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A THEORETICAL STUDY OF CONDITIONALLY STABLE SYSTEMS

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PREFACE

The author is indebted to his wife for her patience during the preparation of this dissertation. In addition, particular thanks goes to Professor Norman Balabanian for the interest he has shown in the work performed, for his helpful suggestions relative to the theroetical development of the dissertation, and for his helpful criticism while reviewing the manuscript.

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CHAPTER I

INTRODUCTION

1.1 Historical Background

In the years prior to World War II the design of feedback control systems was an art that was based on a few isolated facts tempered, of course, by the designer's experience. This resulted in much "cut and try" engineering, with varying degrees of success. During World War II certain groups, notably those at the Radiation Laboratory and Bell Laboratories were called upon to develop certain servomechanisms to meet some of the military requirements. As a result of this concentrated effort, the design of feedback control systems advanced from an art to that of a science, based principally on the works of Bode[1], Draper[2], Hall[3], Harris[4], Hazen[5,6], MacColl[7], and Nyquist[8].

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1. "Network Analysis and Feedback Amplifier Design" (book), D. Van Nostrand, New York, N.Y., 1945, pages 168-169, 451-476.
2. "Design Factors Controlling the Dynamic Performance of Instruments", C.S. Draper and A.P. Bentley, Transactions ASME, Vol. 62, July 1940, pp. 421-432.
3. "The Analysis and Synthesis of Linear Servomechanisms", A.C. Hall, Technology Press, Massachusetts Institute of Technology, May, 1943.
4. "Frequency Response of Automatic Control Systems", Herbert Harris, Electrical Engineering, August-September 1946, pp. 539-546.
5. "Theory of Servomechanisms", H.L. Hazen, Journal of Franklin Institute, Vol. 218, 1934, pp. 279-331.
6. "Design and Test of a High-Performance Servomechanism", H.L. Hazen, Journal of Franklin Institute, Vol. 218 November 1934, pp. 543-580.

With the exception of the work of MacColl[7], the theory developed was a linear theory, that is, one based on the solution of linear differential equations.

In the past decade the linear theories which were developed in the early '40s have been refined and some new methods devised. A considerable amount of effort has also been expended in the areas of nonlinear control system analysis and in the study of sampled-data systems. A review of current literature reveals, however, that an area still exists which has been by-passed by other investigators, namely, a comprehensive study of a class of linear systems commonly referred to as conditionally stable systems. Although individual cases of this class of systems have been treated since H. Nyquist[8] first recognized that such systems do exist, this general class of systems has never received any organized attention.

The word conditionally stable was probably first used by H.W. Bode[1] when he described Nyquist's[8] work in connection with feedback amplifier design. Although not formally defined by either Nyquist[8] or Bode[1] personally, they recognized that some feedback amplifiers exist which are open-loop unstable, but which can be stablized by closing the final feedback loop.

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7. "Fundamental Theory of Servomechanisms" (book) L.A. MacColl, D. Van Nostrand, New York, N.Y. 1945.

8. "Regeneration Theory", H. Nyquist, Bell System Technical Journal, Vol. 11, January, 1932, pp 126-147.

A Conditionally Stable System as defined by Brown and Campbell[9] is "a system which is unstable for a particular gain, but stable for both larger and smaller gain values". A review of literature reveals Herr and Gerst[10], and Travers[11], have studied systems which they also defined as conditionally stable. A closer inspection of the systems studied by them would reveal that their definition of conditional stability does not agree with the one proposed by Brown and Campbell.[9] The systems considered by these latter authors were open-loop unstable. Inspection of the root-locus plots for these systems would reveal that they satisfy the following definition: "a system that is stable for a particular value of gain, but is unstable for both higher and lower gain values". A comparison of these two definitions shows that they are essentially opposite in their definition of the useful regions of operation. In an attempt to clarify this situation, in the work that follows, both types of systems will be discussed.

The author's interest in conditionally stable systems stems from the paper by Travers[11] who stated without proof that using conditionally stable control systems the following advantages could be achieved over the conventional absolutely stable system:

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9. "Principles of Servomechanisms" (book) G. S. Brown, D. P. Campbell, 1948, p. 172, John Wiley & Sons Inc.

10. "The Analysis and an Optimum Synthesis of Linear Servomechanisms", D. Herr, I. Gerst; AIEE Transactions, Vol. 69, 1947, pp. 959-70.

11. "A Note on the Design of Conditionally Stable Feedback Systems", Paul Travers, AIEE Transactions, Vol. 70, 1951,

1. These systems sometime permit larger gain compared to an absolutely stable system of the same bandwidth.
2. These systems sometime permit a smaller bandwidth for the same loop gain.

The reasons for minimizing the bandwidth are:

1. To minimize the transmission of noise by the system.
2. To require the lowest frequency response characteristic for the physical elements of the system.

1.2 Research of Herr and Gerst[10]

Herr and Gerst published an article summarizing their study of a class of servomechanisms having an open-loop frequency response function of the following form:

$$KG(j\omega) = \frac{\left(1 + j\frac{\omega}{\omega_1}\right)^q}{(j\omega)^r \left(1 + j\frac{\omega}{\omega_2}\right)^t} \qquad 1.2\text{-}1$$

where $r \geq 2$ and $t \geq 1$, while $r + t - q = 2$ or 3. The parameters q, r, and t are variables which control the high, medium, and low-frequency attenuations rates, and ω_1 and ω_2 are the corner frequencies associated with the logarithmic representation of the magnitude of the open-loop function (see Figure 1.1). The problem which they treated was: Given A, M_{max}, and the high-frequency attentuation rate, determine that

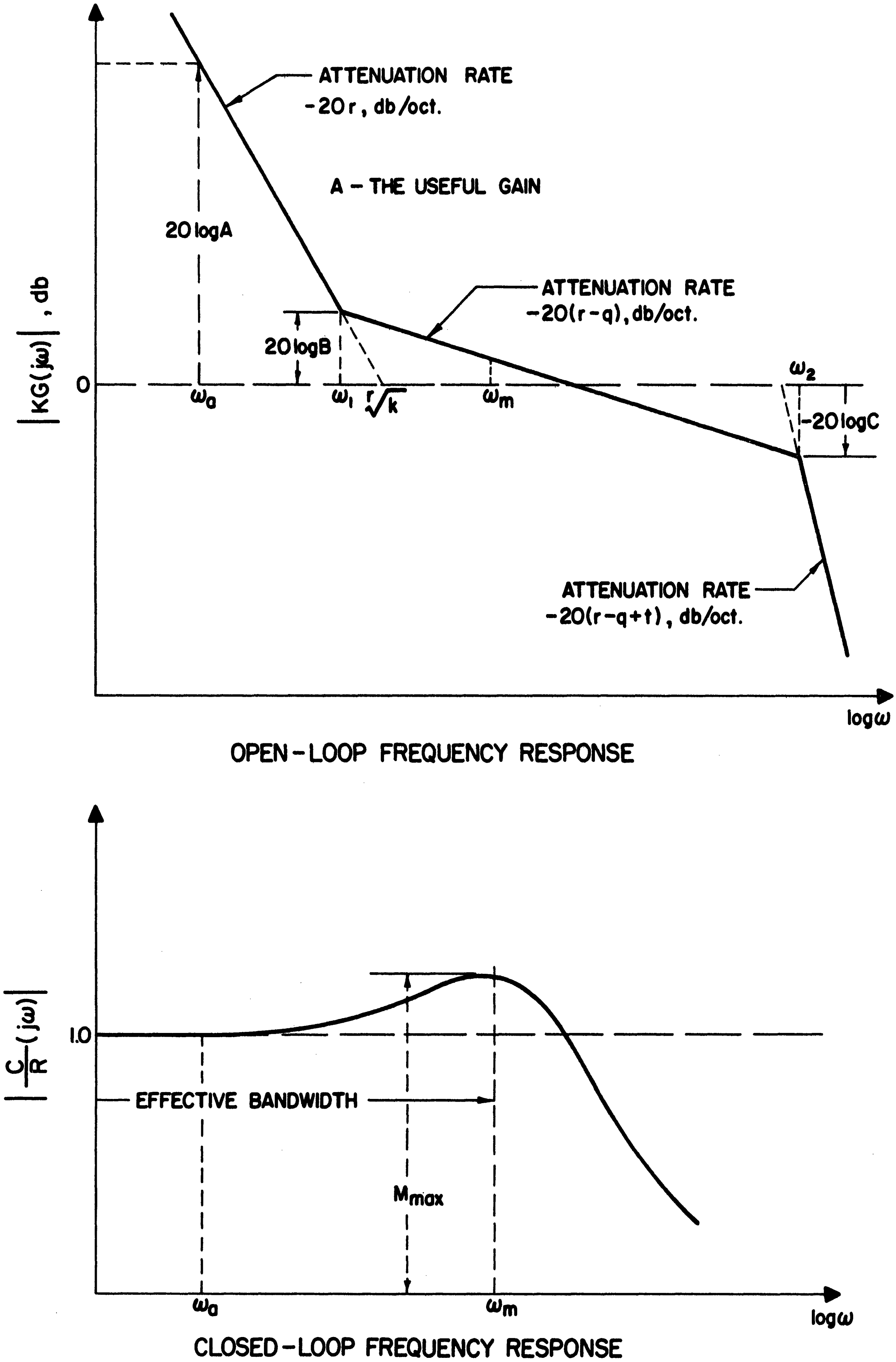

Figure 1.1 Bode Attenuation Diagrams of a Minimum Bandwidth System

attenuation-rate r_o , which minimizes $\frac{\omega_m}{\omega_a}$, the bandwidth ratio.

The authors were trying to minimize the effective bandwidth of the closed-loop functions, since this type of system would: a) minimize the amount of energy required to operate the system, b) minimize extraneous noise present in the system, and c) utilize all the bandwidth possible in a given situation.

The approach they used to solve the above problem was: a) to create a normalized $\frac{C}{E}(s)$ function having the following form:

$$A(j\omega) = \frac{1}{X^r(j\omega)} \quad \frac{\left[1 + a\, X(j\omega)\right]^{\gamma}}{\left[1 + b\, X(j\omega)\right]^{z}} \qquad 1.2\text{-}2$$

where

$$X = \frac{j\omega}{\sqrt[r]{K}} \ , \ a = \frac{\sqrt[r]{K}}{\omega_1} \ , \ \text{and } b = \frac{\sqrt[r]{K}}{\omega_2} \qquad 1.2\text{-}3$$

b) To form a new function

$$C(X,a,b) \triangleq \frac{1}{M^2} - 1 \triangleq \frac{f(X,a,b)}{g(X,a)} \qquad 1.2\text{-}4$$

where C is the $\frac{1}{X}$ intercept of the M contours in the KG(s) plane. M in this case is the magnitude of the system response function. The authors chose to deal with

C rather than M because the analysis could be more readily carried out.

c) To determine where $M > 1$ is a maximum and $C(X,a,b)$ is a minimum in the equation

$$g(X,a)\ C(X,a,b) = f(X,a,b) \qquad 1.2\text{-}5$$

This was done by partially differentiating this equation with respect to X, yielding

$$\left[\frac{\partial}{\partial X} g(X,a)\ C(X,a,b) = \frac{\partial}{\partial X} f(X,a,b)\right]_{X=X_m} \qquad 1.2\text{-}6$$

whose solution, when substituted back into equation 1.2-5 yields

$$g(X_m,a)\ C(X_m,a,b) = f(X_m,a,b) \qquad 1.2\text{-}7$$

d) Now since $C(X_m,a,b) = \frac{1}{M^2} - 1$, assign $M = M_{max}$ (design parameter) and form a second equation

$$C \cdot g(X,a) = f(X,a,b) \qquad 1.2\text{-}8$$

e) Partially differentiating equation 1.2-8 with respect to X yields

$$C \left[\frac{\partial}{\partial X} g(X,a) = \frac{\partial}{\partial X} f(X,a,b) \right]_{X=X_m} \qquad 1.2\text{-}9$$

and upon substitution this value of X back into equation 1.2-8 above yields

$$C\, g(X_m, a) = f(X_m, a, b) \qquad 1.2\text{-}10$$

f) Now equations 1.2-9 and 1.2-10 define b as a function of a . Following their method b is now maximized by differentrating equation 1.2-10 partially with respect to a .

$$C \frac{\partial}{\partial a} g(X_m, a) = \frac{\partial}{\partial a} f(X_m, a, b) \qquad 1.2\text{-}11$$

g) Equation 1.2-9, 1.2-10 , and 1.2-11 form a simultaneous system of equations which can be solved for X_m , a, b. Solutions are restricted to positive values of X_m, a, b, for which $b < a$, for which b is actually maximum, and of course which corresponds to a stable system.

One of the systems treated by the authors was

$$A = \frac{K}{(j\omega)^2} \frac{\left(1 + j\frac{\omega}{\omega_1}\right)^1}{\left(1 + j\frac{\omega}{\omega_2}\right)^2} = \frac{1}{X^2} \frac{(1 + aX)^1}{(1 + bX)^2}$$

In this case

$$g(X,a) = (1 + a^2X^2)$$

$$f(X,a,b) = b^4X^8 + 2b^2X^6 + X^4(1 + 2b^2 - 4ab) - 2X^2$$

The simultaneous set of equations which must be solved is

$$b^4X_m^8 + 2b^2X_m^6 + X_m^4 \ (1 + 2b^2 - 4ab) - 2X_m^2 = C(1 + a^2X_m^2)$$

$$4b^4X_m^6 + 6b^2X_m^4 + 2X_m^2 \ (1 + 2b^2 - 4ab) - 2 = a^2C$$

and

$$aC = -4bX_m^2$$

They Nyquist diagram for this class of systems is shown in Figure 1.2 , while the corresponding root-locus will be found sketched in Figure 1.3 .

1.3 Research of Travers

Utilizing the same form of an open-loop function as proposed by Herr and Gerst, Travers chose Krohn's* criterion as his starting point in developing a set of criteria for minimum bandwidth systems. Noting that Krohn's criterion

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* See reference 12 on following page

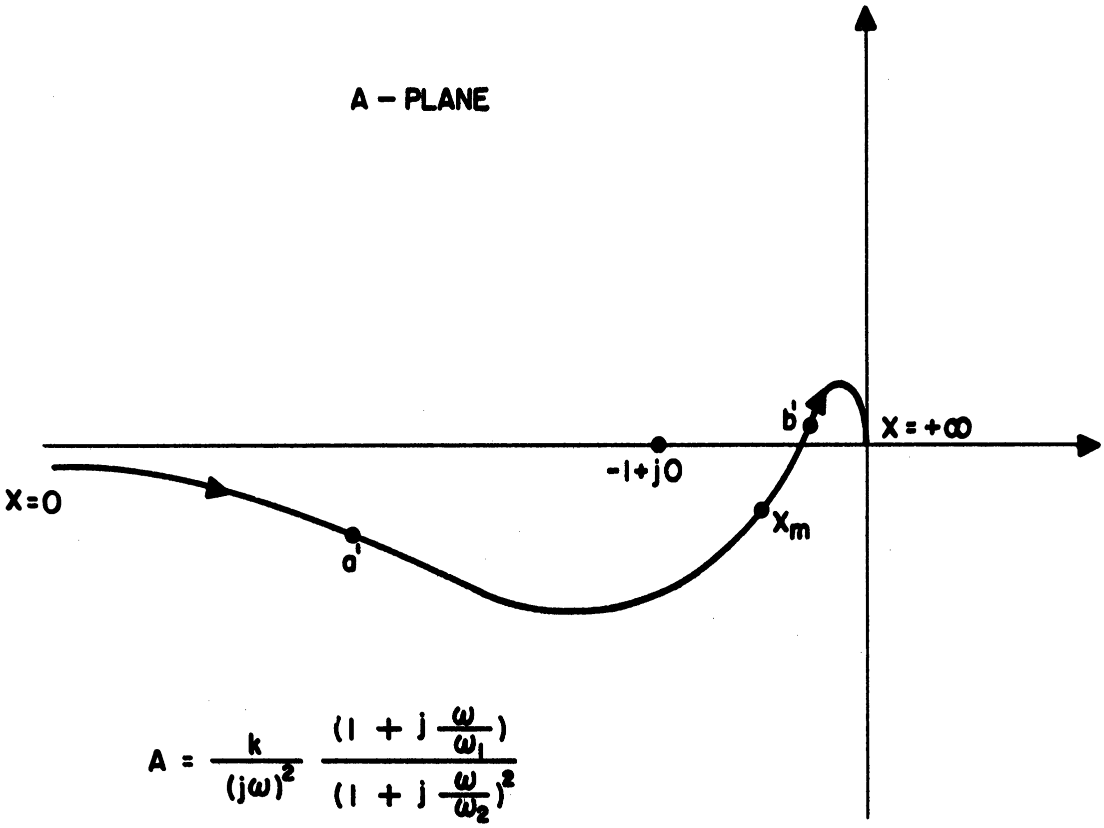

Figure 1.2 Nyquist Diagram of a Minimum Bandwidth System

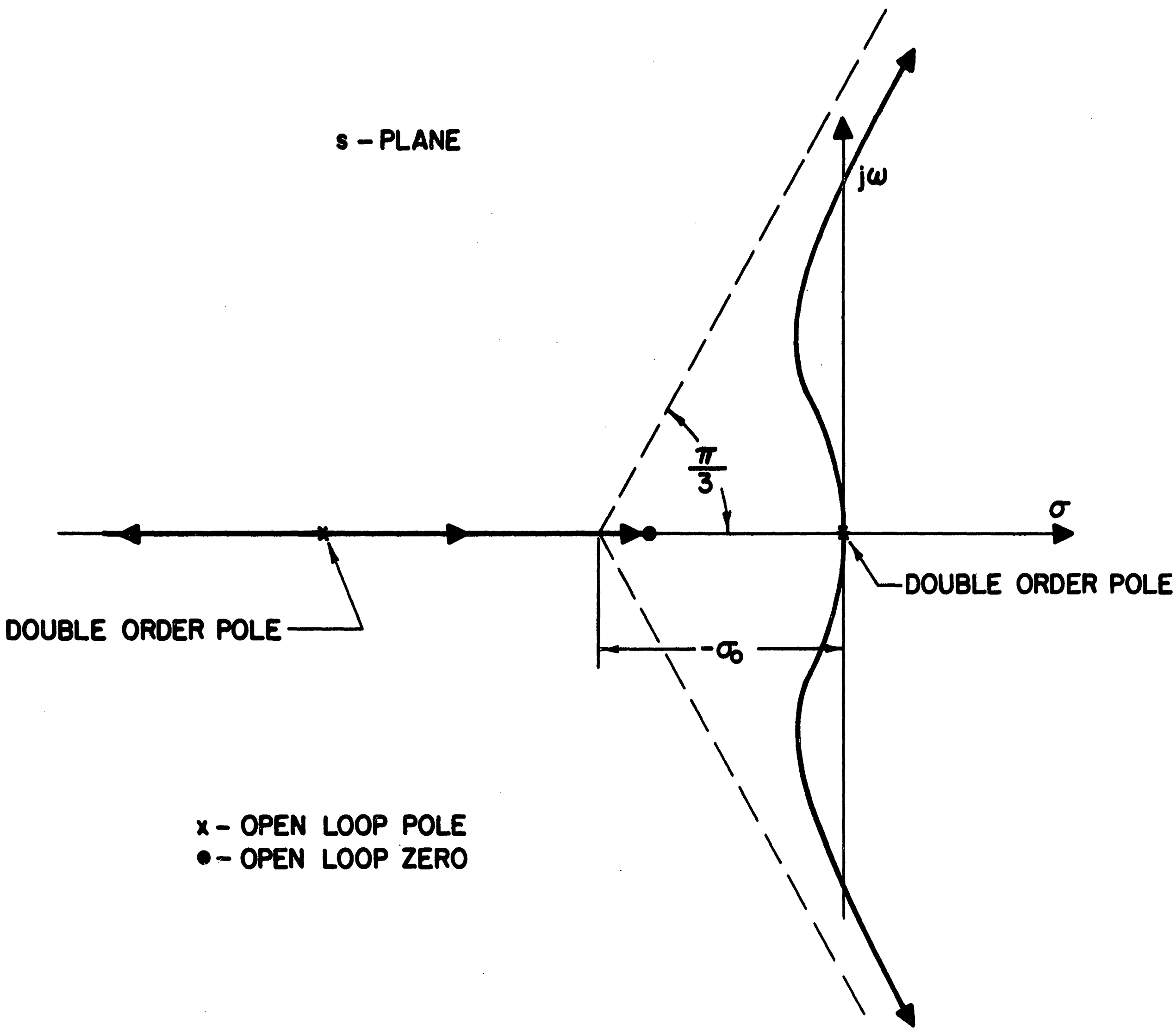

Figure 1.3 Root Locus Sketch of a Minimum Bandwidth System

will be satisfied using this open-loop function if $q = r - 1$, and in addition, making use of the following equations

$$\left| KG(j\omega_m) \right| = \frac{M_{max}}{\sqrt{M_{max}^2 - 1}} \qquad 1.3\text{-}1$$

$$\operatorname{Arg} KG(j\omega_m) = \sin^{-1}\left[\frac{1}{M_{max}}\right] - \pi \qquad 1.3\text{-}2$$

and

$$\frac{\partial}{\partial \omega}\left[\operatorname{Arg} KG(j\omega_m) + \pi\right] = \frac{(r-1)\omega_1}{\omega_1^2 + \omega^2} - \frac{t\omega_2}{\omega_2^2 + \omega^2} = 0 \qquad 1.3\text{-}3$$

Travers established the following relationships:

$$\frac{\omega_2}{\omega_1} = \frac{4(r-1)t}{\left(\cos^{-1}\frac{1}{M_{max}}\right)^2} , \qquad 1.3\text{-}4$$

$$\frac{\omega_m}{\omega_1} = \frac{2(r-1)}{\cos^{-1}\frac{1}{M_{max}}} , \qquad 1.3\text{-}5$$

$$\frac{\omega_2}{\omega_m} = \frac{2t}{\cos^{-1}\frac{1}{M_{max}}} \qquad 1.3\text{-}6$$

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12. "Theory of Servomechanism", (book), James, Nichols, Phillips, McGraw-Hill Book Co., New York, N.Y., 1947, p. 182, Krohn's criterion states that in order to utilize the maximum phase margin of a system, the open-loop transfer function should be tangent to the M-contours at the point of maximum phase margin, providing the curvature of the open-loop function does not exceed the curvature of the M-contours. This curvature requirement is satisfied by making $q = r-1$ in equation 1.2-1.

Inspection of equations 1.3-5 and 1.3-6 will reveal that $\frac{\omega_m}{\omega_1}$ is not dependent on t , the order of the pole away from the origin, and $\frac{\omega_2}{\omega_m}$ is not dependent on r , the order of the pole at the origin, as it was in the case of the work of Herr and Gerst.

The use of Krohn's criterion with this class of loop functions permitted the separation of the high-and low-frequency attenuation rates about ω_m . The effective bandwidth has been made independent of "t". "t" is used only to determine $\frac{\omega_2}{\omega_m}$.

1.4 The Missing Link

A review of literature has therefore shown that a definite gap does exist in the analysis and synthesis of linear systems which are conditionally stable. While in practice such systems do exist, their design and stabilization have been treated as a more complex problem requiring an extension of the design techniques normally used. A class of conditionally stable systems have been discussed by Herr and Gerst[10], and Travers[11]. Also brief mention about such systems will be found in the books by Bode,* James, Nichols, Phillips,** and to a lesser degree in the book by Nixon[13]. In each case the authors have discussed specific examples. However, such questions as:

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* Reference 1, pp. 162-163

** Reference 12, pp. 177-192

13. "Principles of Automatic Controls" (book), F.E. Nixon Prentice-Hall, N.Y., 1953, pp. 114-121.

1. What are necessary and sufficient conditions for an unstable open-loop system to be closed-loop stable?
2. Are all conditionally stable systems open-loop unstable?
3. Are there any advantages in making an open-loop system unstable?
4. Given a prescribed unstable open-loop system, how can it be modified to give satisfactory closed-loop response?

were left unanswered.

An attempt to anwer these questions has led to the subject matter presented in this dissertation.

1.5 Scope of the Research

The objectives of this dissertation are therefore as follows:

1. To study the advantages and disadvantages of systems that are open-loop unstable.
2. To establish the design conditions which permit a feedback control system that is open-loop unstable to become stable on closing the final loop.

CHAPTER II

CONTROL SYSTEM CONCEPTS

2.1 Transfer Function of a Linear System

A complex control system will in general consist of a number of electrical, mechanical, hydraulic and pneumatic elements interconnected in a given physical complex. The physical complex must in turn be broken down into a number of small sections in such a way that each section may be considered as acting independently. Under these conditions, a number of differential equations can be written relating the various sections. In addition, if these equations are linear or can be linearized and still describe the operation of the system in a satisfactory manner, they can be transformed from the real time domain into the complex s-domain using the Laplace transformation. If the initial conditions are now equal to zero, one or more transfer functions $G_i(s)$, $i = 1, 2, 3, \ldots$ will be obtained relating the various components of this system. The procedure for obtaining the transfer function of a physical element is found discussed in such books as Gardner and Barnes[14] or White and Woodson[15].

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14. "Transients in Linear Systems", (book) M. F. Gardner and J. L. Barnes, Vol. 1, 1942, Chapter V.

15. "Electromechanical Energy Conversion," (book) D. C. White and H. H. Woodson, 1959, Chapter II.

In the work that follows, it is assumed that the problem of reducing a physical complex to its equivalent linear transfer function form has been solved.

2.2 Stability of Open-Loop Systems

Attention will now be focused on a typical transfer function, $G_i(s)$, obtained as a result of applying the procedure discussed in the last section. It is a rational function defined as follows:

$$\frac{X_2(s)}{X_1(s)} = G_i(s) = \frac{Z(s)}{P(s)} = \frac{a_0 + a_1 s + - - - a_m s^m}{b_0 + b_1 s + - - - b_n s^n} \qquad 2.2\text{-}1$$

which can be represented in block diagram form as shown in Figure 2.1a . In representing a system in block diagram form, it is implied that the initial conditions operator* is assumed to be zero. In addition, if the transfer function block $G_1(s)$ is to be connected in cascade with a second transfer function $G_2(s) = \frac{X_3}{X_2}(s)$, (see Figure 2.1b), the following assumptions are implied, namely

1) The function $X_2(s)$ is not altered as a result of connecting $G_2(s)$ to it. In other words it assumes that $X_2(s)$ operates into an open circuit or into a system $G_2(s)$ having a much higher impedance level.

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* See reference 14, page 132

2) The impedance level of $G_1(s)$ is such that it causes no change in $X_1(s)$, the excitation function.

In order for $G_1(s)$ to be physically realizable as a voltage ratio, $n \geqslant m$, while to be physically realizable as a transfer impedance, i.e. $\frac{E_1}{I_1}(s)$, $n = m-1$.

Now equation 2.2-1 may be written as

$$X_2(s)=X_1(s)\ F(s) = X_1(s) \left[\frac{(s+z_1)^u (s+z_2)^v - - (s+z_m)^w}{(s+p_1)^{u'} (s+p_2)^{v'} - - (s+p_n)^{w'}} \right]$$

where in general $z_1, z_2, \ldots, p_1, p_2, \ldots$, may be real or complex, and u, v, w, w', v', u', may be of any order as long as they are finite.

If z_i or p_j is complex, however, its complex conjugate z_i* or p_j* must also be present. The following question may be raised regarding the characteristics of $G_1(s)$. "What must be the restrictions on $G_1(s)$ to insure a stable response from $G_1(s)$ which will be independent of the input function $x_1(t)$?"

The following approach is used to answer this question:

In order to determine the stability of the function $G_1(s)$, it is necessary to give the input which is described by $x_1(t)$ an infinitesimal perturbation and note what happens to the output as a function of time. The easiest type of perturbation to give the input is a unit impulse. When $x_1(t)$ represents a delta function input occurring at

$t = 0$, $X_1(s) = 1$, and equation 2.2-3 with this input function may be written in expanded form as follows:

$$X_2(s) = \sum_{i=1}^{u'} \frac{K_{1i}}{(s+P_1)^i} + \sum_{j=1}^{v'} \frac{K_{2j}}{(s+P_2)^j} + \ldots \sum_{\ell=1}^{w'} \frac{K_{n\ell}}{(s+p_n)^\ell} + K_\infty \qquad 2.2\text{-}3$$

Taking inverse transforms of both sides, equation 2.2-3 becomes:

$$x_2(t) = \left[K_{11}\frac{t^{u-1}}{(u-1)!} + \frac{K_{12}\, t^{u-2}}{(u-2)!} + \ldots\right] e^{-p_1 t} +$$

$$+ \left[K_{21}\frac{t^{v-1}}{(v-1)!} + \frac{K_{22}\, t^{v-2}}{(v-2)!} + \ldots\right] e^{-p_2 t} + \qquad 2.2\text{-}4$$

$$+ \left[K_{n1}\frac{t^{W-1}}{(W-1)!} + \frac{K_{n2}\, t^{W-2}}{(W-2)!} + \ldots\right] e^{-p_n t} + \ldots . K_\infty\,\delta(t)$$

From equation 2.2-4 the following restrictions regarding the location of the poles of F(s) in the s-plane can be deduced:

1. $p_1, p_2, \ldots p_i$ cannot be negative real or cannot have a negative real part if complex, for $x_2(t)$ to remain bounded for all values of t.

2. $p_1, p_2, \ldots p_j$ cannot have pure imaginary values, or the system will be continuously oscillatory.

3. A pole or order 2 or more at the origin must be excluded since these lead to functions of the type $K_o t^r$ where $r \geqslant 1$

2.3 Stability of Closed-Loop Systems

Since it is possible to talk about either the stability of the closed-loop function $\frac{C}{R}$ (s) or the stability of the open-loop function $\frac{C}{E}$ (s), the definition of open-loop stability will now be extended to include a closed-loop system of the type shown in Figure 2.1c . It consists of a device known as the error detector which senses the differences between the input and output. Although in an electrical circuit this is usually the difference between two voltages, in the general case, it represents the difference between the input, or the regulated variable, and the output, or controlled variable.

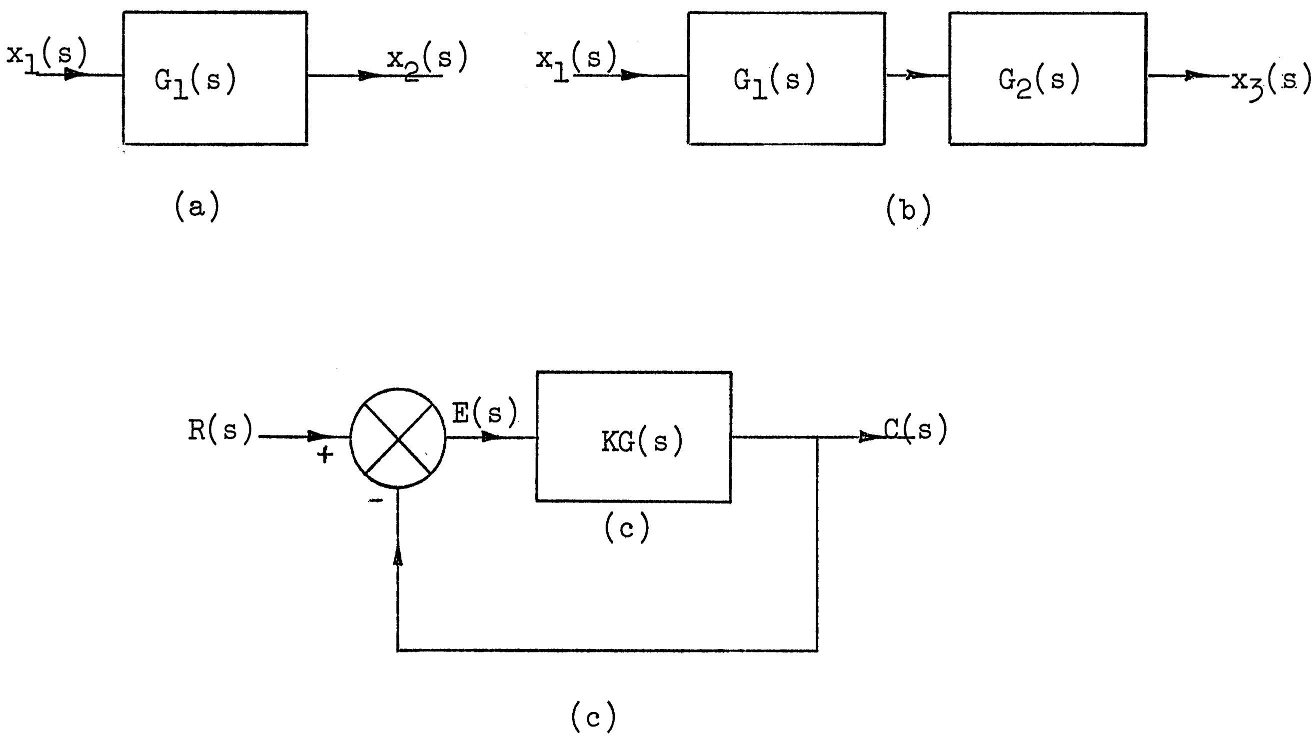

Figure 2.1 Block Diagram Representation of Linear Control System

The closed-loop system, which is represented by the block diagram shown in Figure 2.1c is defined by the following equations:

$$\frac{C}{E}(s) = KG(s) \quad , \tag{2.3-1}$$

and

$$\frac{C}{R}(s) = \frac{KG(s)}{1 + KG(s)} \tag{2.3-2}$$

Equation 2.3-2 can be written in expanded form as follows:

$$\frac{C}{R}(s) = \frac{d_o}{c_o}\left[\frac{c_o + c_1 s + - - - c_p s^p}{d_o + d_1 s + - - - d_q s^q}\right] =$$

$$\frac{\prod_{j=1}^{n} \beta_j}{\prod_{i=1}^{m} \alpha_i}\left[\frac{(s+\alpha_1)(s+\alpha_2) - - - (s+\alpha_p)}{(s+\beta_1)(s+\beta_2) - - - (s+\beta_q)}\right] \tag{2.3-3}$$

where $q \geqslant p$ for $\frac{C}{R}(s)$ to be physically realizable. A comparison of equation 2.2-1 with 2.3-3 reveals the closed-loop function to be identical in form to the open-loop function, and therefore all the restrictions placed on the open-loop function $\frac{C}{E}(s)$, to guarantee its stability, also apply to the closed-loop function, $\frac{C}{R}(s)$.

In the majority of feedback control systems the designer usually possesses a stable open-loop function, and he is called upon to add passive compensation so as to achieve satisfactory closed-loop performance. In the study of control systems a designer may be confronted, however, with an open-loop system that contains a right half-plane pole, and it is therefore open-loop unstable. The following questions must then be answered:

1. Is it ever possible to stabilize the closed-loop system if the open-loop system is of this type?

2. What restrictions must be placed on this type of $\frac{C}{E}(s)$ function in order that the $\frac{C}{R}(s)$ function be stable?

The answer to question 1 will be "yes" if it can be shown that at least one case is possible in which KG(s) is unstable but the closed-loop function $\frac{C}{R}(s)$ is stable. As an example, consider the second-order system having the following form:

$$\frac{C}{E}(s) = KG(s) = \frac{a_o + a_1 s}{b_o + b_2 s^2} \qquad 2.3\text{-}4$$

where $a_o,\ a_1,\ b_o,\ b_2$ are all positive real numbers. Now in this case the poles of KG(s) lie on the $j\omega$ axis and therefore the open-loop function is continuously oscillatory. Equation 2.3-5 represents the closed-loop function of equation 2.3-4.

$$\frac{C}{R}(s) = \frac{KG(s)}{1 + KG(s)} = \frac{a_o + a_1 s}{(a_o + b_o) + a_1 s + b_2 s^2} \qquad 2.3\text{-}5$$

Inspection of equation 2.3-5 reveals that the poles lie in the left-half plane, and therefore $\frac{C}{R}(s)$ will be stable. Question 1 can therefore be answered affirmatively.

The answer to question 2 is not so apparent since all possible types of open-loop functions must be considered. It is therefore discussed in detail in the work that follows.

