THE UNIVERSITY OF MICHIGAN
INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

THE EVALUATION OF THE REFRACTION ANGLES
IN GEODETIC MEASUREMENTS
FROM TWO OBSERVATION STATIONS

By
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SYMBOLS AND ABBREVIATIONS

C = the constant of the perfect Gas Law

g = the acceleration due to the force of gravity

k = a constant which varies slightly with the moisture content of the air

R = the mean radius of curvature of an arc joining two points on the earth's surface

\( h_a \) = the distance between stations A and C

\( h_b \) = the distance between stations B and C

\( \mu \) = the index of refraction of white light in the air

\( \alpha \) = a measured vertical angle

\( \theta \) = an angle subtended by an arc at the center of the earth

\( \Omega \) = the refraction angle

\( T \) = the temperature of the air measured from absolute zero (-273° C.)

\( \rho \) = the pressure of the air

\( \sigma \) = the radius of curvature of a ray of light

\( k \) = the coefficient of refraction

\( h_a \) = the height of Station A above mean sea level

\( h_b \) = the height of Station B above mean sea level

\( h_c \) = the height of Point C above mean sea level

\( \sigma_A \) = the probable error of the refraction angle at Station A

\( \sigma_B \) = the probable error of the refraction angle at Station B

\( \sigma_{h_c} \) = the probable error of the height of Point C computed from Station A

\( \sigma_{h_b} \) = the probable error of the height of Point C computed from Station B
SYMBOLS AND ABBREVIATIONS (CONT'D)

$\delta \alpha_a$ = the probable error of the measured vertical angle at Station A

$\delta \alpha_b$ = the probable error of the measured vertical angle at Station B

$\frac{dT}{dh}$ = the lapse rate of temperature

EST = Eastern Standard Time
INTRODUCTION

Terrestrial refraction is the bending of a ray of light when it travels between any two points within the atmosphere. This phenomenon causes all objects to appear to be at higher elevations than they actually are, except when the air near the earth’s surface is strongly heated, causing the density of air to increase with height so that the objects appear to be at lower elevations.

Terrestrial refraction is a matter of great importance in the subject of trigonometrical leveling. The exact knowledge of it is important in correcting the measured vertical angles resulting from the effect of refraction and reducing them to their actual values.

Previous investigations have dealt with the problem of refraction and some formulas have been developed. These formulas are based on meteorological factors, i.e. pressure, temperature, and lapse-rate of temperature along the path of light; but because of the difficulties of determining the exact values of the pressure, temperature and lapse rate of temperature, the computations arrived at by these formulas are uncertain in most cases.

The present investigation deals with a method of determining the refraction angles in which the meteorological factors are eliminated from the computation.

The method involves observing one point from two observation stations. These observation stations are located in such a way that the paths of light rays from the point to the two stations will be close to
each other; also, the difference between the lengths of the two paths is such as to provide an appreciable discrepancy between the apparent heights of the observed point.

On the basis of this method, new formulas have been developed. A practical application of the new method was evolved and tested by running many experiments in different weather conditions. The results show that an adequate degree of accuracy was achieved.
CHAPTER I

GENERAL STATEMENTS

Two methods exist for determining the difference in elevation between points on the earth's surface. The first is by running lines of spirit level. The principle of this method is well-known and can be found in any surveying text. Moreover, the accuracy of this method is attested by any surveying text. The second method is trigonometrical leveling, by which the differences in elevations are determined from the horizontal or geodetic distances between points and the measured vertical angles after correcting them for the effect of curvature and refraction.

The second method is faster and easier than the first, but it is not as accurate as the first method, mainly because of the uncertainty in determining the refraction angles.

Refraction of light in air has been recognized for a long time, and some coefficients have been introduced; but none of these coefficients can be used with confidence unless the meteorological factors along the path of a ray of light are the same as those under which the coefficients were introduced. Also, previous investigations have produced some formulas for computing the refraction angles, but the accuracy of the results computed from these formulas depends upon exact knowledge of the pressure, temperature, and lapse rate of temperature along the paths of light.

Bomford's Geodesy\textsuperscript{1} gives the following formula for computing

\textsuperscript{1} Briggadier G. Bomford, Geodesy, Great Britain: Oxford at the Clarendon Press, 1952, p. 153, formula 4.15

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the refracting angles: (Figure 1)

$$\mathcal{O} = \frac{i}{L} \int_0^L \frac{L-j}{\alpha} \, dj,$$

(1-1)

Fig. 1 EFFECT OF REFRACTION

where

- $\mathcal{O}$ = the refraction angle,
- $L$ = the length of the path of a ray of light,
- $j$ = a length along the path of light measured from the observation point,
- $\alpha$ = the radius of curvature of the points along a path of a ray of light.
\( \frac{1}{\sigma} \) is given by the following formula:\(^1\)

\[
\frac{1}{\sigma} = \frac{KC\cos \alpha}{\mu} \frac{P}{T^2} \left( \frac{g}{C} + \frac{dT}{dh} \right),
\]

where

- \( K \) = a constant which varies slightly with the moisture of the air,
- \( \alpha \) = the measured vertical angle,
- \( P \) = the pressure,
- \( C \) = the constant of the perfect gas law,
- \( \mu \) = the index of refraction of white light in air,
- \( T \) = the temperature measured from absolute zero (-273°C),
- \( g \) = the acceleration due to the force of gravity,
- \( \frac{dT}{dh} \) = the lapse rate of temperature.

