ADJUSTMENT RULES BASED ON QUALITY CONTROL CHARTS

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When lack of control is detected on a control chart, the usual recommendation is to "search for an assignable cause," or "take remedial action," etc., without specific advice on how to proceed. One important situation is a continuously adjustable machine whose mean output is being monitored by an $\overline{X}$-chart. When lack of control is indicated in this situation, a natural action is to make an adjustment to re-center the process mean. In this paper we propose two simple methods for determining the amount of adjustment, based on usual $\overline{X}$-charts, and report on analytical investigations of their properties. Recommendations as to their effective use are also made.

Key Words: Quality Control, Control Charts, Out-of-Control, Adjustment Amount.

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1. INTRODUCTION

One of the main uses of quality control chart methodology is to detect when a process goes "out of control," as defined by a sample point's falling outside some defined region characterizing an in-control state. Several different quantities associated with the quality of a production process are typically measured, such as mean process performance and variability of the output. When an out-of-control condition is detected by such a chart, a typical recommendation is that "... an investigation is undertaken to find the assignable cause of this extreme variation" (Duncan 1974), or to "... take action when a point falls outside the 'control limits' ..." (Duncan 1974), or to make some adjustment to the process, if possible. There has been less said concerning exactly how one goes about attaining such goals, yet the operator must ultimately make some decision about the appropriate action to take, and the amount of any adjustment. The purpose of this paper is to propose readily-implemented techniques for estimating the amount of adjustment to be made, and to investigate the statistical properties of these techniques. Such an investigation indicates how well these kinds of techniques would perform when applied in an actual production process. Specifically, we investigate three different proposals for specifying adjustments, and pay particular attention to any tendency to *overadjust*, i.e., to take action resulting in being out of control in the opposite direction; in some contexts overadjustment is much more serious than underadjustment, and is thus to be protected against. One of the important conclusions from this study is that the process have an adequate capability index (ratio of tolerance range to six times the standard deviation of an individual item's measurement); only in this case is it possible to gather, without producing appreciable rejectable product, the required information to make an informed decision about the correct action.

As an example, suppose that a process is being monitored by an $\overline{X}$-chart using the standard three-sigma control limits, and a subgroup mean falls outside these limits. This provides fairly strong evidence that the process mean is no longer centered at its desired location, and we thus should attempt to estimate, in some way, an appropriate adjustment amount in order to re-center the process. The methods examined in this paper provide such estimates (in different ways, with different results), and our numerical evaluation in Section 3 indicates how well these methods may be expected to perform in practice; we give indication there of sample sizes required to obtain good performance.

We assume in this paper only that an $\overline{X}$-chart is being maintained, using the standard three-sigma limits. Our interest is in the process mean rather than the variance; an R-chart may well be of interest in itself and could also be constructed, but it is not needed in our techniques.

1.1 Applications

The most important application of the ability to estimate an adjustment amount is clearly in determining an adjustment amount that is subsequently actually made. There are, however, other uses for an estimate of a *shift* amount (see below) in the process mean. In designing control chart sampling plans in an economically optimal way, the inputs are typically various parameters associated with the cost of sampling and of producing rejectable product, and the amount by which the process mean has shifted; see Duncan (1956, 1971), Goel *et. al.* (1968), or Montgomery (1980). Of these inputs, the most critical one is the shift amount, i.e., the sensitivity of the output (being typically subgroup size, sampling frequency, and width factor for the control limits) to misspecification of the cost parameters is far less than to error in specifying the shift amount. With the techniques described below, one could obtain accurate estimates of a typical shift amount for a process to use as input to the economic design of future control chart sampling plans.

Throughout the paper, we attempt to follow the same notation as Grant and Leavenworth (1980): the true (parameter) process mean is $\overline{X'}$, the standard deviation of an individual item produced is σ', the subgroup size is n, and the control limits are set at $\overline{X'} \pm 3\sigma'/\sqrt{n}$.

In an $\overline{X}$-chart, the signal for detection of a change is that a subgroup mean, $\overline{X}$, falls outside the control limits. Although we focus on such a detection rule in this paper, other rules could be used as well, such as runs tests, or CUSUM charts.

1.2 Types of Changes

Given that the process mean has changed, it could in principle have arrived at its new value by any route. It is useful, however, to classify such routes into three categories. A *shift* change is an instantaneous change in the actual process mean from $\overline{X'}$ to some new value $\overline{X''} = \overline{X'} + \delta\sigma'$, where it is assumed to remain; this kind of change is characteristic of the result of suddenly introducing a new raw material lot, or of a catastrophic failure of a machine tool. A *drift* change occurs when the true process mean ceases to be constant over time, and begins to drift in a straight line (constant slope) away from $\overline{X'}$; this is typical of gradual tool wear. All other types of change might be called *erratic*. In this paper we focus on shift changes, partly because it is possible in this situation to carry out the most complete analysis and to make the most definitive recommendations, and because the literature on economic design of control charts focuses on this case. Further, the problem of estimating a drift change might first be addressed by understanding shift changes, and would involve techniques such as regression or forecasting.

1.3 Literature

The problem of detecting a shift in the mean of an ongoing process and related problems have been examined fairly extensively in the statistical literature. Few authors, however, have considered the ultimate adjustment decision made by the operator, the assumption apparently being that once an estimate of process location is made, the best adjustment amount is the difference between the desired process center and the present location. Our literature review reflects this emphasis in the literature and so concentrates on three topics related to estimation of the shift.

The first issue is determining whether a shift in the process location has actually occurred. The quality control literature has provided shift detection rules based on a count of the number of sample points falling in given areas on the control chart, see, e.g., Chapter 18 of Duncan (1974). These rules are derived from the probability of a sample point's falling within a certain number of standard deviations, given that a shift has occurred. A more sophisticated approach is Page's (1954, 1955) CUSUM method, which utilizes *all* sample data.

Assuming that a shift in the location of the process has occurred, another issue is determining at what point in the past it occurred. This "change point problem" has been treated extensively, notably by Quandt (1958, 1960, 1972), who modeled the problem as one of determining when the second of two consecutive regression models takes effect in a stream of data. Knowledge of the change point is important for inspection purposes but important for machine adjustments only if one wishes to include data before the first out-of-control point in estimating the process location, since we assume that the first out-of-control point indicates the shift occurred at a previous time.

The third statistical issue, given that a shift in the location of the process has occurred, is estimating the current location of the process. Some methods of estimation based on classical methods are given in the present paper. Barnard (1959) and Chernoff and Zacks (1966) offer a Bayesian estimator which is essentially a weighted mean of recent observations with the largest weights on the most recent observations.

Only a few writers suggest that the best machine adjustment from an operational viewpoint may not be of equal size and of opposite sign from the estimated amount of the shift. Grubbs (1954) suggests a procedure which requires n adjustments and results in a total adjustment which is unbiased and has the least possible variance of any n-step adjustment. Avoiding overadjustment on any particular adjustment is not a criterion of concern under this procedure, and many adjustments may be necessary to return the process to control. Jackson (1977) provides a statistical evaluation of adjustment amounts, dividing the possible errors in adjustment into four categories and determines the probability of a "bad" adjustment as a function of the shift in mean and the width of the control limits. Jackson suggests a partial adjustment in cases where a full adjustment has a high probability of producing a more severe out-of-control condition.

