CONSTRUCTION ZONE

FIRST CONSTRUCTION;
readers might wish to construct figures to accompany
the electronic text as they read

**Feigenbaum's number: exposition of one case**

Motivated by queries from Michael Woldenberg,
Department of Geography, SUNY Buffalo,
during his visit to Ann Arbor, Summer, 1990.

Here is a description of how Feigenbaum's number arises from a graphical analysis of a simple geometric system [1]. Feigenbaum's original paper is clear and straightforward [1]; this construction is presented to serve as exposure prior to reading Feigenbaum's longer paper [1]. The construction is complicated although individual steps are not generally difficult. Following the construction, a suggestion will be offered as to how to select mathematical constraints within which to choose geographical systems for Feigenbaum-type analysis.

1. Consider the family of parabolas $y = x^2 + c$, where $c$ is an integral constant. This is just the set of parabolas that are like $y = x^2$, slid up or down the $y$-axis. The smaller the value of $c$, the more the parabola opens up (otherwise a lower one would intersect a higher one, creating an algebraic impossibility such as $-1 = 0$) (Figure 1).

2. To begin, consider the particular parabola, $y = x^2 - 1$, obtained by setting $c = -1$. Graph this (Figure 2). Also draw the line $y = x$ on this graph. Now we're going to look at the "orbit" of the value $x = 1/2$ with respect to this parabola (function). By "orbit" is meant simply the iteration string obtained by using $x = 1/2$ as input into $y = x^2 - 1$, then using that output as a new input into $y = x^2 - 1$, then using that output as a new input ... and so forth. In this case, the orbit of $x = 1/2$ is represented as follows, numerically. (Use $.5 \mapsto -0.75$ to mean that the input of .5 is mapped to the output value of $-0.75$ by the function $y = x^2 - 1$.)

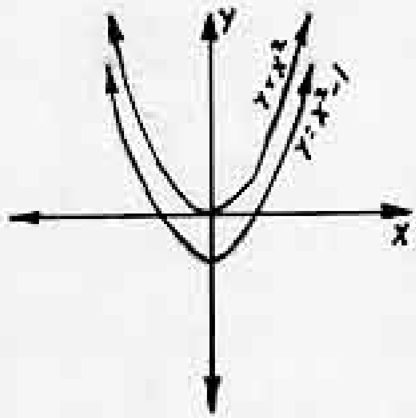
$$0.5 \mapsto -0.75 \mapsto -0.4375 \mapsto -0.8085938$$
$$\mapsto -0.3461761 \mapsto -0.8801621 \mapsto -0.2253147$$
$$\mapsto -0.9492333 \mapsto -0.0989562 \mapsto -0.9902077$$
$$\mapsto -0.019488 \mapsto -0.9996202 \mapsto -0.0007595$$
$$\mapsto -0.9999994 \mapsto -0.0000012 \mapsto -1$$
$$\mapsto 0 \mapsto -1 \mapsto 0 \mapsto \ldots$$

Clearly the values bounce around for awhile, and then eventually settle down to the values, $-1$ and $0$.

3. Let's see what this particular iteration string means geometrically (Figure 3). Locate $x = 0.5$ on the $x$-axis. Drop down to the parabola to read off the corresponding $y$-value (in the usual manner) $-0.75$. Now it is this $y$-value that is to be used as the next input in the iteration string. We could go back up to the $x$-axis and find it and drop back to the parabola, but we won't. Instead execute the following, equivalent transformation—THIS IS THE KEY POINT. Assume your penpoint is on the $y$-value $-0.75$; now slide

horizontally over to the line $y = x$—you want to use the $y$-value in the role of the $x$-value. Thus, treat this point as the new input and drop to the parabola from it as you did in moving from the $x$-axis to the parabola. Then, with your penpoint on the parabola, slide horizontally back to the line $y = x$ and use this as the input; drop to the parabola and keep going. A glance at Figure 2 suggests why economists call this a "cobweb" diagram (presumably looking at fluctuating supply and demand). Follow this diagram long enough, and you will see that eventually values for $x$ fluctuate between 0 and $-1$, around a stationary square cycle. Looking at the "dynamics" of a value, with respect to a function, in this geometrical manner is referred to as (Feigenbaum's) "graphical analysis" [1].

1

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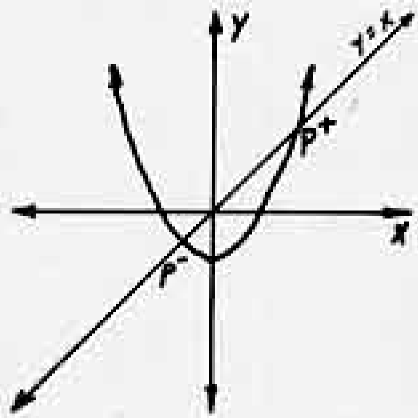