Substituting equation (1-2) in equation (1-1), one gets

\[
\Sigma = \frac{1}{L} \int_0^L \left[ \frac{KC\cos \alpha}{\mu} \frac{P}{T^2} \left( \frac{g}{C} + \frac{dT}{dh} \right) \right] (L - \theta) d\theta. \tag{1-3}
\]

Equation (1-3) is the fundamental formula for computing the refraction angle \( \Sigma \), and in order to determine \( \Sigma \) with adequate accuracy, pressure, temperature, and lapse rate of temperature must be known at all points along the path of a ray of light. Since it is impossible to measure the meteorological factors at so many points, this is an impracticable method of determining the refraction angles, and the method usually followed is to assume that the lapse rate of temperature is constant along the path of a ray of light and that the average temperatures and pressures along the path are those computed or measured at one-third the distance along

\(^1\) Ibid, formula 4.12
the path measured from the observation station. This method of sub-
stitution probably gives good results if the lapse rate of temperature
is determined exactly, and if the assumed values of the pressure and
temperature are equal to the actual values. Practically, the pressure
and the temperature may be determined within one percent or better,¹
but the doubtful item is the lapse rate of temperature. For example,
 Attempts were made in connection with the present investigation to
measure the lapse rate of temperature, and the following results were
obtained: On March 21, 1957, five attempts were made which gave results
that varied from 0.00221 degrees Centigrade per foot to 0.00725 degrees
Centigrade per foot. On March 28, 1957, seven attempts were made which
gave results that varied from 0.00242 to 0.00672 degrees Centigrade per
foot. Such variations make it difficult to determine the exact value
of the lapse rate of temperature; and an uncertainty of 0.001 in the
lapse rate of temperature causes an uncertainty of 10 per cent in the
computed angle of refraction². Therefore, refraction angles computed
from formulas based on the knowledge of the meteorological factors are
uncertain.

Up to the present time there has been no method of determining
the refraction angles with acceptable accuracy and without including the
pressure, the temperature, and the lapse rate of temperature in the
computation. No doubt the discovery of such a method will be a great
benefit in the determination of terrestrial refraction.

¹ Ibid, line 20.
² Ibid, line 28.
CHAPTER II

METHOD AND FORMULAS

The density of air is not constant, but varies with height; therefore a ray of light passing between two points within the atmosphere suffers a continuous change in direction. Thus, the path of a ray of light is a curve in its shape and not a straight line. This phenomenon causes displacement of the apparent positions of the observed objects.

Relative Refraction.

In Figure 2 assume A P C is the path of a ray of light from point A at height $h_a$ to point C at height $h_c$. Assume $\alpha_a$ is the observed angle of elevation C at point A above A D, which is the tangent to the earth's level surface. Assume A C\(_a\) is the tangent to the path A P C at point A. Then the angle C\(_a\) A C (\(\alpha_a\)) will be the angle of refraction and C C\(_a\) the appeared displacement of point C.

\[ \alpha_a = k_a \theta_a, \quad (2-1) \]

where

\[ k_a = \text{the coefficient of refraction,} \]

\[ \theta_a = \text{the angle subtended by the arc A F at the earth's center.} \]

Bomford's Geodesy gives the following equation for determining the coefficient of refraction\(^1\):

\[ k_c = 28000 \frac{\rho}{T^2} \left(0.0104 + \frac{dT}{dh}\right), \quad (2-2) \]

\(^1\) Ibid., p. 159, formula 4.17a

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Fig. 2 RELATIVE POSITIONS OF THE OBSERVATION STATIONS A AND B
where

\( k \) = the coefficient of refraction,
\( P \) = the pressure measured in inches of mercury,
\( T \) = the temperature measured from absolute zero (\(-273^\circ C\)),
\( \frac{dT}{dh} \) = the lapse rate of temperature in Centigrade per foot.

The constant 28000 is given as 27000 while actually it should be 28000.

Substituting equation (2-2) in equation (2-1), one obtains

\[
\Sigma_a = 28000 \frac{\rho_a}{(T_a)^2} \left[ 0.0104 + \left( \frac{dT}{dh} \right)_a \right] \theta_a ,
\]

(2-3)

where

\( \rho_a \) = the average pressure along the path A P C measured in inches of mercury,
\( T_a \) = the average temperature along the path A P C measured from absolute zero (\(-273^\circ C\)),
\( \left( \frac{dT}{dh} \right)_a \) = the lapse rate of temperature along the path A P C measured in Centigrade per foot.

The lengths of sight in geodetic measurements are relatively very short compared with the length of the radius of the earth, and it is safe to assume, therefore, that

\[
\theta_a = \frac{l_a}{R_a + h_a} ,
\]

(2-4)

where

\( l_a \) = the length of the arc AF,
\( R_a \) = the radius of curvature of the earth's surface taken at the middle distance between points A and F.
Substituting equation (2-4) in equation (2-3), we have

$$\zeta_a = 28000 \frac{R_a}{(\varphi_a)^2} \left[ 0.0104 + \left( \frac{dT}{dh} \right)_a \right] \frac{\varphi_a}{R_a + h_a} \tag{2-5}$$

Also, in figure 1, assume point B is another observation station, and let B Q C be the path of a ray of light from point B at height $h_b$ to point C. Assume $\varphi_b$ is the observed angle of elevation of point C at point B above B E, which is the tangent to the earth's surface. Assume B C_b is the tangent to the path B Q C at point B. Then the angle of refraction will be C B C_b($= \zeta_b$), and C C_b is the apparent displacement of point C.

Following the same procedure in deriving equation (2-5), one obtains the following equation for the refraction angle at station B,

$$\zeta_b = 28000 \frac{R_b}{(\varphi_b)^2} \left[ 0.0104 + \left( \frac{dT}{dh} \right)_b \right] \frac{\varphi_b}{R_b + h_b} \tag{2-6},$$

where

$\zeta_b =$ the refraction angle at station B,

$R_b =$ the average pressure along the path B Q C measured in inches of mercury,

$T_b =$ the average temperature along the path B Q C measured from absolute zero (-273° C.),

$$\left( \frac{dT}{dh} \right)_b =$ the lapse rate of temperature along the path B Q C in Centigrade per foot,

$\varphi_b =$ the length of the arc BG,

$R_a =$ the radius of curvature of the earth's surface taken at the middle distance between points B and G.