2. ESTIMATING SHIFT CHANGES

In this section we discuss three methods for estimating shift changes. While it may be tempting to use the first method, we will demonstrate the danger in doing so. The final two methods are more complex (and potentially more costly, depending on the capability index), but will turn out to be greatly preferable in several ways.

2.1 Using the Out-of-Control Point

The first method for estimating a shift change is to use the out of control point itself, minus the desired process center, i.e., the adjustment would be

$$A_0 = -(\overline{X}_{\text{ooc}} - \overline{X'}),$$

where $\overline{X}_{\text{ooc}}$ is the value of the out of control point observed, and $\overline{X'}$ is the desired process mean; see Jackson (1977). Thus, $\overline{X}_{\text{ooc}}$ is a random variable, but $\overline{X'}$ is to be regarded as a constant that would be known in practice. The difficulty with this idea is that, by design, the random variable $\overline{X}_{\text{ooc}}$ is biased away from $\overline{X'}$ since a subgroup mean would not be declared out of control if it were near $\overline{X'}$. In other words, regardless of whether the actual process mean is $\overline{X'}$ or not, $\overline{X}_{\text{ooc}}$ is conditioned on not being inside the control limits, so is not an (unbiased) observation from the population of all subgroup means, and thus misstates the difference between the actual process mean and the desired mean, $\overline{X'}$. In Appendix A, we derive a formula for $\text{E}(\overline{X}_{\text{ooc}})$, the expected value of the out-of-control points, assuming independent and normally distributed individual observations. In Table 1 are selected values of $\text{E}(\overline{X}_{\text{ooc}})$ as functions of the subgroup size n, and of δ, where the process mean has shifted from $\overline{X'}$ to $\overline{X''} = \overline{X'} + \delta\sigma'$. Without loss of generality, we assume that $\overline{X'} = 0$ and $\sigma' = 1$. In this case, a perfect average adjustment would result if $\text{E}(\overline{X}_{\text{ooc}}) = \delta$, and overadjustment if $\text{E}(\overline{X}_{\text{ooc}}) > \delta$. The tendency to overadjust is quite clear, especially for small n or small δ; for example, if $\delta = 0.50$ and $n = 5$ (a typical suggested subgroup size), the average adjustment is more than 200% larger than desired, resulting in a worse position after the adjustment than if no action had been taken at all. Thus, unless it is known that the shift amounts tend to be quite large (relative to σ'), or the subgroup size is large, it appears dangerous to use this type of rule.

2.2 An Unbiased Estimator

Having seen that using the out-of-control point as the sole basis for specifying a shift amount can lead to severe overadjustment, we turn to other methods displaying better performance. Both of these methods involve continuing the process, without adjustment, until m additional subgroup means past the out-of-control point (and not including it) have been observed. Such a plan purposely delays making an adjustment in order to collect information on what that adjustment should be. It is certainly not necessary to maintain the same subgroup size or sampling frequency as before the out-of-control point, under the independent sampling assumption; it may be advisable to alter both during this period to minimize defective production. Furthermore, the attainment of a sufficiently high capability index is very important for the economic feasibility of these sampling plans; this is discussed below with the numerical results.

Let $\overline{X}_1, \ldots, \overline{X}_m$ denote the m subgroup means collected past the out-of-control point; note that since there is no conditioning on these values' being in or outside the control limits, these constitute a random sample from a normal (assumed) distribution having mean $\overline{X''}$ and standard deviation $\sigma'/\sqrt{n}$. Immediately, an unbiased estimator of $\overline{X''}$ is $\overline{\overline{X}}(m)$,the sample mean of $\overline{X}_1, \ldots, \overline{X}_m$, and an unbiased estimator of the ideal adjustment $-\delta\sigma' = -(\overline{X''} - \overline{X'})$ is $A_1 = -\left(\overline{\overline{X}}(m) - \overline{X'}\right)$; recall that $\overline{X'}$ is known, but $\overline{X''}$ is not. A_1 is clearly an unbiased estimator of $-\delta\sigma'$, and is also normally distributed with mean $-\delta\sigma'$ and standard deviation $\sigma'/\sqrt{nm}$; note that the stability of A_1 depends only on nm and not on n and m alone, giving rise to equivalence of alternative sampling plans maintaining constancy of nm. From the statistician's viewpoint, $\overline{\overline{X}}(m)$ is the "best" estimator of $\overline{X''}$, enjoying such additional properties as being a maximum likelihood estimator, and being the minimum variance estimator from the class of all unbiased linear estimators of $\overline{X''}$.

2.3 A Conservative Estimator

One undesirable property of adjustment A_1, however, follows from the symmetry of its distribution about $-\delta\sigma'$. Thus, it is equally likely that A_1 will be above $-\delta\sigma'$ as below it; i.e., in application to an industrial process, A_1 has a 50% chance of resulting in overadjustment. In many applications (e.g., turning operations), overadjustment is much more serious and costly than underadjustment, since overadjustment results in expensive scrap, while underadjustment may result only in less expensive rework. Thus, despite the good statistical properties of A_1, it may be unacceptable due to the high likelihood (50%) of its resulting in overadjustment. Thus, we searched for a "conservative" alternative to A_1; simple rules of thumb, such as adjusting by half of A_1, could be proposed, but would not make good use of available information.

The same subgroup means $\overline{X}_1, \ldots, \overline{X}_m$ used to form a point estimator for $\overline{X''}$ can be used to form a confidence interval (CI) for $\overline{X''}$ as well, by standard methods. An unbiased estimator of $\text{Var}\left(\overline{\overline{X}}(m)\right)$ is $s^2(m)/m$, where

$$s^2(m) = \sum_{i=1}^{m} \left(\overline{X}_i - \overline{\overline{X}}(m)\right)^2 \Big/ (m-1)$$

and again under the random normal sample assumption on the $\overline{X}_i$'s, a $100(1-\alpha)\%$ CI for $\overline{X''}$ is

$$\overline{\overline{X}}(m) \pm t_{m-1,1-\alpha/2}\, s(m)/\sqrt{m}, \tag{1}$$

where $t_{m-1,1-\alpha/2}$ denotes the upper $1-\alpha/2$ critical point of the t distribution with $m-1$ degrees of freedom (DF). Note that $s^2(m)$ is an unbiased estimator of σ'^2/n, so if σ' were known, the CI (1) could be replaced by

$$\overline{\overline{X}}(m) \pm z_{1-\alpha/2}\,\sigma'/\sqrt{nm},$$

where $z_{1-\alpha/2}$ is the upper $1-\alpha/2$ critical point of the standard normal distribution. Our development, however, will be based on the more general assumption that σ' is unknown, since a shift in the process mean may well be accompanied by a change in its variance as well; thus we will use the CI (1). If an R-chart is being kept, examination of it may indicate whether such a change in the variance has occurred.

