ACYCLIC CONTINUA IN THE PLANE AND QUASI-COMPLEXES

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Quasi-complexes, defined by Lefschetz [2, p. 323], form an extensive class of spaces to which the Lefschetz fixed point formula applies. This class includes all finite Euclidean complexes, all parallelotopes, and all snake-like continua [1, p. 667]. One might hope to prove that all acyclic plane continua possess the fixed point property by showing that they are quasi-complexes. It is the purpose of this paper to show that there exist acyclic plane continua which are not quasi-complexes. Indeed, a plane continuum may be T-like and yet fail to be a quasi-complex.

In this paper <u>covering</u> is used to mean <u>finite open covering</u>. An ϵ-covering of a metric space is a covering such that the diameter of each element of the covering is less than ϵ. For coverings A_1, A_2 such that A_1 refines A_2 ($A_1 > A_2$), $N(A_i)$ denotes the complex which is the nerve of the covering A_i and $\pi(1,2)$ denotes one of the projection chain maps

$$\pi(1,2): \quad N(A_1) \to N(A_2)$$

induced by the inclusion map (see [2, pp. 244, 245]). For $\underline{a}$ an element of the covering A, $St\underline{a}$ is the union of all those elements of A which intersect $\underline{a}$. A_1 star refines A_2 ($A_1 \gg A_2$) if the covering C, whose elements are the sets $St\underline{a}$ for all $\underline{a}$ which are elements of A_1, refines A_2.

A simplex σ belongs to a chain C if the coefficient of σ in C is not zero. $K(\sigma)$ denotes the point set which is the intersection of the vertices of σ and $K(C)$ is the union of the sets $K(\sigma)$ for all the simplices which belong to C.

A finite collection of open sets which can be ordered, $a_1, a_2, \ldots a_n$, so that a_i intersects a_j if and only if $|i-j| \leq |$ is called a <u>snake</u>. If the collection is so ordered, we refer to it as an <u>ordered snake</u>. A metric continuum is snake-like if for each positive ϵ it has an ϵ-covering which is a snake; that is, such that the nerve of the covering is an arc. Similarly, a metric continuum is T-like if for each positive ϵ it has an ϵ-covering whose nerve is a "T."

A quasi-complex, to be denoted by QC, is a compact Hausdorff space, X, for which there exists a cofinal family of coverings, say $\{U_\alpha\}$, $\alpha \in A$, partially ordered by refinement, such that in addition to the projection chain maps,

$\pi(\beta,\alpha)$, $\beta > \alpha$, there exists a collection of antiprojection maps

$$\omega(\alpha,\beta):\ N(U_\alpha) \to N(U_\beta),\ \beta > \alpha$$

with the properties:

1) the antiprojections are chain maps,
2) given α, there is an $f(\alpha) > \alpha$ such that an antiprojection $\omega(\alpha,f)$ exists,
3) if $g > f > \alpha$ and there exist antiprojections $\omega(\alpha,f)$ and $\omega(f,g)$, then the composition map $\omega(f,g)\ \omega(\alpha,f)$ is an antiprojection,
4) given two antiprojections $\omega(\alpha,\beta)$ and $\overline{\omega}(\alpha,\beta)$, then $\omega(\alpha,\beta)$ is homologous to $\overline{\omega}(\alpha,\beta)$,
5) $\omega(\alpha,\beta)\ \pi(\beta,\alpha)$ is homologous to the identity chain map,
6) given α, there is a $g=g(\alpha) > \alpha$ and given $\underline{b}$, there is an $h=h(\alpha,g,b) > b,g$ such that antiprojections $\omega(g,h)$ exist and the collection $\{K(\sigma) \cup K[\omega(g,h)\sigma]\}$, for all simplices σ belonging to $N(U_g)$, refines U_α.

(All indices belong to the set A.)

<u>Lemma</u>: Let $\{O_i\}$ denote a cofinal sequence of coverings of the QC X, such that O_{i+1} refines O_i and the maps

$$\pi:\ H[N(O_j)] \to H[N(O_i)]$$

induced by the projection maps, are onto. ($H[N(U)]$ denotes the homology groups of $N(U)$.) If such a sequence $\{O_i\}$ exists, then there exists a subsequence, say $\{V_i\}$, such that antiprojections may be defined among the $N(V_i)$, which with their projections show that X is a QC.

<u>Proof</u>: For a properly chosen subsequence of the $\{O_i\}$ we construct the required antiprojections and show that this system satisfies properties 1)-6).