Dividing equations (2-5) and (2-6), one gets
\[
\frac{\sigma^2_a}{\sigma^2_b} = \frac{28000 \frac{R_a}{(T_a)^2} \left[ 0.0104 + \frac{d\bar{T}}{dh} \right]_a}{28000 \frac{R_b}{(T_b)^2} \left[ 0.0104 + \frac{d\bar{T}}{dh} \right]_b} \frac{\bar{P}_a}{\bar{P}_b} \frac{R_a + h_a}{R_b + h_b}
\]

or

\[
\frac{\sigma^2_a}{\sigma^2_b} = \frac{\bar{P}_a(T_b)^2 \bar{P}_a (R_a + h_a)}{\bar{P}_b(T_a)^2 \bar{P}_b (R_a + h_a)} \left[ 0.0104 + \frac{d\bar{T}}{dh} \right]_a \left[ 0.0104 + \frac{d\bar{T}}{dh} \right]_b
\]  

(2-7)

Positions of the Observation Stations.

In Figure 2, assume A and B are two observation stations, located so that the paths of sight from the stations A and B to point C have nearly the same azimuth. The distance between the stations A and B is about 2 miles, and the elevations of A and B can differ in accordance with the requirements of the terrain (a difference of a few hundred feet does not affect the results). Then the paths A P C and B Q C will be close to each other; hence the lapse rates of temperatures \( \frac{d\bar{T}}{dh} \) and \( \frac{d\bar{T}}{dh} \) \( \text{a} \) and \( \text{b} \) \( \), the average pressures \( \bar{P}_a \), \( \bar{P}_b \), and the average temperatures \( T_a \), \( T_b \) can be assumed equal.

Determination of the Refraction Angles

When the lapse rate of temperatures, the average pressures, and the average temperatures are taken equal, and if these equalities are substituted in equation (2-7), then the equation becomes

\[
\frac{\sigma^2_a}{\sigma^2_b} = \frac{\bar{P}_a(R_b + h_b)}{\bar{P}_b(R_a + h_a)}
\]  

(2-8)

The terms \( (R_a + h_a) \) and \( (R_b + h_b) \) can be assumed equal, so that equation (2-8) becomes
\[ \frac{\varepsilon_a}{\Delta z_b} = \frac{\overline{p}_a}{\overline{p}_b}. \]  

(2-9)

In Figure 1, we have

\[ \varepsilon C_a = \varepsilon_a \overline{p}_a', \]  

(2-10)

where

\[ \varepsilon_a = \text{the refraction angle at point } A, \]

\[ \overline{p}_a' = \text{the length of the path } A \ P \ C; \]

and

\[ \varepsilon C_b = \varepsilon_b \overline{p}_b', \]  

(2-11)

where

\[ \varepsilon_b = \text{the refraction angle at point } B, \]

\[ \overline{p}_b' = \text{the length of the path } B \ Q \ C. \]

Subtracting equation (2-11) from equation (2-10), one obtains

\[ \varepsilon C_a - \varepsilon C_b = \varepsilon_a \overline{p}_a' - \varepsilon_b \overline{p}_b', \]

or

\[ \varepsilon_b C_a = \varepsilon_a \overline{p}_a' - \varepsilon_b \overline{p}_b'. \]  

(2-12)

In Figure 3, we have

\[ \frac{F C_a}{\sin (\alpha_a + \frac{\Theta_a}{2})} = \frac{A F}{\sin \left[ 90 - (\alpha_a + \theta_a) \right]}, \]

or

\[ F C_a = \overline{p}_a \sin \left( \alpha_a + \frac{\Theta_a}{2} \right) \sec (\alpha_a + \theta_a), \]  

(2-13)
and

\[ h_{ca} = F C_a + h_a \]  \hspace{1cm} (2-14)

Substituting equation (2-13) in equation (2-14), one gets

\[ h_{ca} = l_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a) + h_a. \]  \hspace{1cm} (2-15)

In Figure 4, we have

\[ \frac{G C_b}{\sin(\alpha_b + \frac{\theta_b}{2})} = \frac{B G}{\sin[90 - (\alpha_b + \theta_b)]}, \]

or

\[ \frac{C_a C_b}{\sin(\alpha_a + \frac{\theta_a}{2})} = \sec(\alpha_b + \theta_b); \]  \hspace{1cm} (2-16)

and

\[ h_{cb} = G C_b + h_b. \]  \hspace{1cm} (2-17)

Substituting equation (2-16) in equation (2-17), one gets

\[ h_{cb} = l_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) + h_b. \]  \hspace{1cm} (2-18)

Subtracting equation (2-18) from (2-15), one obtains

\[ h_{ca} - h_{cb} = l_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a) + h_a - \left[ l_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) + h_b \right] \]

or

\[ C_a C_b = l_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a) - l_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) \]

\[ + (h_a - h_b). \]  \hspace{1cm} (2-19)
Fig. 3 OBSERVATION FROM STATION A

Fig. 4 OBSERVATION FROM STATION B
Substituting equation (2-19) in equation (2-12) we obtain

\[
\Sigma_a \lambda_a' - \Sigma_b \lambda_b' = \lambda_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a)
- \lambda_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) + (h_a - h_b). \tag{2-20}
\]

From equation (2-9), we have

\[
\Sigma_b = \frac{\lambda_b}{\lambda_a} \Sigma_a. \tag{2-21}
\]

Substituting equation (2-21) in equation (2-20), one gets

\[
(\lambda_a' - \frac{\lambda_b}{\lambda_a} \lambda_b') \Sigma_a = \lambda_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a)
- \lambda_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) + (h_a - h_b),
\]
or

\[
\Sigma_a = \frac{\lambda_a}{\lambda_a' - \lambda_b' \lambda_b} \left[ \lambda_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a)
- \lambda_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) + (h_a - h_b) \right]. \tag{2-22}
\]

Applying the same procedure above and substituting for \(\Sigma_a\) in equation (2-20), one obtains

\[
\Sigma_b = \frac{\lambda_b}{\lambda_a' - \lambda_b' \lambda_b} \left[ \lambda_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a)
- \lambda_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) + (h_a - h_b) \right]. \tag{2-23}
\]