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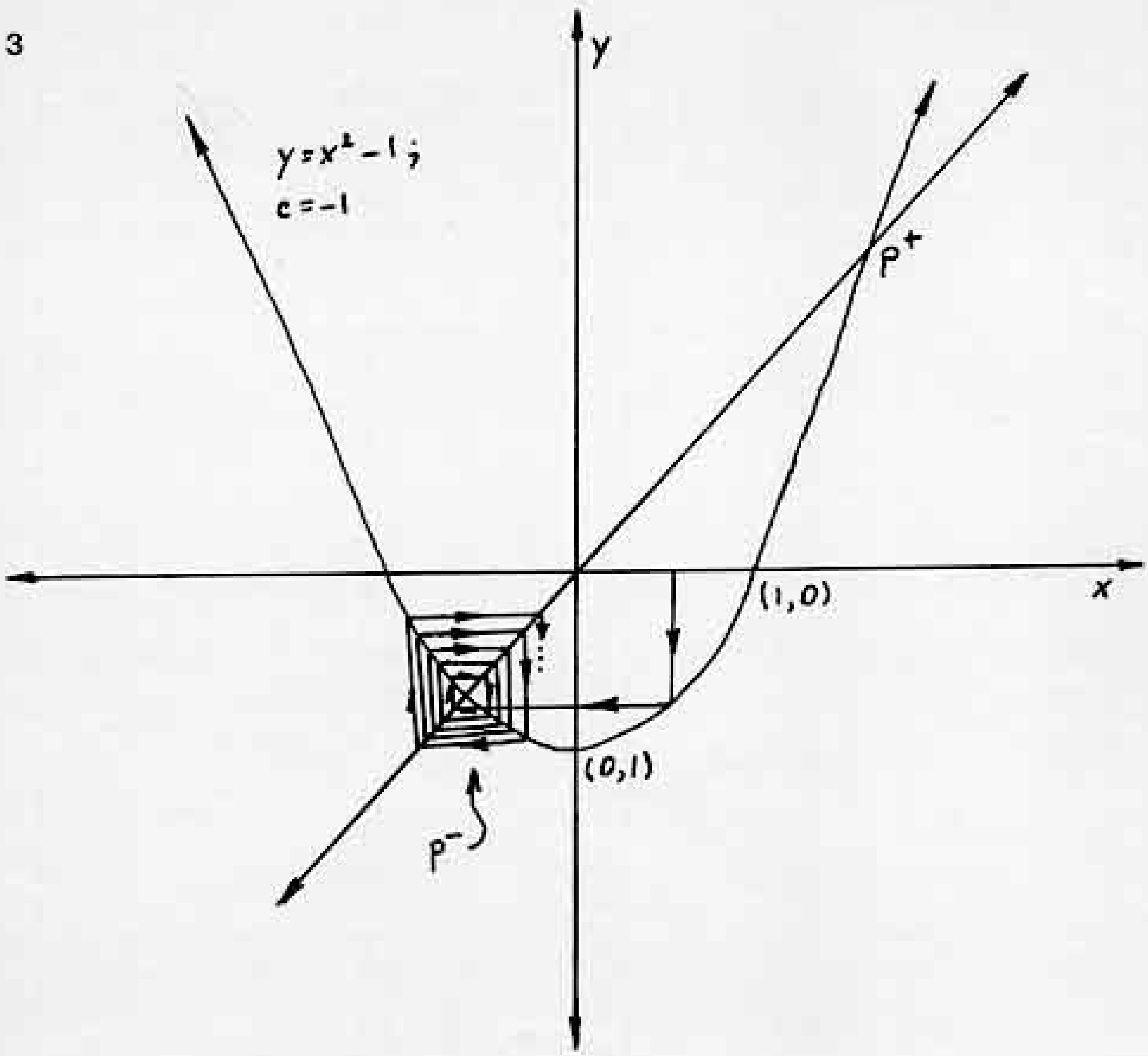


**Figure 1.** Parabolas of the form $y = x^2 + c$. **Figure 2.** The parabola $y = x^2 - 1$ and $y = x$. **Figure 3.** Graphical analysis of $y = x^2 - 1$.

4. So, we have the numerical orbit and the graphical analysis for the value $x = 0.5$ with respect to the function $y = x^2 - 1$. What about calculating these values for starting values of $x$ other than $x = 0.5$. Consider $x = 1.6$. Its orbit is as below, and the

corresponding graphical analysis is given in Figure 4.

$$1.6 \mapsto 1.56 \mapsto 1.4336 \mapsto 1.055209$$

$$\mapsto 0.1134659 \mapsto -0.9871255 \mapsto -0.0255833$$

$$\mapsto -0.9993455 \mapsto -0.0013086 \mapsto -0.9999983$$

$$\mapsto -0.0000034 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto 0 \mapsto \ldots$$

The dynamics of $x = 1.6$ are really very much the same as for $x = 0.5$ with respect to the given function. Let's look at $x = 1.7$.

$$1.7 \mapsto 1.89 \mapsto 2.5721 \mapsto 5.6156984$$

$$\mapsto 30.536069 \mapsto 931.45149 \mapsto 867600.87 \mapsto \ldots \, to \, \infty.$$

Graphical analysis shows this clearly, geometrically, too (Figure 5). This shooting off to infinity is not "interesting" in the way that the cobweb dynamics are. So, for what values of $x$ do you get "interesting" dynamics?