The CI (1) is used as follows to specify an adjustment A_2 serving as an alternative to A_1. Let L and U denote the lower and upper endpoints, respectively, of the CI (1). The idea is to use the "conservative" endpoint of (1), rather than its midpoint (as done for A_1), if possible. There are four cases:

$$A_2 = \begin{cases} -\left(U - \overline{X'}\right) & \text{if } U < \overline{X'} \\ -\left(\overline{\overline{X}}(m) - \overline{X'}\right) & \text{if } \overline{\overline{X}}(m) \le \overline{X'} \le U \\ -\left(L - \overline{X'}\right) & \text{if } \overline{X'} < L \\ -\left(\overline{\overline{X}}(m) - \overline{X'}\right) & \text{if } L \le \overline{X'} < \overline{\overline{X}}(m) \end{cases} \tag{2}$$

Cases 1 and 3 in the definition of A_2 are the "usual" ones in which the end of the CI nearest to $\overline{X'}$ (which lies outside the CI) is used to define the adjustment; this will result in a smaller absolute adjustment than A_1. Cases 2 and 4 are used if $\overline{X'}$ falls inside the CI and are defined so as to avoid adjustment in the wrong direction; in these two cases, $A_2 = A_1$.

Unlike A_1, A_2 will not be unbiased for the ideal adjustment $-\delta\sigma'$; it tends to underadjust due to the conservatism in using the CI endpoints rather than the point estimate $\overline{\overline{X}}(m)$. The reason for introducing A_2 is to decrease the 50% chance of overadjustment inherent in A_1. The choice of A_1 or A_2 would have to be made based on the relative penalty one feels would result from a bias toward underadjustment as opposed to frequent overadjustment. The following section quantifies the performance of the two adjustment rules to aid in making such a choice.

3. ESTIMATOR PERFORMANCE

In this section we evaluate analytically the statistical properties of the adjustment rules A_1 and A_2 proposed in Section 2. We assume without loss of generality that $\sigma' = 1$ and $\overline{X'} = 0$.

The properties of A_1 follow directly from the observation made above that it is normally distributed with mean $-\delta$ and variance $1/(nm)$. Thus, A_1 is always unbiased for the "ideal" adjustment $-\delta$, has standard deviation $1/\sqrt{nm}$, and overadjusts with probability 0.5.

The distribution of A_2, on the other hand, is considerably more difficult to obtain, due to its four-part definition and its (potential) dependence on the CI endpoints. In Appendix B we derive the cumulative distribution function (CDF) $F_{A_2}(a)$ and probability density function (PDF) $f_{A_2}(a)$ of A_2, and discuss computational issues in their evaluation. With $f_{A_2}(a)$, the expectation of A_2 is

$$\mathrm{E}(A_2) = \int_{-\infty}^{\infty} a f_{A_2}(a)\, da;$$

computationally the range of integration was truncated at points beyond which the absolute value of the integrand was less than 10^{-6}. Further,

$$\mathrm{Var}(A_2) = \int_{-\infty}^{\infty} \left(a - \mathrm{E}(A_2)\right)^2 f_{A_2}(a)\, da,$$

with a similar truncation of the range of integration. Both of the above two integrals were evaluated numerically by the IMSL routine DCADRE, based on de Boor's (1971) cautious adaptive Romberg method. Finally, the probability that A_2 results in overadjustment is

$$\mathrm{P}(A_2 < -\delta\sigma') = F_{A_2}(-\delta\sigma') \qquad \text{if } \delta > 0$$

$$\mathrm{P}(A_2 > -\delta\sigma') = 1 - F_{A_2}(-\delta\sigma') \qquad \text{if } \delta < 0$$

First we look at the variability of A_1 and A_2. Figure 1 shows a contour map of the standard deviation of A_1, denoted as $\sigma(A_1)$, as a joint function of n and m; as expected, $\sigma(A_1)$ falls as n or m increase, and has hyperbolic contours. Figure 2 shows a comparable contour map of $\sigma(A_2)$ for the case $\alpha = 0.10$ (i.e., we use a 90% CI) and $\delta = 1.00$ (i.e., there has been a shift in the mean equal to one standard deviation of an individual measurement). From Figure 2, we see that $\sigma(A_2)$ is generally close to $\sigma(A_1)$, with some tendency to be marginally higher, reflecting the fact that A_2 may be defined in terms of a CI endpoint which has an additional source of variability (the variance estimator $s^2(m)$). Contour maps of $\sigma(A_2)$ were also produced for all ten combinations of $\alpha = 0.01$ and 0.10, and $\delta = 0.25, 0.50, 1.00, 1.50$, and 2.00. From examining these plots, $\sigma(A_2)$ always falls with n and m, and rises slightly with δ (due to more frequent use of the CI endpoint in defining A_2). 99% CIs resulted in a somewhat more variable A_2 than did the (shorter) 90% CIs. Overall, A_2 appears to be comparable to A_1 in terms of stability, and depends in anticipated ways on various parameters.

Bearing in mind that A_1 is always unbiased for the ideal shift $-\delta$, we next examine the bias in A_2, measured in terms of percentage of $|-\delta|$,

$$B(A_2) = 100|\mathrm{E}(A_2) + \delta| \Big/ |-\delta|.$$

Figure 3 shows contours of $B(A_2)$ as a function of n and m, again for the case $\alpha = 0.10$ and $\delta = 1.00$. This shows that bias decreases as n or m increase, which was the case for most of the other nine (α, δ) combinations examined. The exceptions occurred for very small δ (0.25 and 0.50) in which case the CI (1) includes $\overline{X'}$ frequently for small n and m; this means that $A_2 \cong A_1$ and is thus much less biased. However, $\sigma(A_2)$ and the probability that A_2 overadjusts were in these cases undesirably high. Further, as the shift δ grows, the percent bias in A_2 tends to fall. Generally, the bias was in the range of 5% to 40% of $|\delta|$ for most of the cases evaluated, and can always be reduced by taking enough data (i.e., increasing n or m).

One of the main reasons for introducing A_2 as an alternative to A_1 is to reduce the probability of overadjustment; A_2 accomplishes this quite well. Figure 4 shows contours of the probability of overadjustment as a function of n and m for the case $\alpha = 0.10$ and $\delta = 0.50$. (Note that this value of δ is different from the $\delta = 1.00$ choice of Figures 2 and 3; for $\delta = 1.00$ the probability of overadjustment was hardly ever greater than 0.05, producing a nearly blank contour plot.) Again, better performance is achieved for higher n and m, and the overadjustment probabilities are substantially lower than the 0.50 inherent in A_1. As δ rises, the overadjustment probability falls rapidly (with $\alpha = 0.10$ and $\delta \geq 1.50$, all probabilities were less than or equal to 0.05); displaying the $\delta = 0.50$ case here rather than the $\delta = 1.00$ case used for Figures 2 and 3 represents a more difficult situation for A_2 in terms of achieving lower probability of overadjustment. Somewhat higher overadjustment probabilities were observed with $\alpha = 0.01$ and small δ, probably due to the wider nature of such 99% CIs and their resulting tendency to contain $\overline{X'}$, making $A_2 \cong A_1$ and thus more likely to overadjust. For this reason, it appears that the 90% CI is preferable to the 99% CI.