When the lines of sight are nearly horizontal, the lengths \(\lambda_a\) and \(\lambda_a'\), \(\lambda_b\) and \(\lambda_b'\) are very nearly equal and may be assumed equal without affecting the results. For the same reason, \(\lambda_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a)\) and \(\lambda_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b)\) can consequently be assumed equal to \(\lambda_a \tan(\alpha_a + \frac{\theta_a}{2})\) and \(\lambda_b \tan(\alpha_b + \frac{\theta_b}{2})\). Applying these equalities in equations
(2-22) and (2-23) we obtain

\[ \omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2} \left[ l_a \tan(\alpha_a + \frac{\theta_a}{2}) - l_b \tan(\alpha_b + \frac{\theta_b}{2}) + (h_a - h_b) \right], \]

(2-24)

and

\[ \omega_b = \frac{l_b}{(l_a)^2 - (l_b)^2} \left[ l_a \tan(\alpha_a + \frac{\theta_a}{2}) - l_b \tan(\alpha_b + \frac{\theta_b}{2}) + (h_a - h_b) \right]. \]

(2-25)

When the lines of sight are considerably inclined, then \( \theta'_a, \theta'_b \) can be computed as follows (See Figure 3):

\[ \frac{AC_a}{\sin(q_0 + \frac{\theta_a}{2})} = \frac{AF}{\sin[q_0 - (\alpha_a + \theta_a)]}, \]

or

\[ \lambda'_a = \lambda_a \cos(\frac{\theta_a}{2}) \sec(\alpha_a + \theta_a). \]

The term \( \cos(\frac{\theta_a}{2}) \) is very small and may be taken to equal one, so that

\[ \lambda'_a = \lambda_a \sec(\alpha_a + \theta_a). \]

(2-26)

Using Figure 4, we have

\[ \frac{BC_b}{\sin(q_0 + \frac{\theta_b}{2})} = \frac{BG}{\sin[q_0 - (\alpha_b + \theta_b)]} \]

or

\[ \lambda'_b = \lambda_b \cos(\frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) \]
The term \( \cos \left( \frac{\Theta}{2} \right) \) is very small and may be taken to equal one so that

\[
l'_b = l_b \sec (\alpha_b + \Theta_b). \tag{2-27}
\]

Substituting equations (2-26) and (2-27) in equations (2-22) and (2-23), we obtain

\[
\begin{align*}
\Sigma_a &= \frac{l_a}{(l_a)^2 \sec (\alpha_a + \Theta_a) - (l_b)^2 \sec (\alpha_b + \Theta_b)} \left[ l_a \sin (\alpha_a + \frac{\Theta_a}{2}) \sec (\alpha_a + \Theta_a) \\
&\quad - l_b \sin (\alpha_b + \frac{\Theta_b}{2}) \sec (\alpha_b + \Theta_b) + (h_a - h_b) \right], \tag{2-28}
\end{align*}
\]

and

\[
\begin{align*}
\Sigma_b &= \frac{l_b}{(l_a)^2 \sec (\alpha_a + \Theta_a) - (l_b)^2 \sec (\alpha_b + \Theta_b)} \left[ l_a \sin (\alpha_a + \frac{\Theta_a}{2}) \sec (\alpha_a + \Theta_a) \\
&\quad - l_b \sin (\alpha_b + \frac{\Theta_b}{2}) \sec (\alpha_b + \Theta_b) + (h_a - h_b) \right]. \tag{2-29}
\end{align*}
\]

**Determination of the Height.**

In Figure 3, we have

\[
h_c = h_{ca} - C_a C. \tag{2-30}
\]

Substituting equations (2-10) and (2-15) in equation (2-30), one obtains

\[
h_c = l_a \sin (\alpha_a + \frac{\Theta_a}{2}) \sec (\alpha_a + \Theta_a) - \Sigma_a l'_b + h_a. \tag{2-31}
\]

Substituting equation (2-26) in equation (2-31), one gets
\[ h_c = l_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a) - \mathcal{R}_a \, l_a \sec(\alpha_a + \theta_a) + h_a \cdot (2-32) \]

For small angles of \( \alpha_a \) and \( \theta_a \), one can reduce equation (2-32) to

\[ h_c = l_a \tan(\alpha_a + \frac{\theta_a}{2}) - \mathcal{R}_a \, l_a + h_a \cdot (2-33) \]

Following the same procedure for the measurements at point B, we obtain

\[ h_c = l_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b) - \mathcal{R}_b \, l_b \sec(\alpha_b + \theta_b) + h_b \cdot (2-34) \]

And for small angles of \( \alpha_b \) and \( \theta_b \), one can reduce equation (2-34) to

\[ h_c = l_b \tan(\alpha_b + \frac{\theta_b}{2}) - \mathcal{R}_b \, l_b + h_b \cdot (2-35) \]

The values of \( \mathcal{R}_a \) and \( \mathcal{R}_b \) are given in equations (2-28) and (2-29).

**Air Density Inversion.**

When the air is strongly heated because of the radiation of heat from the earth's surface, so that the density of the air increases with the height, then the paths of light rays bend downward convexly. This is an exceptional case, but sometimes it does happen on a hot day over sandy terrain. This phenomenon causes the objects to appear to be at lower elevations.

Figure 5 illustrates the courses of the paths of light, and shows the apparent positions of point C when it is observed under such conditions from points A and B. This case can be detected when the position of point C appears to be at a lower elevation from point A than from point B.
Fig. 5 AIR DENSITY INVERSION
The method of calculation is the same as that given previously in this chapter, except in the determination of height, where the error due to refraction must be added to the apparent height, so that equations (2-32) and (2-34) become

\[ h_c = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a + \theta_a) + \tau_2 l_0 \sec(\alpha_a + \theta_a) + h_a, \quad (2-36) \]

and

\[ h_c = l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b + \theta_b) + \tau_2 l_b \sec(\alpha_b + \theta_b) + h_b. \quad (2-37) \]
CHAPTER III

DISCUSSION OF ERRORS

After presenting the method of determining the refraction angles and the height of a point from two observation stations, it is desirable to discuss the errors and the percentage of these errors that might occur from the assumptions used. Also, in order to complete the discussion of errors, it is necessary to derive formulas for computing the probable errors due to the probable errors of measurements.