Since X is a QC, there exists a cofinal family of coverings $\{U_\alpha\}$ and associated projection and antiprojection maps which satisfy properties 1)-6).

We construct a cofinal sequence $\{C_i\}$ of coverings which relate the coverings U_α with the coverings O_i. In this construction, the choices are made so that C_{i+1} refines C_i. C_i is the element $O_{c(i)}$ of the sequence $\{O_i\}$ if $i=4j$ for some integer $\underline{j}$. Otherwise, C_i is an element of the collection $\{U_\alpha\}$. The $\underline{g}$ and $\underline{h}$ functions are those functions given in property 6) of the definition of a quasi-complex.

Select any U_α; call it C_{-5}. Choose C_{-1} such that $-1 = g(-5)$. Let C_0 be any one of the coverings O_i which refines C_{-1}. Choose C_1 so that it is a star refinement of C_0. Let C_2 be chosen so that $2 = g(-1)$. Our conventions require that C_2 refine C_1. This is possible, even though only one index $g(-1)$ is assumed in property 6). For if $\beta > \alpha$ then $g(\beta) > \beta > \alpha$ and the collection $\{K(\sigma) \cup K[\omega(g,h)\sigma]\}$, for all simplices σ belonging to $N(U_g)$, refines U_β which refines U_α. That is, for $\beta > \alpha$, any $g(\beta)$ is also a $g(\alpha)$.

Let $C_3 = C_2$.

Choose C_4 such that C_4 refines C_3 and O_i, $i \leq 4$, and $C(4) > C(0)$.

For $k = 4j$, $j \geq 1$, given C_k, select C_{k+1} such that C_{k+1} is a star refinement of C_k.

Choose C_{k+2} such that $k+2 = h(k-9, k-5, k+1)$.

Choose C_{k+3} such that $k+3 = g(k-1)$.

Choose C_{k+4} such that C_{k+4} refines C_{k+3} and O_i, $i \leq k+4$, and $C(k+4) > C(k)$.

By induction, this selects a sequence $\{C_i\}$. Let $V_i = C_{4i}$. Then $\{V_i\}$ is a cofinal subsequence of $\{O_i\}$. We shall denote projection maps among the C_i:

$$\pi(k,j):\ N(C_i) \to N(C_j) \quad i > j \ .$$

Thus a projection map from $N(V_i)$ to $N(V_j)$ will be written $\pi(4i,4j)$.

Given V_i, V_{i+1} we define an antiprojection map $s(i,i+1)$ from $N(V_i)$ to $N(V_{i+1})$ by

$$s(i,i+1) = \pi(4i+6,\ 4i+4)\ \omega(4i-1,\ 4i+6)\ \pi(4i,\ 4i-1) \ .$$

The antiprojections are all finite products of the maps

$$s(i,i+1); \quad s(i,j) = s(j-1,j) \ldots s(i,i+1) \ .$$

In particular, note that $\omega(4i-1,\ 4i+6)$ exists, since

$$4i+6 = h(4i-5,\ 4i-1,\ 4i+5) \ .$$

These antiprojections are chain maps, since they are defined as a product of chain maps, so that 1) is satisfied.

Requirement 2) is satisfied by letting $f(i) = i+1$.

Requirement 3) follows by the definition of our antiprojections.

To prove 4), suppose that we are given

$$s(i,i+1) = \pi(4i+6,\ 4i+4)\ \omega(4i-1,\ 4i+6)\ \pi(4i,\ 4i-1) \quad \text{and}$$

$$\bar{s}(i,i+1) = \bar{\pi}(4i+6,\ 4i+4)\ \bar{\omega}(4i-1,\ 4i+6)\ \bar{\pi}(4i,\ 4i-1) \ .$$

That $\pi(4i, 4i-1)$ is homologous to $\bar{\pi}(4i, 4i-1)$ **and** $\pi(4i+6, 4i+4)$ is homologous to $\bar{\pi}(4i+6, 4i+4)$ is a well-known property of the projection maps [2, p. 245], and that $\omega(4i-1, 4i+6)$ is homogolous to $\bar{\omega}(4i-1, 4i+6)$ is assured since 4) holds for the $\{U_\alpha\}$ and their antiprojections. Since product maps of homologous maps are homologous, $s(i,i+1)$ is homologous to $\bar{s}(i,i+1)$, and therefore finite products of these antiprojections are homologous. Hence, 4) is satisfied.