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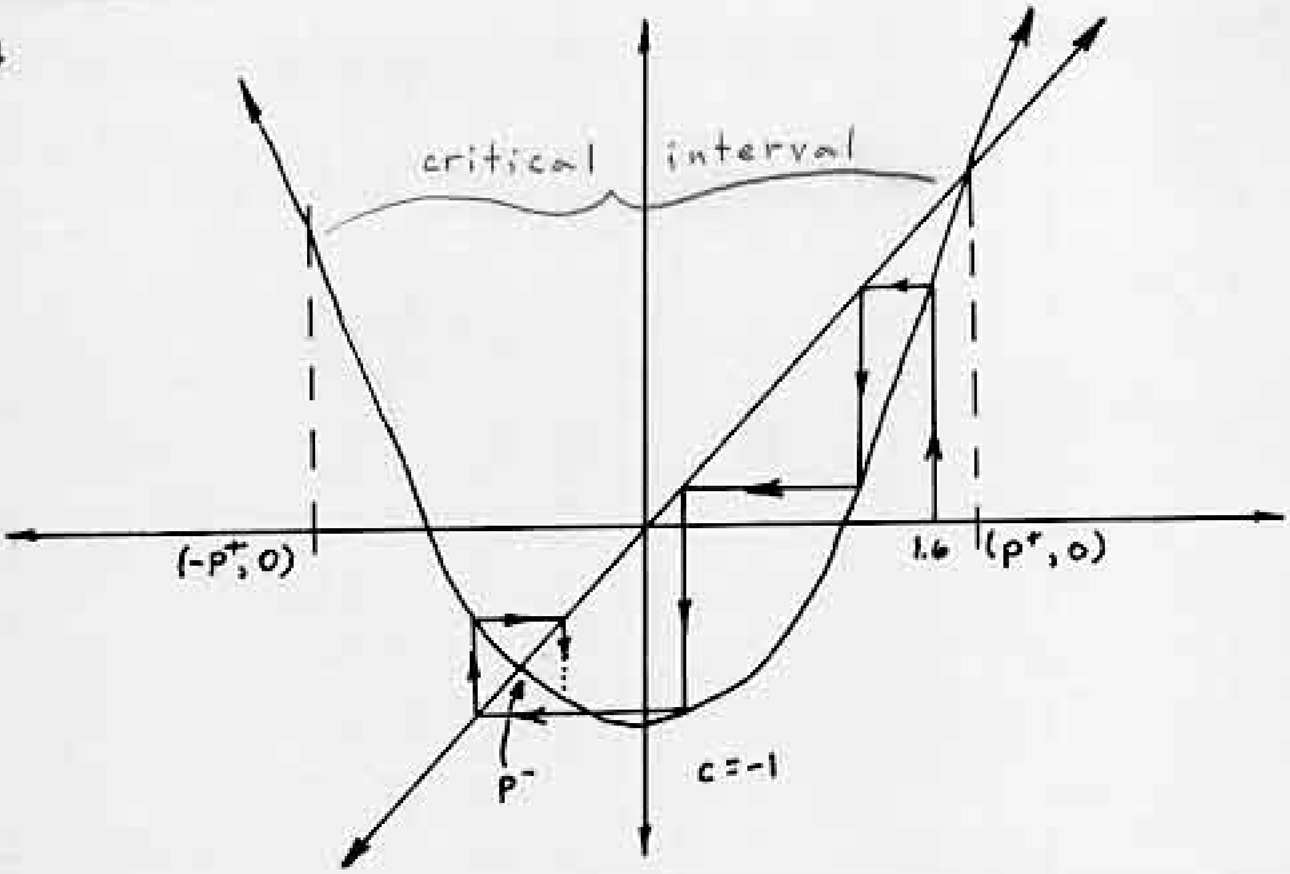

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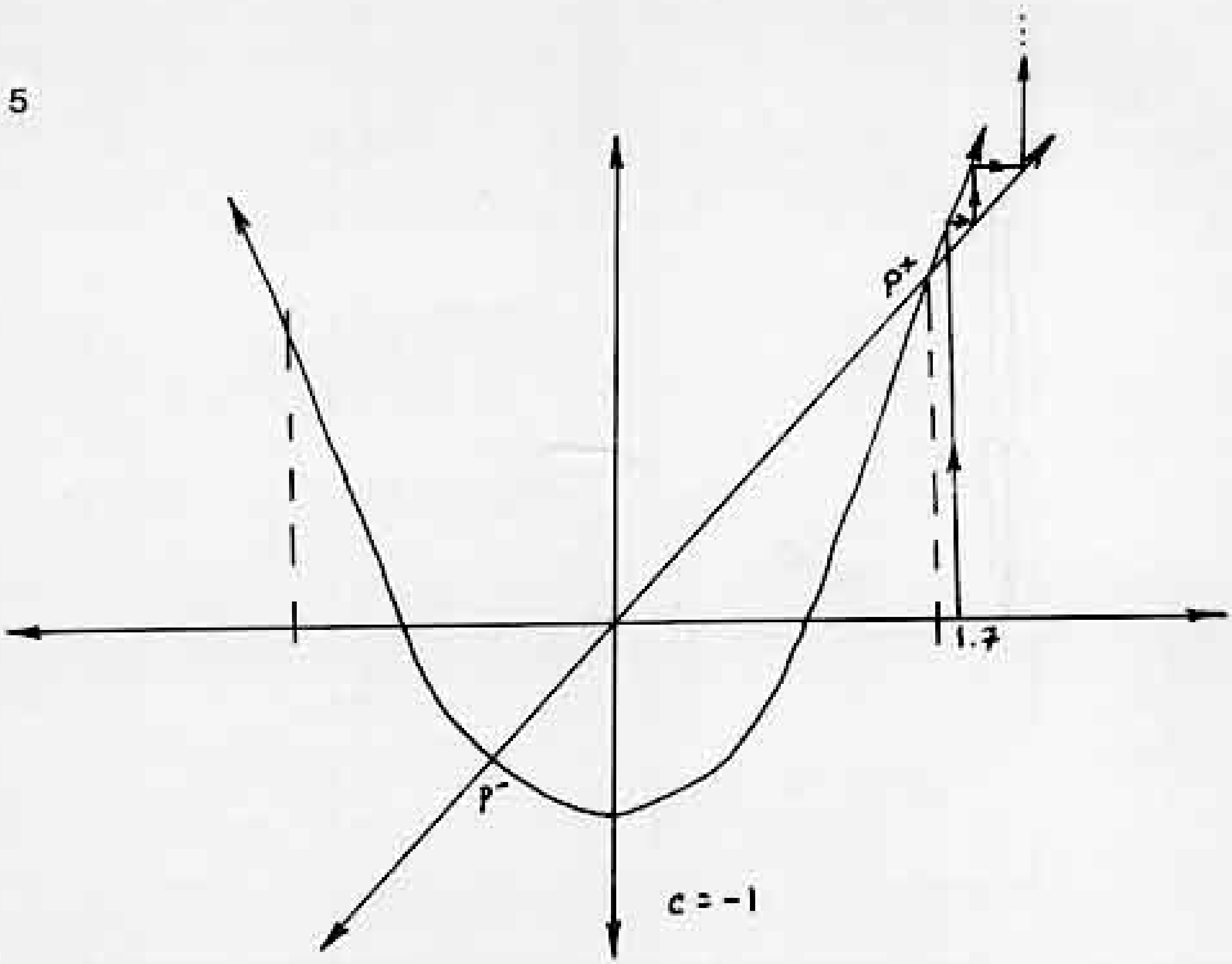