Errors of Assuming Equal Meteorological Factors Along the Paths.

The lapse rate of temperature is the most important factor in the determination of the atmospheric refraction. Its value changes rapidly early in the morning and late in the afternoon, and its variability is least at mid-day, that is from about the 11th hour to about the 14th hour. Therefore, for more accurate work, the measurements must be made during these hours.

In Chapter II, the relative positions of the observation stations are given, and it is pointed out that the two paths are traveling in the same direction, over the same terrain, and close to each other; therefore the lapse rates of temperature along the two paths are equal. For the same reasons the average pressures and temperatures along the paths are equal when the paths coincide, or there will be small discrepancies when they do not coincide, but these have a negligible effect on the results. For example, make the following assumptions: that station A and B in Figure 2 are located at distances of 25 and 23 miles respectively from point C; and that the elevation of station B is lower.
than that of station A by 500 feet; and that the elevation angle at station A is 3 degrees. These assumptions will provide about 170 feet difference in elevations between the two paths at one-third the distance between A and C measured from station A. Then the discrepancy in the pressure is about 0.2 inches of mercury and in the temperature is about 0.5 degrees Centigrade. These discrepancies affect the results by 1 per cent. Therefore it is safe to assume that the average temperatures and pressures are equal along the paths.

Errors in the Difference of the Apparent Elevations.

The difference of the apparent elevations \( C_a C_b \) is an important factor in determining the refraction angles; thus it is important to know the percentage of the probable error in the length \( C_a C_b \).

Table 1 has been prepared to show the percentage of the probable errors for different lengths of sight. The table was prepared on the assumption that angular errors are independent of the lengths of sight; further, it is assumed that the probable error in measuring the vertical angles is 0.3 seconds of arc, and the refraction rate is 3.5 seconds per mile. These two assumptions are based on the results which were obtained in connection with the present investigation under good weather conditions.

The results which are given in column 6 of Table 1 are reasonable; however, to secure better results in long lengths of sight, such as 20 miles or more, the distance between stations A and B may be made more than 2 miles.

1 \( C_a C_b \) is given in Figure 2.
TABLE I

PERCENT OF ERROR IN THE LENGTH C_a C_b

<table>
<thead>
<tr>
<th>Lengths of sight in miles</th>
<th>Refraction errors in feet</th>
<th>Difference C_a C_b</th>
<th>Probable errors in feet</th>
<th>Total Probable Errors in C_a C_b feet</th>
<th>Percent of Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.238</td>
<td>2.149</td>
<td>0.0384</td>
<td>0.066</td>
<td>3</td>
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<td>7</td>
<td>4.387</td>
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<td>0.0537</td>
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<td></td>
</tr>
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<td>10</td>
<td>8.954</td>
<td>3.939</td>
<td>0.0767</td>
<td>0.12</td>
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</tr>
<tr>
<td>12</td>
<td>12.893</td>
<td></td>
<td>0.0921</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>35.816</td>
<td>7.521</td>
<td>0.153</td>
<td>0.23</td>
<td>3</td>
</tr>
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<td>22</td>
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</tr>
<tr>
<td>30</td>
<td>80.586</td>
<td>11.104</td>
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<td>0.34</td>
<td>3</td>
</tr>
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<td></td>
<td>0.246</td>
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</tr>
<tr>
<td>50</td>
<td>223.850</td>
<td>18.270</td>
<td>0.384</td>
<td>0.55</td>
<td>3</td>
</tr>
<tr>
<td>52</td>
<td>242.120</td>
<td></td>
<td>0.399</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Probable Error of the Refraction Angles.

Equation (2-28) is

\[
\Sigma_a = \frac{\rho_a}{(\lambda_a^2 \sec(\alpha_a + \theta_a)^2 - (\lambda_b^2 \sec(\alpha_b + \theta_b))} \left[ \rho_a \sin(\alpha_a + \theta_a) \sec(\alpha_a + \theta_a) \right.

\left. - \rho_b \sin(\alpha_b + \theta_b) \sec(\alpha_b + \theta_b) + (h_a - h_b) \right],
\]
In geodetic measurements the angles \((\alpha_a + \theta_a)\) and \((\alpha_b + \theta_b)\) are usually small and in most cases do not exceed 5 degrees of arc; therefore, for the purpose of computing the probable errors, secants of the angles \((\alpha_a + \theta_a)\) and \((\alpha_b + \theta_b)\) can be assumed equal to one. Also \(l_a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a)\) and \(l_b \sin(\alpha_b + \frac{\theta_b}{2}) \sec(\alpha_b + \theta_b)\) can consequently be assumed equal to \(l_a \tan(\alpha_a + \frac{\theta_a}{2})\) and \(l_b \tan(\alpha_b + \frac{\theta_b}{2})\). Substituting these in equation (2-28), we obtain

\[
\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2} \left[ l_a \tan(\alpha_a + \frac{\theta_a}{2}) - l_b \tan(\alpha_b + \frac{\theta_b}{2}) + (l_a - l_b) \right]
\]

Differentiating equation (3-1), one gets

\[
d\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2} \left[ l_a \sec^2(\alpha_a + \frac{\theta_a}{2}) d\alpha_a - l_b \sec^2(\alpha_b + \frac{\theta_b}{2}) d\alpha_b \right]
\]

For the same reasons which are given above, \(\sec^2(\alpha_a + \frac{\theta_a}{2})\) and \(\sec^2(\alpha_b + \theta_b)\) can be assumed equal to one, therefore equation (3-2) becomes,

\[
d\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2} \left( l_a d\alpha_a - l_b d\alpha_b \right).
\]

