FRACTAL GEOMETRY OF INFINITE PIXEL SEQUENCES: "SUPER-DEFINITION" RESOLUTION?

Sandra Lach Arlinghaus

Introduction

The fractal approach to the geometry of central place theory is particularly powerful because, among other things, it provides numerical proof that the subjective labels of "marketing," "transportation," and "administration" for the $K = 3$, $K = 4$, and $K = 7$ hierarchies are indeed correct [Arlinghaus, 1985] and because it enables solution of all open geometric questions identified by Dacey, Marshall, and others in earlier research [Dacey; Marshall; Arlinghaus and Arlinghaus]. When the problem is wrapped back on itself and the nature of the original, underlying environment is altered—from urban to electronic—the same results, recast in a different light, suggest the degree of improvement in picture resolution that can come from decreasing pixel size.

Curves on cathode ray tubes are formed from a sequence of pixels hooked together at their corners; font designers in word processors offer an easy opportunity to observe these pixel formations (Horstmann, 1986). The pixel sequence merely suggests the curve; it does not actually produce a "correct" curve. Reducing the size of the pixel can improve the resolution of the image representing the curve. The material below uses established results from fractal geometry to evaluate the degree of success, in improving resolution in a raster environment, that results from decreasing pixel size.

Manhattan pixel arrangement

When a square pixel is the fundamental unit, a sequence of pixels has boundaries separating pixels in Manhattan, "city- block" space. When smaller square pixels are introduced, more lines separating pixels are also introduced. The interior of the pixel is what carries the content—not the boundary of the pixel. Thus, it is significant to know what proportion of the space filled with pixels is filled with pixel boundary.

Suppose that, in an effort to produce "high- definition" resolution, the number of square pixels used to cover a fixed area (a cathode ray tube) is substantially increased. One might be tempted to use even more pixels to produce even better resolution and even more beyond that. If the process is carried out infinitely, using a Manhattan grid, the pixel mesh has arbitrarily small cell size and the entire plane region is "filled" with pixel boundary, only; the scale transformation of superimposing finer and finer square mesh on a fixed area has dimension $D = 2$ (Mandelbrot, p. 63, 1983). In this situation, all pixel content is therefore lost. Clearly then, improvement in resolution does not continue, ad infinitum; there is some point at which the tradeoff between fineness in resolution and loss of information content is at its peak. Determining this point is an issue of difficulty and significance. Is this dilemma a universal situation that exists independent of the shape of the fundamental pixel unit?

Hexagonal pixel arrangement

Consider instead an electronic environment in which the fundamental picture element is hexagonal in shape (Rosenfeld; Gibson and Lucas). Such a geometric environment has a number of well-documented advantages, centering on close-packing characteristics (Gibson and Lucas). This environment is examined here along the lines suggested above—to see if improvement in resolution can be carried out infinitely through pixel subdivision.

When a bounded lattice of regular hexagons of uniform cell diameter (on a CRT) is refined as a similar lattice of smaller uniform cell diameter, improvement in resolution results. There are an infinite number of ways in which the lattice of smaller cell-size might be superimposed on the lattice of larger cell size. The geometry of central place theory describes these relative positions of layers. Independent of the orientation selected, when this transformation from larger to smaller cell lattice is iterated infinitely, the bounded space is once again filled (as in the rectangular pixel case) with hexagonal pixel boundary. Thus, in both the case of the rectangular pixel and the hexagonal pixel environments, infinite "improvement" in resolution, brought about by decreasing pixel size, causes a black-hole-like collapse of the original, entire image. However, is this characteristic of the whole necessarily inherited by each of its parts? Any part that does not inherit this collapsing, space-filling characteristic is capable of infinite, "super-definition" resolution. Such a part is invariant (to some extent) under scale transformation.

The fractal approach to central place theory shows that there do exist shapes in the hexagonal pixel environment which, when refined infinitely, do not fill a bounded piece of two dimensional space. Figure 1 shows a hexagon to which a fractal generator has been applied to produce a $K = 4$ hierarchy. Infinite iteration of this self-similarity transformation produces a highly crenulated replacement which **does not** fill a bounded two-dimensional space; in fact, it fills only 1.585 of a two-dimensional space. When the corresponding self-similarity transformation is applied to a square pixel a highly crenulated shape is again the result of infinite iteration; this shape **does** fill a bounded two- dimensional space (Figure 2). The two fractal generators selected are parallel in structure: each is half of the boundary of the fundamental pixel shape.