2.4 Linear System Block Diagram Transformations

The block diagram of a physical system consists of the interconnection of a number of linear blocks of the type described in Section 2.1. If the system is complex, its reduction to a single operational expression can be time consuming. Graybeal[16] has shown that by using eight prototype equivalences given in Figure 2.2, it is possible to change any multiple-loop system containing a number of interconnected linear blocks into a whole host equivalent systems. In order to show that these transformations are equivalent, consider as an example, the case shown in Figure 2.2g. Starting with the left-hand configuration shown, the following equations can be written:

$$\epsilon_1 = x_1 - x_2' \qquad 2.4\text{-}1$$

$$x_2' = G_{31}x_3 \qquad 2.4\text{-}2$$

$$\epsilon_1' = \epsilon_1 \pm x_3G_{32} \qquad 2.4\text{-}3$$

Substituting equations 2.4-1 and 2.4-2 into equation 2.4-3 equations 2.4-5 and 2.4-6 are obtained.

$$\epsilon_1' = x_1 - G_{31}x_3 \pm x_3G_{32} \qquad 2.4\text{-}4$$

$$\epsilon_1' = x_1 - x_3(G_{31} \mp G_{32}) \qquad 2.4\text{-}5$$

$$\epsilon_1' = x_1 - x_3' \qquad 2.4\text{-}6$$

$$x_3' = x_3(G_{31} \mp G_{32}) \qquad 2.4\text{-}7$$

Inspection of equations 2.4-5, 2.4-6, 2.4-7 reveals that they are indeed the defining equations of the right-hand diagram of Figure 2.2g.

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16. Graybeal, T.D., "Block Diagram Network Transformation", Trans. AIEE, Vol. 70, pp. 985-990, 1951.

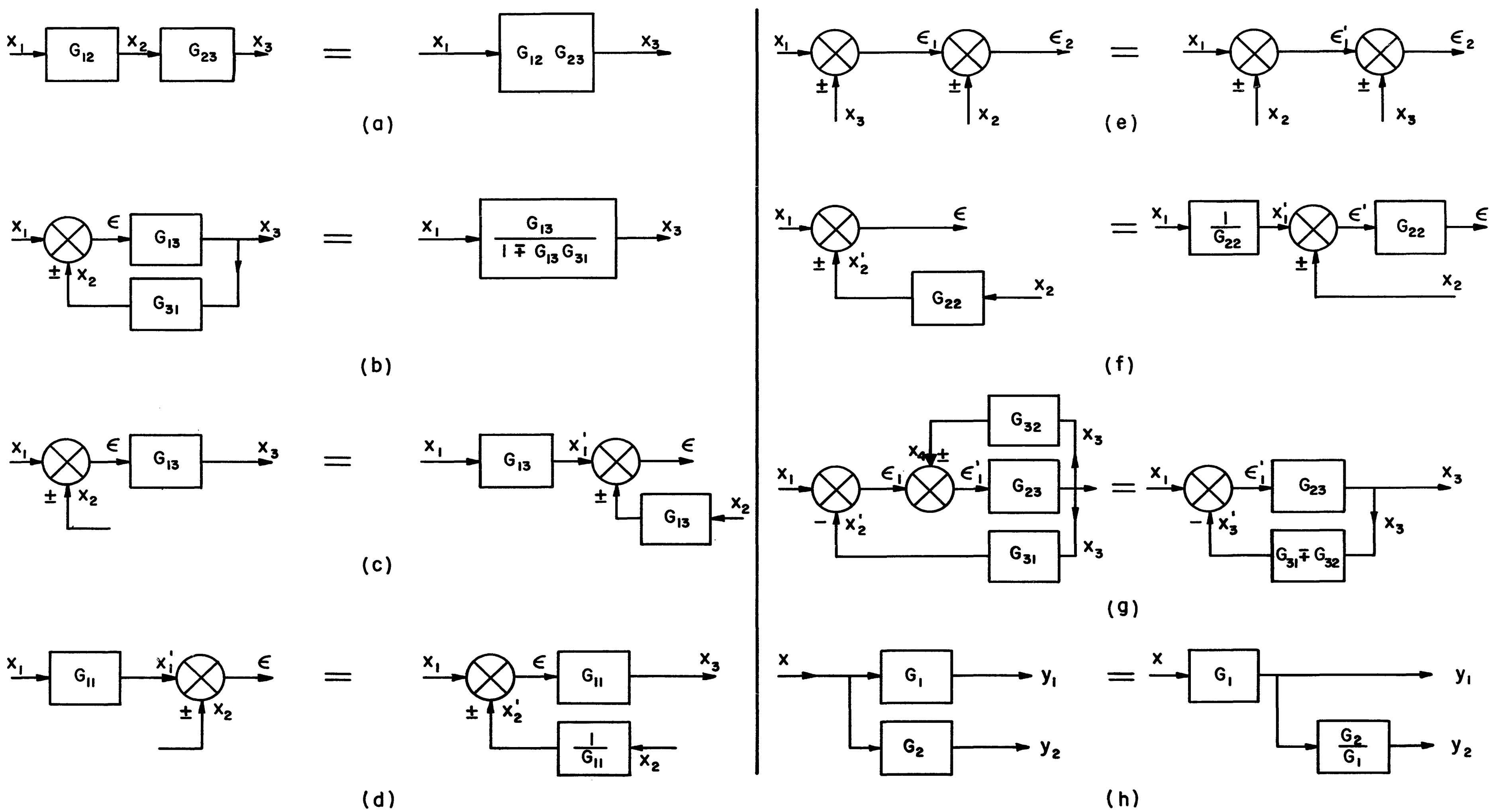

Figure 2.2 Linear Block Diagram Transformations

2.5 The Prototype Configuration

In Section 2.4 the subject of linear equivalences in a feedback control system was introduced. In this section these ideas will be utilized to show that any linear block or group of blocks can be left unaltered when applying these transformations to a random configuration while reducing the remainder of the system to some prescribed configuration. Since, in general, when the transfer functions are combined, a multiple-loop system will result which will also have a random configuration, it is necessary to approach the problem in an organized fashion, and pick a specific configuration which will serve as a prototype for the work of this dissertation. After some consideration, the prototype shown in Figure 2.3 was considered most promising. It consists of two differentials 1 and 2, interconnected by transfer functions $G_{12}(s)$, $G_{20}(s)$, $G_{02}(s)$, $G_{01}(s)$. In this configuration, the inner loop (enclosed by the broken lines) represents a closed-loop system whose transfer function in general takes on any form, stable or unstable, representable by a rational function $p(s)/q(s)$. The function $G_{12}(s)$ and $G_{01}(s)$ represent the remaining portions of any multiple-loop system. In order to see that the configuration of Figure 2.3 is completely general, let us consider the system shown in Figure 2.4a which contains 5 differentials and 6 transfer functions. The inner loop enclosed by the broken lines represents the unstable inner loop in this case. This corresponds to the unstable inner loop in Figure 2.3, which is also shown within the broken lines.

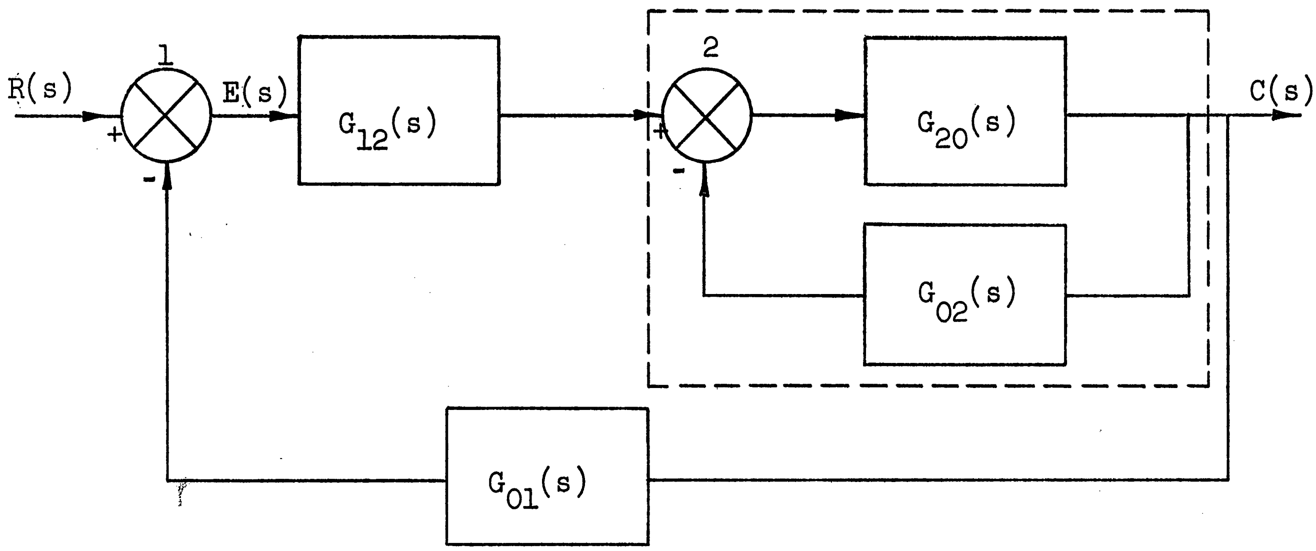

Figure 2.3 The Prototype Configuration

The problem at hand is to take the configuration of Figure 2.4a and reduce it to the prototype of Figure using the equivalent linear transformations shown in Figure 2.3 . First apply transformation "e" of Figure 2.2 and in this way interchange differentials 1 and 5. Next apply transformation "b" of Figure 2.2 and combine G_{34}, G_{43}. The resulting configuration is shown in Figure 2.4b . Next move G_{12} on the other side of differential 2 by application of "d" in Figure 2.2 and interchange differentials 2 and 5 by applying Figure 2.2e . The resulting configuration is shown in Figure 2.4c. Combine G_{12}, G_{45}, and $\frac{G_{34}}{1 + G_{34}G_{43}}$ and its associated differential 5 with the

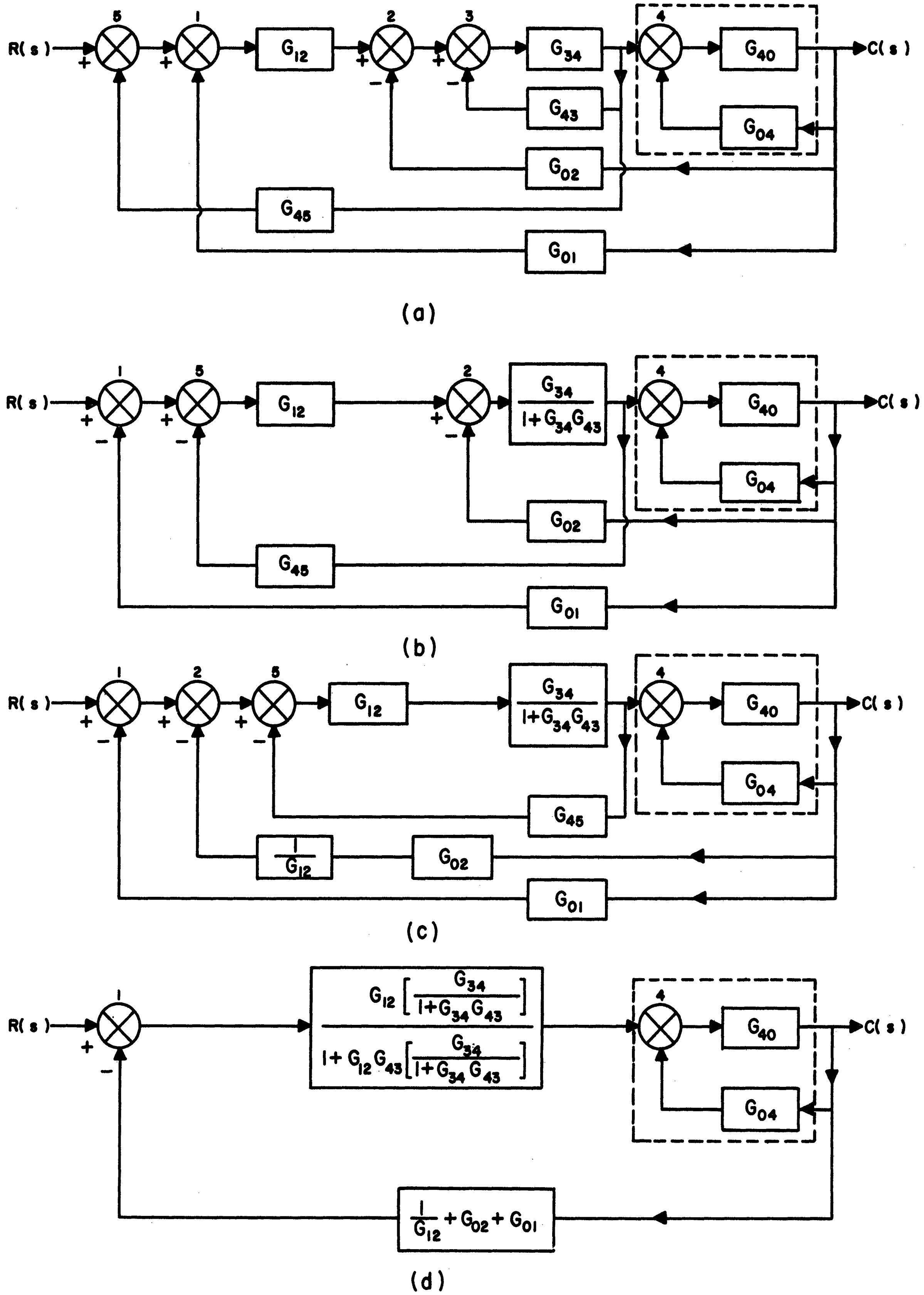

Figure 2.4 Reduction of a Complex System Utilizing Linear Block Transformations

aid of Figure 2.2b. The resulting configuration is shown in Figure 2.4d. Inspection of Figure 2.4d reveals the box in broken lines to have been unaltered in this process and therefore it represents the unstable inner loop of Figure 2.4a. The transformations of Figure 2.2 can be applied in any order to a given linear system to reduce it to the prototype of Figure 2.3. In this way the specific loop under consideration can be separated from the remainder of the system and its effects on the rest of the system can be analyzed.

2.6 Root-Locus Methods

Since the root-locus method of analysis has been used as a major tool in this dessertation, a summary of the well-known properties which govern the behavior of root loci will be discussed. For a more detailed treatment of these aspects of the subject, the reader is referred to a number of excellent papers[17, 18, 19, 20, 21] on the subject. Some

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17. "Graphical Analysis of Control Systems", W. R. Evans, Trans. AIEE, Vol. 67, 1948, pp. 547-557
18. "Servomechanism Analysis", G.J. Thaler and R.G. Brown, (book), McGraw-Hill Book Co., Inc., New York, 1953 Chapter 14.
19. "Control System Dynamics", W.R. Evans, (book) Chapt. 7 and 8. McGraw-Hill Book Company, Inc., New York, 1954.
20. "Control System Synthesis", John G. Truxal, (book), McGraw-Hill Book Company, Inc., 1955, Chapter 4, pp. 224-277.
21. "Feedback Control System" - J.C. Gille, M.J. Pelegrin, P. Decauline, (book), McGraw-Hill Book Company, Inc., pp. 235-255.

of the more recent developments and some of the less known properties have been treated in greater detail.

a) Definition of the Root-Locus

The open-loop transfer function of a linear feedback control system having m zeros and n poles can be described by the following operational equation:

$$\frac{C}{E}(s) = KG(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} \qquad 2.6\text{-}1$$

where z_i and p_j represent the zeros and poles of the open-loop function respectively

The corresponding unity feedback closed-loop function for this system is described by

$$\frac{C}{R}(s) = \frac{KG(s)}{1 \pm KG(s)} = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j) \pm K \prod_{i=1}^{m} (s + z_i)} \qquad 2.6\text{-}2$$

The dynamic behavior of the system is determined, in part, by the location of the poles of $\frac{C}{R}(s)$, which are the roots of the characteristic equation. In terms of the open-loop poles and zeros the roots of the characteristic equation are given by the zeros of

$$\prod_{j=1}^{n} (s + p_j) \pm K \prod_{i=1}^{m} (s + z_i) = 0 \qquad 2.6\text{-}3$$

In equation 2.6-3 the plus and minus signs indicate degenerate and regenerate feedback, respectively. Equation 2.6-3 when refactored will have the following form:

$$\prod_{j=1}^{n} (s + \beta_j) = 0, \qquad \text{for } n \geqslant m \qquad 2.6\text{-}4$$

which puts the roots of the characteristic equation, β_j, in evidence. A comparison of equations 2.6-3 and 2.6-4 reveals that the closed-loop poles are, in general, functions of the open-loop poles, open-loop zeros, and loop gain. In symbolic notation, β_j may be expressed as

$$\beta_j = Q(K, p_1, p_2, \ldots p_n, z_1, z_2, \ldots z_m), \qquad 2.6\text{-}5$$

which puts into evidence all the factors controlling the location of the roots of the characteristic equation. <u>The root-locus is defined as the locus of β_j when K is chosen as a parametric variable with p_j and z_i fixed.</u> Inspection of equation 2.6-5 reveals that it is possible to choose any of the remaining quantities as parametric variables. These have been considered more recently by some authors[22, 23]. Their results will be discussed later in this chapter.

Inspection of equation 2.6-5 reveals that the root locus can also be defined as the zeros of the equation $1 + KG(s) = 0$, or where

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22. "Some Mathematical Properties of Root Loci for Control System Design", F.M. Reza, Transactions AIEE, 1956 Basic Science Paper 56-125.

23. "Synthesis of Feedback Control Systems by Gain-Contour and Root-Contour", V.C. Yeh, Transactions AIEE, Vol.(App & Ind) pp. 85-95, May 1956.

$$KG(s) = \begin{cases} -1 & \text{degenerative feedback} \\ +1 & \text{regenerative feedback} \end{cases} \qquad 2.6\text{-}6$$

Equation 2.6-6 when written in polar form becomes

$$KG(s) = \rho e^{j\varphi} = \begin{cases} e^{j(\pi \pm 2N\pi)} & \text{degenerative feedback} \\ e^{j(0 \pm 2N\pi)} & \text{regenerative feedback} \end{cases} \qquad 2.6\text{-}7$$

Referring to the above definition of the root locus, it is apparent that determining the root locus of a system results in determining where the

$$\text{Arg } KG(s) = \begin{cases} \pi \pm 2N\pi & \text{degenerative feedback} \\ 0 \pm 2N\pi & \text{regenerative feedback} \end{cases}, \qquad 2.6\text{-}8$$

and where $|KG(s)| = 1$.

In the above equations

$N = 0, 1, 2, \ldots$

In constructing the root-locus one therefore maps either the positive or negative real axis (degenerate or regenerate feedback) in the KG(s)-plane into "slits" in the s-plane.

b) A summary of Well Known Root-Locus Properties

In constructing the root-locus the following well known properties are utilized:

1. A branch of the root locus starts from every open-loop pole where $K = 0$ and terminates at an open-loop zero where $K = \infty$.

2. The root locus of a real system is always symmetrical about the real axis.

3. The closed-loop system will become unstable when a branch of the root locus enters the right-half plane.

4. For a realizable transfer function, G(s), there are as many branches of the root-locus as the number of poles of G(s). If the numerator and denominator of G(s) are of degree m and n, respectively, there are n-m zeros of G(s) located at infinity, hence n-m branches of the locus terminate at infinity.

5. The asymptotic center is the intersection point of all linear asymptotes. For a real system this point is on the real axis:

$$\sigma_o = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} \; ; \; \text{Arg } s' = \frac{180^\circ \pm 360^\circ N}{n-m} \qquad 2.6\text{-}9$$

where p_j and z_i are poles and zeros of KG(s), respectively.

6. At a junction point of the branches, the tangents to the locus are equally spaced over 2π radians.

2.7 Gain-Loci and Phase-Loci

The definition of the root-locus as discussed in section 2.6 is in reality a special case of the more general gain-loci and phase-loci.[22,23,24,25] The form of KG(s) as given by equation 2.7-1 namely,

$$\frac{C}{E}(s) = KG(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} \qquad 2.7\text{-}1$$

where z_i and p_j represent the zeros and poles of the open-loop function, respectively, equation 2.7-1 can be written in the following form, which puts in evidence the amplitude and phase relations of this function:

$$KG(s) = \rho\, e^{j\varphi} = \frac{K \prod_{i=1}^{m} r_i\, e^{j\alpha_i}}{\prod_{j=1}^{n} R_j\, e^{j\beta_j}}, \qquad 2.7\text{-}2$$

$$\varphi = \sum_{i=1}^{m} \alpha_i - \sum_{j=1}^{n} \beta_j \qquad 2.7\text{-}3a$$

$$\rho = K \frac{\prod_{i=1}^{m} r_i}{\prod_{j=1}^{n} R_j} \qquad 2.7\text{-}3b$$

While in section 2.6 φ was either zero or some multiple of π, in the case of phase loci φ is allowed to take on all values. Under these conditions a new group of loci will

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22. See reference 22 section 2.6
23. See reference 23 section 2.6
24. "The Study of Transients in Linear Feedback Systems by Conformal Mapping and the Root-Locus Method" Victor C. M. Yeh, Trans. ASME, April 1954, pp. 349-361.
25. "Synthesis of Feedback Control Systems by Phase-Angle Loci" Yaohan Chu, Trans. AIEE, Vol. 71, Part II, 1952 pp. 330-339.

be obtained with φ as a parameter. Correspondingly, by letting ρ take on values other than unity, the value it has in the case of the root-locus, a group of gain-loci will be obtained for each value of ρ.

The concept of gain and phase loci will become readily understandable by considering the following example of a second-order feedback control system containing two left-half plane poles α, β, gain K, and defined by the following open-loop transfer function:

$$\frac{C}{E}(s) = \rho e^{j\varphi} = \frac{K}{(s+\alpha)(s+\beta)} = \frac{K}{r_1 e^{j\theta_1}\, r_2 e^{j\theta_2}} \qquad 2.7\text{-}4$$

or

$$\varphi = \theta_1 - \theta_2 \qquad 2.7\text{-}5$$

and

$$\rho = \frac{K}{r_1\, r_2} \qquad 2.7\text{-}6$$

In order to determine the phase-angle loci, the parameter, φ, has been assigned the following values:

$$\varphi = 0,\ \pm\frac{N\pi}{4} \quad (N = 0,\ 1,\ 2,\ \ldots)$$

in the KG(s) plane, which will be found plotted in Figure 2.5a . The corresponding "slits" produced in the s-plane by allowing K to vary along a prescribed phase-angle locus are shown in Figure 2.5b . Inspection of Figure 2.5a reveals that lines of constant ρ will be concentric

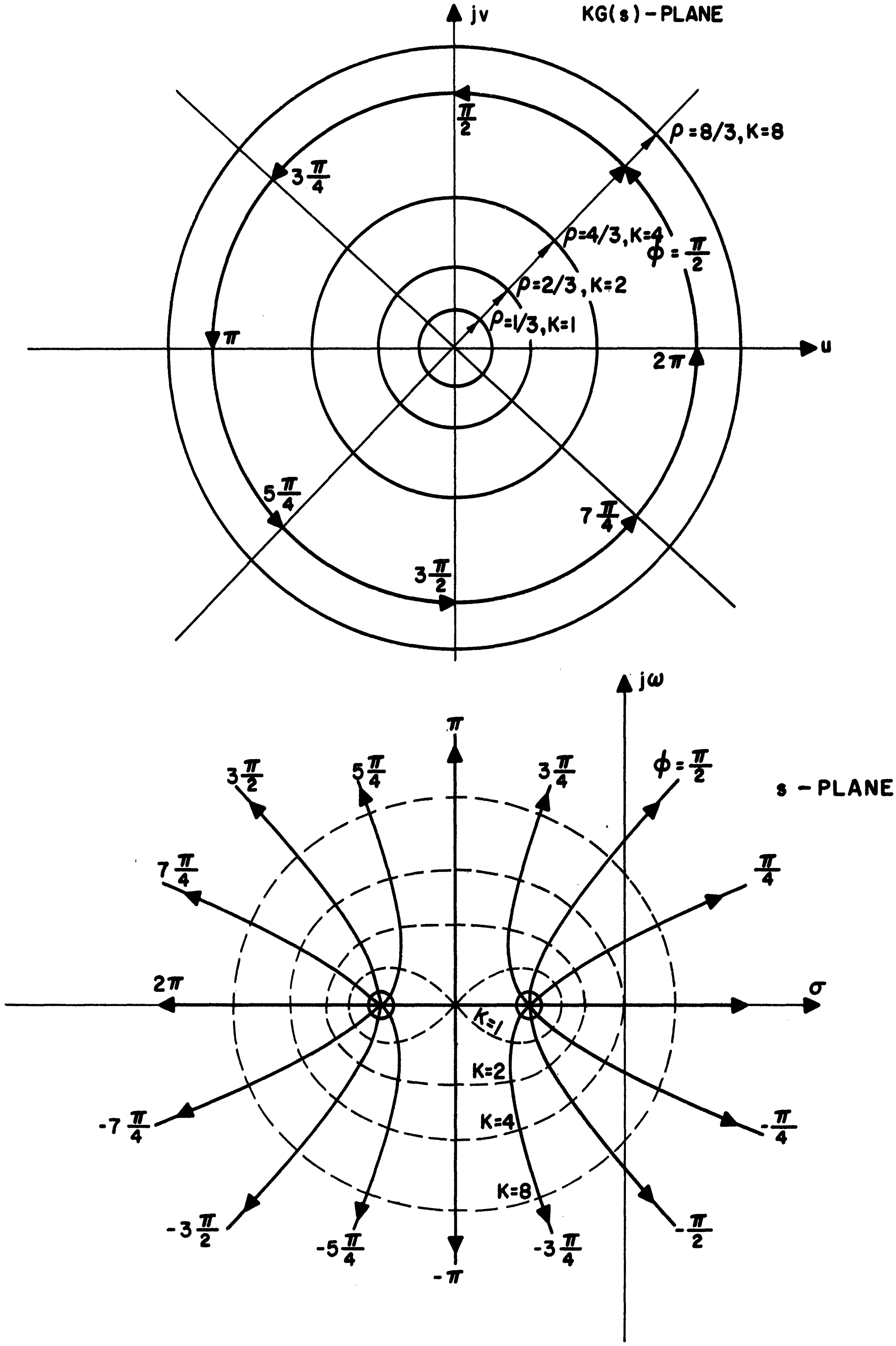

Figure 2.5 Gain and Phase Loci of a Second-Order Control System

circles about the origin of the KG(s)-plane. The corresponding "slits in the s-plane are shown as the dashed lines in Figure 2.5b. Inspection of equation 2.7-6 reveals that for prescribed pole locations of α, and β, varying ρ in KG(s)-plane corresponds to varying K in the s-plane. Thus, in reality the value of ρ and K are related by a constant. Since the system gain, K, is usually a design parameter, it is usually the one that is plotted.

The ideas presented here will be further discussed in section 4.7.

2.8 Method for Determining the Roots of an Nth Degree Polynomial in s.

A new and little publicized method of solving for the roots of a characteristic equation was recently found by the author.* The technique, which utilizes the basic concepts of the root-locus technique, is in no way restricted by the order of the system. The rules governing the behavior and construction of the root-locus also applies in this case.

Given:

$$F(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \text{---} \quad a_n , \qquad 2.8\text{-}1$$

which is an nth order polynomial in the complex variable s, and one whose n roots are desired to graphical accuracy.

The procedure used to determine the roots can be

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* The original method, which was attributed to Walter Evans of Autonetics Division of North American, has not been published to the best of the author's knowledge.

described as follows:

1) The polynomial 2.8-1 is factored by first removing a_n and then factoring an s out of the remaining terms. The following form of equation 2.8-1 results:

$$F(s) = s\left[s^{n-1} + a_1 s^{n-2} + --- a_{n-1}\right] + a_n \qquad 2.8\text{-}2$$

2) Now if within the brackets of 2.8-2 the term a_{n-1} is removed, the resulting terms will again contain a common factor s, which can again be removed. Equation 2.8-1 now takes on the following form:

$$F(s) = s\left[s\ (s^{n-2} + a_1 s^{n-3} + ---a_{n-2}) + a_{n-1}\right] + a_n \qquad 2.8\text{-}3$$

3) The above procedure is repeated until all the coefficients except a_1 are removed. The final form of F(s) is

$$F(s) = s\left[---\left\{s\left[s(s+a_1) + a_2\right] + a_3\right\} + --- a_{n-1}\right] + a_n \qquad 2.8\text{-}4$$

4) Starting with the inner-most factor namely, $s(s+a_1)$, the root-locus for the function $F_1(s) = \dfrac{K_1}{s(s+a_1)}$ is constructed using root locus techniques. Since what is really wanted are the factors of $s^2 + a_1 s + a_2$, attention is focused on locating on the root-locus of $F_1(s)$ those points where $K_1 = |a_2|$. At this point $|s(s + a_1)| = K_1 = a_2$, or, in other words at these points $s^2 + a_1 s + a_2 = 0$. Thus z_1 and z_2 which are the zeros of $s^2 + a_1 s + a_2$, have been found.

5) Knowing the location of these two roots permits one to move on to the next step of the procedure. This consists in making a root locus plot of

$$F_2(s) = \frac{K_2}{s(s + z_1)(s + z_2)}$$

which is the next grouping of factors in the next innermost parentheses.
After the root-locus is constructed, attention is again focused on determining where on the $F_2(s)$ root-locus $K_2 = |a_3|$. This will yield three roots, which in factored form become

$$(s + Z_1)(s + Z_2)(s + Z_3)$$

6) The procedure used in 4) and 5) above is repeated until all the roots of F(s) have been determined and

$$F(s) = (s + A_1)(s + A_2)(s + A_3) \quad . \quad . \quad . \quad . \quad (s + A_n) \qquad 2.8\text{-}5$$

The above procedure can be viewed as determining the root-locus of an n loop linear feedback control system. If $F_1(s) = \frac{K_1}{s(s+a_1)}$ is considered, which is the inner-most function in the inner-most loop, then finding the roots of the polynomial F(s) corresponds to solving for each root-locus of this n-1 loop control system, which in this case possesses a prescribed configuration. This configuration which is shown in Figure 2.6 consists of a single pole on the real axis and a pole at the origin of the inner-most function and a pole at the origin in each of the n-2 external loops. Inspection of Figure 2.6 reveals the gain constants to be ratios of the coefficients of the polynomial. In each case once the angle condition of the root-locus is determined, the closed loop roots are determined as a result of the feedback properties and the prescribed gain constants.

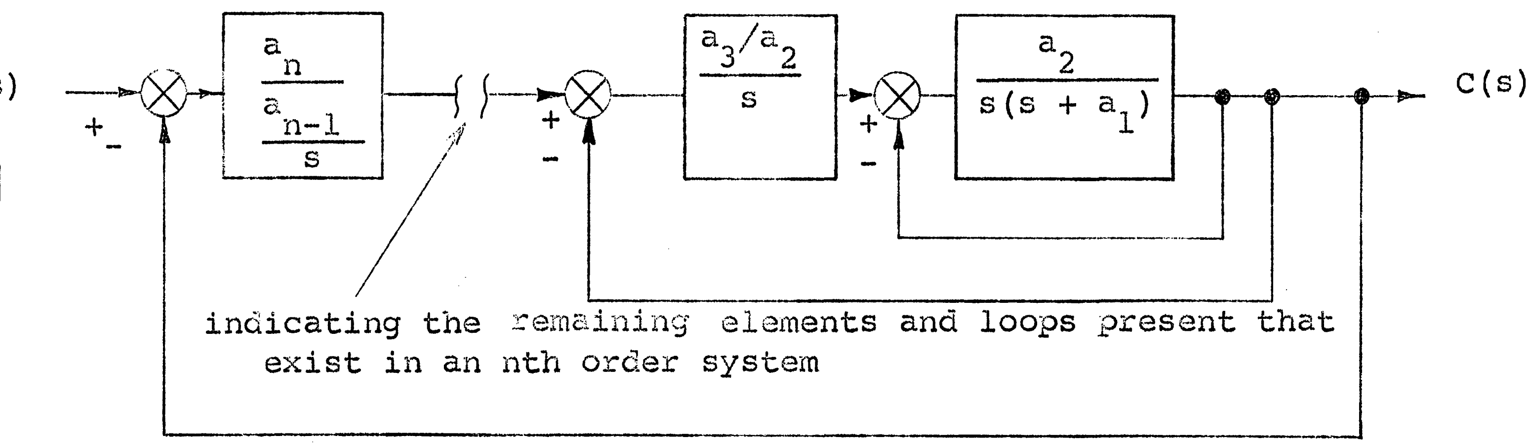


Figure 2.6 Simulation of an n-th Order Polynomial as a Control System

The method will now be illustrated by an example and proceeds as follows:

1. Given $F(s) = s^4 + 12s^3 + 56s^2 + 88s + 68$
2. Factor F(s) as follows:

$$F(s) = s\left[s\left\{ s(s + 12) + 56 \right\} + 88 \right] + 68$$

3. Starting with the inner-most factor, namely $s(s + 12)$ determine the locus of roots of the function $F_1(s) = \dfrac{K_1}{s(s + 12)}$ using root-locus techniques. The resulting locus is shown in

Figure 2.7a

4. Now determine where $|s(s + 12)| = K_1 = 56$. This represents the location of the roots of the polynomial within the braces, which in this case is found graphically to be

$$(s + 6 + j\ 4.43)(s + 6 - j\ 4.43)$$

5. Utilizing the newly found roots at $s = -6 - j\ 4.43$ and $s = -6 + j\ 4.43$, determine the open-loop roots of the next grouping of terms, namely, the closed-loop roots of

$$F_2(s) = \frac{K_2}{s(s + 6 + j\ 4.43)(s + 6 - j\ 4.43)}$$

The locus for this case is shown in Figure 2.7b.

6. Next determine the location on the root-locus of (5) above where

$$|s(s + 6 + j\ 4.43)(s + 6 - j\ 4.43)| = K_2 = 88.$$

As a result of applying these amplitude conditions, the poles of $F_3(s)$ are found to be

$$(s + 3.08)(s + 4.46 + j\ 2.95)(s + 4.46 - j\ 2.95).$$

The above process is again repeated by now writing $F_3(s)$ as follows

$$F_3(s) = \frac{K_3}{s(s + 3.08)(s + 4.46 + j\ 2.95)(s + 4.46 - j\ 2.95)}$$

and first determining the argument conditions on $F_3(s)$, and, then determining where the amplitude conditions are satisfied. From the root-locus of

Figure 2.7c the roots of F(s) are found graphically to be

$$F(s) = (s + .95 + j\,1.1)(s + .95 - j\,1.1)$$
$$(s + 4.98 + j\,3.05)(s + 4.98 - j\,3.05)$$

which compares very favorably with the exact values (known in this case) as

$$F(s) = (s + 1 + j)(s + 1 - j)(s + 5 + j3)(s + 5 - j3)$$

While the above description of the method leads to solutions of graphical accuracy, in the case of higher order systems there may be considerable error utilizing this method. A review of the root locus plots will reveal that if an error is made in finding the locus in the case of the inner-most functions, these errors will accumulate This problem can be circumvented to a certain degree by solving for the roots of the inner binomial $s^2 + 12s + 56$ analytically using the quadratic formula, and then utilize this information in determining the roots of the higher order polynomials. In a similar manner it is possible to extend the analytical approach to the real axis pole from the graphical plot and to utilize this in the following manner:

From root-locus plot (see Figure 2.7b) the real axis root was estimated to be at s = -3.1. Taking this root and dividing it out of the trinomial gives

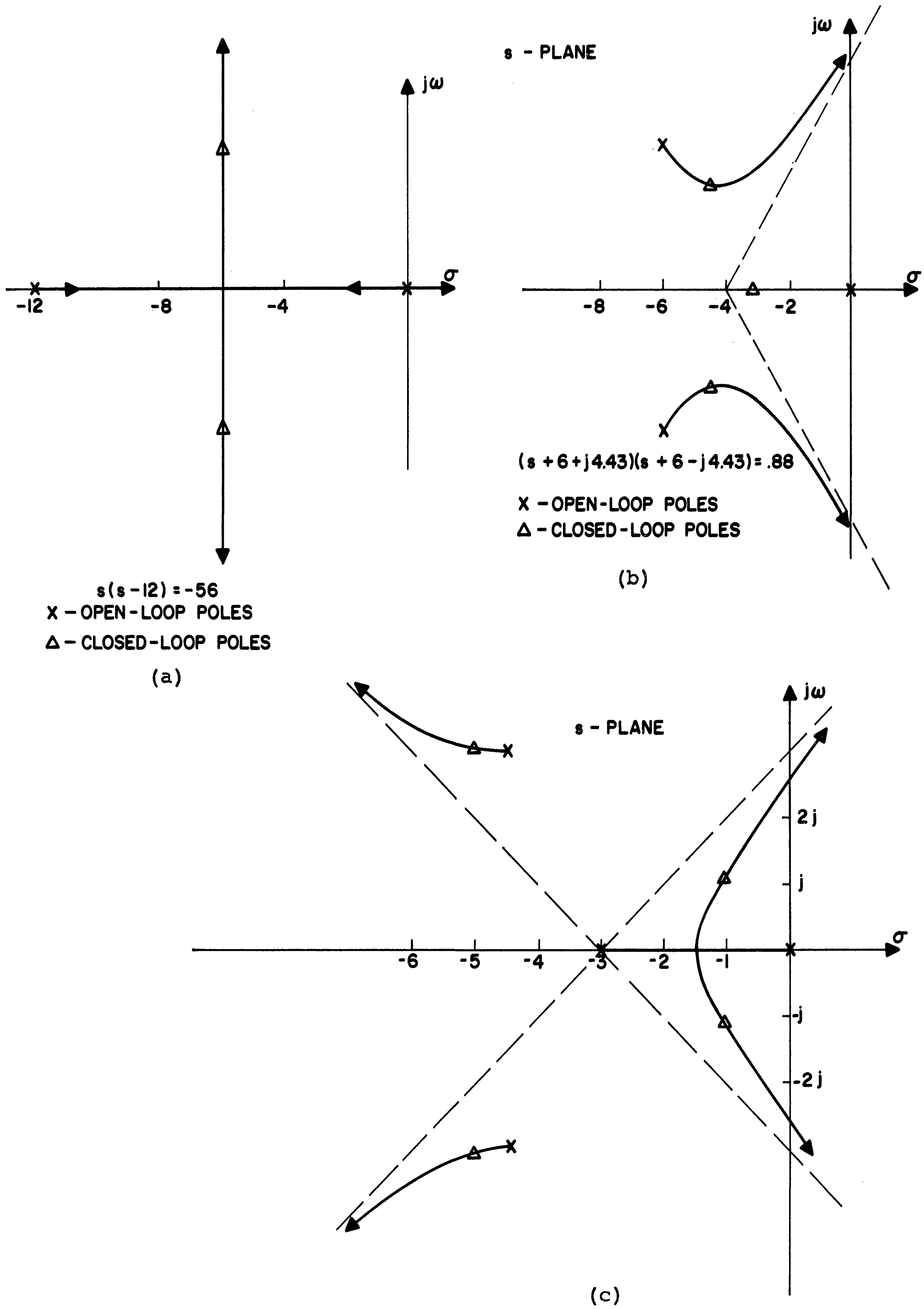


Figure 2.7 Solution of an n-th Order Polynomial by Root Locus Techniques

$$
\begin{array}{r|l}
 & s^2 + 8.9s + 28.41 \\ \hline
s + 3.1 & s^3 + 12s^2 + 56s + K \\
 & \underline{s^3 + 3.1s^2} \\
 & \quad 8.9s^2 + 56\ s \\
 & \quad \underline{8.9s^2 + 27.59s} \\
 & \qquad 28.41s + K \\
 & \qquad \underline{28.41s + 88.071}
\end{array}
$$

Since remainder must be zero $K = 88.071$. However to satisfy the next outer bracket, inspection of $F(s)$ reveals this constant should be 88. It is apparent therefore, that an error does exist and the root at $s = -3.1$ must be modified if greater accuracy is desired. This can be done with the aid of a calculator to any desired degree of accuracy. Once this has been accomplished, the remaining factor being a quadratic can be solved using the quadratic formula. Since for a third order system the approximate location of the <u>real</u> axis root can always be found and the argument condition of root locus position is precisely known, the above procedure can always be carried out. Thus the roots of the <u>cubic</u> can be determined to any desired degree of accuracy. These roots now serve as an accurate starting point by which to locate the roots of the remaining expression.