To prove 5), we use the requirement that the homology maps, induced by the projections, are onto. This is the only use that is made of this requirement. We designate the identity chain map by I. That two maps, or two chains, are homologous is stated by $A \sim B$. We first show that $s(i,i+1)\ \pi(4i+4, 4i) \sim I$. Let Z be any cycle of $N(V_{i+1})$. Since the projection map $\pi(4i+8, 4i+4)$ induces a homology map which is onto, there exists a cycle, say Z_1, of $N(V_{i+2})$ such that

$$\pi\ Z_1 = \pi(4i+6, 4i+4)\ \pi(4i+8, 4i+6)\ Z_1 = \bar{Z} \sim Z \ .$$

Let Z_2 denote $\pi(4i+8, 4i+6)\ Z_1$. Since $\bar{Z} \sim Z$:

$$s(i,i+1)\ \pi(4i+4, 4i)\ \bar{Z} \sim s(i,i+1)\ \pi(4i+4, 4i)\ Z \ .$$

But:

$$s(i,i+1)\ \pi(4i+4, 4i)\ \bar{Z} = \pi(4i+6, 4i+4)\ \omega(4i-1, 4i+6)\ \pi(4i+6, 4i-1)\ Z_2$$

and

$$\omega(4i-1, 4i+6)\ \pi(4i+6, 4i-1) \sim I \quad \text{by 5) for } \{U_\alpha\}.$$

Thus

$$s(i,i+1)\ \pi(4i+4, 4i)\ \bar{Z} \sim \pi(4i+6, 4i+4)\ Z_2 \quad \text{and} \quad \pi(4i+6, 4i+4)\ Z_2 = \bar{Z} \ .$$

Therefore

$$s(i,i+1)\ \pi(4i+4, 4i)\ Z \sim \bar{Z} \sim Z \ .$$

That is,

$$s(i,i+1)\ \pi(4i+4, 4i) \sim I \ .$$

In general:

$$s(j,k)\ \pi(4k, 4j) \sim [s(k-1,k) \dots s(j,j+1)\ \pi(4j+4, 4j) \dots \pi(4k, 4k-4)]$$
$$\sim [s(k-1,k) \dots s(j+2, j+1)\ \pi(4j+8, 4j+4) \dots \pi(4k, 4k-4)] \sim \dots$$
$$\sim [s(k-1,k)\ \pi(4k, 4k-4)] \sim I \ .$$

Thus, the $s(j,k)$ satisfy 5).

To prove 6) we define $g(i) = i+2$, and given V_b let $h(i,g,b)$ be the maximum of $g+1$, b. Then $s(g,h)$ exists since $s(i,j)$ exists for $i < j$. Let σ be an arbitrary, fixed simplex which belongs to $N(V_g)$. The proof that 6) is satisfied will be complete if we can show that there exists an open set $V_{i,o}$ which is an element of V_i such that $K(\sigma) \cup K[s(g,n)\sigma] \subset V_{i,o}$ for all n.

For generic α such that σ_α is an element of $N(V_n)$ and σ_α is also an element of $s(g,n)\sigma$, let $\overline{\sigma}_\alpha$ denote $\pi(4n, 4n-1)\sigma_\alpha$. In addition, σ_α is the image under $\pi(4n+2, 4n)$ of some simplex which is an element both of $N(C_{4n+2})$ and of $\omega(4n-5, 4n+2)\, \pi(4n-4, 4n-5)\, s(g,n-1)\sigma$. Let σ_α' denote such a simplex. Denote $\pi(4g, 4g-1)\sigma$ by $\overline{\sigma}$. Denote elements of C_i by $C_{i,k}$.

Property 6) will be established by induction. First, $\{K(\overline{\sigma}) \cup K[\omega(4g-1, 4g+6)\overline{\sigma}]\}$ refines C_{4g-5} since $4g-1 = g(4g-5)$ and $4g+6 = h(4g-5, 4g-1, 4g+5)$. Thus

$$K(\overline{\sigma}) \cup K[\omega(4g-1, 4g+6)\overline{\sigma}] \subset C_{4g-5,\ell} \quad .$$

Since

$$C_{4g+6} > C_{4g+4} = V_{g+1} > C_{4g+3} > C_{4g-5} \gg C_{4g-8} = V_i \quad ,$$

$$K(\overline{\sigma}) \cup K[\pi(4g+4, 4g+3)\, \pi(4g+6, 4g+4)\, \omega(4g-1, 4g+6)\overline{\sigma}] \subset \mathrm{St}\, C_{4g-5,\ell} \subset V_{i,o} \ .$$