If both geometric environments are then viewed as composed of these highly-crenulated elements (which do fit together to cover the plane), then the hexagonal environment is the one that permits infinite iteration without loss of all pixel content. This approach is akin to that of Barnsley, which stores sets of transformations that are used to drive image production. What is suggested here is a possible way to vastly improve image resolution corresponding,

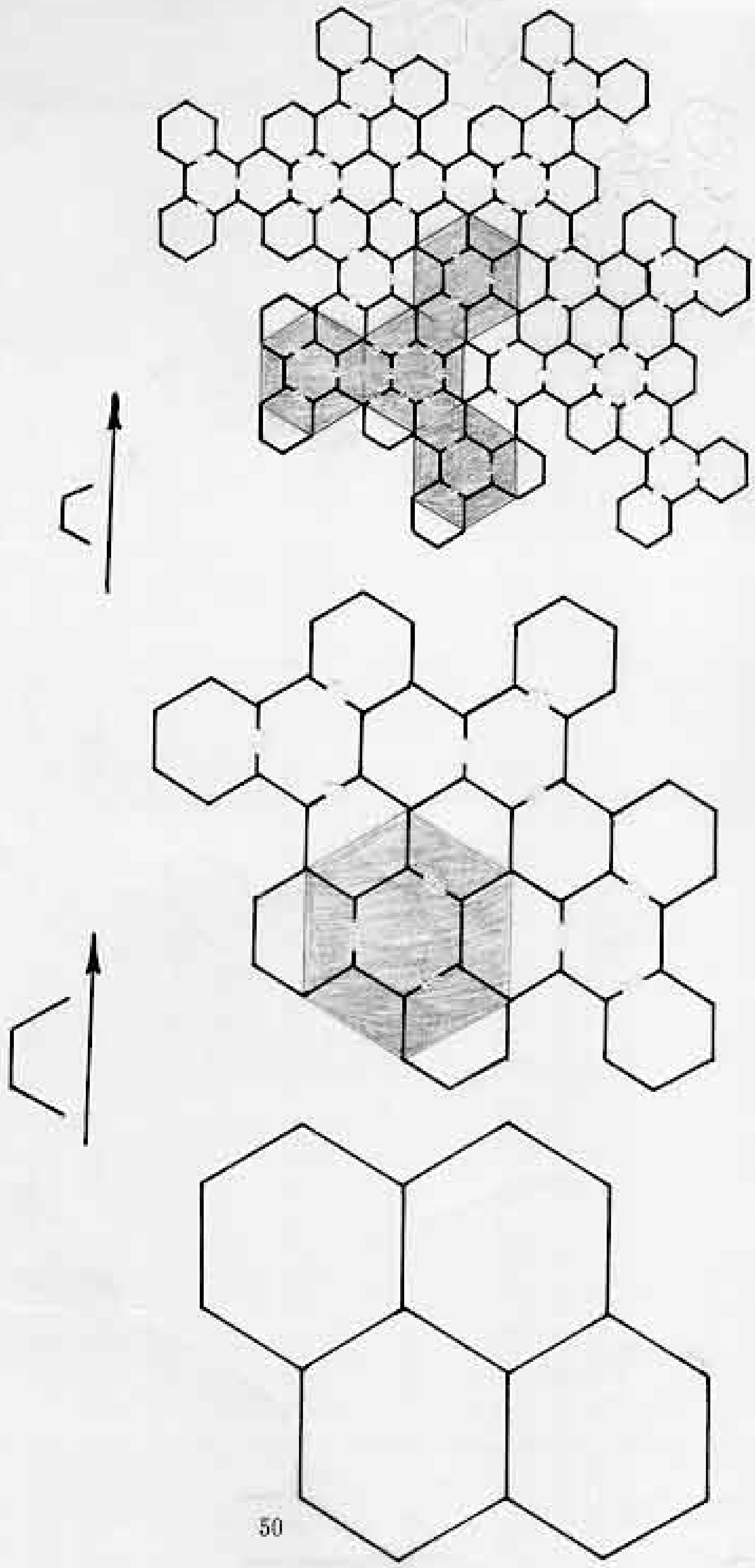

Figure 1. K=4 hierarchy of hexagonal pixels generated fractally.

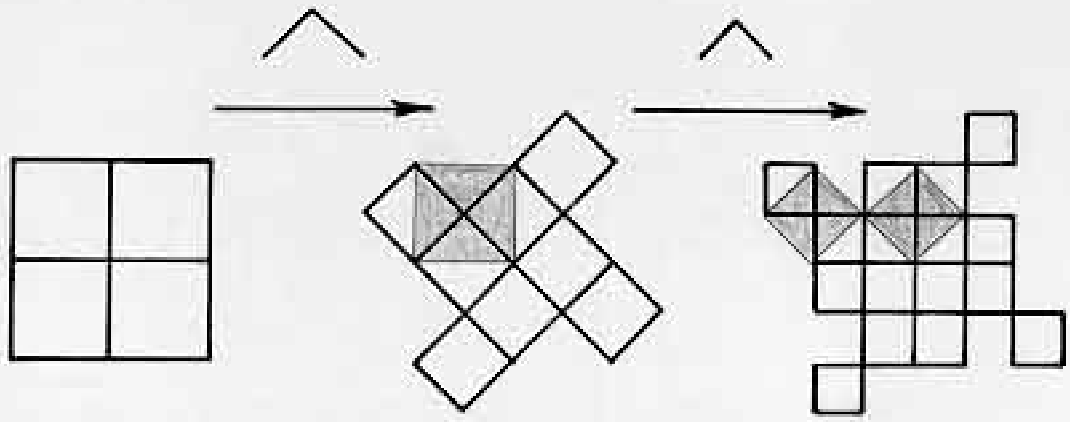

Figure 2. K=4 type of hierarchy generated fractally from square initiators.

to some extent, to Barnsley's successful strategy to improve data compression (Barnsley).