Applying the method of propagation of errors\(^1\), we have

\[
l_a = \pm \frac{l_a}{(l_a)^2 - (l_b)^2} \sqrt{(l_a d\alpha_a)^2 + (l_b d\alpha_b)^2},
\]

where

\(l_a\) = probable error of the refraction angle \(\Omega_a\),

\(l_a\) = the distance between A and C in miles,

\(l_b\) = the distance between B and C in miles,

\(^1\) Mansfield Merriman, Method of Least Square, New York: John Willey and Sons, Inc., 1911, p.79, formula (30)
\[ d\alpha_a = \text{the probable errors of the measured vertical angle } \alpha_a, \]
\[ d\alpha_b = \text{the probable errors of the measured vertical angle } \alpha_b, \]

By the same procedure the probable errors for the refraction angle \( \gamma_b \) will be calculated thus:

\[
R_b = \pm \frac{\frac{b}{(a)^2} - (\frac{b}{b})^2}{(\frac{a}{a})^2 + (\frac{b}{b})^2} \sqrt{(\frac{a}{a})^2 + (\frac{b}{b})^2},
\]

(3-5)

**Probable Error of the Computed Height.**

Equation (2-32) is:

\[
h_c = a \sin(\alpha_a + \frac{\theta_a}{2}) \sec(\alpha_a + \theta_a) - \gamma_a a \sec(\alpha_a + \theta_a) + h_a
\]

For the same reasons as those given in line 5, page 20, one can reduce equation (2-32) to

\[
h_c = a \tan(\alpha_a + \frac{\theta_a}{2}) - a \gamma_a + h_a.
\]

(3-6)

Differentiating equation (3-6), we obtain

\[
dh_c = a \sec^2(\alpha_a + \frac{\theta_a}{2}) d\alpha_a - a d\gamma_a,
\]

(3-7)

\( \sec^2(\alpha_a + \frac{\theta_a}{2}) \) can be assumed equal to one, hence

\[
dh_c = a d\alpha_a - a d\gamma_a.
\]

(3-8)

Applying the method of propagation of errors\(^1\), we have

\[
rh_a = a \sqrt{(d\alpha_a)^2 + (d\gamma_a)^2},
\]

(3-9)

or

\[
rh_a = a \sqrt{0.02558 (d\alpha_a)^2 + (d\gamma_a)^2},
\]

(3-10)

\(^1\) Ibid
where

\( r_a \) = the probable error of the computed height from point A in feet,

\( d\alpha_a \) = the probable error of the measured vertical angle at station A in seconds of arc,

\( \alpha_a \) = the probable error of the refraction angle at station A, and its value as given in equation (3-4),

\( l_a \) = the distance between stations A and point C in miles.

By the same procedure, the probable error of the height computed from point B is calculated as

\[
 r_b = 0.02558 \cdot \sqrt{ (d\alpha_b)^2 + (r_b)^2 },
\]

(3-11)

where

\( r_b \) = the probable error of the computed height from point B in feet,

\( d\alpha_b \) = the probable error of the measured vertical angle at station B in seconds of arc,

\( \alpha_b \) = the probable error of the refraction angle at station A, and its value as given in equation (3-5),

\( l_b \) = the distance between station B and point C in miles.
CHAPTER IV

EXPERIMENTS

The experiments, which were carried out during the months of October, 1956 to May 29, 1957, provided a good verification of the method given in Chapter 2.

Preparations.

The experiments were done in Ann Arbor, Michigan. The area is relatively flat and it is covered with trees which make it difficult to find two observation stations on the ground that fulfill the requirement stated in Chapter II. For this reason the two observation stations selected were located on the roofs of two buildings.

The spot selected as Station A was situated over a 4 inch pipe on the northeast corner of the roof of the Water Treatment Plant in Ann Arbor, Michigan. Station B was located at distances of 6.53 feet north and 31.58 feet west of the flagpole of the hospital of the University of Michigan. And the observed point C was taken to be the middle height of the cap at the top of the water stand pipe in Ypsilanti, Michigan.

The instrument used in measuring the horizontal and vertical angles was the Wild Theodolite Type T₂ No. 12137, by which the second of an angle can be read and the tenth of a second can be estimated by the observer. The theodolite was equipped with a striding level in which one division corresponds to 5 seconds of arc.

The plane coordinates position of Station A was determined from the three known Michigan State coordinate position of Barton Hills Golf

-27-
Club water tank in Ann Arbor Township, the east water tank of the Ford Bomber Plant in Ypsilanti, and the State Hospital water tank near Ypsilanti, and they are equal to

\[ X = 474031.38 \]
\[ Y = 290334.53 \]

The plane coordinates position of Station B was determined from the known Michigan State coordinate position of the flagpole of the University of Michigan Hospital, and they were found equal to

\[ X = 482507.45 \]
\[ Y = 285288.53 \]

The Michigan State coordinates position of point C (water stand pipe in Ypsilanti) was known and equal to

\[ X = 511351.04 \]
\[ Y = 271653.72 \]

The distances between the points A and C, and B and C were computed from the plane coordinates positions. It was found that

- the distance between A and C = 41734.04 feet
- the distance between B and C = 31903.93 feet

These distances are at the mean sea level, and when they were reduced to the average height between the stations above mean sea level, it was found that

- the distance between A and C = 41735.93 feet
- the distance between B and C = 31905.30 feet

The angles subtended by the arc A F(\(\theta_a\)) and by the arc B G(\(\theta_b\)) were computed from the formula \(\theta = \frac{d}{R}\),
where

\[ L = \text{the distance between two points on the earth's surface}, \]
\[ R = \text{the mean radius of curvature of the arc joining the two points on the earth's surface}. \]

The values of the radii of curvature were taken from Table XI which is given in Hosmer's *Geodesy*.\(^1\) The values of the angles are

\[ \alpha = 00^\circ - 06' - 51.048'' \]
\[ \beta = 00^\circ - 05' - 14.206'' \]

Bench marks near stations A and B were located. The elevations of these bench marks were determined from geodetic bench marks by spirit level lines and trigonometry. The elevation of point C was determined from geological bench mark, and the difference between the geodetical and geological data was considered. Each elevation was carried at least twice, and when the discrepancy between two independent elevations was within the margin of probable error, the mean was taken and adopted as the elevation of the bench marks at the station.