**Figure 4.** Orbit of $x = 1.6$. **Figure 5.** Orbit of $x = 1.7$.

5. No doubt you will have noted from the graphical analyses in Figures 4 and 5 that the reason one iteration closes down into a cobweb and the other goes to infinity is that one initial value of $x$ lies to the left of the intersection point of the parabola and the line $y = x$, and the other lies to the right of that intersection point. You might therefore be tempted to guess that all initial values of $x$ that lie between the right hand intersection

point (call it $p^+$) of the parabola and the line and the left hand intersection point (call it $p^-$) of the parabola and the line $y = x$, produce interesting dynamics. (The $x$–coordinates for $p^+$ and $p^-$ are found by solving $y = x$ and $y = x^2 - 1$ simultaneously—that is by solving $x^2 - x - 1 = 0$—the quadratic formula yields $x = (1 \pm \sqrt{5})/2$, or $x = 1.618034$, $x = -0.618034$). Indeed, if you try a number of values intermediate between these you will find that to be the case. However, consider a value of $x$ to the left of $x = -0.62$. Try $x = -1.6$.

$$-1.6 \mapsto 1.56 \mapsto 1.4336 \mapsto 1.055209$$

$$\mapsto 0.1134659 \mapsto -0.9871255 \mapsto -0.0255833$$

$$\mapsto -0.9993455 \mapsto -0.0013086 \mapsto -0.9999983$$

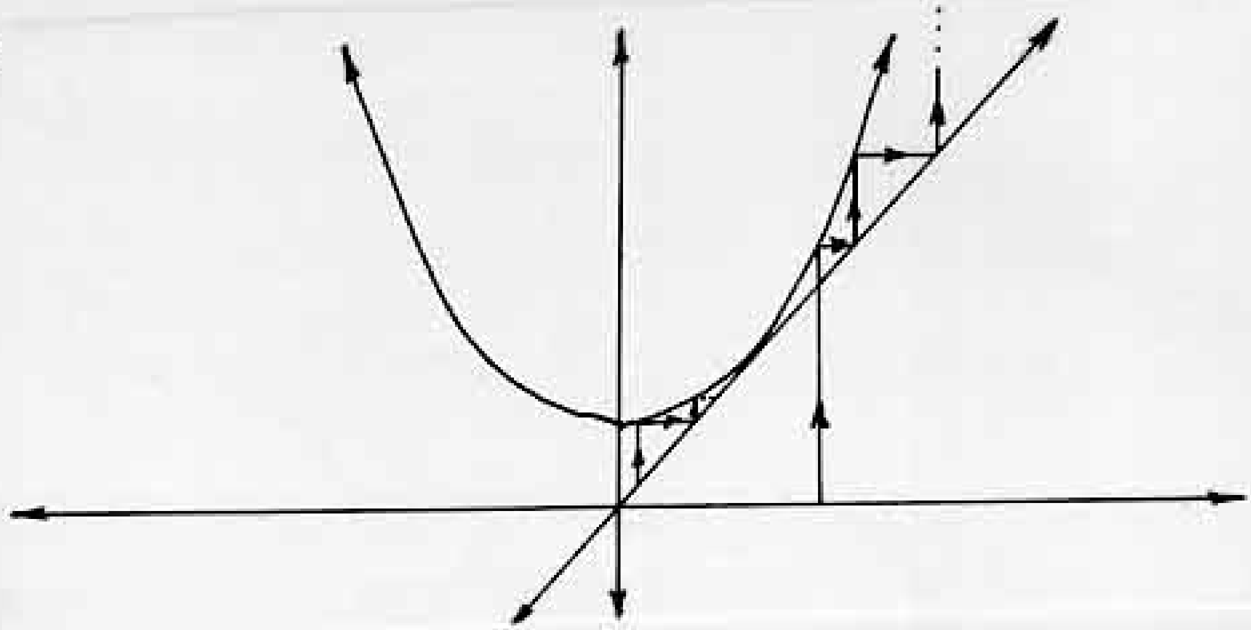
$$\mapsto -0.000003 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto 0 \mapsto \ldots$$

There is obvious bilateral (about the $y$–axis) symmetry in the iteration string, produced by squaring inputs. Clearly, the initial value of $-1.7$ will go to positive infinity, as above. So, the interval of values of $x$ that will produce interesting dynamics is NOT $[p^-, p^+]$, but rather $[-p^+, p^+]$. You might want to draw graphical analyses for $x = -1.6$ and $x = -1.7$ with respect to this function. Call the interval, $[-p^+, p^+]$ the "critical" interval for any given system of parabola and $y = x$. In the case of the system $y = x$ and $y = x^2 - 1$ the critical interval has length 3.236068.

So, now we know something general about the dynamics of input values with respect to the function $y = x^2 - 1$. Recall that we got this function by picking one value, $c = -1$, from the family of parabolas $y = x^2 + c$. Let's see what happens for different values of $c$.

6. Consider $c = 0.25$. For this value of $c$, the line $y = x$ and the parabola $y = x^2 + 0.25$ are tangent to each other. Values of $x$ to the left of the point of tangency (at (0.5, 0.25)) have orbits that converge to 0.5 (Figure 6) while values of $x$ to the right of the point of tangency have orbits that go to positive infinity. Initial inputs to the left of the point of tangency have orbits that are "attracted" to the point of tangency, while initial inputs to the right of the point of tangency have orbits that are "repelled" from the point of tangency. Here, you might view it that $p^+ = p^-$. When $c > 0.25$, the line $y = x$ and the corresponding parabola do not intersect, and so all orbits go to infinity—the dynamics are not interesting (Figure 7). So, we should be looking at parabolas with $c$ less than or equal to 0.25. Let's look at some, in regard to the notions of "attracting" and "repelling."

