Estimated Accuracy: $\pm$ 0.001 for all values not in parentheses

Definition of Dimensionless Variables:

$$x = \frac{x_a}{H} \qquad \text{(V-3)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-4)}$$

$$L = \frac{L_a}{H} \qquad \text{(V-5)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c H^2} \quad \text{(for liquid flow)}$$

$$= \frac{k_1 t_a}{\rho_1 C_{p1} H^2} \quad \text{(for heat conduction)} \qquad \text{(V-15)}$$

All values presented are for $K_2^* = K_2 = \frac{k_2}{k_1}$

L = 5; K2 = 2; c = 0.5

θ	x = 1.0					x = 2.0					x = 3.0				x = 4.0				x = 5.0			
	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.07	(.988)	(.984)	(.965)	(.953)	(.950)																	
.1	(.963)	(.955)	(.925)	(.909)	(.904)																	
.2	.843	.832	.799	.783	.778														(1.)	(1.)	(1.)	1.
.4	.664	.657	.638	.629	.625	.958	.953	.940	.934	.932	.995	.995	.993	.991			.995	.995	1.	(1.)	(1.)	1.
.7	.523	.520	.510	.505	.504	.846	.842	.833	.828	.826	.966	.964	.960	.957	.994	.994	.993	.992	.999	.999	.999	.998
1.	.445	.443	.437	.434	.432	.762	.759	.752	.748	.747	.922	.920	.916	.912	.980	.979	.977	.976	.993	.993	.991	.991
2.	.320	.319	.317	.316	.315	.590	.588	.585	.583	.582	.780	.779	.775	.772	.887	.886	.882	.880	.921	.920	.916	.914
4.	.221	.220	.219	.219	.219	.418	.418	.416	.415	.414	.574	.573	.570	.568	.672	.671	.668	.666	.706	.705	.702	.700
7.	.140	.140	.140	.139	.139	.267	.267	.266	.265	.265	.368	.367	.365	.364	.432	.432	.430	.428	.454	.454	.452	.450
10.	.090	.090	.090	.089	.089	.171	.171	.170	.170	.170	.236	.236	.234	.234	.277	.277	.276	.275	.292	.291	.290	.289
20.	.021	.021	.020	.020	.020	.039	.039	.039	.039	.039	.054	.054	.053	.053	.063	.063	.063	.063	.066	.066	.066	.066
40.	.001	.001	.001	.001	.001	.002	.002	.002	.002	.002	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003	.003

L = 5; K2 = 5; c = 0.5

θ	x = 1.0					x = 2.0					x = 3.0				x = 4.0				x = 5.0			
	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.07	(.972)	(.945)	(.845)	(.823)	(.815)						(1.)	(1.)	(1.)	(1.)								
.1	(.923)	(.888)	(.778)	(.755)	(.748)						(1.)	(1.)	.999	.999								
.2	.743	.714	.631	.616	.611	.998	.981	.929	.919	.916	.996	.994	.987	.985					1.	1.	1.	1.
.4	.534	.521	.482	.475	.473	.848	.836	.799	.791	.789	.961	.955	.938	.934	.990	.989	.984	.983	.998	.997	.994	.994
.7	.399	.393	.376	.373	.371	.701	.693	.670	.665	.664	.875	.869	.851	.847	.953	.950	.939	.936	.974	.972	.964	.961
1.	.332	.328	-	-	.315	.606	.600	-	-	.580	.793	.788	-	.768	.895	.891	-	.874	.927	.923	-	.907
2.	.225	.224	.219	.218	.218	.426	.423	.415	.414	.413	.583	.580	.569	.566	.683	.679	.666	.663	.716	.712	.700	.696
4.	.124	.123	.121	.120	.120	.235	.234	.229	.228	.228	.323	.321	.315	.314	.380	.378	.371	.369	.400	.397	.390	.388
7.	.051	.051	.050	.050	.050	.097	.097	.095	.094	.094	.134	.133	.131	.130	.157	.156	.153	.153	.165	.164	.161	.161
10.	.021	.021	.021	.021	.021	.040	.040	.039	.039	.039	.055	.055	.054	.054	.065	.065	.063	.063	.068	.068	.067	.066
20.	.001	.001	.001	.001	.001	.002	.002	.002	.002	.002	.003	.003	.003	.003	.003	.003	.003	.003	.004	.004	.004	.004

L = 5; K2 = 10; c = 0.5

θ	x = 1.0 y=0	y=.25	y=.5	y=.75	y=1	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 3.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=.75	y=1	x = 5.0 y=0	y=.25	y=.5	y=1
.07	(.943)	(.888)	(.702)	(.683)	(.677)			.976	.965	.962	(1.)	(1.)	(.995)	(.995)	.994									
.1	(.869)	(.809)	(.632)	(.615)	.609			.926	.915	.912	(.999)	(.996)	(.985)	.984	.983						(1.)	(1.)	(1.)	(1.)
.2	.646	.606	.496	.486	.483	.925	.894	.805	.797	.794	.977	.967	.937	.933	.932	.988	.987	.981	.981	.980	.999	.998	.993	.993
.4	.429	.414	.370	.365	.364	.728	.710	.656	.651	.649	.885	.872	.834	.830	.829	.954	.946	.923	.920	.919	.973	.967	.949	.946
.7	.307	.300	.282	.280	.279	.564	.554	.524	.521	.520	.745	.734	.702	.699	.698	.848	.838	.808	.805	.804	.881	.871	.843	.839
1.	.247	.243	.231	.230	.230	.464	.457	.436	.434	.434	.630	.621	.595	.593	.592	.732	.723	.695	.692	.691	.767	.757	.728	.724
2.	.141	.139	.133	.132	.132	.267	.264	.253	.252	.252	.368	.363	.348	.347	.346	.432	.427	.409	.407	.407	.455	.448	.430	.428
4.	.048	.048	.046	.046	.045	.092	.091	.087	.087	.086	.127	.125	.120	.119	.119	.149	.147	.141	.140	.140	.156	.154	.148	.147
7.	.010	.010	.009	.009	.009	.019	.018	.018	.018	.017	.026	.025	.024	.024	.024	.030	.030	.028	.028	.028	.032	.031	.030	.030
10.	.002	.002	.002	.002	.002	.004	.004	.004	.004	.004	.005	.005	.005	.005	.005	.006	.006	.006	.006	.006	.006	.006	.006	.006

L = 5; K2 = 20; c = 0.5

θ	x = 1.0 y=0	y=.25	y=.5	y=.75	y=1	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 3.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=.75	y=1	x = 5.0 y=0	y=.25	y=.5	y=1
.04			(.657)	(.642)	(.637)			.946	.937	.935			(.991)	(.990)	(.990)								(1.)	(1.)
.07	(.902)	(.816)	(.551)	(.538)	(.534)			.856	.847	.845	(.998)	(.992)	(.961)	(.959)	.958						(1.)	(1.)	(.998)	(.998)
.1	(.804)	(.721)	(.489)	.478	.475		.960	.792	.784	.781	(.990)	(.977)	(.927)	.924	.923						(.999)	(.998)	(.991)	.990
.2	.548	.501	.375	.370	.368	.824	.780	.656	.650	.648	.923	.899	.826	.822	.821	.965	.953	.913	.911	.910	.983	.973	.940	.937
.4	.332	.315	.270	.268	.267	.589	.566	.501	.497	.496	.756	.734	.669	.666	.664	.848	.828	.769	.766	.765	.877	.859	.802	.798
.7	.217	.210	.190	.189	.189	.409	.396	.360	.358	.358	.555	.539	.493	.491	.490	.646	.628	.577	.574	.573	.676	.658	.605	.602
1.	.158	.153	.140	.139	.139	.299	.290	.265	.264	.264	.410	.399	.365	.363	.363	.481	.468	.429	.427	.426	.506	.492	.450	.448
2.	.058	.056	.051	.051	.051	.110	.107	.098	.097	.097	.151	.147	.135	.134	.134	.178	.173	.158	.157	.157	.187	.182	.166	.165
4.	.008	.008	.007	.007	.007	.015	.015	.013	.013	.013	.021	.020	.018	.018	.018	.024	.024	.022	.021	.021	.025	.025	.023	.022
7.						.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001

L = 5; K2 = 50; c = 0.5

θ	x = 1.0 y=0	y=.25	y=.5	y=.75	y=1	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 3.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=.75	y=1	x = 5.0 y=0	y=.25	y=.5	y=1
.04			(.462)	(.454)	(.451)			.764	.757	.755			(.911)	(.909)	(.908)	.995	.994	.971	.970	.970			(.987)	(.987)
.07	(.839)	(.718)	(.379)	(.372)	.370		.936	.655	.649	.648	(.981)	(.950)	(.823)	.820	.819	.993	.964	.911	.910	.909	(.997)	(.989)	(.939)	.937
.1	(.715)	(.611)	(.331)	.326	.325	.930	.841	.586	.581	.580	.945	.901	.754	.751	.750	.961	.939	.849	.847	.846	.985	(.964)	(.880)	.878
.2	.430	.378	.240	.237	.236	.666	.606	.440	.437	.436	.782	.732	.586	.583	.583	.849	.806	.675	.672	.671	.876	.834	.704	.701
.4	.209	.192	.144	.143	.142	.379	.351	.271	.269	.269	.500	.466	.370	.368	.367	.574	.537	.432	.430	.429	.599	.561	.453	.450
.7	.098	.091	.072	.071	.071	.185	.172	.136	.135	.135	.253	.236	.187	.186	.186	.296	.276	.220	.219	.218	.311	.290	.231	.230
1.	.049	.046	.036	.036	.036	.093	.087	.069	.069	.069	.128	.11o	.095	.094	.094	.961	.939	.849	.847	.846	.158	.148	.117	.117
2.	.005		.004	.004	.004	.010		.007	.007	.007	.013		.010	.010	.010	.016		.012	.012	012	.016		.012	.012

L = 5; K2 = 100; c = 0.5

θ	x = 1.0 y=0	y=.25	y=.5	y=.75	y=1	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 3.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=.75	y=1	x = 5.0 y=0	y=.25	y=.5	y=1
.04			(.341)	(.337)	(.335)		.993	.603	.600	.598			(.776)	(.773)	(.773)	.993	.980	.873	.871	.871			(.904)	(.903)
.07	(.794)	(.654)	(.272)	.269	.268	.977	.855	.491	.488	.487	(.949)	(.883)	(.647)	.644	.643	.952	.913	.740	.738	.737	(.981)	(.942)	(.771)	.769
.1	.655	(.538)	.229	.226	.225	.853	.738	.417	.414	.413	(.879)	(.800)	.553	.550	.549	.907	.846	.636	.633	.633	.935	(.874)	.664	.661
.2	.347	.290	.141	.140	.140	.531	.459	.263	.261	.260	.623	.553	.354	.352	.351	.681	.612	.411	.409	408	.706	.635	.430	.427
.4	.122	.105	.061	.061	.061	.218	.191	.116	.115	.115	.286	.252	.158	.157	.157	.327	.290	.185	.184	.184	.342	.303	.195	.193
.7	.034	.030	.018	.018	.018	.064	.056	.035	.035	.035	.087	.077	.048	.048	.048	.102	.090	.057	.056	.056	.107	.094	.060	.059
1.	.010	.009	006	.006	.006	.019	.017	.011	.011	.011	.027	.023	.015	.015	.015	.031	.028	.017	.017	.017	.033	.029	.018	.018

L = 10; K2 = 2; c = 0.5

θ	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=1	x = 7.0 y=0	y=.25	y=.5	y=1	x = 10.0 y=0	y=.25	y=.5	y=1
.4	(.943)	(.940)	(.929)		(.922)					(1.)		(1.)	(1.)	(1.)		(1.)	(1.)
.7	.844	.841	.831		.825					1.		1.	1.	1.		1.	1.
1.	.761	.759	.751		.746					1.		1.	1.	1.		1.	1.
2.	.591	.590	.586		.583	.906		.902	.900	.996		.996	.995	1.		1.	1.
4.	.438	.438	.436		.435	.754		.752	.751	.957		.956	.956	.992		.992	.992
7.	.338	.338	.338		.337	.618		.617	.616	.870		.869	.868	.942		.942	.941
10.	.285	.285	.285		.284	.532		.531	.531	.782		.781	.780	.865		.864	.863
20.	.188	.188	.188		.188	.358		.357	.357	.542		.541	.541	.608		.607	.606
40.	.090	.090	.090		.089	.171		.170	.170	.259		.258	.258	.290		.290	.290
70.	.030	.030	.030		.029	.056		.056	.056	.085		.085	.085	.096		.095	.095
100.	.010	.010	.010		.010	.019		.019	.018	.028		.028	.028	.032		.031	.031
200.										.001		.001	.001	.001		.001	.001

L = 10; K2 = 5; c = 0.5

θ	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=1	x = 7.0 y=0	y=.25	y=.5	y=1	x = 10.0 y=0	y=.25	y=.5	y=1
.1					(.973)												
.2		(.951)	(.916)		(.906)												
.4	(.844)	(.832)	(.797)		(.788)					(1.)	(1.)	1.	1.				
.7	.701	.693	.670		.664	.979	.974	.960	.957	.999	.999	.999	.999	1.		1.	1.
1.	.607	.602				.916	.912	.900	.897	.996	.996			1.			
2.	.445	.443	.437		.435	.762	.759	.751	.759	.959	.958	.955	.954	.992		.991	.991
4.	.320	.319	.137		.316	.590	.589	.585	.584	.843	.842	.838	.837	.920		.916	.915
7.	.239	.239	.238		.237	.452	.452	.449	.449	.679	.678	.675	.674	.758		.754	.753
10.	.189	.189	.188		.188	.360	.359	.357	.357	.544	.543	.541	.540	.610		.606	.606
20.	.090	.090	.090		.089	.171	.171	.170	.170	.260	.260	.258	.258	.292		.290	.290
40.	.021	.021	.020		.020	.039	.039	.039	.039	.059	.059	.059	.059	.067		.066	.066
70.	.002	.002	.002		.002	.004	.004	.004	.004	.006	.006	.006	.006	.007		.007	.007
100.										.001	.001	.001	.001	.001		.001	.001

L = 10; K2 = 10; c = 0.5

θ	x = 2.0 y=0	y=.25	y=.5	y=.75	y=1	x = 4.0 y=0	y=.25	y=.5	y=1	x = 7.0 y=0	y=.25	y=.5	y=1	x = 10.0 y=0	y=.25	y=.5	y=1
.07				(.943)	(.942)												
.1			(.909)	(.902)	(.900)												
.2	(.900)	(.875)	(.801)	(.793)	(.791)					(1.)	(1.)	(1.)	(1.)				
.4	(.726)	(.708)	.657	.652	.650	.990	.979	.948	.944	(.999)	(.999)	.997	.997	(1.)	(1.)	(1.)	(1.)
.7	.568	.558	.530	.528	.527	.880	.872	.848	.845	.990	.988	.984	.983	.999	.999	.998	.998
1.	.478	.472	.455	.454	.453	.795	.789	.771	.768	.969	.967	.960	.959	.995	.994	.992	.992
2.	.339	.337	.331	.330	.330	.618	.616	.606	.604	.867	.864	.856	.855	.938	.936	.929	.928
4.	.235	.234	.231	.231	.231	.444	.442	.438	.437	.666	.664	.657	.656	.745	.742	.735	.734
7.	.154	.154	.152	.152	.152	.294	.293	.289	.289	.445	.443	.439	.438	.499	.497	.492	.492
10.	.103	.102	.101	.101	.101	.196	.195	.193	.193	.297	.296	.292	.292	.333	.332	.328	.328
20.	.027	.027	.026	.026	.026	.051	.050	.050	.050	.077	.077	.076	.076	.086	.086	.085	.085
40.	.002	.002	.002	.002	.002	.003	.003	.003	.003	.005	.005	.005	.005	.006	.006	.006	.006

L = 10; K2 = 20; c = 0.5

	x = 2.0					x = 4.0				x = 7.0				x = 10.0			
Θ	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.04				(.921)	(.920)												
.07			(.846)	(.840)	(.838)												
.1			(.786)	(.779)	(.777)							(1.)	(1.)				
.2	(.806)	(.767)	(.655)	(.650)	(.648)			.944	.939	(.999)	(.999)	(.997)	(.996)	(1.)	(1.)	(1.)	(1.)
.4	(.593)	(.571)	.512	.509	.508	.890	.874	.824	.821	(.987)	(.984)	(.975)	.974	(.999)	(.998)	.996	.996
.7	.438	.428	.401	.400	.399	.744	.733	.701	.699	.943	.938	.922	.921	.983	.981	.973	.972
1.	.361	.355	.340	.339	.339	.647	.639	.617	.615	.884	.879	.861	.860	.948	.944	.931	.931
2.	.246	.244	.238	.237	.237	.464	.460	.449	.448	.691	.686	.672	.671	.770	.765	.749	.748
4.	.144	.143	.140	.140	.139	.274	.272	.266	.265	.414	.411	.403	.402	.465	.462	.452	.748
7.	.067	.066	.065	.065	.065	.127	.126	.123	.123	.192	.191	.186	.186	.215	.214	.209	.209
10.	.031	.031	.030	.030	.030	.059	.058	.057	.057	.089	.088	.086	.086	.100	.099	.097	.097
20.	.002	.002	.002	.002	.002	.005	.005	.004	.004	.007	.007	.007	.007	.008	.008	.007	.007

L = 10; K2 = 50; c = 0.5

Θ	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.01				(.964)	(.963)												
.02				(.878)	(.876)												
.04			(.758)	(.753)	(.751)							(1.)	(1.)				
.07			(.652)	(.648)	(.646)			.942	.938			(.997)	(.997)			(1.)	(1.)
.1			(.588)	(.583)	(.582)			.889	.885		(.997)	(.989)	(.989)			(.999)	(.999)
.2	(.658)	(.607)	(.466)	(.463)	(.462)	.922	.882	.768	.765	(.984)	(.976)	(.949)	.948	(.997)	(.995)	(.985)	(.985)
.4	(.426)	(.405)	.347	.346	.346	.711	.687	.621	.619	(.909)	.895	.854	.853	(.959)	(.950)	.921	.920
.7	.293	.285	.261	.261	.261	.537	.525	.488	.487	.771	.757	.717	.716	.845	.832	.794	.792
1.	.230	.225	.211	.210	.210	.433	.424	.398	.398	.644	.632	.598	.597	.717	.705	.668	.667
2.	.121	.119	.112	.112	.112	.231	.226	.214	.213	.349	.343	.324	.323	.392	.385	.364	.363
4.	.036	.035	.033	.033	.033	.068	.066	.063	.063	.102	.101	.095	.095	.115	.113	.107	.106
7.	.006	.006	.005	.005	.005	.011	.011	.010	.010	.016	.016	.016	.015	.018	.018	.017	.017
10.	.001	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.002	.002	.003	.003	.003	.003

L = 10; K2 = 100; c = 0.5

Θ	y=0	y=.25	y=.5	y=.75	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.004					(.978)												
.007			(.926)	(.924)	(.923)												
.01			(.872)	(.869)	(.868)												
.02			(.739)	(.736)	(.735)				.999				(1.)				
.04			(.603)	(.600)	(.598)			.905	.903			(.993)	(.993)			(1.)	(1.)
.07			(.505)	(.502)	(.501)			.810	.807			(.968)	(.967)			(.994)	(.994)
.1			(.449)	(.446)	(.445)		.949	.745	.742		(.981)	(.937)	(.936)			(.980)	(.980)
.2	(.551)	(.495)	(.346)	(.345)	(.344)	.810	.757	.611	.609	(.934)	(.911)	(.837)	.836	(.971)	(.955)	(.904)	(.903)
.4	(.320)	(.300)	.245	.244	.244	.557	.531	.455	.454	(.767)	.741	.665	.664	(.834)	(.809)	.737	.736
.7	.195	.187	.164	.164	.164	.366	.352	.311	.311	.544	.525	.468	.468	.607	.586	.524	.523
1.	.134	.129	.114	.114	.114	.253	.244	.217	.217	.383	.369	.329	.328	.429	.413	.369	.368
2.	.041	.039	.035	.035	.035	.077	.075	.066	.066	.117	.113	.101	.101	.132	.127	.113	.113
4.	.004	.004	.003	.003	.003	.007	.007	.006	.006	.011	.011	.009	.009	.012	.012	.011	.011

L = 20; K2 = 2; c = 0.5

	x = 4				x = 7.0				x = 10				x = 20			
θ	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
1	(.980)		(.978)	(.977)	(1.)		(1.)	(1.)								
2	(.901)		(.897)	(.895)	.996		.995	.995	(1.)		(1.)	(1.)				
4	.754		.752	.750	.957		.956	.956	.996		.996	.996				
7	.619		.617	.617	.874		.873	.873	.971		.971	.971				
10	.536		.535	.534	.800		.799	.798	.933		.932	.932	(1.)		1.	1.
20	.395		.394	.394	.634		.634	.634	.804		.803	.803	.980		.980	.980
40	.284		.284	.284	.475		.475	.474	.633		.633	.632	.864		.864	.864
70	.207		.207	.207	.349		.349	.349	.472		.472	.472	.665		.665	.665
100	.156		.156	.156	.264		.264	.264	.357		.357	.357	.505		.505	.505
200	.062		.062	.062	.105		.105	.105	.142		.141	.141	.200		.200	.200
400	.010		.010	.010	.016		.016	.016	.022		.022	.022	.031		.031	.031
700	.001		.001	.001	.001		.001	.001	.001		.001	.001	.002		.002	.002

L = 20; K2 = 5; c = 0.5

	x = 4				x = 7.0				x = 10				x = 20			
θ	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
0.7	(.957)	(.957)	(.945)	(.941)	(1.)	(1.)	(.999)	(.999)	(1.)		(1.)	(1.)				
1	(.908)	(.905)			(.996)	(.996)			(1.)							
2	.761	.759	.751	.750	.959	.958	.955	.954	.996		.995	.995				
4	.591	.590	.586	.585	.851	.850	.846	.845	.960		.958	.957	1.		1.	1.
7	.466	.465	.463	.463	.723	.722	.720	.719	.879		.877	.876	.996		.996	.996
10	.396	.396	.395	.394	.636	.635	.634	.633	.805		.803	.803	.981		.980	.980
20	.285	.285	.284	.284	.476	.475	.475	.474	.634		.632	.632	.865		.864	.864
40	.188	.188	.188	.188	.318	.318	.318	.318	.430		.430	.430	.608		.607	.607
70	.108	.108	.108	.108	.182	.182	.182	.182	.247		.247	.246	.349		.349	.349
100	.062	.062	.062	.062	.105	.105	.105	.105	.142		.142	.141	.200		.200	.200
200	.010	.010	.010	.010	.016	.016	.016	.016	.022		.022	.022	.032		.031	.031

L = 20; K2 = 10; c = 0.5

	x = 4				x = 7.0				x = 10				x = 20			
θ	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
0.4		(.951)	(.931)	(.928)	(.998)	(.998)	(.997)	(.997)		(1.)	(1.)	(1.)				
0.7	(.874)	(.866)	(.845)	(.842)	(.990)	(.988)	(.984)	(.983)	(1.)	(1.)	(.999)	(.999)				
1	(.794)	(.788)	.770	.768	.969	.967	.961	.960	(.997)	(.997)	.996	.996				
2	.619	.616	.007	.605	.872	.870	.862	.861	.969	.968	.965	.964	1.		1.	1.
4	.459	.458	.454	.453	.715	.713	.709	.708	.872	.871	.867	.867	.995		.994	.994
7	.354	.354	.352	.351	.578	.578	.575	.575	.748	.747	.745	.744	.955		.953	.953
10	.298	.298	.297	.296	.496	.496	.494	.494	.658	.657	.655	.655	.888		.886	.886
20	.201	.201	.200	.200	.340	.339	.338	.338	.459	.459	.457	.457	.647		.645	.645
40	.102	.102	.101	.101	.172	.172	.171	.171	.232	.232	.232	.232	.329		.328	.328
70	.037	.037	.037	.037	.062	.062	.062	.062	.084	.084	.084	.084	.119		.119	.119
100	.013	.013	.013	.013	.022	.022	.022	.022	.030	.030	.030	.030	.043		.043	.043
200					.001	.001	.001	.001	.001	.001	.001	.001	.001		.001	.001

L = 20; K2 = 20; c = 0.5

θ	x = 4 y=0	x = 4 y=.25	x = 4 y=.5	x = 4 y=1	x = 7 y=0	x = 7 y=.25	x = 7 y=.5	x = 7 y=1	x = 10 y=0	x = 10 y=.25	x = 10 y=.5	x = 10 y=1	x = 20 y=0	x = 20 y=.25	x = 20 y=.5	x = 20 y=1
0.1							(.998)	(.998)								
0.2			(.924)	(.921)	(.994)	(.995)	(.996)	(.996)			(.999)	(1.)				
0.4	(.876)	(.862)	(.820)	(.817)	(.987)	(.984)	(.975)	(.974)	(.999)	(.999)	(.998)	(.998)				
0.7	(.743)	(.732)	(.701)	(.700)	(.944)	(.939)	.924	.923	(.992)	(.990)	(.986)	(.986)				
1	.647	.640	.618	.616	.889	.884	.868	.867	.974	.972	.966	.965	1.		1.	1.
2	.476	.473	.464	.463	.733	.730	.719	.719	.885	.883	.875	.874	.996		.994	.994
4	.343	.341	.338	.338	.562	.560	.555	.555	.730	.738	.723	.723	.944		.939	.939
7	.258	.258	.256	.256	.433	.433	.430	.430	.581	.580	.577	.577	.806		.800	.800
10	.208	.208	.207	.207	.352	.351	.349	.349	.475	.474	.472	.472	.670		.664	.664
20	.109	.108	.108	.108	.184	.183	.182	.182	.248	.248	.247	.247	.351		.349	.349
40	.030	.030	.030	.030	.050	.050	.050	.050	.068	.068	.068	.068	.097		.096	.096
70	.004	.004	.004	.004	.007	.007	.007	.007	.010	.010	.010	.010	.014		.014	.014
100	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.002		.002	.002

L = 20; K2 = 50; c = 0.5

θ	x = 4 y=0	x = 4 y=.25	x = 4 y=.5	x = 4 y=1	x = 7 y=0	x = 7 y=.25	x = 7 y=.5	x = 7 y=1	x = 10 y=0	x = 10 y=.25	x = 10 y=.5	x = 10 y=1	x = 20 y=0	x = 20 y=.25	x = 20 y=.5	x = 20 y=1
.07				(.920)			(.996)	(.996)								
.1			(.875)	(.873)			(.988)	(.988)			(.999)	(.999)				(1.)
.2			(.763)	(.760)	(.980)	(.973)	(.949)	(.949)			(.993)	(.992)			(1.)	(1.)
.4	(.705)	(.684)	(.622)	(.620)	(.912)	(.900)	(.863)	(.862)	(.979)	(.975)	(.961)	(.960)	(1.)	(1.)	(1.)	(1.)
.7	(.544)	(.533)	.500	.499	.796	.785	.754	.753	(.922)	(.916)	.897	.896	(.998)	(.997)	.996	.996
1.	.454	.447	.427	.427	.704	.696	.672	.672	.858	.852	.834	.833	.990	.988	.984	.984
2	.318	.316	.309	.308	.526	.522	.511	.511	.690	.686	.674	.673	.911	.908	.898	.898
4	.215	.214	.211	.211	.363	.361	.356	.356	.489	.487	.480	.480	.688	.685	.675	.675
7	.113	.133	.131	.131	.226	.225	.221	.221	.305	.304	.300	.299	.432	.430	.424	.423
10	.083	.083	.082	.082	.141	.140	.138	.138	.191	.190	.187	.187	.270	.270	.265	.265
20	.017	.017	.017	.017	.030	.029	.029	.029	.040	.040	.039	.039	.057	.056	.056	.055
40	.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002	.002	.002	.002	.002

L = 20; K2 = 100; c = 0.5

θ	x = 4 y=0	x = 4 y=.25	x = 4 y=.5	x = 4 y=1	x = 7 y=0	x = 7 y=.25	x = 7 y=.5	x = 7 y=1	x = 10 y=0	x = 10 y=.25	x = 10 y=.5	x = 10 y=1	x = 20 y=0	x = 20 y=.25	x = 20 y=.5	x = 20 y=1
.04			(.892)	(.890)			(.992)	(.992)			(.999)	(.999)				(1.)
.07			(.802)	(.800)			(.968)	(.967)			(.997)	(.997)			(1.)	(1.)
.1			(.739)	(.737)			(.938)	(.937)			(.990)	(.990)			(1.)	(1.)
.2			(.612)	(.610)	(.933)	(.913)	(.849)	(.848)			(.952)	(.951)			(1.)	(1.)
.4	.563	(.539)	(.474)	(.474)	(.797)	(.777)	.721	.720	(.917)	(.905)	(.868)	(.868)	(.996)	(.995)	(.990)	(.999)
.7	.409	(.399)	.370	.369	.643	.631	.595	.595	(.802)	(.791)	.760	.759	(.969)	(.965)	.952	.952
1	.336	.328	.311	.311	.545	.537	.514	.514	.708	.700	.675	.674	.918	.912	.894	.894
2	.221	.219	.212	.212	.372	.369	.358	.357	.502	.497	.482	.482	.702	.696	.677	.677
4	.118	.117	.114	.113	.199	.198	.192	.192	.270	.267	.260	.260	.382	.378	.367	.367
7	.047	.046	.045	.045	.079	.078	.076	.076	.107	.106	.103	.103	.152	.150	.146	.146
10	.018	.018	.018	.018	.032	.031	.030	.030	.042	.042	.041	.041	.060	.060	.058	.058
20	.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003

L = 50; K2 = 2; c = 0.5

	x = 4.0				x = 7.0				x = 10.0				x = 20.0				x = 50.0			
θ	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
1 x 10	(.536)	(.536)	(.535)	(.534)	(.800)	(.799)	(.799)	(.798)	(.933)	(.932)	(.932)	(.932)								
2	.395	.395	.394	.394	.634	.634	.634	.634	.804	.804	.803	.803								
4	.285	.285	.285	.285	.477	.477	.477	.477	.639	.639	.629	.639	.942	.942	.942	.942	1.	1.	1.	1.
7	.218	.218	.217	.217	.371	.371	.371	.371	.510	.510	.510	.510	.834	.834	.834	.833	.999	.999	.999	.999
1 x 10^2	.183	.183	.183	.183	.314	.314	.314	.314	.436	.436	.436	.436	.752	.752	.752	.752	.992	.992	.992	.992
2	.130	.130	.130	.130	.225	.225	.225	.225	.317	.317	.317	.317	.585	.585	.585	.585	.918	.918	.918	.918
4	.089	.089	.089	.089	.155	.155	.155	.155	.219	.219	.219	.219	.416	.416	.416	.416	.702	.702	.702	.702
7	.057	.057	.057	.057	.099	.099	.099	.099	.140	.140	.140	.140	.266	.266	.266	.266	.452	.452	.452	.452
1 x 10^3	.036	.036	.036	.036	.063	.063	.063	.063	.090	.090	.090	.090	.170	.170	.170	.170	.290	.290	.290	.290
2	.008	.008	.008	.008	.014	.014	.014	.014	.020	.020	.020	.020	.039	.039	.039	.039	.066	.066	.066	.066
4					.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003

L = 50; K2 = 5; c = 0.5

7 x 10^0	.466	.465	.463	.463	(.723)	(.722)	(.720)	(.719)	(.879)	(.878)	(.877)	(.876)								
1 x 10	.396	.396	.395	.394	.636	.635	.634	.633	.805	.805	.803	.803								
2	.286	.285	.285	.285	.478	.478	.477	.477	.640	.639	.639	.639	.943	.943	.942	.942	1.	1.	1.	1.
4	.204	.204	.204	.204	.349	.349	.349	.349	.482	.482	.481	.481	.804	.804	.804	.804	.997	.997	.997	.997
7	.155	.155	.155	.155	.267	.267	.267	.267	.375	.375	.374	.374	.671	.671	.671	.671	.971	.971	.971	.971
1 x 10^2	.130	.130	.130	.130	.225	.225	.225	.225	.317	.317	.317	.317	.585	.585	.585	.585	.918	.918	.917	.917
2	.089	.089	.089	.089	.155	.155	.155	.155	.219	.219	.219	.219	.416	.416	.416	.416	.702	.702	.702	.702
4	.049	.049	.049	.049	.085	.085	.085	.085	.120	.120	.120	.120	.229	.229	.229	.229	.390	.390	.390	.390
7	.020	.020	.020	.020	.035	.035	.035	.035	.050	.050	.050	.050	.094	.094	.094	.094	.160	.160	.160	.160
1 x 10^3	.008	.008	.008	.008	.014	.014	.014	.014	.020	.020	.020	.020	.039	.039	.039	.039	.066	.066	.066	.066
2					.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003

L = 50; K2 = 10; c = 0.5

4	.459	.458	.454	.453	.715	.713	.709	.708	.872	.871	.867	.867								
7	.354	.354	.352	.352	.579	.578	.575	.575	.749	.748	.745	.745								
1 x 10	.299	.298	.297	.297	.498	.497	.496	.495	.662	.662	.660	.659	.958	.957	.956	.956	1.	1.	1.	1.
2	.213	.213	.213	.213	.364	.364	.363	.363	.501	.501	.500	.500	.824	.824	.823	.823	.998	.998	.998	.998
4	.151	.151	.151	.151	.262	.262	.261	.261	.367	.367	.366	.366	.660	.660	.660	.659	.966	.966	.966	.965
7	.114	.114	.114	.114	.199	.199	.198	.198	.281	.280	.280	.280	.525	.525	.525	.525	.857	.857	.857	.857
1 x 10^2	.094	.094	.094	.094	.163	.163	.163	.163	.231	.231	.231	.231	.438	.438	.438	.438	.737	.737	.737	.737
2	.054	.054	.054	.054	.094	.094	.094	.094	.133	.133	.133	.133	.253	.253	.253	.253	.430	.430	.430	.430
4	.018	.018	.018	.018	.032	.032	.032	.032	.045	.045	.045	.045	.085	.085	.085	.085	.145	.145	.145	.145
7	.004	.004	.004	.004	.006	.006	.006	.006	.009	.009	.009	.009	.017	.017	.017	.017	.029	.029	.029	.029
1 x 10^3	.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003	.006	.006	.006	.006