Finally, we can use our results on the bias in A_2 to address an important operational issue in application of these techniques. The total number of items sampled past the out-of-control point (but before any adjustment) for the purpose of estimating the adjustment is nm, consisting of m subgroups of n individual items each. If we can choose n freely and view the value of nm as a constraint, many alternative sampling plans may be possible; for example, if we are willing to let the process run for $nm = 20$ more individual items, we could form $m = 4$ subgroups of size $n = 5$, or form $m = 10$ subgroups of size $n = 2$ each, and so on. The splitting up of nm into n and m can affect A_2, in particular its bias. Figure 5 shows plots of $B(A_2)$ as a function of n *alone*, for several different levels of nm; again we take $\alpha = 0.10$ and $\delta = 1.00$ for this illustration. It appears that, except for nm very small, it is better to have many subgroups of small size (i.e., choose n to be small), than a few large subgroups. In fact, bias is minimized in these cases if $n = 1$, i.e., there is no subgrouping at all. These observations hold also for other cases of α and δ, except when δ is very small and nm is small as well. Thus, for situations in which experience indicates that shifts are large (and thus of more practical importance), the post-out-of-control sampling should be done as quickly as possible, and with subgroups of small size.

As an example of the use of these kinds of observations in practice, suppose first that overadjustment is no more serious than underadjustment, so that estimator A_1 is preferred. One goal in sampling for adjustment specification in this case is to obtain an adjustment with prespecified standard deviation, e.g., we want $\sigma(A_1) = 0.10$ or less. From the formula for $\sigma(A_1)$ (or from Figure 1), this requires $nm \geq 100$; for example, $m = 10$ subgroups of size $n = 10$ individuals each would do. (This is true regardless of the actual [unknown] shift amount $\delta\sigma'$.) On the other hand, if overadjustment is to be avoided, rule A_2 would probably be preferable, and from Figure 2 we see that, with subgroups of size $n = 10$, approximately $m = 11$ would be needed to attain a standard deviation of 0.10; note that this is slighlty more data than were required for rule A_1 to attain this stability. For these choices of n and m, Figure 3 shows that A_2 will be approximately 17% biased; however, in this case A_2 will overadjust with probability of only about 0.05. While these values are for the case of a shift of one standard deviation in the process mean, the observation holds true across a wide range of potential shifts that A_2 is only slightly more variable than A_1, may entail some bias (especially for small n and m), but has a very much smaller chance of overadjustment than does A_1.

4. CONCLUSIONS

In this paper we have examined three specific methods for quantifying the action to be taken when there is evidence that the process has shifted to an out-of-control state. The first adjustment A_0 uses the out-of-control point itself and thus involves no additional sampling, but unfortunately displays a bias toward (sometimes severe) overadjustment, which in many situations is much more costly than underadjustment. Two other estimators, A_1 and A_2, require that the process be left in operation for some time after the out-of-control indication, to obtain unbiased information on the current process mean. A_1 is an unbiased adjustment, but has a 50% chance of overadjustment. A_2 is somewhat biased toward underadjustment, but has a very low probability of overadjustment, and is therefore probably to be recommended in many situations. Since both A_1 and A_2 involve sampling during a statistically out-of-control state, it is important that the process have a reasonably high capability index to prevent production during this period of a large number of defectives. In this way we can afford to obtain the information necessary to attempt an accurate re-centering of the process.

APPENDIX A: EXPECTATION OF $\overline{X}_{\text{ooc}}$

Here we derive the expectation of $\overline{X}_{\text{ooc}}$, an out-of-control point. More generally, suppose we are sampling $Y_1, Y_2, \ldots$ independent and identically distributed (IID) random variables (RV's) from a continuous distribution with CDF F and PDF f. Let a and b be fixed real numbers with $a < b$, and Y^C denote the RV obtained from the Y_i's, except conditioned on *not* being in the interval $[a, b]$; that is, to observe Y^C, we sample the Y_i's and throw out any falling in $[a, b]$. Then it is easily seen that the CDF of Y^C is, for any real y,

$$F^C(y) = \mathrm{P}(Y^C \le y) = \frac{\mathrm{P}\left(Y_i \le \min\{y, a\}\right) + \mathrm{P}\left(b < Y_i \le y\right)}{1 - F(b) + F(a)},$$

and by considering three cases on the relationship among $y, a,$ and b, the PDF of Y^C is

$$f^C(y) = \begin{cases} f(y)/\left(1 - F(b) + F(a)\right) & \text{if } y < a \\ 0 & \text{if } a \le y \le b \\ f(y)/\left(1 - F(b) + F(a)\right) & \text{if } b < y \end{cases}$$

Thus, f^C has the same shape as f, except that it vanishes on $[a, b]$, and is inflated elsewhere to remain a density. Further, the expectation of Y^C is

$$\mathrm{E}(Y^C) = \int_{-\infty}^{\infty} y f^C(y)\, dy = \frac{\int_{-\infty}^{a} y f(y)\, dy + \int_{b}^{\infty} y f(y)\, dy}{1 - F(b) + F(a)} = \frac{\mathrm{E}(Y_i) - \int_a^b y f(y)\, dy}{1 - F(b) + F(a)} \qquad (A.1)$$

Assuming normal sampling, suppose the Y_i's are IID normal with mean μ and variance σ^2, and let Φ and ϕ respectively denote the CDF and PDF of the standard normal distribution. In our quality control application, the Y_i's are subgroup means, a and b are the lower and upper control limits (symmetric about $\overline{X'}$, but not about the actual process mean, if a shift has occurred), and $Y^C = \overline{X}_{\text{ooc}}$; further, $\mu = \overline{X''}$, the new process mean, and $\sigma = \sigma'/\sqrt{n}$. In this case, (A.1) becomes

$$\mathrm{E}(\overline{X}_{\text{ooc}}) = \frac{\overline{X''} - (\sqrt{n}/\sigma') \int_a^b y\, \phi\left((y - \overline{X''})/(\sigma'/\sqrt{n})\right)\, dy}{1 - \Phi\left((b - \overline{X''})/(\sigma'/\sqrt{n})\right) + \Phi\left((a - \overline{X''})/(\sigma'/\sqrt{n})\right)}. \qquad (A.2)$$

(A.2) was used to obtain the values in Table 1; the integral was evaluated by IMSL routine DCADRE, and Φ was evaluated by formula 26.2.17 of Abramowitz and Stegun (1964).

APPENDIX B: DISTRIBUTION OF A_2

The purpose here is to derive the CDF and PDF of A_2, the "conservative" adjustment in (2). Let C_k denote the event that case k occurs in (2), for $k = 1, 2, 3,$ and 4; the cases are numbered in the order listed in (2). Then, since the events $C_1, \ldots, C_4$ are mutually exclusive and exhaustive, the CDF of A_2 is

$$F_{A_2}(a) = \mathrm{P}(A_2 \le a) = \sum_{k=1}^{4} \mathrm{P}(A_2 \le a \mid C_k)\, \mathrm{P}(C_k), \qquad (B.1)$$

for any real a. The first step in computing (B.1) is to find the $\mathrm{P}(C_k)$'s.