6



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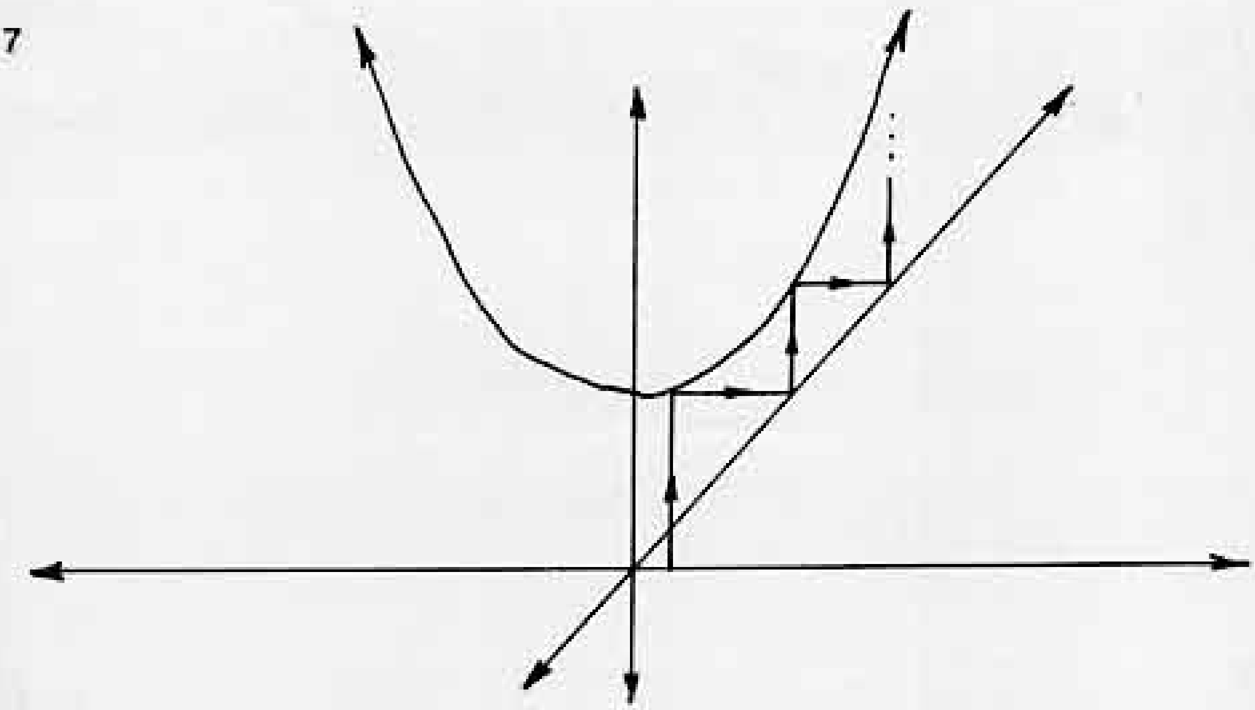

**Figure 6.** The case for $c = 1/4$. **Figure 7.** The case for $c > 1/4$.

7. Consider $c = 0.24$—system: $y = x$, $y = x^2 + 0.24$ (Figure 8). Use graphical analysis to study the dynamics (Figure 8). An orbit of 0.5 is

$$0.5 \mapsto .3025 \mapsto .3315063 \mapsto .3498964$$

$$\mapsto .362427 \mapsto .3713537 \mapsto .3779036$$

$$\mapsto .3828111 \mapsto .3865443 \mapsto .3894165$$

$$\mapsto .3916452 \mapsto .393386 \mapsto .3947525 \mapsto \ldots \mapsto 0.4.$$

The orbit converges to the $x$-value of $p^-$ which is found as 0.4 by solving the system using the quadratic formula. Here, $p^-$ is an attracting fixed point of the system, and $p^+$ is a repelling fixed point of the system. There is convergence of orbits to a single value within the zone $[-p^+, p^+]$. Notice a kind of doubling effect as one moves from the system with $c = 0.25$ to the one with $c = 0.26$ (period-doubling).

8. Consider $c = -0.74$. The system is: $y = x$, $y = x^2 - 0.74$. Graphical analysis (Figure 9) shows that this system behaves similarly to the one for $c = 0.24$; $p^-$ is attracting and $p^+$ is repelling for all $x$ in $[-p^+, p^+]$. The values of $p^-$ and $p^+$ are respectively $-0.4949874$ and $1.4949874$. Look at the orbit of 0.5, for example.