L = 50; K2 = 20; c = 0.5

θ	x = 4.0				x = 7.0				x = 10.0				x = 20.0				x = 50.0			
	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
1	.647	.640	.618	.616	.889	.884	.868	.867	.973	.971	.965	.965								
2	.476	.473	.464	.463	.733	.730	.719	.719	.885	.883	.875	.874								
4	.343	.342	.338	.338	.562	.560	.555	.555	.731	.730	.725	.724	.999	.998	.995	.995				
7	.261	.260	.259	.259	.440	.439	.437	.436	.595	.594	.591	.590	.909	.908	.906	.905	1.	1.	1.	1.
1 x 10	.219	.219	.218	.218	.373	.373	.317	.371	.513	.512	.510	.510	.836	.835	.833	.833	.999	.999	.999	.999
2	.155	.155	.155	.155	.268	.268	.267	.267	.376	.375	.375	.374	.672	.672	.671	.671	.971	.971	.970	.970
4	.109	.109	.109	.109	.190	.190	.190	.190	.269	.268	.268	.268	.505	.505	.504	.504	.832	.831	.831	.831
7	.078	.078	.077	.077	.135	.135	.135	.135	.191	.191	.191	.191	.363	.363	.363	.363	.616	.616	.616	.616
1 x 10^2	.057	.057	.057	.057	.099	.099	.099	.099	.140	.140	.140	.140	.266	.266	.266	.266	.452	.452	.452	.452
2	.020	.020	.020	.020	.035	.035	.035	.035	.050	.050	.050	.050	.094	.094	.094	.094	.160	.160	.160	.160
4	.003	.003	.003	.003	.004	.004	.004	.004	.006	.006	.006	.006	.012	.012	.012	.012	.020	.020	.020	.020
7													.001	.001	.001	.001	.001	.001	.001	.001

L = 50; K2 = 50; c = 0.5

θ	x = 4.0				x = 7.0				x = 10.0				x = 20.0				x = 50.0			
	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.4	(.713)	(.690)	(.623)	(.621)			(.864)	(.863)				(.959)								
.7	(.545)	(.533)	(.500)	(.499)	(.796)	(.785)	(.754)	(.754)	(.922)	(.916)	(.897)	(.896)								
1.	(.454)	(.447)	(.427)	(.427)	(.704)	(.696)	(.672)	(.672)	(.858)	(.852)	(.834)	(833)								
2.	.319	.316	.309	.309	.527	.523	.513	.513	.693	.689	.678	.677	.975	.973	.966	.966	1.	1.	1.	(1.)
4.	.224	.224	.221	.221	.382	.381	.377	.377	.523	.522	.517	.517	.846	.844	.839	.839	.999	.999	.999	.999
7.	.169	.169	.168	.168	.292	.291	.289	.289	.407	.406	.404	.404	.714	.713	.710	.710	.984	.983	.983	.983
1 x 10	.142	.141	.141	.141	.245	.245	.244	.244	.344	.344	.342	.342	.627	.626	.624	.624	.947	.947	.945	.945
2	.098	.098	.098	.098	.171	.171	.171	.171	.242	.242	.242	.242	.458	.458	.457	.457	.767	.766	.765	.765
4	.059	.058	.058	.058	.102	.102	.102	.102	.144	.144	.144	.144	.274	.274	.274	.274	.467	.466	.465	.465
7	.028	.027	.027	.027	.048	.048	.048	.048	.068	.068	.068	.068	.129	.129	.129	.129	.220	.219	.219	.219
1 x 10^2	.013	.013	.013	.013	.023	.023	.022	.022	.032	.032	.032	.032	.061	.061	.061	.060	.103	.103	.103	.103
2	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003	.005	.005	.005	.005	.008	.008	.008	.008

L = 50; K2 = 100; c = 0.5

θ	x = 4.0				x = 7.0				x = 10.0				x = 20.0				x = 50.0			
	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1	y=0	y=.25	y=.5	y=1
.1				(.744)																
.2			(.614)	(.612)				(.848)				(.950)								
.4	(.568)	(.543)	(.475)	(.474)			(.721)	(.720)	(.915)	(.903)	(.868)	(.868)								
.7	(.409)	(.399)	(.370)	(.369)	(.643)	(.631)	(.596)	(.595)	(.803)	(.792)	(.761)	(.760)								
1.	(.334)	(.328)	.312	.312	(.547)	(.539)	(.516)	(.516)	(.711)	(.703)	(.680)	(.679)	.984	.980	.966	.965	(1.)	(1.)	(1.)	(1.)
2.	.230	.228	.223	.223	.391	.388	.379	.379	.534	.530	.519	.519	.854	.850	.840	.840	.999	.999	.998	.998
4.	.161	.160	.158	.158	.277	.276	.273	.273	.387	.386	.382	.382	.688	.686	.680	.680	.975	.974	.972	.972
7.	.121	.120	.119	.119	.209	.209	.207	.207	.295	.295	.293	.293	.550	.549	.545	.545	.882	.881	.878	.878
10.	.099	.099	.099	.099	.173	.173	.172	.172	.245	.245	.243	.243	.462	.461	.459	.459	.772	.771	.768	.768
20.	.059	.059	.059	.059	.103	.103	.103	.103	.146	.146	.145	.145	.278	.278	.276	.276	.473	.472	.470	.470
40.	.022	.022	.022	.022	.038	.038	.038	.038	.054	.054	.054	.054	.103	.103	.102	.102	.175	.175	.174	.174
70.	.005	.005	.005	.005	.009	.009	.009	.009	.012	.012	.012	.012	.023	.023	.023	.023	.039	.039	.039	.039
100.	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003	.005	.005	.005	.005	.009	.009	.009	.009

θ	x = 4			x = 7			x = 10			x = 20			x = 50			x = 100		
	y=0	y=.5	y=1.0	y=0	y=.5	y=1.0	y=0	y=.5	y=1.0	y=0	y=.5	y=1.0	y=0	y=.5	y=1.0	y=0	y=.5	y=1.0
L = 100; K2 = 2; c = 0.5																		
100	.183	.183	.183	.314	.314	.314	.436	.436	.436	.752	.752	.752	.996	.996	.996			
200	.130	.130	.130	.225	.225	.225	.317	.317	.317	.586	.586	.586	.959	.959	.959	1.	1.	1.
400	.092	.092	.092	.160	.160	.160	.227	.227	.227	.436	.436	.436	.851	.851	.851	.992	.992	.992
700	.070	.070	.070	.121	.121	.121	.173	.173	.173	.337	.337	.337	.724	.724	.724	.942	.942	.942
1000	.058	.058	.058	.101	.101	.101	.144	.144	.144	.284	.284	.284	.633	.633	.633	.864	.864	.864
2000	.038	.038	.038	.067	.067	.067	.095	.095	.095	.188	.188	.188	.430	.430	.430	.607	.607	.607
4000	.018	.018	.018	.032	.032	.032	.045	.045	.045	.090	.090	.090	.205	.205	.205	.290	.290	.290
7000	.006	.006	.006	.010	.010	.010	.015	.015	.015	.029	.029	.029	.067	.067	.067	.095	.095	.095
10000	.002	.002	.002	.003	.003	.003	.005	.005	.005	.010	.010	.010	.022	.022	.022	.031	.031	.031
20000													.001	.001	.001	.001	.001	.001
L = 100; K2 = 5; c = 0.5																		
70	.155	.155	.155	.267	.267	.267	.375	.374	.374	.671	.671	.671	.985	.985	.985	1.	1.	1.
100	.130	.130	.130	.225	.225	.225	.317	.317	.317	.586	.586	.586	.959	.959	.959	1.	1.	1.
200	.092	.092	.092	.160	.160	.160	.227	.227	.227	.436	.436	.436	.851	.851	.851	.992	.992	.992
700	.048	.048	.048	.085	.085	.085	.120	.120	.120	.238	.238	.238	.539	.539	.539	.754	.754	.754
1000	.038	.038	.038	.067	.067	.067	.095	.095	.095	.188	.188	.188	.430	.430	.430	.607	.607	.607
2000	.018	.018	.018	.032	.032	.032	.045	.045	.045	.090	.090	.090	.205	.205	.205	.290	.290	.290
4000	.004	.004	.004	.001	.001	.001	.010	.010	.010	.020	.020	.020	.047	.047	.047	.066	.066	.066
7000				.001	.001	.001	.001	.001	.001	.002	.002	.002	.005	.005	.005	.007	.007	.007
10000													.001	.001	.001	.001	.001	.001
L = 100; K2 = 10; c = 0.5																		
40	.151	.151	.151	.262	.261	.261	.367	.366	.366	.660	.660	.660	.983	.983	.983	1.	1.	1.
70	.115	.115	.115	.199	.199	.199	.282	.281	.281	.529	.529	.529	.929	.928	.928	.999	.999	.999
100	.096	.096	.096	.167	.167	.167	.237	.237	.237	.454	.454	.454	.868	.868	.868	.995	.995	.995
200	.068	.068	.068	.119	.119	.119	.169	.169	.169	.330	.330	.330	.712	.712	.712	.934	.934	.934
400	.047	.047	.047	.082	.082	.082	.117	.117	.117	.231	.231	.231	.526	.525	.525	.737	.737	.737
700	.031	.031	.031	.054	.054	.054	.077	.077	.077	.152	.152	.152	.348	.348	.348	.492	.492	.492
1000	.021	.021	.021	.036	.036	.036	.051	.051	.051	.101	.101	.101	.232	.232	.232	.328	.328	.328
2000	.005	.005	.005	.009	.009	.009	.013	.013	.013	.026	.026	.026	.060	.060	.060	.084	.084	.084
4000				.001	.001	.001	.001	.001	.001	.002	.002	.002	.004	.004	.004	.006	.006	.006

L = 100; K2 = 20; c = 0.5

θ	x = 4 y=0	x = 4 y=.5	x = 4 y=1.0	x = 7 y=0	x = 7 y=.5	x = 7 y=1.0	x = 10 y=0	x = 10 y=.5	x = 10 y=1.0	x = 20 y=0	x = 20 y=.5	x = 20 y=1.0	x = 50 y=0	x = 50 y=.5	x = 50 y=1.0	x = 100 y=0	x = 100 y=.5	x = 100 y=1.0
10	.219	.218	.218	.373	.371	.371	.513	.510	.510	.835	.832	.832	.999	.999	.999			
20	.155	.155	.155	.268	.267	.267	.376	.375	.374	.672	.671	.671	.985	.985	.985	1.	1.	1.
40	.110	.110	.110	.191	.191	.191	.270	.270	.270	.511	.510	.510	.916	.915	.915	.999	.999	.999
70	.083	.083	.085	.145	.145	.145	.206	.206	.206	.398	.398	.398	.808	.808	.808	.982	.982	.982
100	.070	.070	.070	.121	.121	.121	.173	.173	.173	.338	.337	.337	.724	.724	.724	.942	.942	.942
200	.048	.048	.048	.085	.086	.085	.120	.120	.120	.238	.238	.238	.539	.539	.539	.755	.754	.754
400	.028	.028	.028	.050	.050	.050	.071	.071	.071	.140	.140	.140	.320	.319	.319	.452	.452	.452
700	.013	.013	.013	.023	.023	.023	.032	.032	.032	.064	.064	.064	.147	.147	.147	.208	.208	.208
1000	.006	.006	.006	.010	.010	.010	.015	.015	.015	.030	.029	.029	.068	.067	.067	.095	.095	.095
2000				.001	.001	.001	.001	.001	.001	.002	.002	.002	.005	.005	.005	.007	.007	.007

L = 100; K2 = 50; c = 0.5

θ	x = 4 y=0	x = 4 y=.5	x = 4 y=1.0	x = 7 y=0	x = 7 y=.5	x = 7 y=1.0	x = 10 y=0	x = 10 y=.5	x = 10 y=1.0	x = 20 y=0	x = 20 y=.5	x = 20 y=1.0	x = 50 y=0	x = 50 y=.5	x = 50 y=1.0	x = 100 y=0	x = 100 y=.5	x = 100 y=1.0
4	.224	.221	.221	.382	.377	.376	.523	.517	.517	.844	.838	.837	.999	.999	.999			
7	.169	.168	.168	.292	.289	.289	.407	.404	.404	.714	.710	.710	.992	.991	.991			
10	.142	.141	.141	.245	.244	.244	.344	.342	.342	.627	.624	.624	.973	.973	.973	1.	1.	1.
20	.100	.100	.100	.174	.174	.174	.247	.246	.246	.470	.469	.469	.883	.882	.882	.996	.996	.996
40	.071	.071	.071	.123	.123	.123	.175	.175	.175	.343	.342	.342	.731	.731	.731	.947	.946	.946
70	.053	.053	.053	.093	.093	.093	.132	.132	.132	.260	.260	.260	.586	.585	.585	.812	.811	.811
100	.043	.043	.043	.075	.075	.075	.107	.107	.107	.211	.211	.211	.481	.481	.481	.678	.677	.677
200	.023	.023	.023	.040	.040	.040	.057	.057	.057	.112	.112	.112	.256	.256	.256	.362	.362	.362
400	.006	.006	.006	.011	.011	.011	.016	.016	.016	.032	.032	.032	.073	.073	.073	.103	.103	.103
700	.001	.001	.001	.002	.002	.002	.002	.002	.002	.005	.005	.005	.011	.011	.011	.016	.016	.016
1000										.001	.001	.001	.002	.002	.002	.002	.002	.002

L = 100; K2 = 100; c = 0.5

θ	x = 4 y=0	x = 4 y=.5	x = 4 y=1.0	x = 7 y=0	x = 7 y=.5	x = 7 y=1.0	x = 10 y=0	x = 10 y=.5	x = 10 y=1.0	x = 20 y=0	x = 20 y=.5	x = 20 y=1.0	x = 50 y=0	x = 50 y=.5	x = 50 y=1.0	x = 100 y=0	x = 100 y=.5	x = 100 y=1.0
1	(.333)	(.312)	(.312)	(.547)	(.516)	(.516)	(.712)	(.680)	(.679)	(.959)	(.947)	(.947)	(1.)	(1.)	(1.)			
2	(.230)	(.223)	(.223)	(.391)	(.379)	(.379)	(.534)	(.519)	(.519)	(.851)	(.838)	(.838)	(.999)	(.999)	(.999)			
4	.161	.158	.158	.277	.273	.273	.387	.382	.382	.688	.680	.680	.987	.986	.986			
7	.121	.120	.120	.210	.208	.208	.296	.294	.293	.552	.548	.548	.941	.939	.939	1.	1.	1.
10	.101	.100	.100	.176	.174	.174	.249	.247	.247	.474	.471	.471	.886	.884	.884	.997	.996	.996
20	.071	.071	.071	.124	.124	.124	.177	.176	.176	.345	.344	.344	.735	.733	.733	.948	.947	.947
40	.050	.049	.049	.087	.086	.086	.123	.123	.123	.243	.243	.243	.551	.550	.550	.770	.768	.768
70	.034	.033	.033	.059	.058	.058	.083	.083	.083	.165	.165	.165	.377	.376	.376	.533	.532	.532
100	.023	.023	.023	.040	.040	.040	.057	.057	.057	.113	.113	.113	.259	.259	.259	.367	.366	.366
200	.007	.007	.007	.012	.012	.012	.017	.016	.016	.033	.033	.033	.075	.075	.075	.106	.105	.105
400	.001	.001	.001	.001	.001	.001	.001	.001	.001	.003	.003	.003	.006	.006	.006	.009	.009	.009

L = 10^3; K2 = 2; c = 0.5

θ	= 10 y=0	= 10 y=1.0	= 50 y=0	= 50 y=1.0	= 100 y=0	= 100 y=1.0	= 10^3 y=0	= 10^3 y=1.0
10^4	.046	.046	.227	.227	.436	.436		
2 x 10^4	.033	.033	.162	.162	.317	.317	1.	1.
4 x 10^4	.023	.023	.115	.115	.227	.227	.992	.992
7 x 10^4	.017	.017	.087	.087	.173	.173	.942	.942
10^5	.015	.015	.073	.073	.144	.144	.864	.864
2 x 10^5	.010	.010	.048	.048	.095	.095	.607	.607
4 x 10^5	.005	.005	.023	.023	.045	.045	.290	.290
7 x 10^5	.001	.001	.007	.007	.015	.015	.095	.095
10^6	.002	.002	.005	.005	.031	.031		
2 x 10^6					.001	.001		

L = 10^3; K2 = 5; c = 0.5

θ	= 10 y=0	= 10 y=1.0	= 50 y=0	= 50 y=1.0	= 100 y=0	= 100 y=1.0	= 10^3 y=0	= 10^3 y=1.0
7 x 10^3	.039	.039	.193	.193	.374	.374		
10^4	.033	.033	.162	.162	.317	.317	1.	1.
2 x 10^4	.023	.023	.115	.115	.227	.227	.992	.992
4 x 10^4	.016	.016	.081	.081	.162	.162	.918	.918
7 x 10^4	.012	.012	.060	.060	.120	.120	.754	.754
10^5	.010	.010	.048	.048	.095	.095	.607	.607
2 x 10^5	.005	.005	.023	.023	.045	.045	.290	.290
4 x 10^5	.001	.001	.005	.005	.010	.010	.066	.066
7 x 10^5			.001	.001	.001	.001	.007	.007
10^6							.001	.001

L = 10^3; K2 = 10; c = 0.5

θ	= 10 y=0	= 10 y=1.0	= 50 y=0	= 50 y=1.0	= 100 y=0	= 100 y=1.0	= 10^3 y=0	= 10^3 y=1.0
4 x 10^3	.038	.038	.188	.188	.366	.366	1.	1.
7 x 10^3	.029	.029	.143	.143	.281	.281	.999	.999
10^4	.024	.024	.120	.120	.237	.237	.995	.995
2 x 10^4	.017	.017	.085	.085	.169	.169	.934	.934
4 x 10^4	.012	.012	.059	.059	.117	.117	.737	.737
7 x 10^4	.008	.008	.039	.039	.077	.077	.492	.492
10^5	.005	.005	.026	.026	.051	.051	.328	.328
2 x 10^5	.001	.001	.007	.007	.013	.013	.084	.084
4 x 10^5					.001	.001	.006	.006

L = 10^2; K2 = 20; c = 0.5

θ	= 10 y=0	= 10 y=1.0	= 50 y=0	= 50 y=1.0	= 100 y=0	= 100 y=1.0	= 10^3 y=0	= 10^3 y=1.0
10^3	.055	.055	.270	.270	.510	.510		
2 x 10^3	.039	.039	.193	.193	.374	.374	1.	1.
4 x 10^3	.028	.028	.137	.137	.270	.270	.999	.999
7 x 10^3	.021	.021	.104	.104	.206	.206	.982	.982
10^4	.017	.017	.087	.087	.173	.173	.942	.942
2 x 10^4	.012	.012	.060	.060	.120	.120	.754	.754
4 x 10^4	.007	.007	.035	.035	.071	.071	.452	.452
7 x 10^4	.003	.003	.016	.016	.032	.032	.208	.208
10^5	.001	.001	.007	.007	.015	.015	.095	.095
2 x 10^5			.001	.001	.001	.001	.007	.007

L = 10^3; K2 = 50; c = 0.5

θ	= 10 y=0	= 10 y=1.0	= 50 y=0	= 50 y=1.0	= 100 y=0	= 100 y=1.0	= 10^3 y=0	= 10^3 y=1.0
4 x 10^2	.056	.056	.274	.274	.516	.516		
7 x 10^2	.042	.042	.209	.209	.403	.403		
10^3	.035	.035	.175	.175	.342	.342	1.	1.
2 x 10^3	.025	.025	.124	.124	.246	.246	.997	.997
4 x 10^3	.018	.018	.088	.088	.175	.175	.946	.946
7 x 10^3	.013	.013	.066	.066	.132	.132	.812	.812
10^4	.011	.011	.054	.054	.107	.107	.677	.677
2 x 10^4	.006	.006	.028	.028	.057	.057	.362	.362
4 x 10^4	.002	.002	.008	.008	.016	.016	.103	.103
7 x 10^4			.001	.001	.002	.002	.016	.016
10^5							.002	.002

L = 10^3; K2 = 100; c = 0.5

θ	= 10 y=0	= 10 y=1.0	= 50 y=0	= 50 y=1.0	= 100 y=0	= 100 y=1.0	= 10^3 y=0	= 10^3 y=1.0
10^2	(.079)	(.079)	(.381)	(.381)	(.681)	(.680)		
2 x 10^2	.056	.056	.275	.275	.518	.518		
4 x 10^2	.040	.040	.196	.916	.381	.381		
7 x 10^2	.030	.030	.149	.149	.293	.293	1.	1.
10^3	.025	.025	.125	.125	.247	.247	.997	.997
2 x 10^3	.018	.018	.089	.089	.176	.176	.948	.948
4 x 10^3	.012	.012	.062	.062	.123	.123	.769	.769
7 x 10^3	.008	.008	.042	.042	.083	.083	.532	.532
10^4	.006	.006	.029	.029	.057	.057	.366	.366
2 x 10^4	.002	.002	.008	.008	.016	.016	.105	.105
4 x 10^4			.001	.001	.001	.001	.009	.009

L = 20; K2 = 2; c = 0.1

	x = 4			x = 7			x = 10			x = 20		
θ	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0
1.	(.961)	(.961)	(.959)	(1.000)	(1.000)	(1.000)						
2.	.854	.854	.853	.989	.989	.989	1.000	1.000	1.000			
7.	.562	.562	.562	.826	.826	.825	.948	.948	.947	1.000	1.000	1.000
10.	.484	.484	.484	.744	.744	.744	.895	.895	.895	.998	.998	.998
20.	.354	.354	.354	.578	.578	.578	.748	.748	.748	.956	.956	.956
40.	.251	.251	.251	.423	.423	.422	.568	.568	.568	.791	.791	.790
70.	.173	.173	.173	.293	.293	.293	.397	.397	.397	.560	.560	.560
100.	.122	.122	.122	.206	.206	.206	.279	.279	.279	.394	.394	.394
200.	.038	.038	.038	.064	.064	.064	.086	.086	.086	.122	.122	.122
400.	.004	.004	.004	.006	.006	.006	.008	.008	.008	.117	.117	.117

L = 20; K2 = 5; c = 0.1

θ	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0
.7	(.887)	(.886)	(.884)	.994	.994	.994	(1.000)	(1.000)	(1.000)			
1.	.815	.814	.812	.979	.979	.979	.999	.999	.999			
2.	.650	.649	.649	.898	.898	.897	.980	.980	.980	1.000	1.000	1.000
4.	.491	.491	.490	.752	.752	.751	.901	.901	.901	.988	.988	.988
7.	.382	.382	.382	.617	.617	.617	.787	.787	.787	.975	.975	.975
10.	.323	.323	.323	.534	.534	.534	.701	.701	.701	.926	.926	.926
20.	.225	.225	.225	.380	.380	.380	.512	.512	.512	.719	.719	.719
40.	.127	.126	.126	.214	.214	.214	.289	.289	.289	.409	.409	.409
70.	.054	.054	.054	.091	.091	.091	.124	.124	.124	.175	.175	.175
100.	.023	.023	.023	.039	.039	.039	.053	.053	.053	.075	.075	.075
200.	.001	.001	.001	.002	.002	.002	.003	.003	.003	.004	.004	.004

L = 20; K2 = 10; c = 0.1

θ	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0
.4	.865	.863	.861	.991	.991	.990	1.000	1.000	1.000			
.7	.740	.738	.737	.951	.950	.950	.995	.995	.995			
1.	.653	.652	.651	.900	.900	.899	.981	.981	.981	1.000	1.000	1.000
2.	.493	.493	.493	.755	.754	.754	.903	.903	.902	.998	.998	.998
4.	.361	.361	.361	.588	.588	.588	.759	.759	.758	.962	.962	.962
7.	.276	.276	.276	.462	.462	.461	.617	.617	.616	.847	.847	.847
10.	.227	.227	.227	.382	.382	.382	.516	.516	.515	.724	.724	.724
20.	.128	.128	.128	.217	.217	.216	.293	.293	.293	.414	.414	.414
40.	.042	.042	.042	.070	.070	.070	.095	.095	.095	.135	.135	.135
70.	.008	.008	.008	.013	.013	.013	.018	.018	.018	.025	.025	.025
100.	.001	.001	.001	.002	.002	.002	.003	.003	.003	.005	.005	.005

L = 20; K2 = 20; c = 0.1

	x = 4					x = 7					x = 10			x = 20		
Θ	y=0	y=.05	y=0.1	y=.55	y=1.0	y=0	y=.05	y=0.1	y=.55	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0
.1	(.969)	(.968)	(.965)	(.964)	(.963)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)						
.2	869	.867	.864	.862	.862	.992	.991	.991	.990	.990	1.000	1.000	1.000			
.4	.711	.710	.708	.707	.706	.936	.936	.934	.934	.934	.992	.991	.991			
.7	.575	.575	.574	.573	.573	.837	.837	.836	.836	.835	.954	.953	.953	1.000	1.000	1.000
1.	.495	.495	.491	.494	.494	.757	.757	.756	.755	.755	.904	.904	.903	.998	.998	.998
2.	.362	.362	.362	.362	.362	.590	.590	.589	.589	.589	.760	.760	.760	.963	.963	.962
4.	.258	.258	.258	.258	.258	.434	.434	.433	.433	.433	.582	.582	.581	.807	.807	.807
7.	.181	.180	.180	.180	.180	.305	.305	.305	.305	.305	.413	.412	.412	.583	.582	.582
10.	.129	.129	.129	.129	.129	.218	.218	.218	.218	.218	.295	.295	.295	.417	.417	.417
20.	.042	.042	.042	.042	.042	.071	.071	.071	.071	.071	.097	.097	.097	.137	.137	.137
40.	.005	.005	.005	.005	.005	.008	.008	.008	.008	.008	.010	.010	.010	.015	.015	.015
70.											.000	.000	.000	.001	.001	.001

L = 20; K2 = 50; c = 0.1

	x = 4					x = 7					x = 10			x = 20		
Θ	y=0	y=.05	y=0.1	y=.55	y=1.0	y=0	y=.05	y=0.1	y=.55	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0
.07	(.902)	(.899)	(.889)	(.888)	.887	.996	.996	.994	.994	.994	(1.000)	(1.000)	(1.000)			
.1	.829	.826	.818	.817	.816						.999	.999	.999			
.2	.660	.659	.654	.654	.653	.905	.904	.901	.900	.900	.983	.982	.981	1.000	1.000	1.000
.4	.498	.497	.495	.495	.494	.759	.759	.756	.756	.756	.906	.904	.904	.998	.998	.998
.7	.387	.386	.385	.385	.385	.624	.623	.622	.622	.622	.793	.792	.791	.977	.976	.976
1.	.327	.327	.326	.326	.326	.540	.539	.539	.538	.538	.707	.706	.706	.930	.930	.929
2.	.228	.228	.228	.228	.228	.385	.385	.384	.384	.384	.519	.518	.518	.728	.727	.727
4.	.130	.129	.129	.129	.129	.219	.219	.219	.219	.219	.296	.296	.296	.419	.418	.418
7.	.056	.056	.056	.056	.056	.095	.095	.095	.095	.095	.129	.128	.128	.182	.182	.182
10.	.024	.024	.024	.024	.024	.041	.041	.041	.041	.041	.056	.056	.056	.079	.079	.079
20.	.002	.002	.002	.002	.002	.003	.003	.003	.003	.003	.003	.003	.003	.005	.005	.005

L = 20; K2 = 100; c = 0.1

	x = 4					x = 7					x = 10			x = 20		
Θ	y=0	y=.05	y=0.1	y=.55	y=1.0	y=0	y=.05	y=0.1	y=.55	y=1.0	y=0	y=0.1	y=1.0	y=0	y=0.1	y=1.0
.07			.741	.740				.951	.950			.995				
.1	.667	.664	.655	.654	.654	.909	.907	.901	.900	.900	.984	.981	.981	1.000	1.000	1.000
.2	.501	.499	.495	.495	.495	.763	.761	.757	.756	.756	.908	.904	.904	.998	.998	.998
.4	.365	.364	.363	.362	.362	.593	.593	.590	.590	.590	.764	.761	.761	.964	.963	.963
.7	.278	.278	.277	.277	.277	.465	.465	.464	.464	.464	.621	.619	.619	.852	.850	.850
1.	.229	.229	.228	.228	.228	.386	.385	.385	.384	.384	.520	.519	.518	.729	.728	.727
2.	.130	.130	.130	.129	.129	.220	.219	.219	.219	.219	.297	.296	.296	.420	.419	.419
4.	.043	.043	.043	.043	.043	.072	.072	.072	.072	.072	.098	.098	.097	.138	.138	.138
7.	.008	.008	.008	.008	.008	.014	.014	.014	.014	.014	.018	.018	.018	.026	.026	.026
10.	.002	.002	.002	.002	.002	.003	.003	.003	.003	.003	.003	.003	.003	.005	.005	.005

L = 20; K2 = 2; c = 0.3

θ	x = 4 y=0	x = 4 y=.15	x = 4 y=.3	x = 4 y=.65	x = 4 y=1.0	x = 7 y=0	x = 7 y=.15	x = 7 y=.3	x = 7 y=.65	x = 7 y=1.0	x = 10 y=0	x = 10 y=.3	x = 10 y=1.0	x = 20 y=0	x = 20 y=.3	x = 20 y=1.0
1.	(.972)	(.971)	(.970	(.970)	(.970)	(1.000)	(1.000)	(1.000)	(1.000)	(1.000)						
2.	.877	.877	.876	.874	.874	.993	.993	.993	.992	.992	1.000	1.000	1.000			
4.	.724	.723	.722	.722	.721	.943	.943	.942	.942	.942	.993	.993	.993			
7.	.589	.589	.588	.588	.587	.849	.849	.849	.848	.848	.960	.960	.959	1.000	1.000	1.000
10.	.508	.508	.508	.507	.507	.771	.771	.770	.770	.770	.914	.914	.913	.999	.999	.999
20.	.373	.373	.372	.372	.372	.604	.604	.604	.604	.604	.775	.775	.774	.970	.969	.969
40.	.267	.267	.267	.267	.266	.447	.447	.447	.447	.447	.599	.599	.599	.828	.827	.827
70.	.189	.189	.189	.189	.189	.320	.320	.320	.320	.320	.433	.433	.432	.611	.611	.610
100.	.138	.138	.138	.138	.138	.233	.233	.233	.233	.233	.316	.316	.315	.446	.446	.446
200.	.048	.048	.048	.048	.048	.082	.082	.082	.082	.082	.111	.111	.111	.156	.156	.156
400.	.006	.006	.006	.066	.066	.010	.010	.010	.010	.010	.014	.014	.014	.019	.019	.019
700.											.001	.001	.001	.001	.001	.001

L=20; K2 = 5; c= .3

θ	x = 4 y=0	x = 4 y=.15	x = 4 y=.3	x = 4 y=.65	x = 4 y=1.0	x = 7 y=0	x = 7 y=.15	x = 7 y=.3	x = 7 y=.65	x = 7 y=1.0	x = 10 y=0	x = 10 y=.3	x = 10 y=1.0	x = 20 y=0	x = 20 y=.3	x = 20 y=1.0
.7	(.924)		(.917)		(.914)	(.998)		(.997)		(.997)						
1.	.861	.859	.854	.852	.851	.990	.989	.988	.988	.988	1.000	1.000	1.000			
2.	.700	.699	.696	.695	.694	.930	.929	.927	.927	.926	.990	.989	.989	1.000	1.000	1.000
4.	.534	.534	.532	.532	.532	.798	.797	.796	.795	.795	.931	.930	.930	.999	.999	.999
7.	.418	.418	.417	.417	.416	.664	.664	.663	.663	.663	.831	.830	.829	.988	.988	.988
10.	.354	.354	.354	.354	.354	.579	.579	.578	.578	.578	.749	.748	.748	.957	.956	.956
20.	.252	.251	.251	.251	.251	.423	.423	.423	.422	.422	.568	.568	.568	.791	.791	.790
40.	.154	.154	.154	.154	.154	.261	.261	.261	.261	.261	.353	.353	.353	.499	.499	.498
70.	.076	.076	.076	.076	.076	.129	.129	.129	.129	.129	.175	.175	.174	.247	.247	.247
100.	.038	.038	.038	.038	.038	.064	.064	.064	.064	.064	.086	.086	.086	.122	.122	.122
200.	.004	.004	.004	.004	.004	.006	.006	.006	.006	.006	.008	.008	.008	.012	.012	.012

L = 20; K2 = 10; c = .3

θ	x = 4 y=0	x = 4 y=.15	x = 4 y=.3	x = 4 y=.65	x = 4 y=1.0	x = 7 y=0	x = 7 y=.15	x = 7 y=.3	x = 7 y=.65	x = 7 y=1.0	x = 10 y=0	x = 10 y=.3	x = 10 y=1.0	x = 20 y=0	x = 20 y=.3	x = 20 y=1.0
.4	(.916)	(.913)	(.901)	(.898)	(.897)	(.997)	(.996)	(.995)	(.995)	(.995)	(1.000)	(1.000)	(1.000)			
.7	.803	.799	.790	.788	.787	.974	.973	.970	.969	.969	.998	.998	.998			
1.	.715	.713	.706	.704	.704	.938	.936	.932	.931	.931	.992	.990	.990			
2.	.546	.545	.542	.541	.541	.809	.808	.805	.804	.804	.938	.935	.935	1.000	.999	.999
4.	.401	.401	.400	.399	.399	.643	.642	.641	.640	.640	.811	.809	.809	.983	.982	.982
7.	.308	.308	.307	.307	.307	.512	.511	.510	.510	.510	.676	.675	.674	.905	.904	.904
10.	.257	.257	.257	.257	.257	.432	.432	.432	.431	.431	.580	.579	.579	.805	.804	.804
20.	.160	.160	.160	.160	.160	.271	.271	.270	.270	.270	.367	.366	.366	.518	.517	.517
40.	.065	.065	.065	.065	.065	.110	.110	.110	.110	.110	.149	.149	.194	.211	.210	.210
70.	.017	.017	.017	.017	.017	.029	.029	.028	.028	.028	.039	.039	.039	.055	.055	.054
100.	.004	.004	.004	.004	.004	.007	.007	.007	.007	.007	.010	.010	.010	.014	.014	.014