In the case when a fourth degree polynomial has real axis roots, the above technique can be extended to the fourth order equation. If the roots (as determined by the root-locus) are complex, the above technique cannot be carried out since the <u>precise location of the root locus curves is not accurately known.</u>

CHAPTER III

SECOND ORDER CONDITIONALLY STABLE CONTROL SYSTEMS

3.1 Background

An idealized feedback control system represented by the following closed loop transfer function:

$$\frac{C}{R}(s) = \frac{d_2 s^2 + d_1 s + d_o}{e_2 s^2 + e_1 s + e_o} \qquad 3.1\text{-}1$$

has been considered as the logical starting point for this investigation. While second-order systems seldom exist in practice, they lend themselves readily to analysis. If d_2, d_1, d_o are allowed to assume positive, negative, and zero values, it is apparent that all second order systems that are closed loop stable are being considered. In addition, if in Figure 2.3 $G_{12} = G_{o1} = 1$, the configuration of Figure 2.3 reduces to that of Figure 3.1. The defining equations then become

$$\frac{C}{E}(s) = KG(s)$$

$$\frac{C}{R}(s) = \frac{KG(s)}{1 + KG(s)} \qquad 3.1\text{-}2$$

and

$$\frac{E}{R}(s) = \frac{1}{1 + KG(s)}$$

where KG(s) may in this case contain its own feedback loop. This, of course, is necessary to generate unstable $\frac{C}{E}(s)$ functions.

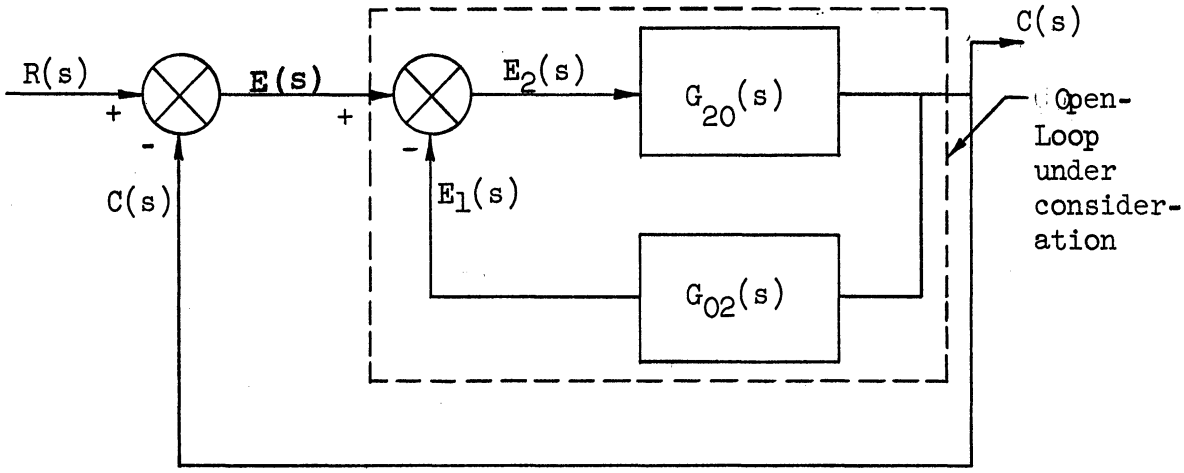


Figure 3.1 The Prototype Configuration

Now let

$$\frac{E}{R}(s) = \frac{1}{1 + KG(s)} = \frac{b_2 s^2 + b_1 s + b_o}{e_2 s^2 + e_1 s + e_o} \qquad 3.1\text{-}3$$

then

$$\frac{C}{E}(s) = KG(s) = \frac{(e_2 - b_2)\, s^2 + (e_1 - b_1)s + (e_o - b_o)}{b_2 s^2 + b_1 s + b_o} \qquad 3.1\text{-}4$$

and

$$\frac{C}{R}(s) = \frac{KG(s)}{1 + KG(s)} = \frac{(e_2 - b_2)s^2 + (e_1 - b_1)s + (e_o - b_o)}{e_2 s^2 + e_1 s + e_o} \qquad 3.1\text{-}5$$

If

$$d_2 = e_2 - b_2 \quad , \; d_1 = e_1 - b_1 \; , \text{ and } d_o = e_o - b_o$$

then equation 3.1-5 is equivalent to equation 3.1-1.

Inspection of equation 3.1-4 reveals the following information:

1. The zeros of $\frac{C}{E}(s)$ and $\frac{C}{R}(s)$ can be in the right or left half plane depending upon the relative magnitudes of $b_o,\ b_1,\ b_2$ and $e_o,\ e_1,\ e_2$.
2. Any missing powers of s in the denominator polynomial of $\frac{C}{E}(s)$ must be present in the numerator for a system to be stable.
3. In addition consider the special case when

$$\frac{C}{E}(s) = \frac{K(s - z_1)(s + z_2)}{(s - p_1)(s + p_2)} \qquad 3.1\text{-}6$$

where $z_1,\ z_2,\ p_1,\ p_2,$ are real and positive. Under these conditions

$$\frac{C}{E}(s) = \frac{K(s - z_1)(s + z_2)}{(s - p_1)(s + p_2) + K(s - z_1)(s + z_2)} \qquad 3.1\text{-}7$$

$$= \frac{K(s - z_1)(s + z_2)}{(1+K)s^2 + \left[(p_2 + Kz_2) - (p_1 + Kz_1)\right]s - \left[p_1 p_2 + Kz_1 z_2\right]} \qquad 3.1\text{-}8$$

Solving for the closed-loop poles of equation 3.1-8 reveals that the closed-loop function contains one right-half plane pole and therefore yields an unstable closed-loop system.

3.2 A Comparison of the Two Commonly Encountered Definitions of Conditionally Stable Systems

Conditionally stable systems have been treated in detail by few authors. Review of the literature reveals that in many cases authors have specifically excluded this class of feedback control systems from their discussion. As stated in Section 1.1 Brown and Campbell called a conditionally stable system a system which is unstable for a particular gain, but stable for both larger and smaller gain values. While the definition used by other authors is a system that is stable for a particular value of gain, but is unstable for both higher and lower gain values. To get a clear insight into these definitions the root-locus of a typical system from each catagory is compared. In Figure 3.2a is shown the root-locus for a system satisfying Brown and Campbell's definition. Inspection of the open loop function which characterizes this root-locus reveals that it contains only left half plane poles and at least two finite left half plane zeros. As the gain is increased, it is apparent that a pair of closed loop poles move into the right half plane for a prescribed range of gain, but with further increase in gain the root-locus and therefore the closed loop poles return to the left half plane.

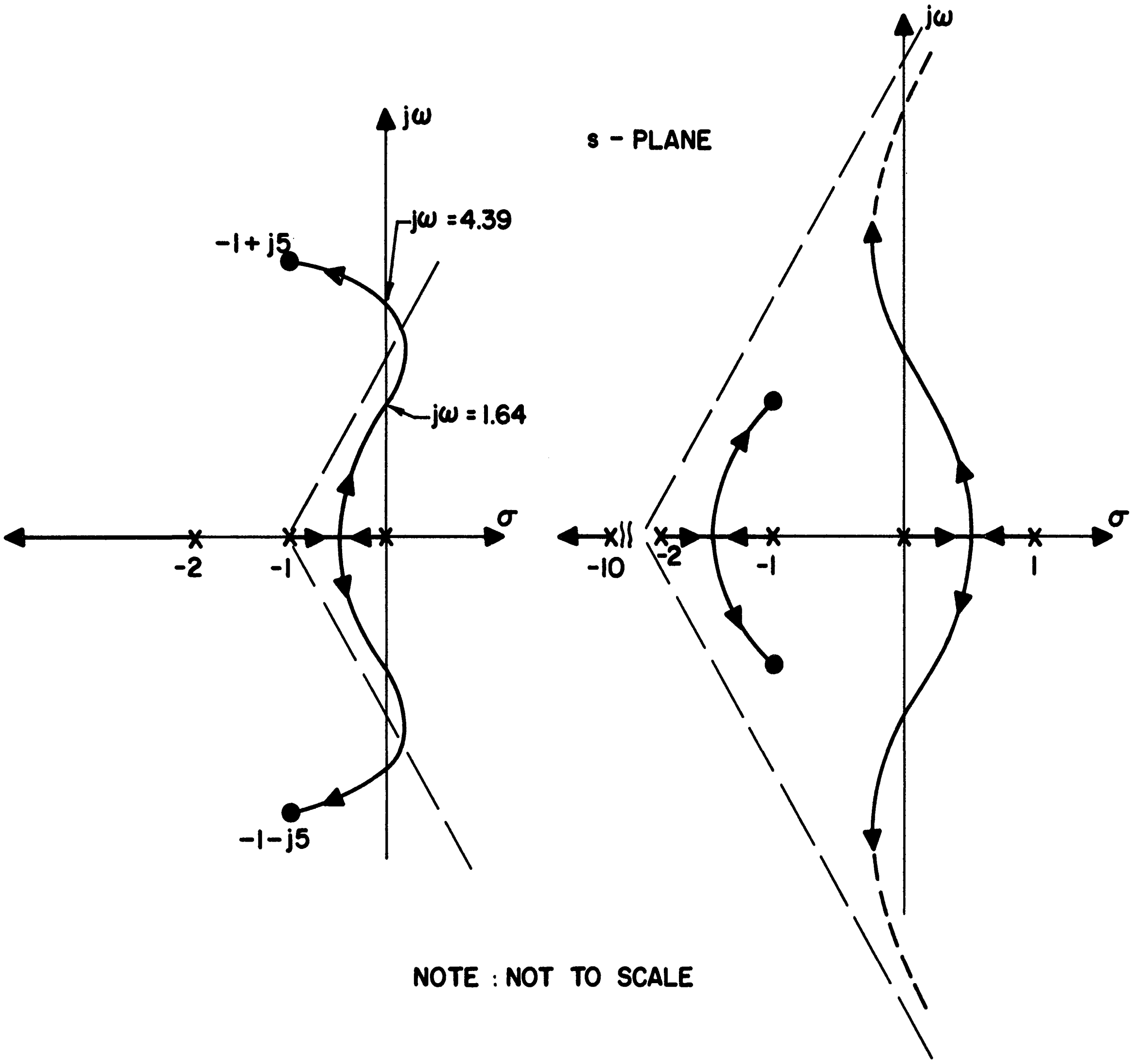
On the other hand, the system whose root-locus is shown in Figure 3.2b possesses a pole of order 2 or more at the origin, or one or more right half plane open loop poles. In this case the closed loop system is unstable for small gain since the locus lies in the right half plane, but becomes stable when the closed loop poles move into the left half plane. Depending upon the order of the system, as the gain is increased one or more branches of the locus return to the right half plane, thus indicating unstable closed loop behavior.

Both types of systems are ones which require care in their design and are therefore avoided by many control system designers.

3.3 Conditionally Stable Second-Order Systems

In section 3.2 two different types of conditionally stable systems were discussed. Inspection of Figures 3.2a and 3.2b reveals that neither of these systems is second-order in nature, and therefore it is necessary to modify the above definitions in order to include all possible types of conditionally stable systems.

Conditionally stable systems as used in this dissertation is that set of linear feedback control systems, whose internal zeros all lie in the left half plane, whose root-locus contains one or more branches that cross the $j\omega$-axis more than once, and/or whose root-locus contains one or more right half plane open-loop poles. This result would be cbtained for the system represented by Figure 3.2b if two additional



$$\frac{C}{E}(s) = \frac{K(s^2+2s+26)}{s(s+1)(s+2)} \qquad \frac{C}{E}(s) = \frac{K(s^2+2s+2)}{s(s-1)(s+1)(s+2)(s+10)}$$

Figure 3.2 Root-Locus Plots Representing Two Types of Conditionally Stable Systems finite left half plane zeros were added to the system. In this investigation all mathematically feasible second-order $\frac{C}{R}(s)$ functions were investigated using principally the root-locus method of analysis. As a result of this analysis it was found that conditionally stable second-order systems may arise when the open-loop transfer function contains one of the following:

1) One or two right half plane poles, one or two left half plane zeros.
2) A negative gain constant in the error channel; one or two finite left half plane zeros, two poles located anywhere.
3) A pole of order two at the origin. This represents the limiting case of a conditionally stable system, since in this case for zero gain constant the root locus approaches the imaginary axis.

As an example of 1) above, consider the open loop function

$$KG(s) = \frac{+K(s^2 + s + 1)}{(s^2 - 1.4s + .98)} = \frac{+K(s + .5 + j.866)(s + .5 - j.866)}{(s - .7 + j.7)(s - .7 - j.7)} \qquad 3.3\text{-}1$$

This function is characterized by two right half plane poles and two left half plane zeros. The root-locus for this function is shown in Figure 3.3 . Inspection of the root-locus reveals that for small gains the closed-loop function contains two complex right-half plane poles and therefore the system is unstable. As the system gain is increased from zero to infinity, the closed-loop poles move from the right half plane into the left half plane. In this case for $K = 1.4$ the closed-loop system will be continuously oscillatory, while for $K > 1.4$ the closed-loop system will be stable. In this case as the gain of KG(s) increases from zero to infinity, the poles of the closed loop function always remain complex.

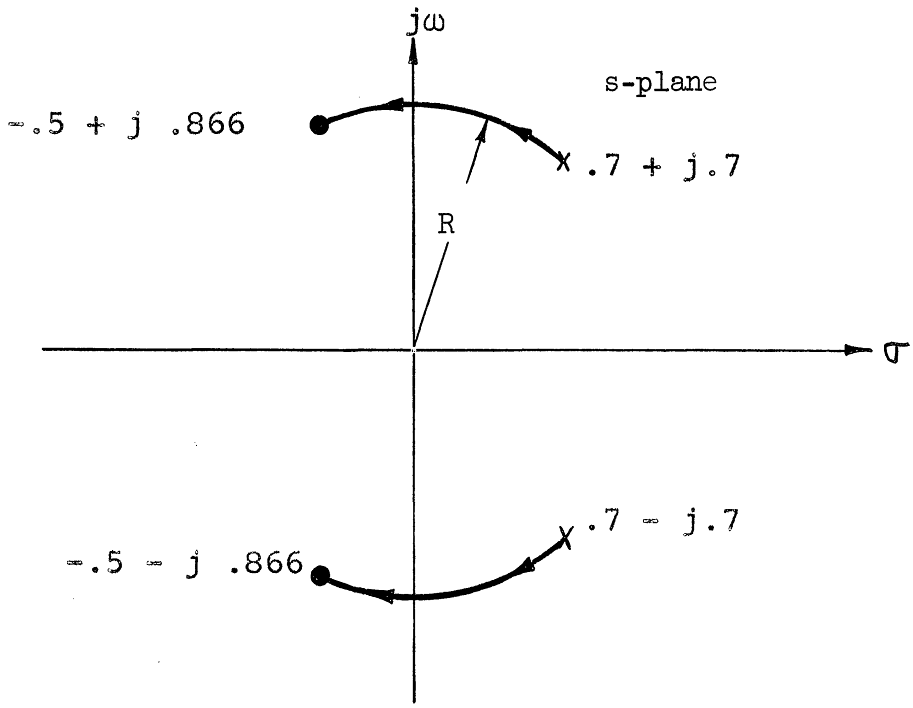


Figure 3.3 Root-Locus Plot of a Conditionally Stable System with Two Complex Right-Half Plane Poles

As an example of 2) above, consider the open loop function

$$KG(s) = \frac{-K(s^2 + 1.4s + .98)}{(s^2 + s + 1)} = \frac{-K(s+.7+j.7)(s+.7-j.7)}{(s+.5+j.866)(s+.5-j.866)} \qquad 3.3\text{-}2$$

This open loop function is characterized by two left half plane poles, two left half plane zeros and a negative gain constant. The root-locus for this function is shown in Figure 3.4. Inspection of the locus reveals that for small system gain the closed loop function contains two left half plane poles which yield a stable oscillatory system. As the system gain is increased, the closed loop poles move into the right half plane. Still larger system gain, will yield a closed loop function having two real right half plane poles. As the gain is made still greater the closed loop roots move again into the left half plane. As the gain tends to infinity, the closed loop roots become complex and lie in the left half plane. In this case since the poles and zeros of KG(s) lie in the left half plane, the open loop function is stable although the closed loop function may be stable or unstable depending upon the system gain. More specifically the system will be unstable for $1.01 > K > .715$

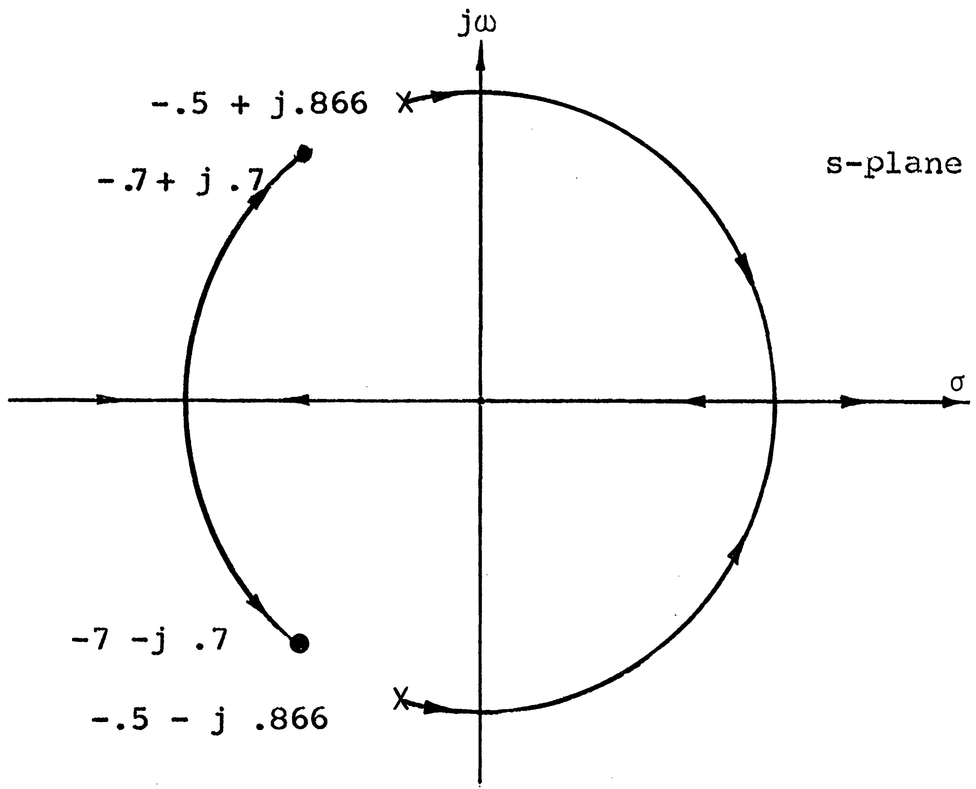


Figure 3.4 Root-Locus Plot of a Second-Order System with a Negative Gain Constant

As an example of the third class of functions, consider the open loop function

$$KG(s) = \frac{K(s^2 + s + 1)}{s^2} = \frac{K(s + .5 + j.866)(s + .5 - j.866)}{s^2} \quad 3.3\text{-}3$$

The open loop function is characterized by a pole of order two at the origin, and 2 internal left half plane zeros. The root-locus for this function is shown in Figure 3.5 . Inspection of this locus reveals that for positive gain this system cannot be rightly referred to as a conditionally stable system, since as the system gain is changed from zero to infinity, the poles of the closed loop function always remain in the left half plane. However, it does represent the limiting case of a conditionally stable system, since with the addition of another pole it can be shown (see Section 4.5) that it is possible for the locus of this system to move into the right half plane before finally terminating at the zeros that lie in the left half plane.

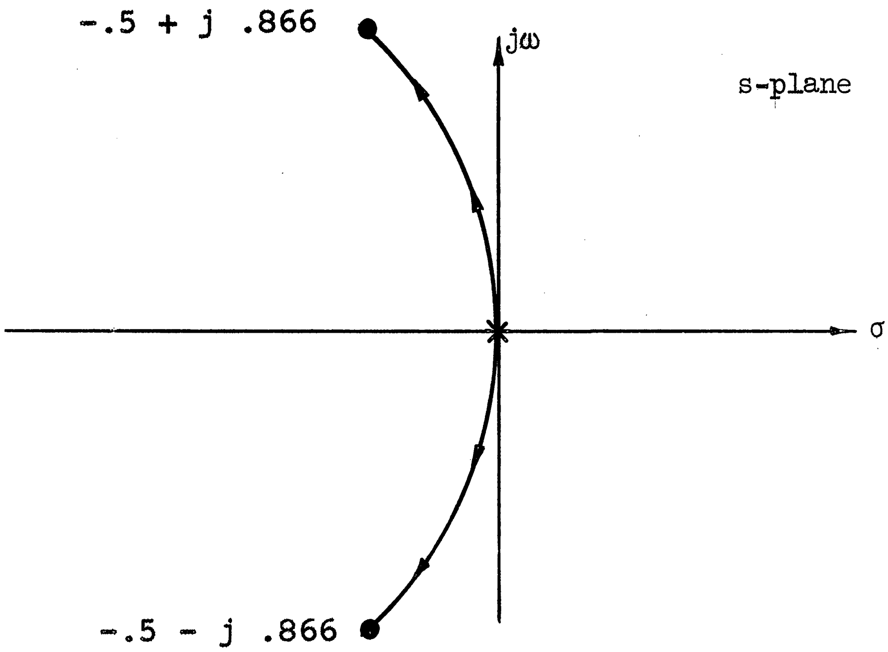


Figure 3.5 The Limiting Case of a Conditionally Stable System

3.4 Mathematical Derivations for the Root Locus of Second Order Feedback Control Systems

In the construction of the root locus for real values of $\frac{C}{E}(s)$ the locus must lie on the real axis. The construction therefore imposes no real problem. When the roots of $\frac{C}{E}(s)$ are complex, in the usual case, the construction of the root locus is a time consuming procedure which must be done with care, if acceptable results are to be obtained. In order to facilitate the plotting of the locus and to add mathematical credence to the locus plots, the equations that govern the location of the roots of $\frac{C}{R}(s)$ when they are complex have been determined. As an example of this technique, consider the second order transfer function having the following form:

$$KG(s) = \frac{(s + z_1)(s + z_2)}{\mathcal{k}\,(s + p_1)(s + p_2)} \qquad 3.4\text{-}1$$

which is the form of the second example on page 51 . In this case $z_1 = \sigma_1 + j\,\omega_1$, $p_1 = \sigma_2 + j\,\omega_2$, and $z_2 = z_1^*$, $p_2 = p_1^*$ but $\mathcal{k}$ * of course, is real. The problem at hand is to find the location of all the complex roots of $\frac{C}{R}(s)$, as the gain $\mathcal{k}$ is varied. Now since $\frac{C}{R}(s) = \frac{KG(s)}{1 + KG(s)}$ what is really desired are the zeros of $1 + KG(s)$ for all values of gain $\mathcal{k}$. Thus

$$1 + KG(s) = 0 \qquad 3.4\text{-}2$$

or

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* note $\mathcal{k}$ is used in this derivation in place of $K = \frac{1}{\mathcal{k}}$ as previously defined.

$$KG(s) = -1 = \frac{(s + z_1)(s + z_2)}{k(s + p_1)(s + p_2)} \qquad 3.4\text{-}3$$

Utilizing the complex components of KG(s), and at the same time solving for k, the following equation is obtained

$$k = - \frac{\left[(\sigma + \sigma_1) + j(\omega + \omega_1)\right] \left[(\sigma + \sigma_1) + j(\omega - \omega_1)\right]}{\left[(\sigma + \sigma_2) + j(\omega + \omega_2)\right] \left[(\sigma + \sigma_2) + j(\omega - \omega_2)\right]} \qquad 3.4\text{-}4$$

Now since k must be real, its imaginary part must be equal to zero. In other words

$$\mathcal{I}[k] = \mathcal{I}\left\{ \frac{\left[(\sigma + \sigma_1) + j(\omega + \omega_1)\right] \left[(\sigma + \sigma_1) + j(\omega - \omega_1)\right]}{\left[(\sigma + \sigma_2) + j(\omega + \omega_2)\right] \left[(\sigma + \sigma_2) + j(\omega - \omega_2)\right]} \right\} = 0 \qquad 3.4\text{-}5$$

After expanding and collecting terms this can be written in the following form:

$$\left[\sigma + \frac{\sigma_2^2 - \sigma_1^2 + \omega_2^2 - \omega_1^2}{2(\sigma_2 - \sigma_1)}\right]^2 + \omega^2 =$$

$$\left[\frac{\sigma_2\omega_1^2 - \sigma_1\omega_2^2 - \sigma_1\sigma_2^2 + \sigma_2\sigma_1^2}{\sigma_2 - \sigma_1}\right] + \left[\frac{\sigma_2^2 - \sigma_1^2 + \omega_2^2 - \omega_1^2}{2(\sigma_2 - \sigma_1)}\right]^2 \qquad 3.4\text{-}6$$

Equation 3.4-6 will be recognized as the equation of a circle with center at

$$\omega_c = 0 \quad ; \quad \sigma_c = - \frac{\sigma_2^2 - \sigma_1^2 + \omega_2^2 - \omega_1^2}{2(\sigma_2 - \sigma_1)} \qquad 3.4\text{-}7$$

and radius

$$\rho = \sqrt{\frac{\sigma_2 \, \omega_1^2 - \sigma_1 \, \omega_2^2 - \sigma_1 \sigma_2^2 + \sigma_2 \sigma_1^2}{\sigma_2 - \sigma_1} + \sigma_c^2} \qquad 3.4\text{-}8$$

The root locus when the roots of $\frac{C}{R}(s)$ are complex can thus be found using equations 3.4-7 and 3.4-8 . When the closed loop roots are real, they will fall on the real axis, and their root locus can readily be constructed. The above results will now be applied to a specific example. Let

$$KG(s) = - \frac{(s + .7 + j\,.7)(s + .7 - j\,.7)}{K(s + .5 + j.866)(s + .5 - j.866)} \qquad 3.4\text{-}9$$

By application of equation 3.4-7 the center of this circle is found to be

$$\omega_c = 0 \quad ; \quad \sigma_c = - \frac{.25 - .49 + .75 - .49}{2(.5 - .7)} = + .05 \qquad 3.4\text{-}10$$

By application of equation 3.4-8 the radius of the locus is found to be

$$\rho = \sqrt{\frac{.49 - .70}{-.2} + (.05)^2} = 1.02$$

Using a compass and a straight edge the complete locus for this function can readily be constructed (see Figure 3.6). The accuracy obtained by this method exceeds that obtained by the usual "cut and try" technique.

The author developed equations which would allow rapid construction of the root-locus for all second-order systems. These will be found listed in Figure 3.7.

Since this unpublished work was carried out, Yeh[24] published a similar list. However, his list also included equations for the root-locus of some third and fourth order systems as well. The emphasis of Yeh's research was to develop the form of the root-locus equations and little mention was made as to its application.

3.5 Computer Study of Second-Order Systems

In order to verify the theoretical results on the second-order system, a computer study was made of a number of second-order systems that were open loop unstable. Although it would be desirable to test the open loop as

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24. See reference 24, Section 2.7

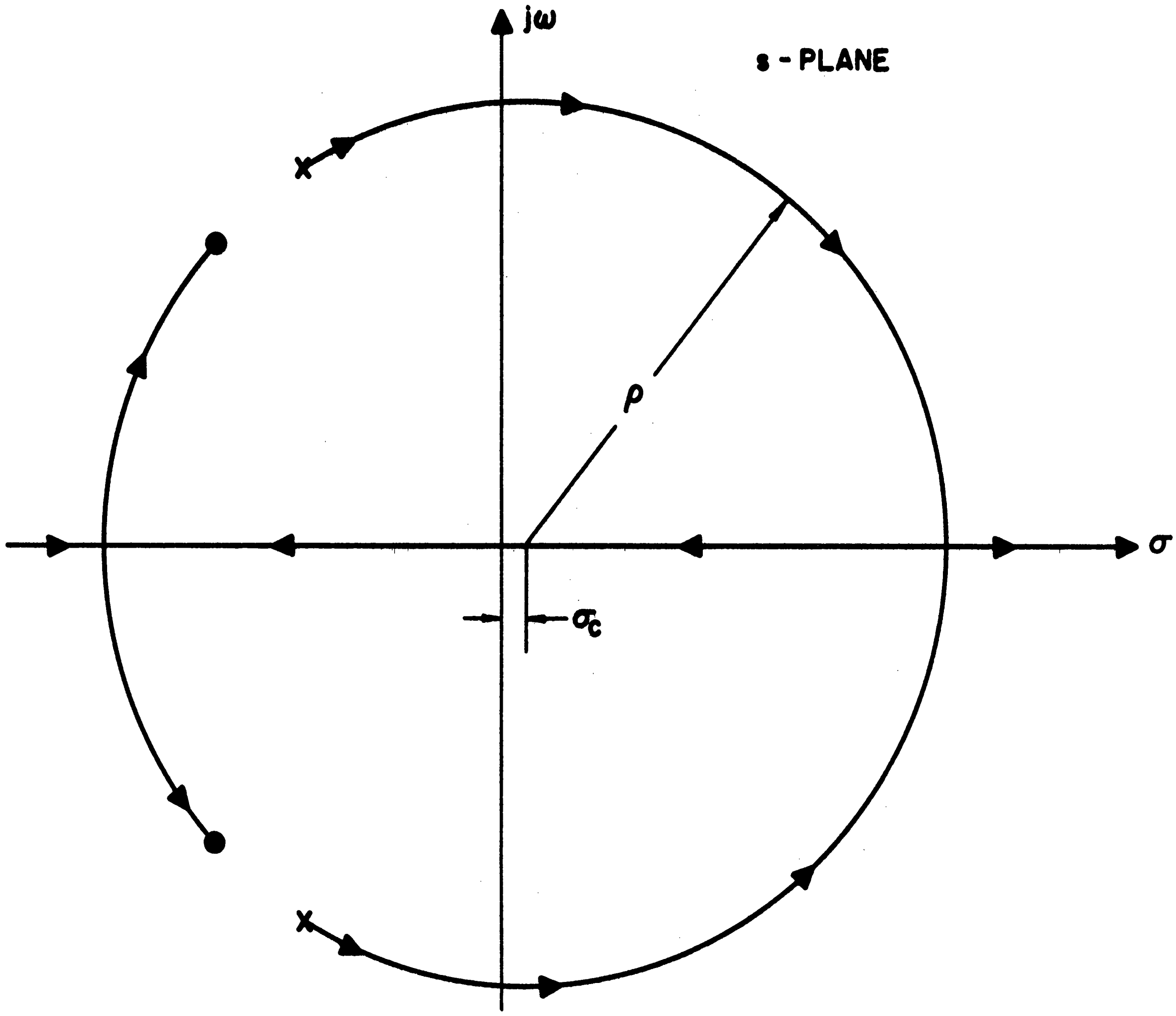


Figure 3.6 Example of Root-Locus Construction Utilizing Only a Compass and a Ruler