$$0.5 \mapsto -0.49 \mapsto -0.4999 \mapsto -0.4901$$

$$\mapsto -0.499802 \mapsto -0.490198 \mapsto \ldots \mapsto -0.4949874$$

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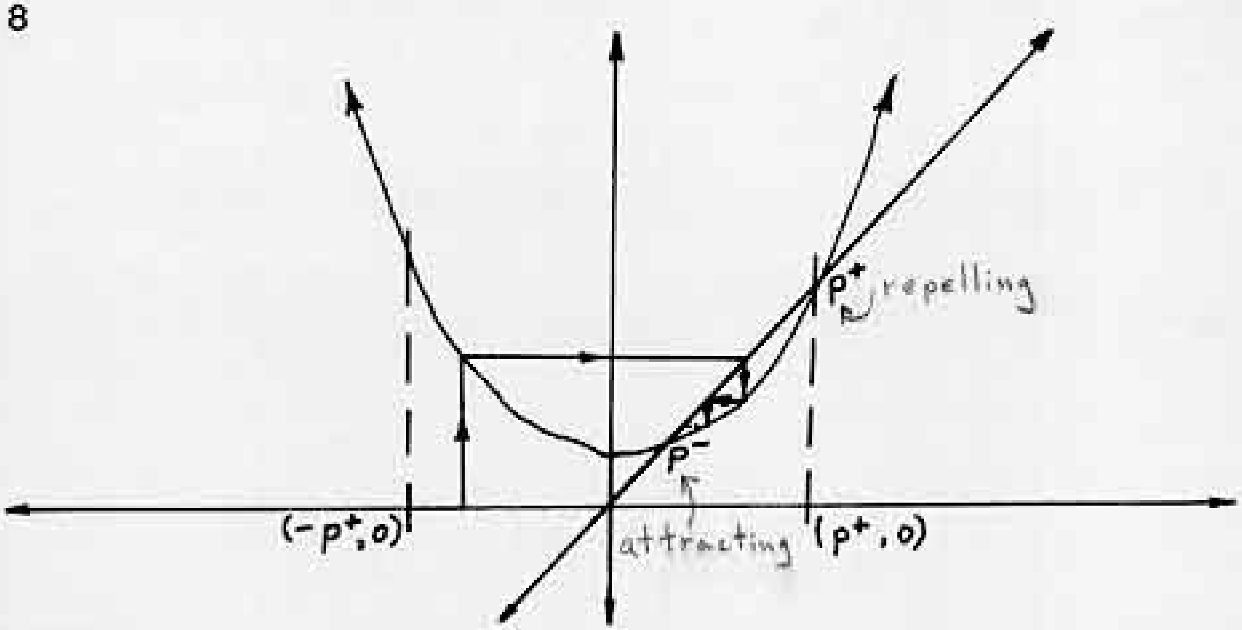

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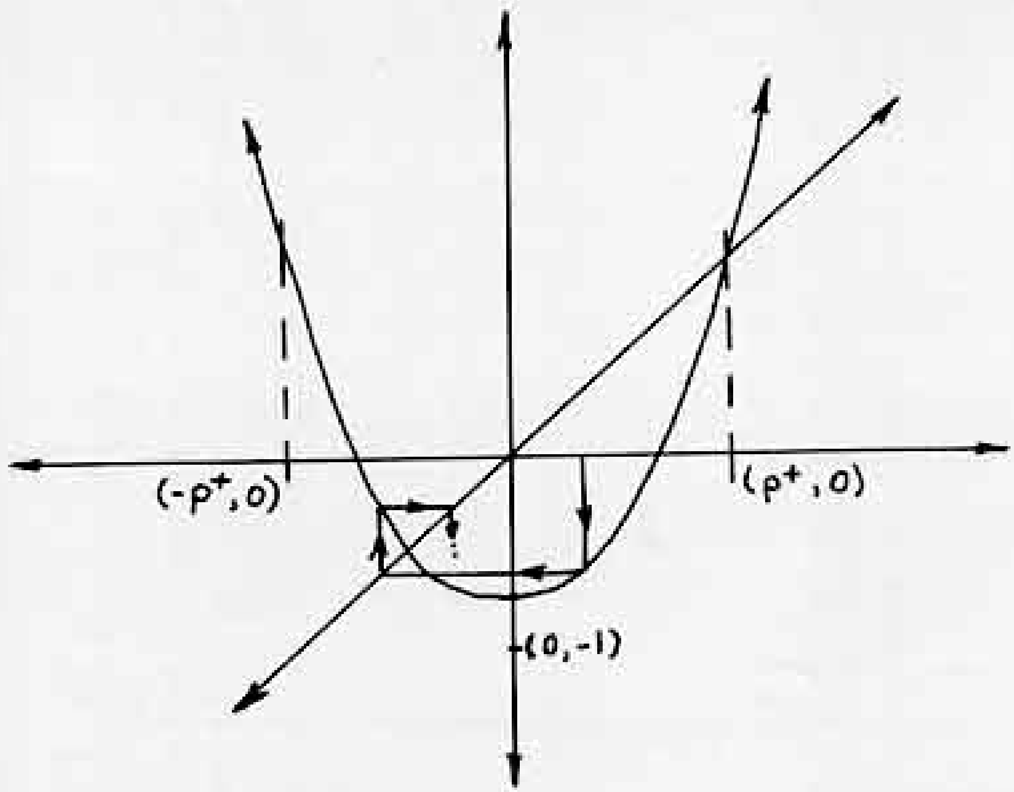


**Figure 8.** The case for $c = 0.24$. **Figure 9.** The case for $c = -0.74$.

9. Consider $c = -0.75$. The system is: $y = x$, $y = x^2 - 0.75$. This is not at all the same sort of system as those in 7 and 8 above. Here, $p^-$ and $p^+$ are respectively $-0.5$ and 1.5. Consider the orbit of 0.5.

$$0.5 \mapsto -0.5 \mapsto -0.5 \mapsto -0.5 \mapsto \ldots$$

Consider the orbit of 0.1:

$$0.1 \mapsto -0.74 \mapsto -0.2024 \mapsto -0.7090342$$

$$\mapsto -0.2472704 \mapsto -.6888573 \mapsto -.2754756$$

$$\mapsto -.6741132 \mapsto -.2955714 \mapsto -.6626376 \mapsto -.3109115 \mapsto \ldots$$

here, one might see this closing in, from above and below, very slowly on $-0.5$. Or, there might be two points the orbit is fluctuating toward getting close to. Consider the orbit of 1.4:

$$1.4 \mapsto 1.21 \mapsto .7141 \mapsto -.2400612 \mapsto -.6923706 \mapsto \ldots$$

Again, the same sort of thing as above. The behavior of this system is suggestive of that of the tangent case when $c = 0.25$.

10. So, we might suspect some sort of shift in the dynamics for values of $c$ less than $-0.75$. Indeed, we have already looked at the case $c = -1$. In that case, the point $p^-$ is repelling, rather than attracting (as it was for $0.25 < c < -0.75$). Also, the length of the period over which an orbit stabilizes has doubled — lands on two values, instead of converging to one. Again, there is a sort of bifurcation of dynamical process at $c = -0.75$, much as there was at $c = 0.25$. The next value of c at which there is bifurcation of process is at $c = -1.25$ (analysis not shown). Values of $c$ slightly less than $-1.25$ produce systems with orbits for initial $x$-values in the critical interval that settle down to fluctuating among four values; the point $p^-$, which had been repelling for $-0.75 < c < -1.25$ now becomes attracting. And so this continues— another bifurcation near 1.37, and another somewhere near 1.4. The values for $c$ at which successive bifurcations occur come faster and faster.
11. A summary of this material appears below.