L = 20; K2 = 20; c = .3

θ	x = 4 y=0	x = 4 y=.15	x = 4 y=.3	x = 4 y=.65	x = 4 y=1.0	x = 7 y=0	x = 7 y=.15	x = 7 y=.3	x = 7 y=.65	x = 7 y=1.0	x = 10 y=0	x = 10 y=.15	x = 10 y=.3	x = 10 y=1.0	x = 20 y=0	x = 20 y=.15	x = 20 y=.3	x = 20 y=1.0
.1					(.972)			(.999)	(.999)	(.999)								
.2	(.931)	(.923)	(.900)	(.897)	(.896)	(.997)	(.996)	(.995)	(.994)	(.994)	(1.000)	(1.000)	(1.000)	(1.000)				
.4	(.789)	(.782)	(.763)	.761	.760	.968	.966	.959	.958	.958	(.997)	(.997)	.996	.996				
.7	.645	.641	.630	.629	.628	.892	.890	.881	.880	.880	.997	.976	.973	.973	1.000	1.000	1.000	1.000
1.	.556	.554	.547	.546	.545	.819	.816	.809	.808	.808	.943	.941	.937	.937	1.000	1.000	.999	.999
2.	.408	.407	.404	.403	.403	.651	.650	.646	.645	.645	.818	.817	.814	.813	.984	.984	.983	.983
4.	.293	.292	.291	.291	.291	.488	.487	.485	.485	.485	.648	.647	.645	.645	.879	.878	.876	.876
7.	.214	.214	.214	.213	.213	.362	.361	.360	.360	.360	.489	.488	.487	.487	.687	.687	.685	.685
10.	.164	.163	.163	.163	.163	.277	.276	.276	.275	.275	.374	.374	.373	.373	.529	.528	.527	.527
20.	.068	.068	.067	.067	.067	.115	.114	.114	.114	.114	.155	.155	.154	.154	.219	.219	.218	.218
40.	.012	.012	.012	.012	.012	.020	.020	.020	.020	.020	.027	.027	.026	.026	.038	.038	.038	.037
70.	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.003

L = 20; K2 = 50; c = .3

θ	x = 4 y=0	x = 4 y=.15	x = 4 y=.3	x = 4 y=.65	x = 4 y=1.0	x = 7 y=0	x = 7 y=.15	x = 7 y=.3	x = 7 y=.65	x = 7 y=1.0	x = 10 y=0	x = 10 y=.15	x = 10 y=.3	x = 10 y=1.0	x = 20 y=0	x = 20 y=.15	x = 20 y=.3	x = 20 y=1.0
.07			(.910)	(.908)	(.907)			(.995)	(.995)	(.995)			(1.000)	(1.000)				
.1			(.854)	(.852)	(.851)	(.992)	(.991)	(.986)	(.985)	(.985)			(1.000)	(.999)				
.2	(.770)	(.755)	(.711)	(.709)	(.709)	(.955)	(.949)	(.931)	(.930)	(.929)	(.994)	(.993)	(.989)	(.989)	(1.000)	(1.000)	(1.000)	(1.000)
.4	.578	.571	.550	.549	.549	.835	.829	.811	.810	.809	.950	.947	.937	.936	(1.000)	1.000	.999	.999
.7	.445	.442	.432	.431	.431	.697	.693	.681	.680	.680	.857	.854	.844	.843	.992	.991	.990	.989
1.	.375	.373	.367	.366	.366	.606	.604	.596	.595	.595	.776	.773	.765	.765	.821	.819	.814	.814
2.	.265	.264	.262	.261	.261	.444	.443	.439	.439	.439	.594	.593	.588	.588	.821	.819	.814	.814
4.	.166	.166	.165	.165	.165	.281	.281	.279	.279	.278	.381	.380	.377	.377	.538	.537	.533	.532
7.	.087	.086	.086	.086	.086	.146	.146	.145	.145	.145	.198	.198	.196	.196	.280	.280	.278	.278
10.	.045	.045	.045	.045	.045	.076	.076	.076	.076	.076	.103	.103	.102	.102	.146	.146	.145	.145
20.	.005	.005	.005	.005	.005	.009	.009	.009	.009	.009	.012	.012	.012	.012	.117	.017	.016	.016

L = 20; K2 = 100; c = .3

θ	x = 4 y=0	x = 4 y=.15	x = 4 y=.3	x = 4 y=.65	x = 4 y=1.0	x = 7 y=0	x = 7 y=.15	x = 7 y=.3	x = 7 y=.65	x = 7 y=1.0	x = 10 y=0	x = 10 y=.15	x = 10 y=.3	x = 10 y=1.0	x = 20 y=0	x = 20 y=.15	x = 20 y=.3	x = 20 y=1.0
.07			(.780)	(.777)	(.777)	(.983)	(.979)	(.961)	(.961)	(.961)			(.996)	(.996)				
.1			(.706)	(.704)	(.703)	(.965)	(.956)	(.924)	(.924)	(.923)			(.987)	(.987)			(1.000)	(1.000)
.2	(.614)	(.598)	(.551)	(.550)	(.550)	(.857)	(.845)	.808	.807	.805	(.958)	(.952)	(.933)	.933	(1.000)	(1.000)	(.999)	(.999)
.4	.432	.426	.408	.408	.408	.679	.672	.650	.649	.649	.840	.834	.815	.815	.987	.986	.981	.981
.7	.324	.322	.314	.314	.314	.535	.531	.519	.519	.519	.700	.696	.684	.683	.921	.918	.908	.908
1.	.269	.267	.262	.262	.262	.450	.447	.440	.440	.440	.602	.599	.589	.589	.828	.825	.814	.814
2.	.169	.168	.165	.165	.165	.284	.284	.280	.280	.280	.386	.384	.378	.378	.545	.542	.534	.534
4.	.071	.071	.070	.070	.070	.120	.119	.118	.118	.118	.162	.162	.159	.159	.230	.229	.225	.225
7.	.019	.018	.019	.019	.019	.033	.033	.032	.032	.032	.044	.044	.044	.044	.063	.063	.062	.062
10.	.005	.005	.005	.005	.005	.009	.009	.009	.009	.009	.012	.012	.012	.012	.017	.017	.017	.017

θ	x = 4 y=0	x = 4 y=.35	x = 4 y=.7	x = 4 y=1.0	x = 7 y=0	x = 7 y=.35	x = 7 y=.7	x = 7 y=1.0	x = 10 y=0	x = 10 y=.7	x = 10 y=1.0	x = 20 y=0	x = 20 y=.7	x = 20 y=1.0
L = 20; K2 = 2; c = .7														
1.	(.987)	(.987)	(.985)	(.984)	(.999)	(.999)	(1.000)	1.000						
2.	(.923)	(.922)	(.919)	(.918)	(.998)	(.988)	(.998)	.998	(1.000)	(1.000)	(1.000)			
4.	.787	.787	.784	.783	.971	.970	.969	.969	.998	.998	.998			
7.	.653	.652	.651	.651	.900	.900	.899	.898	.981	.981	.981			
10.	.568	.568	.567	.567	.831	.831	.830	.830	.950	.950	.950	1.000	1.000	1.000
20.	.421	.421	.421	.421	.669	.669	.668	.668	.835	.834	.834	.989	.989	.989
40.	.305	.305	.305	.305	.507	.506	.506	.506	.670	.670	.670	.900	.900	.900
70.	.227	.227	.227	.227	.382	.382	.382	.382	.516	.515	.515	.724	.723	.723
100.	.177	.177	.177	.177	.299	.299	.299	.299	.404	.404	.404	.571	.571	.571
200.	.079	.079	.079	.079	.134	.134	.134	.134	.181	.181	.181	.256	.256	.256
400.	.016	.016	.016	.016	.027	.027	.027	.027	.036	.036	.036	.052	.052	.052
700.	.001	.001	.001	.001	.002	.002	.002	.002	.003	.003	.003	.005	.005	.005
L = 20; K2 = 5; c = .7														
.7		(.997)	(.968)	(.967)	(.999)	(.999)	(.999)	(.999)						
1.	(.950)	(.947)	(.935)	(.934)	(.999)	(.999)	(.998)	(.998)	(1.000)	(1.000)	(1.000)			
2.	.833	.829	.818	.817	.982	.982	.979	.978	.999	.999	.999			
4.	.666	.664	.658	.658	.908	.907	.902	.902	.983	.981	.981	1.000	1.000	1.000
7.	.533		.528	.528	.796		.792	.791	.930	.927	.927	.999	.999	.999
10.	.456	.455	.453	.453	.711	.711	.708	.708	.870	.867	.867	.995	.994	.994
20.	.331	.331	.330	.330	.545	.545	.544	.544	.713	.712	.712	.934	.933	.933
40.	.232	.231	.231	.231	.390	.390	.390	.390	.526	.525	.525	.738	.736	.736
70.	.152	.152	.152	.152	.258	.258	.257	.257	.349	.348	.348	.493	.492	.492
100.	.101	.101	.101	.101	.172	.171	.171	.171	.232	.232	.232	.328	.328	.328
200.	.026	.026	.026	.026	.044	.044	.044	.044	.060	.060	.060	.085	.084	.084
400.	.002	.002	.002	.002	.003	.003	.003	.003	.004	.004	.004	.006	.006	.006
L = 20; K2 = 10; c = .7														
.4				(.958)				(.998)						
.7	(.937)	(.929)	(.904)	(.902)	(.997)	(.996)	(.994)	(.994)	(1.000)	(1.000)	1.000			
1.	(.880)	(.872)	(.847)	(.845)	(.990)	(.988)	(.983)	(.983)	(.999)	(.999)	.999			
2.	.721	.715	.698	.698	.937	.934	.925	.924	.990	.987	.987	1.000	1.000	1.000
4.	.548	.545	.537	.537	.809	.807	.799	.799	.937	.931	.931	.999	.999	.999
7.	.427	.425	.422	.421	.675	.673	.669	.668	.839	.834	.834	.989	.988	.988
10.	.361	.360	.358	.358	.588	.587	.584	.584	.758	.754	.753	.960	.960	.958
20.	.256	.256	.255	.255	.430	.430	.428	.428	.577	.575	.575	.801	.798	.798
40.	.159	.158	.158	.158	.268	.268	.267	.267	.363	.361	.361	.513	.511	.511
70.	.080	.080	.080	.080	.135	.135	.135	.135	.183	.182	.182	.259	.258	.258
100.	.040	.040	.040	.040	.068	.068	.068	.068	.092	.092	.092	.131	.131	.130
200.	.004	.004	.004	.004	.007	.007	.007	.007	.009	.009	.009	.013	.013	.013

L = 20; K2 = 20; c = .7

θ	x = 4 y=0	x = 4 y=.35	x = 4 y=.7	x = 4 y=1.0	x = 7 y=0	x = 7 y=.35	x = 7 y=.7	x = 7 y=1.0	x = 10 y=0	x = 10 y=.35	x = 10 v=.7	x = 10 y=1.0	x = 20 y=0	x = 20 y=.35	x = 20 y=.7	x = 20 y=1.0
.2							(.997)	(.997)								
.4			(.881)	(.879)	(.994)	(.993)	(.988)	(.988)			(.999)	(.999)				
.7	(.853)	(.837)	(.791)	(.790)	(.980)	(.975)	(.963)	(.962)	(.998)	(.997)	(.995)	(.995)				
1.	(.769)	(.756)	(.718)	(.717)	(.952)	(.947)	.930	.929	(.993)	(.991)	(.987)	(.987)	(1.000)	(1.000)	(1.000)	(1.000)
2.	.586	.560	.560	.560	.840	.835	.818	.818	.951	.948	.939	.939	.999	.999	.999	.999
4.	.426		.416	.416	.673		.660	.660	.836		.825	.825	.987		.984	.984
7.	.325	.324	.321	.320	.536	.535	.429	.529	.703	.701	.695	.695	.924	.922	.918	.918
10.	.271	.271	.269	.269	.455	.454	.450	.450	.608	.060	.602	.602	.835	.834	.829	.829
20.	.174	.173	.172	.172	.294	.293	.291	.291	.397	.396	.394	.394	.561	.560	.556	.556
40.	.076	.076	.075	.075	.129	.128	.128	.128	.174	.174	.173	.173	.246	.246	.244	.244
70.	.022	.022	.022	.022	.037	.037	.037	.037	.051	.051	.050	.050	.072	.071	.071	.071
100.	.006	.006	.006	.006	.011	.011	.011	.011	.015	.015	.015	.015	.021	.021	.021	.021

L = 20; K2 = 50; c = .7

θ	x = 4 y=0	x = 4 y=.35	x = 4 y=.7	x = 4 y=1.0	x = 7 y=0	x = 7 y=.35	x = 7 y=.7	x = 7 y=1.0	x = 10 y=0	x = 10 y=.35	x = 10 v=.7	x = 10 y=1.0	x = 20 y=0	x = 20 y=.35	x = 20 y=.7	x = 20 y=1.0
.07								(.997)								
.1							(.992)	(.992)								
.2			(.820)	(.819)		(.983)	(.969)	(.969)			(.996)	(.996)				
.4			(.712)	(.711)	(.964)	(.953)	(.917)	(.916)			(.981)	(.981)			(1.000)	(1.000)
.7	(.691)	(.668)	(.601)	(.601)	(.898)	(.884)	(.843)	.842	(.971)	(.965)	(.947)	(.947)	(1.000)	(.999)	(.999)	(.999)
1.	(.588)	(.572)	(.526)	.526	(.828)	(.815)	.777	.777	(.938)	(.931)	.909	.909	(.998)	(.998)	(.996)	(.996)
2.	.413	.407	.389	.389	.652	.645	.623	.622	.813	.807	.787	.787	.975	.973	.966	.966
4.	.287	.285	.278	.278	.478	.474	.464	.464	.634	.630	.618	.618	.859	.855	.843	.843
7.	.204		.200	.200	.346		.338	.338	.467		.456	.456	.657		.642	.642
10.	.153	.152	.149	.149	.258	.257	.252	.252	.349	.347	.342	.342	.494	.491	.483	.483
20.	.058	.058	.057	.057	.099	.098	.097	.097	.134	.133	.131	.131	.189	.188	.185	.185
40.	.009	.009	.008	.008	.015	.014	.014	.014	.020	.020	.019	.019	.028	.028	.027	.027
70.					.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002

L = 20; K2 = 100; c = .7

θ	x = 4 y=0	x = 4 y=.35	x = 4 y=.7	x = 4 y=1.0	x = 7 y=0	x = 7 y=.35	x = 7 y=.7	x = 7 y=1.0	x = 10 y=0	x = 10 y=.35	x = 10 v=.7	x = 10 y=1.0	x = 20 y=0	x = 20 y=.35	x = 20 y=.7	x = 20 y=1.0
.07			(.839)	(.838)			(.978)	(.977)			(.998)	(.998)				
.1			(.787)	(.786)			(.957)	(.956)			(.994)	(.994)				
.2			(.683)	(.682)		(.953)	(.894)	(.894)			(.971)	(.971)			(1.000)	(1.000)
.4			(.567)	(.567)	(.901)	(.876)	(.804)	(.803)			(.921)	(.921)			(.997)	(.997)
.7	(.558)	(.533)	(.462)	(.462)	(.784)	(.763)	.702	.702	(.904)	(.890)	(.850)	(.849)	(.993)	(.990)	(.982)	(.982)
1.	(.455)	(.440)	.396	.396	(.689)	(.673)	.626	.626	(.836	(.823)	.785	.785	(.976)	(.972)	.959	.959
2.	.303	.298	.283	.282	.500	.492	.470	.470	.656	.647	.622	.622	.871	.863	.840	.840
4.	.193	.191	.184	.184	.326	.322	.311	.311	.440	.435	.420	.420	.619	.612	.592	.592
7.	.110	.109	.105	.105	.186	.184	.178	.178	.251	.249	.240	.240	.356	.352	.340	.340
10.	.063	.062	.060	.060	.106	.105	.102	.102	.144	.143	.138	.138	.204	.202	.195	.195
20.	.010	.010	.009	.009	.017	.017	.016	.016	.023	.022	.022	.022	.032	.032	.030	.030
40.									.001	.001	.001	.001	.001	.001	.001	.001

L = 20; K2 = 2; c = 0.9

	x = 4				x = 7				x = 10			x = 20		
Θ	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.9	y=1.0	y=0	y=.9	y=1.0
2.	(.944)	(.944)	(.942)	(.942)	(.999)	(.999)	(.999)	(.999)	(1.000)	(1.000)	(1.000)			
4.	.823	.823	.821	.821	.982	.982	.981	.981	.999	.999	.999			
7.	.693		.691	.691	.926		.925	.925	.989	.989	.989			
10.	.607	.606	.606	.606	.865	.865	.864	.864	.967	.967	.967	1.000	1.000	1.000
20.	.454	.454	.453	.453	.709	.709	.708	.708	.868	.868	.868	.995	.995	.995
40.	.330	.330	.330	.330	.544	.544	.544	.544	.712	.712	.712	.934	.934	.934
70.	.250	.249	.249	.249	.420	.420	.420	.420	.564	.564	.564	.786	.786	.786
100.	.200	.200	.200	.200	.339	.338	.338	.338	.458	.457	.457	.645	.645	.645
200.	.101	.101	.101	.101	.171	.171	.171	.171	.232	.232	.232	.328	.328	.328
400.	.026	.026	.026	.206	.044	.044	.044	.044	.060	.060	.060	.084	.084	.084
700.	.003	.003	.003	.003	.006	.006	.006	.006	.008	.008	.008	.011	.011	.011
1000.					.001	.001	.001	.001	.001	.001	.001	.001	.001	.001

L = 20; K2 = 5; c = .9

	x = 4				x = 7				x = 10			x = 20		
Θ	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.9	y=1.0	y=0	y=.9	y=1.0
1.	(.984)	(.982)	(.978)	(.978)	(1.000)	(1.000)	(1.000)	(1.000)						
2.	(.913)	(.911)	(.904)	.903	(.997)	.997	.996	.996	(1.000)	(1.000)	(1.000)			
4.	.772	.770	.765	.765	.964	.964	.962	.962	.997	.997	.997			
7.	.636		.632	.632	.887		.885	.884	.976	.975	.975			
10.	.552	.551	.549	.549	.815	.815	.813	.813	.942	.941	.940	1.000	1.000	1.000
20.	.408	.407	.407	.407	.651	.651	.650	.650	.819	.818	.818	.985	.985	.985
40.	.294	.294	.294	.294	.490	.490	.490	.490	.651	.650	.650	.883	.882	.882
70.	.217	.217	.216	.216	.366	.365	.365	.365	.493	.493	.493	.694	.693	.693
100.	.166	.166	.166	.166	.281	.281	.281	.281	.380	.380	.380	.537	.536	.536
200.	.070	.070	.070	.070	.118	.118	.118	.118	.160	.160	.160	.227	.226	.226
400.	.012	.012	.012	.012	.021	.021	.021	.021	.028	.028	.028	.040	.040	.040
700.	.001	.001	.001	.001	.002	.002	.002	.002	.002	.002	.002	.003	.003	.003

L = 20; K2 = 10; c = .9

	x = 4				x = 7				x = 10			x = 20		
Θ	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.9	y=1.0	y=0	y=.9	y=1.0
.7			(.973)	(.973)			(.999)	(.999)						
1.	(.964)	(.960)	(.946)	(.946)	(.999)	(.999)	(.998)	(.998)	(1.000)	(1.000)	(1.000)			
2.	(.863)	(.858)	(.844)	(.844)	.989	.988	.985	.985	(1.000)	(.999)	(.999)			
4.	.703	.700	.691	.691	.930	.928	.923	.923	.990	.988	.988			
7.	.567	.565	.560	.560	.829	.827	.823	.823	.948	.945	.945	1.000	1.000	1.000
10.	.487	.486	.483	.483	.747	.746	.742	.742	.897	.893	.893	.997	.997	.997
20.	.355	.354	.353	.353	.579	.579	.577	.577	.749	.747	.747	.957	.955	.955
40.	.252	.252	.251	.251	.423	.423	.422	.422	.569	.567	.567	.791	.790	.790
70.	.174	.174	.173	.173	.294	.293	.293	.293	.397	.396	.396	.561	.560	.560
100.	.122	.122	.122	.122	.206	.206	.206	.206	.279	.279	.279	.395	.394	.394
200.	.038	.038	.038	.038	.064	.064	.064	.064	.087	.086	.086	.122	.122	.122
400.	.004	.004	.004	.004	.006	.006	.006	.006	.008	.008	.008	.012	.012	.012

L = 20; K2 = 20; c = .9

θ	x = 4				x = 7				x = 10				x = 20			
	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0
.4							(.998)	(.998)								
.7			(.924)	(.924)	(.998)	(.997)	(.995)	(.995)			(1.000)	(1.000)				
1.	(.920)	(.910)	(.880)	(.880)	(.995)	(.993)	(.989)	(.989)	(1.000)	(1.000)	(.999)	(.999)				
2.	(.781	.773	.750	.750	.962	.959	.950	.950	.996	.995	.993	.993				
4.	.607	.603	.591	.591	.861	.858	.848	.848	.963	.961	.957	.957	1.000	1.000	1.000	1.000
7.	.477		.469	.469	.735		.726	.726	.887		.880	.880	.996		.995	.995
10.	.405	.404	.400	.400	.647	.646	.641	.641	.814	.813	.809	.809	.982	.982	.980	.980
20.	.291	.290	.289	.289	.486	.484	.482	.482	.644	.643	.640	.640	.874	.874	.871	.871
40.	.194	.194	.193	.193	.327	.327	.326	.325	.442	.442	.440	.440	.624	.623	.621	.621
70.	.113	.113	.112	.112	.191	.191	.190	.190	.258	.258	.257	.257	.366	.365	.364	.364
100.	.006	.006	.006	.006	.112	.112	111	.111	.151	.151	.151	.151	.214	.214	.213	.213
200.	.011	.011	.011	.011	.019	.019	.019	.019	.025	.025	.025	.025	.036	.036	.036	.036
400.					.001	.001	.001	.001	.001	.001	.001	.001				

L = 20; K2 = 50; c = .9

θ	x = 4				x = 7				x = 10				x = 20			
	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0
.1			(.996)	(.996)												
.2			(.994)	(.994)												
.4			(.873)	(.872)	(.992)	(.990)	(.983)	(.983)			(.998)	(.998)				
.7	(.883)	(.861)	(.798)	(.798)	(.982)	(.977)	(.960)	(.960)	(.997)	(.997)	(.994)	(.994)				
1.	(.810)	(.791)	(.735)	(.735)	(.961)	(.954)	(.933)	(.933)	(.994)	(.992)	(.986)	(.986)	(1.000)	(1.000)	(1.000)	(1.000)
2.	.630	.619	.587	.587	.869	.861	.837	.837	.962	.958	.947	.947	1.000	.999	.999	.999
4.	.458	.454	.440	.440	.710	.704	.689	.689	.864	.860	.848	.848	.991	.990	.987	.987
7.	.349	.347	.341	.341	.570	.567	.558	.558	.737	.734	.725	.725	.944	.942	.937	.937
10.	.292	.290	.287	.287	.485	.483	.478	.478	.644	.642	.635	.635	.871	.869	.863	.863
20.	.193	.192	.191	.190	.325	.324	.321	.321	.439	.438	.434	.434	.620	.618,	.612	.612
40.	.093	.093	.092	.092	.157	.157	.155	.155	.213	.212	.210	.210	.301	.300	.297	.297
70.	.031	.031	.031	.031	.053	.053	.052	.052	.072	.072	.071	.071	.102	.101	.100	.100
100.	.011	.011	.010	.010	.018	.018	.018	.018	.024	.024	.024	.024	.034	.034	.034	.034
200.									.001	.001	.001	.001	.001	.001	.001	.001

L = 20; K2 = 100; c = .9

θ	x = 4				x = 7				x = 10				x = 20			
	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0	y=0	y=.45	y=.9	y=1.0
.07							(.994)	(.994)								
.1							(.989)	(.989)								
.2			(.838)	(.838)			(.970)	(.970)			(.995)	(.995)				
.4			(.757)	(.757)	(.978)	(.968)	(.935)	(.935)			(.985)	(.985)			(1.000)	(1.000)
.7	(.791)	(.759)	(.668)	(.668)	(.941)	(.927)	(.884)	(.884)	(.985)	(.980)	(.965)	(.965)	(1.000)	(1.000)	(.999)	(.999)
1.	(.697)	(.671)	(.600)	(.600)	(.895)	(.880)	.836	.836	(.968)	(.961)	(.941)	(.941)	(.999)	(.999)	(.998)	(.998)
2.	.503	.491	.458	.458	.749	.738	.704	.704	.886	.878	.855	.855	.991	.990	.985	.985
4.	.350	.346	.333	.333	.569	.563	.546	.546	.733	.727	.710	.710	.935	.931	.920	.920
7.	.258	.256	.251	.251	.433	.430	.421	.421	.579	.575	.564	.564	.798	.793	.780	.780
10.	.206	.205	.201	.201	.348	.346	.339	.339	.470	.467	.458	.458	.660	.656	.644	.644
20.	.105	.105	.103	.103	.178	.177	.174	.174	.241	.240	.235	.235	.341	.339	.333	.333
40.	.028	.028	.027	.027	.047	.047	.046	.046	.064	.064	.062	.062	.091	.090	.088	.088
70.	.004	.004	.004	.004	.006	.006	.006	.006	.009	.009	.009	.009	.012	.012	.012	.012
100.	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001	.002	.002	.002	.002

APPENDIX K

DIMENSIONLESS PRESSURE DISTRIBUTION, $P(r,y,\theta)$, FOR TWO-LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Tables valid for all $r_b/H > 10.0$, for <u>all</u> values of y at any r, θ

Calculated from:

$$P(r,y,\theta) = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} C_{mn} e^{-a^2\theta} Y_{mn} \right]_m F(b)U(br) \qquad \text{(V-188)}$$

Eigenvalues b from: Appendix B

Eigenvalues a from: Appendix E

Truncation Test: $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$ for m series

Only one term used in n series

Estimated Accuracy: ± 0.001

Definition of Dimensionless Variables

$$r = \frac{r_a}{r_b} \qquad \text{(V-135)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(V-136)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-137)}$$

$$\theta = \frac{k_1 t_a}{\mu \phi_1 c r_b^2} \quad \text{(for liquid flow)}$$

$$= \frac{k_1 t_a}{\rho_1 C_{p1} r_b^2} \quad \text{(for heat conduction)} \qquad \text{(V-140)}$$

All values values presented are for $K_2 = K_2^* = \frac{k_2}{k_1}$.

R = 5; K2 = 2; c = .5

θ	r=2	r=3	r=4	r=5
0.5	.698	.939	.993	.999
0.6	.666	.918	.987	.997
0.7	.641	.899	.980	.994
0.8	.619	.881	.972	.990
1	.584	.849	.954	.979
2	.486	.733	.855	.890
3	.426	.648	.760	.793
4	.378	.574	.674	.704
5	.335	.509	.598	.625
6	.297	.452	.531	.555
7	.264	.401	.471	.492
8	.234	.356	.418	.437
10	.184	.280	.329	.344
20	.056	.085	.100	.104
30	.017	.026	.030	.031
40	.005	.008	.009	.010
50	.002	.002	.003	.003
60	.000	.001	.001	.001
70	.000	.000	.000	.000

R = 5; K2 = 5; c = .5

θ	r=2	r=3	r=4	r=5
0.2	.737	.960	.997	1.000
0.3	.666	.918	.987	.997
0.4	.619	.881	.972	.990
0.5	.584	.849	.954	.979
0.6	.557	.821	.935	.965
0.7	.535	.796	.915	.948
0.8	.895	.773	.895	.929
1	.486	.733	.855	.890
2	.378	.574	.674	.704
3	.297	.452	.531	.555
4	.234	.356	.418	.437
5	.184	.280	.329	.344
6	.145	.221	.259	.271
7	.114	.174	.204	.213
8	.090	.137	.161	.168
10	.056	.085	.100	.104
20	.005	.008	.009	.010
30	.000	.001	.001	.001
40	.000	.000	.000	.000

R = 5; K2 = 10; c = .5

θ	r=2	r=3	r=4	r=5
0.1	.753	.966	.998	1.000
0.2	.633	.893	.977	.993
0.3	.570	.835	.945	.972
0.4	.528	.788	.904	.942
0.5	.498	.749	.872	.907
0.6	.472	.715	.835	.871
0.7	.450	.683	.800	.835
0.8	.430	.653	.766	.799
1	.393	.598	.702	.733
2	.253	.385	.453	.473
3	.163	249	.292	.305
4	.105	.160	.188	.197
5	.068	.103	.121	.127
6	.044	.067	.078	.082
7	.028	.043	.051	.053
8	.018	.028	.033	.034
10	.008	.012	.014	.014
20	.000	.000	.000	.000

R = 5; K2 = 20; c = .5

θ	r=2	r=3	r=4	r=5
.05	.761	.970	.998	1.000
.06	.729	.956	.996	1.000
.07	.702	.941	.993	.999
.08	.678	.927	.989	.998
.1	.641	.899	.980	.994
.2	.535	.796	.915	.948
.3	.479	.724	.845	.881
.4	.437	.663	.778	.812
.5	.401	.610	.716	.748
.6	.369	.561	.658	.688
.7	.339	.515	.605	.632
.8	.312	.474	.557	.582
1	.264	.401	.471	.492
2	.114	.174	.204	.213
3	.049	.075	.088	.092
4	.021	.033	.038	.040
5	.009	.014	.017	.017
6	.004	.006	.007	.007
7	.002	.003	.003	.003
8	.001	.001	.001	.001
10	.000	.000	.000	.000

R = 5; K2 = 50; c = .5

θ	r=2	r=3	r=4	r=5
.02	.767	-	-	-
.03	.695	.937	.992	.999
.04	.646	.903	.981	.995
.05	.609	.873	.968	.988
.06	.581	.846	.952	.978
.07	.558	.822	.936	.965
.08	.539	.801	.919	.951
.1	.508	.763	.885	.920
.2	.406	.617	.724	.757
.3	.331	.503	.591	.618
.4	.270	.411	.482	.504
.5	.220	.335	.394	.411
.6	.180	.274	.321	.336
.7	.147	.223	.262	.274
.8	.120	.182	.214	.223
1	.080	.121	.142	.149
2	.010	.016	.019	.019
3	.001	.002	.002	.003
4	.000	.000	.000	.000

R = 5; K2 = 100; c = .5

θ	r=2	r=3	r=4	r=5
.01	.769	-	-	-
.02	.647	.904	.982	.995
.03	.583	.848	.953	.979
.04	.541	.803	.921	.952
.05	.509	.765	.887	.921
.06	.484	.731	.853	.889
.07	.463	.701	.820	.855
.08	.443	.672	.788	.822
.1	.408	.620	.727	.760
.2	.272	.414	.486	.508
.3	.182	.277	325	.340
.4	.122	.185	.217	.227
.5	.081	.124	.145	.152
.6	.054	.083	097	.102
.7	.036	.055	.065	.068
.8	.024	.027	.043	.045
1	.011	.017	.019	.020
2	.000	.000	.000	.000

R = 10; K2 = 2; c = .5

θ	r=2	r=4	r=7	r=10
2	.489	.881	.994	1.000
3	.443	.827	.981	.998
4	.414	.786	.965	.993
5	.393	.755	.948	.985
6	.377	.730	.931	.974
7	.365	.709	.914	.961
8	.354	.691	.897	.947
10	.337	.660	.865	.917
20	.277	.545	.721	.768
30	.231	.454	.601	.640
40	.192	.378	.500	.533
50	.160	.315	.417	.444
60	.134	.263	.347	.370
70	.111	.219	.290	.308
80	.093	.182	.241	.257
100	.064	.127	.168	.178
200	.010	.020	.027	.029
300	.002	.003	.004	.005
400	.000	.001	.001	.001
500	.000	000	.000	.000

R = 10; K2 = 5; c = .5

θ	r=2	r=4	r=7	r=10
1	.489	.881	.994	1.000
2	.414	.786	.965	.993
3	.377	.730	.931	.974
4	.354	.691	.987	.947
5	.337	.660	.865	.917
6	.232	.633	.834	.886
7	.310	.609	.804	.856
8	.298	.587	.775	.835
10	.277	.545	.721	.768
20	.192	.378	.500	.533
30	.134	.263	.347	.370
40	.093	.182	.241	.257
50	.064	.127	.168	.178
60	.045	.088	.116	.124
70	.031	.061	.081	.086
80	.022	.042	.056	.060
100	.010	.020	.027	.029
200	.000	.001	.001	.001
300	.000	.000	.000	.000

R = 10; K2 = 10; c = .5

θ	r=2	r=4	r=7	r=10
.5	-	.892	.996	1.000
.6	.478	.869	.992	1.000
.7	.460	.848	.988	.999
.8	.445	.830	.982	.998
1	.422	.799	.971	.995
2	.361	.703	.908	.957
3	.330	.646	.849	.902
4	.306	.602	.794	.845
5	.286	.562	.743	.791
6	.267	.525	.695	.740
7	.250	.491	.650	.692
8	.234	.460	.608	.648
10	.204	.402	.532	.566
20	.105	.206	.272	.290
30	.054	.106	.140	.149
40	.027	.054	.072	.076
50	.014	.028	.037	.039
60	.007	.014	.019	.020
70	.004	.007	.010	.010
80	.002	.004	.005	.005
100	.000	.001	.001	.001
200	.000	.000	.000	.000

R = 10; K2 = 20; c = .5

θ	r=2	r=4	r=7	r=10
.2	-	-	.999	-
.3	.483	.875	.993	1.000
.4	.450	.836	.984	.999
.5	.427	.805	.973	.996
.6	.409	.780	.962	.992
.7	.395	.758	.950	.986
.8	.383	.740	.938	.979
1	.365	.709	.914	.961
2	.310	.609	.804	.856
3	.272	.535	.708	.754
4	.239	.471	.623	.663
5	.211	.414	.548	.584
6	.186	.365	.482	.514
7	.163	.321	.425	.452
8	.144	.283	.374	.398
10	.111	.219	.290	.308
20	.031	.061	.081	.086
30	.009	.017	.023	.024
40	.002	.005	.006	.007
50	.001	.001	.002	.002
60	.000	.000	.000	.001
70			.000	.000

R = 10; K2 = 50; c = .5

θ	r=2	r=4	r=7	r=10
.1	-	-	.997	1.000
.2	.430	.809	.975	.997
.3	.391	.753	.946	.984
.4	.367	.713	.917	.964
.5	.349	.683	.889	.940
.6	.335	.657	.862	.914
.7	.323	.635	.835	.888
.8	.313	.614	.810	.862
1	.293	.576	.761	.811
2	.215	.422	.558	.595
3	.157	.310	.409	.436
4	.115	.227	.300	.320
5	.085	.167	.220	.235
6	.062	.122	.162	.172
7	.046	.090	.118	.126
8	.033	.066	.087	.093
10	.018	.035	.047	.050
20	.001	.002	.002	.002
30	.000	.000	.000	.000

R = 10; K2 = 100; c = .5

θ	r=2	r=4	r=7	r=10
.05	-	-	.997	1.000
.06	.488	.880	.994	1.000
.07	.470	.860	.990	.999
.08	.455	.842	.986	.999
.1	.431	.811	.976	.997
.2	.368	.714	.918	.965
.3	.336	.658	.863	.916
.4	.313	.615	.812	.864
.5	.294	.578	.764	.813
.6	.276	543	.718	.765
.7	.260	.511	.675	.719
.8	.244	.480	.635	.677
1	.216	.425	.562	.598
2	.117	.230	.304	.324
3	.063	.124	.164	.175
4	.034	.067	.089	.095
5	.019	.036	.048	.051
6	.010	.020	.026	.028
7	.005	.011	.014	.015
8	.003	.006	.008	.008
10	.001	.002	.002	.002
20	.000	.000	.000	.000