No.	Open Loop Functions as defined in equations 3.1-3, 3.1-4 and 3.1-5	Open Loop Functions in Factored Form	Arg cond	Radius, ρ, and intercept for complex loci
1	$\frac{e_2 s^2 + e_1 s + (e_o - b_o)}{b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K}$	180°	$\rho = \infty$, $\sigma_c = -\sigma_1$, $\omega_c = 0$ locus straight vertical lines
2	$\frac{e_2 s^2 + (e_1 - b_1)s + e_o}{b_1 s}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{Ks}$	180°	$\sigma_c = 0$, $\omega_c = 0$, $\rho = \sqrt{\omega_1^2 + \sigma_1^2}$ circle with center at origin
3	$\frac{(e_2 - b_2) s^2 + e_1 s + e_o}{b_2 s^2}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{Ks^2}$	180°	$\sigma_c = -\frac{\omega_1^2 + \sigma_1^2}{2\sigma_1}$, $\omega_c = 0$, $\rho = \frac{\omega_1^2 + \sigma_1^2}{2\sigma_1}$
4	$\frac{e_2 s^2 + e_1 s + (b_o + e_o)}{-b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K}$	0°	$\rho = \infty$, $\sigma_c = -\sigma_1$, $\omega_c = 0$ locus straight vertical lines
5	$\frac{e_2 s^2 + (e_1 + b_1)s + e_o}{b_1 s}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-Ks}$	0°	$\sigma_c = 0$, $\omega_c = 0$, $\rho = \sqrt{\omega_1^2 + \sigma_1^2}$
6	$\frac{(e_2 + b_2)s^2 + e_1 s + e_o}{-b_2 s^2}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-Ks^2}$	0°	$\sigma_c = -\frac{\omega_1^2 + \sigma_1^2}{2\sigma_1}$, $\omega_c = 0$, $\rho = \frac{\omega_1^2 + \sigma_1^2}{2\sigma_1}$
7	$\frac{e_2 s^2 + (e_1 - b_1)s + (e_o - b_o)}{b_1 s + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + \sigma_o)}$	180°	$\sigma_c = -\sigma_o$, $\omega_c = 0$, $\rho = \sqrt{\sigma_o^2 + \sigma_1^2 + \omega_1^2 - 2\sigma_1\sigma_o}$
8	$\frac{(e_2 - b_2)s^2 + e_1 s + (e_o - b_o)}{b_2 s^2 + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s^2 + \omega_o^2)}$	180°	$\sigma_c = -\frac{\sigma_1^2 + \omega_1^2}{2\sigma_1 - \omega_o}$, $\omega_c = 0$, $\rho = \sqrt{\sigma_c(\omega_o + \sigma_c)}$
9	$\frac{(e_2 - b_2)s^2 + (e_1 - b_1) s + e_o}{b_2 s^2 + b_1 s}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{Ks(s + \sigma_o)}$	180°	$\sigma_c = +\frac{\sigma_1^2 + \omega_1^2}{\sigma_o - 2\sigma_1}$, $\omega_c = 0$, $\rho = \sqrt{\sigma_c^2 + \frac{(\sigma_1^2 + \omega_1^2)\sigma_o}{\sigma_o - 2\sigma_1}}$
10	$\frac{e_2 s^2 + (e_1 + b_1) s + (e_o + b_o)}{-b_1 s - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + \sigma_o)}$	0°	$\sigma_c = -\sigma_o$, $\omega_c = 0$, $\rho = \sqrt{\sigma_o^2 + \sigma_1^2 + \omega_1^2 - 2\sigma_1\sigma_o}$
11	$\frac{(e_2 + b_2)s^2 + (e_1 + b_1)s + e_o}{b_2 s^2 - b_1 s}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-Ks(s + \sigma_o)}$	0°	$\sigma_c = -\frac{\sigma_1^2 + \omega_1^2}{2\sigma_1 - \sigma_o}$, $\omega_c = 0$, $\rho = \sqrt{\sigma_c(\sigma_o + \sigma_c)}$
12	$\frac{(e_2 + b_2) s^2 + e_1 s + (e_o + b_o)}{-b_2 s^2 - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s^2 + \omega_o^2)}$	0°	$\sigma_c = +\frac{\omega_o^2 - \sigma_1^2 - \omega_1^2}{2\sigma_1}$, $\omega_c = 0$, $\rho = \sqrt{\omega_o^2 + \sigma_c^2}$
13	$\frac{(e_2 - b_2)s^2 + (e_1 - b_1)s + (e_o + b_o)}{b_2 s^2 + b_1 s - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + \sigma_2)(s + \sigma_3)}$	180°	$\sigma_c = -\left[\frac{\sigma_2\sigma_3 - \sigma_1^2 - \omega_1^2}{\sigma_2 + \sigma_3 - 2\sigma_1}\right]$, $\omega_c = 0$ $\rho = \sqrt{\frac{(\sigma_1^2 + \omega_1^2)(\sigma_2 + \sigma_3) - 2\sigma_1\sigma_2\sigma_3}{\sigma_2 + \sigma_3 - 2\sigma_1} + \sigma_c^2}$
14	$\frac{(e_2 - b_2)s^2 + (e_1 + b_1)s + e_o - b_o}{b_2 s^2 - b_1 s + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + \sigma_2 + j\omega_2)(s + \sigma_2 - j\omega_2)}$	180°	$\omega_c = 0$ $\sigma_c = -\left[\frac{\sigma_2^2 - \sigma_1^2 + \omega_2^2 - \omega_1^2}{2(\sigma_2 - \sigma_1)}\right]$ $\rho = \sqrt{\sigma_c^2 + \frac{\sigma_2\omega_1^2 - \sigma_1\omega_2^2 + \sigma_2\sigma_1^2 - \sigma_1\sigma_2^2}{\sigma_2 - \sigma_1}}$
15	$\frac{(e_2 + b_2)s^2 + (e_1 - b_1)s + e_o - b_o}{-b_2 s^2 + b_1 s + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + \sigma_2)(s + \sigma_3)}$	0°	Same as no. 13
16	$\frac{(e_2 - b_2)s^2 + (e_1 + b_1)s + e_o + b_o}{b_2 s^2 - b_1 s - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + \sigma_2)(s + \sigma_3)}$	0°	Same as no. 13
17	$\frac{(e_2 + b_2)s^2 + (e_1 + b_1)s + (e_o - b_o)}{-b_2 s^2 - b_1 s + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + \sigma_2)(s + \sigma_3)}$	0°	Same as no. 13
18	$\frac{(e_2 + b_2)s^2 + (e_1 - b_1)s + (e_o + b_o)}{-b_2 s^2 + b_1 s - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + s_2)(s + s_3)}$	0°	depending on b_o, b_1, b_2 will be either no. 13 or 14
19	$\frac{(e_2 - b_2)s^2 + (e_1 - b_1)s + (e_o - b_o)}{b_2 s^2 + b_1 s + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + s_2)(s + s_3)}$	180°	depending on b_o, b_1, b_2 will be either no. 13 or 14
20	$\frac{(e_2 + b_2)s^2 + (e_1 + b_1)s + (e_o + b_o)}{-b_2 s^2 - b_1 s - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + s_2)(s + s_3)}$	0°	depending on b_o, b_1, b_2 will be either no. 13 or 14
21	$\frac{(e_2 + b_2)s^2 + (e_1 - b_1)s + e_o}{-b_2 s^2 + b_1 s}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-Ks(s + \sigma_o)}$	0°	Same as no. 10
22	$\frac{(e_2 - b_2)s^2 + (e_1 + b_1) s + e_o}{b_2 s^2 - b_1 s}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{Ks(s + \sigma_o)}$	0°	Same as no. 10
23	$\frac{(e_2 + b_2)s^2 + e_1 s + (e_o - b_o)}{-b_2 s^2 + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + \sigma_2)(s + \sigma_3)}$	0°	Same as no. 13
24	$\frac{(e_2 - b_2)s^2 + e_1 s + (e_o + b_o)}{b_2 s^2 - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + \sigma_2)(s + \sigma_3)}$	180°	Same as no. 13
25	$\frac{e_2 s^2 + (e_1 - b_1)s + (e_o + b_o)}{b_1 s - b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{K(s + \sigma_o)}$	180°	Same as no. 7
26	$\frac{e_2 s^2 + (e_1 + b_1)s + (e_o - b_o)}{-b_1 s + b_o}$	$\frac{(s + \sigma_1 + j\omega_1)(s + \sigma_1 - j\omega_1)}{-K(s + \sigma_o)}$	0°	Same as no. 7

Figure 3.7 Equations for Constructing the Root-Locus of Second-Order Systems When the Roots are Complex.

well as the closed loop character of these systems, due to their unstable open loop character, the computer study was limited to the closed loop functions only. The following computer "road map" was utilized for the simulation of these systems which have the form

$$\frac{C}{R}(s) = \frac{d_2 s^2 + d_1 s + d_o}{s^2 + e_1 s + e_o} \qquad 3.5\text{-}1$$

Equation 3.5-1 may be rewritten in the following fashion:

$$C(s) = \left[\frac{R(s)}{s^2 + e_1 s + e_o} \right] (d_2 s^2 + d_1 s + d_o) \qquad 3.5\text{-}2$$

Letting

$$C_x(s) = \frac{R(s)}{s^2 + e_1 s + e_o} \qquad 3.5\text{-}3$$

equation 3.5-2 can be put in the following form:

$$C(s) = (d_2 s^2 + d_1 s + d_o) \quad C_x(s) \qquad 3.5\text{-}4$$

Now since the setup that is used to solve for $C_x(s)$ will also contain $s^2 C_x(s)$ and $s\ C_x(s)$ terms, the right hand side of equation 3.5-4 may be solved to obtain C(s). The "road map" for the simulation of this equation is shown in Figure 3.8. The advantage of this "road map" over others being that it is possible for the constants e_1, e_o, d_2, d_1, d_o to take on any positive value. In this way it is possible to simulate either real or complex poles and zeros lying in the left-half plane.

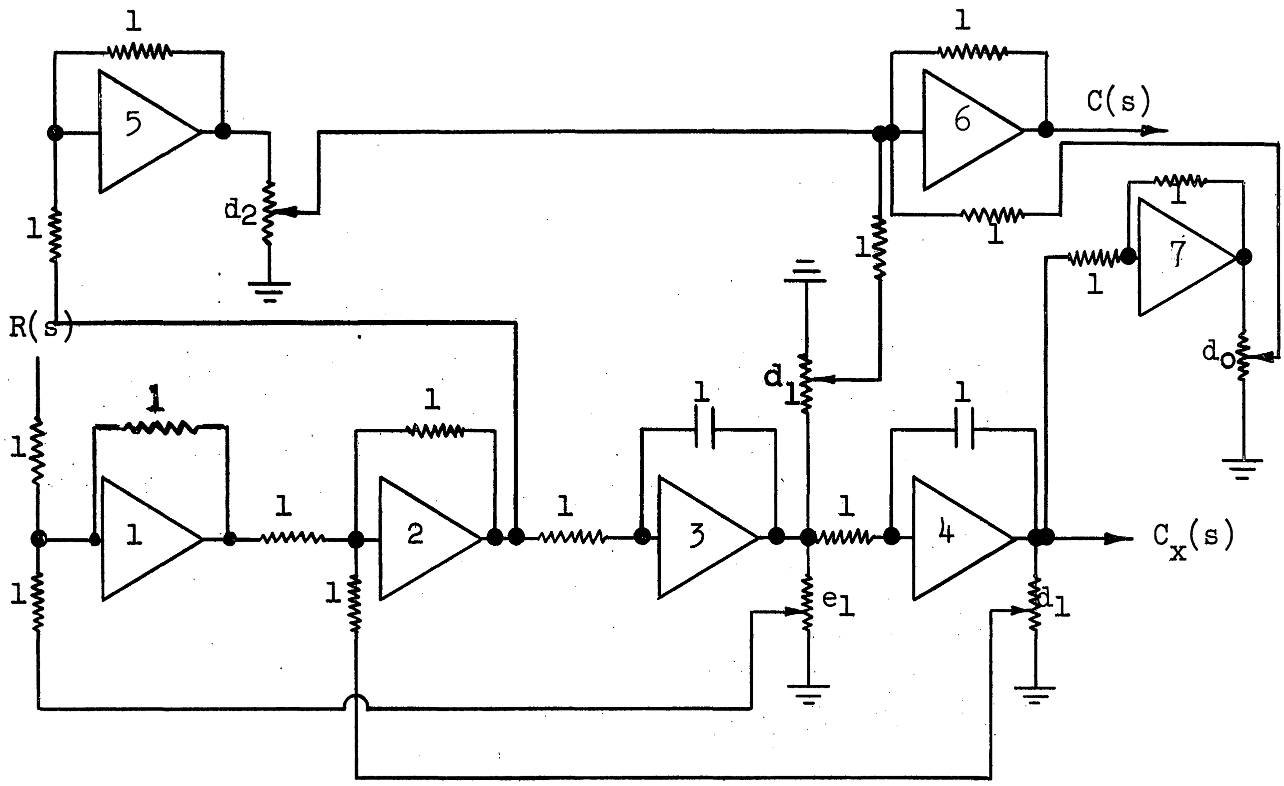
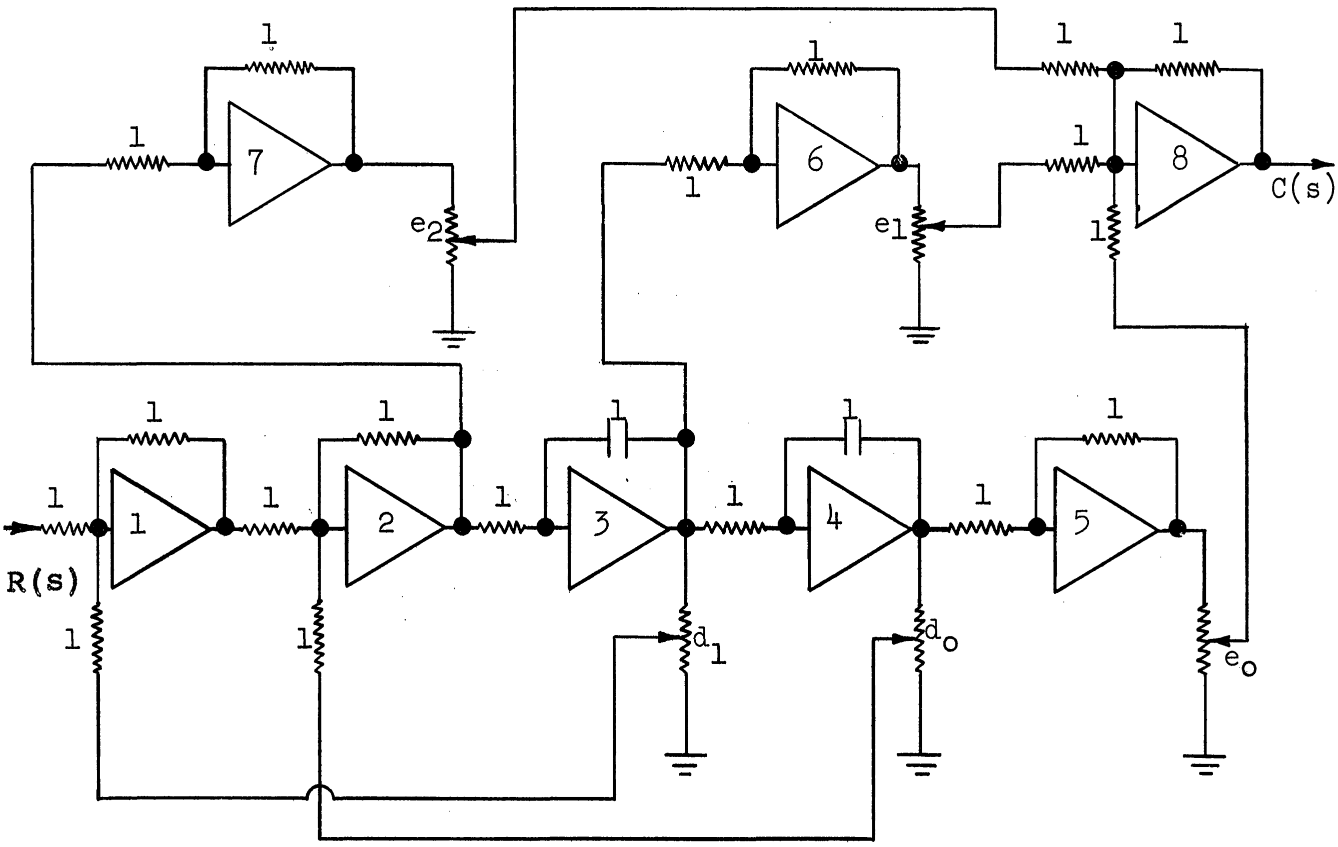


Figure 3.8 Computer Diagram Used for the Study of Conditionally Stable Systems with Left-Half Plane Closed Loop Zeros

In addition, if d_2, d_1, d_o take on negative values, $\frac{C}{R}(s)$ will contain one or more right half plane zeros. It is also possible by a modification of Figure 3.8 to simulate these non-minimum phase functions. A road map to simulate non-minimum phase functions is shown in Figure 3.9 .



All resistance have a value of 1 meg. ohm.
All condensers have a value of 1 microfarad

Figure 3.9 Computer Diagram Used for Studying Conditionally Stable Systems Containing Right Half Plane Closed Loop Zeros

In Table I are listed the necessary modifications that must be made in the "road map" of Figure 3.9 to simulate any prescribed second-order non-minium phase $\frac{C}{R}(s)$.

TABLE I.

Closed Loop Functions	Modifications Made in Figure
a) $\frac{C}{R}(s) = \frac{d_2 s^2 - d_1 s - d_o}{s^2 + e_1 s + e_o}$	remove amplifier 5

Closed Loop Functions (con't)	Modifications Made in Figure (con't)
b) $\frac{C}{R}(s) = \frac{d_2 s^2 + d_1 s - d_o}{s^2 + e_1 s + e_o}$	remove amplifiers 5 and 6
c) $\frac{C}{R}(s) = \frac{-d_2 s^2 + d_1 s + d_o}{s^2 + e_1 s + e_o}$	remove amplifiers 6 and 7
d) $\frac{C}{R}(s) = \frac{-d_2 s^2 + d_1 s - d_o}{s^2 + e_1 s + e_o}$	remove amplifiers 5, 6, and 7
e) $\frac{C}{R}(s) = \frac{-d_2 s^2 - d_1 s + d_o}{s^2 + e_1 s + e_o}$	remove amplifier 7
f) $\frac{C}{R}(s) = \frac{-d_2 s^2 - d_1 s - d_o}{s^2 + e_1 s + e_o}$	remove amplifiers 5 and 7

As an example of the simulation of a second-order system, a computer study was made of a system defined by the following open loop equation:

$$\frac{C}{E}(s) = KG(s) = \frac{K(s + 4)}{s^2 - 3s + 5}$$

The closed loop function for this system becomes

$$\frac{C}{R}(s) = \frac{K(s + 4)}{s^2 + s(K-3) + (5 + 4K)}$$

A root locus plot of this system is shown in Figure 3.10.

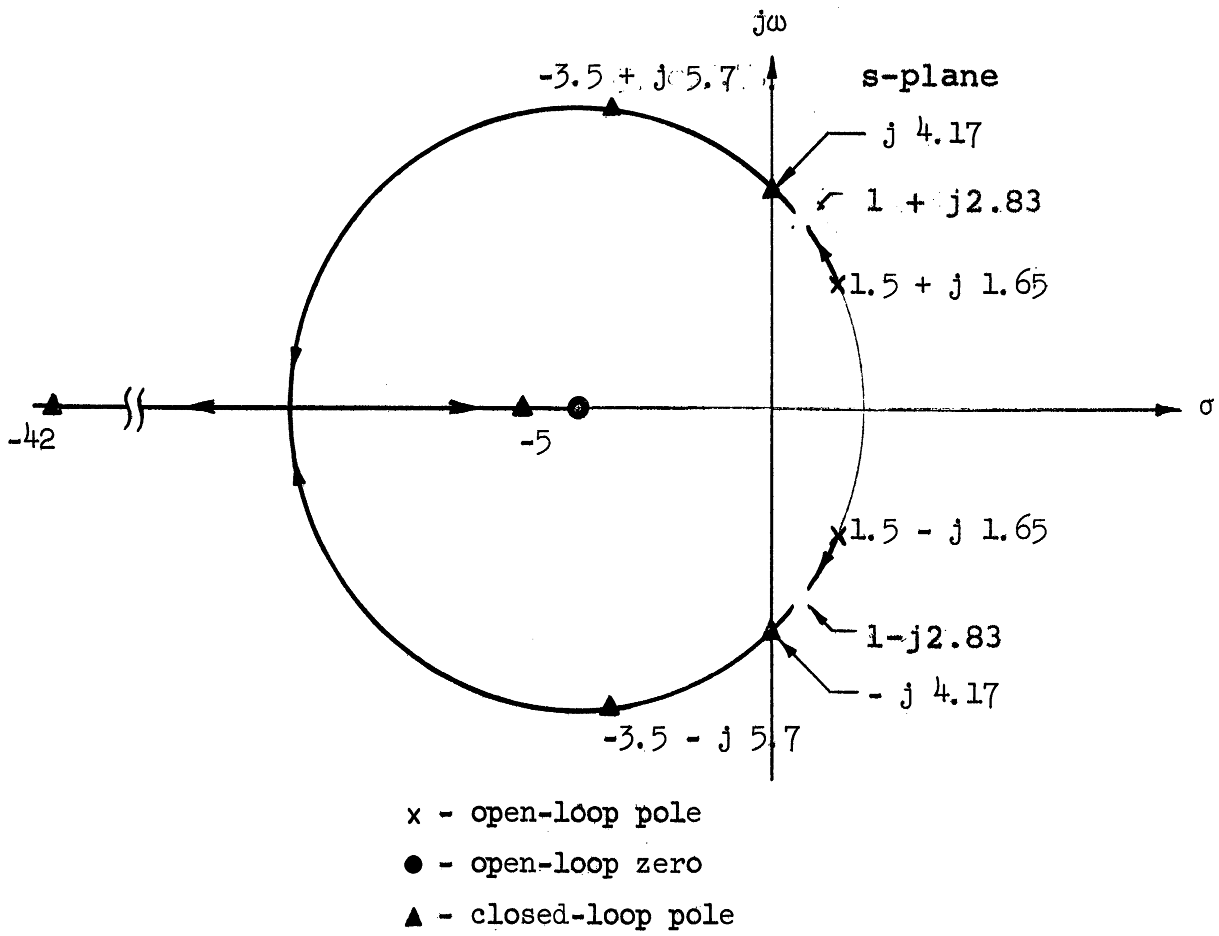


Figure 3.10 Root-Locus Plot of the System Studied on an Analog Computer

Inspection of the root locus reveals that for K=1, the closed loop system is unstable. In Figure 3.11 are recordings of the system response to a step input. It is apparent that this system is unstable for this gain. Now if the gain is increased until K=3, inspection of the root locus reveals that the closed loop system possesses two poles on the $j\omega$ axis. The system will be therefore continuously oscillatory; ω_n = 4.12 rad/sec. being the undamped natural frequency. The computer recordings in Figure 3.12 for this value of gain clearly indicate the stable oscillatory nature of this system. For K greater than 3 and less than 22 the closed loop poles will be complex and lie in the left half plane. Inspection of the recordings of Figure 3.13 indicate the damped oscillatory nature of the system for K = 10; the damping constant for the system in this case being ζ = .52. For still larger gains the closed loop roots become two real left half plane roots. In Figure 3.14 will be found the recordings for K = 50.

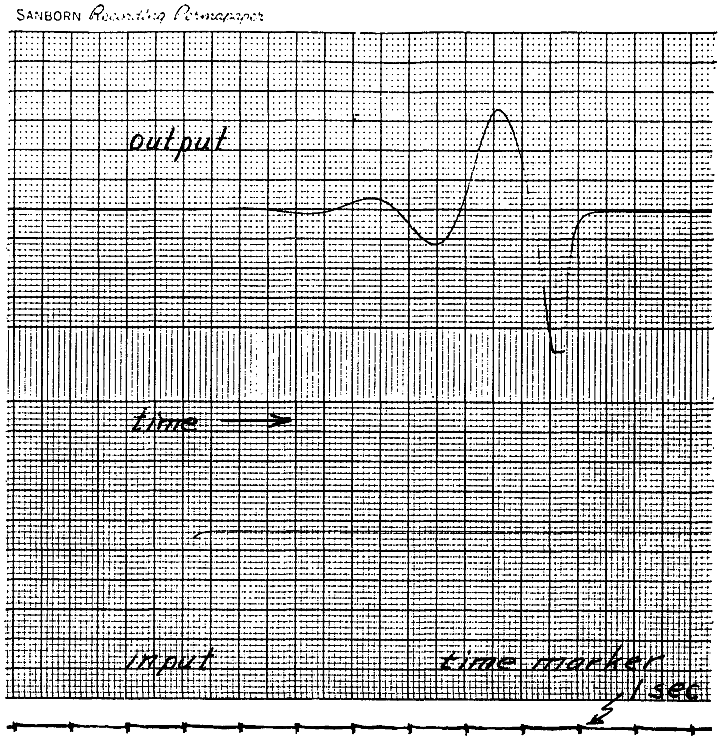


Figure 3.11 Time Response Curves for the System of Figure 3.10 with K = 1

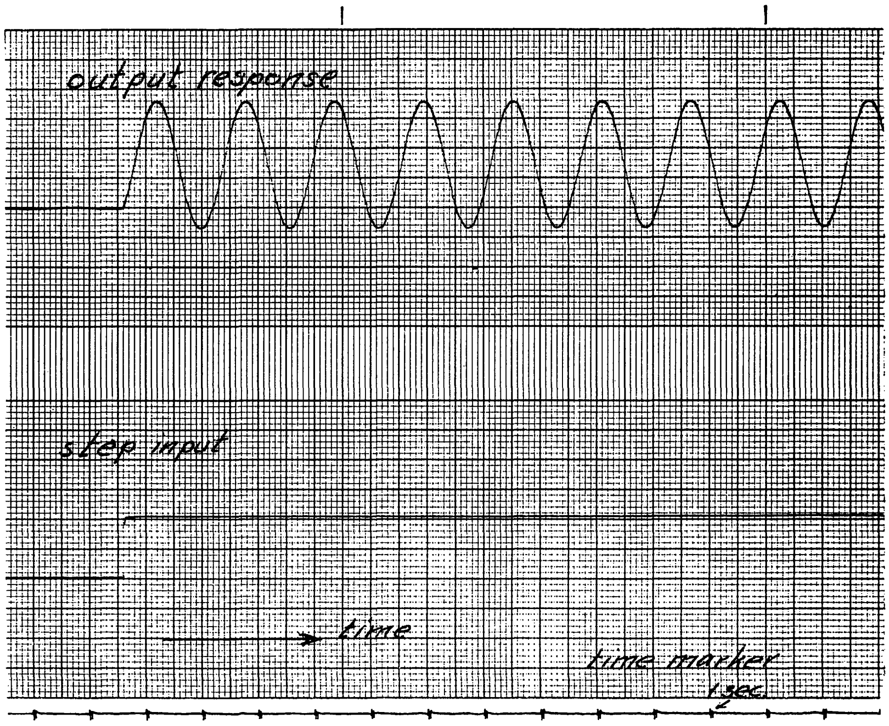


Figure 3.12 Time Response Curves for the System of Figure 3.10 with K = 3

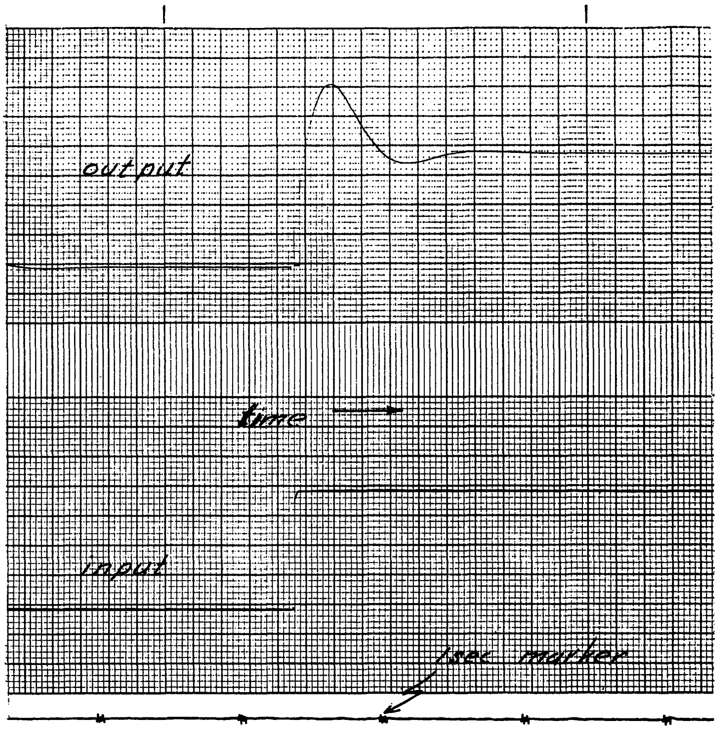


Figure 3.13 Time Response Curves for the System of Figure 3.10 with K = 10

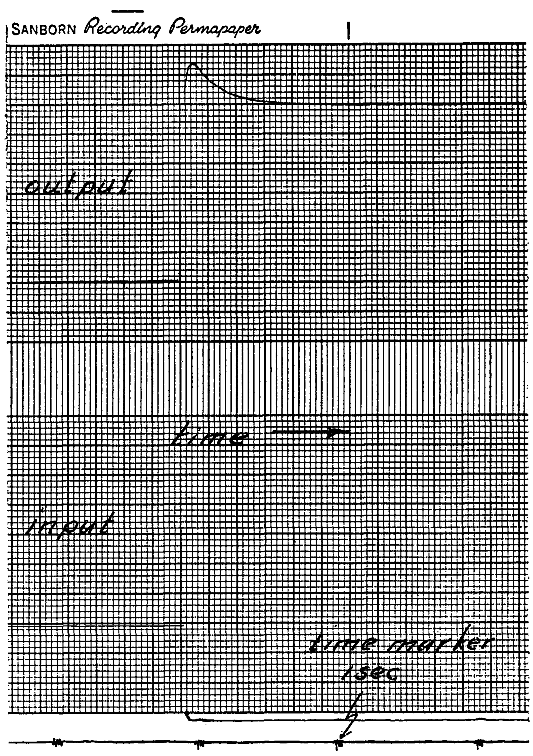


Figure 3.14 Time Response Curves for the System of Figure 3.10 with K = 50

CHAPTER IV
HIGHER ORDER SYSTEMS

4.1 Background

In chapter III the root locus method of analysis was used to analyze second order control systems that are conditionally stable. The ideas developed in that chapter can be extended directly to higher order systems. However, a more general approach is desirable, since the complexity and the number of possible system combinations increases rapidly as the order of the system increases. Attention in this chapter is therefore directed toward the more general properties of the root locus as applied to conditionally stable systems. These include the following topics:

1. The behavior of the root locus for large values of s.
2. The sufficient conditions which will insure that the locus terminate in the left half plane.
3. The possible pole-zero configurations of KG(s).
4. Minimum or non-minimum phase closed-loop systems.

4.2 Minimum or Non-minimum Phase Closed-Loop Systems

Up to the present time no specific mention has been made as to what characteristics the closed-loop function must possess. In this section the characteristics of these systems are discussed and will serve as a justification for the material in the following sections.

Consider the open-loop function

$$\frac{C}{E}(s) = KG(s) = \frac{K(s + z_1) - - - (s + z_1) - - - (s + z_m)}{(s + p_1) - - - (s + p_j) - - - (s + p_n)} \qquad 4.2\text{-}1$$

and the closed-loop function

$$\frac{C}{R}(s) = \frac{K(s + z_1) - - (s + z_i) - - (s + z_m)}{(s + p_1) - - (s + p_n) + K(s + z_1) - - (s + z_m)} \qquad 4.2\text{-}2$$

which in this form focuses attention on the relation between the poles and zeros of the open and closed-loop functions. A comparison of equations 4.2-1 and 4.2-2 reveals that the zeros of the open-loop function are also the zeros of the closed-loop function, while the poles of the closed-loop function are dependent upon the zeros and the poles of the open-loop function plus the loop gain, K.

In order to investigate the effect a zero has at various locations in the s-plane, two examples will be studied.

Example of a third-order system with one real zero

Consider first the simple third-order system shown in Figure 4.1a and let a finite zero be added on the real axis in the extreme left half plane. The behavior of the root-locus will now be investigated as this zero, $z_{\prime}$, moves along the real axis. As $z_{\prime} \rightarrow p_3$ the system becomes more like a second-order system with vertical asymptotes located

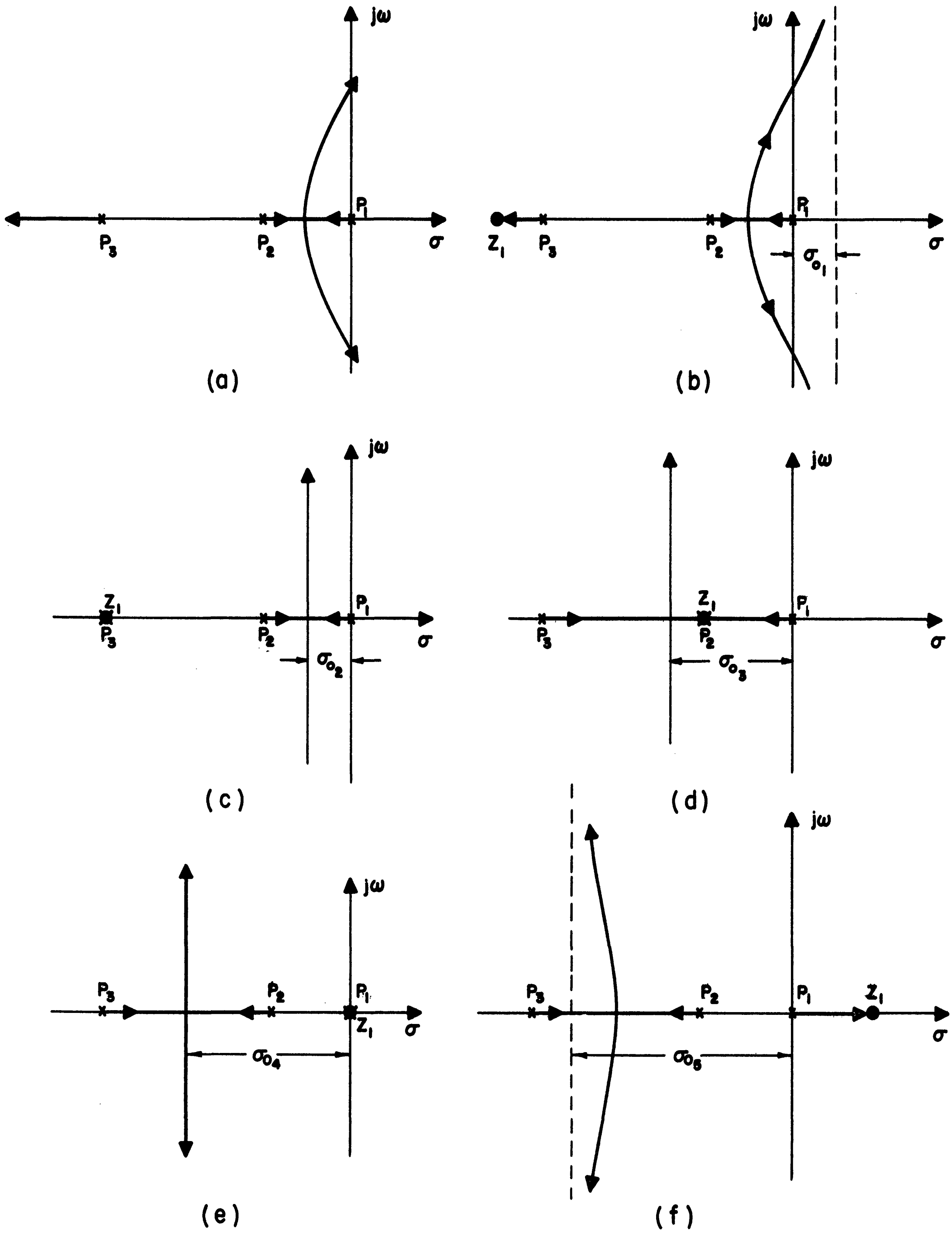


Figure 4.1 s-Plane Plots Showing How the Root-Locus Changes as a Function of the Location of z_1

at σ_o (see Figure 4.1b). The case when z_1 is located so as to cancel p_3 is shown in Figure 4.1c .

As z_1 is allowed to move toward the origin, successively different root-loci are obtained which, of course, lead to a different set of closed-loop poles. When z_1 cancels p_2 a second-order system results with vertical asymptotes at σ_{o_3} . When z_1 cancels the pole at the origin, another second-order system results with a vertical asymptotes at σ_{o_4} .

Now let z_1 enter the right half plane. Inspection of the root locus (see Figure 4.1f) reveals that no value of positive real gain exists that would lead to a stable closed-loop system, since one of the branches of the root-locus always lies in the right half plane.

Example of a third-order system with complex zeros

As a second example consider a simple third order system, but now let a pair of complex zeros be introduced far out in the left half plane. Under these conditions, the system will behave similar to a third-order system. However, the addition of the complex zeros will force the complex branches back into the left half plane for some large gain, and therefore adding these zeros has made it a conditionally stable system of a type described by Brown and Campbell, namely, a system that is stable for small gains but which becomes unstable for larger gains and then becomes stable again as the gain tends to infinity.