This approach is also similar, in general strategy to that employed by Hall and Gökmen; both seek transformations, applied in an electronic environment, under which some properties are preserved. Hall and Gökmen focus on transformations linking hexagonal and rectangular pixel space whereas the transformations employed here function entirely within a single type of geometric environment (using one on the other appears to be of interest). Additionally, this approach offers a systematic characterization, in the infinite, for the aggregate 7-kernels of hexagons, at various levels of aggregation, suggested only as finite sequences in Gibson and Lucas. Finally, Tobler's maps of Swiss migration patterns at three levels of spatial resolution suggest a methodological handle of an attractivity function to implement ideas involving spatial resolution in an electronic environment. Deeper analysis, of the sort represented in the works mentioned here, is beyond the scope of this particular short piece.

Table 1 shows a set of fractal dimensions for selected Löschian numbers.

Table 1

(derived from a Table in Arlinghaus and Arlinghaus, 1989)

K=3, D=1.262;	K=12, D=1.116;	K=27, D=1.087;	K=48, D=1.074;	...
K=7, D=1.129;	K=19, D=1.093;	K=37, D=1.078;	K=61, D=1.069;	...
K=4, D=1.585;	K=13, D=1.255;	K=28, D=1.168;	K=49, D=1.129;	...

The line of Löschian numbers that begins with $K = 4$, those that are organized according to an "transportation" principle, are the ones that fill two dimensional space most thickly. Thus, when introducing smaller and smaller hexagonal cells to improve resolution in the quality of curve representation, or when "zooming in," it would appear appropriate to let the orientation of successive layers of smaller and smaller cells correspond to the $K = 4$ type of hierarchy. Clutter would not enter as fast as in the Manhattan environment, even in this densest arrangement. "Super," rather than "high," definition of resolution could therefore fall naturally from an underlying hexagonal pixel geometry with measures of clutter and information content determined using fractal dimensions.

Shortest paths

At an even broader scale, one might also look for this sort of application in hooking computers together as parallel processing units. When "central places" are thought of as central processing units, not of urban information, but rather of electronic information, then an underlying geometry for finding "shortest" paths through networks linking multiple points might emerge. For in an electronic environment with the hexagonal pixel as the fundamental unit, the 120° intersection points would correspond exactly to the requirements for finding Steiner networks, as "shortest" networks linking multiple locations. Steiner points in an electronic configuration might then correspond to locations at which to "jump" from one hexagonal lattice of fixed cell-size to another of different cell size (from one machine to another), where cell size is prescribed by "lengths" (in whatever metric) between "transmission times" between adjacent Steiner points.

References

Arlinghaus, S. (1985). Fractals take a central place. *Geografiska Annaler*, 67B, 2, 83-88.

Arlinghaus, S. and Arlinghaus, W. (1989). The fractal theory of central place geometry: A Diophantine analysis of fractal generators for arbitrary Löschian numbers. *Geographical Analysis* 21, 2, 103-121.

Barnsley, M. F. *Fractals Everywhere*. San Diego: Academic Press, 1988.

Dacey, M. F. The geometry of central place theory. *Geografiska Annaler*. 47: 111-124.

Gibson, L. and Lucas D., Vectorization of raster images using hierarchical methods. Paper: Interactive Systems Corporation, 5500 South Sycamore Street, Littleton, Colorado, 80120.

Hall, R. W. and M. Gökmen. Rectangular/hexagonal tesselation transforms and parallel shrinking. Paper: Department of Electrical Engineering, University of Pittsburgh, Pittsburgh, PA 15261, TR-SP-90-004, June, 1990. Presented: Summer Conference on General Topology and Applications. Long Island University, 1990.

Horstmann, C. (1986). *ChiWriter: the scientific/multifont word processor for the IBM-P.C. (and compatibles)*. Ann Arbor: Horstmann Software Design.

Mandelbrot, B. (1983). *The Fractal Geometry of Nature*. San Francisco: W. H. Freeman.

Marshall, J. U. 1975. The Löschian numbers as a problem in number theory. *Geographical Analysis*. 7: 421-426.

Rosenfeld, A. (1990). Session on Digital Topology, National meetings of the American Mathematical Society, Louiville, KY, January, 1990.

Tobler, W. R. Frame independent spatial analysis, in Goodchild, M. F. and Gopal, *The Accuracy of Spatial Databases*. London: Taylor and Francis, 1990.

* The author wishes to thank Michael Goodchild for constructive comments on a 1989 version of this paper. Much of this content content has been presented previously: before national meetings of the American Mathematical Society in August of 1990; before national

meetings of the Association of American Geographers in April of 1990; and, before a classroom audience at The University of Michigan in the Winter Semester of 1989/90.