Let $F_{\nu,\xi}$ denote the CDF of the noncentral t distribution with ν DF and noncentrality parameter ξ, and let

$$\eta(b) = \left(\overline{X''} - b\right) / \left(\sigma'/\sqrt{nm}\right)$$

for any real b; also write simply $t = t_{m-1,1-\alpha/2}$, $s^2 = s^2(m)$, and $\overline{\overline{X}} = \overline{\overline{X}}(m)$.

Proposition B1:

$$\mathrm{P}(C_1) = F_{m-1,\eta(\overline{X'})}(-t)$$

$$\mathrm{P}(C_2) = \Phi\left(-\eta(\overline{X'})\right) - \mathrm{P}(C_1)$$

$$\mathrm{P}(C_3) = 1 - F_{m-1,\eta(\overline{X'})}(t)$$

$$\mathrm{P}(C_4) = \Phi\left(\eta(\overline{X'})\right) - \mathrm{P}(C_3)$$

Proof: It is convenient to note first that

$$\mathrm{P}\left(\overline{\overline{X}} \le \overline{X'}\right) = \mathrm{P}\left(Z \le -\eta(\overline{X'})\right) = \Phi\left(-\eta(\overline{X'})\right),$$

where

$$Z = \left(\overline{\overline{X}} - \overline{X''}\right) / \left(\sigma'/\sqrt{nm}\right) \sim N(0,1);$$

$N(0,1)$ denotes the standard normal distribution. Since $(m-1)s^2/(\sigma'^2/n) \sim \chi^2_{m-1}$ (χ^2_ν denotes a chi-squared distribution with ν DF) and is independent of $\overline{\overline{X}}$,

$$Q(b) = \frac{\overline{\overline{X}} - b}{\sigma'/\sqrt{nm}} \Big/ \sqrt{(m-1)s^2/((\sigma'^2/n)(m-1))} = \left(\overline{\overline{X}} - b\right) / \left(s/\sqrt{m}\right)$$

has a noncentral t distribution with $m-1$ DF and noncentrality parameter $\eta(b)$, for any real b. Thus,

$$\mathrm{P}(C_1) = \mathrm{P}(U < \overline{X'}) = \mathrm{P}(\overline{\overline{X}} + ts/\sqrt{m} - \overline{X'} < 0) = \mathrm{P}(Q(\overline{X'}) < -t) = F_{m-1,\eta(\overline{X'})}(-t),$$

as desired. This is used to obtain

$$\mathrm{P}(C_2) = \mathrm{P}(\overline{X''} \le \overline{X'} \le U) = \mathrm{P}(\overline{\overline{X}} \le \overline{X'}) - \mathrm{P}(U < \overline{X'}) = \Phi(-\eta(\overline{X'})) - \mathrm{P}(C_1).$$

Next,

$$\mathrm{P}(C_3) = \mathrm{P}(\overline{X'} < L) = \mathrm{P}(Q(\overline{X'}) > t) = 1 - F_{m-1,\eta(\overline{X'})}(t).$$

Finally,

$$\mathrm{P}(C_4) = \mathrm{P}(L \le \overline{X'} < \overline{\overline{X}}) = \mathrm{P}(\overline{\overline{X}} > \overline{X'}) - \mathrm{P}(\overline{X'} < L) = 1 - \Phi(-\eta(\overline{X'})) - \mathrm{P}(C_3).$$

As a check, note that $\sum_{k=1}^4 \mathrm{P}(C_k) = 1$.

The four conditional probabilities in (B.1) involve more complicated derivations, so will be stated separately.

Proposition B2:

$$P(A_2 \le a \mid C_1) = \begin{cases} 1 - F_{m-1,\eta(\overline{X'}-a)}(-t)/P(C_1) & \text{if } a > 0 \\ 0 & \text{if } a \le 0 \end{cases}$$

Proof: Under C_1, i.e., that $U < \overline{X'}$, we have $A_2 = \overline{X'} - U > 0$, showing the second case of the proposition; thus, assume $a > 0$. Now for any real u,

$$P(U \le u) = P(Q(u) \le -t) = F_{m-1,\eta(u)}(-t),$$

so

$$\begin{aligned} P(A_2 \le a \mid C_1) &= P(U > \overline{X'} - a \mid U < \overline{X'}) \\ &= 1 - P(U \le \overline{X'} - a,\ U < \overline{X'})/P(U < \overline{X'}) \\ &= 1 - P(U \le \overline{X'} - a)/P(U < \overline{X'}) \\ &= 1 - F_{m-1,\eta(\overline{X'}-a)}(-t)/F_{m-1,\eta(\overline{X'})}(-t), \end{aligned}$$

since $a > 0$ implies $\overline{X'} - a \le \overline{X'}$.

Proposition B3: Let $d = a^2 nm(m-1)/(t^2\sigma'^2)$, $\eta = \eta(\overline{X'})$, and let g_ν and G_ν respectively denote the PDF and CDF of the χ^2_ν distribution. Then $P(A_2 \le a | C_2) = 0$ if $a \le 0$, and for $a > 0$,

$$\begin{aligned} P(A_2 \le a \mid C_2) = \Big\{ &\Phi(-\eta) - \Phi(-\eta - a\sqrt{nm}/\sigma')\Big(1 - G_{m-1}(d)\Big) \\ &- \int_0^d \Phi\big(-\eta - t\sqrt{c/(m-1)}\big)\, g_{m-1}(c)\, dc \Big\} \Big/ P(C_2). \end{aligned}$$

Proof: As before, C_2 implies $A_2 > 0$, so assume $a > 0$. Now

$$\begin{aligned} P(A_2 \le a,\ C_2) &= P(\overline{X'} - \overline{\overline{X}} \le a,\ \overline{\overline{X}} \le \overline{X'} \le \overline{\overline{X}} + ts/\sqrt{m}) \\ &= P(\overline{X'} - \min\{a, ts/\sqrt{m}\} \le \overline{\overline{X}} \le \overline{X'}) \\ &= p_1 + p_2, \end{aligned} \tag{B.2}$$

where

$$p_1 = P(\overline{X'} - a \le \overline{\overline{X}} \le \overline{X'},\ a \le ts/\sqrt{m})$$

and

$$p_2 = P(\overline{X'} - ts/\sqrt{m} \le \overline{\overline{X}} \le \overline{X'},\ ts/\sqrt{m} \le a).$$

By normality, $\overline{\overline{X}} \sim N(\overline{X''}, \sigma'^2/(nm))$, $n(m-1)s^2/\sigma'^2 \sim \chi^2_{m-1}$, and $\overline{\overline{X}}$ and s^2 are independent, so that

$$\begin{aligned} p_1 &= P(\overline{X'} - a \le \overline{\overline{X}} \le \overline{X'})\, P(a \le ts/\sqrt{m}) \\ &= \left(\Phi(-\eta) - \Phi(-\eta - a\sqrt{nm}/\sigma')\right)\left(1 - G_{m-1}(d)\right). \end{aligned} \tag{B.3}$$

To evaluate p_2, we condition on $n(m-1)s^2/\sigma'^2 = c$ and integrate with respect to g_{m-1}; noting that the event $ts/\sqrt{m} \le a$ is equivalent to $c \le d$,

$$\begin{aligned} p_2 &= \int_0^d \mathrm{P}(\overline{X'} - t\sigma'\sqrt{c/(nm(m-1))} \le \overline{\overline{X}} \le \overline{X'})g_{m-1}(c)\,dc \\ &= \int_0^d \left(\Phi(-\eta) - \Phi(-\eta - t\sqrt{c/(m-1)})\right) g_{m-1}(c)\,dc \\ &= \Phi(-\eta)G_{m-1}(d) - \int_0^d \Phi(-\eta - t\sqrt{c/(m-1)})g_{m-1}(c)\,dc. \end{aligned} \tag{B.4}$$

Substituting (B.3) and (B.4) into (B.2), simplifying, and using the definition of conditional probability completes the proof.