Thus

$$K(\sigma) \cup K[s(g,g+1)\sigma] \subset V_{i,o} \quad .$$

Let σ_j' be an element of $\omega(4g-1, 4g+6)\overline{\sigma}$. Then

$$K(\sigma_j') \subset C_{4g-5,\ell} \quad .$$

Also

$$K(\overline{\sigma}_j) \cup K[\omega(4g+3, 4g+10)\overline{\sigma}_j] \subset C_{4g-1,a(j)}$$

since

$$4g+3 = g(4g-1)$$

and

$$4g+10 = h(4g-1, 4g+3, 4g+9) \quad .$$

Since

$$C_{4g+10} > C_{4g+8} > C_{4g+7} > C_{4g-1} \gg C_{4g-5} \quad :$$

$$K(\overline{\sigma}_j) \cup K[\pi(4g+8,4g+7)\ \pi(4g+10,4g+8)\ \omega(4g+3,4g+10)\overline{\sigma}_j] \subset \mathrm{St}\ C_{4g-1,a(j)} \subset C_{4g-5,b(j)}\ .$$

But $K(\sigma_j')$ is contained in the intersection of $C_{4g-5,b(j)}$ and $C_{4g-5,\ell}$. Therefore

$$C_{4g-5,b(j)} \subset \mathrm{St}\ C_{4g-5,\ell} \subset V_{i,o}\ .$$

Thus

$$K(\sigma) \cup \bigcup_{j\in P} \{K(\sigma_j) \cup K[s(g+1,\ g+2)\sigma_j]\} \subset V_{i,o}$$

where P is the set of all those simplices σ_j which belong to $s(g,g+1)\sigma$.

That is, $K(\sigma) \cup K[s(g,g+2)\sigma] \subset V_{i,o}$.

We are now ready for the general inductive step. Assume that $K(\sigma) \cup K[s(g,n)\sigma] \subset V_{i,o}$ where if σ_2 is an element of $s(g,n)\sigma$, there is a simplex σ_1, which belongs to $s(g,n-1)\sigma$, for which σ_2 belongs to $s(n-1,n)\sigma_1$, and is such that

$$K(\sigma_1) \cup K(\sigma_2') \subset C_{4n-9,a(\sigma_1)} \quad \text{and}$$

$$K(\overline{\sigma}_1) \cup K(\overline{\sigma}_2) \subset \mathrm{St}\ C_{4n-9,a(\sigma_1)} \subset C_{4n-13,b(\sigma_1)} \subset V_{i,o}\ .$$

($n \geq g + 2$. We have seen that these assumptions are satisfied for $n = g+2$.)

$$K(\overline{\sigma}_2) \cup K[\omega(4n-1,\ 4n+6)\overline{\sigma}_2] \subset C_{4n-5,a(\sigma_2)}$$

since

$$4n+6 = h(4n-5,\ 4n-1,\ 4n+5)\ .$$

Since

$$C_{4n+6} > C_{4n+4} > C_{4n+3} > C_{4n-5} \gg C_{4n-9}\ ,$$

for σ_3' an element of $\omega(4n-1,\ 4n+6)\overline{\sigma}_2$, we have:

$$K(\overline{\sigma}_2) \cup K(\overline{\sigma}_3) \subset \mathrm{St}\ C_{4n-5,a(\sigma_2)} \subset C_{4n-9,b(\sigma_2)}\ .$$

But

$$K(\sigma_2') \subset C_{4n-9,b(\sigma_2)} \cap C_{4n-9,a(\sigma_1)}\ ,$$

or

$$K(\overline{\sigma}_2) \cup K(\overline{\sigma}_3) \subset \mathrm{St}\ C_{4n-9,a(\sigma_1)} \subset V_{i,o}\ .$$

Thus

$$K(\sigma) \cup K[s(n,n+1)\ s(g,n)\sigma] = K(\sigma) \cup K[s(g,n+1)\sigma] \subset V_{i,o} .$$

By replacing n by $n+1$, σ_1 by σ_2, and σ_2 by σ_3 in the induction assumptions, we see that all conditions are fulfilled, and the induction is complete.

This completes the proof of the lemma.

Corollary 1: The maps

$$\pi\colon\ H[N(V_j)] \to H[N(V_i)] ,$$

which are required to be onto maps, are actually isomorphisms.