In Figure 4.2 is shown how the root-locus changes with changes in the location of the zeros. Attention is

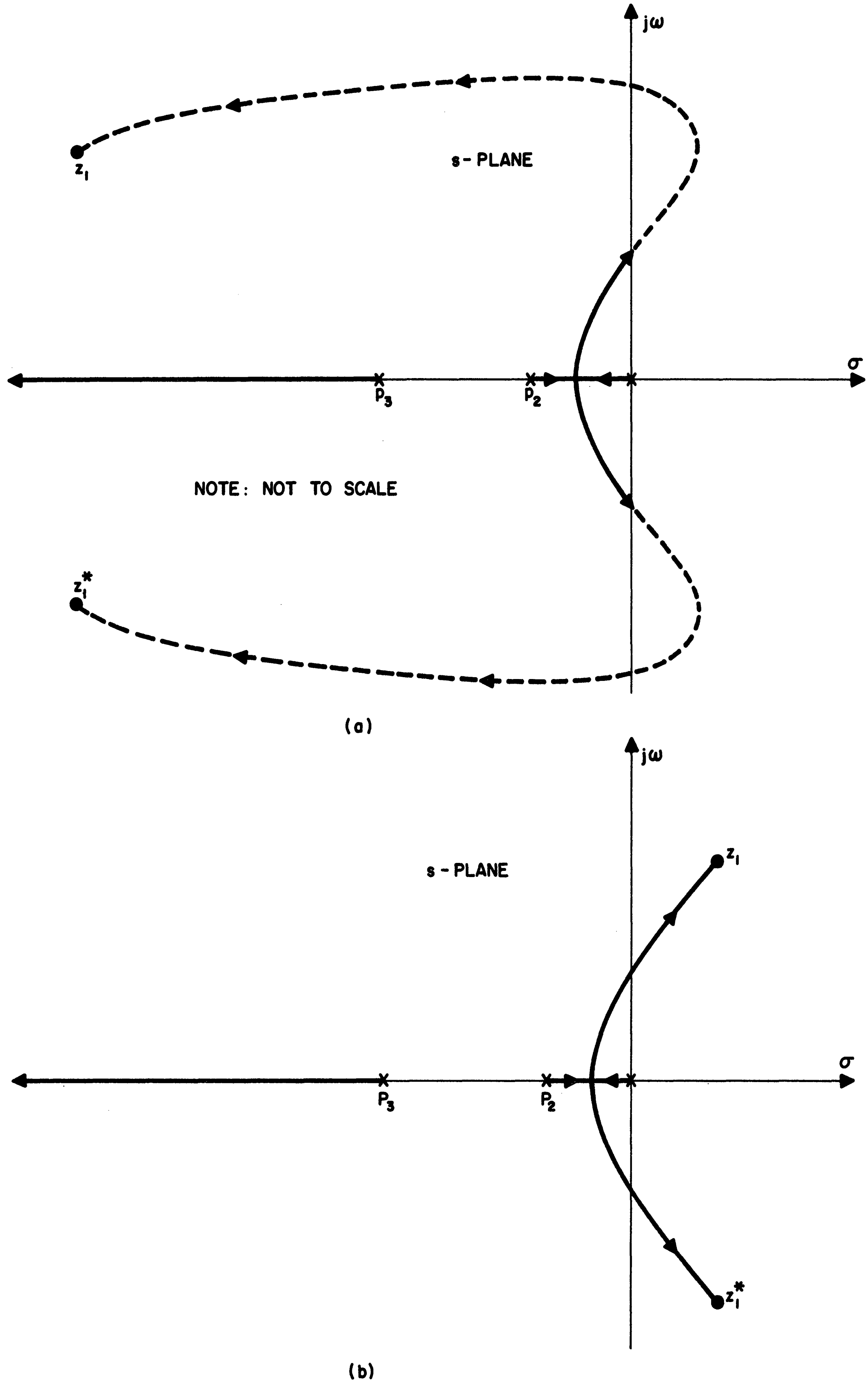


Figure 4.2 Third Order System with Complex Zeros

directed toward Figure 4.2b where the complex zeros are allowed to move into the right half plane. Under these conditions the system will become unstable for some gain. Truly, this is an undesirable condition, because the closed-loop performance has not been basically improved by the addition of the complex zeros.

As a result of the above observations the author has restricted the work that follows to systems whose finite zeros all lie in the left half plane, or in other words, to closed-loop minimum phase systems.

4.3 Extension of the Root-Locus Techniques

In section 2.6 some of the well-known properties of the root-locus method of analysis were presented. However, in the application of the root-locus method of analysis to n-th order systems the author has found the multiplicity relationship between the s and KG(s) functions not clearly defined. Consider, for example, the simple open-loop function

$$KG(s) = \frac{K}{s^3} \qquad 4.3\text{-}1$$

and let

$$KG(s) = U + j\,V = \rho e^{j\varphi} \quad \text{and } s = \sigma + j\omega = r e^{j\theta}$$

Substituting these quantities into equation 4.3-1 and then solving for r and θ yields

$$r = \sqrt[3]{\frac{1}{\rho}} \qquad \text{and} \qquad \theta = \frac{-\varphi}{3}$$

The root-locus in the KG(s) plane (see Figure 4.3a) is investigated by letting $0 < \rho < \infty$ and $\varphi = \pi \pm 2N\pi$, while N, which acts as a parameter, takes on all integer values. For each new value of N, KG(s) will always map the negative real axis. However, a study of the root-locus in the s-plane (see Figure 4.3b) will reveal that

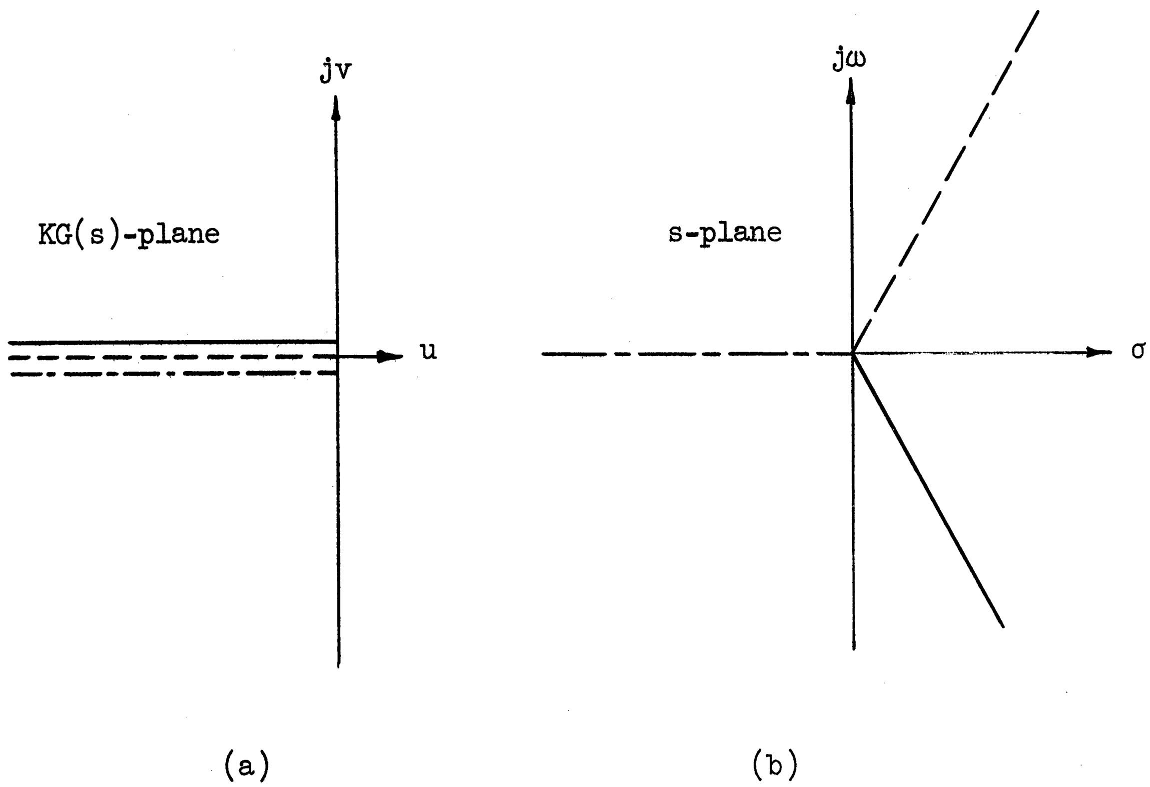


Figure 4.3 Showing the Multiple Mapping Property in the s and KG(s) Planes

for

$N = 0 \qquad \theta = -\frac{\pi}{3}$

$N = 1 \qquad \theta = -\pi$

$N = 2 \qquad \theta = -\frac{5\pi}{3}$

$N = 3 \qquad \theta = -\frac{7\pi}{3} = -\frac{\pi}{3}$

Thus, we arrive at the conclusion that associated with each N there exists a certain curve or branch of the root-locus in the s-plane. In this case, for $N \geq 3$ no new branches of the root-locus are obtained. In general, however, the number of branches that occur depend upon the number of poles and zeros in the KG(s) function. The multiple mapping of the negative real axis in the KG(s) plane results in "slits" being mapped in the s-plane. An example of a more complex root-locus diagram is shown in Figures 4.4a and 4.4b . The different values of N being distinctly labelled.

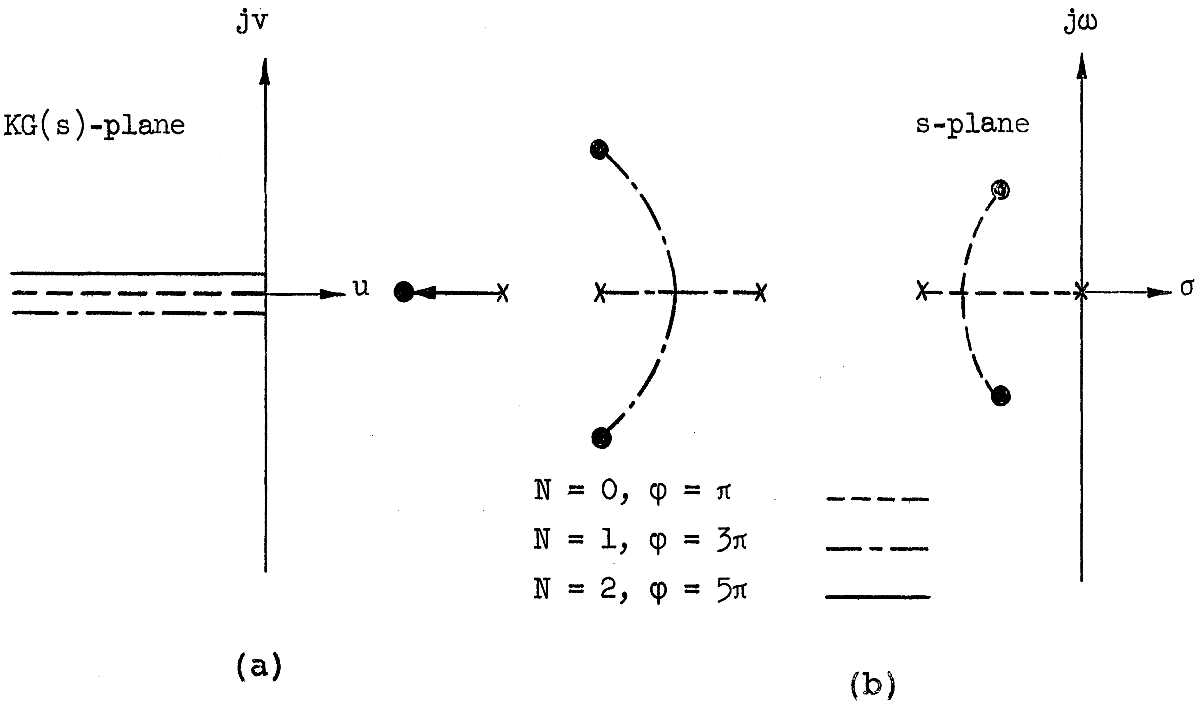


Figure 4.4 An Example Showing the Multiple Mapping Property of a Higher-Order System

If, in place of letting φ take on fixed values in equation 4.3-2 it is allowed to be a variable so that a sector in the KG(s) plane is mapped, there will be a corresponding sector in the s-plane which is defined by $\theta = -\frac{\varphi}{3}$. These are shown in Figures 4.5a and 4.5b . In a more complex case a sector in the KG(s) plane will map into an odd shaped region in the s-plane.

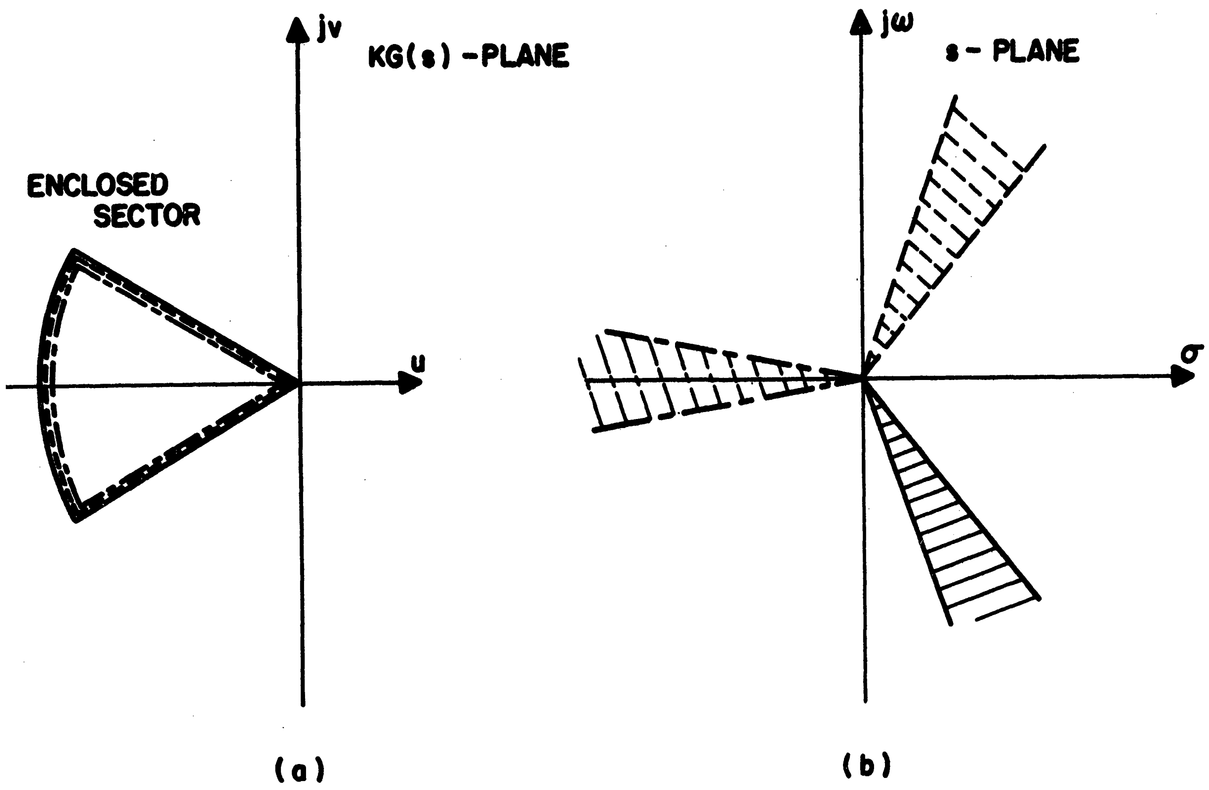


Figure 4.5 An Example Showing the Multiple Mapping Property Utilizing Phase-Angle Loci

As an example of a somewhat more complex plot, consider the open-loop function

$$KG(s) = \frac{K(s + z_1)(s + z_2)(s + z_3)(s + z_4)}{s(s + p_1)(s + p_2)(s + p_3)(s + p_4)} \qquad 4.3\text{-}3$$

where

$$|z_4| > |z_3| > |z_2| > |z_1| > |p_3| > |p_2| > |p_1|$$

A plot of the root-locus for this system is shown in solid lines in Figure 4.6, while the cross-hatched area represents the region associated with a sector in the KG(s) plane (in this case around $\varphi = \pi \pm 2N\pi$).

As the number of poles and zeros of KG(s) increase and as their locations change the sectors in the s-plane may take on odd shapes. However, for a fixed zero-pole configuration each section will always start at a pole and terminate at a zero of KG(s).

4.4 Crossing or Intersection of Root-Locus Branches

In the study of n-th order systems sooner or later the following question must be answered:

"Is it possible for the branches of the root-locus of a given system to cross or intersect?"

In section 4.3 it was shown that once the zero-pole configuration of KG(s) is given, a definite angular condition exists over the entire surface of the s-plane. Any fixed angular value in the KG(s)-plane will map "slits" in the s-plane, which start at each pole of KG(s) and terminate

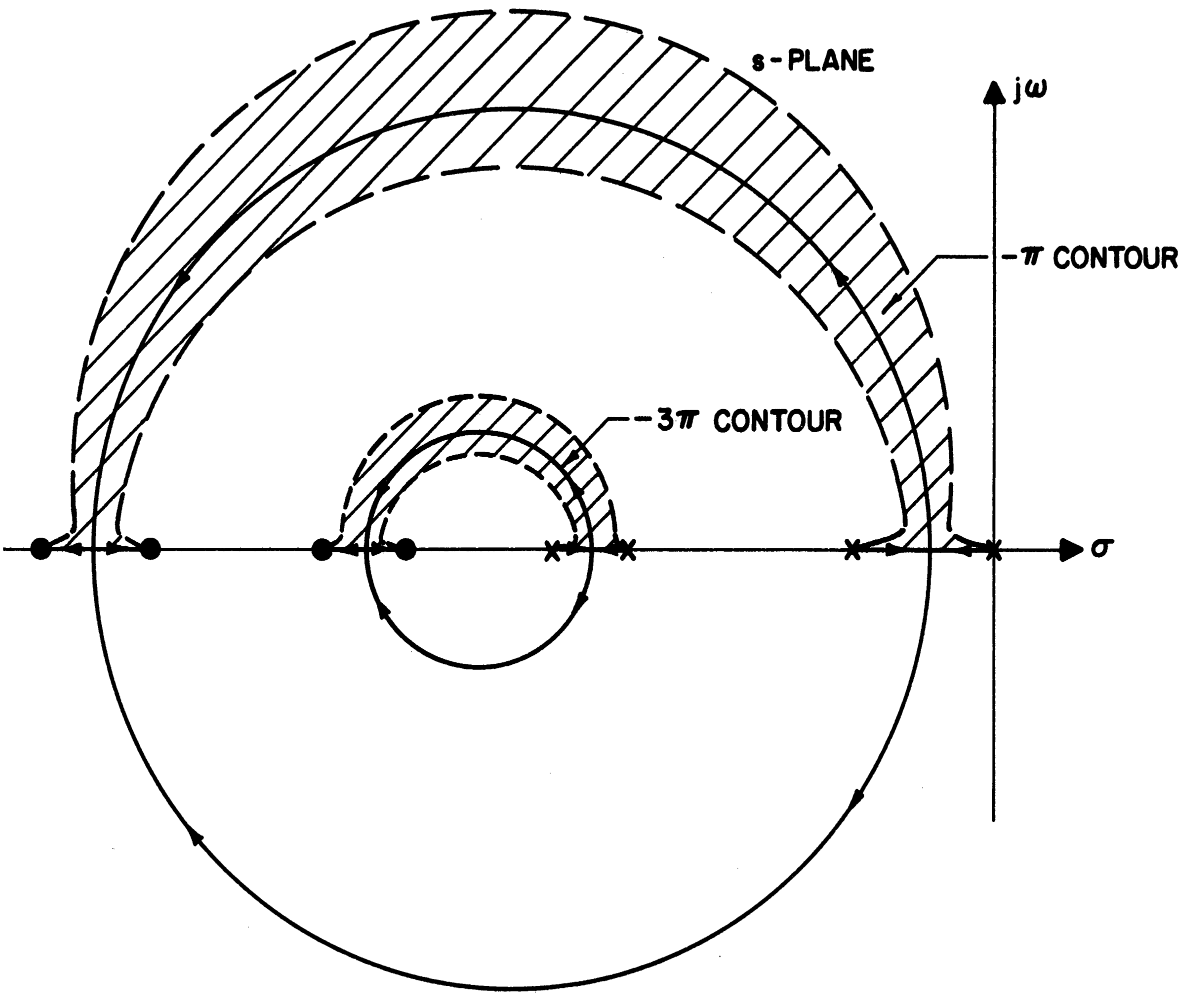


Figure 4.6 An Example of Multiple Mapping Utilizing Phase Angle Loci for a Complex System

at each zero of the system as the value of ρ is increased from zero to infinity.

Now, in order for two distinct branches of a root-locus to cross it is necessary for this crossing point in the s-plane to possess two different argument values. The only place such a condition can exist for a rational function KG(s) is at its poles and zeros. Therefore it is impossible for two branches of the root-locus which possess different argument values to intersect anywhere in the s-plane except at the poles and zeros of the KG(s) function.

The other type of intersection can be brought about by the intersection of two branches of the root-locus having the same argument value. Inspection of the system shown in Figure 4.7 will reveal it to be such a system. It represents the one of a family of zero-pole configurations obtainable from this type (fourth-order) of system, that yields a unique angular condition everywhere on the four branches of the locus, which in this case is $\varphi = -\pi$. A slight movement of the poles p_1, p_1^*, to the right will result in the zero-pole configuration shown in Figure 4.8, while a slight movement of the poles, p_1, p_1^*, to the left gives the zero-pole configuration shown in Figure 4.9.

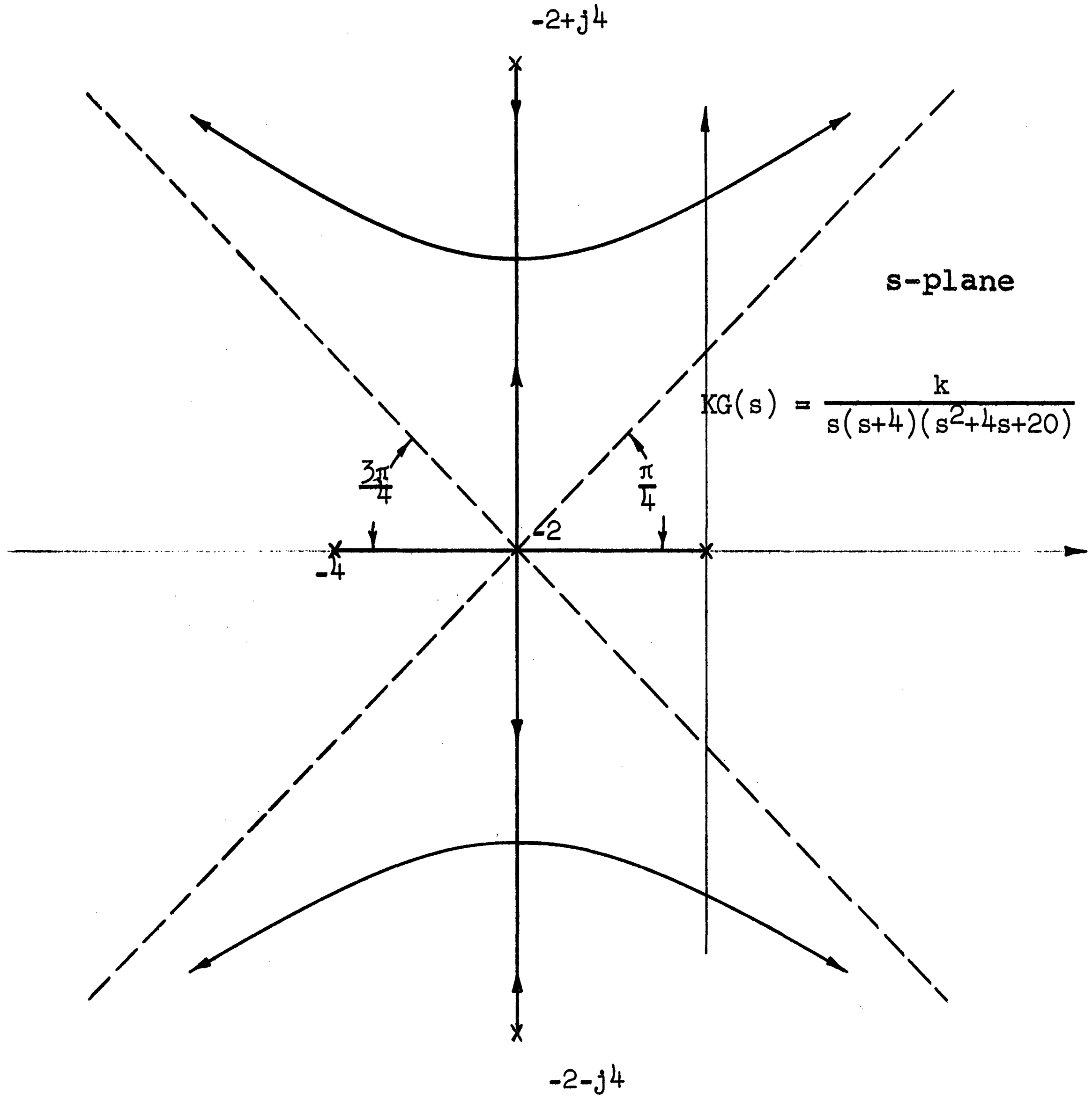


Figure 4.7 An Example of a System Possessing Three Breakaway Points

A more detailed investigation into the characteristics of systems which possess intersecting root-locus branches will reveal, that it is a characteristic of even the simplest second-order system. These systems which contain two closed-

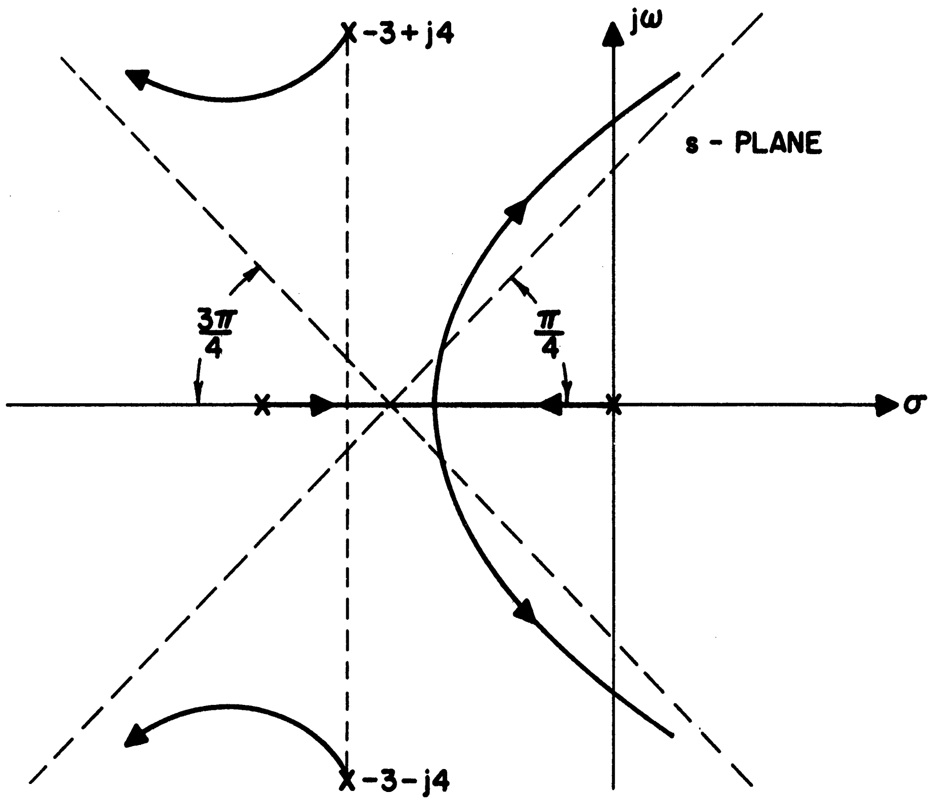


Figure 4.8 System of Figure 4.7 with Complex Plane Poles Moved Further Into the Left-Half Plane

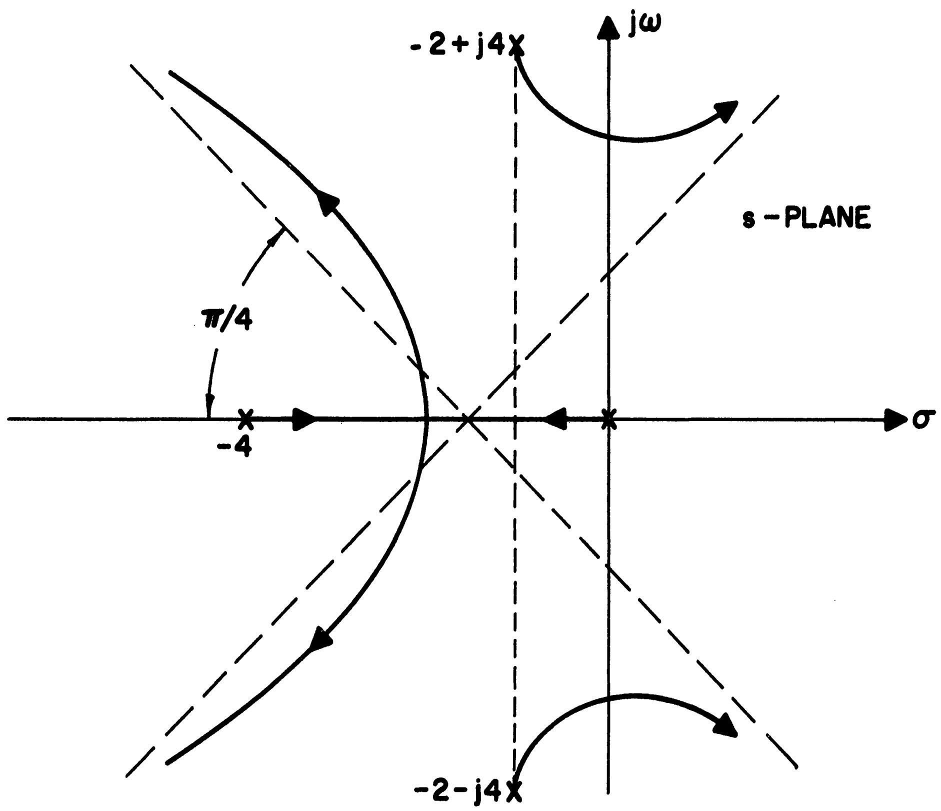


Figure 4.9 System of Figure 4.7 with Complex Plane Poles Moved Toward the Right Half Plane

loop poles that lie on the negative real axis for small loop gains, but which become two critically damped poles for some larger gain, and then two complex poles for still larger gains. This in effect represents the same type of argument condition as the example shown in Figure 4.7 . In the case of the second-order system, however, there exists only one point where the closed-loop system has two identical roots, while in the case of the fourth-order system for small gains (i.e. $K_v = .8$) the closed-loop system possesses two real roots, while for $K_v = 1.25$ the system possesses two pair of complex conjugate roots at $s = -2 \pm j\,2.45$.

In section 4.7, it will be shown that the points in the system where two roots occur are in reality <u>saddle points</u> or <u>points of stagnation</u> of the KG(s) function.

4.5 Conditionally Stable n-th Order Systems

In Chapter 3 the subject of conditionally stable second-order systems was introduced and a number of second-order systems were studied. One of the topics considered at that time was that of stability. In attempting to extend the ideas presented to a general class of n-th order systems, it is necessary to return to the n-th order open-loop function

$$KG(s) = \frac{K(s + z_1)(s + z_2) - - (s + z_i) - (s + z_m)}{(s + p_1)(s + p_2) - - (s + p_j) - - (s + p_n)} \qquad 2.6\text{-}1$$

which was discussed in detail in section 2.6 .

In order to define a stability criterion for a general class of systems, it has been found necessary to classify the general n-th order system and to obtain an answer to the following question: "Given a rational open-loop function, KG(s), which satisfies the definition of a conditionally stable function, what set of sufficient conditions can be placed on KG(s) which will insure stability of the closed-loop function, $\frac{C}{R}(s)$?" As a result of this research three classes of open-loop functions were found which will always lead to a stable closed-loop function for some real gain, K. These will be discussed in turn:

<u>Class A Conditionally Stable System</u> is defined as one possessing the following characteristics:

1) In equation 2.6-1 $(n - m) = 1$, or in other words, the open-loop function contains a single order zero at infinity,
2) KG(s) contains $(n - 1)$ left half plane zeros, and
3) KG(s) contains one or more right half plane poles, or, a pole of order 2 or more at the origin.

<u>Class B Conditionally Stable System</u> is defined as one possessing the following characteristics:

1) In equation 2.6-1 $(n-m) = 2$, or in other words, the open-loop function contains a double order zero at infinity,
2) KG(s) contains $(n - 2)$ left half plane zeros,

3) KG(s) will contain one or more right half plane poles, or, a pole of order 2 or more at the origin.

<u>Class C Conditionally Stable Systems</u> are defined as ones possessing the following characteristics:

1) In equation 2.6-1 $(n-m) \geqslant 3$ or in other words, the open-loop function contains a zero of order three or more at infinity,
2) KG(s) contains (n - 3) or less left hand plane zeros,
3) KG(s) will contain one or more right half plane poles, or a pole of two or more at the origin.

The conditions that must exist for each class of systems to be stable will now be discussed.

Consider first systems that satisfy the conditions defined under class A. The open-loop function will be of the form

$$KG(s) = \frac{K(s + z_1) - - - (s + z_{n-1})}{(s + p_1) - - - (s + p_n)} \qquad 4.5\text{-}1$$

where all the z_i lie in the left half plane, while p_j can fall in either the right or left half plane. To satisfy the conditionally stable requirement, it is necessary that at least some p_j be in the right half plane, or, as a limiting case a pole of order 2 exist at the origin. In the above class of functions, however, all the open-loop poles may lie in the right half plane.

Now since all the (n - 1) internal zeros lie in the left half plane, there will be (n - 1) branches of the root locus which terminate at these zeros. The remaining external zero terminates at infinity along the negative real axis, and therefore some gain can always be found which will place all the closed-loop poles in the left half plane.

Stability Criterion for a Class A Conditionally Stable System

The stability requirement of the class A type system can now be stated: A system meeting the requirements given under Class A above can always be stabilized for some finite real gain, K.

Examples of root-locus plots for Class A conditionally stable systems are shown in Figure 4.10. Inspection of the root-locus and the corresponding open-loop function, will reveal that a number of distinct zero-pole configurations have been included.

Consider next systems that satisfy the conditions defined under class B. The open-loop function will be of the form

$$KG(s) = \frac{K\ (s+z_1)\ \text{-----}(s+z_{n-2})}{(s+p_1)\text{--------}(s+p_n)} \qquad 4.5\text{-}2$$

where all the z_i lie in the left half plane, while p_j can fall in either the right or left half plane. However, to satisfy the conditionally stable requirement, item no. 3)

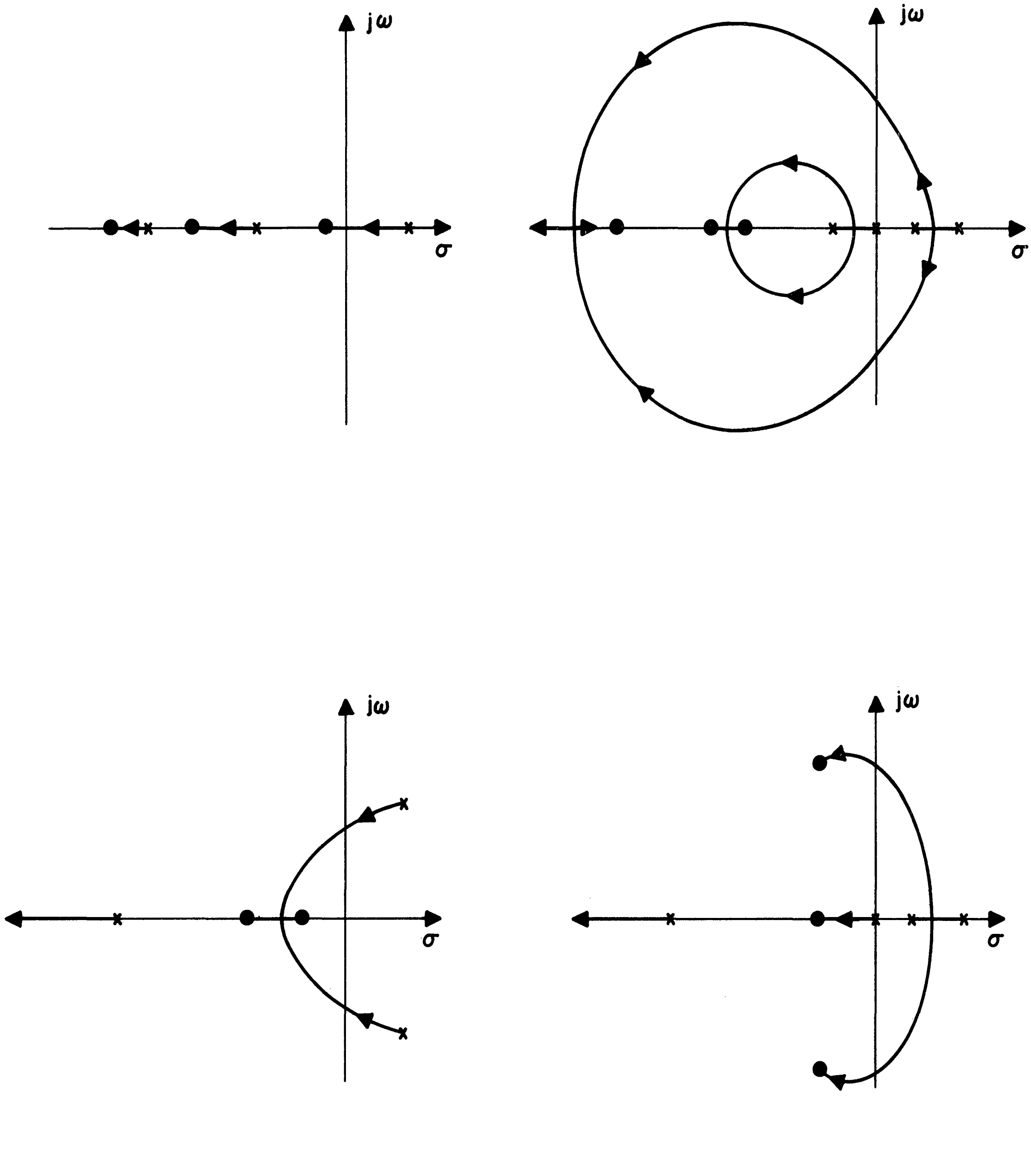


Figure 4.10 Typical Type A Conditionally Stable Systems

listed under this class of systems must be added. Now since all the (n - 2) internal zeros lie in the left half plane, there will be (n - 2) branches of the root-locus which will terminate at these zeros as the loop gain tends to infinity. For these branches of the root-locus, therefore, some gain can be found which will place (n - 2) of the closed-loop poles in the left half plane. It is now necessary to investigate what happens to the remaining 2 branches of the locus which tend to infinity along the asymptotes, which were defined by equation 2.6-9 in section 2.6 and which is repeated here for convenience.

$$\sigma_o = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} \; ; \quad \text{Arg } s' = \frac{\pm \pi \pm 2N\pi}{n - m} \qquad 2.6\text{-}9$$

For the case of (n - m) = 2, these asymptotes have an argument condition of $-\frac{\pi}{2} \pm N\pi$, which makes the remaining branches of the root-locus tend to infinity in the $j\omega$ direction. Now, if in addition, the pole - zero configuration is such as to make $\sigma_o < 0$, these asymptotes will lie in the left half plane, and so will the remaining 2 closed-loop roots.