The bench mark near Station A was located on the north edge of the sill of the east window of the upper room and was found equal to 1039.911 feet above mean sea level. The bench mark near Station B was located on the lower side of the iron belt around the flagpole and its elevation was equal to 1007.857 feet above mean sea level. The height of point C was found equal to 928.552 feet above mean sea level.

**Procedure of Measuring the Vertical Angles.**

The vertical angles were measured consecutively from one station to another. Each set of measurements was made by taking sixteen readings,

Fig. 6 RELATIVE POSITIONS OF STATIONS A, B, AND POINT C
Scale: 1/8" = 1000'

Fig. 7 RELATIVE ELEVATIONS OF THE PATHS AC AND BC
Scale: Vert. 1" = 100'
Horiz. 1/8" = 1000'
Figure 8. SET-UP AT STATION A
Figure 9. SET-UP AT STATION B
Figure 10. THE OBSERVED POINT C
eight direct and eight inverted.

In the first eleven experiments, one set of measurements was taken, and it was found that the probable errors were high. Then the method of measurement was revised and four sets of measurements were taken at each station. It was found that the probable errors were reduced considerably in this second set of measurements.

Results and Discussion of Results.

The measurements and the results of the computations are all compiled in Table 2. A detailed explanation of Table 2 and a typical example of the computation are given in Appendix A, page 39.

In Table 2, column 8 gives the values of the refraction angles computed by the new method, and column 10 gives the actual values of the same angles. If the observation which is made on October 11, 1956, is disregarded because it was gained under very hazy visibility, then the results show that the discrepancies between the computed and the actual values are well within the probable errors. Also, if one considers the errors, then the results show that the percentage of errors varies between 0.2 per cent and 5.1 per cent, and the average percentage of the errors is 2.1 per cent.

Examination of the results given in column 12 and column 13 show that the errors in the computed height are well within the margin of the probable error.

Arrangements were made to measure the pressure, the temperature, and the lapse-rate of temperature while three of the observations were in progress, (observations on March 21, March 28, and May 29). Then the
refraction angles \( \mathbf{\Omega}_a \) and \( \mathbf{\Omega}_b \) were computed independently by adopting the new and the old method.\(^1\) The results indicate that the accuracy of the new method is greater than that of the older. For example, the average percentage of the errors of the three mentioned measurements computed by using the new method was 2.6 per cent, and that computed by adopting the old method was 20.5 per cent.

\(^1\) See Table 3 in Appendix B, page 45.
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<th>DATE</th>
<th>Oct.10</th>
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<th>29.28</th>
<th>51</th>
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<th>00°-12'0-09.5&quot;</th>
<th>+0.61&quot;</th>
<th>28.193&quot;</th>
<th>± 2.83&quot;</th>
<th>28.437&quot;</th>
<th>+0.479</th>
<th>+0.308</th>
<th>928.399</th>
<th>+0.153</th>
<th>Partial Clouds</th>
<th>Strong Wind</th>
<th>Bed Visibility</th>
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</thead>
<tbody>
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<td>± 1.7&quot;</td>
<td>21.328&quot;</td>
<td></td>
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<td>Very Hazy</td>
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<td>23.855&quot;</td>
<td>± 1.91&quot;</td>
<td>21.237&quot;</td>
<td></td>
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<td></td>
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<td>Cloudy</td>
<td>Moderate Wind</td>
<td></td>
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<tr>
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<td>± 1.35&quot;</td>
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<td>29.300&quot;</td>
<td>+1.56&quot;</td>
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<td>Pressure in inches of mercury</td>
<td>Temp. in °F</td>
<td>Rod Readings in ft.</td>
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<td>Computed Refraction Angles ( \delta \alpha )</td>
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CONCLUSION

The determination of the refraction angles from two observation stations as presented in this investigation has been conducted by both theoretical and experimental procedures. It has been shown that in the new method extra work is needed in locating a secondary station and making observations at this station, but that this work is simple since the two observation stations are close to each other.

The new method provides a way to determine the refraction angles without using pressure, temperature, and lapse rate of temperature in the computation. It thus eliminates the necessity of measuring these factors, and by doing so, the small uncertainty of lapse rate of temperature and its high effect on the computed refraction angles is avoided.

The results of the experiments that are determined by adopting the new method are satisfactory and of relatively high accuracy; for example, the percentage of the errors of the refraction angles that are determined at station A between March 21, 1957 and May 29, 1957 vary between 1.2 per cent and 2.9 per cent, whereas the percentage of the errors of the refraction angles that are determined by adopting the method of including pressure, temperature, and lapse rate of temperature for the same observations were between 14 per cent and 32 per cent.

Therefore the new method can be used with confidence in determining the heights of high points, and in establishing bench marks in regions that can not be reached, or which are very difficult to reach by lines of spirit leveling.
APPENDIX A

EXPLANATION OF TABLE 2

The measurements and the results of the computations are all compiled in Table 2, and it is felt that an explanation of this table will be helpful to the reader. In order to make the explanation simple, the measurements which were made on March 21, 1957 and the necessary computations are carried out as a typical example.

Columns 1 and 2 give the date and the time at which the observations were made.

Columns 3 and 4 give the average pressure and temperature, which were measured at stations A and B.

Column 5 gives the level rod readings. The ( / ) sign indicates that the line of sight is higher than the bench mark and the ( - ) sign indicates that it is lower.