Proposition B4:

$$\mathrm{P}(A_2 \le a \mid C_3) = \begin{cases} 1 & \text{if } a \ge 0 \\ (1 - F_{m-1,\eta(\overline{X'}-a)}(t))/\mathrm{P}(C_3) & \text{if } a < 0 \end{cases}$$

Proof: Under C_3, $A_2 = \overline{X'} - L < 0$, so assume $a < 0$. For any real ℓ,

$$\mathrm{P}(L \ge \ell) = \mathrm{P}(Q(\ell) \ge t) = 1 - F_{m-1,\eta(\ell)}(t), \tag{B.5}$$

where $Q(\ell)$ is defined above. Thus,

$$\mathrm{P}(A_2 \le a \mid C_3) = \mathrm{P}(\overline{X'} - L \le a,\, \overline{X'} < L)/\mathrm{P}(\overline{X'} < L) = \mathrm{P}(L \ge \overline{X'} - a)/\mathrm{P}(L > \overline{X'}),$$

which is as desired after using (B.5) with $\ell = \overline{X'} - a$ and again with $\ell = \overline{X'}$.

Proposition B5: With d, η, g_ν, and G_ν as in Proposition B3, $\mathrm{P}(A_2 \le a | C_4) = 1$ if $a \ge 0$, and if $a < 0$,

$$\begin{aligned} \mathrm{P}(A_2 \le a \mid C_4) = 1 + \Big\{ &\Phi(-\eta) - \Phi(-\eta - a\sqrt{nm}/\sigma')(1 - G_{m-1}(d)) \\ &- \int_0^d \Phi(-\eta + t\sqrt{c/(m-1)})g_{m-1}(c)\,dc\Big\}\Big/\mathrm{P}(C_4) \end{aligned}$$

Proof: Under C_4, $A_2 = \overline{X'} - \overline{\overline{X}} < 0$, so assume $a < 0$. Then

$$\begin{aligned} \mathrm{P}(A_2 > a \mid C_4) &= \mathrm{P}(\overline{X'} - \overline{\overline{X}} > a,\, \overline{\overline{X}} > \overline{X'},\, \overline{\overline{X}} \le \overline{X'} + ts/\sqrt{m}) \\ &= \mathrm{P}(\overline{X'} < \overline{\overline{X}} \le \overline{X'} + \min\{-a, ts/\sqrt{m}\}) \\ &= p_1 + p_2, \end{aligned} \tag{B.6}$$

with

$$p_1 = \mathrm{P}(\overline{X'} < \overline{\overline{X}} \le \overline{X'} - a,\, -a < ts/\sqrt{m}) = \left(\Phi(-\eta - a\sqrt{nm}/\sigma') - \Phi(-\eta)\right)(1 - G_{m-1}(d)), \tag{B.7}$$

by arguing as in the proof of Proposition B3. Further arguing as in that proof,

$$p_2 = \int_0^d \Phi(-\eta + t\sqrt{c/(m-1)})g_{m-1}(c)\,dc - \Phi(-\eta)G_{m-1}(d). \qquad (B.8)$$

Substituting (B.7) and (B.8) into (B.6) yields the desired result upon simplification.

We can now substitute the results of Propositions B1–B5 into (B.1) to obtain the following more efficient and stable computational form. Defining d, η, g_ν, and G_ν as in Proposition B3, let

$$\beta = \Phi(-\eta) - \Phi(-\eta - a\sqrt{nm}/\sigma')\big(1 - G_{m-1}(d)\big)$$

and, for $\tau = \pm 1$, let

$$I(\tau) = \int_0^d \Phi\big(-\eta + \tau t\sqrt{c/(m-1)}\big)\,g_{m-1}(c)\,dc.$$

Then

$$F_{A_2}(a) = \begin{cases} \mathrm{P}(C_4) + \beta + 1 - F_{m-1,\eta(\overline{X'}-a)}(t) - I(+1) & \text{if } a < 0 \\ \mathrm{P}(C_4) + \mathrm{P}(C_3) & \text{if } a = 0 \\ \mathrm{P}(C_4) + \beta + \mathrm{P}(C_1) + \mathrm{P}(C_3) - F_{m-1,\eta(\overline{X'}-a)}(-t) - I(-1) & \text{if } a > 0 \end{cases} \qquad (B.9)$$

(Evaluating the $a < 0$ case or the $a > 0$ case at $a = 0$ leads to $\mathrm{P}(C_4) + \mathrm{P}(C_3)$, so that F_{A_2} is continuous.) These results have been implemented and confirmed by simulations consisting of 1000 observations on A_2 in each of some 24 cases of particular parameter combinations; close agreement was noted. The noncentral t CDF $F_{\nu,\eta}$ was evaluated by the methods of Owen (1963), and the integral $I(\tau)$ was evaluated by IMSL subroutine DCADRE. (DCADRE was also used to evaluate the integral appearing in Owen's formula for computation of $F_{\nu,\eta}$.)

To obtain the PDF f_{A_2} of A_2, we differentiate F_{A_2} from (B.9) with respect to a. The $\mathrm{P}(C_k)$'s do not depend on a, but $\beta, F_{m-1,\eta(\overline{X'}-a)}(\pm t)$, and $I(\pm 1)$ do depend on a. For $a < 0$ the expression $\beta - I(+1)$ appears in $F_{A_2}(a)$, while for $a > 0$ the expression $\beta - I(-1)$ appears. The derivative with respect to a in either of these cases is

$$(\sqrt{nm}/\sigma')\,\phi(-\eta - a\sqrt{nm}/\sigma')\left(1 - G_{m-1}\left(a^2 nm(m-1)/(t^2\sigma'^2)\right)\right). \qquad (B.10)$$

Thus, it remains to differentiate $F_{m-1,\eta(\overline{X'}-a)}(\pm t)$ with respect to a, which appears as a parameter of the noncentrality parameter. The derivative of the noncentral t CDF $F_{\nu,\xi}(x)$ with respect to ξ is given in Owen (1963), and this is evaluated at $\nu = m-1, \xi = \eta(\overline{X'} - a)$, and $x = \pm t$ as the case may be; using the chain rule we multiply this by

$$\frac{\partial}{\partial a}\eta(\overline{X'} - a) = \sqrt{nm}/\sigma',$$

and subtract the result from (B.10) to obtain finally

$$f_{A_2}(a) = \frac{\sqrt{nm}}{\sigma'}\phi(-\eta - a\frac{\sqrt{nm}}{\sigma'})\left(1 - G_{m-1}\left(a^2 nm(m-1)/(t^2\sigma'^2)\right)\right) - \frac{\sqrt{nm}}{\sigma'}\frac{\partial}{\partial\xi}F_{m-1,\eta(\overline{X'}-a)}(\pm t),$$

taking $+t$ if $a < 0$ and $-t$ if $a > 0$.