R = 20; K2 =2; c = .5

θ	r=4	r=7	r=10	r=20
10	.664	.880	.963	1.000
20	.585	.799	.906	.992
30	.545	.751	.863	.973
40	.518	.717	.829	.948
50	.497	.690	.800	.920
60	.480	.666	.773	.892
70	.464	.644	.748	.864
80	.448	.623	.724	.836
100	.420	.584	.678	.784
200	.304	.422	.490	.567
300	.220	.305	.354	.410
400	.159	.220	.256	.296
500	.115	.159	.185	.214
600	.083	.115	.134	.155
700	.060	.083	.097	.112
800	.043	.060	.070	.081
1000	.023	.031	.037	.042
2000	.001	.001	.001	.002

R = 20; K2 = 5; c = .5

θ	r=4	r=7	r=10	r=20
5	.664	.880	.963	1.000
6	.642	.859	.951	.999
7	.624	.841	.938	.998
8	.609	.825	.927	.997
10	.585	.799	.906	.992
20	.518	.717	.829	.948
30	.480	.666	.773	.892
40	.448	.623	.724	.836
50	.420	.584	.678	.784
60	.394	.547	.636	.735
70	.369	.512	.596	.689
80	.346	.480	.558	.645
100	.304	.422	.490	.567
200	.159	.220	.256	.296
300	.083	.115	.134	.155
400	.043	.060	.070	.081
500	.023	.031	.037	.042
600	.012	.016	.019	.022
700	.006	.009	.010	.012
800	.003	.004	.005	.006
1000	.001	.001	.001	.002
2000	.000	.000	.000	.000

R = 20; K2 = 10; c = .5

θ	r=4	r=7	r=10	r=20
2	-	-	.980	1.000
3	.652	.869	.957	.999
4	.619	.835	.934	.998
5	.594	.809	.914	.994
6	.575	.787	.896	.989
7	.560	.769	.880	.982
8	.547	.754	.866	.974
10	.526	.727	.840	.956
20	.458	.637	.740	.855
30	.407	.565	.657	.759
40	.361	.501	.583	.674
50	.321	.445	.518	.598
60	.285	.395	.459	.531
70	.253	.351	.408	.472
80	.224	.312	.362	.419
100	.177	.246	.286	.330
200	.054	.075	.087	.100
300	.016	.023	.026	.031
400	.005	.007	.008	.009
500	.002	.002	.002	.003
600	.000	.001	.001	.001
700	.000	.000	.001	.000

R = 20; K2 = 20; c = .5

θ	r=4	r=7	r=10	r=20
1	-	-	.982	-
2	.624	.841	.938	.998
3	.580	.793	.901	.991
4	.551	.759	.871	.977
5	.530	.733	.845	.961
6	.514	.711	.823	.942
7	.499	.692	.803	.923
8	.487	.675	.784	.903
10	.464	.644	.748	.864
20	.369	.512	.596	.689
30	.294	.408	.475	.549
40	.234	.325	.378	.437
50	.187	.259	.301	.348
60	.149	.207	.240	.278
70	.119	.165	.191	.221
80	.094	.131	.152	.176
100	.060	.083	.097	.112
200	.006	.009	.010	.012
300	.001	.001	.001	.001
400	.000	.000	.000	.000

R = 20; K2 = 50; c = .5

θ	r=4	r=7	r=10	r=20
.5	.685	.898	.973	1.000
.6	.662	.878	.962	1.000
.7	.643	.860	.951	.999
.8	.627	.844	.941	.998
1	.602	.818	.921	.996
2	.533	.736	.849	.963
3	.496	.687	.797	.917
4	.467	.648	.753	.869
5	.441	.613	.712	.823
6	.417	.580	.674	.779
7	.395	.548	.638	.737
8	.374	.519	.603	.698
10	.335	.465	.540	.625
20	.193	.268	.311	.360
30	.111	.154	.179	.207
40	.064	.089	.103	.119
50	.037	.051	.060	.069
60	.021	.029	.034	.040
70	.012	.017	.020	.023
80	.007	.010	.011	.013
100	.002	.003	.004	.004
200	.000	.000	.000	.000

R = 20; K2 = 100; c = .5

θ	r=4	r=7	r=10	r=20
.2	-	-	.984	-
.3	.663	.879	.963	1.000
.4	.629	.846	.942	.998
.5	.603	.819	.922	.996
.6	.584	.797	.905	.992
.7	.568	.779	.889	.986
.8	.555	.763	.875	.980
1	.534	.737	.850	.964
2	.468	.650	.755	.871
3	.419	.582	.676	.781
4	.375	.521	.606	.701
5	.336	.467	.543	.628
6	.302	.419	.487	.563
7	.270	.376	.437	.505
8	.242	.337	.392	.453
10	.195	.271	.315	.364
20	.065	.091	.106	.122
30	.022	.030	.035	.041
40	.007	.010	.012	.014
50	.002	.003	.004	.005
60	.001	.001	.001	.002
70	.000	.000	.000	.000

R = 50; K2 = 2; c = 0.5

θ	r=7	r=10	r=20	r=50
100	.627	.736	.915	.992
200	.569	.671	.853	.980
300	.538	.635	.813	.950
400	.515	.608	.780	.917
500	.495	.585	.751	.884
600	.477	.563	.723	.852
700	.459	.542	.696	.820
800	.442	.522	.671	.790
1000	.410	.485	.622	.733
2000	.282	.333	.428	.505
3000	.194	.229	.295	.347
4000	.134	.158	.203	.239
5000	.092	.109	.139	.164
6000	.063	.075	.096	.113
7000	.044	.051	.066	.078
8000	.030	.035	.045	.054
10000	.014	.017	.022	.025

R = 50; K2 = 5; c = 0.5

θ	r=7	r=10	r=20	r=50
40	.648	.759	.933	.999
50	.627	.736	.915	.998
60	.611	.718	.900	.996
70	.598	.704	.886	.993
80	.587	.691	.874	.989
100	.569	.671	.853	.980
200	.515	.608	.780	.917
300	.477	.563	.723	.852
400	.442	.522	.671	.790
500	.410	.485	.622	.733
600	.381	.450	.577	.681
700	.353	.417	.536	.632
800	.328	.387	.497	.586
1000	.282	.333	.428	.505
2000	.134	.158	.203	.239
3000	.063	.075	.096	.113
4000	.030	.035	.045	.054
5000	.014	.017	.022	.025
6000	.007	.008	.010	.012
7000	.003	.004	.005	.006
8000	.002	.002	.002	.003
10000	.000	.000	.001	.001

R = 50; K2 = 10; c = 0.5

θ	r=7	r=10	r=20	r=50
20	.656	.768	.940	1.000
30	.619	.727	.907	.997
40	.594	.699	.882	.992
50	.576	.679	.861	.984
60	.562	.662	.844	.974
70	.550	.649	.829	.964
80	.540	.637	.815	.952
100	.522	.617	.791	.928
200	.453	.535	.688	.810
300	.395	.467	.599	.707
400	.345	.407	.523	.616
500	.301	.355	.456	.537
600	.262	.309	.397	.468
700	.228	.270	.346	.408
800	.199	.235	.302	.346
1000	.151	.179	.230	.270
2000	.038	.045	.058	.069
3000	.010	.012	.015	.017
4000	.002	.003	.004	.004
5000	.001	.001	.001	.001
6000	.000	.000	.000	.000

R = 50; K2 = 20; c = 0.5

θ	r=7	r=10	r=20	r=50
10	.661	.773	.943	1.000
20	.598	.704	.886	.993
30	.565	.667	.848	.977
40	.543	.641	.820	.956
50	.526	.621	.796	.934
60	.511	.603	.774	.910
70	.497	.587	.753	.887
30	.484	.572	.734	.864
100	.459	.542	.696	.820
200	.353	.417	.536	.632
300	.272	.321	.412	.486
400	.209	.247	.317	.374
500	.161	.190	.244	.288
600	.124	.146	.188	.222
700	.095	.113	.145	.171
800	.073	.087	.111	.131
1000	.044	.051	.066	.078
2000	.003	.004	.005	.006
3000	.000	.000	.000	.000

R = 50; K2 = 50; c = 0.5

θ	r=7	r=10	r=20	r=50
4	.664	.776	.945	1.000
5	.642	.753	.929	.999
6	.626	.734	.914	.998
7	.612	.719	.901	.996
8	.600	.706	.889	.994
10	.582	.686	.868	.987
20	.528	.624	.799	.937
30	.493	.583	.748	.881
40	.463	.546	.701	.827
50	.434	.513	.658	.776
60	.407	.481	.618	.728
70	.382	.451	.580	.683
80	.359	.424	.544	.641
100	.316	.373	.479	.565
200	.167	.198	.254	.299
300	.089	.105	.134	.158
400	.047	.055	071	.084
500	.025	.029	038	.044
600	.013	.016	.020	.024
700	.007	.008	.011	.012
800	.004	.004	.006	.007
1000	.001	.001	.002	.002
2000	.000	.000	.000	.000

R = 50; K2 = 100; c = 0.5

θ	r=7	r=10	r=20	r=50
2	.665	.777	.946	1.000
3	.626	.735	.915	.998
4	.601	.707	.890	.994
5	.583	.686	.869	.987
6	.568	.670	.852	.979
7	.556	.656	.837	.970
8	.546	.645	.824	.960
10	.529	.625	.800	.938
20	.464	.548	.703	.829
30	.409	.483	.620	.731
40	.361	.426	.547	.644
50	.318	.375	.482	.568
60	.280	.331	.425	.501
70	.247	.292	.375	.442
80	.218	.257	.330	.389
100	.169	.200	.257	.303
200	.048	.057	.073	.086
300	.014	.016	.021	.024
400	.004	.005	.006	.007
500	.001	.001	.002	.002
600	.000	.000	.000	.001

R = 100; K2 = 2; c = 0.5

θ	r=7	r=10	r=20	r=50	r=100
$5x10^2$	.505	.597	.770	.952	.996
$6x10^2$	.494	.584	.755	.941	.993
$7x10^2$	.485	.573	.742	.931	.989
$8x10^2$	.477	.564	.731	.921	.984
10^3	.465	.550	.712	.904	.973
$2x10^3$	.424	.502	.651	.832	.905
$3x10^3$	.392	.464	.602	.771	.838
$4x10^3$	.363	.430	.558	.713	.776
$5x10^3$	.336	.398	.516	.660	.719
$6x10^3$	.311	.368	.478	.611	.665
$7x10^3$	.288	.341	.443	.566	.616
$8x10^3$	.267	.316	.410	.524	.570
10^4	.229	.271	.351	.449	.489
$2x10^4$	.106	.125	.163	.208	.226
$3x10^4$	.049	.058	.073	.096	.105
$4x10^4$	.023	.027	.035	.045	.048
$5x10^4$	.010	.012	.016	.021	.022
$6x10^4$	.005	.006	.007	.010	.010
7x10	.002	.003	.003	.004	.005
$8x10^4$	.001	.001	.002	.002	.002
10^5	.000	.000	.000	.000	.000

R = 100; K2 = 5; c = 0.5

θ	r=7	r=10	r=20	r=50	r=100
$2x10^2$	.520	.614	.790	.965	.999
$3x10^2$	.494	.584	.755	.941	.993
$4x10^2$	.477	.564	.731	.921	.984
$5x10^2$	.465	.550	.712	.904	.973
$6x10^2$	.455	.538	.698	.888	.960
$7x10^2$	.446	.528	.685	.873	.947
$8x10^2$	.439	.519	.673	.859	.933
10^3	.424	.402	.651	.833	.905
$2x10^3$	.363	.430	.558	.713	.776
$3x10^3$	.311	.368	.478	.612	.665
$4x10^3$	.267	.316	.410	.524	.570
$5x10^3$	.229	.271	.351	.449	.489
$6x10^3$	.196	.232	.301	.385	.419
$7x10^3$	.168	.199	.258	.330	.359
$8x10^3$	.144	.170	.221	.283	.308
10^4	.106	.125	.163	.208	.226
$2x10^4$	.023	.027	.035	.045	.048
$3x10^4$	.005	.006	.007	.010	.010
$4x10^4$	.001	.001	.002	.002	.002
$5x10^4$	.000	.000	.000	.000	.000

R = 100; K2 = 10; c = 0.5

θ	r=7	r=10	r=20	r=50	r=100
10^2	.525	.621	.798	.969	.999
$2x10^2$	.482	.570	.738	.927	.988
$3x10^2$	.460	.544	.705	.896	.967
$4x10^2$	.444	.525	.681	.868	.942
$5x10^2$	.430	.509	.660	.843	.917
$6x10^2$	.418	.494	.641	.820	.891
$7x10^2$	.406	.480	.623	.797	.867
$8x10^2$	.395	.467	.605	.775	.843
10^3	.373	.441	.572	.732	.796
$2x10^3$	.281	.332	.431	.552	.600
$3x10^3$	.212	.251	.325	.416	.453
$4x10^3$	.160	.189	.245	.314	.341
$5x10^3$	.120	.142	.185	.236	.257
$6x10^3$	.091	.107	.139	.178	.194
$7x10^3$	.068	.081	.105	.134	.146
$8x10^3$	.052	.061	.079	.101	.110
10^4	.029	.035	.045	.058	.063
$2x10^4$	.002	.002	.003	.003	.004
$3x10^4$	.000	.000	.000	.000	.000

R = 100; K2 = 20; c = 0.5

θ	r=7	r=10	r=20	r=50	r=100
50	.529	.624	.802	.971	.999
60	.516	.610	.786	.962	.998
70	.507	.599	.772	.957	.997
80	.498	.589	.761	.945	.995
10^2	.485	.573	.742	.931	.989
$2x10^2$	.446	.528	.685	.873	.947
$3x10^2$	.421	.498	.646	.826	.898
$4x10^2$	.399	.471	.612	.783	.851
$5x10^2$	.378	.447	.580	.741	.807
$6x10^2$	.358	.423	.549	.702	.764
$7x10^2$	.339	.401	.520	.666	.724
$8x10^2$	.321	.380	.493	.631	.686
10^3	.288	.341	.443	.566	.616
$2x10^3$	.168	.199	.258	.330	.359
$3x10^3$	.098	.116	.151	.193	.209
$4x10^3$	.057	.068	.088	.112	.122
$5x10^3$	.033	.039	.051	.065	.071
$6x10^3$	.019	.023	.030	.038	.043
$7x10^3$	.011	.013	.017	.022	.024
$8x10^3$	.007	.008	.010	.013	.014
10^4	.002	.003	.003	.004	.005
$2x10^4$	.000	.000	.000	.000	.000

R = 100; K2 = 50; c = 0.5

θ	r=7	r=10	r=20	r=50	r=100
20	.531	.626	.805	.973	.999
30	.504	.596	.769	.951	.996
40	.487	.575	.744	.933	.990
50	.474	.560	.726	.916	.982
60	.464	.549	.711	.902	.972
70	.455	.539	.698	.889	.961
80	.448	.530	.687	.876	.949
10^2	.435	.514	.667	.852	.926
$2x10^2$	.381	.450	.584	.747	.813
$3x10^2$	.334	.395	.512	.655	.713
$4x10^2$	.293	.346	.449	.575	.626
$5x10^2$	.257	.304	.394	.504	.549
$6x10^2$	.225	.267	.346	.442	.481
$7x10^2$	.198	.234	.303	.388	.422
$8x10^2$	.173	.205	.266	.341	.370
10^3	.133	.158	.205	.262	.285
$2x10^3$	.036	.043	.055	.071	.077
$3x10^3$	.010	.012	.015	.019	.021
$4x10^3$	.003	.003	.004	.005	.006
$5x10^3$	.001	.001	.001	.001	.002
$6x10^3$	.000	.000	.000	.000	.000

R = 100; K2 = 100; c = 0.5

θ	r=7	r=10	r=20	r=50	r=100
10	.531	.627	.806	.973	.999
20	.487	.576	.745	.933	.990
30	.464	.549	.712	.903	.972
40	.449	.530	.688	.877	.950
50	.435	.515	.668	.854	.927
60	.424	.501	.650	.831	.904
70	.413	.488	.633	.810	.881
80	.402	.475	.617	.789	.858
10^2	.382	.451	.586	.749	.815
$2x10^2$	.294	.348	.452	.578	.629
$3x10^2$	.227	.269	.349	.446	.485
$4x10^2$	.175	.207	.269	.344	.374
$5x10^2$	.135	.160	.207	.265	.289
$6x10^2$	.104	.123	.160	.205	.223
$7x10^2$	.080	.095	.123	.158	.172
$8x10^2$	.062	.073	.095	.122	.133
10^3	.037	.044	.057	.073	.079
$2x10^3$	.003	.003	.004	.005	.006
$3x10^3$	.000	.000	.000	.000	.000

R = 500; K2 = 2; c = .5

θ	r=10	r=20	r=50	r=100	r=200	r=500
10^4	.434	.564	.735	.858	.956	.998
2×10^4	.408	.530	.692	.811	.918	.989
3×10^4	.393	.512	.668	.784	.891	.972
4×10^4	.383	.498	.650	.764	.870	.953
5×10^4	.374	.487	.635	.746	.850	.933
6×10^4	.366	.476	.621	.729	.832	.913
7×10^4	.358	.465	.607	.714	.814	.893
8×10^4	.350	.455	.594	.698	.796	.874
10^5	.335	.436	.569	.668	.762	.837
2×10^5	.269	.350	.457	.537	.612	.673
3×10^5	.216	.282	.368	.432	.492	.541
4×10^5	.174	.226	.295	347	.396	.435
5×10^5	.140	.182	.237	.279	.318	.349
6×10^5	.112	.146	.191	.224	.256	.281
7×10^5	.090	.118	.153	.180	.206	.226
8×10^5	.072	.095	.123	.145	.165	.181
10^6	.047	.061	.080	.094	.107	.117
2×10^6	.005	.007	.009	.011	.012	.013

R = 500; K2 = 5; c = .5

θ	r=10	r=20	r=50	r=100	r=200	r=500
5×10^3	.434	.565	.736	.858	.957	.999
6×10^3	.427	.555	.724	.846	.947	.998
7×10^3	.421	.548	.714	.835	.939	.997
8×10^3	.416	.541	.706	.826	.931	.995
10^4	.408	.531	.692	.811	.918	.990
2×10^4	.383	.499	.651	.764	.870	.953
3×10^4	.366	.476	.621	.730	.832	.913
4×10^4	.350	.456	.595	.698	.796	.874
5×10^4	.335	.436	.569	.669	.762	.837
6×10^4	.321	.417	545	.640	.730	.801
7×10^4	.307	.400	.522	.613	.699	.767
8×10^4	.294	.383	.499	.586	.669	.734
10^5	.269	.351	.457	.537	.613	.673
2×10^5	.174	.226	.296	.347	.396	.435
3×10^5	.112	.146	.191	.224	.256	.281
4×10^5	.073	.095	.123	.145	.165	.181
5×10^5	.047	.061	.080	.094	.107	.117
6×10^5	.030	.039	.051	.060	.069	.076
7×10^5	.020	.025	.033	.036	.045	.049
8×10^5	.013	.016	.021	.025	.029	.032
10^6	.005	.007	.009	.011	.012	.013
2×10^6	.000	.000	.000	.000	.000	.000

R = 500; K2 = 10; c = .5

θ	r=10	r=20	r=50	r=100	r=200	r=500
2×10^3	.447	.581	.756	.879	.970	.999
3×10^3	.430	.559	.729	.851	.951	.998
4×10^3	.419	.545	.711	.832	.936	.995
5×10^3	.411	.534	.697	.817	.923	.991
6×10^3	.404	.526	.686	.805	.912	.986
7×10^3	.399	.519	.677	.794	.902	.980
8×10^3	.394	.513	.669	.785	.893	.973
10^4	.386	.503	.656	.770	.877	.959
2×10^4	.355	.462	.603	.708	.808	.887
3×10^4	.328	.427	557	.654	.746	.819
4×10^4	.303	.394	.514	.604	.688	.756
5×10^4	.279	.363	.474	.557	.635	.698
6×10^4	.258	.336	.438	.514	.586	.644
7×10^4	.238	.310	.404	.475	.541	.594
8×10^4	.220	.286	.373	.438	.500	.549
10^5	.187	.244	.318	.373	426	.468
2×10^5	.084	.109	.143	.168	.191	.210
3×10^5	.038	.049	.064	.075	.086	.094
4×10^5	.017	.022	.029	.034	.039	.042
5×10^5	.008	.010	.013	.015	.017	.019
6×10^5	.003	.004	.006	.007	.008	.009
7×10^5	.002	.002	.003	.003	.004	.004
8×10^5	.001	.001	.001	.001	.002	.002
10^6	.000	.000	.000	.000	.000	.000

R = 500; K2 = 20; c = .5

θ	r=10	r=20	r=50	r=100	r=200	r=500
10^3	.449	.584	.760	.883	.973	1.000
2×10^3	.421	.548	.714	.835	.939	.997
3×10^3	.406	.528	.689	.808	.915	.988
4×10^3	.396	.515	.672	.789	.897	.976
5×10^3	.388	.505	.659	.774	.881	.963
6×10^3	.381	.496	.648	.760	.866	.949
7×10^3	.375	.488	.637	.748	.853	.935
8×10^3	.369	.480	.627	.736	.840	.921
10^4	.358	.466	.608	.714	.814	.894
2×10^4	.307	.400	.522	.613	.699	.767
3×10^4	.264	.343	.448	.526	.600	.658
4×10^4	.226	.294	.384	.451	.515	.565
5×10^4	.194	.253	.330	.387	.442	.485
6×10^4	.167	.217	.283	.332	.379	.416
7×10^4	.143	.186	.243	.285	.325	.357
8×10^4	.123	.160	.208	.245	.279	.306
10^5	.090	.118	.153	.180	.296	.226
2×10^5	.020	.025	.033	.039	.045	.049
3×10^5	.004	.006	.007	.008	.010	.011
4×10^5	.001	.001	.002	.002	.002	.002
5×10^5	.000	.000	.000	.000	.000	.000

R = 500; K2 = 50; c = .5

θ	r=10	r=20	r=50	r=100	r=200	r=500
5×10^2	.441	.573	.746	.869	.964	.999
6×10^2	.433	.563	.734	.857	.955	.999
7×10^2	.427	.556	.724	.846	.947	.998
8×10^2	.422	.549	.716	.837	.940	.996
10^3	.414	.538	.702	.822	.928	.993
2×10^3	.389	.506	.661	.776	.883	.965
3×10^3	.373	.486	.634	.745	.849	.931
4×10^3	.359	.468	.610	.717	.817	.897
5×10^3	.346	.451	.588	.691	.788	.865
6×10^3	.334	.434	.567	.666	.759	.833
7×10^3	.322	.418	.546	.641	.731	.803
8×10^3	.310	.403	.526	.618	.705	774
10^4	.288	.374	.488	.574	.654	.718
2×10^4	.199	.258	.337	.396	.451	.496
3×10^4	.137	.178	.233	.273	.311	.342
4×10^4	.095	.123	.160	.188	.215	.236
5×10^4	.065	.085	.111	.130	.148	.163
6×10^4	.045	.059	.076	.090	.102	.112
7×10^4	.031	.040	.053	.062	.071	.078
8×10^4	.021	.029	.036	.043	.049	.053
10^5	.010	.013	.017	.020	.023	.025
2×10^5	.000	.000	.000	.000	.001	.001

R = 500; K2 = 100; c = .5

θ	r=10	r=20	r=50	r=100	r=200	r=500
2×10^2	.451	.586	.763	.886	.975	1.000
3×10^2	.434	.564	.735	.858	.967	1.000
4×10^2	.423	.550	.717	.838	.942	.998
5×10^2	.414	.539	.703	.823	.929	.994
6×10^2	.408	.531	.692	.811	.918	.990
7×10^2	.402	.523	.683	.801	.909	.985
8×10^2	.398	.517	.675	.792	.900	.979
10^3	.390	.507	.662	.777	.884	.966
2×10^3	.360	.469	.612	.719	.819	.900
3×10^3	.335	.435	.568	.668	.761	.836
4×10^3	.311	.405	.528	.620	.707	.777
5×10^3	.289	.376	.491	.576	.657	.722
6×10^3	.268	.349	.456	.535	.610	.670
7×10^3	.249	.324	.423	.497	.567	.623
8×10^3	.232	.301	.393	.462	.527	.579
10^4	.200	.260	.340	.399	.455	.500
2×10^4	.096	.125	.163	.191	.218	.239
3×10^4	.046	.060	.078	.092	.105	.115
4×10^4	.022	.029	.037	.044	.051	.055
5×10^4	.011	.014	.018	.021	.024	.026
6×10^4	.005	.007	.009	.010	.012	.013
7×10^4	.002	.003	.004	.005	.006	.006
8×10^4	.001	.002	.002	.002	.003	.003
10^5	.000	.000	.000	.001	.001	.001

R = 10^3; K2 = 2; c = 0.5

θ	r=50	r=100	r=200	r=500	r=1000
5x10^4	.641	.753	.861	.971	.997
6x10^4	.631	.742	.850	.964	.995
7x10^4	.624	.734	.840	.958	.992
8x10^4	.617	.726	.832	.951	.990
10^5	.607	.714	.818	.940	.983
2x10^5	.572	.673	.772	.892	.940
3x10^5	.544	.640	.735	.850	.896
4x10^5	.518	.610	.700	.810	.853
5x10^5	.494	.581	.667	.771	.813
6x10^5	.470	.554	.636	.735	.774
7x10^5	.448	.527	.606	.700	.738
8x10^5	.427	.502	.577	.667	.703
10^6	.386	.456	.524	.605	.638
2x10^6	.239	.281	.322	.373	.393
3x10^6	.147	.173	.199	.230	.242
4x10^6	.091	.106	.122	.141	.149
5x10^6	.056	.066	.075	.087	.092
6x10^6	.034	.040	.046	.054	.056
7x10^6	.021	.025	.29	.033	.035
8x10^6	.013	.015	.018	.020	.021
10^7	.005	.006	.007	.008	.008

R = 10^3; K2 = 5; c = 0.5

θ	r=50	r=100	r=200	r=500	r=1000
2x10^4	.652	.766	.874	.978	.997
3x10^4	.631	.742	.849	.964	.994
4x10^4	.617	.726	.832	.951	.989
5x10^4	.606	.713	.818	.940	.982
6x10^4	.598	.703	.807	.929	.974
7x10^4	.590	.695	.797	.919	.966
8x10^4	.584	.687	.788	.910	.957
10^5	.571	.672	.772	.892	.940
2x10^5	.518	.610	.700	.810	.853
3x10^5	.470	.554	.636	.735	.774
4x10^5	.427	.502	.577	.667	.703
5x10^5	.388	.456	.524	.606	.638
6x10^5	.352	.414	.475	.550	.579
7x10^5	.319	.376	.432	.499	.526
8x10^5	.290	.341	.392	.453	.477
10^6	.239	.281	.323	.373	.393
2x10^6	.091	.107	.123	.142	.149
3x10^6	.034	.041	.047	.054	.057
4x10^6	.013	.015	.018	.020	.022
5x10^6	.005	.006	.007	.008	.008
6x10^6	.002	.002	.003	.003	.003
7x10^6	.001	.001	.001	.001	.001
8x10^6	.000	.000	.000	.000	.000

R = 10^3; K2 = 10; c = 0.5

θ	r=50	r=100	r=200	r=500	r=1000
10^4	.658	.773	.881	.983	.999
2x10^4	.622	.732	.838	.957	.993
3x10^4	.603	.709	.814	.936	.980
4x10^4	.589	.693	.796	.918	.965
5x10^4	.577	.679	.780	.901	.949
6x10^4	.567	.667	.766	.885	.932
7x10^4	.557	.655	.752	.869	.916
8x10^4	.547	.643	.739	.854	.900
10^5	.527	.621	.713	.824	.868
2x10^5	.442	.519	.596	.690	.727
3x10^5	.370	.435	.499	.577	.608
4x10^5	.309	.364	.418	.483	.509
5x10^5	.259	.305	.350	.404	.426
6x10^5	.217	.255	.293	.338	.357
7x10^5	.181	.213	.245	.283	.299
8x10^5	.152	.179	.205	.237	.250
10^6	.106	.125	.144	.166	.175
2x10^6	.018	.021	.024	.028	.030
3x10^6	.003	.004	.004	.005	.005
4x10^6	.001	.001	.001	.001	.001
5x10^6	.000	.000	.000	.000	.000

R = 10^3; K2 = 20; c = 0.5

θ	r=50	r=100	r=200	r=500	r=1000
5x10^3	.660	.775	.883	.983	.998
6x10^3	650	.764	.872	.977	.998
7x10^3	.642	.754	.862	.972	.997
8x10^3	.635	.747	.854	.967	.996
10^4	.624	.734	.840	.958	.993
2x10^4	.591	.695	.798	.920	.967
3x10^4	.569	.669	.769	.888	.936
4x10^4	.550	.648	.743	.859	.905
5x10^4	.531	.625	.718	.830	.875
6x10^4	.514	.604	.694	.802	.845
7x10^4	.496	.584	.671	.775	.817
8x10^4	.480	.565	.648	.750	.790
10^5	.448	.528	.606	.701	.738
2x10^5	.320	.376	.432	.499	.526
3x10^5	.228	.268	.308	.356	.375
4x10^5	.162	.191	.219	.253	.267
5x10^5	.116	.136	.156	.181	.190
6x10^5	.082	.097	.111	.129	.136
7x10^5	.059	.069	.079	.092	.097
8x10^5	.042	.049	.057	.065	.069
10^6	.021	025	.209	.033	.035
2x10^6	.001	.001	.001	.001	.001
3x10^6	.000	.000	.000	.000	.000

R = 10^3; K2 = 50; c = 0.5

θ	r=50	r=100	r=200	r=500	r=1000
2x10^3	.661	.777	.885	.984	.998
3x10^3	.640	.752	.860	.971	.997
4x10^3	.625	.735	.842	.959	.993
5x10^3	.614	.723	.828	.948	.988
6x10^3	.606	.713	.817	.939	.982
7x10^3	.599	.704	.808	.930	.975
8x10^3	.592	.697	.799	.922	.968
10^4	.581	.683	.784	.906	.953
2x10^4	.534	.628	.721	.834	.879
3x10^4	.492	.579	.664	.768	.809
4x10^4	.453	.533	.612	.707	.745
5x10^4	.417	.491	.563	.651	.687
6x10^4	.384	.451	.519	.600	.632
7x10^4	.354	.416	.478	.553	.582
8x10^4	.326	.383	.440	.509	.526
10^5	.276	.325	.373	.432	.455
2x10^5	.121	.143	.164	.190	.200
3x10^5	.053	.063	.072	.083	.088
4x10^5	.023	.028	.032	.037	.039
5x10^5	.010	.012	.014	.016	.017
6x10^5	.005	.005	.006	.007	.007
7x10^5	.002	.002	.003	.003	.003
8x10^5	.001	.001	.001	.001	.001
10^6	.000	.000	.000	.000	.000

R = 10^3; K2 = 100; c = 0.5

θ	r=50	r=100	r=200	r=500	r=1000
10^3	.662	.778	.886	.985	.999
2x10^3	.626	.736	.843	.960	.994
3x10^3	.607	.714	.819	.940	.983
4x10^3	.593	.698	.801	.923	.969
5x10^3	.582	.684	.786	.907	.955
6x10^3	.572	.673	.772	.892	.940
7x10^3	.562	.661	.759	.878	.925
8x10^3	.553	.650	.747	.863	.910
10^4	.535	.629	.723	.836	.881
2x10^4	.454	.535	.614	.710	.748
3x10^4	.386	.454	.522	.603	.635
4x10^4	.328	.386	.443	.512	.540
5x10^4	.279	.328	.376	.435	.459
6x10^4	.237	.278	.320	.370	.389
7x10^4	.201	.236	.272	.314	.331
8x10^4	.171	.201	.231	.267	.281
10^5	.123	.145	.166	.192	.203
2x10^5	.024	.028	.033	.038	.040
3x10^5	.005	.006	.006	.007	.008
4x10^5	.001	.001	.001	.001	.002
5x10^5	.000	.000	.000	.000	.000

R = 20; K2 = 2; c = 0.1

θ	r=4	r=7	r=10	r=20
	y = .1			
10	.636	.853	.946	.999
20	.561	.771	.882	.983
30	.523	.723	.836	.953
40	.496	.688	.798	.918
50	.474	.658	.765	.882
60	.454	.631	.733	.847
70	.436	.605	.704	.813
80	.418	.581	.675	.781
100	.385	.535	.622	.719
200	.255	.355	.412	.477
300	.169	.235	.273	.316
400	.112	.156	.181	.210
500	.074	.103	.120	.139
600	.049	.069	.080	.092
700	.033	.045	.053	.061
800	.022	.030	.035	.040
1000	.010	.013	.015	.018
2000	.000	.000	.000	.000

R = 20; K2 = 5; c = 0.1

θ	r=4	r=7	r=10	r=20
	y = .1			
5	.614	.830	.931	.997
6	.594	.808	.914	.994
7	.578	.790	.899	.990
8	.564	.774	.885	.984
10	.543	.748	.861	.971
20	.478	.663	.770	.888
30	.431	.599	.696	.805
40	.390	.542	.630	.728
50	.353	.491	.570	.659
60	.320	.444	.516	.597
70	.290	.402	.467	.540
80	.262	.364	.423	.489
100	.215	.298	.347	.401
200	.079	.110	.128	.148
300	.029	.041	.047	.055
400	.011	.015	.018	.020
500	.004	.006	.006	.007
600	.001	.002	.002	.003
700	.001	.001	.001	.001
800	.000	.000	.000	.000

R = 20; K2 = 10; c = 0.1

θ	r=4	r=7	r=10	r=20
	y = .1			
2	.641	.858	.950	.999
3	.595	.810	.915	.994
4	.565	.776	.886	.985
5	.544	.750	.862	.972
6	.527	.728	.841	.957
7	.513	.710	.822	.941
8	.500	.694	.804	.924
10	.479	.664	.772	.890
20	.392	.544	.633	.732
30	.322	.447	.520	.601
40	.264	.367	.427	.494
50	.217	.302	.351	.405
60	.178	.248	.288	.333
70	.146	.203	.237	.273
80	.120	.167	.194	.225
100	.081	.113	.131	.152
200	.011	.016	.018	.212
300	.002	.002	.003	.003
400	.000	.000	.000	.000