<u>Stability Criterion for a Class B Conditionally Stable System</u>

The stability requirement of the class B type of system can now be stated: <u>A system meeting the requirements given under Class B, above, can always be stabilized</u>

for some finite real gain, K, if in addition to these requirements, a restriction is place on the zero-pole configuration so as to make σ_o lie in the left half plane.

Examples of root-locus plots for Class B conditionally stable systems are shown in Figure 4.11. They were chosen because of their distinct zero-pole character.

Consider now the systems which satisfy the conditions defined under Class C. They represent a number of distinct types as (n - m) takes on various values greater than three. In order to investigate the behavior of these systems, attention will be focused on the specific class defined by (n - m) = 3. Under these conditions the open-loop function will be of the form:

$$KG(s) = \frac{k(s + z_1) - - - - (s + z_{n-3})}{(s + p_1) - - - - (s + p_n)} \qquad 4.5\text{-}3$$

where again all the z_i lie in the left half plane, while p_j can fall in either the right or left half plane. However, to satisfy the conditionally stable requirements, item no. 3) listed under this class of systems must be added. Now in this case there are (n - 3) internal zeros which lie in the left half plane, and correspondingly there are (n - 3) branches of the root-locus which terminate at these zeros as the loop gain tends to infinity. For these branches of the root-locus it is therefore apparent that some gain can be found which will place

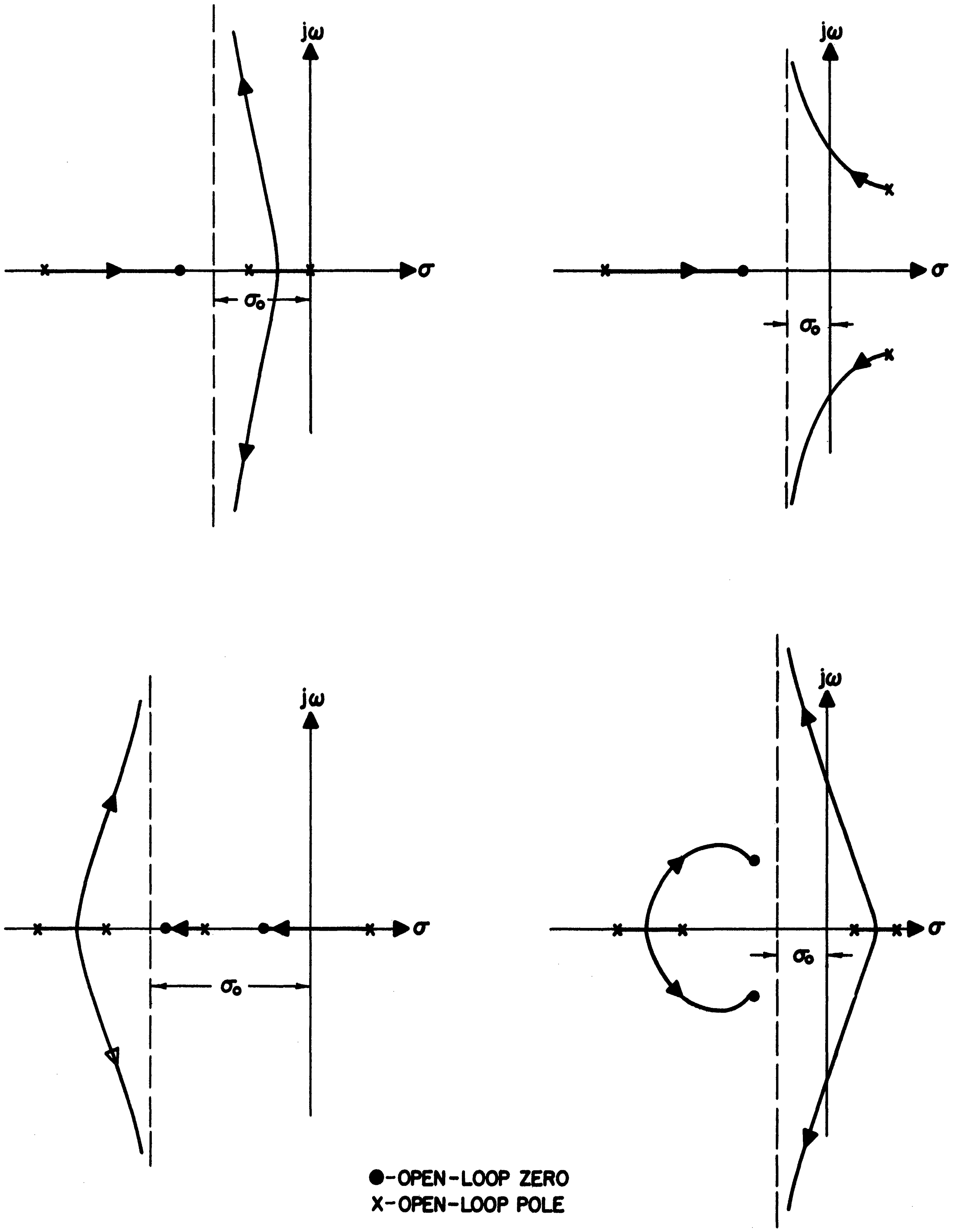


Figure 4.11 Typical Type B Conditionally Stable Systems

(n − 3) of the closed-loop poles in the left half plane. It is now necessary to investigate what happens to the remaining 3 branches of the root-locus, which tend to infinity along the asymptotes as defined for this class of functions by

$$\sigma_o = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{n-3} z_i}{3} \qquad 4.5\text{-}4$$

and

$$\text{Arg} \quad s' = \frac{-\pi \pm 2N\pi}{n - m} = \frac{-\pi \pm 2N\pi}{3} \qquad 4.5\text{-}5$$

Inspection of equation 4.5-5 reveals that the asymptote of the remaining branches of the root-locus tend to infinity along lines which make an angle of $\pm \frac{\pi}{3}$ and $-\pi$ radians with respect to the positive real axis. Thus, it is apparent that for this class of systems, if the gain is made large enough the system will always become unstable. Intuitively at first sight, it appears that a system possessing one or more right half plane open-loop poles and having two of its asymptotes terminating in the right half plane will always be unstable. However, a closer investigation into various zero-pole configurations will reveal that it is possible to have a zero-pole configuration which will lead to a system that satisfies the requirement of being conditionally stable, namely, of being unstable for small gains and then becoming stable as the gain of the system is increased. However, for this class of system, if the gain is made sufficiently large these systems will always become unstable. It is apparent that the asymptote requirement, at least by itself, does not produce sufficient conditions for stability in the case of Class C systems.

The above discussion on Class C (n - m = 3) type of systems can in general be extended to systems in which $(n - m > 3)$. The Arg s' will of course be different for each change in n - m. For example, for (n - m = 4) the Arg s' $= \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$ radians. This change in the Arg s' requirement will not basically alter the ideas developed for Class (n - m = 3) type of systems, and with slight modification they can be extended to these higher order systems. As (n - m) becomes larger, the possible range of K or the choice of a suitable zero-pole configuration will probably be severely restricted.

To date no general criterion has been found which can be applied to Class C type systems. However, certain statements can be made regarding this class of systems which will intutively help the designer when confronted with such a system. They are:

1) Since in this type of system it is the branches of the root-locus which are farthest in the right half plane that must be controlled, the one way possible to "bend" these root-locus branches back into the left half plane is to cause a portion of

the Arg s' asymptote to lie in the left half plane.

2) The addition of one or more left half plane zeros near the origin will aid in causing the locus to move into the left half plane.

3) Routh's Stability criterion can then be applied to determine whether the root-locus has really crossed the $j\omega$ axis.

Numerous conditionally stable systems which are of the $(n - m = 2)$ or $(n - m = 3)$ type can be found in practice. However, no practical applications have been found where $(n - m > 3)$.

In Figure 4.12 will be found a number of typical zero-pole configurations for the Class C $(n - m = 3)$ type of conditionally stable system. Inspection will reveal that they are all for systems possessing few poles and zeros. For more complex systems, a wide variety of configurations is possible. One such example is discussed in detail in Section 4.8.

4.6 A Comparison of Defining Characteristics Between Conditionally Stable and Absolutely Stable Systems

In previous sections attention was directed toward defining what is meant by a conditionally stable system and toward discussing the stability aspects of such a system. In this section attention will be directed toward answering such questions as:

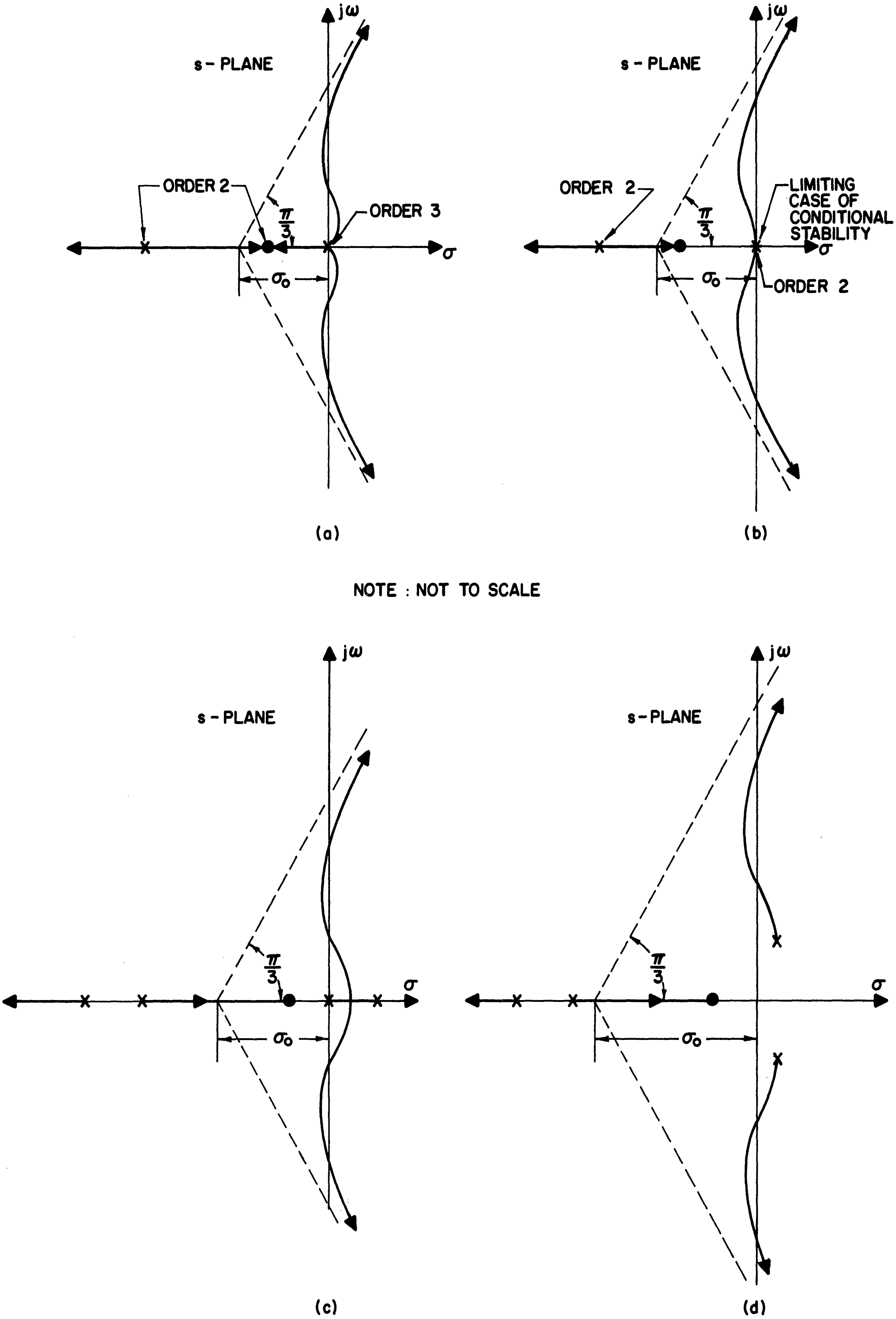


Figure 4.12 Typical Type C Conditionally Stable Systems

1) How do the closed-loop frequency responses of these systems compare for the same magnitude of K_v?

2) What frequency response characteristics do the closed-loop systems possess if the open-loop amplitude characteristics are identical?

3) How does the closed-loop time response of a conditionally stable system compare with that of an absolutely stable system?

Since in general the answers to these questions depend upon the zero-pole configuration of the functions involved, it was necessary to define the relative zero-pole position of the functions utilized in the investigation. After some deliberation, it was decided to utilize the closed-loop zero-pole configuration first proposed by Guillemin[26] and later exploited by Truxal[27]. However, both authors restricted their discussion to systems possessing left half plane poles. In the work that follows, all types of systems will be considered. As mentioned by Guillemin[26], his choice of a closed-loop function leads to open-loop systems which have negative real axis poles and zeros, and can therefore be readily synthesized as R-C networks, providing of course, that the "plant" possesses real axis poles and zeros. The closed-loop function utilized in his study was

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26. J. G. Truxal "Servomechanism Synthesis through Pole-Zero Configurations", MIT Research Laboratory of Electronics Tech. Report 162, August, 1950

27. J. G. Truxal, "Automatic Feedback Control System Synthesis", (book) McGraw-Hill Book Company, Inc., 1955, Chapter 5.

$$\frac{C}{R}(s) = \frac{\dfrac{p_1\omega_n^2}{z_1}\ (s + z_1)}{(s^2 + 2\,\zeta\,\omega_n\,s + \omega_n{}^2)(s + p_1)}\,, \qquad 4.6\text{-}1$$

which contains two control poles and a real axis pole and zero. The values of z_1, p_1, ω_n, and ζ are chosen in such a manner so as to meet the closed-loop requirements of frequency response, time response, specified K_v, and etc., or some combination of these quantities.

One of the reasons for choosing this function in this study is that an explicit solution of the open-loop function can be obtained directly. Carrying out this procedure leads, after some manipulation, to:

$$\frac{C}{E}(s) = \frac{\dfrac{p_1\,\omega_n^2}{z_1}\ (s + z_1)}{s\left[\,s^2 + (2\quad \omega_n + p_1)s + (\omega_n^2 + \omega\quad \omega_n p_1 - \dfrac{p_1\omega_n^2}{z_1}\,)\right]} \qquad 4.6\text{-}2$$

The roots of the quadratic term with the brackets are:

$$s_1,\ s_2 = \frac{-\ 2\zeta\,\omega_n + p_1}{2} \pm$$

$$\sqrt{\left[\frac{2\quad \omega_n + p_1}{2}\right]^2 - \left[\omega_n^2 + 2\quad \omega_n\ p_1 - \frac{p_1\ \omega_n^2}{z_1}\right]} \qquad 4.6\text{-}3$$

Thus, given the closed-loop poles and zeros, it is possible to substitute these values directly into equation 4.6-3 and obtain the location of the open-loop poles, which in general are not located at the origin. Inspection of equation 4.6-3 reveals that depending upon the relative magnitudes of z_1, p_1, ζ, and ω_n this closed-loop function can possess the following types of open-loop poles:

1) For the case when $\left[\omega_n^2 + 2\zeta\omega_n p_1 - \frac{p_1\omega_n^2}{z_1}\right] > 0$,
   s_1 and s_2 are two left half plane open loop poles; s_3 is located at the origin.

2) When $\left[\omega_n^2 + 2\zeta\omega_n p_1 - \frac{p_1\omega_n^2}{z_1}\right] < 0$,
   $\frac{C}{E}(s)$ will contain one right and one left half half plane pole; s_3 is located at origin.

3) When $\left[\omega_n^2 + 2\zeta\omega_n p_1 - \frac{p_1\omega_n^2}{z_1}\right] = 0$,
   $\frac{C}{E}(s)$ will possess a double order pole at the origin and one left half plane pole.

Further investigation into the conditions which lead to the open-loop pole configuration of 3) above, will reveal that it is directly related to Truxal's* equation which relates the K_v of a unity feedback control system to the zero-pole configuration of the closed-loop function. As derived by Truxal, this is

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* See reference No. 27, page 284, equation 5.21

$$\frac{1}{K_v} = \sum_{j=1}^{n} \frac{1}{p_j} - \sum_{i=1}^{m} \frac{1}{z_i} \qquad 4.6\text{-}4$$

For the system defined by equation 4.6-1 this reduces to

$$\frac{1}{K_v} = \frac{2\zeta}{\omega_n} + \frac{1}{p_1} - \frac{1}{z_1} \qquad 4.6\text{-}5$$

Now for $K_v = \infty$, it is apparent that

$$\frac{2\zeta}{\omega_n} + \frac{1}{p_1} - \frac{1}{z_1} = 0 \qquad 4.6\text{-}6$$

or

$$\omega_n^2 + 2\zeta\omega_n p_1 - \frac{p_1 \omega_n^2}{z_1} = 0. \qquad 4.6\text{-}7$$

The right side of equation 4.6-5 is the constant term in the quadratic factor of equation 4.6-3. Thus, condition 3) above corresponds to the case of infinite K_v , which is in agreement with the other definition, namely,

$$K_v = \lim_{s \to 0} s \frac{C}{E}(s) = \lim_{s \to 0} s \left[\frac{\frac{p_1 \omega_n^2}{z_1} (s + z_1)}{s^2 (s + s_1)} \right]$$

$$= \infty \quad \text{for } s \to 0$$

It is also apparent from 2) above that certain values of z_1 , p_1 , ζ , and ω_n , correspond to certain closed-loop requirements. These requirements can be met only if a right half plane open-loop pole is utilized. By dividing the expression in 2) above by p_1 and ω_n^2 (which of course assumes neither to be zero) the following expression is obtained:

$$\frac{1}{p_1} + \frac{2}{\omega_n} - \frac{1}{z_1} < 0 \qquad 4.6\text{-}8$$

Thus, if the sum of the reciprocal of the closed-loop poles is less than the reciprocal of the left half plane zero, the open-loop function will contain a right half plane pole. In addition, it should be pointed out that whenever a closed-loop zero-pole configuration is of this type, it will always lead to a system which contains a negative K_v , (a characteristic of systems possessing an odd number of right half plane open-loop poles).

The relative position of the poles can be better observed if in the expression 4.6-8 the substitution $A = \frac{p_1}{w_n}$ and $B = \frac{p_1}{z_1}$ is made. The following expression results:

$$1 + 2\gamma A - B < 0. \qquad 4.6\text{-}9$$

In Figure 4.13 will be found a plot of this expression for A and B greater than zero, and with γ as a parameter. Thus for a specified γ when B falls in the shaded area, a right half plane open-loop pole results.

In order to gain further insight into this subject, a comparison is made between two systems whose closed-loop functions are of the form given by equation 4.6-1, but whose zero-pole configurations are such that one of

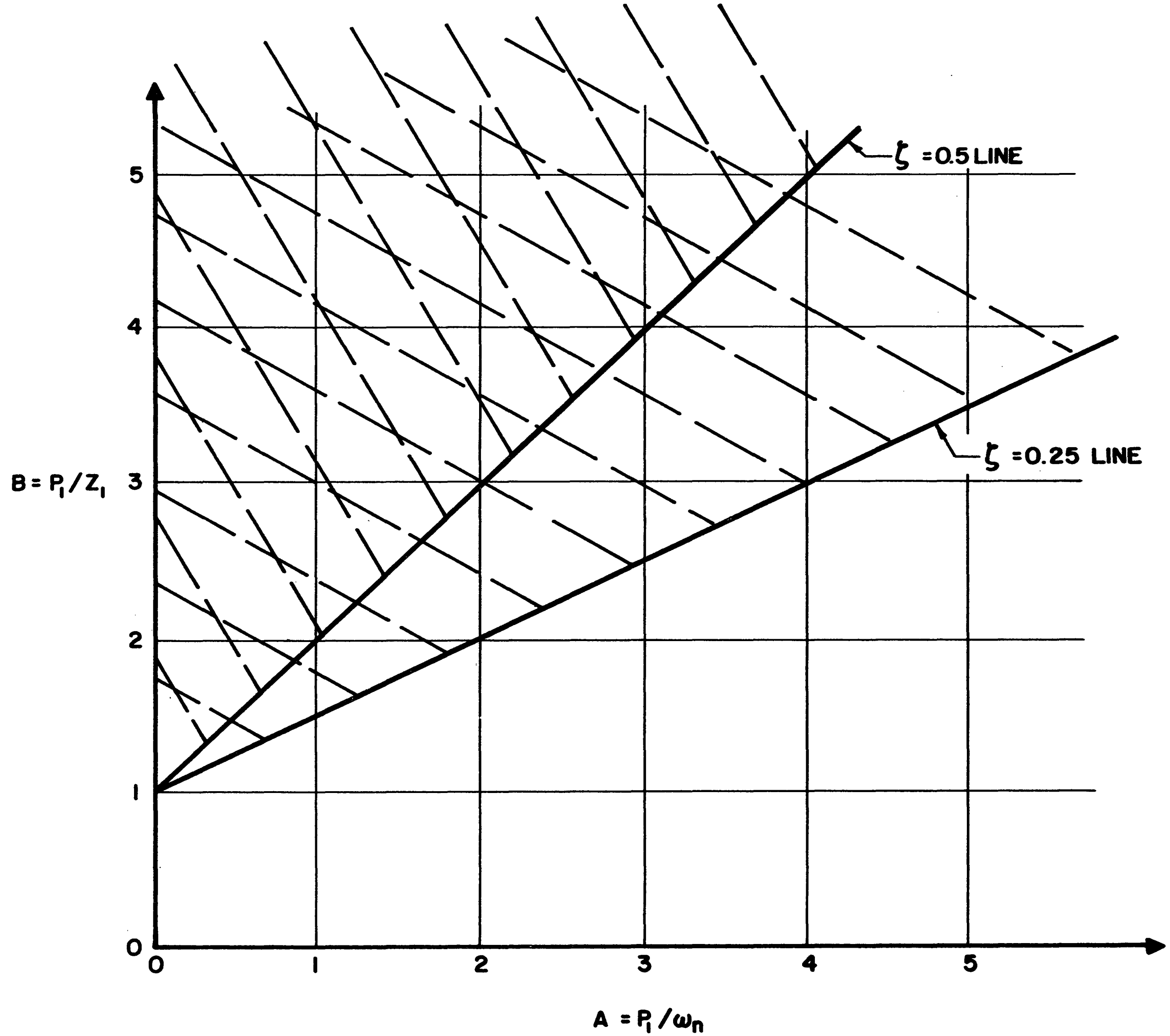

Figure 4.13 A Graph of the B vs. A for Two Values of ζ .

the systems possesses a right half plane open-loop pole, while the other system possesses all left half plane open-loop poles. In addition, a further constraint is placed on the open-loop transfer functions, namely, that they possess the same magnitude of velocity constant, K_v. The two systems considered possess the following transfer functions:

System I

$$\left[\frac{C}{R}(s)\right]_I = \frac{75\ (s + 5/3)}{(s + 5)(s^2 + 5s + 25)} \qquad 4.6\text{-}10$$

and

$$\left[\frac{C}{E}(s)\right]_I = \frac{75\ (s + 5/3)}{s(s + 12.07)(s - 2.07)} \ . \qquad 4.6\text{-}11$$

System II

$$\left[\frac{C}{R}(s)\right]_{II} = \frac{75\ (s + 5/3)}{(s + 1.55)(s^2 + 12.59s + 80.5)} , \qquad 4.6\text{-}12$$

and

$$\left[\frac{C}{E}(s)\right]_{II} = \frac{75(s + 5/3)}{s(s + 12.07)(s + 2.07)} \ . \qquad 4.6\text{-}13$$

In system I above $\frac{p_1}{\omega_n} = 1$ and $\frac{p_1}{z_1} = 3$, while in system II $\frac{p_1}{\omega_n} = .173$ and $\frac{p_1}{z_1} = .928$. It is apparent from equation 4.6-11 that system I contains an open-loop right half plane pole, while a study of equation 4.6-13 reveals that system II does not.

Taking the inverse transforms of equations 4.6-10 and 4.6-12 gives equations 4.6-14 and 4.6-15, respectively.

$$c(t) = 1 + 2e^{-5t} + 3.16\, e^{-2.5t} \sin(4.34t - 79.2^{\circ}) \qquad 4.6\text{-}14$$

and

$$c(t) = 1 - .0915\, e^{-1.55t} + 1.29e^{-6.3t} \sin(6.4t - 135.4^{\circ}) \qquad 4.6\text{-}15$$

A comparison of the time responses of these systems will be found plotted in Figure 4.14. Inspection of these curves reveal that the time response of system I possesses a high degree of overshoot. This seems to be a characteristic of systems possessing right half plane open-loop poles. On the other hand however, if the criterion of performance is to be the rise-time, it is apparent that the rise-time of system I is considerable less than that of system II.

It is left as a future investigation to make a detailed comparison regarding the defining characteristics.

4.7 Relation Between Saddle Points and Root-Locus Plots

In previous parts of this dissertation various aspects of root-loci and phase-angle loci were discussed. In this section it will be shown that the breakaway point on a root-locus plot is in reality a saddle point of the KG(s) function.

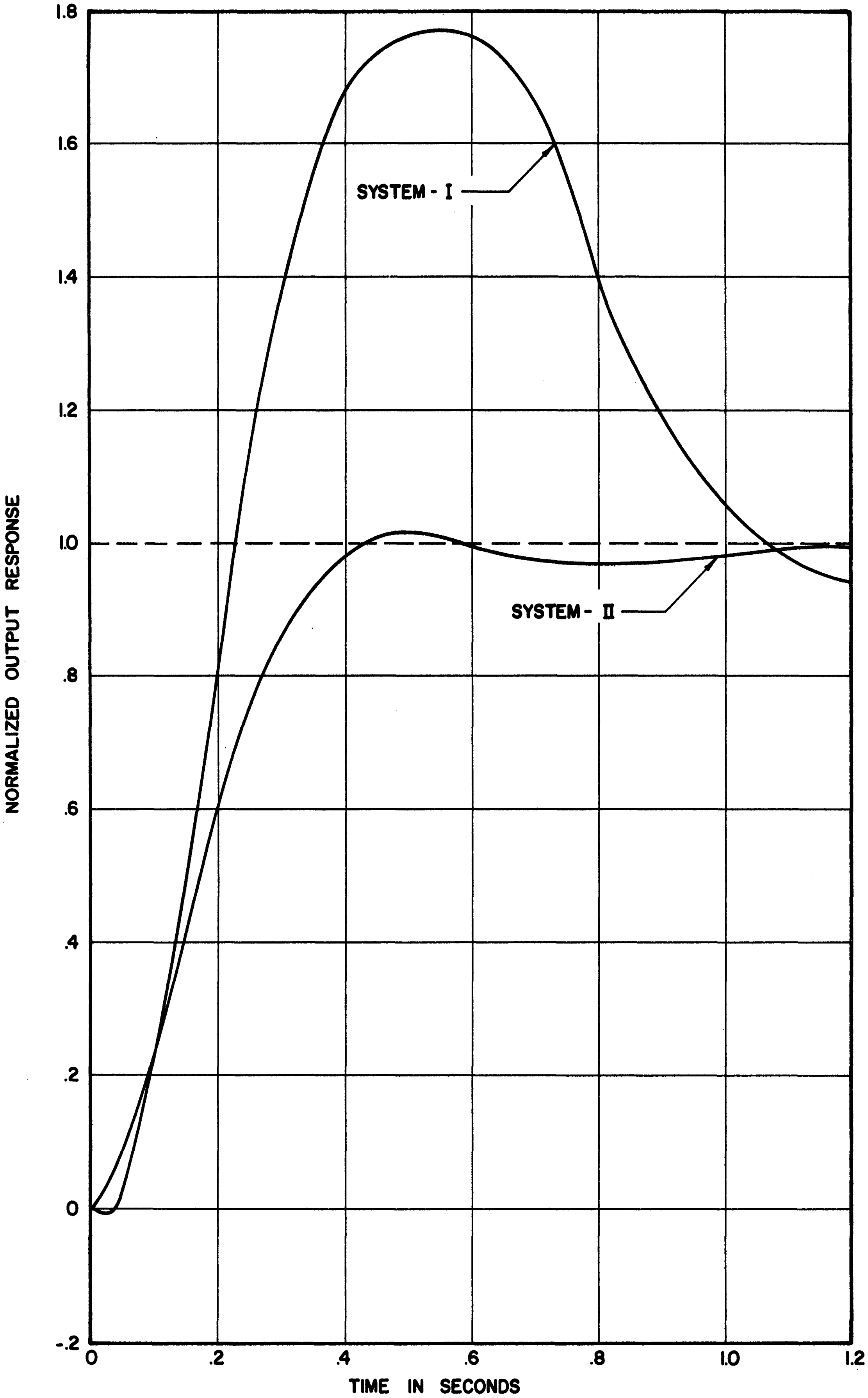

Figure 4.14 Transient Response Characteristics of Two Typical Systems

As discussed in Chapter 2, the root-locus plot in the s-plane graphically represents where the Arg KG(s)= $\pi \pm 2N\pi$ radians. It was shown in section 2.8 that the root-locus is a special case of the more general phase-angle loci where the Arg KG(s) $= \varphi \pm 2N\pi$ radians, in which φ takes on all positive real values. In place of having Arg KG(s) as a parameter it is possible to consider $|KG(s)|$ as a parameter. In the KG(s) - plane this corresponds to a series of concentric circles about the origin. In the s-plane the corresponding contours will take on some odd shape depending upon the zero-pole configuration of the system. In Figure 2.5a will be found amplitude and phase-angle plots of KG(s) in the KG(s)-plane, while in Figure 2.5b will be found the corresponding plots in the s-plane for a simple second-order system containing two real poles. Additional information can be obtained however, if a three dimensional plot is made which involves the magnitude of KG(s) as an axis perpendicular to the σ and $j\omega$ axes of the s-plane. The plot $|KG(j\omega)|$ vs. ω is a special case in this multi-dimensional plot which can be obtained by passing a plane through the $|KG(s)|$ and $j\omega$ axes, and then letting $s = j\omega$ be the variable. If however $|KG(s)|$ is plotted as s is allowed to take on all complex values, a three-dimensional plot is obtained which looks like a "relief" map. On such a map, constant amplitude lines of KG(s) would appear as level contours of constant elevation. In Figure 4.15 is shown a three dimensional plot of the

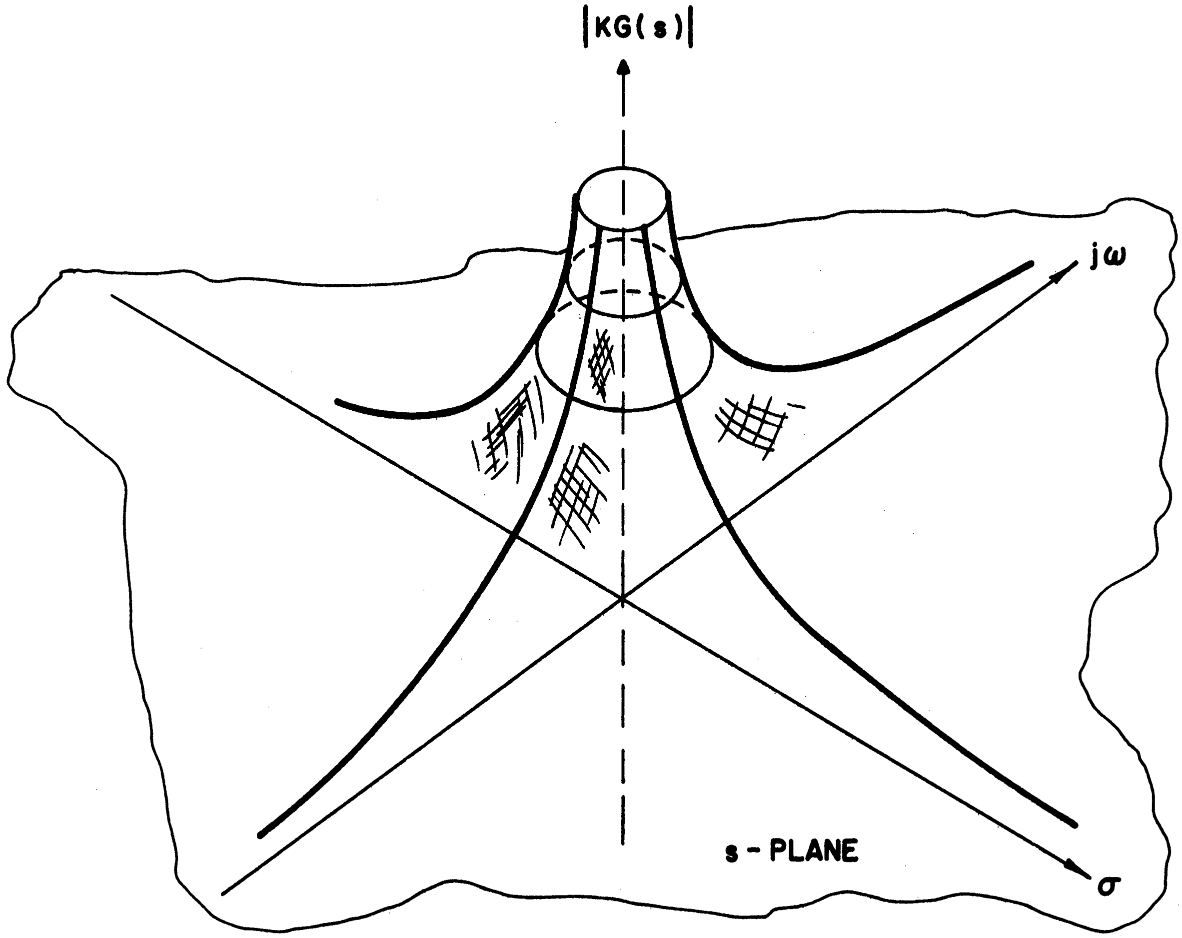

Figure 4.15 "Relief" Map of the Function $KG(s) = \frac{1}{s}$

function $\left| KG(s) \right| = \frac{1}{s}$. Inspection of this plot shows a pronounced characteristic, namely, it looks like a "pole", which in this case is located at the origin. Common usage of the term "pole" for a root which lies in the denominator of a function, F(s) dates back to other fields of applied mathematics where this type of function is often found plotted. However, to date, in the area of control systems little work has been published which utilizes the multi-dimensional contours in the analysis or synthesis of feedback control systems. In this section attention is directed toward relating the breakaway points on a root-locus to saddle points as defined in potential theory.

Consider now a simple second-order system of the type depicted by the phase-angle and amplitude loci of Figure 2.81a and 2.81b. The root-locus plot which is a special case of the more general phase-angle loci is given in the Figure by Arg KG(s) $= \mathcal{f} = -\pi$.