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Table 1. Expectations of Out-of-Control Points $\overline{X}_{occ}$.

δ	n = 1	2	3	4	5	6	7	8	9	10
0.5	3.09	2.32	1.93	1.68	1.51	1.39	1.29	1.21	1.15	1.09
1.0	3.36	2.42	2.01	1.76	1.60	1.48	1.39	1.32	1.27	1.22
1.5	3.44	2.51	2.12	1.90	1.76	1.67	1.61	1.57	1.55	1.53
2.0	3.53	2.64	2.30	2.14	2.06	2.03	2.01	2.00	2.00	2.00

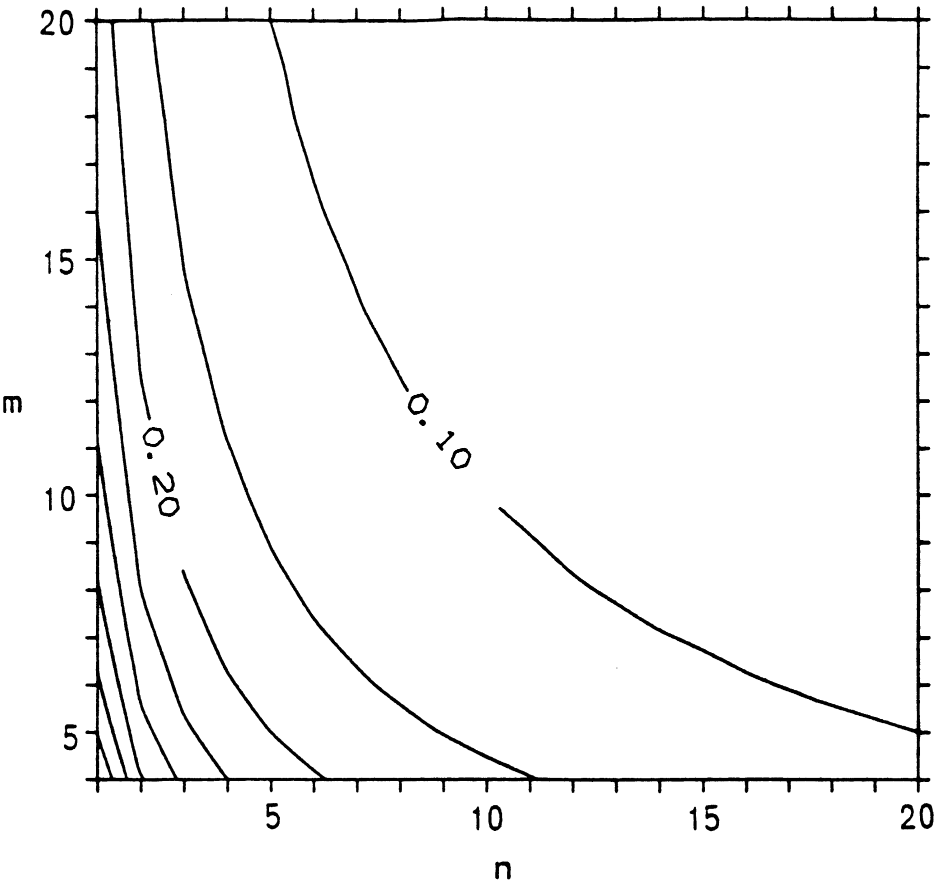

Figure 1. Contours of $\sigma(A_1)$.

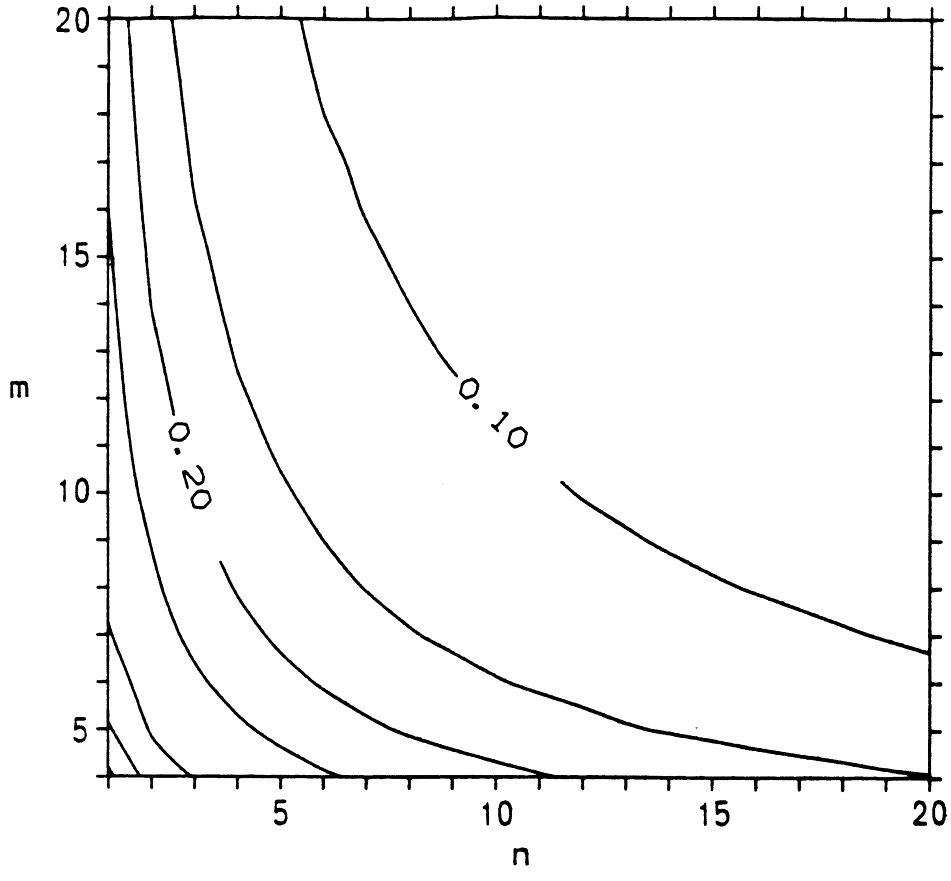

Figure 2. Contours of $\sigma(A_2)$ for $\alpha = 0.10$, $\delta = 1.00$.