Proof: We have $s\pi \sim I$. The isomorphism can fail only if there exists a nonzero homology class Z such that $\pi Z \sim 0$. Suppose so. Then $s\pi Z = s(\pi Z) = s(0) = 0$, and $s\pi Z \sim IZ = Z$. But $Z \not\sim 0$. Therefore, $\pi Z \sim 0$ only if $Z \sim 0$, and π is an isomorphism.

Let

$$T_1 = \{(x,y) \mid x = 0,\ 0 \leqq y \leqq 2\} ,$$

$$T_2 = \{(x,y) \mid 0 \leqq x \leqq 2,\ y = 2\} ,$$

$$T_3 = \{(x,y) \mid -2 \leqq x \leqq 0,\ y = 2\} ,$$

$$T = T_1 \cup T_2 \cup T_3 \subset E^2 \text{, the plane.}$$

Let R denote a ray which spirals down on T, $R \subset E^2$.

$$\begin{aligned} R = {} & (2^{-1} \leqq x \leqq 2,\ y = -2^{-1}) \cup \bigcup_{n=1}^{\infty} \{(x = 2^{-n},\ -2^{-n} \leqq y \leqq 2 - 2^{-n}) \\ & \cup (2^{-n} \leqq x \leqq 2 + 2^{-n},\ y = 2 - 2^{-n}) \cup (x = 2+2^{-n},\ 2-2^{-n} \leqq y \leqq 2+2^{-n}) \\ & \cup (-2-2^{-n} \leqq x \leqq 2+2^{-n},\ y = 2+2^{-n}) \cup (x = -2-2^{-n},\ 2-2^{-n} \leqq y \leqq 2+2^{-n}) \\ & \cup (-2-2^{-n} \leqq x \leqq -2^{-n},\ y = 2-2^{-n}) \cup (x = -2^{-n},\ -2^{-(n+1)} \leqq y \leqq 2-2^{-n}) \\ & \cup (-2^{-n} \leqq x \leqq 2^{-(n+1)},\ y = -2^{-(n+1)})\} . \end{aligned}$$

Obviously $\bar{R} = R \cup T$. Denote $\bar{R}$ by X.

Corollary 2: There exist a T-like plane continuum which is not a QC.

Proof: X is obviously a T-like plane continuum. Under the assumption that X is a QC we will arrive at a contradiction. Let

$$P_0 = (2, -2^{-1}) ,$$

$$P_1 = (1, -2^{-1}) ,$$

$$P_2 = (0, 0) ,$$

$$P_3 = (0, 1) ,$$

$$P_4 = (0, 2) ,$$

$$P_5 = (1, 2) ,$$

$$P_6 = (2, 2) ,$$

$$P_7 = (-1, 2), \text{ and}$$

$$P_8 = (-2, 2) .$$

It is clear that for each positive ϵ, there is an ϵ-covering of X, say $U_\epsilon = U_{\epsilon,1} \cdots U_{\epsilon,n(\epsilon)}$ such that:

i) $P_i \in U_{\epsilon,P(i,e)}$, $P(i,\epsilon) < P(i+1,\epsilon)$, $P(0,\epsilon) = 1$, $P(8,\epsilon) = n(\epsilon)$;

ii) $U_{\epsilon,1} \cdots U_{\epsilon,P(2,\epsilon)}$ is an ordered snake;

iii) $U_{\epsilon,P(2,\epsilon)} \cdots U_{\epsilon,P(4,\epsilon)}$ is an ordered snake which covers T_1;

iv) $U_{\epsilon,P(4,\epsilon)} \cdots U_{\epsilon,P(6,\epsilon)}$ is an ordered snake which covers T_2;

v) $U_{\epsilon,P(6,\epsilon)+1} \cdots U_{\epsilon,n(\epsilon)}$ is an ordered snake which covers T_3; and

vi) the nerve of U_ϵ is a "T."

Obviously, $\{U_{2^{-n}}\}$ is a cofinal sequence of coverings of X for which the homology maps induced by the projections are onto maps, since each nerve is acyclic. Thus, by the lemma, some subsequence of $\{U_{2^{-n}}\}$ carries the QC structure of X. Denote this subsequence by V_i, which is assumed chosen, as in the proof of the lemma, such that $i > j$ implies that V_i refines V_j. We adopt the notation

$$V_i = V_{i,1} \cdots V_{i,n(i)}, \; P_j \in V_{i,P(j,i)}$$

satisfying i)-vi) above. For $b > a$ and $k \leq P(3,b)$ let:

$V_{b,P(n;1,1;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(n,a)}$;