If s is allowed to take on all complex values and $\left| KG(s) \right|$ vs. s is plotted in a three-dimensional fashion as described above, Figure 4.16 will result. Inspection of this Figure reveals that the "relief" map for this function contains two "poles" which occur at each of the singular points of KG(s). In addition, it is apparent that a trough or minimum point exists at point, s_m, along the real axis between these two poles. Since there are points removed from the real axis for which

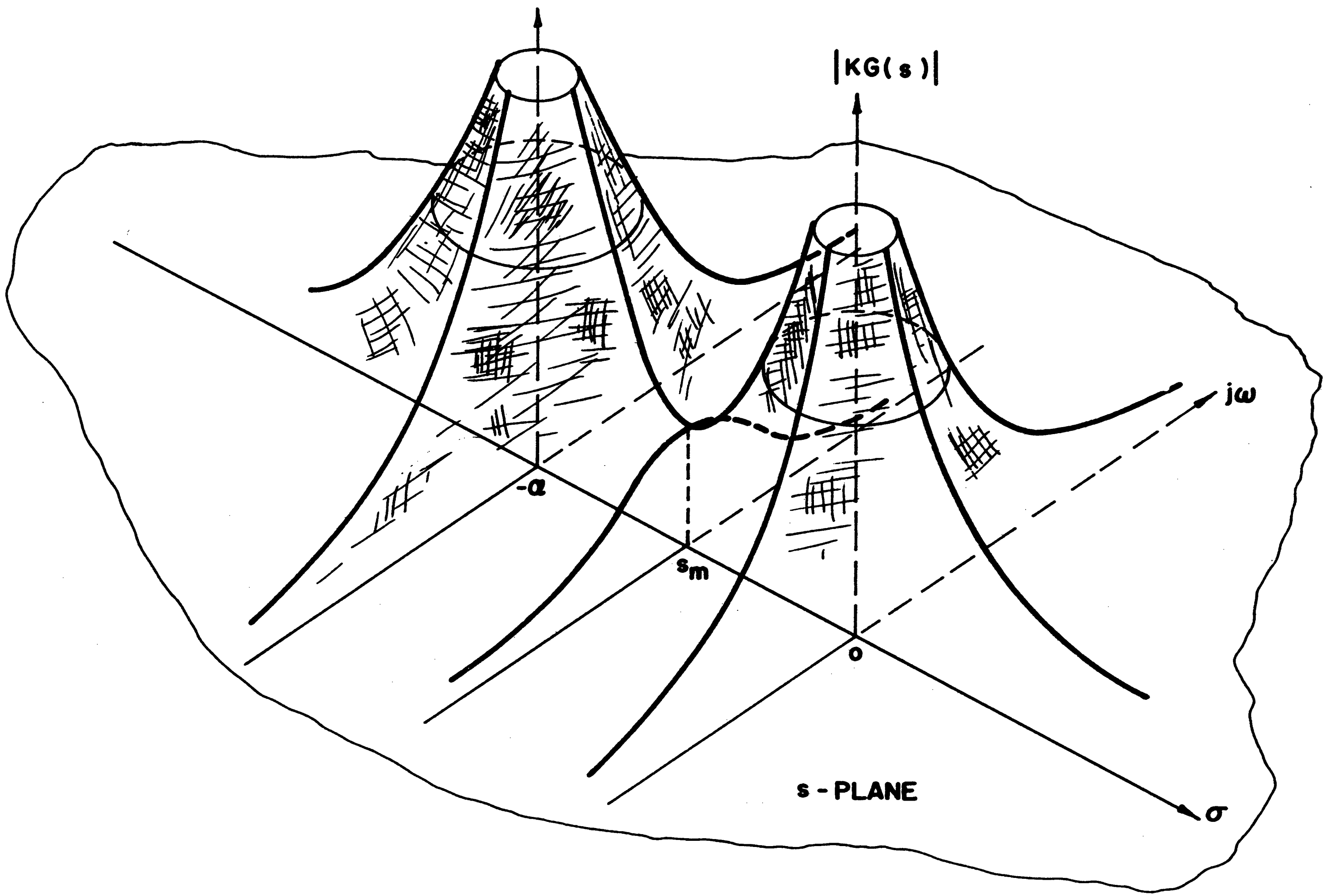

Figure 4.16 "Relief" Map of the Function $KG(s) = \dfrac{K}{s(s + \alpha)}$

$|KG(s)| < |KG(s_m)|$, in reality this trough looks rather like a "mountain pass" or has the familar shape of a "saddle". Thus, point s_m is referred to as a saddle point of KG(s). Since s_m is a minimum point for $s = \sigma$, it is possible to determine the minimum value by setting KG(s)' = 0. It should be noted that since Arg KG(s) is not plotted in this case, it is only a coincidence that this point occurs along the root-locus curve. In this particular case it is part of the negative real axis that lies between the poles.

Sometimes in finding KG(s)' = 0, it is possible to obtain more than one value for s_m. While inspection of a root-locus plot may reveal there is only one possible breakaway point. The following questions then arise:

1) Which, if any, of the values of s_m is correct?
2) What do the other values obtained as a result of setting KG(s)' = 0 represent?

In an attempt to answer these questions a third example will be considered, namely, a second-order system that contains a left half plane open-loop zero. In Figure 4.17 is shown the "relief map" for the system defined by the following open-loop function

$$\frac{C}{E}(s) = KG(s) = \frac{K(s + 2)}{s(s + 1)} \quad . \qquad 4.7\text{-}1$$

A study of Figure 4.17 reveals that the addition of a real left half plane zero has altered the characteristics of the "relief map" of the simple second-order system of

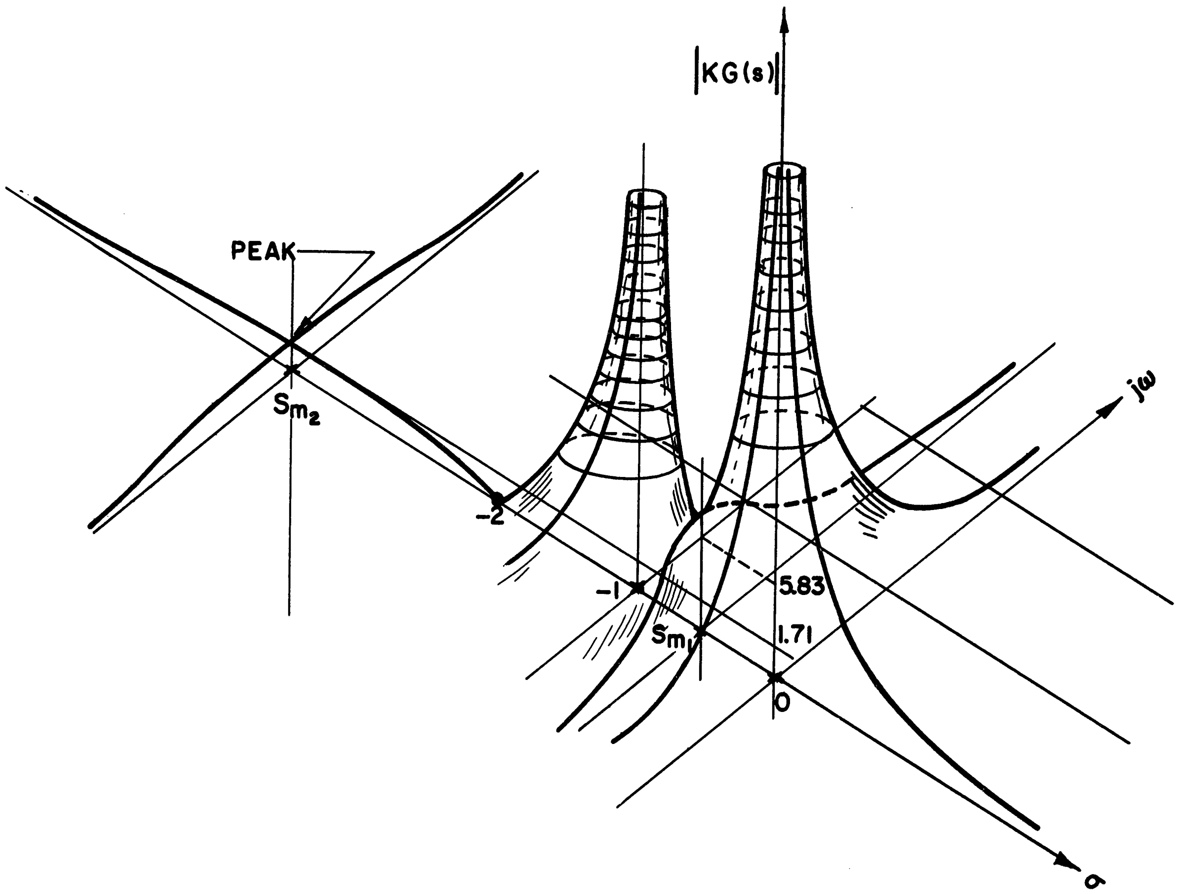

Figure 4.17 "Relief" Map of the Control System defined by equation 4.7-1.

Figure 4.16 . Setting $KG(s)' = 0$ in this case yields two values of s_m , namely, $s_m = -.59$ and $s_m = -3.41$. From Figure 4.17 it is seen that these values define where a minimum and a maximum occur in $KG(s)$ as s is allowed to vary along the negative real axis. A study of the root-locus for this system, which is shown in Figure 4.18, shows that the minimum or saddle point of $KG(s)$ occurs at the one breakaway point of the root-locus, while the maximum point occurs at the other breakaway point where the complex root-locus returns to the real axis.

As a final case, the system whose root-locus is given by Figure 4.7 will be investigated. As it will be recalled, this is the system which contains intersecting root-locus branches. In Figure 4.19 will be found the "relief map" for this system. A study of the figure reveals that this system contains three saddle points, one on the negative real axis mid-way between the real axis poles and two others which occur at the complex values of s, namely at $s_m = -2 \pm j\,2.5$. In addition, as mentioned previously, the root-locus does not in general indicate the bottom of the valley between two poles.

The characteristics relating the breakaway point of a root-locus plot to the saddle point commonly used in potential theory will now be summarized:

1) Setting $KG(s)' = 0$ is a convenient method for locating the breakaway points on the root-locus of a low-order transfer function.

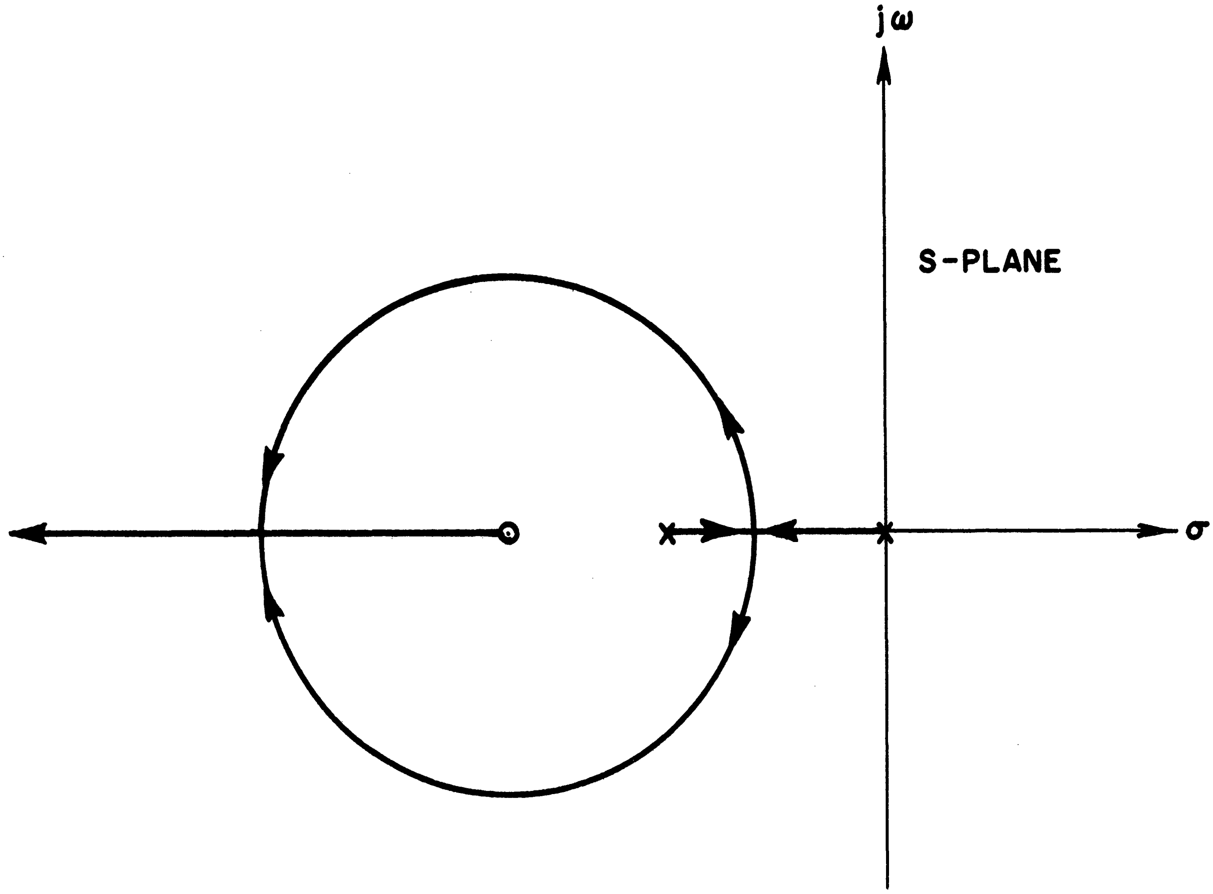

Figure 4.18 Root-Locus Plot of the System defined by equation 4.7-1

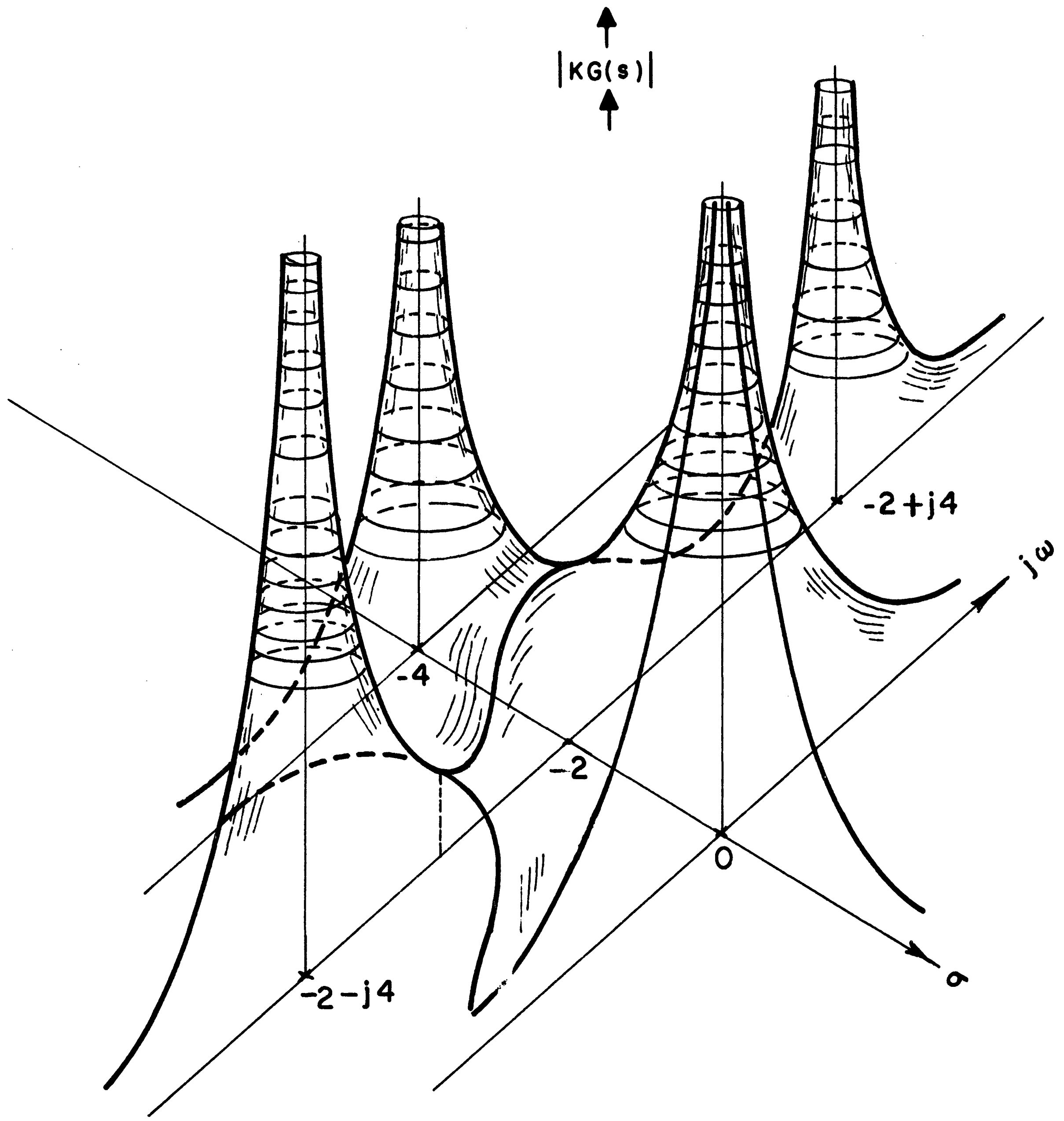

Figure 4.19 "Relief" Map of a Control System with Four Open-Loop Poles

2) Since not all values of s_m determined in this fashion are breakaway points, it is necessary to utilize the Arg conditions of $1 + KG(s) = 0$ to determine which values of s_m apply.
3) For higher order systems the determination of s_m requires the solution of a high order polynomial in s. This leaves some question as to the desirability of utilizing this approach if extreme accuracy is not needed.

4.8 Practical Examples of Conditionally Stable Systems

As mentioned in Section 3.2 conditionally stable systems can occur in two ways, namely, either as the result of the designer being "given a plant" that contains one or more right half plane open-loop poles and which is therefore open-loop unstable, or as the result of his changing an inner loop of a complex system in order to achieve some desired performance that couldn't be obtained by using an absolutely stable inner loop. Under the former conditions the closed-loop system will always be conditionally stable, and care in design must be exercised if the closed loop function is to be a stable one.

One of the prevalent examples of the former type can be found in the missile industry at the present time, where the dynamic behavior of the missile in the vertical plane in inherently unstable in uncontrolled flight, and suitable control must always be devised. A typical example of such an open-loop function is the following:

$$KG(s) = \frac{K(s + .025)}{(s^2 + .473s + 9.43)(s - .063)} \qquad 4.8\text{-}1$$

It is apparent that without feedback the system is unstable, since it possesses a right half plane pole. From the root-locus plot of the uncompensated unity feedback system shown in Figure 4.20, it is also apparent that the closed-loop system will also be unstable for $K < 23.8$. In addition since the complex closed-loop poles possess very low damping the system will be highly oscillatory even for $K > 23.8$. Inspection of equation 4.8-1 and/or the root-locus in Figure 4.20 reveals it to be an $n - m = 2$ system possessing one right half plane pole and it is therefore a class B conditionally stable system. Utilizing the ideas developed in previous sections, but principally the results of Section 4.5, it is found that with a series compensating network of the form $k\,\frac{s+3}{s+30}$ it is possible to modify the characteristics of this system and thus improve its performance by moving the complex poles away from the $j\omega$ axis to the $\zeta = .5$ line, thereby allowing an increase in K_v and allowing the system to have a larger bandwidth. From the root-locus of modified system which is shown in Figure 4.21, it is apparent that the addition of the series compensation network has forced the asymptote line from $\sigma_o = -.18$ to $\sigma_o = -13.65$. In the normal case this would permit a range of loop gains to be utilized, unless very specific bandwidth requirements are specified.

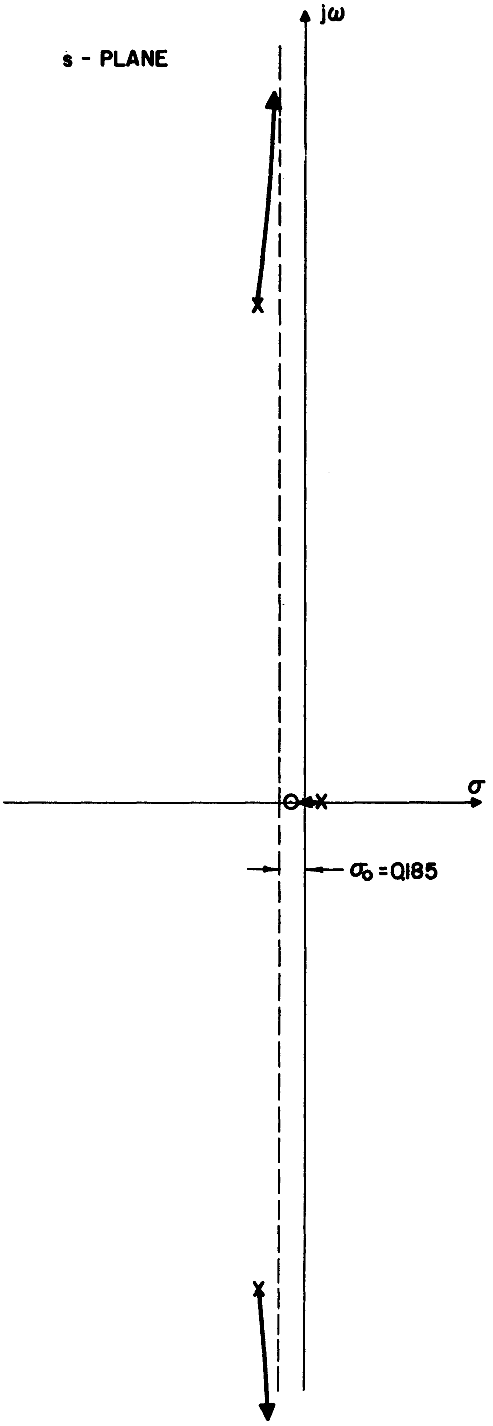

Figure 4.20 Root-Locus Plot of Uncompensated System

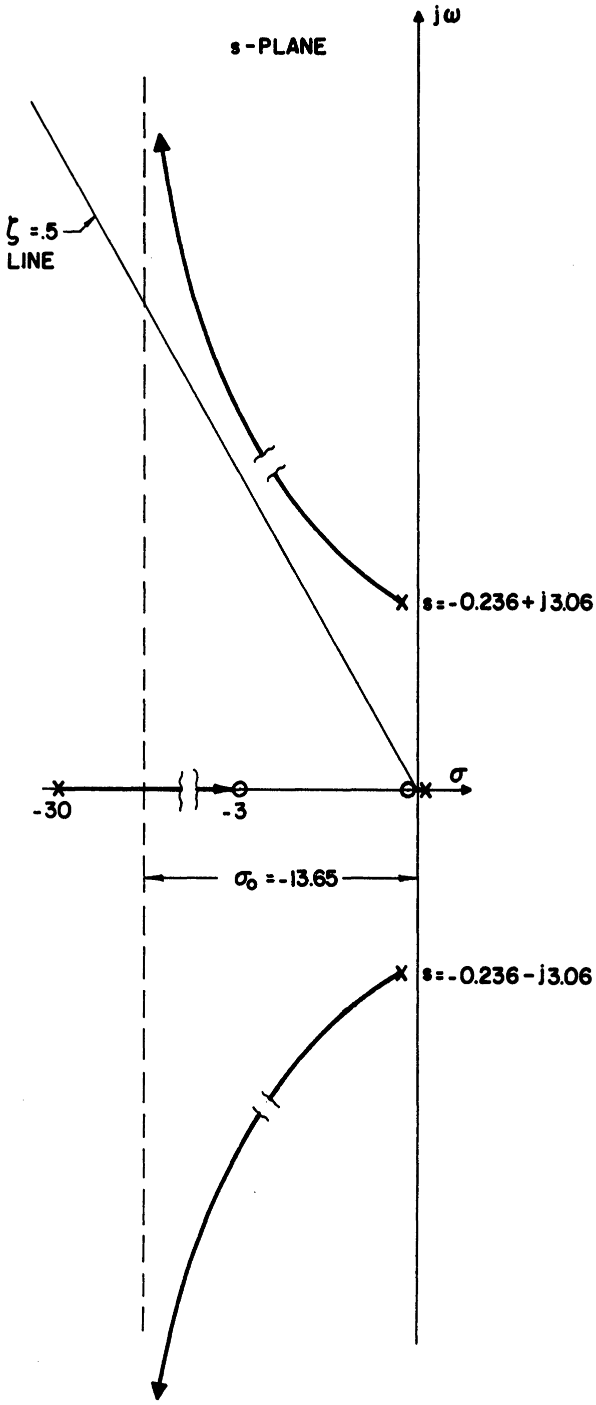

Figure 4.21 Root-Locus Plot of Compensated System

As an example of the advantages of utilizing a conditionally stable system to achieve a prescribed performance, consider the double-loop system shown in block diagram form in Figure 4.22. The physical form of this system may consist of the following possible types of components:

K_1 = an amplifier 4.8-2

$$\frac{K_2}{s(s+1)} = \text{a field controlled motor} \qquad 4.8\text{-}3$$

sK_t = an electric tachometer 4.8-4

$$\frac{s^3 K_3}{(s+1)^2(1+.25s)} = \text{an active filter} \qquad 4.8\text{-}5$$

The performance of this system will be studied under various operating conditions. First consider the system, if it were operated without the tachometer feedback loop. This is equivalent to opening the tachometer loop at point, p, (see Figure 4.22). The resulting system is now a single-loop second-order system, whose performance is defined by the following open-loop and closed-loop functions:

$$\frac{C}{E}(s) = \frac{K_1 K_2}{s(s+1)} \qquad 4.8\text{-}6$$

$$\frac{C}{R}(s) = \frac{K_1 K_2}{s^2 + s + K_1 K_2} \qquad 4.8\text{-}7$$

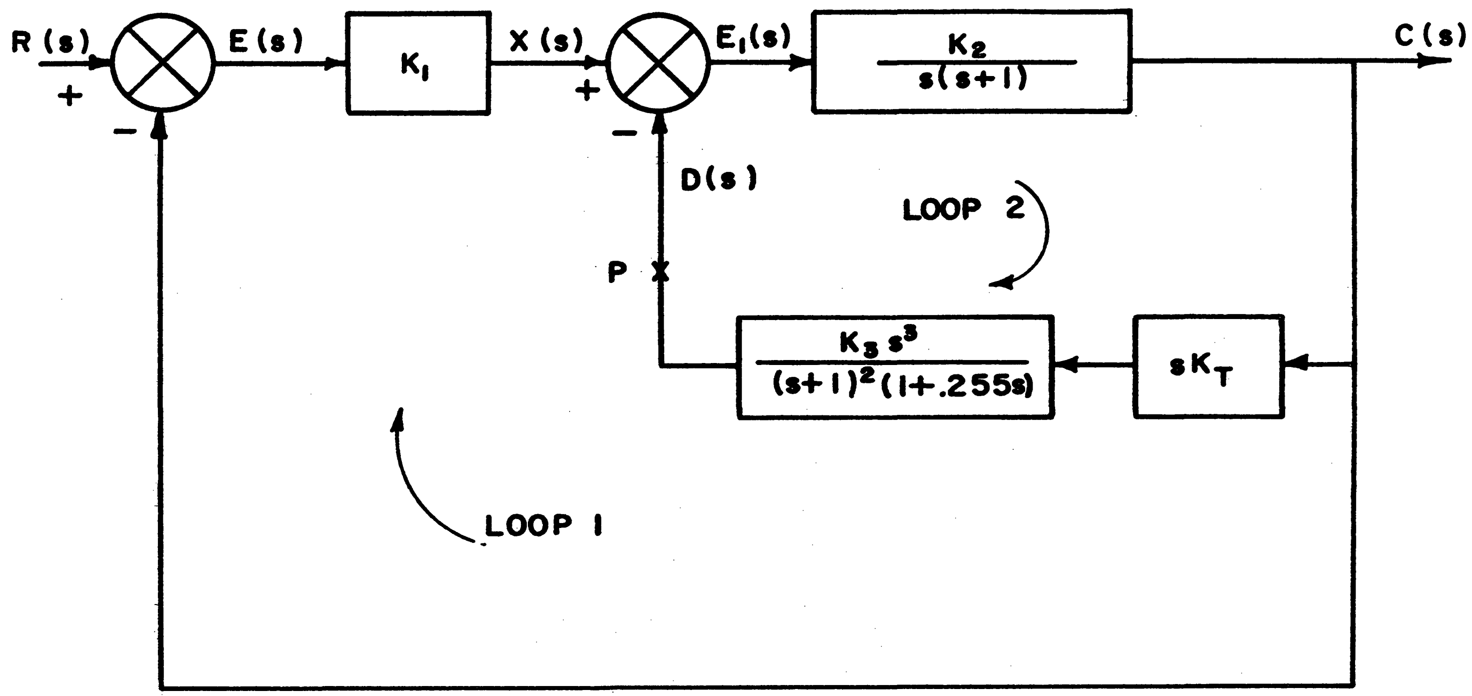

Figure 4.22 Block Diagram of a Two-Loop Control System

In order to study the system's performance two methods of analysis will be applied, namely, the Nyquist diagram and the root-locus technique. The results obtained in this fashion are shown in Figures 4.23 and 4.24. From Figure 4.23 the following information is obtained: for an $M_p = 1.3$, $K_v = 1.37\ sec^{-1}$ and $\omega_r = 0.9$. From the root-locus plot in Figure 4.24, utilizing the same K_v, the closed loop poles are found to be located at $s = -\frac{1}{2} + j1.05$ and $s = -\frac{1}{2} - j1.05$. The addition of the normal passive compensation network in cascade with the open-loop function of equation 4.8-6 will result in only a mild increase in the system K_v and ω_r, for the same M_p. Attempts to improve materially the K_v or ω_r using only a single-loop system are hopeless. If the feedback loop is now closed at point, p, an entirely different approach is being made to the problem. The system now contains two closed loops (labelled 1 and 2 in Figure 4.22). It is now possible to vary the over-all system performance over a wide range by adjusting the gain, K_4, (see equation 4.8-8) in loop 2. In order to demonstrate the type of performance that may be obtained, the over-all performance will be evaluated for two specific values of K_4.

Consider now the two-loop system shown in Figure 4.22 with point, p, closed. The equations governing the behavior of the system under these conditions are:

$$\frac{D}{E_1}(s) = \frac{K_2\ K_2\ K_t\ s^3}{(s+1)^3\ (1 + .25\ s)} \qquad \text{or}$$

$$\frac{D}{E_1}(s) = \frac{K_4\ s^3}{(s+1)^3\ (s+4)} \qquad 4.8\text{-}8$$

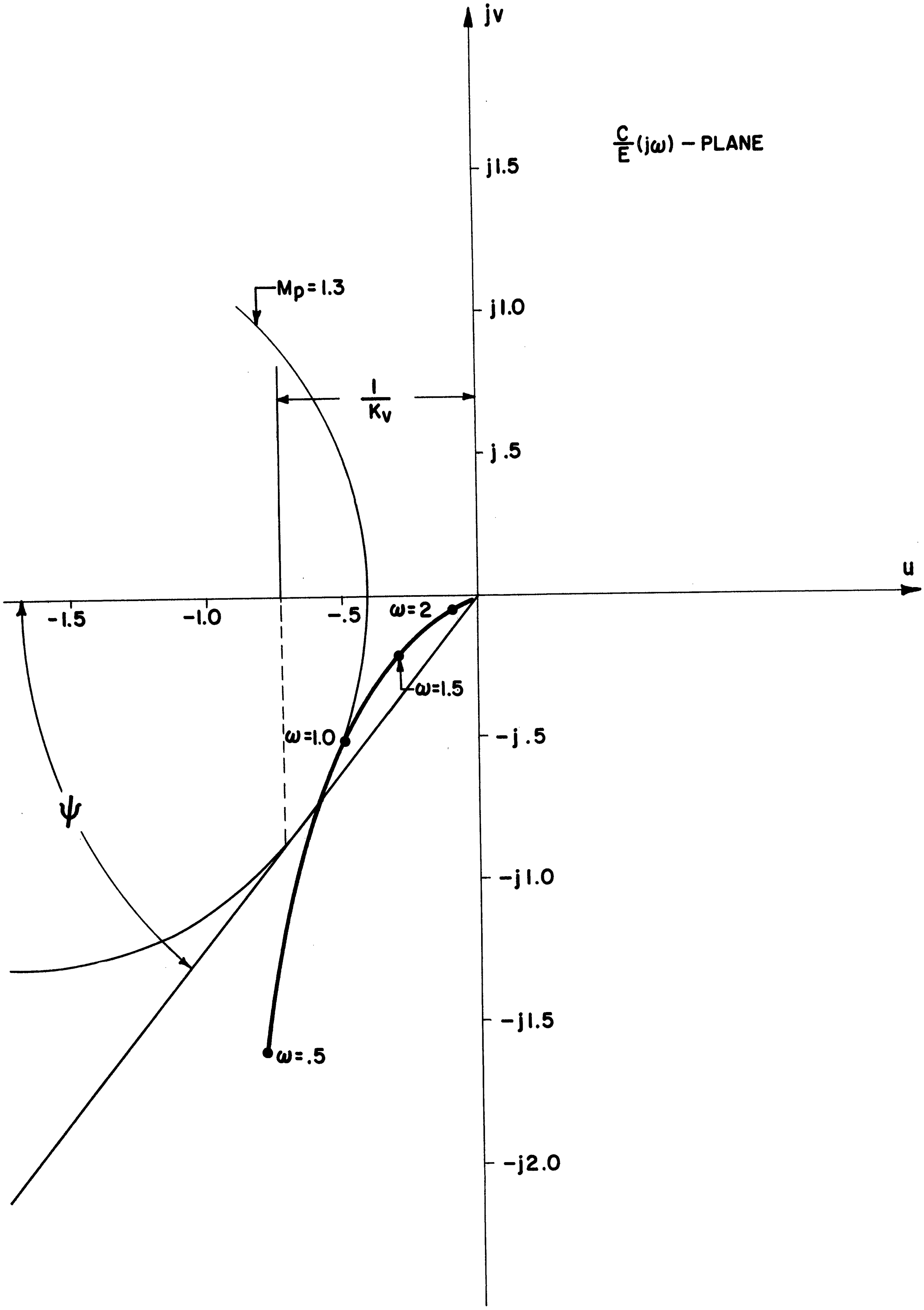

Figure 4.24 Nyquist Diagram for System of Figure 4.22 with p open

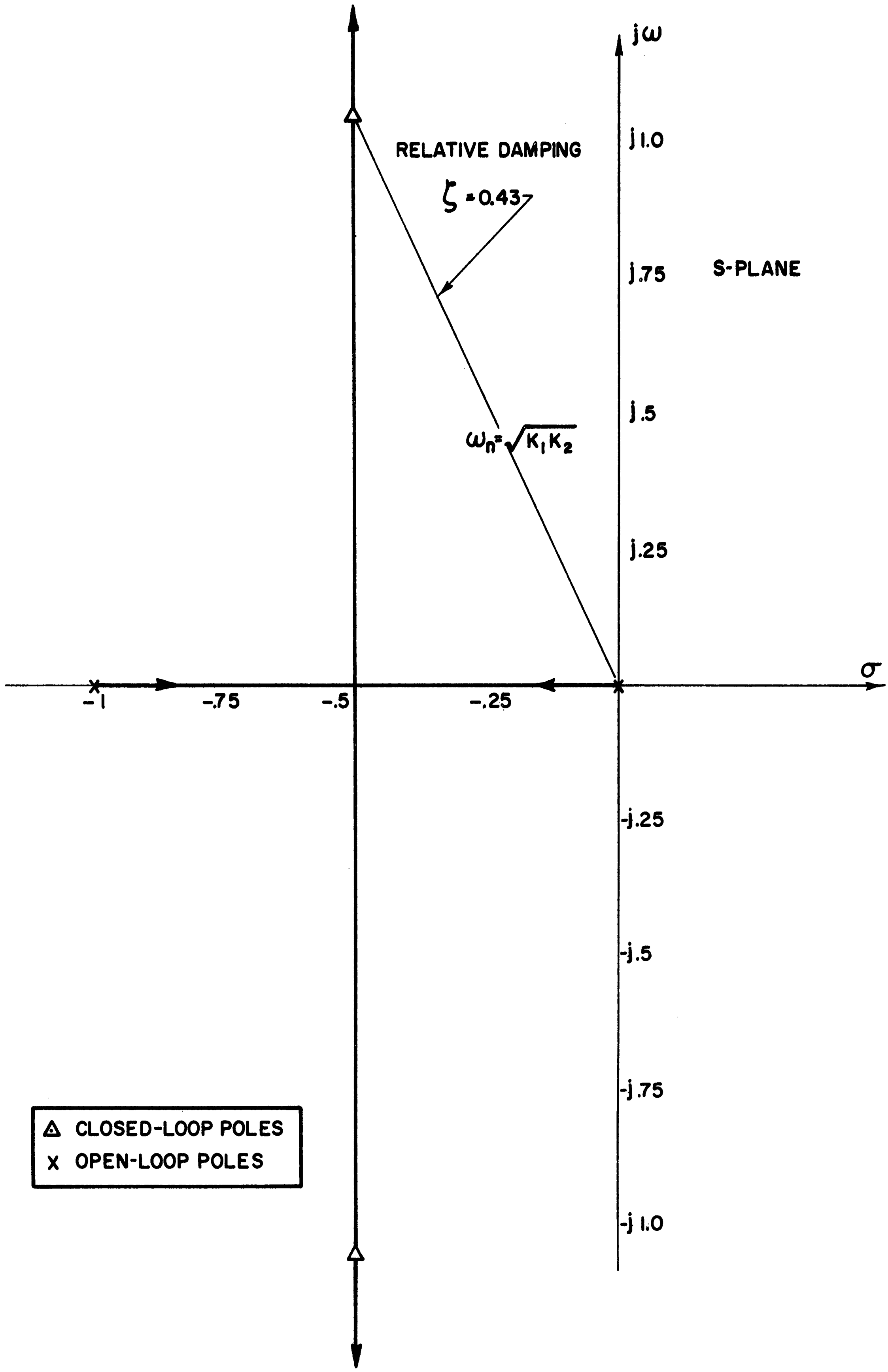

Figure 4.23 Root-Locus Plot of System of Figure 4.22 with p open

and

$$\frac{C}{E}(s) = \frac{K\,(s+1)^2\,(s+4)}{s\left[s^4 + (7+K_4)\,s^3 + 15\,s^2 + 13\,s + 4\right]} \qquad 4.8\text{-}9$$

The behavior of the system is now investigated by first studying the behavior of the inner loop using a Nyquist diagram, Bode diagrams, and the root-locus plot. These plots are shown in Figures 4.25, 4.26a, 4.26b, 4.26c, and 4.27. The three methods of analysis are carried along more or less in a parallel fashion as in the case of the single loop, primarily for comparison.