R = 20; K2 = 20; c = 0.1

θ	r=5	r=7	r=10	r=20
	y = .1			
1	.641	.858	.950	.999
2	.566	.776	.887	.985
3	.527	.729	.841	.958
4	.501	.694	.805	.925
5	.479	.665	.772	.891
6	.460	.639	.742	.857
7	.442	.614	.713	.824
8	.425	.590	.686	.793
10	.393	.546	.634	.733
20	.266	.369	.429	.496
30	.180	.249	.290	.335
40	.121	.169	.196	.227
50	.082	.114	.132	.153
60	.055	.077	.090	.104
70	.037	.052	.061	.070
80	.025	.035	.041	.047
100	.012	.016	.019	.022
200	.000	.000	.000	.000

R = 20; K2 = 50; c = 0.1

θ	r=4	r=7	r=10	r=20
	y = .1			
0.5	.616	.832	.932	.997
0.6	.596	.811	.916	.995
0.7	.580	.792	.901	.990
0.8	.566	.777	.887	.985
1	.545	.751	.863	.973
2	.480	.666	.773	.891
3	.434	.602	.700	.809
4	.393	.546	.635	.734
5	.357	.496	.576	.666
6	.324	.449	.522	.604
7	.293	.408	.474	.548
8	.266	.370	.430	.497
10	.219	.304	.354	.409
20	.083	.115	.133	.154
30	.031	.043	.050	.058
40	.012	.016	.019	.022
50	.004	.006	.007	.008
60	.002	.002	.003	.003
70	.001	.001	.001	.001

R = 20; K2 = 100; c = 0.1

θ	r=4	r=7	r=10	r=20
	y = .1			
0.2	.642	.859	.951	.999
0.3	.596	.811	.916	.995
0.4	.566	.777	.887	.985
0.5	.545	.751	.863	.973
0.6	.528	.729	.842	.958
0.7	.514	.711	.823	.942
0.8	.501	.695	.805	.925
1	.480	.666	.773	.892
2	.393	.547	.635	.734
3	.324	.450	.523	.604
4	.266	.370	.430	.497
5	.219	.305	.354	.409
6	.180	.251	.291	.337
7	.148	.206	.240	.277
8	.122	.170	.197	.228
10	.083	.115	.134	.154
20	.012	.016	.019	.022
30	.002	.002	.003	.003
40	.000	.000	.000	.000

R = 20; K2 = 2; c = 0.3

θ	r=4	r=7	r=10	r=20
	y = .3			
10	.649	.866	.955	.999
20	.572	.784	.893	.988
30	.533	.736	.849	.963
40	.507	.702	.813	.933
50	.485	.674	.782	.901
60	.467	.648	.753	.869
70	.449	.624	.725	.838
80	.433	.602	.699	.808
100	.402	.559	.649	.751
200	.278	.387	.450	.520
300	.193	.268	.311	.360
400	.133	.185	.216	.249
500	.092	.128	.149	.173
600	.064	.089	.103	.119
700	.044	.062	.072	.083
800	.031	.043	.050	.057
1000	.015	.020	.024	.027
2000	.000	.001	.001	.001

R = 20; K2 = 5; c = 0.3

θ	r=4	r=7	r=10	r=20
	y = .3			
5	.636	.853	.946	.999
6	.615	.831	.931	.997
7	.598	.813	.917	.995
8	.584	.797	.904	.992
10	.561	.771	.882	.983
20	.496	.688	.798	.918
30	.454	.631	.733	.847
40	.418	.581	.675	.781
50	.385	.535	.622	.719
60	.355	.493	.573	.662
70	.327	.454	.528	.610
80	.301	.418	.486	.562
100	.255	.355	.412	.477
200	.112	.156	.181	.210
300	.049	.069	.080	.092
400	.022	.030	.035	.040
500	.010	.013	.015	.018
600	.004	.006	.007	.008
700	.002	.003	.003	.003
800	.001	.001	.001	.002
1000	.000	.000	.000	.000

R = 20; K2 = 10; c = 0.3

θ	r=4	r=7	r=10	r=20
	y = .3			
2	.667	.883	.965	1.000
3	.619	.836	.935	.998
4	.588	.802	.908	.993
5	.565	.775	.886	.985
6	.547	.754	.866	.975
7	.533	.736	.849	.963
8	.521	.720	.833	.950
10	.500	.693	.804	.924
20	.424	.589	.684	.791
30	.362	.503	.584	.675
40	.309	.429	.499	.577
50	.264	.366	.426	.492
60	.225	.313	.364	.421
70	.192	.267	.311	.359
80	.164	.228	.265	.307
100	.120	.166	.193	.224
200	.025	.034	.040	.046
300	.005	.007	.008	.010
400	.001	.001	.002	.002
500	.000	.000	.000	.000

R = 20; K2 = 20; c = 0.3

θ	r=4	r=7	r=10	r=20
	y = .3			
1	.670	.885	.996	1.000
2	.590	.804	.910	.993
3	.549	.756	.868	.976
4	.523	.723	.835	.953
5	.502	.696	.807	.926
6	.485	.672	.781	.900
7	.469	.651	.756	.873
8	.454	.631	.733	.847
10	.427	.592	.689	.796
20	.313	.435	.505	.584
30	.230	.319	.371	.429
40	.169	.234	.272	.315
50	.124	.172	.200	.231
60	.001	.126	.147	.170
70	.067	.093	.108	.124
80	.049	.068	.079	.091
100	.026	.037	.043	.049
200	.001	.002	.002	.002
300	.000	.000	.000	.000

R = 20; K2 = 50; c = 0.3

θ	r=4	r=7	r=10	r=20
	y = .3			
0.5	.644	.861	.952	.999
0.6	.623	.840	.938	.998
0.7	.606	.822	.924	.996
0.8	.591	.806	.912	.994
1	.568	.779	.889	.986
2	.503	.697	.808	.928
3	.463	.643	.747	.862
4	.428	.595	.691	.799
5	.397	.551	.640	.740
6	.367	.510	.593	.686
7	.340	.473	.550	.636
8	.315	.438	.509	.589
10	.271	.376	.437	.505
20	.126	.175	.204	.236
30	.059	.082	.095	.110
40	.027	.038	.044	.051
50	.013	.018	.021	.024
60	.006	.008	.010	.011
70	.003	.004	.004	.005
80	.001	.002	.002	.002
100	.000	.000	.000	.001

R = 20; K2 = 100; c = 0.3

θ	r=4	r=7	r=10	r=20
	y = .3			
0.2	.672	.887	.967	1.000
0.3	.624	.840	.938	.998
0.4	.592	.806	.912	.994
0.5	.569	.780	.890	.986
0.6	.551	.758	.870	.977
0.7	.536	.740	.853	.966
0.8	.524	.725	.837	.954
1	.504	.698	.809	.929
2	.429	.596	.692	.800
3	.368	.511	.594	.687
4	.316	.439	.511	.590
5	.272	.377	.439	.507
6	.233	.324	.377	.436
7	.200	.278	.324	.374
8	.172	.239	.278	.321
10	.127	.176	.205	.237
20	.028	.039	.045	.052
30	.006	.008	.010	.011
40	.001	.002	.002	.002
50	.000	.000	.000	.001

R = 20; K2 = 2; c = 0.7

θ	r=4 y=.7	r=7 y=.7	r=10 y=.7	r=20 y=.7
10	.682	.896	.972	1.000
20	.600	.816	.919	.995
30	.559	.768	.879	.982
40	.531	.734	.847	.962
50	.511	.708	.819	.939
60	.494	.685	.794	.914
70	.479	.664	.771	.890
80	.465	.645	.750	.866
100	.439	.609	.708	.819
200	.331	.460	.535	.618
300	.250	.347	.404	.467
400	.189	.262	.305	.352
500	.142	.198	.230	.266
600	.108	.149	.174	.201
700	.081	.113	.131	.152
800	.061	.085	.099	.114
1000	.035	.049	.056	.065
2000	.002	.003	.003	.004

R = 20; K2 = 5; c = 0.7

θ	r=4 y=.7	r=7 y=.7	r=10 y=.7	r=20 y=.7
5	-	-	.980	1.000
6	.680	.894	.971	1.000
7	.661	.877	.962	1.000
8	.645	.862	.952	.999
10	.619	.835	.934	.998
20	.547	.754	.866	.974
30	.509	.706	.817	.937
40	.482	.669	.777	.895
50	.458	.637	.740	.855
60	.437	.607	.705	.815
70	.416	.578	.672	.77
80	.397	.552	.641	.741
100	.361	.501	.583	.674
200	.224	.312	.362	.419
300	.139	.194	.225	.260
400	.087	.120	.140	.162
500	.054	.075	.087	.100
600	.033	.046	.054	.062
700	.021	.029	.034	.039
800	.013	.018	.021	.024
1000	.005	.007	.008	.009
2000	.000	.000	.000	.000

R = 20; K2 = 10; c = 0.7

θ	r=4 y=.7	r=7 y=.7	r=10 y=.7	r=20 y=.7
2	-	-	-	-
3	-	-	.980	1.000
4	.666	.882	.964	1.000
5	.639	.856	.949	.99
6	.618	.834	.934	.998
7	.601	.816	.920	.995
8	.586	.800	.907	.992
10	.564	.774	.884	.984
20	.499	.692	.802	.922
30	.457	.635	.738	.853
40	.422	.586	.681	.787
50	.389	.541	.629	.727
60	.359	.499	.580	.671
70	.332	.461	.536	.619
80	.306	.425	.495	.572
100	.261	.363	.421	.487
200	.117	.163	.189	.219
300	.053	.073	.085	.098
400	.024	.033	.038	.044
500	.001	.015	.017	.020
600	.005	.007	.008	.009
700	.002	.003	.003	.004
800	.001	.001	.002	.002
1000	.000	.000	.000	.000

R = 20; K2 = 20; c = 0.7

θ	r=4 y=.7	r=7 y=.7	r=10 y=.7	r=20 y=.7
1	-	-	-	-
2	.678	.893	.970	1.000
3	.629	.846	.942	.999
4	.597	.812	.917	.995
5	.574	.785	.895	.988
6	.556	.764	.876	.980
7	.541	.746	.858	.970
8	.529	.730	.843	.959
10	.508	.704	.815	.935
20	.435	.604	.702	.812
30	.376	.523	.607	.702
40	.325	.452	.525	.607
50	.282	.391	.455	.526
60	.244	.338	.393	.455
70	.211	.293	.340	.393
80	.182	.253	.294	.340
100	.136	.189	.220	.255
200	.032	.044	.052	.060
300	.008	.010	.012	.014
400	.002	.002	.003	.003
500	.000	.001	.001	.001
600	.000	.000	.000	.000

R = 20; K2 = 50; c = 0.7

θ	r=4 y=.7	r=7 y=.7	r=10 y=.7	r=20 y=.7
0.5	-	-	-	-
0.6	-	-	-	-
0.7	-	-	.980	1.000
0.8	.686	.900	.974	1.000
1	.658	.875	.960	1.000
2	.580	.793	.901	.991
3	.541	.746	.858	.970
4	.514	.712	.824	.943
5	.493	.684	.794	.913
6	.475	.660	.766	.884
7	.459	.637	.740	.855
8	.443	.615	.715	.826
10	.414	.575	.668	.772
20	.295	.409	.476	.550
30	.210	.291	.339	.392
40	.149	.207	.241	.279
50	.106	.148	.172	.199
60	.076	.105	.122	.141
70	.054	.075	.087	.101
80	.038	.053	.062	.072
100	.019	.027	.031	.036
200	.001	.001	.001	.001
300	.000	.000	.000	.000

R = 20; K2 = 100; c = 0.7

θ	r=4 y=.7	r=7 y=.7	r=10 y=.7	r=20 y=.7
0.2	-	-	-	-
0.3	-	-	-	
0.4	.689	.902	.975	1.000
0.5	.661	.877	.962	1.000
0.6	.639	.856	.949	.999
0.7	.621	.838	.936	.998
0.8	.607	.822	.925	.996
1	.583	.796	.903	.991
2	.516	.714	.826	.945
3	.477	.663	.770	.888
4	.446	.619	.719	.831
5	.417	.579	.673	.778
6	.390	.542	.630	.728
7	.365	.507	.589	.681
8	.342	.474	.551	.633
10	.299	.415	.483	.448
20	.154	.214	.249	.287
30	.079	.110	.128	.148
40	.041	.057	.066	.076
50	.021	.029	.034	.039
60	.011	.015	.017	.020
70	.006	.008	.009	.010
80	.003	.004	.005	.005
100	.001	.001	.001	.001
200	.000	.000	.000	.000

R = 20; K2 = 10; c = 0.9

θ	r=4 y=0.9	r=7 y=0.9	r=10 y=0.9	r=20 y=0.9
2	-	-	-	-
3	-	-	-	-
4	-	-	-	-
5	-	-	-	-
6	-	-	.979	1.000
7	.679	.893	.971	1.000
8	.662	.879	.963	1.000
10	.636	.853	.946	.999
20	.561	.771	.882	.983
30	.523	.723	.836	.953
40	.496	.688	.798	.918
50	.474	.658	.765	.882
60	.454	.631	.733	.847
70	.436	.605	.704	.813
80	.418	.581	.675	.781
100	.385	.535	.622	.719
200	.255	.355	.412	.477
300	.169	.235	.274	.316
400	.112	.156	.181	.210
500	.074	.103	.120	.139
600	.049	.069	.080	.092
700	.033	.045	.053	.061
800	.022	.030	.035	.041
1000	.010	.013	.015	.018
2000	.000	.000	.000	.000

R = 20; K2 = 5; c = 0.9

θ	r=4 y=0.9	r=7 y=0.9	r=10 y=0.9	r=20 y=0.9
5	-	-	-	-
6	-	-	-	-
7	-	-	-	-
8	-	-	.980	1.000
10	-	-	.980	1.000
20	.592	.807	.912	.994
30	.551	.759	.871	.977
40	.524	.725	.838	.955
50	.504	.699	.809	.929
60	.487	.675	.784	.903
70	.471	.654	.760	.877
80	.456	.634	.737	.851
100	.429	.596	.693	.801
200	.317	.440	.512	.592
300	.234	.325	.378	.437
400	.173	.240	.279	.323
500	.128	.178	.206	.239
600	.094	.131	.152	.176
700	.070	.097	.113	.130
800	.052	.072	.083	.096
1000	.028	.039	.045	.053
2000	.001	.002	.002	.003

R = 20; K2 = 2; c = 0.9

θ	r=4 y=0.9	r=7 y=0.9	r=10 y=0.9	r=20 y=0.9
10	-	-	.980	1.000
20	.619	.835	.934	.998
30	.575	.787	.896	.989
40	.547	.754	.866	.974
40	.526	.727	.840	.956
60	.509	.706	.817	.937
70	.495	.687	.796	.916
80	.482	.669	.777	.895
100	.458	.637	.740	.855
200	.361	.501	.583	.674
300	.285	.395	.459	.531
400	.224	.312	.362	.419
500	.177	.246	.286	.330
600	.139	.194	.225	.260
700	.110	.153	.177	.205
800	.087	.120	.140	.162
1000	.054	.075	.087	.100
2000	.005	.007	.008	.009

R = 20; K2 = 20; c = 0.9

θ	r=4 y=0.9	r=7 y=0.9	r=10 y=0.9	r=20 y=0.9
1	-	-	-	-
2	-	-	-	-
3	-	-	-	-
4	-	-	.978	1.000
5	.668	.884	.966	1.000
6	.646	.863	.953	.999
7	.628	.845	.941	.998
8	.613	.829	.930	.997
10	.589	.803	.909	.993
20	.521	.721	.833	.951
30	.483	.671	.778	.897
40	.452	.628	.730	.844
50	.425	.590	.686	.793
60	.399	.554	.644	.744
70	.375	.520	.605	.699
80	.352	.489	.568	.657
100	.310	.431	.501	.579
200	.166	.230	.268	.309
300	.089	.123	.143	.165
400	.047	.066	.076	.088
500	.025	.035	.041	.047
600	.013	.019	.022	.025
700	.007	.010	.012	.013
800	.004	.005	.006	.007
1000	.001	.002	.002	.002
2000	.000	.000	.000	.000

R = 20; K2 = 50; c = 0.9

θ	r=4 y=0.9	r=7 y=0.9	r=10 y=0.9	r=20 y=0.9
0.5	-	-	-	-
0.6	-	-	-	-
0.7	-	-	-	-
0.8	-	-	-	-
1	-	-	-	-
2	.694	.906	.977	1.000
3	.644	.861	.952	.999
4	.611	.827	.928	.997
5	.587	.801	.907	.992
6	.568	.779	.889	.986
7	.553	.761	.873	.978
8	.540	.745	.858	.969
10	.520	.719	.831	.949
20	.450	.626	.727	.840
30	.396	.550	.640	.740
40	.349	.484	.563	.651
50	.307	.426	.496	.573
60	.270	.375	.436	.504
70	.238	.330	.384	.444
80	.209	.291	.338	.391
100	.162	.225	.262	.303
200	.045	.063	.073	.085
300	.013	.018	.020	.024
400	.004	.005	.006	.007
500	.001	.001	.002	.002
600	.000	.000	.000	.001

R = 20; K2 = 100; c = 0.9

θ	r=4 y=0.9	r=7 y=0.9	r=10 y=0.9	r=20 y=0.9
0.5	-	-	-	-
0.3	-	-	-	-
0.4	-	-	-	-
0.5	-	-	-	-
0.6	-	-	-	-
0.7	-	-		-
0.8	-	-	-	-
1	-	-	.981	1.000
2	.620	.837	.935	.998
3	.576	.788	.897	.989
4	.548	.755	.867	.975
5	.527	.729	.841	.957
6	.510	.707	.818	.938
7	.496	.688	.798	.917
8	.483	.670	.778	.897
10	.460	.638	.741	.856
20	.363	.504	.585	.677
30	.286	.398	.462	.535
40	.226	.314	.365	.422
50	.179	.248	.289	.334
60	.141	.196	.228	.264
70	.112	.155	.180	.208
80	.088	.122	.142	.165
100	.055	.076	.089	.103
200	.005	.007	.008	.010
300	.000	.001	.001	.001
400	.000	.000	.000	.000

APPENDIX L

INDIVIDUAL INSTANTANEOUS DIMENSIONLESS FLUXES, q_1 AND q_2, FOR TWO-LAYER LINEAR FLOW, CONSTANT TERMINAL PRESSURE

Calculated from:

$$q_1 = \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{\sin \gamma c}{\gamma}\right] \qquad \text{(V-114)}$$

$$q_2 = \frac{2}{L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)}\right] \qquad \text{(V-115)}$$

Eigenvalues b from:

$$b = \left(\frac{2m-1}{2}\right)\frac{\pi}{L} \; ; \quad m = 1,2,3,\ldots \qquad \text{(V-43)}$$

Eigenvalues a from: Appendix D

Truncation Test (Both Series): $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$

Estimated Accuracy for q_1 and q_2: $\pm$ 1 part in 10^4

Definition of Dimensionless Variables:

$$x = \frac{x_a}{H} \qquad \text{(V-3)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-4)}$$

$$L = \frac{L_a}{H} \qquad \text{(V-5)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c H^2} \quad \text{(for liquid flow)}$$

$$= \frac{k_1 t_a}{\rho_1 C_{p1} H^2} \quad \text{(for heat conduction)} \qquad \text{(V-15)}$$

All values presented are for $K_2^* = K_2 = \frac{k_2}{k_1}$

L = 5; c = 0.5

θ	K2 = 2.0 q1	K2 = 2.0 q2	K2 = 2.0 q1/(q1+q2)	K2 = 5.0 q1	K2 = 5.0 q2	K2 = 5.0 q1/(q1+q2)	K2 = 10.0 q1	K2 = 10.0 q2	K2 = 10.0 q1/(q1+q2)
.2	1.1021	.9955	.52543	.5423	.4421	.55086	.3910	.2951	.56989
.3	.8832	.8223	.51786	.4202	.3665	.53411	.2927	.2441	.54520
.4	.7564	.7166	51352	.3530	.3195	.52497	.2412	.2122	.53203
.5	.6717	.6433	.51080	.3098	.2866	.51946	.2093	.1898	.52443
.6	.6102	.5887	.50897	.2792	.2620	.51588	.1871	.1729	.51974
.8	.5251	.5112	.50669	.2380	.2272	.51156	.1573	.1483	.51470
1.0	.4679	.4580	.50533	.2107	.2032	.50914	.1370	.1303	.51239
1.5	.3800	.3747	.50353	.1687	.1645	.50635	.1023	.0980	.51059
2	.3281	.3246	.50266	.1420	.1389	.50542	.0780	.0749	.51030
3	.2653	.2633	.50190	.1046	.1025	.50500	.0457	.0439	.51024
4	.2242	.2227	.50166	.0779	.0763	.50495	.0268	.0257	.51024
5	.1921	.1909	.50158	.0580	.0569	.50495	.0157	.0151	.51024
6	.1654	.1643	.50155	.0432	.0424	.50495	.0092	.0088	.51024
8	.1229	.1222	.50154	.0240	.0235	.50495	.0032	.0030	.51024
10	.0914	.0909	.50154	.0133	.0131	.50495	.0011	.0010	.51024
15	.0436	.0434	.50154	.0031	.0030	.50495	.0001	.001	.51024
20	.0208	.0207	.50154	.0007	.0007	.50495			.51024
30	.0047	.0047	.50154						
40	.0011	.0011	.50154						
50	.0002	.0002	.50154						
60	.0001	.001	.50154						

θ	K2 = 20 q1	K2 = 20 q2	K2 = 20 q1/(q1+q2)	K2 = 50 q1	K2 = 50 q2	K2 = 50 q1/(q1+q2)	K2 = 100 q1	K2 = 100 q2	K2 = 100 q1/(q1+q2)
.2	.2975	.2057	.59118	.2144	.1251	.63146	.1640	.0728	.69269
.3	.2129	.1690	.55746	.1372	.0953	.59020	.0884	.0473	.65134
.4	.1706	.1451	.54039	.0987	.0743	.57054	.0532	.0314	.62914
.5	.1445	.1274	.53153	.0750	.0587	.56129	.0338	.0209	.61763
.6	.1262	.1133	.52679	.0585	.0465	.55690	.0221	.0140	.61169
.8	.1001	.0914	.52273	.0366	.0295	.55375	.0098	.0063	.60700
1.0	.0811	.0745	.52143	.0232	.0187	.55297	.0044	.0029	.60569
1.5	.0490	.0451	.52082	.0075	.0060	.55272	.0007	.0004	.60517
2	.0298	.0274	.52078	.0024	.0020	.55271	.0001	.0001	.60514
3	.0110	.0101	.52077	.0003	.0002	.55271			
4	.0040	.0037	.52077						
5	.0015	.0014	.52077						
6	.0006	.0005	.52077						
8	.0001	.0001	.52077						
10			.52077						
15									
20									
30									
40									
50									
60									

L = 10.0; c = 0.5

θ	K2 = 2.0 q_1	K2 = 2.0 q_2	K2 = 2.0 $\frac{q_1}{q_1+q_2}$	K2 = 5.0 q_1	K2 = 5.0 q_2	K2 = 5.0 $\frac{q_1}{q_1+q_2}$	K2 = 10 q_1	K2 = 10 q_2	K2 = 10 $\frac{q_1}{q_1+q_2}$
.3							.2925	.2441	.54510
.4				.3530	.3195	.52496	.2413	.2123	.53198
.5	.6717	.6433	.51080	.3098	.2866	.51946	.2094	.1900	.52428
.6	.6102	.5887	.50897	.2792	.2620	.51587	.1875	.1735	.51941
.8	.5251	.5112	.50669	.2380	.2273	.51154	.1586	.1501	.51375
1.0	.4679	.4580	.50533	.2109	.2034	.50905	.1400	.1342	.51062
1.5	.3801	.3747	.50353	.1701	.1661	.50587	.1124	.1094	.50678
2	.3283	.3248	.50264	.1464	.1439	.50434	.0966	.0947	.50500
3	.2673	.2655	.50175	.1189	.1175	.50286	.0779	.0768	.50343
4	.2312	.2300	.50131	.1026	.1017	.50214	.0661	.0653	.50287
5	.2067	.2058	.50105	.0914	.0908	.50175	.0571	.0565	.50266
6	.1885	.1879	.50087	.0829	.0824	.50153	.0497	.0492	.50259
8	.1631	.1627	.50066	.0700	.0697	.50133	.0379	.0375	.50255
10	.1455	.1452	.50054	.0600	.0597	.50126	.0289	.0286	.50255
15	.1163	.1161	.50042	.0414	.0412	.50124	.0147	.0146	.50255
20	.0958	.0956	.50039	.0286	.0284	.50123	.0075	.0074	.50255
30	.0660	.0659	.50039	.0137	.0136	.50123	.0019	.0019	.50255
40	.0456	.0455	.50039	.0065	.0065	.50123	.0005	.0005	.50255
50	.0315	.0314	.50039	.0031	.0031	.50123	.0001	.0001	.50255
60	.0217	.0217	.50039	.0015	.0015	.50123			
80	.0104	.0104	.50039	.0003	.0003	.50123			
100	.0049	.0049	.50039	.0001	.0001	.50123			
150	.0008	.0008	.50039						
200	.0001	.0001	.50039						

θ	K2 = 20 q_1	K2 = 20 q_2	K2 = 20 $\frac{q_1}{q_1+q_2}$	K2 = 50 q_1	K2 = 50 q_2	K2 = 50 $\frac{q_1}{q_1+q_2}$	K2 = 100 q_1	K2 = 100 q_2	K2 = 100 $\frac{q_1}{q_1+q_2}$
.2			.62300	.2156	.1305	.65317	.1744	.0926	.65317
.3	.2131	.1697	.55670	.1445	.1071	.57440	.1091	.0751	.59241
.4	.1717	.1471	.53862	.1119	.0923	.54797	.0806	.0634	.55965
.5	.1472	.1314	.52837	.0939	.0821	.53373	.0652	.0548	.54321
.6	.1308	.1197	.52212	.0824	.0743	.52578	.0551	.0479	.53502
.8	.1099	.1034	.51523	.0678	.0630	.51828	.0418	.0373	.52871
1	.0967	.0923	.51163	.0581	.0547	.51532	.0326	.0293	.52693
1.5	.0771	.0748	.50759	.0418	.0397	.51331	.0180	.0162	.52618
2	.0652	.0637	.50611	.0307	.0291	.51301	.0100	.0090	.52614
3	.0494	.0484	.50530	.0166	.0158	.51296	.0031	.0027	.52613
4	.0381	.0373	.50517	.0090	.0085	.51296	.0009	.0008	.52613
5	.0295	.0289	.50515	.0049	.0046	.51296	.0003	.0003	.52613
6	.0228	.0223	.50515	.0026	.0025	.51296			
7				.0008	.0007	.51296			
8	.0136	.0134	.50515	.0002	.0002	.51296			
10	.0082	.0080	.50515						
15	.0023	.0022	.50515						
20	.0006	.0006	.50515						
30									
40									
50									

L = 20; c = 0.5

θ	K2 = 2			K2 = 5			K2 = 10		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
.8							.1586	.1501	.51373
1							.1400	.1342	.51062
1.5							.1124	.1094	.50677
2	.3283	.3249	.50264	.1464	.1439	.50434	.0966	.0947	.50497
3	.2673	.2655	.50175	.1189	.1175	.50285	.0783	.0773	.50325
4	.2312	.2300	.50131	.1027	.1018	.50213	.0675	.0669	.50241
5	.2067	.2058	.50105	.0917	.0910	.50169	.0603	.0598	.50192
6	.1886	.1879	.50087	.0836	.0831	.50141	.0549	.0546	.50159
8	.1632	.1628	.50065	.0723	.0720	.50105	.0475	.0473	.50119
10	.1459	.1456	.50052	.0646	.0644	.50084	.0424	.0422	.50097
15	.1191	.1189	.50035	.0527	.0525	.50056	.0341	.0340	.50073
20	.1031	.1030	.50026	.0455	.0454	.50043	.0284	.0283	.50066
30	.0841	.0841	.50017	.0363	.0363	.50034	.0201	.0201	.50064
40	.0727	.0726	.50013	.0299	.0299	.50032	.0143	.0143	.50064
50	.0645	.0645	.50012	.0248	.0248	.50031	.0102	.0102	.50064
60	.0581	.0581	.50011	.0206	.0206	.50031	.0073	.0073	.50064
80	.0478	.0478	.50010	.0142	.0142	.50031	.0037	.0037	.50064
100	.0397	.0397	.50010	.0098	.0098	.50031	.0019	.0019	.50064
150	.0250	.0250	.50010	.0039	.0039	.50031	.0003	.0003	.50064
200	.0157	.0157	.50010	.0015	.0015	.50031			
300	.0062	.0062	.50010	.0002	.0002	.50031			
400	.0025	.0025	.50010						
500	.0010	.0010	.50010						
600	.0004	.0004	.50010						

θ	K2 = 20			K2 = 50			K2 = 100		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
.4							.0811	.0653	.55373
.5				.0939	.0823	.53285	.0673	.0581	.53664
.6				.0827	.0749	.52486	.0588	.0528	.52679
.8	.1099	.1034	.51520	.0690	.0646	.51643	.0487	0455	.51710
1	.0967	.0923	.51158	.0605	.0576	.51228	.0426	.0405	.51273
1.5	.0774	.0752	.50727	.0483	.0468	.50761	.0336	.0325	.50846
2	.0664	.0650	.50531	.0414	.0404	.50560	.0280	.0273	.50713
3	.0538	.0530	.50345	.0331	.0326	.50396	.0204	.0198	.50653
4	.0464	.0459	.50257	.0277	.0274	.50345	.0149	.0146	.50647
5	.0413	0410	50206	.0236	.0233	.50329	.0110	.0107	.50646
6	.0376	.0373	50176	.0201	.0199	.50324	.0081	.0079	.50646
8	.0320	.0318	.50146	.0147	.0145	.50322	.0044	.0043	.50646
10	.0278	.0277	.50135	.0108	.0106	.50322	.0024	.0023	.50646
15	.0200	.0199	.50129	.0049	.0049	.50322	.0005	.0005	.50646
20	.0145	.0144	.50128	.0023	.0022	.50322	.0001	.0001	.50646
30	.0076	.0076	.50128	.0005	.0005	.50322			
40	.0040	.0040	.50128						
50	.0021	.0021	.50128						
60	.0011	.0011	.50128						
80	.0003	.0003	.50128						

L = 50; c = 0.5

θ	K2 = 2 q_1	K2 = 2 q_2	K2 = 2 $\frac{q_1}{q_1+q_2}$	K2 = 5 q_1	K2 = 5 q_2	K2 = 5 $\frac{q_1}{q_1+q_2}$	K2 = 10 q_1	K2 = 10 q_2	K2 = 10 $\frac{q_1}{q_1+q_2}$
4							.06754	.06690	.50241
5							.06028	.05982	.50192
6				.08358	.08311	.50141	.05495	.05460	.50159
8				.07228	.07198	.50105	.04750	.04728	.50119
10				.06460	.06438	.50084	.04244	.04228	.50095
15				.05268	.05257	.50056	.03461	.03452	.50063
20	.1031	.1030	.50026	.04560	.04552	.50042	.02995	.02989	.50047
30	.0841	.0841	.50017	.03721	.03717	.50028	.02444	.02440	.50031
40	.0729	.0728	.50013	.03222	03219	.50021	.02115	02113	.50024
50	.0652	.0651	.50010	.02881	.02879	.50017	.01891	01890	.50019
60	.0595	.0595	.50009	.02630	.02628	.50014	.01725	.01724	.50016
80	.0515	.0515	.50007	.02277	.02276	.50010	.01485	.01484	.50013
100	.0461	.0461	.50005	.02036	.02035	.50008	.01309	.01308	.50011
150	.0376	.0376	.50004	.01650	.01649	.50006	.00986	.00986	.50010
200	.0326	.0326	.50003	.01395	.01395	.50005	.00751	.00751	.50010
300	.0264	.0264	.50002	.01029	.01029	.50005	.00436	.00436	.50010
400	.0223	.0223	.50002	.00765	.00765	.50005	.00254	.00253	.50010
500	.0191	.0191	.50002	.00569	.00569	.50005	.00147	.00147	.50010
600	.0165	.0165	.50002	.00423	.00423	.50005	.00086	.00086	.50010
800	.0122	.0122	.50002	.00234	.00234	.50005	.00029	.00029	.50010
1000	.0091	.0091	.50002	.00129	.00129	.50005	.00010	.00010	.50010
1500				.00029	.00029	.50005	.00001	.00001	.50010
2000	.0043	.0043	.50002	.00007	.00007	.50005			
3000	.0021	.0021	.50005						
4000	.0005	.0005	.50005						
5000	.0001	.0001	.50005						

θ	K2 = 20 q_1	K2 = 20 q_2	K2 = 20 $\frac{q_1}{q_1+q_2}$	K2 = 50 q_1	K2 = 50 q_2	K2 = 50 $\frac{q_1}{q_1+q_2}$	K2 = 100 q_1	K2 = 100 q_2	K2 = 100 $\frac{q_1}{q_1+q_2}$
1.5	.07738	.07518	.50724	.04829	.04685	.50760	.03400	.03297	.50771
2	.06643	.06503	.50531	.04140	.04050	.50552	.02915	.02850	.50560
3	.05377	.05303	.50345	.03349	.03301	.50358	.02357	.02323	.50362
4	.04637	.04590	.50256	.02887	.02857	.50265	.02031	.02010	.50268
5	.04138	.04104	.50203	.02576	.02554	.50210	.01812	.01796	.50213
6	.03771	.03746	.50169	.02347	.02331	.50174	.01650	.01639	.50178
8	.03259	.03243	.50126	.02028	.02018	.50130	.01421	.01413	.50138
10	.02912	.02900	.50100	.01811	.01804	.50104	.01256	.01250	.50119
15	.02374	.02367	.50067	.01472	.01468	.50071	.00963	.00959	.50105
20	.02054	.02050	.50050	.01259	.01256	.50059	.00749	.00746	.50103
30	.01674	.01672	.50034	.00963	.00961	.50053	.00455	.00453	.50103
40	.01443	.01441	.50026	.00748	.00746	.50052	.00277	.00276	.50103
50	.01275	.01274	.50023	.00581	.00580	.50051	.00168	.00168	.50103
60	.01139	.01138	.50022	.00452	.00451	.50051	.00102	.00102	.50103
80	.00921	.00920	.50021	.00273	.00273	.50051	.00038	.00038	.50103
100	.00748	.00747	.50021	.00165	.00165	.50051	.00014	.00014	.50103
150	.00445	.00445	.50021	.00047	.00047	.50051			
200	.00265	.00265	.50021	.00013	.00013	.50051	.00001	.00001	.50103
300	.00094	.00094	.50021	.00001	.00001	.50051			
400	.00033	.00033	.50021						
500	.00012	.00012	.50021						
600	.00004	.00004	.50021						

