THE HEDETNIEMI MATRIX SUM: A REAL–WORLD APPLICATION

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In a recent paper, we presented an algorithm for finding the shortest distance between any two nodes in a network of n nodes when given only distances between adjacent nodes [Arlinghaus, Arlinghaus, Nystuen, 1990(b)]. In that previous research, we applied the algorithm to the generalized road network graph surrounding San Francisco Bay. The resulting matrices are repeated here (Figure 1), in order to examine consequent changes in matrix entries when the underlying adjacency pattern of the road network was altered by the 1989 earthquake that closed the San Francisco–Oakland Bay Bridge. Thus, we test the algorithm against a changed adjacency configuration and interpret the results with the benefit of hindsight from an actual event. Figure 1 shows a graph, with edges weighted with time–distances, representing the general expressway linkage pattern joining selected cities surrounding San Francisco Bay. The matrix A displays these time–distances in tabular form; an asterisk indicates that there is no direct linkage between corresponding entries. Thus, an asterisk in entry a_{13} indicates that there is no single edge of the graph linking San Francisco and San Jose (all paths have 2 or more edges). Higher powers of the matrix A count numbers of paths of longer length—A^2 counts paths of 2 edges as well as those of one edge. Thus, one expects in A^2 to see a number measuring time–distance between San Francisco and San Jose; indeed, there are two such paths, one of length 30+50=80, and one of length 30+25=55. The Hedetniemi matrix operator always selects the shortest. Readers wishing to understand the mechanics of this algorithm should refer to the other references related to this topic in the list at the end [Arlinghaus, Arlinghaus, and Nystuen; W. Arlinghaus]. It is sufficieint here simply to understand generally how the procedure works, as described above.

When a recent earthquake caused a disastrous collapse of a span on the San Francisco–Oakland Bay Bridge, forcing the closing of the bridge, municipal authorities managed to keep the city moving using a well–balanced combination of added ferry boats, media messages urging people to stay off the roads, and dispersal of information concerning alternate route strategies. National telecasts showed a city on the move, albeit slowly, although outside forecasters of doom were predicting a massive grid–lock that never occured. What would the Hedetniemi algorithm have forecast in this situation?

To find out, we compare the matrices of Figure 1 to those of Figure 2, derived from the graph of Figure 1 with the link between San Francisco and Oakland removed; that is, the edge linking vertex 4 to vertex 1 is removed — the results show in the matrix entries a_{14} and a_{41}. Thus in Figure 2, the adjacency matrix A, describing 1–step edge linkages differs from that of Figure 1 only in the a_{14} (a_{41}) position. The value of * replaces the time–distance of 30 minutes in that graph because the bridge connection was destroyed. When 2–edge paths are counted, there is spread of increased time–distances across these paths, as well. What used to take 30 minutes, under conditions of normal traffic, to go from San Francisco to Oakland now takes 70 minutes, under conditions of normal traffic, going by way of San Mateo. The trip from San Francisco to Walnut Creek had been possible along a 2–edge path passing through Oakland (and taking a total of 60 minutes); the asterisk in A^2 in the a_{15} entry indicates that that path no longer exists. The journey from San Francisco to Richmond, along a 2–edge path, increased in time–distance from 50 to 60 minutes—going around the "longer" side of the rectangle. Note that what is being evaluated here is change in trip–time under "normal" circumstances, according to whether or not routing exists; congestion fluctuates but actual road lengths do not (once in place). These values therefore form a set of benchmarks against which to measure time–distance changes resulting from more variable quantities, such as increased congestion.

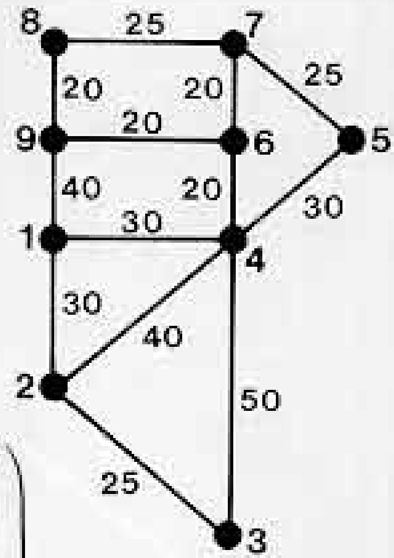
When three–edged paths are brought into the system, in A^3 (Figure 2), the trip from San Francisco to Walnut Creek now becomes possible, but takes 100 rather than 60 minutes. Also, the trip from San Francisco to Vallejo now becomes possible (in both pre- and post-earthquake systems) although it takes 10 minutes longer with removal of the bridge. When paths of length four are introduced, no changes occur in these entries; the system is stable and the effects are confined to locations "close" to the bridge that was removed. The relatively small number of changes in the basic underlying route choices, forced by the removal of the Bay Bridge, suggest why it was possible, with swift action by municipal authorities and citizens to control congestion, to avert a situation that appeared destined to lead to gridlock.