Inspection of Figure 4.25 which contains the Nyquist diagram, reveals that for values of gain $K_4 > 12$ the system will encircle the $-1 + j0$ point in the $\frac{D}{E_1}(j\omega)$ plane two times as ω is varied from $-\infty$ to $+\infty$. Applying Nyquist's criterion in its most general form, namely, $Z = N-P$. to this diagram indicates that the inner loop system contains two right half plane closed-loop poles.

Figure 4.27 shows that this closed loop system contains two real left half plane poles and two complex poles which move from the left half plane to the right half plane as the system gain, K_4, is increased from 0 to ∞. For the value of $K = 12$, two branches of the root-locus lie on the $j\omega$ axis. This corresponds to the $\frac{D}{E_1}(j\omega)$ locus passing through the $-1 + j0$ point in the Nyquist diagram.

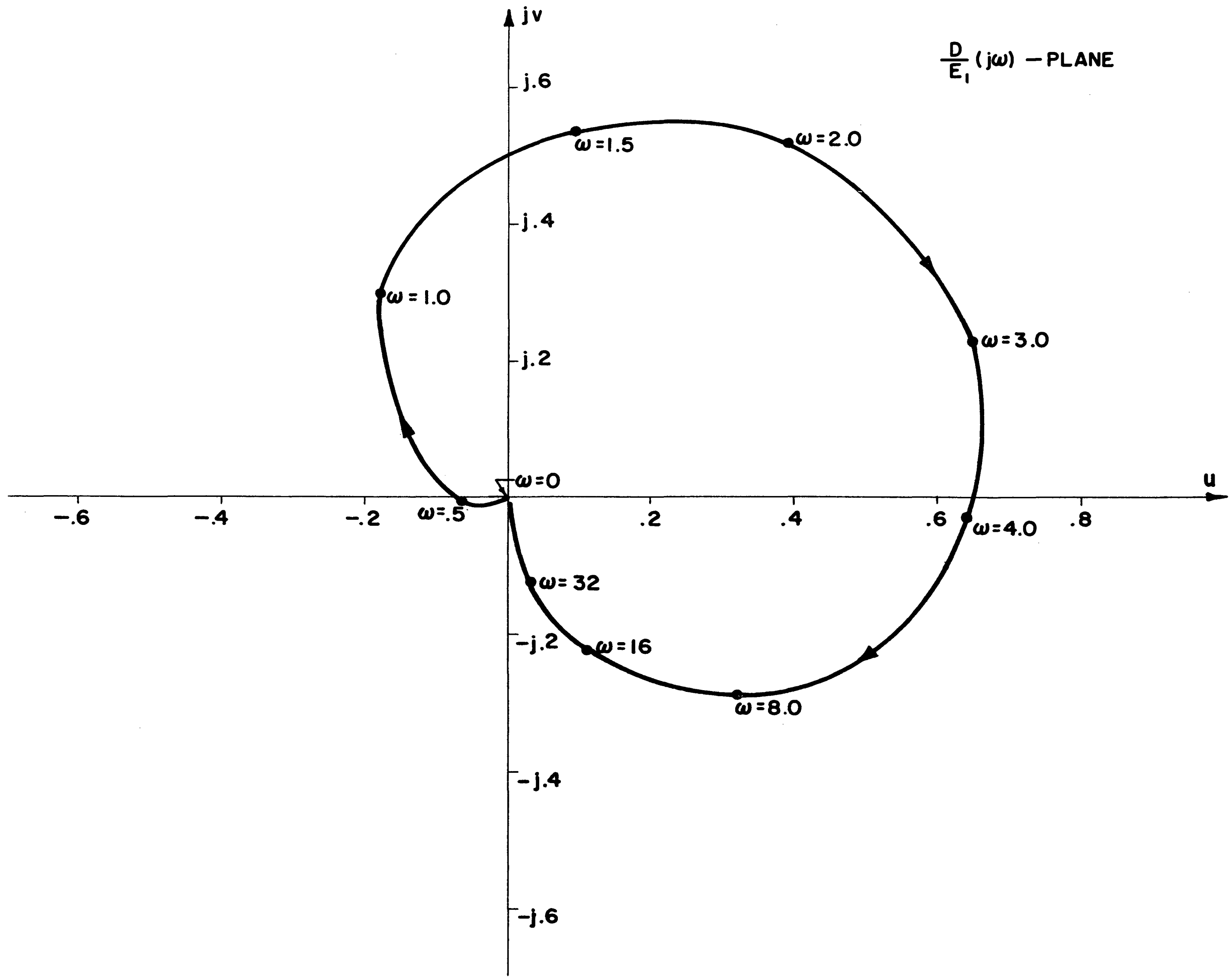

Figure 4.25 Nyquist Diagram of $\frac{D}{E_1}(j\omega)$

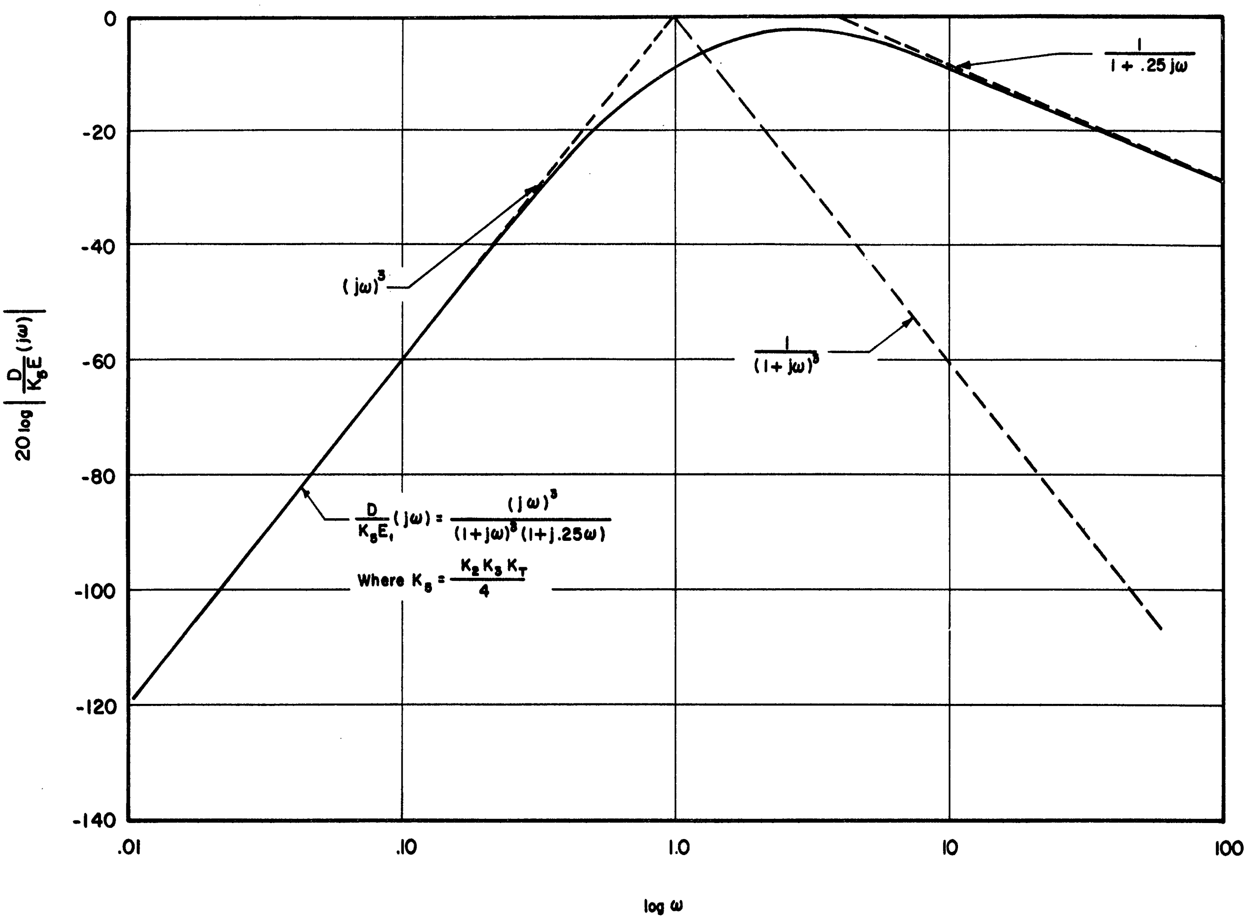

Figure 4.26a Log Amplitude Response of $\frac{D}{K_5E_1}$ (jω) vs. Log ω

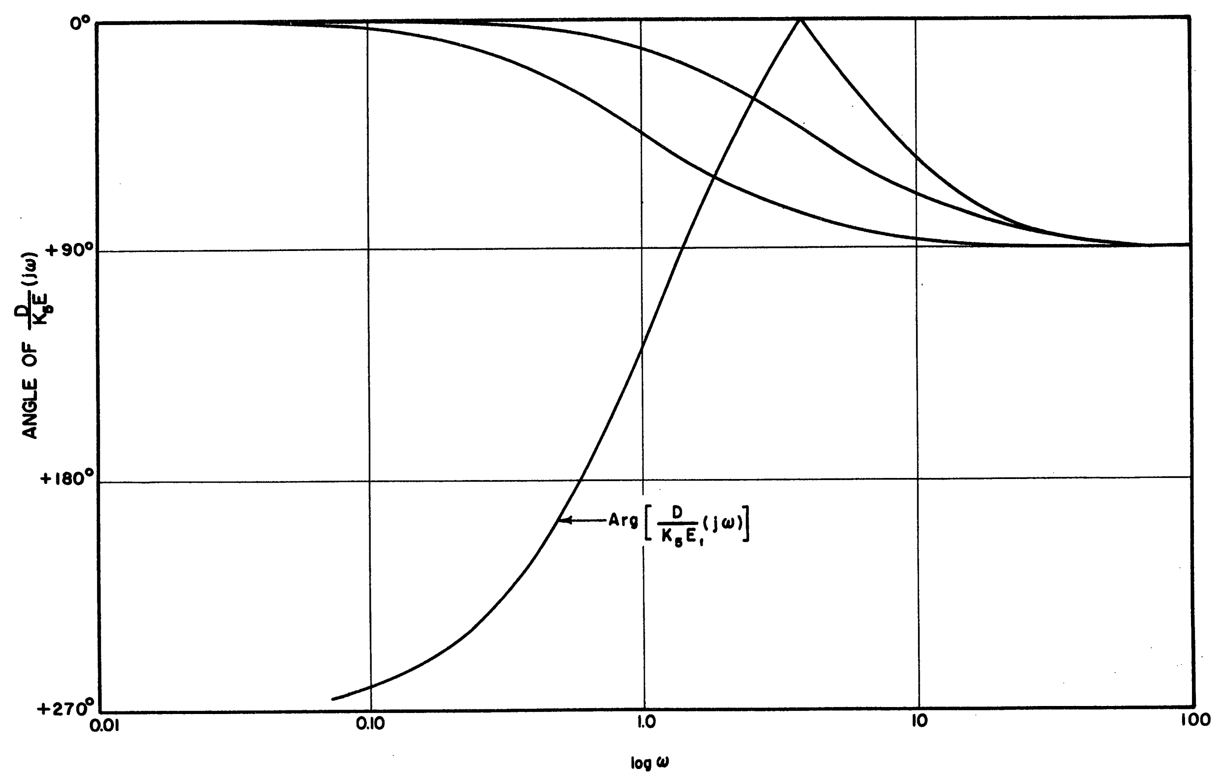

Figure 4.26b Argument of $\frac{D}{K_5E_1}$ (jω) vs. Log ω

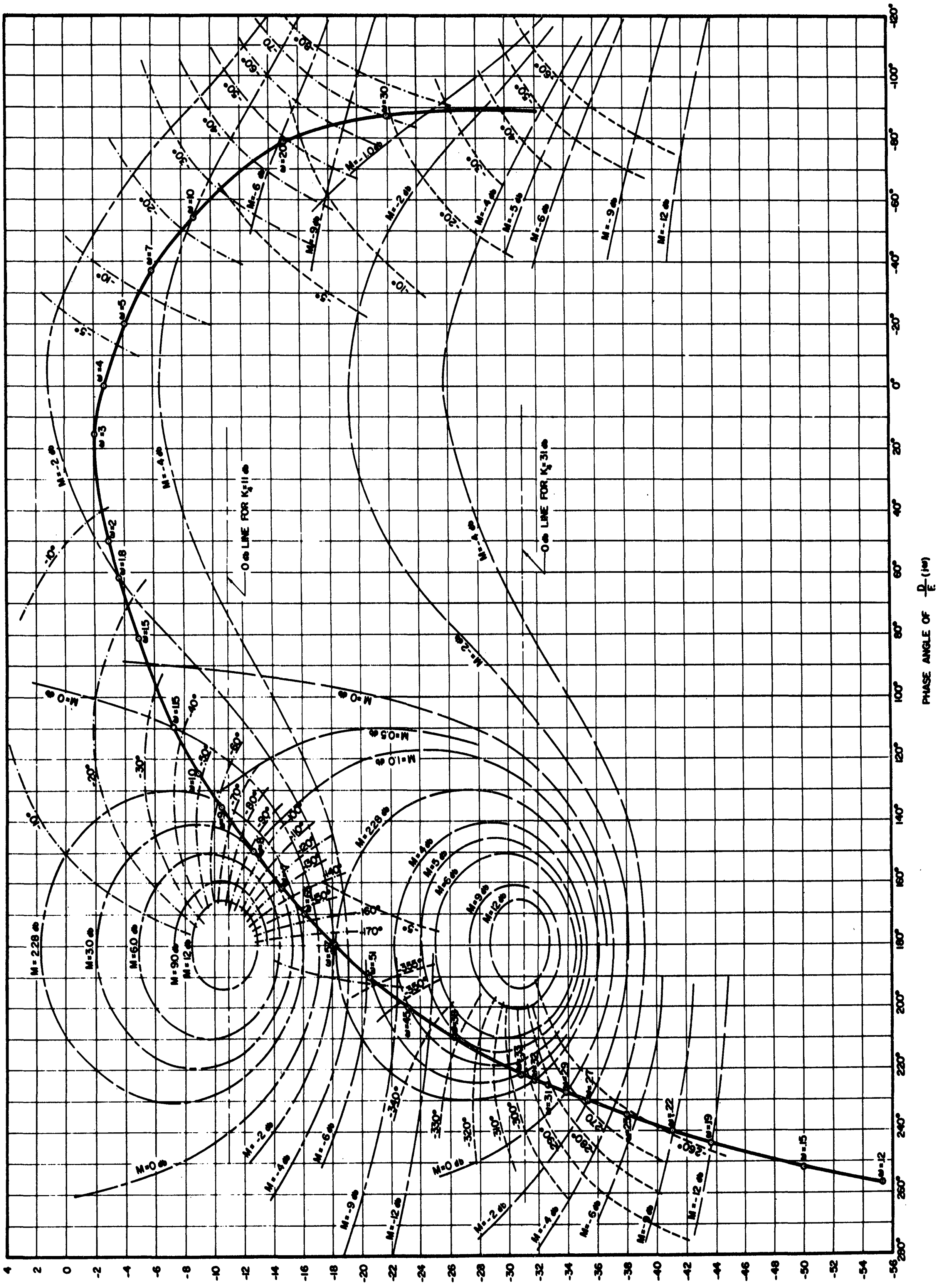

Figure 4.26c Nichols Diagram of $\frac{D}{E_1}(j\omega)$ for Two Different Values of K_4

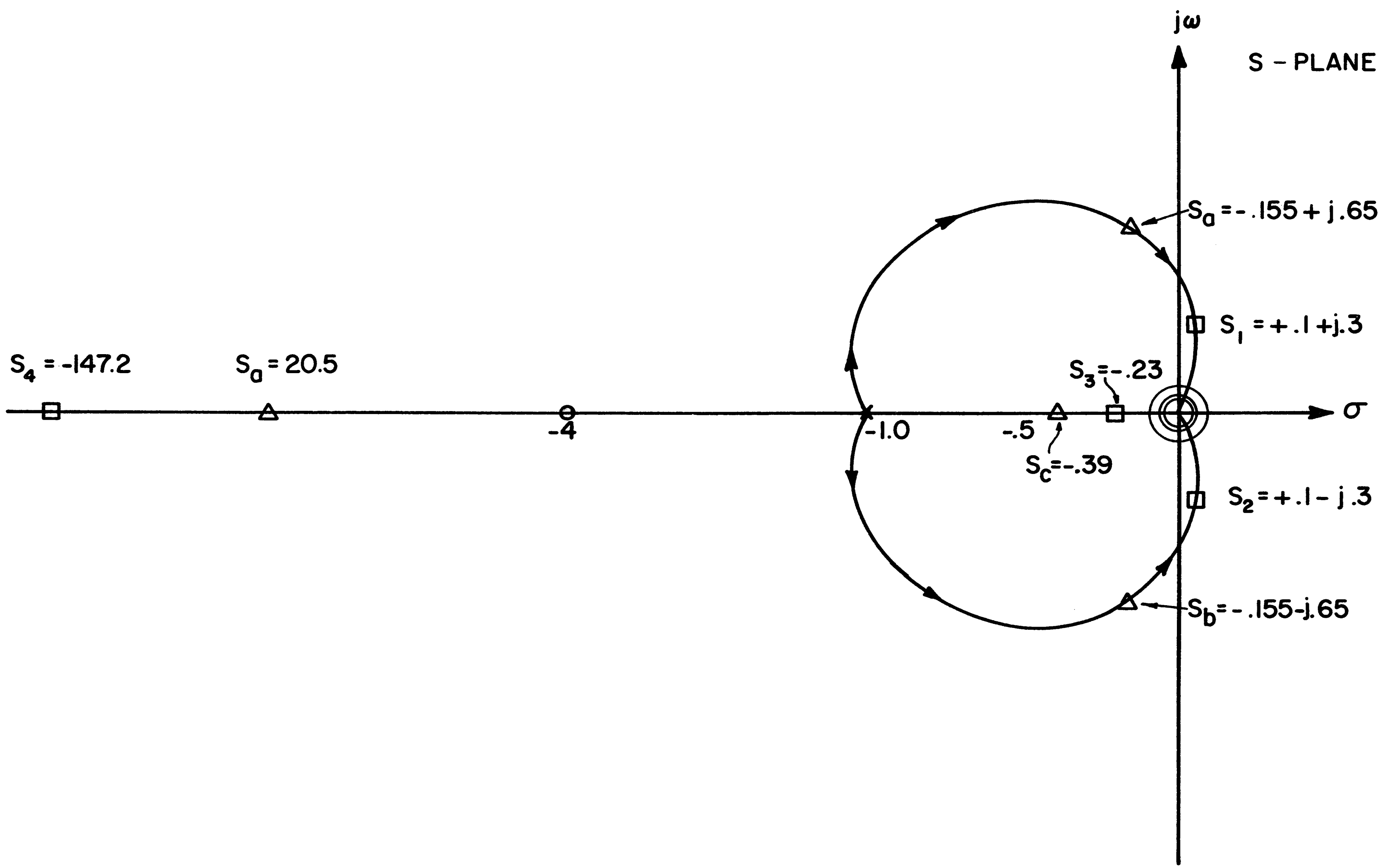

Figure 4.27 Root-Locus Plot of the Inner Loop Function $\frac{D_1}{E_1}$ (s)

The performance of the over-all system will now be studied for two specific values of inner-loop gain, namely, $K = 3.55$ and $K = 35.5$. For $K = 3.55$ equation 4.8-9 in factored form becomes:

$$\frac{C}{E}(s) = \frac{K(s+1)^2(s+4)}{s(s+20.5)(s+.155+j.65)(s+.155-j.65)} \qquad 4.8\text{-}10$$

while for $K = 35.5$ equation 4.8-9 takes on the following form

$$\frac{C}{E}(s) = \frac{K(s+1)^2(s+4)}{s(s+147.2)(s-.1+j.3)(s-.1-j.3)} \qquad 4.8\text{-}11$$

A study of equation 4.8-10 and 4.8-11 reveals that changing the gain by 20 db has resulted in the following modifications:

1) Moving the complex poles of $\frac{C}{E}(s)$ from the left half plane to the right half plane.
2) Moving the real axis pole much farther into the left half plane.

In Figure 4.28 will be found the Nyquist diagrams corresponding to the two values of gain, K_4. It is apparent from an inspection of the diagrams that the overall open-loop behavior is completely different in the two cases. This is due to the pair of right half plane poles in the open-loop function in equation 4.8-11 which causes a large amount of phase shift in the $\frac{C}{E}(j\omega)$ function.

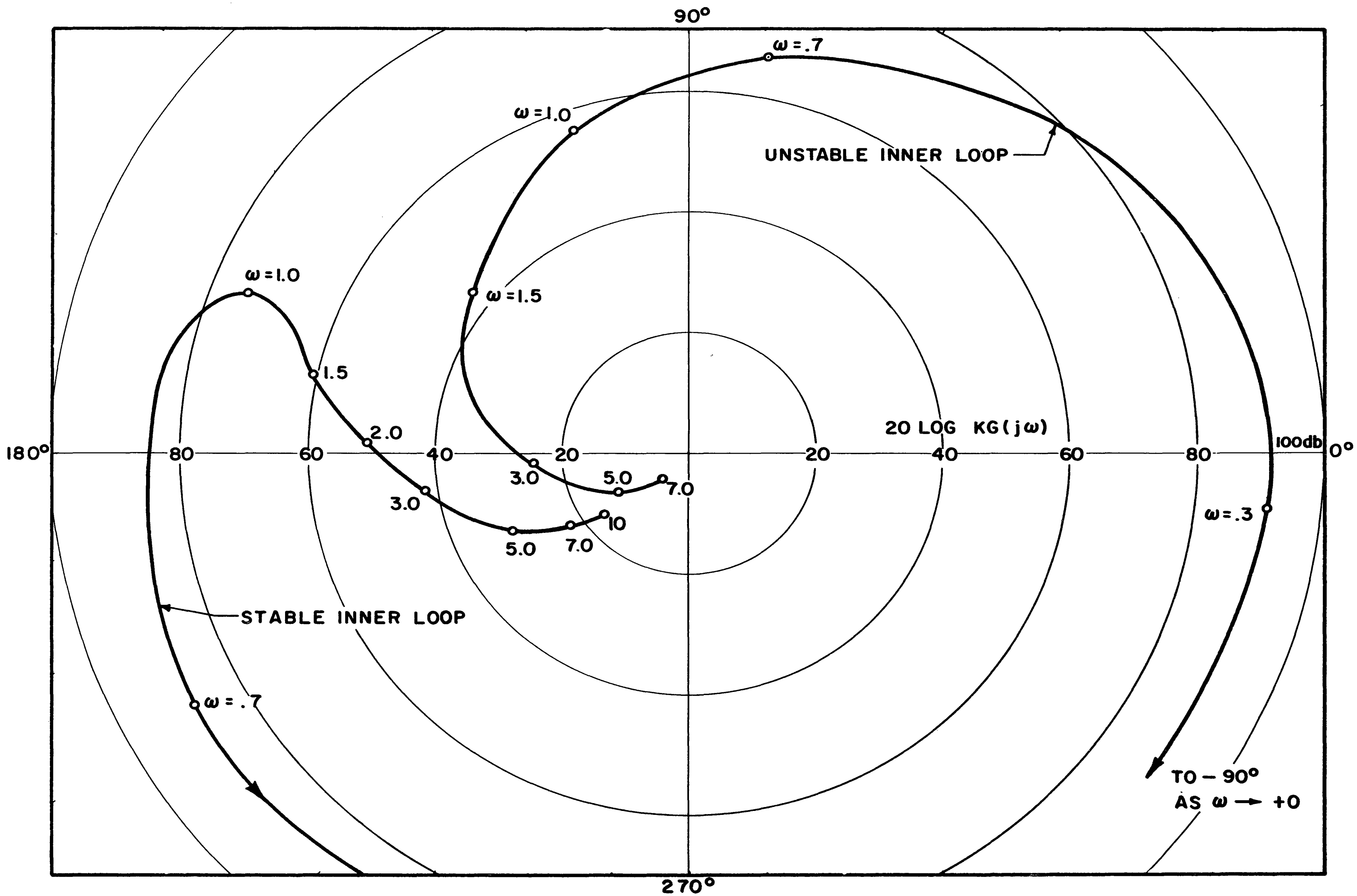

Figure 4.28 Nyquist Diagram of $\frac{C}{E}(j\omega)$ for System of equation 4.8-9

The closed-loop behavior of the over-all system will not be investigated using the Nyquist diagram, and the associated M circles. The closed-loop frequency response for these two values of K_4 gain setting are shown in Figure 4.28 for K adjusted to give an $M_p = 1.3$ in each case. The velocity constant for this system when adjusted in this fashion is $K_v = 150 \text{ sec}^{-1}$, for $K_4 = 3.55$ and $K_v = 10^4 \text{sec}^{-1}$ for $K_4 = 35.5$. Thus, by adding an unstable inner loop, the K_v of the system has been increased by more than sixty fold. From Figure 4.28 it is apparent that the resonant frequency of the two systems is also widely different.

In Figures 4.29a and 4.29b will be found the root-locus diagram for the system when operating under the above conditions. Inspection of the diagrams indicates the presence of the right half plane open-loop poles cause the complex branches of the root-locus to move rapidly far into the left half plane as the gain, K, is increased. A constant damping line, $\zeta = .5$ for the two cases indicate that much larger values of K_1 can be used before the system's complex closed-loop poles cross this line for the case of $K_4 = 35.5$ than for $K_4 = 3.55$. This yields, therefore, a large velocity constant and a correspondingly smaller steady-state velocity error. In addition, moving the closed-loop complex poles farther into the left-half plane results in a system having a large bandwidth.

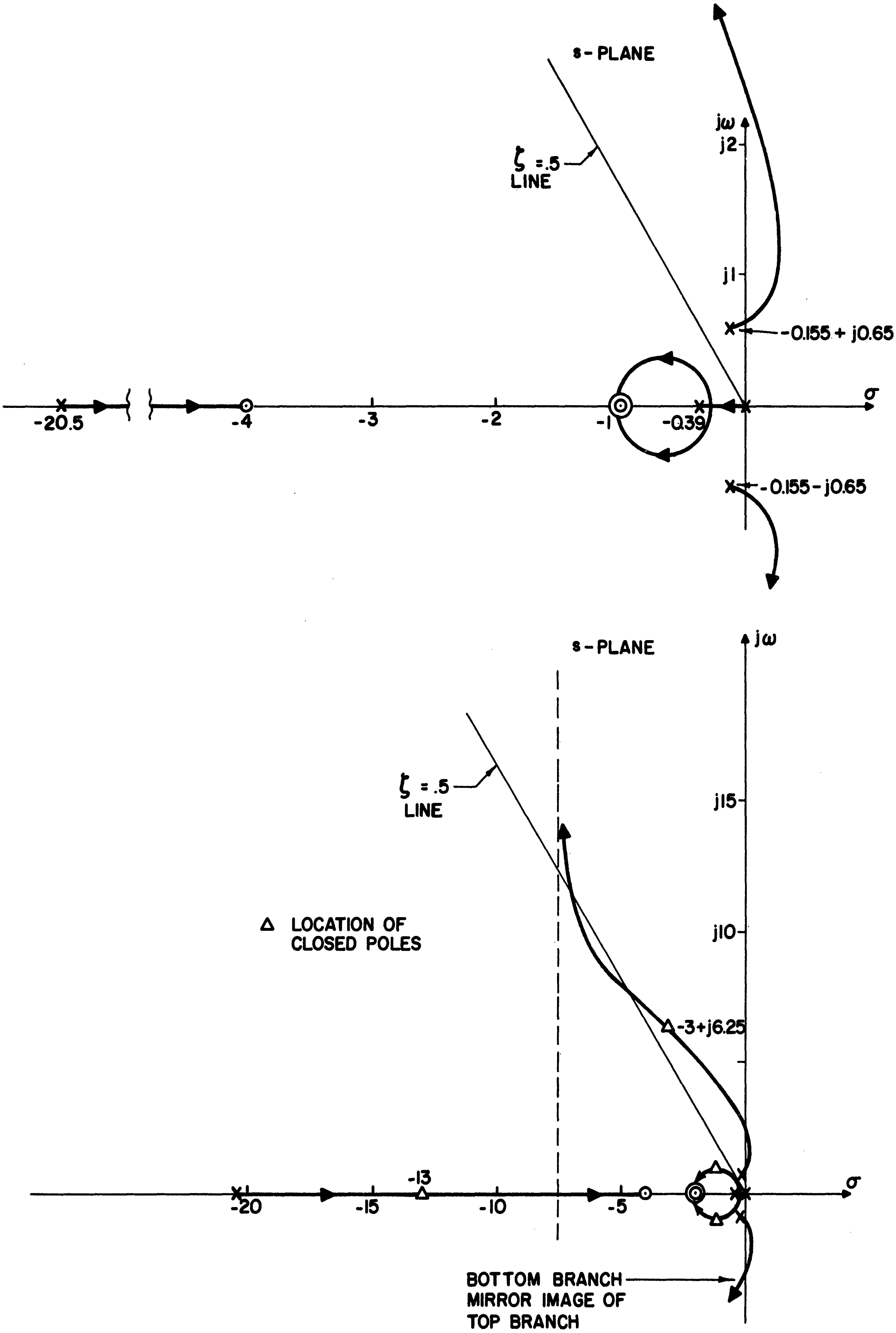

Figure 4.29a Root Locus Plot for System of Equation 4.8-9 with $K_4 = 3.55$

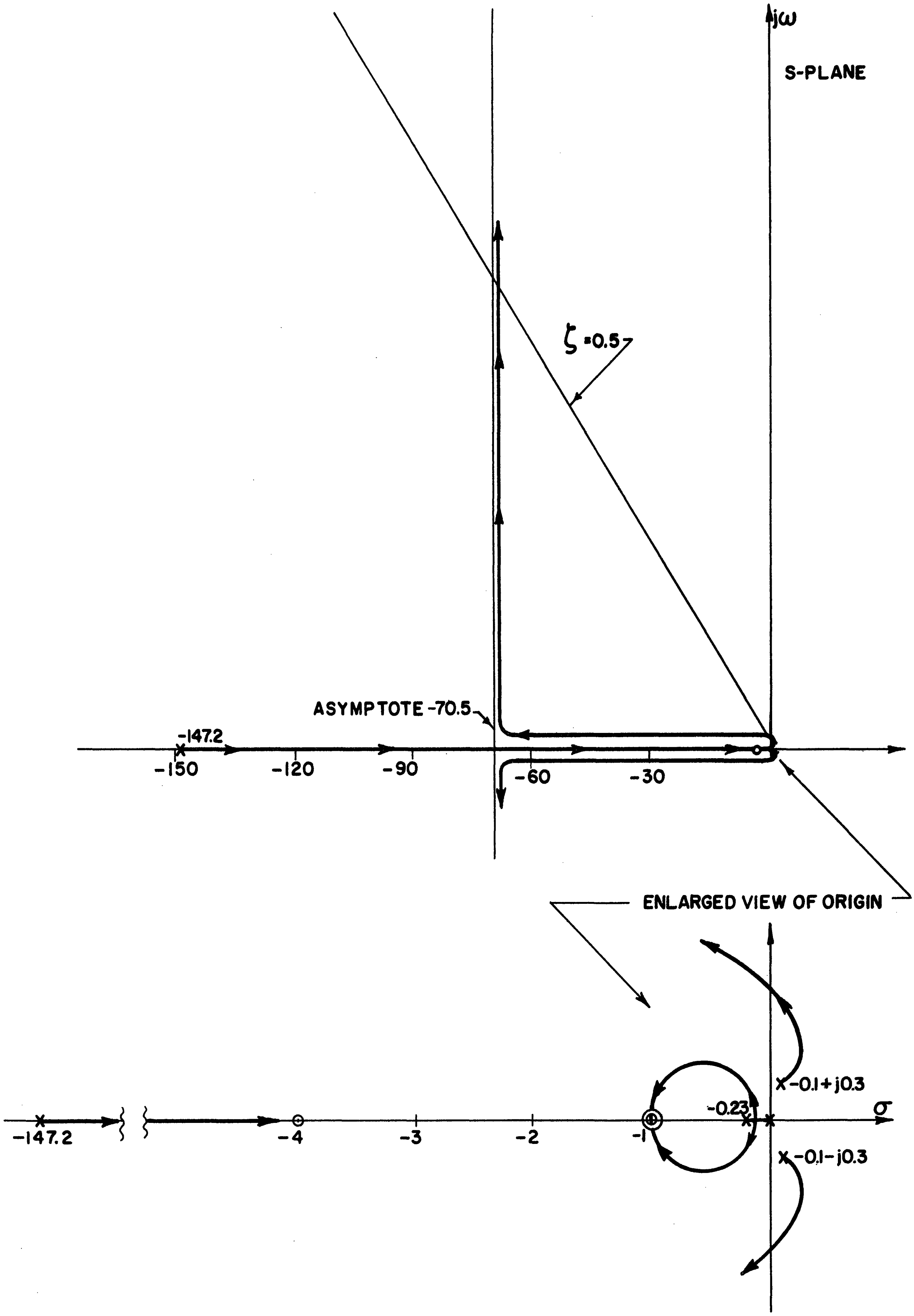

Figure 4.29b Root Locus Plot for System of equation 4.8-9 with K_4 = 35.5

CHAPTER V
CONCLUSIONS

The major objective of this dissertation was to establish a better understanding regarding the characteristics of conditionally stable systems. Although these systems are found discussed in technical articles and texts on feedback control systems, no organized treatment of the subject was to be found. The subject matter was introduced with a brief discussion of control system terminology and methodology. This was followed by a discussion (see Section 3.2) of the meaning of a conditionally stable system, where it was shown that the word conditionally stable has been used to define two types of systems possessing widely different characteristics. In this work the definition of a conditionally stable system is <u>one which becomes unstable as the loop gain is decreased from some maximum value to zero.</u> Utilizing this definition allows it to include systems that possess both of the above characteristics. Also to be found in this chapter are the results of an extensive study of conditionally stable second-order systems. The insight in studying these systems was then utilized in the study of n-th order systems, which culminated in a set of sufficient conditions for conditionally stable n-th order systems. More specifically, it was shown that if the closed-loop function is constrained to be a minimum phase function, the open-loop function could be catagorized into three types, A, B, C. Sufficient conditions were found for types A and B systems which would guarantee stable operation of these systems for some specified loop gain, K.

In addition, in Section 2.8 it will be found a description of a novel method for solving for the roots of an n-th degree polynomial utilizing a modified root-locus technique, which to the best of the author's knowledge has not been published. While in Section 4.4 will be found a discussion regarding the intersection of root-locus branches, where it was shown that it was impossible for root-locus branches not possessing the same argument condition to cross except at a pole or zero of the open-loop transfer function. Section 4.7 discusses the relation between saddle points of potential theory and the breakaway points of the root-locus. Three dimensional contour plots, indicate that breakaway points of a root-locus may be either maxima or minima of the $\frac{C}{E}(s)$ function. It was shown that setting $\frac{C}{E}(s)' = 0$ is a convenient method for locating the breakaway points on the root-locus of a low-order transfer function. Since not all the values of s determined in this fashion are breakaway points, the approximate shape of the root-locus must be known beforehand. For higher order systems the determination of s requires the solution of a high order polynomial in s.

This leaves some question as to the desirability of utilizing this approach if extreme accuracy is not needed. Although much of the emphasis in the control system theory at the present time is directed toward nonlinear systems, there are numerous topics in the realm of linear control theory which have not been adaquately treated. These include

1) An adequate method of describing the requirements about a given physical system, so that the synthesis problem becomes unique,
2) Extending the ideas of the root-locus so that it can be more readily adapted to higher order systems,
3) To utilize more of the complex variable theory which is probable available and known to the mathematicans in order to enhance our understanding of systems.

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