L = 100; c = 0.5

K2 = 2

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
60	.05959	.05947	.50009
80	.05151	.05150	.50007
100	.04607	.04606	.50005
150	.03762	.03761	.50004
200	.03258	.03257	.50003
300	.02660	.02660	.50002
400	.02303	.02303	.50001
500	.02060	.02060	.50001
600	.01881	.01881	.50001
800	.01628	.01628	.50001
1000	.01453	.01453	.50001
1500	.01161	.01161	.50001
2000	.00957	.00957	.50001
3000	.00659	.00659	.50001
4000	.00455	.00455	.50001
5000	.00314	.00314	.50001
6000	.00217	.00217	.50001
8000	.00104	.00104	.50001
10^4	.00049	.00049	.50001
1.5×10^4	.00008	.00008	.50001
2×10^4	.00001	.00001	.50001

K2 = 5

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
60	.02630	.02628	.50014
80	.02277	.02276	.50010
10^2	.02037	.02036	.50008
1.5×10^2	.01663	.01662	.50006
2×10^2	.01440	.01440	.50004
3×10^2	.01175	.01175	.50003
4×10^2	.01018	.01017	.50002
5×10^2	.00908	.00908	.50002
6×10^2	.00825	.00825	.50002
8×10^2	.00697	.00697	.50001
10^3	.00598	.00598	.50001
1.5×10^3	.00412	.00412	.50001
2×10^3	.00284	.00284	.50001
3×10^3	.00136	.00136	.50001
4×10^3	.00065	.00065	.50001
5×10^3	.00031	.00031	.50001
6×10^3	.00015	.00015	.50001
8×10^3	.00003	.00003	.50001

K2 = 10

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
20	.02995	.02990	.50047
30	.02444	.02440	.50031
40	.02115	.02113	.50024
50	.01892	.01890	.50019
60	.01727	.01726	.50016
80	.01495	.01494	.50012
10^2	.01337	.01337	.50009
1.5×10^2	.01092	.01091	.50006
2×10^2	.00945	.00945	.50005
3×10^2	.00768	.00768	.50003
4×10^2	.00654	.00654	.50003
5×10^2	.00566	.00566	.50003
6×10^2	.00493	.00493	.50003
8×10^2	.00375	.00375	.50003
10^3	.00286	.00286	.50003
1.5×10^3	.00145	.00145	.50003
2×10^3	.00074	.00074	.50003
3×10^3	.00019	.00019	.50003
4×10^3	.00004	.00004	.50003
5×10^3	.00001	.00001	.50003

K2 = 20

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
10	.02912	.02900	.50100
15	.02374	.02367	.50067
20	.02054	.02050	.50050
30	.01676	.01673	.50033
40	.01451	.01449	.50025
50	.01297	.01296	.50020
60	.01184	.01183	.50017
80	.01025	.01025	.50013
10^2	.00917	.00916	.50010
1.5×10^2	.00746	.00746	.50007
2×10^2	.00637	.00637	.50006
3×10^2	.00485	.00485	.50005
4×10^2	.00374	.00374	.50005
5×10^2	.00288	.00288	.50005
6×10^2	.00222	.00222	.50005
8×10^2	.00133	.00132	.50005
10^3	.00079	.00079	.50005
1.5×10^3	.00022	.00022	.50005
2×10^3	.00006	.00006	.50005

K2 = 50

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
4	.02887	.02857	.50265
5	.02576	.02554	.50210
6	.02347	.02331	.50174
8	.02028	.02018	.50130
10	.01812	.01804	.50104
15	.01477	.01473	.50069
20	.01278	.01275	.50052
30	.01042	.01041	.50034
40	.00902	.00901	.50026
50	.00806	.00806	.50021
60	.00734	.00734	.50018
80	.00628	.00628	.50015
100	.00548	.00548	.50014
150	.00398	.00397	.50013
200	.00290	.00290	.50013
300	.00155	.00155	.50013
400	.00084	.00084	.50013
500	.00044	.00044	.50013
600	.00023	.00023	.50013
800	.00007	.00007	.50013
1000	.00002	.00002	.50013

K2 = 100

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
2	.02914	.02850	.50559
3	.02357	.02323	.50362
4	.02032	.02010	.50268
5	.01812	.01797	.50213
6	.01651	.01640	.50176
8	.01427	.01419	.50131
10	.01275	.01269	.50105
15	.01039	.0136	.50070
20	.00899	.00897	.50052
30	.00731	.00730	.50036
40	.00626	.00625	.50030
50	.00546	.00545	.50027
60	.00480	.00479	.50026
80	.00373	.00373	.50026
100	.00291	.00291	.50026
200	.00156	.00156	.50026
300	.00084	.00084	.50026
400	.00024	.00024	.50026
500	.00007	.00007	.50026
600	.00002	.00002	.50026

$L = 10^3; \quad c = 0.5$

K2 = 2

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
10^4	.004607	.004607	.50000
1.5×10^4	.003761	.003761	.50000
2×10^4	.003257	.003257	.50000
3×10^4	.002660	.002660	.50000
4×10^4	.002303	.002303	.50000
5×10^4	.002060	.002060	.50000
6×10^4	.001881	.001881	.50000
8×10^4	.001628	.001628	.50000
10^5	.001453	.001453	.50000
1.5×10^5	.001161	.001161	.50000
2×10^5	.000957	.000957	.50000
3×10^5	.000659	.000659	.50000
4×10^5	.000455	.000455	.50000
5×10^5	.000314	.000314	.50000
6×10^5	.000217	.000217	.50000
8×10^5	.000104	.000104	.50000
10^6	.000049	.000049	.50000
1.5×10^6	.000008	.000008	.50000
2×10^6	.000001	.000001	.50000

K2 = 5

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
6×10^3	.002628	.002628	.50000
8×10^3	.002276	.002276	.50000
10^4	.002036	.002036	.50000
1.5×10^4	.001662	.001662	.50000
2×10^4	.001440	.001440	.50000
3×10^4	.001175	.001175	.50000
4×10^4	.001017	.001017	.50000
5×10^4	.000908	.000908	.50000
6×10^4	.000825	.000825	.50000
8×10^4	.000697	.000697	.50000
10^5	.000598	.000598	.50000
1.5×10^5	.000412	.000412	.50000
2×10^5	.000284	.000284	.50000
3×10^5	.000136	.000136	.50000
4×10^5	.000065	.000065	.50000
5×10^5	.000031	.000031	.50000
6×10^5	.000015	.000015	.50000
8×10^5	.000003	.000003	.50000
10^6	.000001	.000001	.50000

K2 = 10

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
2×10^3	.002989	.002988	.50001
3×10^3	.002440	.002440	.50001
4×10^3	.002113	.002113	.50001
5×10^3	.001890	.001890	.50001
6×10^3	.001725	.001725	.50001
8×10^3	.001494	.001494	.50001
10^4	.001336	.001336	.50001
1.5×10^4	.001091	.001091	.50001
2×10^4	.000945	.000945	.50001
3×10^4	.000768	.000768	.50001
4×10^4	.000654	.000654	.50001
5×10^4	.000566	.000566	.50001
6×10^4	.000493	.000493	.50001
8×10^4	.000375	.000375	.50001
10^5	.000286	.000286	.50001
1.5×10^5	.000145	.000145	.50001
2×10^5	.000074	.000074	.50001
3×10^5	.000019	.000019	.50001
4×10^5	.000005	.000005	.50001
5×10^5	.000001	.000001	.50001

K2 = 20

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
10^3	.002898	.002898	.50001
1.5×10^3	.002366	.002366	.50001
2×10^3	.002049	.002049	.50001
3×10^3	.001673	.001673	.50001
4×10^3	.001449	.001449	.50001
5×10^3	.001296	.001296	.50001
6×10^3	.001183	.001183	.50001
8×10^3	.001025	.001025	.50001
10^4	.000916	.000916	.50001
1.5×10^4	.000746	.000746	.50001
2×10^4	.000637	.000637	.50001
3×10^4	.000485	.000485	.50001
4×10^4	.000374	.000374	.50001
5×10^4	.000288	.000288	.50001
6×10^4	.000222	.000222	.50001
8×10^4	.000132	.000132	.50001
10^5	.000079	.000079	.50001
1.5×10^5	.000022	.000022	.50001
2×10^5	.000006	.000006	.50001

K2 = 50

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
4×10^2	.002850	.002850	.50003
5×10^2	.002549	.002549	.50003
6×10^2	.002327	.002327	.50002
8×10^2	.002015	.002015	.50002
10^3	.001803	.001803	.50002
1.5×10^3	.001472	.001472	.50001
2×10^3	.001275	.001275	.50001
3×10^3	.001041	.001041	.50001
4×10^3	.000901	.000901	.50001
5×10^3	.000806	.000806	.50001
6×10^3	.000734	.000734	.50001
8×10^3	.000628	.000628	.50001
10^4	.000547	.000547	.50001
1.5×10^4	.000397	.000397	.50001
2×10^4	.000290	.000290	.50001
3×10^4	.000155	.000155	.50001
4×10^4	.000082	.000082	.50001
5×10^4	.000044	.000044	.50001
6×10^4	.000023	.000023	.50001
8×10^4	.000007	.000007	.50001
10^5	.000002	.000002	.50001

K2 = 100

θ	q_1	q_2	$\frac{q_1}{q_1+q_2}$
2×10^2	.002836	.002835	.50006
3×10^2	.002315	.002315	.50004
4×10^2	.002005	.002005	.50003
5×10^2	.001793	.001793	.50003
6×10^2	.001637	.001637	.50002
8×10^2	.001418	.001418	.50002
10^3	.001268	.001268	.50002
1.5×10^3	.001035	.001035	.50001
2×10^3	.000897	.000896	.50001
3×10^3	.000730	.000730	.50001
4×10^3	.000625	.000625	.50001
5×10^3	.000525	.000525	.50001
6×10^3	.000479	.000479	.50001
8×10^3	.000373	.000373	.50001
10^4	.000291	.000291	.50001
1.5×10^4	.000156	.000156	.50001
2×10^4	.000084	.000084	.50001
3×10^4	.000024	.000024	.50001
4×10^4	.000007	.000007	.50001
5×10^4	.000002	.000002	.50001

L = 20; c = 0.1

θ	K2 = 2			K2 = 5.0			K2 = 10		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
.6							.02694	.24149	.10043
.8							.02332	.20913	.10032
1				.03301	.29595	.10033	.02084	.18704	.10026
1.5				.02691	.24164	.10022	.01700	.15272	.10017
2	.05801	.52093	.10020	.02329	.20927	.10017	.01472	.13225	.10013
3	.04733	.42535	.10013	.01901	.17086	.10011	.01201	.10798	.10008
4	.04097	.36837	.10010	.01646	.14797	.10008	.01040	.09351	.10006
5	.03664	.32948	.10008	.01472	.13235	.10007	.00930	.08362	.10005
6	.03344	.30077	.10007	.01343	.12082	.10005	.00848	.07625	.10004
8	.02896	.26048	.10005	.01163	.10463	.10004	.00729	.06558	.10003
10	.02590	.23298	.10004	.01040	.09355	.10003	.00641	.05768	.10003
15	.02114	.19023	.10003	.00844	.07595	.10002	.00479	.04314	.10003
20	.01831	.16473	.10002	.00716	.06446	.10002	.00362	.03255	.10003
30	.01492	.13427	.10001	.00534	.04808	.10002	.00206	.01856	.10003
40	.01281	.11528	.10001	.00402	.03616	.10002	.00118	.01059	.10003
50	.01123	.10110	.10001	.00303	.02723	.10002	.00067	.00604	.10003
60	.00994	.08942	.10001	.00228	.02050	.10002	.00038	.00345	.10003
80	.00784	.07052	.10001	.00129	01162	.10002	.00012	.00112	.10003
100	.00620	.05576	.10001	.00073	.00659	.10002	.00004	.00036	.10003
150	.00345	.03103	.10001	.00018	.00159	.10002			
200	.00192	.01727	.10001	.00004	.00039	.10002			
300	.00059	.00535	.10001						
400	.00018	.00166	.10001						
500	.00006	.00051	.10001						
600	.00002	.00016	.10000						

θ	K2 = 20			K2 = 50			K2 = 100		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
.08							.02177	.19127	.10221
.1				.02770	.24421	.10186	.01937	.17103	.10173
.15				.02245	.19933	.10121	.01571	.13960	.10113
.2				.01937	.17260	.10090	.01356	.12088	.10084
.3	.02569	.22945	.10070	.01576	.14090	.10059	.01103	.09868	.10055
.4	.02221	.19869	.10052	.01362	.12202	.10044	.00954	.08545	.10041
.5	.01984	.17771	.10042	.01217	.10913	.10035	.00852	.07641	.10033
.6	.01809	.16222	.10035	.01110	.09962	.10029	.00777	.06968	.10028
.8	.01565	.14048	.10026	.00961	.08627	.10022	.00668	.05994	.10022
1	.01499	.12565	.10021	.00859	.07714	.10018	.00587	.05276	.10020
1.5	.01142	.10259	.10014	.00697	.06266	.10012	.00440	.03954	.10018
2	.00988	.08884	.10010	.00592	.05326	.10011	.00333	.02992	.10018
3	.00806	.07244	.10007	.00444	.03991	.10010	.00191	.01716	.10018
4	.00693	.06232	.10006	.00336	.03019	.10010	.00110	.00985	.10018
5	.00610	.05483	.10005	.00254	.02285	.10010	.00063	.00565	.10018
6	.00542	.04871	.10005	.00192	.01730	.10010	.00036	.00324	.10018
8	.00431	.03881	.10005	.00110	.00992	.10010	.00012	.00107	.10018
10	.00345	.03102	.10005	.00063	.00569	.10010	.00004	.00035	.10018
15	.00197	.01775	.10005	.00016	.00142	.10010			
20	.00113	.01016	.10005	.00004	.00035	.10010			
30	.00037	.00333	.10005						
40	.00012	.00109	.10005						
50	.00004	.00036	.10005						
60	.00001	.00012	.10005						

L = 20; c = 0.3

θ	K2 = 2			K2 = 5			K2 = 10		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
1							.0709	.1626	.30354
1.5				.0896	.2068	.30234	.0575	.1327	.30232
2				.0774	.1791	.30174	.0497	.1149	.30172
3	.1505	.3490	.30094	.0630	.1462	.30115	.0404	.0938	.30114
4	.1302	.3028	.30071	.0545	.1266	.30086	.0350	.0812	.30085
5	.1164	.2708	.30057	0487	.1133	.30069	.0312	.0727	.30068
6	.1062	.2473	.30047	.0444	.1034	.30057	.0285	.0663	.30057
8	.0919	.2141	.30035	.0385	.0895	.30043	.0246	.0573	.30043
10	.0822	.1915	.30028	.0344	.0801	.30034	.0219	.0509	.30037
15	.0671	.1564	.30019	.0280	.0653	.30023	.0171	.0398	.30031
20	.0581	.1354	.30014	.0240	.0560	.30019	.0136	.0316	.30030
30	.0474	.1105	.30010	.0186	.0435	.30016	.0086	.0201	.30030
40	.0408	.0952	.30008	.0147	.0343	.30016	.0055	.0128	.30030
50	.0361	.0841	.30007	.0116	.0271	.30016	.0035	.0082	.30030
60	.0322	.0751	.30006	.0092	.0214	.30016	.0022	.0052	.30030
80	.0260	.0606	.30006	.0009	.0021	.30030	.0009	.0021	.30030
100	.0210	.0491	.30006	.0036	.0084	.30016	.0004	.0009	.30030
150	.0124	.0290	.30006	.0011	.0026	.30016			
200	.0074	.0172	.30006	.0003	.0008	.30016			
300	.0026	.0060	.30006						
400	.0009	.0021	.30006						
500	.0003	.0007	.30006						

θ	K2 = 20			K2 = 50			K2 = 100		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
.5	.0693	.1561	.30740	.0427	.0964	.30728	.0300	.0676	.30724
.6	.0628	.1424	.30602	.0387	.0879	.30592	.0272	.0616	.30589
.8	.0539	.1232	.30439	.0333	.0760	.30431	.0233	.0532	.30437
1	.0480	.1101	.30346	.0296	.0680	.30340	.0206	.0473	.30361
1.5	.0389	.0899	.30226	.0240	.0554	.30225	.0161	.0371	.30294
2	.0336	.0778	.30168	.0206	.0477	.30177	.0129	.0297	.30281
3	.0274	.0635	.30111	.0161	.0374	.30147	.0084	.0193	.30278
4	.0236	.0549	.30085	.0129	.0299	.30141	.0054	.0125	.30278
5	.0210	.0488	.30072	.0104	.0241	.30140	.0035	.0081	.30278
6	.0189	.0440	.30065	.0084	.0194	.30140	.0023	.0053	.30278
8	.0157	.0365	.30060	.0054	.0125	.30140	.0010	.0022	.30278
10	.0131	.0305	.30058	.0035	.0081	.30140	.0004	.0009	.30278
15	.0084	.0196	.30058	.0012	.0027	.30140			
20	.0054	.0126	.30058	.0004	.0009	.30140			
30	.0022	.0052	.30058						
40	.0009	.0022	.30058						
50	.0004	.0009	.30058						
60	.0002	.0004	.30058						

$L=20$; $\sigma=0.7$

θ	K2 = 2 q_1	K2 = 2 q_2	K2 = 2 $\frac{q_1}{q_1+q_2}$	K2 = 5 q_1	K2 = 5 q_2	K2 = 5 $\frac{q_1}{q_1+q_2}$	K2 = 10 q_1	K2 = 10 q_2	K2 = 10 $\frac{q_1}{q_1+q_2}$
1.5							.1953	.0799	.70963
2							.1671	.0692	.70705
3				.1949	.0822	.70347	.1348	.0565	.70458
4	.3474	.1481	.70120	.1682	.0712	.70259	.1160	.0489	.70339
5	.3106	.1325	.70096	.1501	.0637	.70206	.1034	.0438	.70269
6	.2834	.1210	.70080	.1368	.0582	.70171	.0942	.0399	.70223
8	.2453	.1048	.70060	.1183	.0504	.70128	.0813	.0346	.70166
10	.2193	.0938	.70048	.1057	.0451	.70102	.0726	.0309	.70133
15	.1790	.0766	.70032	.0862	.0368	.70068	.0591	.0252	.70090
20	.1550	.0664	.70024	.0746	.0319	.70051	.0507	.0217	.70072
30	.1265	.0542	.70016	.0606	.0259	.70036	.0395	.0169	.70062
40	.1095	.0469	.70012	.0516	.0221	.70030	.0313	.0134	.70060
50	.0976	.0418	.70010	.0446	.0191	.70029	.0249	.0106	.70060
60	.0884	.0379	.70009	.0389	.0166	.70028	.0198	.0085	.70060
80	.0742	.0318	.70008	.0296	.0127	.70028	.0126	.0054	.70060
100	.0629	.0269	.70008	.0225	.0097	.70028	.0080	.0034	.70060
150	.0421	.0180	.70008	.0114	.0049	.70028	.0026	.0011	.70060
200	.0282	.0121	.70008	.0058	.0025	.70028	.0008	.0003	.70060
300	.0126	.0054	.70008	.0015	.0006	.70028			
400	.0057	.0024	.70008	.0004	.0002	.70028			
500	.0025	.0011	.70008						
600	.0011	.0005	.70008						
800	.0002	.0001	.70008						

θ	K2 = 20 q_1	K2 = 20 q_2	K2 = 20 $\frac{q_1}{q_1+q_2}$	K2 = 50 q_1	K2 = 50 q_2	K2 = 50 $\frac{q_1}{q_1+q_2}$	K2 = 100 q_1	K2 = 100 q_2	K2 = 100 $\frac{q_1}{q_1+q_2}$
1.5	.1399	.0567	.71166	.0901	.0362	.71356	.0642	.0256	.71468
2	.1190	.0490	.70838	.0762	.0312	.70950	.0539	.0220	.71038
3	.0956	.0399	.70536	.0608	.0253	.70601	.0420	.0174	.70741
4	.0821	.0345	.70394	.0520	.0218	.70455	.0342	.0142	.70660
5	.0731	.0309	.70312	.0458	.0193	.70384	.0283	.0118	.70637
6	.0665	.0282	.70258	.0410	.0173	.70348	.0234	.0097	.70630
8	.0573	.0243	.70195	.0334	.0141	.70322	.0162	.0067	.70627
10	.0509	.0217	.70162	.0275	.0116	.70315	.0112	.0046	.70627
15	.0401	.0171	.70131	.0170	.0072	.70313	.0044	.0018	.70627
20	.0324	.0138	.70125	.0105	.0044	.70313	.0017	.0007	.70627
30	.0215	.0091	.70123	.0040	.0017	.70313	.0003	.0001	.70627
40	.0142	.0061	.70123	.0015	.0007	.70313			
50	.0094	.0040	.70123	.0006	.0003	.70313			
60	.0062	.0027	.70123	.0002	.0001	.70313			
80	.0027	.0012	.70123						
100	.0012	.0005	.70123						
200	.0002	.0001	.70123						

L = 20; c = 0.9

θ	K2 = 2			K2 = 5			K2 = 10		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
3	.5596	.0619	90039	31162	.03417	90118	.24004	.02609	.90196
4	.4845	.0537	.90029	.26947	.02965	.90088	.20708	.02263	.90147
5	.4333	.0580	.90023	.24080	.02655	.90071	.18479	.02026	.90117
6	.3955	.0438	.90019	.21969	.02425	.90059	.16842	.01851	.90098
8	.3425	.0380	.90015	.19011	.02102	.90044	.14557	.01604	.90073
10	.3063	.0340	.90012	.16996	.01881	.90035	.13005	.01436	.90058
15	.2501	.0278	.90008	.13868	.01537	.90024	.10602	.01173	.90039
20	.2165	.0240	.90006	.12006	.01331	.90018	.09174	.01016	.90029
30	.1768	.0196	.90004	.09799	.01087	.90012	.07471	.00828	.90020
40	.1531	.0170	.90003	.08472	.00940	.90009	.06411	.00711	.90016
50	.1368	.0152	.90002	.07539	.00837	.90008	.05621	.00624	.90015
60	.1244	.0138	.90002	.06809	.00756	.90007	.04971	.00552	.90014
80	.1060	.0118	.90002	.05662	.00629	.90006	.03921	.00435	.90014
100	.0917	.0102	.90002	.04729	.00527	.90006	.03101	.00344	.90014
150	.0651	.0072	.90002	.03081	.00342	.90006	.01726	.00191	.90014
200	.0463	.0051	.90002	.02001	.00222	.90006	.00961	.00107	.90014
300	.0235	.0026	.90002	.00844	.00094	.90006	.00298	.00033	.90014
400	.0119	.0013	.90002	.00356	.00040	.90006	.00092	.00010	.90014
500	.0061	.0007	.90002	.00150	.00017	.90006	.00029	.00003	.90014
600	.0031	.0003	.90002	.00063	.00007	.90006	.00009	.00001	.90014
800	.0008	.0001	.90002	.00011	.00001	.90006			

θ	K2 = 20			K2 = 50			K2 = 100		
	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$	q_1	q_2	$\frac{q_1}{q_1+q_2}$
2							.12141	.01245	.90699
3	.18668	.02010	.90280	.12946	.01379	.90375	.09542	.01010	.90429
4	.16046	.01742	.90208	.11063	.01192	.90274	.08118	.00871	.90310
5	.14288	.01559	.90165	.09819	.01065	.90216	.07176	.00776	.90247
6	.13004	.01423	.90137	.08918	.00971	.90178	.06484	.00704	.90209
8	.11220	.01233	.90102	.07670	.00840	.90133	.05480	.00597	.90171
10	.10013	.01103	.90081	.06820	.00749	.90108	.04732	.00517	.90157
15	.08149	.00900	.90054	.05430	.00598	.90082	.03369	.00368	.90148
20	.07033	.00778	.90042	.04477	.00493	.90076	.02416	.00264	.90147
30	.05629	.00623	.90032	.03106	.00342	.90073	.01244	.00136	.90147
40	.04656	.00516	.90029	.02163	.00238	.90073	.00641	.00070	.90147
50	.03884	.00430	.90029	.01506	.00166	.90073	.00330	.00036	.90147
60	.03247	.00360	.90029	.01049	.00116	.90073	.00170	.00019	.90147
80	.02271	.00252	.90029	.00509	.00056	.90073	.00045	.00005	.90147
100	.01589	.00176	.90029	.00247	.00027	.90073	.00012	.00001	.90147
150	.00651	.00072	.90029	.00040	.00004	.90073			
200	.00267	.00030	.90029	.00007	.00001	.90073			
300	.00045	.00005	.90029						
400	.00007	.00001	.90029						

APPENDIX M

INDIVIDUAL INSTANTANEOUS DIMENSIONLESS FLUXES, q_1 AND q_2, FOR TWO-LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Tables valid for all $r_b/H > 10.0$

Calculated from:

$$q_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{2b^2V^2(b)}{b^2V^2(b) - \frac{4}{\pi^2}}\right] \frac{\sin \gamma c}{\gamma} \qquad \text{(V-189)}$$

$$q_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{2b^2V^2(b)}{b^2V^2(b) - \frac{4}{\pi^2}}\right] \left[\frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)}\right] \qquad \text{(V-190)}$$

Eigenvalues b from: Appendix B

Eigenvalues a from: Appendix E

Truncation Test: $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$ for m series

Only one term used in n series

Estimated Accuracy for q_1 and q_2: $\pm$ 1 part in 10^4

Definition of Dimensionless Variables:

$$r = \frac{r_a}{r_b} \qquad \text{(V-135)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(V-136)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-137)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c r_b^2} \quad \text{(for liquid flow)}$$

$$= \frac{k_1 t_a}{\rho_1 C_{p1} r_b^2} \quad \text{(for heat conduction)} \qquad \text{(V-140)}$$

All values presented are for $K_2^* = K_2 = \frac{k_2}{k_1}$

R = 5; c = 0.5

θ	K2 = 2: q_1,q_2	θ	K2 = 5: q_1,q_2	θ	K2 = 10: q_1,q_2	θ	K2 = 20: q_1,q_2	θ	K2 = 50: q_1,q_2	θ	K2 = 100: q_1,q_2	K2 = 100: q_1,q_2
.5	.5390	.3	.5084	.15	.5227	.08	.5198	.03	.5358	.015	.5377	.5373
.6	.5084	.4	.4650	.2	.4776	.1	.4845	.04	.4890	.02	.4907	.4904
.8	.4650	.5	.4350	.3	.4230	.15	.4288	.05	.4567	.03	.4339	.4337
1	.4350	.6	.4124	.4	.3894	.2	.3946	.06	.4325	.04	.3991	.3990
1.5	.3869	.8	.3799	.5	.3653	.3	.3510	.08	.3979	.05	.3745	.3744
2	.3561	1	.3561	.6	.3460	.4	.3197	.1	.3734	.06	.3551	.3550
3	.3118	1.5	.3118	.8	.3144	.5	.2933	.15	.3301	.08	.3241	.3241
4	.2761	2	.2761	1	.2874	.6	.2695	.2	.2969	.1	.2981	.2981
5	.2449	3	.2173	1.5	.2307	.8	.2297	.3	.2420	.15	.2434	.2434
6	.2173	4	.1710	2	.1852	1	.1928	.4	.1975	.2	.1990	.1990
8	.1710	5	.1347	3	.1195	1.5	.1268	.5	.1611	.3	.1331	.1331
10	.1347	6	.1060	4	.0771	2	.0835	.6	.1315	.4	.0890	.0890
15	.0741	8	.0657	5	.0497	3	.0361	.8	.0876	.5	.0595	.0595
20	.0407	10	.0407	6	.0321	4	.0156	1	.0583	.6	.0398	.0398
30	.0123	15	.0123	8	.0133	5	.0068	1.5	.0211	.8	.0178	.0178
40	.0037	20	.0037	10	.0055	6	.0029	2	.0076	1	.0079	.0079
50	.0011	30	.0003	15	.0006	8	.0005	3	.0010	1.5	.0011	.0011
60	.0003			20	.0001	10	.0001	4	.0001	2	.0001	.0001

R = 10; c = 0.5

θ	K2 = 2: q_1,q_2	θ	K2 = 5: q_1,q_2	θ	K2 = 10: q_1,q_2	θ	K2 = 20: q_1,q_2	θ	K2 = 50: q_1,q_2	θ	K2 = 100: q_1,q_2
2	.3581	1	.3581	.6	.3492	.3	.3535	.15	.3360	.06	.3572
3	.3224	1.5	.3224	.8	.3243	.4	.3281	.2	.3126	.08	.3314
4	.3004	2	.3004	1	.3068	.5	.3103	.3	.2837	.1	.3133
5	.2850	3	.2733	1.5	.2788	.6	.2970	.4	.2657	.15	.2844
6	.2733	4	.2562	2	.2612	.8	.2776	.5	.2528	.2	.2663
8	.2562	5	.2436	3	.2383	1	.2640	.6	.2425	.3	.2431
10	.2436	6	.2333	4	.2213	1.5	.2409	.8	.2259	.4	.2265
15	.2199	8	.2158	5	.2066	2	2242	1	.2118	.5	.2124
20	.2003	10	.2003	6	.1931	3	.1967	1.5	.1811	.6	.1996
30	.1669	15	.1669	8	.1689	4	.1731	2	.1551	.8	.1765
40	.1390	20	.1390	10	.1478	5	.1523	3	.1138	1	.1561
50	.1159	30	.0965	15	.1058	6	.1341	4	.0834	1.5	.1148
60	.0965	40	.0670	20	.0757	8	.1039	5	.0612	2	.0845
80	.0670	50	.0465	30	.0388	10	.0805	6	.0449	3	.0457
100	.0465	60	.0323	40	.0199	15	.0425	8	.0241	4	.0247
150	.0187	80	.0156	50	.0102	20	.0224	10	.0130	5	.0134
200	.0075	100	.0075	60	.0052	30	.0063	15	.0028	6	.0072
300	.0012	150	.0012	80	.0014	40	.0014	20	.0006	8	.0021
400	.0002	200	.0002	100	.0004	50	.0005			10	.0006
						60	.0001				

R = 20; c = 0.5

K2 = 2		K2 = 5		K2 = 10		K2 = 20		K2 = 50		K2 = 100	
θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2
10	.2446	5	.2446	3	.2398	1.5	.2421	.5	.2531	.3	.2441
15	.2252	6	.2355	4	.2262	2	.2283	.6	.2436	.4	.2301
20	.2130	8	.2224	5	.2166	3	.2111	.8	.2296	.5	.2202
30	.1977	10	.2130	6	.2093	4	.2002	1	.2198	.6	.2126
40	.1877	15	.1977	8	.1985	5	.1923	1.5	.2036	.8	.2016
50	.1801	20	.1877	10	.1907	6	.1860	2	.1933	1	.1936
60	.1737	30	.1737	15	.1768	8	.1762	3	.1794	1.5	.1798
80	.1623	40	.1623	20	.1659	10	.1678	4	.1689	2	.1693
100	.1520	50	.1520	30	.1471	15	.1495	5	.1596	3	.1515
150	.1292	60	.1424	40	.1306	20	.1335	6	.1510	4	.1358
200	.1099	80	.1251	50	.1160	30	.1064	8	.1352	5	.1218
300	.0794	100	.1099	60	.1130	40	.0848	10	.1211	6	.1092
400	.0574	150	.0794	80	.0812	50	.0675	15	.0919	8	.0877
500	.0415	200	.0574	100	.0640	60	.0538	20	.0698	10	.0705
600	.0300	300	.0300	150	.0353	80	.0342	30	.0402	15	.0408
800	.0157	400	.0157	200	.0195	100	.0217	40	.0232	20	.0237
1000	.0082	500	.0082	300	.0059	150	.0070	50	.0133	30	.0079
1500	.0016	600	.0043	400	.0018	200	.0022	60	.0077	40	.0027
		800	.0012	500	.0005	300	.0002	80	.0025	50	.0009
		1000	.0003	600	.0002			100	.0008	60	.0003
								150	.0001		

R = 50; c = 0.5

K2 = 2		K2 = 5		K2 = 10		K2 = 20		K2 = 50		K2 = 100	
θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2	θ	q_1,q_2
10^2	.1622	40	.1678	20	.1702	10	.1714	4	.1722	2	.1725
1.5×10^2	.1527	50	.1622	30	.1599	15	.1610	5	.1663	3	.1619
2×10^2	.1466	60	.1578	40	.1532	20	.1543	6	.1617	4	.1552
3×10^2	.1385	80	.1513	50	.1484	30	.1456	8	.1549	5	.1502
4×10^2	.1326	100	.1466	60	.1447	40	.1399	10	.1500	6	.1464
5×10^2	.1274	150	.1385	80	.1389	50	.1354	15	.1417	8	.1406
6×10^2	.1227	200	.1326	100	.1344	60	.1315	20	.1360	10	.1362
8×10^2	.1138	300	.1227	150	.1250	80	.1245	30	.1269	15	.1272
10^3	.1056	400	.1138	200	.1167	100	.1181	40	.1190	20	.1193
1.5×10^3	.0876	500	.1056	300	.1017	150	.1036	50	.1117	30	.1052
2×10^3	.0726	600	.0980	400	.0887	200	.0909	60	.1048	40	.0928
3×10^3	.0500	800	.0844	500	.0773	300	.0700	80	.0923	50	.0818
4×10^3	.0344	1000	.0727	600	.0674	400	.0539	100	.0813	60	.0721
5×10^3	.0237	1500	.0500	800	.0513	500	.0415	150	.0591	80	.0561
6×10^3	.0163	2000	.0344	1000	.0390	600	.0319	200	.0430	100	.0436
8×10^3	.0077	3000	.0163	1500	.0196	800	.0189	300	.0228	150	.0232
10^4	.0036	4000	.0077	2000	.0099	1000	.0112	400	.0121	200	.0124
		5000	.0037	3000	.0025	1500	.0030	500	.0064	300	.0035
		6000	.0017	4000	.0006	2000	.0008	600	.0034	400	.0010
		8000	.0004	5000	.0002	3000	.0001	800	.0010	500	.0003
		10^4	.0001					1000	.0003	600	.0001