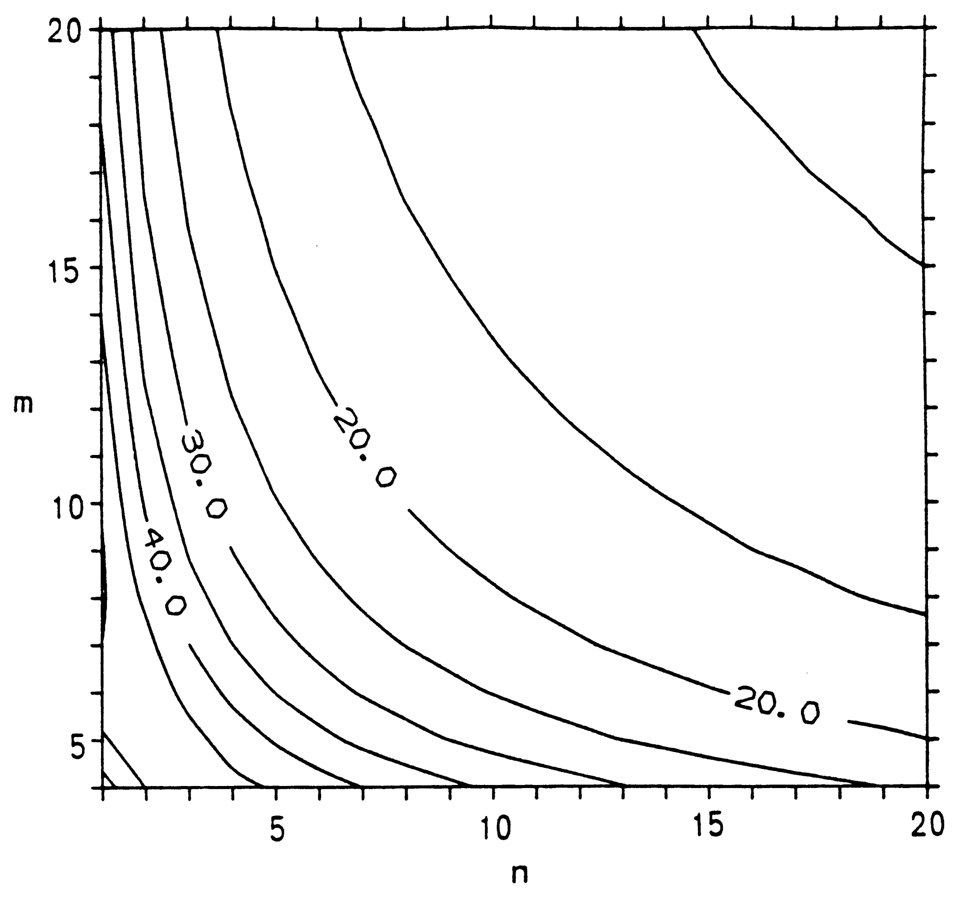

Figure 3. Contours of $B(A_2)$ for $\alpha = 0.10$, $\delta = 1.00$.

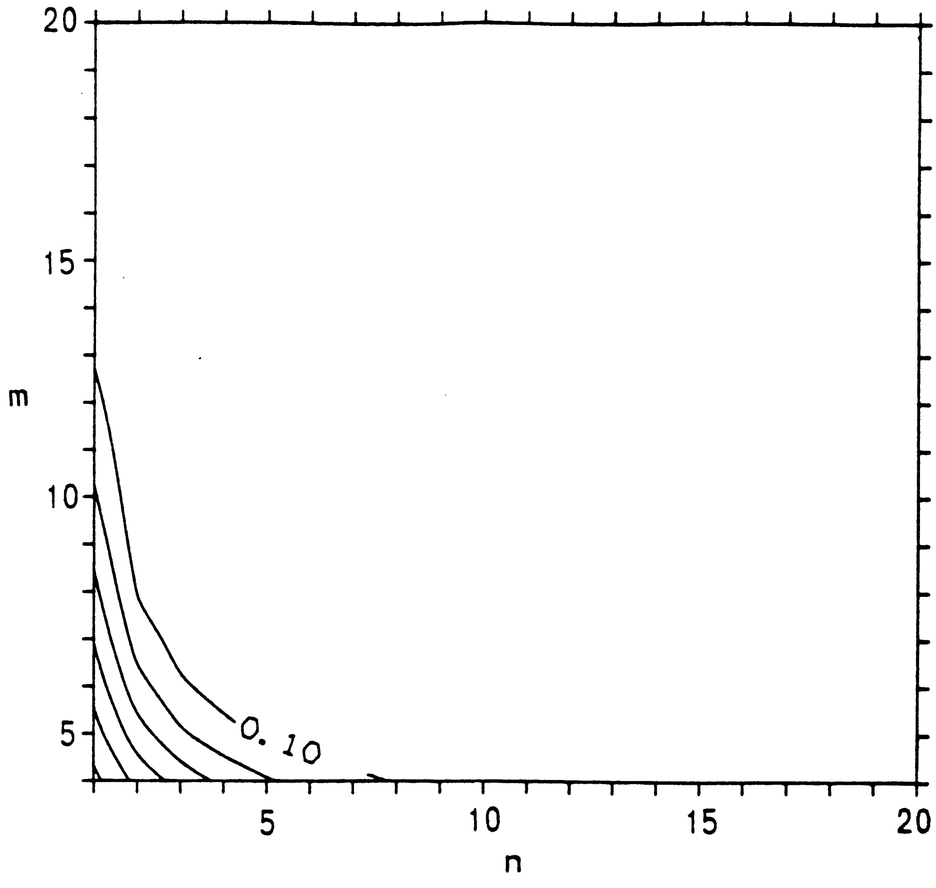

Figure 4. Contours of the Probability that A_2 Overadjusts for $\alpha = 0.10$, $\delta = 0.5$

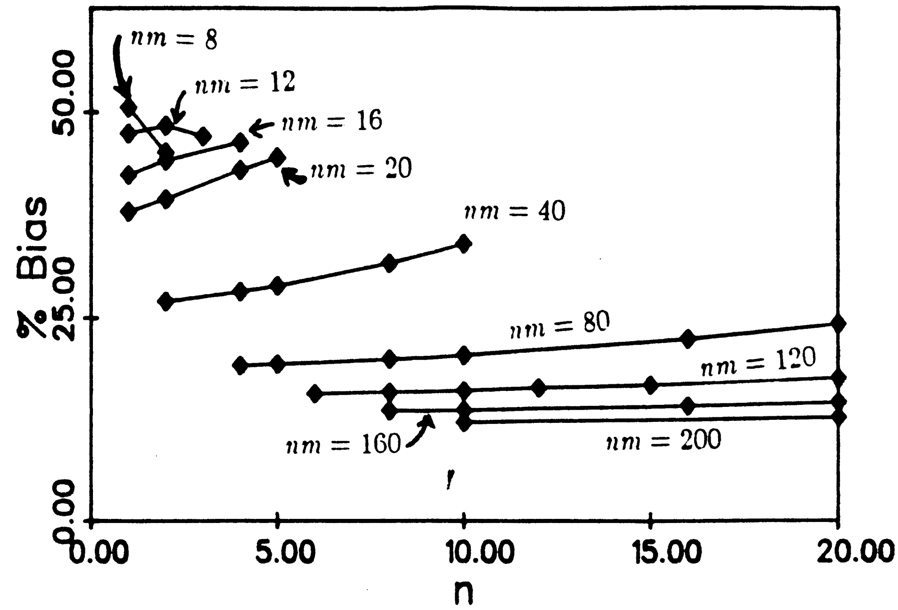

Figure 5. Plots of $B(A_2)$ as Functions of n with nm Fixed, for $\alpha = 0.10$, $\delta = 1.00$.