Bifurcation values, $b$:

$$c = 0.25 - - - b = 1$$

$$c = -0.75 - - - b = 2$$

$$c = -1.25 - - - b = 3$$

$$c = -1.37 - - - b = 4$$

derived from empirical evidence of examining the orbit dynamics of the corresponding systems of parabolas and $y = x$. Lengths of critical intervals, $I_b$, $[-p^+, p^+]$, associated with the system corresponding to each bifurcation value, $b$.

$c = 0.25$; Solve: $y = x$, $y = x^2 + .25$; use quadratic formula—
$x = (1 \pm \sqrt{(1 - 4 \times 0.25)})/2 = 0.5$. Thus, $p^+ = 0.5$ so

$$I_1 = 2 \times 0.5 = 1.0$$

$c = -0.75$. Solve: $y = x$, $y = x^2 - .75$. $x = (1 \pm \sqrt{(1 + 4 \times 0.75)})/2 = 1.5$ or $-0.5$. Thus, $p^+ = 1.5$ so

$$I_2 = 2 \times 1.5 = 3.0$$

$c = -1.25$. Solve: $y = x$, $y = x^2 - 1.25$. $x = (1 \pm \sqrt{(1 + 4 \times 1.25)})/2 = 1.7247449$ or $-0.7247449$. So,

$$I_3 = 3.4494898$$

$c = -1.37$. Solve: $y = x$, $y = x^2 - 1.37$. $x = (1 \pm \sqrt{(1 + 4 \times 1.37)})/2 = 1.7727922$ or $-0.7727922$. So,

$$I_4 = 3.5455844$$

Now, suppose we find the successive differences between these interval lengths:

$$D_1 = I_2 - I_1 = 3 - 1 = 2$$

$$D_2 = I_3 - I_2 = 3.4494898 - 3 = 0.4494898$$

$$D_3 = I_4 - I_3 = 3.5455844 - 3.4494898 = 0.0960946$$

Then, form successive ratios of these differences, larger over smaller:

$$D_1/D_2 = 2/0.4494898 = 4.4494892$$

$$D_2/D_3 = .4494898/.0960946 = 4.6775761$$

This set of ratios converges to Feigenbaum's number, 4.6692016...

12. Apparently, empirical evidence suggests that any parabola–like system exhibits the same sorts of dynamics and the corresponding sets of ratios converge to Feigenbaum's number. For example, this appears to be the case, from literature, for the system $y = x$ and $y = c(sinx)$ and for the system involving the logistic curve, $y = x$ and $y = cx(1 - x)$ [1].

13. However, when the curved piece of the system is not parabola–like, different constants may occur. (A different curve might be a parabola with the vertex squared off— singularities are introduced—where the derivative is undefined) [1].

14. Obviously, many geographical systems can be characterized by a curve with fluctuations that are somewhat parabolic. Of course, we often do not know the equation of the curve. But, Simpson's rule from calculus, that pieces together parabolic slabs to approximate the area under a curve, generally gives a good approximation to the area of such curves. Thus, geographic systems that give rise to curves for which Simpson's rule provides a good areal approximation are ones that might be reasonable to explore in connection with Feigenbaum's number.

15. Steps 1 to 11 show how Feigenbaum's "universal" number can be generated. Steps 12 to 14 give a systematic way to select geographical systems to examine with respect to this constant.

## REFERENCE

Feigenbaum, Mitchell J. "Universal behavior in non–linear systems." *Los Alamos Science*, Summer, 1980, pp. 4-27.

SECOND CONSTRUCTION

A three-axis coordinatization of the plane

Motivated by a question from Richard Weinand

Department of Computer Science, Wayne State University

1. Triangulate the plane using equilateral triangles. Then, choose any triangle as a triangle of reference—this triangle is to serve as an "origin" for a coordinate system (an area-origin rather than a conventional point-origin—this is like homogeneous coordinates in projective geometry *e.g.* H. S. M. Coxeter, *The Real Projective Plane*). Each side of the triangle is an axis—$x=0$, $y=0$, $z=0$ (Figure 10-draw to match text).

2. Each vertex of a triangle has unique representation as an ordered triple with reference to the origin-triangle (but, not every ordered triple of integers corresponds to a lattice point— there is no point $(x,x,x)$) (Figure 10).

3. Assign an orientation (clockwise or counterclockwise) to the origin-triangle, and mark the edges of the triangle with arrowheads to correspond to this orientation. This then determines the orientation of all the remaining triangles.

4. Now suppose that a triangle is picked out at random. Suppose it has orientation the same as the reference triangle (clockwise, say). The coordinates of its vertices, in general, will be (choosing $(x,y,z)$ to be the lower left-hand corner):

$$(x,y,z);(x+1,y,z-1);(x,y+1,z-1)$$

and those of triangles sharing a common edge with it (and of opposite orientation to it) will have coordinates:

$$\text{left}:(x,y,z);(x+1,y,z-1);(x+1,y-1,z)$$

$$\text{right}:(x+1,y,z-1);(x,y+1,z-1);(x+1,y+1,z-2)$$

$$\text{bottom}:(x,y+1,z-1);(x,y,z);(x-1,y+1,z)$$

Suppose the arbitrarily selected triangle has orientation opposite that of the reference triangle (counterclockwise). The coordinates of its vertices, in general, will be (choosing $(x,y,z)$ to be the upper left-hand corner):

$$(x,y,z);(x-1,y+1,z);(x,y+1,z-1)$$

and those of triangles sharing a common edge with it (and of opposite orientation to it (clockwise)) will have coordinates:

$$\text{left}:(x,y,z);(x-1,y+1,z);(x-1,y,z+1)$$

$$\text{right}:(x-1,y+1,z);(x,y+1,z-1);(x-1,y+2,z-1)$$

$$\text{top}:(x,y,z);(x+1,y,z-1);(x,y+1,z-1)$$

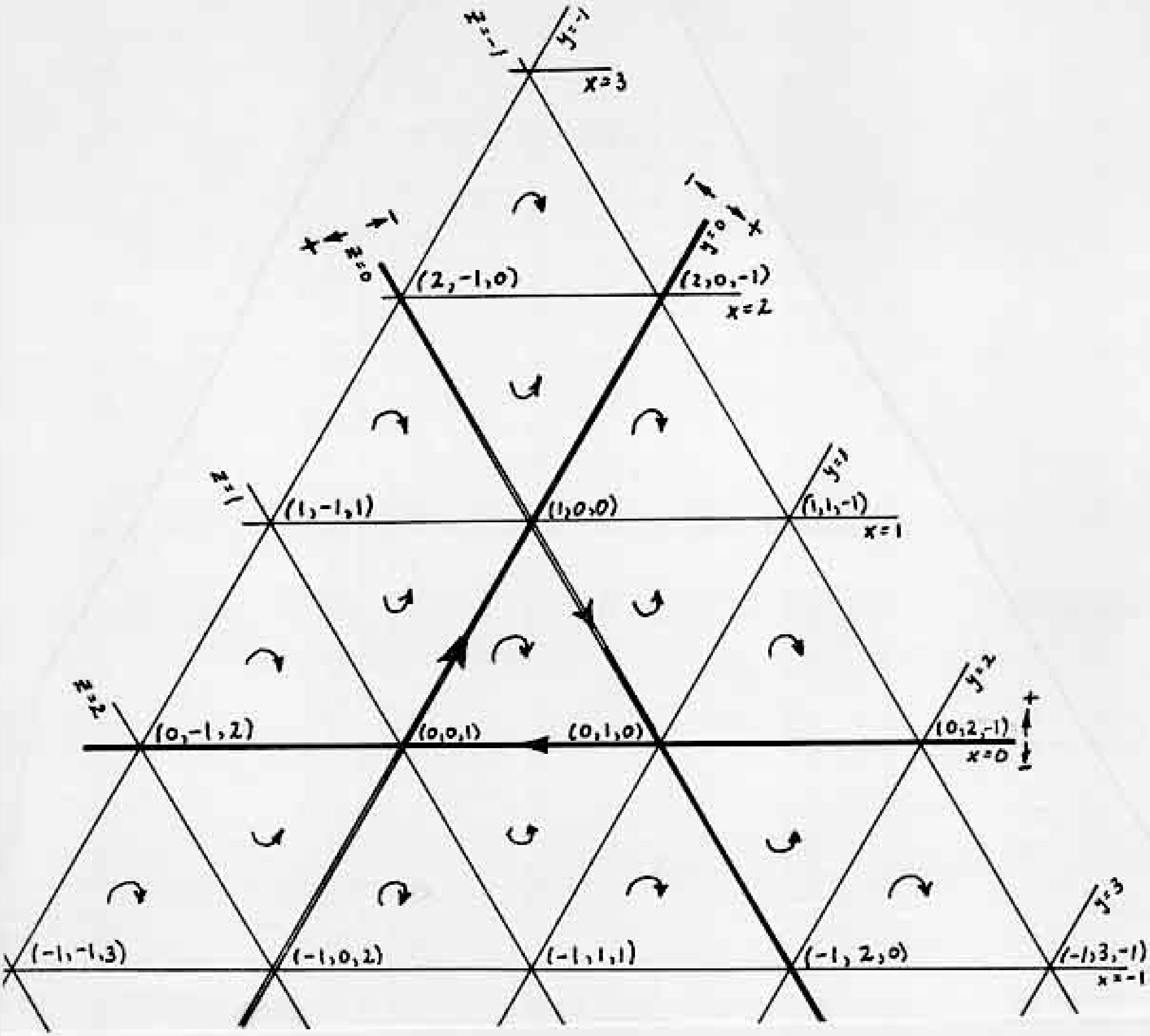


Figure 10. Three-axis coordinate system for the plane.

5. Coordinates of triangles sharing a point-boundary (and of the same orientation as the arbitrarily selected triangle) might also be read off in a similar fashion.

6. Naturally, six of these triangles form a hexagon. So, this could be considered from the viewpoint of an hexagonal tesselation, as well. Choose an arbitrary hexagon and read off coordinates of adjacent hexagonal regions in a similar manner.

7. In a current *College Mathematics Journal*, Vol 21, No. 4, September, 1990, there is an article by David Singmaster (of Rubik's Cube fame) which also employs triangular coordinates of the sort mentioned above (pages 278-285— "Triangles with integer sides and sharing barrels").

8. This strategy would seem to work for any developable surface (cylinder, torus, Möbius strip, Klein bottle—all can be cut apart into a plane). Triangles were chosen because procedure involving them might be extended to simplicial complexes (triangle=simplex).

9. One way to triangulate a sphere is to project an icosahedron, inscribed in the sphere, onto the surface of the sphere (conversation with Jerrold Grossman, Dep't. of Mathematics, Oakland University). This procedure will produce 20 triangular regions of equal size (under suitable transformation). But, more triangles may be desirable. Alternately, one might subdivide the triangular faces of the icosahedron into, say, three triangles of equal area, and project the point that produces this subdivision (a barycentric subdivision, for example) onto the sphere (using gnomonic projection (from the sphere's center)). (Subdividing all of them a second time would produce 180 triangles of equal area and shape covering the sphere.) Subdivision centers on opposite sides of the icosahedron appear to lie on a single diameter of the sphere; therefore, when their images are projected onto the sphere they will be antipodal points. In that event, a coordinate system similar to the one described for developable surfaces might work.