R = 100; c = 0.5

θ	K2 = 2 q_1, q_2	θ	K2 = 5 q_1, q_2	θ	K2 = 10 q_1, q_2
5×10^2	.1300	2×10^2	.1337	10^2	.1352
6×10^2	.1271	3×10^2	.1271	1.5×10^2	.1285
8×10^2	.1228	4×10^2	.1228	2×10^2	.1240
10^3	.1195	5×10^2	.1195	3×10^2	.1182
1.5×10^3	.1137	6×10^2	.1170	4×10^2	.1140
2×10^3	.1091	8×10^2	.1127	5×10^2	.1106
3×10^3	.1009	10^3	.1091	6×10^2	.1074
4×10^3	.0934	1.5×10^3	.1009	8×10^2	.1014
5×10^3	.0865	2×10^3	.0934	10^3	.0958
6×10^3	.0801	3×10^3	.0801	1.5×10^3	.0832
8×10^3	.0686	4×10^3	.0686	2×10^3	.0722
10^4	.0588	5×10^3	.0588	3×10^3	.0545
1.5×10^4	.0400	6×10^3	.0504	4×10^3	.0411
2×10^4	.0272	8×10^3	.0371	5×10^3	.0310
3×10^4	.0126	10^4	.0272	6×10^3	.0233
4×10^4	.0058	1.5×10^4	.0126	8×10^3	.0133
5×10^4	.0027	2×10^4	.0058	10^4	.0075
6×10^4	.0012	3×10^4	.0012	1.5×10^4	.0018
8×10^4	.0003	4×10^4	.0003	2×10^4	.0004
10^5	.0001	5×10^4	.0001		

θ	K2 = 20 q_1, q_2	θ	K2 = 50 q_1, q_2	θ	K2 = 100 q_1, q_2
50	.1360	20	.1366	10	.1367
60	.1329	30	.1297	15	.1298
80	.1282	40	.1252	20	.1253
10^2	.1247	50	.1219	30	.1194
1.5×10^2	.1189	60	.1193	40	.1153
2×10^2	.1147	80	.1152	50	.1119
3×10^2	.1082	10^2	.1118	60	.1089
4×10^2	.1025	1.5×10^2	.1045	80	.1033
5×10^2	.0971	2×10^2	.0978	10^2	.0981
6×10^2	.0920	3×10^2	.0858	1.5×10^2	.0861
8×10^2	.0826	4×10^2	.0753	2×10^2	.0757
10^3	.0741	5×10^2	.0660	3×10^2	.0584
1.5×10^3	.0566	6×10^2	.0579	4×10^2	.0450
2×10^3	.0432	8×10^2	.0446	5×10^2	.0347
3×10^3	.0252	10^3	.0343	6×10^2	.0268
4×10^3	.0147	1.5×10^3	.0178	8×10^2	.0160
5×10^3	.0086	2×10^3	.0093	10^3	.0095
6×10^3	.0050	3×10^3	.0025	1.5×10^3	.0026
8×10^3	.0017	4×10^3	.0007	2×10^3	.0007
10^4	.0006	5×10^3	.0002	3×10^3	.0001

R = 500; c = 0.5

K2 = 2

θ	q_1	q_2
10^4	.09423	.09393
1.5×10^4	.09082	.09053
2×10^4	.08854	.08825
3×10^4	.08544	.08517
4×10^4	.08318	.08291
5×10^4	.08124	.08097
6×10^4	.07943	.07917
8×10^4	.07601	.07576
10^5	.07276	.07252
1.5×10^5	.05623	.06502
2×10^5	.05848	.05829
3×10^5	.04701	.04686
4×10^5	.03779	.03766
5×10^5	.03037	.03027
6×10^5	.02441	.02434
8×10^5	.01577	.01572
10^6	.01019	.01016
1.5×10^6	.00342	.00341

K2 = 5

θ	q_1	q_2
5×10^3	.09428	.09422
6×10^3	.09272	.09265
$.8 \times 10^3$	.09035	.09029
10^4	.08859	.08853
1.5×10^4	.08550	.08544
2×10^4	.08323	.08317
3×10^4	.07948	.07942
4×10^4	.07605	.07600
5×10^4	.07605	.07600
6×10^4	.06969	.06964
8×10^4	.06386	.06381
10^5	.05851	.05847
1.5×10^5	.04703	.04700
2×10^5	.03780	.03777
3×10^5	.02442	.02440
4×10^5	.01578	.01576
5×10^5	.01019	.01018
6×10^5	.00658	.00658
8×10^5	.00275	.00275
10^6	.00115	.00115
1.5×10^6	.00013	.00013
2×10^6	.00001	.00001

K2 = 10

θ	q_1	q_2
2×10^3	.09707	.09671
3×10^3	.09340	.09311
4×10^3	.09100	.09071
5×10^3	.08922	.08893
6×10^3	.08780	.08752
8×10^3	.08562	.08534
10^4	.08389	.08361
1.5×10^4	.08033	.08006
2×10^4	.07714	.07688
3×10^4	.07120	.07096
4×10^4	.06572	.06551
5×10^4	.06067	.06047
6×10^4	.05600	.05582
8×10^4	.04772	.04757
10^5	.04067	.04053
1.5×10^5	.02726	.02717
2×10^5	.01827	.01827
3×10^5	.00821	.00821
4×10^5	.00369	.00368
5×10^5	.00166	.00165
6×10^5	.00074	.00074
8×10^5	.00015	.00015
10^6	.00003	.00003

K2 = 20

θ	q_1	q_2
10^3	.09750	.09744
1.5×10^3	.09386	.09381
2×10^3	.09144	.09139
3×10^3	.08822	.08817
4×10^3	.08602	.08598
5×10^3	.08431	.08426
6×10^3	.08283	.08278
8×10^3	.08019	.08015
10^4	.07774	.07770
1.5×10^4	.07201	.07197
2×10^4	.06671	.06667
3×10^4	.05725	.05722
4×10^4	.04913	.04910
5×10^4	.04217	.04214
6×10^4	.03619	.03617
8×10^4	.02665	.02664
10^5	.01963	.01962
1.5×10^5	.00914	.00913
2×10^5	.00425	.00425
3×10^5	.00092	.00092
4×10^5	.00020	.00020
5×10^5	.00004	.00004
6×10^5	.00001	.00001

K2 = 50

θ	q_1	q_2
5×10^2	.09568	.09548
6×10^2	.09407	.09387
8×10^2	.09164	.09144
10^3	.08983	.08963
1.5×10^3	.08670	08650
2×10^3	.08450	.08431
3×10^3	.08108	.08089
4×10^3	.07806	.07788
5×10^3	.07521	.07503
6×10^3	.07247	.07230
8×10^3	.06728	.06713
10^4	.06247	.06232
1.5×10^4	.05189	.05177
2×10^4	.04311	.04301
3×10^4	.02974	.02967
4×10^4	.02052	.02048
5×10^4	.01416	.01413
6×10^4	.00977	.00975
8×10^4	.00465	.00464
10^5	.00221	.00221
1.5×10^5	.00035	.00034
2×10^5	.00005	.00005

K2 = 100

θ	q_1	q_2
2×10^2	.09791	.09759
3×10^2	.09425	.09393
4×10^2	.09181	.09150
5×10^2	.09000	.08969
6×10^2	.08856	.08826
8×10^2	.08627	.08607
10^3	.08466	.08436
1.5×10^3	.08124	.08096
2×10^3	.07825	.07797
3×10^3	.07269	.07243
4×10^3	.06753	.06730
5×10^3	.06275	.06253
6×10^3	.05830	.05809
8×10^3	.05032	.05015
10^4	.04344	.04329
1.5×10^4	.03008	.02997
2×10^4	.02082	.02075
3×10^4	.00998	.00995
4×10^4	.00479	.00477
5×10^4	.00229	.00229
6×10^4	.00110	.00110
8×10^4	.00025	.00025
10^5	.00006	.00006

R = 10^3; c = 0.5

K2 = 2

θ	q_1	q_2
5×10^4	.08193	.08146
6×10^4	.08074	.08047
8×10^4	.07893	.07846
10^5	.07756	.07710
1.5×10^5	.07506	.07461
2×10^5	.07308	.07308
3×10^5	.06956	.06914
4×10^5	.06626	.06626
5×10^5	.06313	.06275
6×10^5	.06014	.05978
8×10^5	.05458	.05426
10^6	.04954	.04924
1.5×10^6	.03887	.03864
2×10^6	.03051	.03032
3×10^6	.01879	.01867
4×10^6	.01157	.01150
5×10^6	.00712	.00708
6×10^6	.00439	.00436
8×10^6	.00166	.00165
10^7	.00063	.00063
1.5×10^7	.00006	.00006

K2 = 5

θ	q_1	q_2
2×10^4	.08341	.08279
3×10^4	.08071	.08010
4×10^4	.07890	.07829
5×10^4	.07753	.07693
6×10^4	.07642	.07583
8×10^4	.07462	.07403
10^5	.07306	.07249
1.5×10^5	.06954	.06900
2×10^5	.06625	.06574
3×10^5	.06014	.05967
4×10^5	.05459	.05416
5×10^5	.04955	.04917
6×10^5	.04498	.04463
8×10^5	.03706	.03677
10^6	.03054	.03030
1.5×10^6	.01882	.01867
2×10^6	.01160	.01151
3×10^6	.00441	.00437
4×10^6	.00167	.00166
5×10^6	.00064	.00063
6×10^6	.00024	.00024
8×10^6	.00003	.00003

K2 = 10

θ	q_1	q_2
10^4	.08415	.08413
1.5×10^4	.08142	.08139
2×10^4	.07958	.07955
3×10^4	.07709	.07706
4×10^4	.07531	.07528
5×10^4	.07381	.07379
6×10^4	.07244	.07242
8×10^4	.06988	.06985
10^5	.06743	.06741
1.5×10^5	.06169	.06167
2×10^5	.05644	.05642
3×10^5	.04724	.04722
4×10^5	.03954	.03953
5×10^5	.03309	.03308
6×10^5	.02770	.02769
8×10^5	.01941	.01940
10^6	.01360	.01359
$.5 \times 10^6$	.00559	.00558
2×10^6	.00229	.00229
3×10^6	.00039	.00039
4×10^6	.00007	.00007
5×10^6	.00001	.00001

K2 = 20

θ	q_1	q_2
5×10^3	.08438	.08395
6×10^3	.08312	.08269
8×10^3	.08120	.08077
10^4	.07977	.07935
1.5×10^4	.07728	.07686
2×10^4	.07552	.07511
3×10^4	.07273	.07234
4×10^4	.07026	.06988
5×10^4	.06792	.06755
6×10^4	.06565	.06529
8×10^4	.06135	.06101
10^5	.05733	.05702
1.5×10^5	.04839	.04813
2×10^5	.04085	04063
3×10^5	.02911	.02895
4×10^5	.02074	.02063
5×10^5	.01478	.01470
6×10^5	.01053	.01048
8×10^5	.00532	.00532
10^6	.00272	.00270
1.5×10^6	.00050	.00050
2×10^6	.00009	.00009

K2 = 50

θ	q_1	q_2
2×10^3	.08458	.08914
3×10^3	.08181	.08134
4×10^3	.07995	.07948
5×10^3	.07856	.07810
6×10^3	.07745	.07699
8×10^3	.07569	.07524
10^4	.07425	.07380
1.5×10^4	.07112	.07070
2×10^4	.06824	.06783
3×10^4	.06285	.06247
4×10^4	.05789	.05754
5×10^4	.05332	.05300
6×10^4	.04911	.04881
8×10^4	.04166	.04141
10^5	.03534	.03513
1.5×10^5	.02342	.02328
2×10^5	.01552	.01543
3×10^5	.00682	.00678
4×10^5	.00300	.00298
5×10^5	.00132	.00313
6×10^5	.00058	.00057
8×10^5	.00011	.00011
10^6	.00002	.00002

K2 = 100

θ	q_1	q_2
10^3	.08471	.08448
1.5×10^3	.08193	.08171
2×10^3	.08007	.07985
3×10^3	.07756	.07735
4×10^3	.07580	.07559
5×10^3	.07436	.07416
6×10^3	.07307	.07287
8×10^3	.07067	.07047
10^4	.06839	.06820
1.5×10^4	.06303	.06286
2×10^4	.05809	.05793
3×10^4	.04935	04921
4×10^4	.04192	04180
5×10^4	.03561	.03551
6×10^4	.03025	.03016
8×10^4	.02183	.02176
10^5	.01575	.01570
1.5×10^5	.00697	.00695
2×10^5	.00308	.00307
3×10^5	.00060	.00060
4×10^5	.00012	.00012
5×10^5	.00002	.00002

R = 20; c = 0.1

K2 = 2

θ	q_1	q_2
10	.04659	.41932
15	.04303	.38723
20	.04077	.36696
30	.03790	.34107
40	.03593	.32340
50	.03433	.30897
60	.03289	.29602
80	.03027	.27242
100	.02788	.25089
150	.02270	.20429
200	.01848	.16634
300	.01225	.11029
400	.00812	.07312
500	.00539	.04848
600	.00357	.03214
800	.00157	.01413
1000	.00069	.00621
1500	.00009	.00080

K2 = 5

θ	q_1	q_2
5	.04485	.40361
6	.04329	.38958
8	.04101	.36910
10	.03938	.35442
15	.03660	.32935
20	.03457	.31110
30	.03120	.28081
40	.02824	.25415
50	.02557	.23008
60	.02314	.20829
80	.01897	.17070
100	.01555	.13990
150	.00945	.08507
200	.00575	.05173
300	.00213	.01913
400	.00079	.00707
500	.00029	.00261
600	.00011	.00097
800	.00001	.00013

K2 = 10

θ	q_1	q_2
2	.04700	.4230
3	.04338	.3904
4	.04110	.3698
5	.03946	.3551
6	.03819	.3437
8	.03623	.3260
10	.03465	.3118
15	.03131	.2817
20	.02836	.2663
30	.02330	.2096
40	.01913	.1722
50	.01572	.1414
60	.01291	.1162
80	.00871	.0784
100	.00587	.0529
150	.00220	.0198
200	.00082	.0074
300	.00011	.0010
400	.00002	.0001

K2 = 20

θ	q_1	q_2
1	.04705	.4234
1.5	.04343	.3908
2	.04114	.3702
3	.03823	.3440
4	.03627	.3264
5	.03469	.3122
6	.03329	.2996
8	.03075	.2767
10	.20843	.2558
15	.02337	.2103
20	.01922	.1729
30	.01299	.1169
40	.00878	.0790
50	.00594	.0534
60	.00401	.0361
80	.00183	.0165
100	.00084	.0075
150	.00012	.0011
200	.00002	.0002

K2 = 50

θ	q_1	q_2
.5	.04502	.4052
.6	.04345	.3910
.8	.04116	.3704
1	.03952	.3557
1.5	.03673	.3305
2	.03471	.3124
3	.03139	.2825
4	.02846	.2561
5	.02582	.2323
6	.02342	.2107
8	.01927	.1734
10	.01585	.1426
15	.00973	.0876
20	.00598	.0538
30	.00225	.0203
40	.00085	.0076
50	.00032	.0029
60	.00012	.0011
80	.00002	.0002

K2 = 100

θ	q_1	q_2
.2	.04709	.4238
.3	.04346	.3911
.4	.04117	.3705
.5	.03953	.3558
.6	.03827	.3443
.8	.03630	.3267
1	.03472	.3125
1.5	.03140	.2826
2	.02847	.2563
3	.02343	.2109
4	.01928	.1735
5	.01587	.1428
6	.01306	.1175
8	.00884	.0796
10	.00599	.0539
15	.00226	.0203
20	.00085	.0077
30	.00012	.0011
40	.00002	.0002

R = 20; c = 0.3

K2 = 2			K2 = 5			K2 = 10		
θ	q_1	q_2	θ	q_1	q_2	θ	q_1	q_2
10	.1430	.3336	5	.1398	.3261	2	.1476	.3443
15	.1319	.3077	6	.1348	.3145	3	.1359	.3170
20	.1249	.2913	8	.1275	.2957	4	.1285	.2998
30	.1160	.2706	10	.1223	.2854	5	.1232	.2875
40	.1101	.2569	15	.1137	.2653	6	.1192	.2781
50	.1054	.2460	20	.1078	.2515	8	.1132	.2642
60	.1014	.2365	30	.0987	.2302	10	.1086	.2535
80	.0940	.2194	40	.0908	.2119	15	.0997	.2325
100	.0873	.2038	50	.0836	.1951	20	.0920	.2146
150	.0727	.1695	60	.0770	.1797	30	.0785	.1833
200	.0605	.1411	80	.0654	.1525	40	.0671	.1565
300	.0419	.0977	100	.0554	.1234	50	.0573	.1336
400	.0290	.0676	150	.0368	.0858	60	.0489	.1141
500	.0201	.0468	200	.0244	.0569	80	.0357	.0832
600	.0139	.0324	300	.0107	.0250	100	.0260	.0607
800	.0067	.0155	400	.0047	.0110	150	.0118	.0276
1000	.0032	.0074	500	.0021	.0048	200	.0054	.0125
1500	.0005	.0012	600	.0009	.0021	300	.0011	.0026
			800	.0002	.0004	400	.0002	.0005

K2 = 20			K2 = 50			K2 = 100		
θ	q_1	q_2	θ	q_1	q_2	θ	q_1	q_2
1	.1482	.3458	.5	.1419	.3311	.2	.1488	.3471
1.5	.1364	.3183	.6	.1368	.3191	.3	.1369	.3194
2	.1290	.3010	.8	.1293	.3017	.4	.1294	.3019
3	.1196	.2792	1	.1240	.2893	.5	.1241	.2895
4	.1136	.2651	1.5	.1152	.2688	.6	.1200	.2800
5	.1091	.2545	2	.1093	.2551	.8	.1140	.2659
6	.1052	.2455	3	.1005	.2344	1	.1094	.2553
8	.0986	.2300	4	.0930	.2169	1.5	.1006	.2347
10	.0926	.2161	5	.0861	.2009	2	.0931	.2172
15	.0793	.1851	6	.0798	.1861	3	.0799	.1865
20	.0679	.1585	8	.0685	.1598	4	.0687	.1602
30	.0499	.1164	10	.0588	.1372	5	.0590	.1376
40	.0366	.0854	15	.0401	.0936	6	.0507	.1182
50	.0269	.0627	20	.0274	.0639	8	.0374	.0872
60	.0197	.0460	30	.0128	.0298	10	.0276	.0643
80	.0106	.0248	40	.0059	.0139	15	.0129	.0301
100	.0057	.0134	50	.0028	.0065	20	.0060	.0141
150	.0012	.0028	60	.0013	.0030	30	.0013	.0031
200	.0003	.0006	80	.0003	.0007	40	.0003	.0007
			100	.0001	.0001	50	.0001	.0001

R = 20; c = 0.7

K2 = 2			K2 = 5			K2 = 10		
θ	q_1	q_2	θ	q_1	q_2	θ	q_1	q_2
10	.3529	.1512	6	.3518	.1507	4	.3434	.1471
15	.3244	.1390	8	.3312	.1419	5	.3279	.1405
20	.3065	.1314	10	.3167	.1357	6	.3161	.1355
30	.2841	.1217	15	.2930	.1255	8	.2990	.1281
40	.2697	.1156	20	.2779	.1191	10	.2868	.1229
50	.2590	.1110	30	.2583	.1107	15	.2665	.1142
60	.2503	.1072	40	.2443	.1047	20	.2528	.1083
80	.2354	.1009	50	.2323	.0995	30	.2318	.0993
100	.2223	.0952	60	.2213	.0948	40	.2137	.0916
150	.1930	.0827	80	.2011	.0862	50	.1973	.0845
200	.1677	.0719	100	.1829	.0784	60	.1821	.0780
300	.1266	.0543	150	.1442	.0618	80	.1552	.0665
400	.0956	.0410	200	.1136	.0487	100	.1322	.0567
500	.0722	.0309	300	.0706	.0303	150	.0886	.0380
600	.0545	.0233	400	.0439	.0188	200	.0594	.0255
800	.0310	.0133	500	.0273	.0117	300	.0267	.0114
1000	.0177	.0076	600	.0169	.0073	400	.0120	.0051
1500	.0043	.0019	800	.0065	.0028	500	.0054	.0023
			1000	.0025	.0011	600	.0024	.0010
			1500	.0002	.0001	800	.0005	.0002

K2 = 20			K2 = 50			K2 = 100		
θ	q_1	q_2	θ	q_1	q_2	θ	q_1	q_2
2	.3507	.1503	.8	.3555	.1524	.5	.3408	.1460
3	.3225	.1382	1	.3392	.1453	.6	.3282	.1406
4	.3047	.1306	1.5	.3125	.1339	.8	.3099	.1328
5	.2921	.1252	2	.2957	.1267	1	.2970	.1273
6	.2825	.1211	3	.2745	.1176	1.5	.2756	.1181
8	.2682	.1149	4	.2607	.1117	2	.2617	.1121
10	.2576	.1104	5	.2500	.1071	3	.2419	.1037
15	.2373	.1017	6	.2408	.1032	4	.2258	.0968
20	.2203	.0944	8	.2244	.0962	5	.2112	.0905
30	.1906	.0817	10	.2096	.0898	6	.1976	.0847
40	.1648	.0706	15	.1769	.0758	8	.1730	.0742
50	.1426	.0611	20	.1492	.0640	10	.1515	.0649
60	.1234	.0529	30	.1063	.0455	15	.1087	.0466
80	.0923	.0396	40	.0757	.0324	20	.0780	.0334
100	.0691	.0296	50	.0539	.0231	30	.0402	.0172
150	.0335	.0143	60	.0384	.0164	40	.0207	.0089
200	.0162	.0070	80	.0195	.0083	50	.0106	.0046
300	.0038	.0016	100	.0099	.0042	60	.0055	.0023
400	.0009	.0004	150	.0018	.0008	80	.0015	.0006
500	.0002	.0001	200	.0003	.0001	100	.0004	.0002

R = 20; c = 0.9

K2 = 2

θ	q_1	q_2
15	.4316	.04794
20	.4072	.04523
30	.3767	.04184
40	.3573	.03969
50	.3433	.03813
60	.3321	.03689
80	.3141	.03488
100	.2987	.03317
150	.2648	.02942
200	.2351	.02612
300	.1853	.02059
400	.1461	.01623
500	.1152	.01279
600	.0908	.01008
800	.0564	.00627
1000	.0351	.00389
1500	.0107	.00118

K2 = 5

θ	q_1	q_2
10	.4467	.04962
15	.4110	.04566
20	.3885	.04317
30	.3603	.04003
40	.3422	.03801
50	.3285	.03650
60	.3171	.03523
80	.2973	.03303
100	.2796	.03107
150	.2403	.02669
200	.2065	.02294
300	.1526	.01695
400	.1127	.01252
500	.0833	.00925
600	.0615	.00683
800	.0336	.00373
1000	.0183	.00204
1500	.0040	.00045
2000	.0009	.00010

K2 = 10

θ	q_1	q_2
8	.4390	.04876
10	.4193	.04658
15	.3872	.04301
20	.3670	.04076
30	.3411	.03788
40	.3234	.03592
50	.3090	.03432
60	.2960	.03288
80	.2724	.03026
100	.2509	.02787
150	.2043	.02269
200	.1664	.01848
300	.1103	.01225
400	.0731	.00812
500	.0485	.00539
600	.0322	.00357
800	.0141	.00157
1000	.0062	.00069
1500	.0008	.00009
2000	.0001	.00001

K2 = 20

θ	q_1	q_2
5	.4434	.04925
6	.4269	.04742
8	.4030	.04476
10	.3859	.04287
15	.3580	.03977
20	.3400	.03777
30	.3148	.03497
40	.2947	.03274
50	.2766	.03072
60	.2598	.02885
80	.2291	.02545
100	.2021	.02245
150	.1477	.01641
200	.1080	.01199
300	.0577	.00640
400	.0308	.00342
500	.0164	.00183
600	.0088	.00098
800	.0025	.00028
1000	.0007	.00008

K2 = 50

θ	q_1	q_2
3	.4254	.04726
4	.4016	.04462
5	.3847	.04274
6	.3718	.04131
8	.3528	.03920
10	.3390	.03766
15	.3137	.03485
20	.2934	.03260
30	.2581	.02867
40	.2271	.02524
50	.1999	.02221
60	.1760	.01955
80	.1363	.01515
100	.1056	.01174
150	.0558	.00620
200	.0295	.00328
300	.0082	.00091
400	.0023	.00026
500	.0006	.00007
600	.0002	.00002

K2 = 100

θ	q_1	q_2
1.5	.4324	.04804
2	.4079	.04532
3	.3773	.04192
4	.3579	.03976
5	.3438	.03820
6	.3327	.03696
8	.3147	.03496
10	.2993	.03326
15	.2657	.02952
20	.2362	.02624
30	.1866	.02073
40	.1474	.01637
50	.1164	.01294
60	.0920	.01022
80	.0574	.00638
100	.0358	.00398
150	.0110	.00122
200	.0034	.00038
300	.0003	.00004

APPENDIX N

TOTAL CUMULATIVE DIMENSIONLESS FLUX, $Q_{(t)}$, FOR BOTH LAYERS, FOR TWO-LAYER LINEAR FLOW, CONSTANT TERMINAL PRESSURE

Calculated from:

$$Q_{(t)} = L(1-\epsilon) \qquad \text{(V-126)}$$

$$\epsilon = \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left(\frac{2}{b^2L^2}\right) \left[\frac{\sin \gamma c}{\gamma} + \frac{\phi_2}{\phi_1} \frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)}\right]}{\left[c + \frac{\phi_2}{\phi_1}(1-c)\right]} \qquad \text{(V-134)}$$

Eigenvalues b from: Equation (V-43)

Eigenvalues a from: Appendix D

Truncation Test (Both Series): $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$

Estimated Accuracy: $\pm$ 1 part in 10^4

Definition of Dimensionless Variables

$$x = \frac{x_a}{H} \qquad \text{(V-3)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-4)}$$

$$L = \frac{L_a}{H} \qquad \text{(V-5)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c H^2} \quad \text{(for liquid flow)}$$

$$= \frac{k_1 t_a}{\rho_1 C_{p1} H^2} \quad \text{(for heat conduction)} \qquad \text{(V-15)}$$

All values are for $\phi_1 = \phi_2$, $K_2^* = K_2 = \frac{k_2}{k_1}$

L = 5.0; c = 0.5

θ	K2 = 2.0	5.0	10.	20.	50.	100.
.04				.431	.916	1.345
.05				.518	1.047	1.526
.06			.118	.598	1.167	1.692
.08			.228	.741	1.384	1.989
.10			.325	.869	1.578	2.250
.15			.535	1.146	1.996	2.788
.2			.714	1.382	2.347	3.207
.3		.066	1.015	1.778	2.910	3.809
.4		.275	1.268	2.110	3.343	4.203
.5		.459	1.491	2.397	3.682	4.465
.6		.625	1.692	2.651	3.950	4.641
.8		.920	2.046	3.081	4.333	4.837
1.0		1.179	2.353	3.429	4.576	4.926
1.5		1.729	2.978	4.047	4.863	4.990
2.		2.184	3.452	4.421	4.956	4.999
3.		2.904	4.092	4.786	4.995	5.000
4.	.509	3.438	4.468	4.921	5.000	
5.	1.129	3.836	4.688	4.971		
6.	1.662	4.133	4.817	4.989		
8.	2.517	4.518	4.937	4.999		
10.	3.153	4.733	4.978	5.000		
15.	4.118	4.939	4.998			
20.	4.579	4.986	5.000			
30.	4.904	4.999				
40.	4.978	5.000				
50.	4.995					
60.	4.999					

L = 10; c = 0.5

θ	K2 = 2.0	5.0	10.	20.	50.	100.
.010						.593
.015						.745
.02					.518	.877
.03					.681	1.104
.04					.823	1.303
.05					.951	1.482
.06				.339	1.069	1.647
.08				.481	1.284	1.948
.10			-	.607	1.478	2.219
.15			-	.883	1.898	2.809
.2		-	.158	1.119	2.258	3.313
.3		-	.459	1.516	2.865	4.158
.4	-	-	.713	1.850	3.373	4.857
.5	-	-	.936	2.144	3.818	5.453
.6	-	-	1.137	2.408	4.217	5.972
.8	-	-	1.493	2.876	4.916	6.828
1.0	-	-	1.806	3.287	5.515	7.498
1.5	-	.481	2.471	4.157	6.707	8.615
2.	-	.945	3.031	4.881	7.577	9.233
3.	-	1.724	3.966	6.048	8.688	9.765
4.	-	2.379	4.746	6.944	9.290	9.928
5.	-	2.956	5.415	7.636	9.615	9.978
6.	-	3.475	5.996	8.171	9.792	9.993
8.	-	4.384	6.945	8.905	9.939	
10.	-	5.159	7.669	9.345	9.982	
15.	.681	6.656	8.814	9.818	9.999	
20.	2.263	7.689	9.396	9.950	10.000	
30.	4.658	8.896	9.844	9.996		
40.	6.310	9.473	9.960	10.000		
50.	7.451	9.748	9.990			
60.	8.239	9.880	9.997			
80.	9.160	9.973				
100.	9.599	9.994				
150.	9.937	10.000				
200.	9.990					
300.	10.000					

L = 20; c = 0.5

θ	K2: 2.0	5.0	10.	20.	50	100.
.04						1.24
.05						1.41
.06						1.57
.08						1.89
.10					1.29	2.13
.15					1.70	2.71
.2					2.06	3.22
.3					2.66	4.06
.4				1.32	3.17	4.77
.5				1.62	3.62	5.40
.6			.026	1.88	4.02	5.96
.8			.382	2.35	4.73	6.95
1.0			.695	2.76	5.35	7.81
1.5			1.36	3.63	6.66	9.64
2.	-	-	1.92	4.37	7.77	11.14
3.	-	-	2.86	5.59	9.62	13.49
4.	-	-	3.65	6.63	11.14	15.22
5.	-	.457	4.34	7.54	12.43	16.48
6.	-	.979	4.97	8.36	13.52	17.41
8.	-	1.91	6.09	9.80	15.26	18.60
10.	-	2.72	7.07	11.05	16.54	19.24
15.	-	4.46	9.14	13.53	18.41	19.84
20.	-	5.92	10.85	15.31	19.27	19.97
30.	-	8.35	13.48	17.54	19.85	20.00
40.	-	10.33	15.36	18.71	19.97	
50.	-	11.96	16.69	19.33	19.99	
60.	1.37	13.32	17.64	19.65	20.00	
80.	4.53	15.39	18.80	19.90		
100.	7.15	16.81	19.39	19.97		
150.	11.91	18.74	19.89	20.00		
200.	14.90	19.50	19.98			
300.	17.98	19.92	20.00			
400.	19.20	19.99				
500.	19.68	20.00				
600.	19.87					
800.	19.98					
1000.	20.00					

L = 50; c = 0.5

θ	K2: 2.0	5.0	10.	20.	50.	100.
.2						2.95
.3						3.77
.4					2.57	4.48
.5					3.01	5.10
.6					3.41	5.65
.8					4.11	6.64
1.0				1.18	4.73	7.51
1.5				2.05	6.05	9.35
2.				2.79	7.16	10.89
3.				4.01	9.02	13.48
4.			.31	5.05	10.58	15.65
5.			1.01	5.96	11.96	17.57
6.		-	1.64	6.78	13.20	19.30
8.		-	2.75	8.24	15.40	22.37
10.		-	3.74	9.53	17.35	25.05
15.		-	5.83	12.27	21.48	30.58
20.	-	-	7.59	14.57	24.94	34.86
30.	-	.88	10.55	18.44	30.54	40.79
40.	-	2.95	13.04	21.70	34.87	44.40
50.	-	4.77	15.23	24.54	38.24	46.60
60.	-	6.42	17.22	27.07	40.85	47.93
80.	-	9.35	20.73	31.37	44.47	49.23
100.	-	11.93	23.79	34.86	46.65	49.72
150.	-	17.40	30.05	40.98	49.05	49.98
200.	-	21.95	34.79	44.63	49.73	50.00
300.	-	29.16	41.16	48.09	49.98	
400.	5.12	34.50	44.86	49.32	50.00	
500.	11.32	38.47	47.01	49.76		
600.	16.65	41.43	48.27	49.91		
800.	25.20	45.26	49.41	49.99		
1000.	31.56	47.38	49.80	50.00		
1500.	41.20	49.40	49.99			
2000.	45.80	49.86	50.00			
3000.	49.05	49.99				
4000.	49.78	50.00				
5000.	49.95					
6000.	49.99					

L = 100; c = 0.5

θ	K2 2.0	5.0	10.	20.	50.	100.
2.						10.39
3.						12.97
4.					9.56	15.15
5.					10.94	17.06
6.					12.18	18.80
8.					14.38	21.87
10.				6.90	16.33	24.58
15.				9.63	20.46	30.34
20.			2.03	11.94	23.95	35.19
30.			4.99	15.81	29.80	43.33
40.			7.48	19.07	34.72	50.14
50.			9.68	21.95	39.06	56.04
60.	-	-	11.66	24.55	42.98	61.21
80.	-	-	15.19	29.16	49.89	69.77
100.	-	-	18.29	33.22	55.87	76.43
150.	-	4.92	24.90	41.86	67.80	87.36
200.	-	9.55	30.47	49.09	76.49	93.22
300.	-	17.31	39.80	60.77	87.47	98.05
400.	-	23.86	47.59	69.73	93.32	99.44
500.	-	29.62	54.28	76.64	96.44	99.84
600.	-	34.81	60.10	81.97	98.10	99.95
800.	-	43.90	69.59	89.26	99.46	100.00
1000.	-	51.65	76.82	93.60	99.85	
1500.	6.83	66.62	88.24	98.25	99.99	
2000.	22.65	76.95	94.03	99.52	100.00	
3000.	46.59	89.00	98.46	99.96		
4000.	63.11	94.75	99.60	100.00		
5000.	74.52	97.50	99.90			
6000.	82.40	98.81	99.97			
8000.	91.61	99.73				
10,000.	96.00	99.94				
15,000.	99.37	100.00				
20,000.	99.90					
30,000.	100.00					

L = 1000; c = 0.5

θ	K2 2.0	5.0	10.	20.	50.	100.
2×10^2						104.5
3						130.2
4					95.9	151.9
5					109.6	171.0
6					122.0	188.3
8					144.1	219.0
10				69.1	163.5	246.0
15				96.5	204.8	303.6
2×10^3			20.4	119.5	239.6	352.1
3			50.0	158.2	298.1	433.5
4			74.9	190.8	347.3	501.6
5			96.8	219.5	390.7	560.6
6		-	116.7	245.5	429.9	612.2
8		-	151.9	291.6	499.0	697.8
10	-	-	183.0	332.3	558.8	764.5
15	-	49.2	249.0	418.7	678.1	873.7
2×10^4	-	95.5	304.7	490.9	765.0	932.3
3	-	173.2	398.0	607.7	874.7	980.5
4	-	238.6	475.9	697.3	933.2	994.4
5	-	296.2	542.8	766.4	964.4	998.4
6	-	348.1	601.0	819.7	981.0	999.5
8	-	439.0	695.9	892.6	994.6	1000.0
10	-	516.6	768.2	936.0	998.5	
15	68.3	666.2	882.4	982.5	999.9	
2×10^5	226.5	769.5	940.3	995.2	1000.0	
3	465.9	890.0	984.6	999.6		
4	631.1	947.5	996.0	1000.0		
5	745.2	975.0	998.9			
6	824.0	988.1	999.7			
8	916.1	997.3	1000.0			
10	960.0	999.4				
15	993.7	1000.0				
2×10^6	999.0					
3	1000.0					

L = 20; c = 0.1

θ	K2: 2.	5.	10.	20.	50.	100.
.06					1.48	2.44
.08					1.78	2.83
.10				.544	2.03	3.22
.15				.903	2.58	3.99
.2				1.21	3.05	4.63
.3				1.71	3.83	5.72
.4			.169	2.14	4.48	6.64
.5			.451	2.52	5.06	7.45
.6		-	.393	2.41	4.87	7.17
.8		-	1.16	3.47	6.51	9.47
1.0	-	-	1.56	4.00	7.32	10.59
1.5	-	-	2.41	5.14	9.06	12.88
2.	-	-	3.13	6.09	10.50	14.61
3.	-	.239	4.33	7.70	12.82	16.91
4.	-	1.05	5.34	9.05	14.56	18.23
5.	-	1.76	6.23	10.22	15.88	18.98
6.	-	2.41	7.04	11.26	16.88	19.42
8.	-	3.56	8.47	13.01	18.21	19.81
10.	-	4.57	9.71	14.41	18.98	19.94
15.	-	6.71	12.24	16.80	19.75	20.00
20.	-	8.50	14.14	18.17	19.94	
30.	-	11.35	16.66	19.40	20.00	
40.	-	13.49	18.09	19.80		
50.	1.94	15.10	18.91	19.94		
60.	3.94	16.31	19.38	19.98		
80.	7.30	17.91	19.80	20.00		
100.	9.96	18.81	19.93			
150.	14.41	19.71	20.00			
200.	16.89	19.93				
300.	19.04	20.00				
400.	19.70					
500.	19.91					
600.	19.97					
800.	20.00					
1000.						