What if the Golden Gate Bridge had been removed rather than the San Francisco–Oakland Bay Bridge? Figure 3 shows that the same sort of clustered, localized results follow. When both bridges are removed (Figure 4), the position of affected matrix entries is identical to the union of the positions of entries in Figures 1 and 2, but the magnitude of time–distances has been magnified by the combined removal.

With hindsight, the test seems to be reasonable. One direction for a larger application might therefore be to consider historical evidence in which bridge bombing (or some such) was critical to associated circulation patterns. When large data sets are entered into a computer, and manipulated using the Hedetniemi matrix algorithm, previously unnoticed historical associations might emerge and maps showing alternate possibilities could be produced. In short, this might serve as a tool useful in historical discovery. Other important directions for application of the Hedetniemi algorithm involve those in a discrete mathematical setting that focus on tracing actual paths [W. Arlinghaus, 1990—includes program for algorithm], and those using the Hedetniemi algorithm in the computer architecture of parallel processing [Romeijn and Smith].

SAN FRANCISCO BAY AREA; GRAPH OF TIME–DISTANCES
(in minutes)
LEGEND: numeral attached to city is its node number in the corresponding, underlying, graph.

1. SAN FRANCISCO
2. SAN MATEO COUNTY
3. SAN JOSE
4. OAKLAND
5. WALNUT CREEK
6. RICHMOND
7. VALLEJO
8. NOVATO
9. SAN RAFAEL (MARIN COUNTY)



$$A=\begin{pmatrix} 0 & 30 & * & 30 & * & * & * & * & 40\\ 30 & 0 & 25 & 40 & * & * & * & * & *\\ * & 25 & 0 & 50 & * & * & * & * & *\\ 30 & 40 & 50 & 0 & 30 & 20 & * & * & *\\ * & * & * & 30 & 0 & * & 25 & * & *\\ * & * & * & 20 & * & 0 & 20 & * & 20\\ * & * & * & * & 25 & 20 & 0 & 25 & *\\ * & * & * & * & * & * & 25 & 0 & 20\\ 40 & * & * & * & * & 20 & * & 20 & 0 \end{pmatrix}$$

$$A^2=\begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & * & 60 & 40\\ 30 & 0 & 25 & 40 & 70 & 60 & * & * & 70\\ 55 & 25 & 0 & 50 & 80 & 70 & * & * & *\\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & * & 40\\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & *\\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20\\ * & * & * & 40 & 25 & 20 & 0 & 25 & 40\\ 60 & * & * & * & 50 & 40 & 25 & 0 & 20\\ 40 & 70 & * & 40 & * & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3=\begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 60 & 40\\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70\\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & * & 90\\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40\\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65\\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20\\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40\\ 60 & 90 & * & 60 & 50 & 40 & 25 & 0 & 20\\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

Figure 1. Pre-earthquake matrix sequence.

POST–EARTHQUAKE SAN FRANCISCO BAY AREA
SAN FRANCISCO–OAKLAND BAY BRIDGE IS REMOVED.
GRAPH OF TIME–DISTANCES (in minutes)
Adjustment is made for change in time–distance
in a "normal" situation–not for resultant fluctuation in congestion

$$A = \begin{pmatrix} 0 & 30 & \ast & \ast & \ast & \ast & \ast & \ast & 40 \\ 30 & 0 & 25 & 40 & \ast & \ast & \ast & \ast & \ast \\ \ast & 25 & 0 & 50 & \ast & \ast & \ast & \ast & \ast \\ \ast & 40 & 50 & 0 & 30 & 20 & \ast & \ast & \ast \\ \ast & \ast & \ast & 30 & 0 & \ast & 25 & \ast & \ast \\ \ast & \ast & \ast & 20 & \ast & 0 & 20 & \ast & 20 \\ \ast & \ast & \ast & \ast & 25 & 20 & 0 & 25 & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & 25 & 0 & 20 \\ 40 & \ast & \ast & \ast & \ast & 20 & \ast & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 70 & \ast & 60 & \ast & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & \ast & \ast & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & \ast & \ast & \ast \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & \ast & 40 \\ \ast & 70 & 80 & 30 & 0 & 45 & 25 & 50 & \ast \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & \ast & \ast & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & \ast & \ast & \ast & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & \ast & 40 & \ast & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 60 & 80 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & \ast & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 80 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & \ast & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 60 & 80 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 80 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 60 & 80 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 80 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

Figure 2. Matrix sequence with San Francisco–Oakland Bay Bridge removed.