$V_{b,P(5;j,2;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(5,a)}$ with $k > P(6;j,1;b,a)$;

$V_{b,P(4;j,2;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(4,a)}$ with $k > P(5;j,2;b,a)$;

$V_{b,P(7;j,2;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(7,a)}$ with $k > P(8;j,1;b,a)$;

$V_{b,P(4;j,3;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(4,a)}$ with $k > P(7;j,2;b,a)$;

$V_{b,P(3;j,2;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(3,a)}$ with $k > P(4;j,3;b,a)$;

$V_{b,P(n;j,1;b,a)}$ be the first $V_{b,k}$ such that $V_{b,k}$ intersects $V_{a,P(n,a)}$ with $k > P(3;j-1,2;b,a)$;

and $N(b,a)$ be the maximum j such that $P(2;j,1;b,a)$ is defined by the above procedure.

It is obvious that for each pair (b,a), $N(b,a)$ exists and that given an integer M and an index a, there exists an index b such that $N(b,a) > M$.

Let V_a be an element of $\{V_i\}$ such that V_a refines $U_{2^{-3}}$; $g = g(a)$; and $h = h(a,g,b)$ where b is chosen so that $N(b,a) > N(g,a) + 4$. We assume the existence of an antiprojection ω: $N(V_g) \to N(V_h)$ and show that ω cannot satisfy all of the properties 1)-6).

For any zero chain,

$$\sum_{i=1}^{n} a_i V_{n,i} \quad ,$$

define $KI(\Sigma a_i V_{n,i}) = \Sigma a_i$. For any one chain A, $KI(\partial A) = 0$ where ∂ denotes the usual boundary operator. By property 5) $KI[\omega V_{g,i}] = 1$. Let $V_{n,i} \to V_{n,j}$ denote the one chain $V_{n,i} V_{n,i+1} + V_{n,i+1} V_{n,i+2} + \dots + V_{n,j-1} V_{n,j}$ where $V_{n,\ell}$ intersects $V_{n,k}$ if and only if $|\ell-k| \leq 1$, $i \leq \ell$, $k \leq j$. By property 6)

$$K(\omega V_{g,P(i,g)}) \subset St\, V_{a,P(i,a)} \ ;$$

$$K(\omega V_{g,P(i;j,k;g,a)}) \subset St\, V_{a;P(i,a)} \ ; \text{ and}$$

$$K[\omega(V_{g,1} \to V_{g,P(2;1,1;g,a)})] \subset \bigcup_{i=1}^{P(2,a)} St\, V_{a,i} \quad .$$

Given the complex K with n-dimensional simplices $\sigma_1 \ldots \sigma_r$ and the n-chains

$$C_1 = \sum_{i=1}^{r} a_i \sigma_i , \quad C_2 = \sum_{i=1}^{r} b_i \sigma_i , \quad C_1 * C_2$$

denotes the n-chain

$$C_3 = \sum_{i=1}^{r} c_i \sigma_i$$

where $c_i = a_i$ if $b_i \neq 0$, $c_i = b_i$ if $b_i = 0$. Two n-chains are said to intersect if there exists an n-simplex whose coefficient in each chain is nonzero. A 1-chain between $V_{n,i}$ and $V_{n,j}$ is a 1-chain C such that $\partial C = a_1 V_{n,i} + a_2 V_{n,j}$.

Since the distances $|P_0P_1|$, $|P_1P_2|$ and $|P_0P_2|$ are each greater than 1/2:

$$\omega V_{g,P(2;1,1;g,a)} = \partial\omega(V_{g,1} \to V_{g,P(2;1,1;g,a)})* \sum_{i=P(1,h)}^{n(h)} V_{h,i} .$$

Also

$$\omega(V_{g,1} \to V_{g,P(2;1,1;g,a)})$$

does not intersect

$$\sum_{n=1}^{N(h,a)-1} (V_{h,P(3;n,1;h,a)} \to V_{h,P(3;n,2;h,a)}) + \sum_{i=P(3,h)}^{n(h)} V_{h,i} V_{h,k} .$$

Therefore:

$$KI[\omega V_{g,P(2;1,1;g,a)}* \sum_{i=P(3;n,2;h,a)}^{P(3;n+1,1;h,a)} V_{h,i}] = 0 \quad n=1\ldots N(h,a)-2 ,$$

and

$$KI[\omega V_{g,P(2;1,1;g,a)}* \sum_{i=P(3;N(h,a)-1,2;h,a)}^{P(3,h)} V_{h,i}] = 0 .$$