L = 20; c = 0.3

θ	K2: 2.	5.	10.	20.	50.	100.
.06					1.21	2.44
.08					1.47	2.42
.10				.339	1.70	2.74
.15				.661	2.20	3.44
.2				.993	2.61	4.02
.3				1.39	3.31	4.99
.4			-	1.77	3.89	5.81
.5			.163	2.11	4.41	6.52
.6		-	.707	2.86	5.58	8.18
.8		-	.800	2.95	5.69	8.32
1.0	-	-	1.16	3.43	6.42	9.33
1.5	-	-	1.92	4.44	7.96	11.43
2.	-	-	2.56	5.29	9.25	13.10
3.	-	-	3.64	6.72	11.38	15.52
4.	-	.497	4.55	7.92	13.06	17.09
5.	-	1.15	5.35	8.98	14.42	18.11
6.	-	1.73	6.07	9.93	15.51	18.77
8.	-	2.78	7.36	11.57	17.09	19.48
10.	-	3.69	8.48	12.93	18.12	19.78
15.	-	5.65	10.83	15.45	19.36	19.97
20.	-	7.29	12.68	17.07	19.79	20.00
30.	-	9.96	15.33	18.79	19.98	
40.	-	12.06	17.02	19.50	20.00	
50	.775	13.72	18.10	19.80		
60.	2.70	15.03	18.79	19.91		
80	5.99	16.89	19.51	19.99		
100.	8.64	18.06	19.80	20.00		
150.	13.27	19.40	19.98			
200.	16.02	19.81	20.00			
300.	18.60	19.98				
400.	19.51	20.00				
500.	19.83					
600.	19.94					
800.	19.99					
1000.	20.00					

L = 20; c = 0.7

Θ	K2: 2.	5.	10.	20.	50.	100.
.06					.571	1.09
.08					.717	1.29
.10				-	.852	1.48
.15				.049	1.15	1.91
.2				.224	1.42	2.28
.3				.531	1.89	2.94
.4			-	.796	2.29	3.50
.5			-	1.03	2.65	4.00
.6		-	-	1.25	2.97	4.45
.8		-	-	1.62	3.53	5.24
1.0	-	-	.152	1.96	4.03	5.93
1.5	-	-	.700	2.66	5.08	7.39
2.	-	-	1.16	3.25	5.96	8.60
3.	-	-	1.93	4.23	7.43	10.59
4.	-	-	2.58	5.06	8.65	12.20
5.	-	-	3.15	5.79	9.73	13.52
6.	-	.117	3.67	6.45	10.68	14.62
8.	-	.910	4.59	7.62	12.32	16.29
10.	-	1.61	5.39	8.64	13.66	17.44
15.	-	3.10	7.11	10.79	16.08	18.99
20.	-	4.35	8.55	12.51	17.57	19.60
30.	-	6.45	10.90	15.04	19.07	19.94
40.	-	8.20	12.76	16.71	19.64	19.99
50.	-	9.71	14.24	17.82	19.86	20.00
60.	-	11.02	15.41	18.56	19.95	
80.	2.92	13.15	17.09	19.37	19.99	
100.	5.46	14.78	18.16	19.72	20.00	
150.	10.26	17.35	19.41	19.96		
200.	13.48	18.66	19.81	20.00		
300.	17.08	19.65	19.98			
400.	18.69	19.91	20.00			
500.	19.41	19.98				
600.	19.74	19.99				
800.	19.95	20.00				
1000.	19.99					
1500.	20.00					
2000.						

L = 20; c = 0.9

Θ	K2: 2.	5.	10.	20.	50.	100.
.06					.231	.571
.08					.309	.679
.10				-	.382	.780
.15				-	.550	1.01
.2				-	.702	1.22
.3				-	.972	1.59
.4			-	.168	1.21	1.91
.5			-	.319	1.42	2.21
.6		-	-	.457	1.62	2.47
.8		-	-	.704	1.97	2.95
1.0	-	-	-	.921	2.27	3.37
1.5	-	-	-	1.38	2.29	4.25
2.	-	-	.206	1.77	3.46	4.98
3.	-	-	.758	2.42	4.37	6.20
4.	-	-	1.22	2.97	5.13	7.23
5.	-	-	1.63	3.45	5.79	8.12
6.	-	-	2.00	3.88	6.39	8.93
8.	-	-	2.66	4.65	7.46	10.34
10.	-	.275	3.24	5.33	8.40	11.56
15.	-	1.46	4.46	6.77	10.37	13.95
20.	-	2.46	5.50	7.98	11.97	15.66
30.	-	4.14	7.24	10.00	14.41	17.77
40.	-	5.55	8.70	11.64	16.11	18.85
50.	-	6.79	9.96	13.01	17.29	19.41
60.	-	7.91	11.08	14.15	18.11	19.69
80.	1.13	9.84	12.94	15.91	19.08	19.92
100.	3.54	11.45	14.42	17.14	19.56	19.98
150.	8.28	14.45	16.89	18.83	19.93	
200.	11.65	16.40	18.27	19.52	19.99	
300.	15.77	18.48	19.46	19.92		
400.	17.85	19.36	19.83	19.99		
500.	18.91	19.73	19.95			
600.	19.45	19.89	19.98			
800.	19.86	19.98				
1000.	19.96	20.00				
1500.	20.00					
2000.						

APPENDIX O

TOTAL CUMULATIVE DIMENSIONLESS FLUX, $Q_{(t)}$, FOR BOTH LAYERS, FOR TWO-LAYER RADIAL FLOW, CONSTANT TERMINAL PRESSURE

Tables valid for all $r_b/H > 10.0$

Calculated from:

$$Q_{(t)} = \left(\frac{R^2-1}{2}\right)(1-\epsilon) \qquad \text{(V-193)}$$

$$\epsilon = \frac{\left(\frac{2}{R^2-1}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-a^2\Theta} \left[\frac{\sin \gamma c}{\gamma} + \frac{\phi_2}{\phi_1} \frac{\cos \gamma c \sin \beta(1-c)}{\beta \cos \beta(1-c)}\right] \left[\frac{2V^2(b)}{\frac{4}{\pi^2} - b^2V^2(b)}\right]}{\left[c + \frac{\phi_2}{\phi_1}(1-c)\right]} \qquad \text{(V-194)}$$

Eigenvalues b from: Appendix B

Eigenvalues a from: Appendix D

Truncation Test: $|\text{Term}| < \left|\frac{\text{Sum}}{10^8}\right|$ for m series

Only one term used in n series

Definition of Dimensionless Variables:

$$r = \frac{r_a}{r_b} \qquad \text{(V-135)}$$

$$R = \frac{r_e}{r_b} \qquad \text{(V-136)}$$

$$y = \frac{y_a}{H} \qquad \text{(V-137)}$$

$$\Theta = \frac{k_1 t_a}{\mu \phi_1 c r_b^2} \quad \text{(for liquid flow)}$$

$$= \frac{k_1 t_a}{\rho_1 C_{p1} r_b^2} \quad \text{(for heat conduction)} \qquad \text{(V-140)}$$

Estimated Accuarcy: $\pm$ 1 part in 10^4

All values are for $\phi_1 = \phi_2$, $K_2^* = K_2 = \frac{k_2}{k_1}$

R = 5; c = 0.5

θ	$K_2 = 2$	$K_2=5$	$K_2=10$	$K_2=20$	$K_2=50$	$K_2=100$
.010						1.030
.015						1.319
.02					1.037	1.578
.03					1.327	2.042
.04					1.588	2.461
.05				1.055	1.829	2.851
.06				1.178	2.055	3.219
.08				1.407	2.477	3.904
.1			1.085	1.617	2.870	4.532
.15			1.391	2.094	3.763	5.894
.2		1.144	1.665	2.525	4.562	7.007
.3		1.468	2.158	3.304	5.930	8.662
.4		1.759	2.603	4.007	7.047	9.768
.5	1.311	2.029	3.018	4.650	7.958	10.51
.6	1.468	2.283	3.409	5.241	8.702	11.00
.8	1.760	2.757	4.134	6.283	9.804	11.55
1	2.029	3.198	4.795	7.164	10.54	11.80
1.5	2.642	4.197	.6.214	8.818	11.47	11.97
2	3.198	5.077	7.353	9.906	11.81	12.00
3	4.197	6.550	9.003	11.09	11.97	
4	5.077	7.709	10.07	11.61	12.00	
5	5.857	8.622	10.75	11.83		
6	6.550	9.341	11.20	11.93		
8	7.709	10.35	11.67	11.99		
10	8.622	10.98	11.86	12.00		
15	10.14	11.69	11.98			
20	10.98	11.91	12.00			
30	11.69	11.99				
40	11.91	12.00				
50	11.97					
60	11.99					
80	12.00					
100						
150						

R = 10; c = 0.5

θ	$K_2=2$	$K_2=5$	$K_2=10$	$K_2=20$	$K_2=50$	$K_2=100$
.05						2.852
.06						3.221
.08						3.915
.1					2.871	4.565
.15					3.772	6.067
.2				2.526	4.597	7.455
.3				3.307	6.110	10.02
.4				4.021	7.508	12.39
.5			3.019	4.691	8.829	14.60
.6			3.413	5.328	10.09	16.68
.8			4.152	6.532	12.48	20.47
1		3.200	4.845	7.668	14.71	23.83
1.5		4.216	6.448	10.31	19.71	30.62
2	3.200	5.149	7.930	12.75	23.98	35.61
3	4.215	6.862	10.67	17.16	30.79	41.98
4	5.147	8.448	13.19	21.03	35.78	45.43
5	6.024	9.946	15.54	24.45	39.43	47.30
6	6.861	11.38	17.74	27.45	42.12	48.31
8	8.447	14.07	21.72	32.42	45.53	49.15
10	9.945	16.56	25.20	36.27	47.36	49.40
15	13.41	22.05	32.10	42.51	49.05	49.50
20	16.56	26.63	37.05	45.81	49.40	
30	22.53	33.62	43.12	48.47	49.50	
40	26.63	38.47	46.23	49.21		
50	30.44	41.84	47.83	49.42		
60	33.62	44.18	48.64	49.48		
80	38.47	46.94	49.27	49.50		
100	41.84	48.26	49.44			
150	46.42	49.30	49.50			
200	48.26	49.47				
300	49.30	49.50				
400	49.47					
500	49.49					

R = 20; c = 0.5

θ	$K_2=2$	$K_2=5$	$K_2=10$	$K_2=20$	$K_2=50$	$K_2=100$
.2						7.463
.3						10.03
.4						12.42
.5					8.837	14.69
.6					10.10	16.88
.8					12.51	21.05
1				7.672	14.80	25.04
1.5				10.32	20.18	34.44
2			7.937	12.78	25.23	43.25
3			10.68	17.38	34.71	59.43
4			13.24	21.69	43.59	73.92
5		9.958	15.67	25.81	51.96	86.92
6		11.40	18.01	29.78	59.88	98.57
8		14.14	22.49	37.38	74.46	118.4
10		16.75	26.77	44.60	87.52	134.3
15		22.89	36.85	61.23	114.5	161.7
20		28.66	46.27	76.08	135.0	177.6
30		39.48	63.46	101.2	162.3	192.2
40		49.55	78.72	121.1	178.1	197.0
50		58.97	92.26	137.1	187.2	198.7
60		67.80	104.3	149.7	192.4	199.2
80		83.83	124.4	167.9	197.1	199.5
100		97.91	140.3	179.4	198.7	
150		126.1	166.9	193.1	199.5	
200		146.4	181.5	197.4		
300		171.8	194.0	199.3		
400		185.0	197.8	199.5		
500		191.9	199.0			
600		195.5	199.3			
800		198.4	199.5			
1000		199.2				
1500		199.5				
2000						

R = 50; c = 0.5

θ	K2 = 2	K2 = 5	K2 = 10	K2 = 20	K2 = 50	K2 = 100
2						43.47
3						60.31
4					43.85	76.30
5					52.47	91.71
6					60.83	1.067×10^{2}
8					76.96	1.356×10^{2}
10				44.82	92.50	1.636×10^{2}
15				62.22	1.296×10^{2}	2.300×10^{2}
20			46.71	78.75	1.650×10^{2}	2.922×10^{2}
30			64.81	1.102×10^{2}	2.319×10^{2}	4.054×10^{2}
40		50.05	82.00	1.401×10^{2}	2.946×10^{2}	5.053×10^{2}
50		59.94	98.58	1.690×10^{2}	3.534×10^{2}	5.933×10^{2}
60		69.53	1.147×10^{2}	1.970×10^{2}	4.086×10^{2}	6.709×10^{2}
80		88.05	1.458×10^{2}	2.507×10^{2}	5.090×10^{2}	7.997×10^{2}
100	59.89	1.059×10^{2}	1.759×10^{2}	3.017×10^{2}	5.974×10^{2}	8.998×10^{2}
150	83.45	1.486×10^{2}	2.471×10^{2}	4.149×10^{2}	7.749×10^{2}	1.063×10^{3}
200	1.059×10^{2}	1.892×10^{2}	3.136×10^{2}	5.199×10^{2}	9.041×10^{2}	1.150×10^{3}
300	1.485×10^{2}	2.657×10^{2}	4.334×10^{2}	6.879×10^{2}	1.067×10^{3}	1.221×10^{3}
400	1.892×10^{2}	3.366×10^{2}	5.380×10^{2}	8.173×10^{2}	1.153×10^{3}	1.241×10^{3}
500	2.281×10^{2}	4.023×10^{2}	6.291×10^{2}	9.168×10^{2}	1.198×10^{3}	1.247×10^{3}
600	2.657×10^{2}	4.634×10^{2}	7.086×10^{2}	9.934×10^{2}	1.222×10^{3}	1.249×10^{3}
800	3.365×10^{2}	5.726×10^{2}	8.383×10^{2}	1.098×10^{3}	1.242×10^{3}	
1000	4.023×10^{2}	6.666×10^{2}	9.369×10^{2}	1.160×10^{3}	1.247×10^{3}	
1500	5.468×10^{2}	8.484×10^{2}	1.092×10^{3}	1.225×10^{3}	1.249×10^{3}	
2000	6.666×10^{2}	9.735×10^{2}	1.170×10^{3}	1.243×10^{3}		
3000	8.484×10^{2}	1.119×10^{3}	1.229×10^{3}	1.249×10^{3}		
4000	9.735×10^{2}	1.188×10^{3}	1.244×10^{3}			
5000	1.060×10^{3}	1.220×10^{3}	1.248×10^{3}			
6000	1.119×10^{3}	1.236×10^{3}	1.249×10^{3}			
8000	1.188×10^{3}	1.246×10^{3}				
10000	1.220×10^{3}	1.249×10^{2}				
15000						

R = 100 ; c = 0.5

θ	K2 = 2	K2 = 5	K2 = 10	K2 = 20	K2 = 50	K2 = 100
10						1.638×10^{2}
15						2.310×10^{2}
20					1.661×10^{2}	2.953×10^{2}
30					2.338×10^{2}	4.187×10^{2}
40					2.987×10^{2}	5.371×10^{2}
50				1.703×10^{2}	3.616×10^{2}	6.518×10^{2}
60				1.985×10^{2}	4.231×10^{2}	$7.633.10^{2}$
80				2.532×10^{2}	5.425×10^{2}	9.776×10^{2}
100			1.765×10^{2}	3.063×10^{2}	6.582×10^{2}	1.181×10^{3}
150			2.488×10^{2}	4.339×10^{2}	9.336×10^{2}	1.645×10^{3}
200		1.895×10^{2}	3.182×10^{2}	5.564×10^{2}	1.191×10^{3}	2.053×10^{3}
300		2.676×10^{2}	4.512×10^{2}	7.902×10^{2}	1.659×10^{3}	2.726×10^{3}
400		3.425×10^{2}	5.788×10^{2}	1.011×10^{3}	2.069×10^{3}	3.246×10^{3}
500	2.305×10^{2}	4.151×10^{2}	7.023×10^{2}	1.221×10^{3}	2.429×10^{3}	3.646×10^{3}
600	2.690×10^{2}	4.861×10^{2}	8.221×10^{2}	1.419×10^{3}	2.744×10^{3}	3.956×10^{3}
800	3.439×10^{2}	6.238×10^{2}	1.052×10^{3}	1.785×10^{3}	3.264×10^{3}	4.378×10^{3}
10^{3}	4.165×10^{2}	7.568×10^{2}	1.268×10^{3}	2.114×10^{3}	3.664×10^{3}	4.630×10^{3}
1.5×10^{3}	5.911×10^{2}	1.072×10^{3}	1.760×10^{3}	2.796×10^{3}	4.306×10^{3}	4.898×10^{3}
2×10^{3}	7.581×10^{2}	1.363×10^{3}	2.187×10^{3}	3.317×10^{3}	4.639×10^{3}	4.972×10^{3}
3×10^{3}	1.073×10^{3}	1.882×10^{3}	2.879×10^{3}	4.018×10^{3}	4.902×10^{3}	4.997×10^{3}
4×10^{3}	1.364×10^{3}	2.327×10^{3}	3.401×10^{3}	4.427×10^{3}	4.973×10^{3}	4.999×10^{3}
5×10^{3}	1.633×10^{3}	2.709×10^{3}	3.794×10^{3}	4.666×10^{3}	4.992×10^{3}	
6×10^{3}	1.883×10^{3}	3.036×10^{3}	4.091×10^{3}	4.805×10^{3}	4.998×10^{3}	
8×10^{3}	2.328×10^{3}	3.557×10^{3}	4.483×10^{3}	4.933×10^{3}		
10^{4}	2.710×10^{3}	3.939×10^{3}	4.706×10^{3}	4.977×10^{3}		
1.5×10^{4}	3.442×10^{3}	4.509×10^{3}	4.928×10^{3}	4.998×10^{3}		
2×10^{4}	3.940×10^{3}	4.772×10^{3}	4.983×10^{3}	4.999×10^{3}		
3×10^{4}	4.509×10^{3}	4.951×10^{3}	4.998×10^{3}			
4×10^{4}	4.772×10^{3}	4.989×10^{3}	4.999×10^{3}			
5×10^{4}	4.894×10^{3}	4.997×10^{3}				
6×10^{4}	4.951×10^{3}	4.999×10^{3}				
8×10^{4}	4.989×10^{3}					
10^{5}	4.997×10^{3}					
1.5×10^{5}	4.999×10^{3}					

R = 500 ; c = 0.5

θ	K2 = 2	K2 = 5	K2 = 10	K2 = 20	K2 = 50	K2 = 100
2×10^{2}						2.382×10^{3}
3×10^{2}						3.349×10^{3}
4×10^{2}						4.286×10^{3}
5×10^{2}					2.939×10^{3}	5.202×10^{3}
6×10^{2}					3.422×10^{3}	6.101×10^{3}
8×10^{2}					4.367×10^{3}	7.863×10^{3}
10^{3}				2.347×10^{3}	5.290×10^{3}	9.586×10^{3}
1.5×10^{3}				3.350×10^{3}	7.532×10^{3}	1.376×10^{4}
2×10^{3}			2.694×10^{3}	4.322×10^{3}	9.709×10^{3}	1.778×10^{4}
3×10^{3}			3.737×10^{3}	6.204×10^{3}	1.392×10^{4}	2.538×10^{4}
4×10^{3}			4.747×10^{3}	8.032×10^{3}	1.797×10^{4}	3.244×10^{4}
5×10^{3}		3.222×10^{3}	5.735×10^{3}	9.819×10^{3}	2.187×10^{4}	3.901×10^{4}
6×10^{3}		3.783×10^{3}	6.705×10^{3}	1.157×10^{4}	2.562×10^{4}	4.510×10^{4}
8×10^{3}		4.880×10^{3}	8.606×10^{3}	1.499×10^{4}	3.273×10^{4}	5.603×10^{4}
10^{4}	3.453×10^{3}	5.952×10^{3}	1.046×10^{4}	1.831×10^{4}	3.933×10^{4}	6.546×10^{4}
1.5×10^{4}	4.836×10^{3}	8.559×10^{3}	1.496×10^{4}	2.616×10^{4}	5.384×10^{4}	8.378×10^{4}
2×10^{4}	6.177×10^{3}	1.109×10^{4}	1.928×10^{4}	3.344×10^{4}	6.589×10^{4}	9.646×10^{4}
3×10^{4}	8.778×10^{3}	1.596×10^{4}	2.741×10^{4}	4.642×10^{4}	8.421×10^{4}	1.113×10^{5}
4×10^{4}	1.130×10^{4}	2.062×10^{4}	3.491×10^{4}	5.756×10^{4}	9.686×10^{4}	1.184×10^{5}
5×10^{4}	1.376×10^{4}	2.509×10^{4}	4.184×10^{4}	6.713×10^{4}	1.056×10^{5}	1.219×10^{5}
6×10^{4}	1.617×10^{4}	2.936×10^{4}	4.824×10^{4}	7.533×10^{4}	1.116×10^{5}	1.235×10^{5}
8×10^{4}	2.082×10^{4}	3.736×10^{4}	5.959×10^{4}	8.842×10^{4}	1.186×10^{5}	1.247×10^{5}
10^{5}	2.527×10^{4}	4.469×10^{4}	6.926×10^{4}	9.806×10^{4}	1.220×10^{5}	1.249×10^{5}
1.5×10^{5}	3.559×10^{4}	6.045×10^{4}	8.764×10^{4}	1.125×10^{5}	1.245×10^{5}	1.250×10^{5}
2×10^{5}	4.484×10^{4}	7.312×10^{4}	9.996×10^{4}	1.192×10^{5}	1.249×10^{5}	
3×10^{5}	6.056×10^{4}	9.148×10^{4}	1.137×10^{5}	1.237×10^{5}		
4×10^{5}	7.321×10^{4}	1.033×10^{5}	1.199×10^{5}	1.247×10^{5}		
5×10^{5}	8.337×10^{4}	1.110×10^{5}	1.227×10^{5}	1.249×10^{5}		
6×10^{5}	9.153×10^{4}	1.160×10^{5}	1.240×10^{5}	1.250×10^{5}		
8×10^{5}	1.034×10^{5}	1.212×10^{5}	1.248×10^{5}			
10^{6}	1.110×10^{5}	1.250×10^{5}				
1.5×10^{6}	1.203×10^{5}	1.248×10^{5}				
2×10^{6}		1.250×10^{5}				

R = 10^{3}; c = 0.5

θ	K2 = 2	K2 = 5	K2 = 10	K2 = 20	K2 = 50	K2 = 100
10^{3}						1.045×10^{4}
1.5×10^{3}						1.464×10^{4}
2×10^{3}					1.169×10^{4}	1.871×10^{4}
3×10^{3}					1.591×10^{4}	2.664×10^{4}
4×10^{3}					2.001×10^{4}	3.436×10^{4}
5×10^{3}				1.176×10^{4}	2.402×10^{4}	4.192×10^{4}
6×10^{3}				1.351×10^{4}	2.798×10^{4}	4.935×10^{4}
8×10^{3}				1.694×10^{4}	3.574×10^{4}	6.382×10^{4}
10^{4}			1.031×10^{4}	2.030×10^{4}	4.334×10^{4}	7.783×10^{4}
1.5×10^{4}			1.486×10^{4}	2.849×10^{4}	6.176×10^{4}	1.109×10^{5}
2×10^{4}		1.387×10^{4}	1.928×10^{4}	3.647×10^{4}	7.942×10^{4}	1.414×10^{5}
3×10^{4}		1.876×10^{4}	2.789×10^{4}	5.195×10^{4}	1.126×10^{5}	1.954×10^{5}
4×10^{4}		2.351×10^{4}	3.626×10^{4}	6.688×10^{4}	1.432×10^{5}	2.412×10^{5}
5×10^{4}	1.569×10^{4}	2.817×10^{4}	4.446×10^{4}	8.131×10^{4}	1.714×10^{5}	2.802×10^{5}
6×10^{4}	1.812×10^{4}	3.276×10^{4}	5.250×10^{4}	9.526×10^{4}	1.973×10^{5}	3.133×10^{5}
8×10^{4}	2.288×10^{4}	4.176×10^{4}	6.815×10^{4}	1.218×10^{5}	2.433×10^{5}	3.653×10^{5}
10^{5}	2.756×10^{4}	5.056×10^{4}	8.325×10^{4}	1.466×10^{5}	2.822×10^{5}	4.028×10^{5}
1.5×10^{5}	3.895×10^{4}	7.181×10^{4}	1.187×10^{5}	2.017×10^{5}	3.557×10^{5}	4.570×10^{5}
2×10^{5}	5.001×10^{4}	9.204×10^{4}	1.512×10^{5}	2.482×10^{5}	4.043×10^{5}	4.810×10^{5}
3×10^{5}	7.132×10^{4}	1.297×10^{5}	2.080×10^{5}	3.205×10^{5}	4.580×10^{5}	4.963×10^{5}
4×10^{5}	9.160×10^{4}	1.638×10^{5}	2.556×10^{5}	3.721×10^{5}	4.815×10^{5}	4.993×10^{5}
5×10^{5}	1.109×10^{5}	1.949×10^{5}	2.954×10^{5}	4.089×10^{5}	4.919×10^{5}	4.999×10^{5}
6×10^{5}	1.293×10^{5}	2.230×10^{5}	3.288×10^{5}	4.351×10^{5}	4.964×10^{5}	5.000×10^{5}
8×10^{5}	1.636×10^{5}	2.718×10^{5}	3.801×10^{5}	4.670×10^{5}	4.993×10^{5}	
10^{6}	1.947×10^{5}	3.119×10^{5}	4.160×10^{5}	4.833×10^{5}	4.999×10^{5}	
1.5×10^{6}	2.604×10^{5}	3.841×10^{5}	4.655×10^{5}	4.969×10^{5}	5.000×10^{5}	
2×10^{6}	3.120×10^{5}	4.286×10^{5}	4.858×10^{5}	4.994×10^{5}		
3×10^{6}	3.842×10^{5}	4.729×10^{5}	4.976×10^{5}	5.000×10^{5}		
4×10^{6}	4.287×10^{5}	4.897×10^{5}	4.996×10^{5}			
5×10^{6}	4.561×10^{5}	4.961×10^{5}	4.999×10^{5}			
6×10^{6}	4.730×10^{5}	4.985×10^{5}	5.000×10^{5}			
8×10^{6}	4.897×10^{5}	4.998×10^{5}				
10^{7}	4.961×10^{5}	5.000×10^{5}				
1.5×10^{7}	4.997×10^{5}					

R = 20; c = 0.3

θ	K2=2	K2=5	K2=10	K2=20	K2=50	K2=100
.2						9.494
.3						12.83
.4						15.94
.5					11.22	18.91
.6					12.86	21.77
.8					15.99	27.24
1				9.612	18.96	32.47
1.5				12.99	25.98	44.75
2			9.763	16.15	32.57	56.09
3			13.20	22.05	44.90	76.32
4			16.41	27.60	56.27	93.69
5		11.86	19.47	32.91	66.80	108.6
6		13.60	22.41	38.01	76.55	121.4
8		16.91	28.06	47.72	93.96	141.9
10	10.93	20.07	33.45	56.83	108.9	157.0
15	14.81	27.52	46.10	77.27	137.7	179.6
20	18.44	34.53	57.75	94.79	157.3	190.2
30	25.24	47.59	78.46	122.6	179.8	197.5
40	31.64	59.58	96.14	143.1	190.3	199.1
50	37.74	70.62	111.2	158.1	195.2	199.4
60	43.60	80.79	124.1	169.1	197.5	199.5
80	54.66	98.78	144.5	183.1	199.1	
100	64.93	114.0	159.4	190.7	199.4	
150	87.53	142.8	181.3	197.6	199.5	
200	106.9	161.9	191.2	199.1		
300	135.0	183.0	197.8	199.5		
400	154.8	192.2	199.1			
500	168.6	196.3	199.4			
600	178.1	198.1	199.5			
800	189.2	199.2				
1000	194.6	199.4				
1500	198.7	199.5				

R = 20; c = 0.1

θ	K2=2	K2=5	K2=10	K2=20	K2=50	K2=100
.2						11.41
.3						15.48
.4						19.28
.5					13.51	22.91
.6					15.50	26.42
.8					19.31	33.12
1				11.45	22.95	39.51
1.5				15.53	31.52	54.39
2			11.50	19.35	39.57	67.87
3			15.59	26.51	54.45	91.17
4			19.43	33.25	67.93	110.4
5		13.70	23.09	39.66	80.16	126.1
6		15.72	26.62	45.81	91.25	139.1
8		19.60	33.39	57.39	110.4	158.6
10	11.86	23.29	39.83	68.10	126.2	171.8
15	16.10	32.00	54.82	91.46	154.5	189.1
20	20.07	40.18	68.38	110.7	171.9	195.6
30	27.52	55.28	91.81	139.4	189.1	198.9
40	34.53	68.94	111.0	158.9	195.6	199.4
50	41.20	81.31	126.9	172.1	198.0	199.5
60	47.58	92.50	139.8	180.9	198.9	
80	59.58	111.8	159.2	191.0	199.4	
100	70.62	127.6	172.3	195.6	199.5	
150	94.56	155.8	189.4	199.0		
200	114.0	172.9	195.7	199.4		
300	142.8	189.7	199.0	199.5		
400	161.9	195.9	199.4			
500	174.6	198.2	199.5			
600	183.0	199.0				
800	192.2	199.4				
1000	196.3	199.5				
1500	199.1					

R = 20; c = 0.7

θ	K2=2	K2=5	K2=10	K2=20	K2=50	K2=100
.2						5.237
.3						6.980
.4						8.595
.5					6.228	10.12
.6					7.094	11.59
.8					8.738	14.38
1				5.567	10.29	17.04
1.5				7.432	13.93	23.30
2			5.974	9.160	17.34	29.18
3			7.988	12.37	23.71	40.20
4			9.857	15.37	29.70	50.45
5		7.941	11.63	12.82	35.43	60.03
6		9.068	13.33	20.97	40.93	68.99
8		11.21	16.58	26.23	51.35	85.22
10	8.968	13.24	19.67	31.26	61.08	99.43
15	12.10	18.02	26.96	43.08	62.70	127.7
20	15.02	22.49	33.81	54.02	100.9	148.0
30	20.49	30.90	46.60	73.65	129.3	173.0
40	25.62	38.78	58.36	90.63	149.5	185.8
50	30.53	46.27	69.22	105.3	163.9	192.5
60	35.25	53.39	79.24	118.0	174.2	195.9
80	44.27	66.66	97.02	138.5	186.7	198.5
100	52.76	78.72	112.2	153.9	193.0	199.2
150	72.01	104.3	141.0	177.4	198.3	199.5
200	88.73	124.4	160.3	188.8	199.3	
300	115.9	152.9	181.9	197.0	199.5	
400	136.4	170.5	191.6	198.9		
500	151.8	181.5	195.9	199.4		
600	163.5	188.3	197.9	199.5		
800	179.0	195.2	199.2			
1000	187.8	197.8	199.4			
1500	196.6	199.3	199.5			
2000		199.5				

R = 20; c = 0.9

θ	K2=2	K2=5	K2=10	K2=20	K2=50	K2=100
.2						2.625
.3						3.405
.4						4.131
.5					3.183	4.816
.6					3.590	5.470
.8					4.362	6.709
1				3.155	5.090	7.878
1.5				4.132	6.781	10.60
2			3.772	5.037	8.346	13.14
3			4.977	6.705	11.25	17.88
4			6.091	8.249	13.95	22.33
5		5.740	7.142	9.711	16.53	26.57
6		6.534	8.147	11.11	19.01	30.67
8		8.037	10.05	13.75	23.75	38.50
10	7.938	9.459	11.86	16.32	28.28	45.93
15	10.68	12.78	16.11	22.29	38.95	63.01
20	13.24	15.88	20.08	27.90	48.89	78.19
30	18.01	21.69	27.53	38.43	66.94	103.7
40	22.49	27.14	34.53	48.24	82.82	123.8
50	26.77	32.36	41.20	57.44	96.80	139.7
60	30.89	37.38	47.59	66.07	109.1	152.3
80	38.78	46.93	59.58	81.80	129.5	170.0
100	46.27	55.90	70.62	95.68	145.2	181.1
150	63.46	76.07	94.55	123.6	170.8	193.8
200	78.72	93.41	114.0	144.0	184.4	197.8
300	104.3	121.1	142.8	169.9	195.3	199.3
400	124.5	141.6	161.9	183.7	198.3	199.5
500	140.3	156.7	174.6	191.0	199.2	
600	152.9	167.9	183.0	195.0	199.4	
800	170.5	182.3	192.2	198.2	199.5	
1000	181.5	190.1	196.3	199.1		
1500	194.0	197.4	199.1	199.5		
2000		199.0	199.4			