POST-EARTHQUAKE SAN FRANCISCO BAY AREA
GOLDEN GATE BRIDGE IS REMOVED.
GRAPH OF TIME-DISTANCES (in minutes)
Adjustment is made for change in time-distance
in a "normal" situation—not for resultant fluctuation in congestion

$$A = \begin{pmatrix} 0 & 30 & \ast & 30 & \ast & \ast & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & \ast & \ast & \ast & \ast & \ast \\ \ast & 25 & 0 & 50 & \ast & \ast & \ast & \ast & \ast \\ 30 & 40 & 50 & 0 & 30 & 20 & \ast & \ast & \ast \\ \ast & \ast & \ast & 30 & 0 & \ast & 25 & \ast & \ast \\ \ast & \ast & \ast & 20 & \ast & 0 & 20 & \ast & 20 \\ \ast & \ast & \ast & \ast & 25 & 20 & 0 & 25 & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & 25 & 0 & 20 \\ \ast & \ast & \ast & \ast & \ast & 20 & \ast & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & 70 & 60 & \ast & \ast & \ast \\ 55 & 25 & 0 & 50 & 80 & 70 & \ast & \ast & \ast \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & \ast & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & \ast \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & \ast & \ast & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & \ast & \ast & \ast & 50 & 40 & 25 & 0 & 20 \\ \ast & \ast & \ast & 40 & \ast & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & \ast & 70 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & \ast & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & \ast & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & \ast & \ast & 60 & 50 & 40 & 25 & 0 & 20 \\ 70 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 90 & 70 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 90 & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 70 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 90 & 70 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 90 & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 70 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

Figure 3. Matrix sequence with the Golden Gate Bridge removed.

POST–EARTHQUAKE SAN FRANCISCO BAY AREA
BAY BRIDGE AND GOLDEN GATE BRIDGE ARE BOTH REMOVED.
GRAPH OF TIME–DISTANCES (in minutes)
Adjustment is made for change in time–distance
in a "normal" situation—not for resultant fluctuation in congestion

$$A = \begin{pmatrix} 0 & 30 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & \ast & \ast & \ast & \ast & \ast \\ \ast & 25 & 0 & 50 & \ast & \ast & \ast & \ast & \ast \\ \ast & 40 & 50 & 0 & 30 & 20 & \ast & \ast & \ast \\ \ast & \ast & \ast & 30 & 0 & \ast & 25 & \ast & \ast \\ \ast & \ast & \ast & 20 & \ast & 0 & 20 & \ast & 20 \\ \ast & \ast & \ast & \ast & 25 & 20 & 0 & 25 & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & 25 & 0 & 20 \\ \ast & \ast & \ast & \ast & \ast & 20 & \ast & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 70 & \ast & \ast & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & 70 & 60 & \ast & \ast & \ast \\ 55 & 25 & 0 & 50 & 80 & 70 & \ast & \ast & \ast \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & \ast & 40 \\ \ast & 70 & 80 & 30 & 0 & 45 & 25 & 50 & \ast \\ \ast & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & \ast & \ast & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & \ast & \ast & \ast & 50 & 40 & 25 & 0 & 20 \\ \ast & \ast & \ast & 40 & \ast & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 90 & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & \ast & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & \ast & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 90 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & 80 & \ast & 60 & 50 & 40 & 25 & 0 & 20 \\ \ast & \ast & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 90 & 110 & \ast & 110 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 90 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 110 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 110 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 90 & 110 & 130 & 110 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 90 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 110 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 130 & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 110 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

Figure 4. Matrix sequence with both the Golden Gate and the Bay bridges removed.

References.

Arlinghaus, S. L.; W. C. Arlinghaus; J. D. Nystuen. 1990. Poster—*Elements of Geometric Routing Theory–II*. Association of American Geographers, National Meetings, Toronto, Ontario, April.

Arlinghaus, S. L.; W. C. Arlinghaus; J. D. Nystuen. 1990. "The Hedetniemi Matrix Sum: An Algorithm for Shortest Path and Shortest Distance." *Geographical Analysis*. 22: 351-360.

Arlinghaus, W. C. "Shortest Path Problems," invited chapter in *Applications of Discrete Mathematics*, edited by Kenneth H. Rosen and John Michaels. March 11, 1990. In press, McGraw–Hill.

Romeijn, H. E. and R. L. Smith. "Notes on Parallel Algorithms and Aggregation for Solving Shortest Path Problems." Unpublished, October, 1990.