Hence:

$$KI[\omega V_{g,P(2;1,1;g,a)}* \sum_{i=P(1,h)}^{P(3;1,1;h,a)} V_{h,i}] = 1 .$$

Assume that

$$KI[\omega V_{g,P(2;n,1;g,a)} * \sum_{i=P(3;j-1,2;h,a)}^{P(3;j,1;h,a)} V_{h,i}] = \delta_{n,j}$$

for $n < N(g,a) = N$, where $\delta_{n,j} = 0$, $n \neq j$; $\delta_{n,j} = 1$, $n = j$. We have seen that this assumption is valid for $n=1$, if we let $P(1,h) = P(i;0,j;h,a)$. Now consider the 1-chain

$$C = V_{g,P(2;n,1;g,a)} \to V_{g,P(4,n,1;g,a)} \quad .$$

By property 6):

$$K[\omega C] \subset \bigcup_{i=P(2,a)}^{P(4,a)} St\ V_{a,i} \quad .$$

Thus:

$$\omega C * [V_{h,P(5;j,1;h,a)} \to V_{h,P(5;j,2;h,a)} + V_{h,P(7;j,1;h,a)} \to V_{h,P(7;j,2;h,a)}] = 0 \quad .$$

This implies that

$$KI[\partial\omega C * \sum_{i=P(7,j-1,2;h,a)}^{P(5;j,1;h,a)} V_{h,i}] = 0 \quad ,$$

and

$$KI[\partial\omega C * \sum_{i=P(5;j,2;h,a)}^{P(7;j,1;h,a)} V_{h,i}] = 0 \quad .$$

Since also

$$K[\omega V_{g,P(2;n,1;g,a)}] \subset St\ V_{a,P(2,a)}$$

we have:

$$KI[\omega V_{g,P(4;n,1;g,a)} * \sum_{i=P(7;j-1,2;h,a)}^{P(5;j,1;h,a)} V_{h,i}] = \delta_{n,j}$$

and

$$KI[\omega V_{g,P(4;n,1;g,a)} * \sum_{i=P(7;j-1,2;h,a)}^{P(3;j-1,2;h,a)} V_{h,i}] =$$

$$-KI[\omega V_{g,P(4;n,1;g,a)} * \sum_{i=P(3;j,1;h,a)}^{P(5;j,1;h,a)} V_{h,i}] \text{ for } j \neq n \quad .$$

Assume that

$$KI[\omega V_{g,P(4;n,1;g,a)} * \sum_{i=P(7;j;2;h,a)}^{P(3;j,2;h,a)} V_{h,i}] \neq 0 \quad .$$

In this case

$$\omega(V_{g,P(4;n,1;g,a)} \to V_{g,P(6;n,1;g,a)})$$

must contain a 1-chain between an element in

$$\sum_{i=P(7;j,2;h,a)}^{P(3;j,2;h,a)} V_{h,i}$$

and an element either in

$$\sum_{i=1}^{P(7;j,1;h,a)} V_{h,i}$$

or in

$$\sum_{i=P(3;j+1,1;h,a)}^{n(h)} V_{h,i} \quad .$$

But:

$$K[\omega(V_{g,P(4;n,1;g,a)} \to V_{g,P(6;n,1;g,a)})] \subset \bigcup_{i=P(4,a)}^{P(6,a)} St\ V_{a,i} \quad ,$$

and no such 1-chain s exists with the property that

$$K(s) \subset \bigcup_{i=P(4,a)}^{P(6,a)} St\ V_{a,i} \quad .$$

Therefore:

$$KI[\omega V_{g,P(4;n,1;g,a)} * \sum_{i=P(7;j,2;h,a)}^{P(3;j,2;h,a)} V_{h,i}] = 0 \quad ,$$

$$KI[\omega V_{g,P(4;n,1;g,a)} * \sum_{i=P(5;j,2;h,a)}^{P(7;j,1;h,a)} V_{h,i}] = 0 \quad ,$$

and

$$KI[\omega V_{g,P(4;n,1;g,a)}* \sum_{i=P(3;j,1;h,a)}^{P(5;j,1;h,a)} V_{h,i}] = \delta_{n,j} \quad .$$

In this same manner, successively considering:

$$\omega(V_{g,P(4;n,1;g,a)} \to V_{g,P(6;n,1;g,a)})$$

which does not intersect

$$V_{h,P(7;j,1;h,a)} \to V_{h,P(7;j,2;h,a)}$$

nor

$$V_{h,P(3;j,2;h,a)} \to V_{h,P(3;j+1,1;h,a)} \ ;$$

$$\omega(V_{g,P(6;n,1;g,a)} \to V_{g,P(8;n,1;g,a)})$$

which does not intersect

$$V_{h,P(3;j,2;h,a)} \to V_{h,P(3;j+1,1;h,a)} \ ;$$

$$\omega(V_{g,P(8;n,1;g,a)} \to V_{g,P(4;n,3;g,a)})$$

which does not intersect

$$V_{h,P(3;j,2;h,a)} \to V_{h,P(3;j+1,1;h,a)}$$

nor

$$V_{h,P(5;j,1;h,a)} \to V_{h,P(5;j,2;h,a)} \ ;$$

and

$$\omega(V_{g,P(4;n,3;g,a)} \to V_{g,P(2;n+1,1;g,a)})$$

which does not intersect

$$V_{h,P(7;j,1;h,a)} \to V_{h,P(7;j,2;h,a)}$$

nor

$$V_{h,P(5;j,1;h,a)} \to V_{h,P(5;j,2;h,a)} \ ;$$

we finally conclude that

$$KI[\omega V_{g,P(2;n+1,1;g,a)}* \sum_{i=P(3;j,2;h,a)}^{P(3;j+1,1;h,a)} V_{h,i}] = \delta_{n,j} \quad .$$

In particular, this holds for $n+1 = N$, by induction.

Since

$$K[\omega(V_{g,P(2;N,1;g,a)} \to V_{g,P(4,g)})] \subset \bigcup_{i=P(2,a)}^{P(4,a)} \mathrm{St}\, V_{a,i} ,$$

we have:

$$KI[\partial\omega(V_{g,P(2;N,1;g,a)} \to V_{g,P(4,g)})^* \sum_{i=P(7;N-1,2;h,a)}^{P(5;N,1;h,a)} V_{h,i}] = 0 .$$

Therefore, either:

$$\text{(A)} \qquad KI[\, V_{g,P(4,g)}{}^* \sum_{i=P(7;N-1,2;h,a)}^{P(3;N-1,2;h,a)} V_{h,i}] \neq 0 ,$$

or

$$\text{(B)} \qquad KI[\omega V_{g,P(4,g)}{}^* \sum_{i=P(3;N,1;h,a)}^{P(5;N,1;h,a)} V_{h,i}] \neq 0 .$$

Suppose that (A) holds. Since

$$K[\omega(V_{g,P(4,g)} \to V_{g,P(6,g)})] \subset \bigcup_{i=P(4,a)}^{P(6,a)} \mathrm{St}\, V_{a,i} ,$$

$$\omega(V_{g,P(4,g)} \to V_{g,P(6,g)})$$

must contain a 1-chain s between some element in

$$\sum_{i=P(7;N-1,2;h,a)}^{P(3;N-1,2;h,a)} V_{h,i}$$

and some element either in

$$\sum_{i=1}^{P(7;N-1,1;h,a)} V_{h,i}$$

or in

$$\sum_{i=P(3;N,1;h,a)}^{n(h)} V_{h,i} .$$

But no such s exists with the property that

$$K(s) \subset \bigcup_{i=P(4,a)}^{P(6,a)} \mathrm{St}\, V_{a,i} \ .$$

Hence (A) does not hold.

By analogously considering

$$\omega(V_{g,P(4,g)}\ V_{g,P(6,g)+1} + V_{g,P(6,g)+1} \to V_{g,P(8,g)})$$

which, by property 6) has the property:

$$K[\omega(V_{g,P(4,g)}\ V_{g,P(6,g)+1} + V_{g,P(6,g)+1} \to V_{g,P(8,g)})]$$

$$\subset \bigcup_{i=P(6,a)+1}^{P(8,a)} \mathrm{St}\, V_{a,i} \cup \mathrm{St}\, V_{a,P(4,a)} \ ,$$

we see that (B) does not hold. That is, the required antiprojections do not exist.

Therefore, X is not a quasi-complex.

Using polar coordinates in the plane, let $R_1 = \{(r,\theta) \mid \theta = 1/r-1,\ 1 < r \leq 2\}$, $C = \{(r,\theta) \mid |r| = 1\}$, $D = \{(r,\theta) \mid |r| \leq 1\}$. The above methods can be used to show that neither $R_1 \cup C$ nor $R_1 \cup D$ is a QC.

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