Columns 6 and 7 give the mean of the measured vertical angles and the probable errors of the mean. The following example illustrates the method of computation:

At station A

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<th>Measured zenith angle</th>
<th>Probable error in seconds of arc</th>
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<td>17</td>
<td>123.25</td>
</tr>
</tbody>
</table>

Total 58 408.38
mean = \frac{408.38}{58} = 7.04''

probable error of the mean = 0.59 \sqrt{\frac{20}{58}} = \pm 0.35''

zenith angle = 90° - 12' - 07.04'' \pm 0.35''

vertical angle = (-) 00° - 12' - 07.04'' \pm 0.35''

Column 8 gives the computed values of the refraction angles \( \alpha_a \) and \( \alpha_b \). The following illustrate the computations: equation (2-24) is

\[
\alpha_a = \frac{I_a}{(I_a)^2 - (I_b)^2} - \frac{I_b \tan(\alpha_a + \frac{\Theta_a}{2}) + (I_a - I_b)}{I_a \tan(\alpha_b + \frac{\Theta_b}{2}) + (I_a - I_b)}
\]

\[
\frac{I_a}{(I_a)^2 - (I_b)^2} = \frac{41735.93}{(41735.93)^2 - (31905.30)^2}
\]

\[
= 0.0000576511
\]

\[
\alpha_a = -00 - 12 - 07.04
\]

\[
\frac{\Theta_a}{2} = 00 - 03 - 25.52
\]

\[
\alpha_a + \frac{\Theta_a}{2} = -00 - 08 - 41.52
\]

\[
\log \tan(\alpha_a + \frac{\Theta_a}{2}) = 7.4028466
\]

\[
\log I_a = 4.6205101
\]

\[
\log \left[I_a \tan(\alpha_a + \frac{\Theta_a}{2})\right] = 2.0233567
\]

\[
I_a \tan(\alpha_a + \frac{\Theta_a}{2}) = -105.525
\]

\[
\alpha_b = -00 - 10 - 29.86
\]

\[
\frac{\Theta_b}{2} = 00 - 02 - 37.10
\]

\[
(\alpha_b + \frac{\Theta_b}{2}) = -00 - 07 - 52.76
\]
\[
\log \tan (\alpha_b + \frac{\theta_b}{2}) = 7.3602162
\]
\[
\log b = 4.5038628
\]
\[
\log \left[ b \tan (\alpha_b + \frac{\theta_b}{2}) \right] = 1.8640790
\]
\[
b \tan (\alpha_b + \frac{\theta_b}{2}) = -73.127
\]

\[\begin{align*}
ha &= 1039.911 \div 0.529 = 1040.440^1 \\
hb &= 1007.857 - 2.490 = 1005.367^2 \\
ha - hb &= 35.073
\end{align*}\]

Substituting values in equation (2-24), one gets

\[\begin{align*}
\gamma_a &= 0.0000576511 (-105.525 \div 73.127 \div 35.073) \\
\gamma_a &= 0.0001542167 \text{ radians} \\
\gamma_a &= 31.809 \text{ seconds of arc}
\end{align*}\]

Applying equation (2-25) and following the same procedure as above, one obtains

\[\begin{align*}
\gamma_b &= 0.0001178919 \text{ radians} \\
\gamma_b &= 24.317 \text{ seconds of arc}
\end{align*}\]

Column 9 gives the probable error of the computed refraction angles \(\gamma_a\) and \(\gamma_b\). The following illustrate the computation: equation (3-4) is

\[\begin{align*}
K_a &= \frac{\lambda_a}{(\lambda_a)^2 - (\lambda_b)^2} \sqrt{(\lambda_a d\alpha_a)^2 + (\lambda_b d\alpha_b)^2}.
\end{align*}\]

Substituting values, \(\lambda_a\) and \(\lambda_b\) are in miles, we have

1 1039.911 is the height of the bench mark near station A.
2 1006.857 is the height of the bench mark near station B.
\[ r_a = t \frac{7.904}{(7.904)^2 - (6.042)^2} \sqrt{(7.904 \times 0.35)^2 - (6.042 \times 0.38)^2} \]
\[ r_b = t \]
\[ r_a = t \quad 0.304 \times 3.60 \]
\[ r_a = t \quad 1.09 \text{ seconds of arc} \]

Applying equation (3-5) and following the same procedure as above, we obtain

\[ r_b = t \quad 0.84 \text{ seconds of arc} \]

Column 10 gives the actual values of the refraction angles \( \gamma_a \) and \( \gamma_b \), they are computed as follows:

\[ h_a = 1040.440 \]
\[ l_a \tan(\alpha_a + \frac{\theta_a}{2}) = -105.525 \]

observed height = \( 934.915 \)
actual height = \( 928.552 \)
effect of refraction = \( 6.363 \)

\[ \gamma_a = \frac{6.363}{41735.93} \times 206264.8 \]
\[ \gamma_a = 31.447 \text{ seconds of arc} \]

and

\[ h_b = 1005.367 \]
\[ l_b \tan(\alpha_b + \frac{\theta_b}{2}) = -73.127 \]

observed height = \( 932.240 \)
actual height = \( 928.552 \)
effect of refraction = \( 3.688 \)
\[ \Omega_b = \frac{3.688}{31905.30} \times 206264.8 \]

\[ \Omega_b = 23.842 \text{ seconds of arc} \]

Column 11 gives the computed height of point C. The following illustrate the computations: formula (2-33) is

\[ h_c = l_a \tan \left( \alpha_a + \frac{\Theta_a}{2} \right) - \Omega_a l_a + h_a. \]

Substituting values from page 42, we have

\[ h_c = 1040.440 - 105.525 - 41735.93 \times 0.0001542167 \]

or

\[ h_c = 928.478 \text{ feet} \]

Column 12 gives the probable error of the height \( h_c \). The following illustrate the computations: equation (3-10) is

\[ r_{ha} = \pm 0.02558 \frac{l_a}{\sqrt{\left( d \alpha_a \right)^2 + \left( r_a \right)^2}} \]

Substituting values, we obtain

\[ r_{ha} = \pm 0.02558 \times 7.904 \sqrt{\left( 0.35 \right)^2 + \left( 1.09 \right)^2} \]

\[ r_{ha} = \pm 0.231 \text{ feet} \]

Applying formula (3-11) and substituting values, we get

\[ r_{hb} = \pm 0.142 \text{ feet} \]

Column 13 gives the actual discrepancy between the computed and the actual values. The actual height of point C is 928.552 feet.

Column 14 gives the weather conditions when the observations were made.
APPENDIX B

DETERMINATION OF THE REFRACTION ANGLES BY INCLUDING THE METEOROLOGICAL FACTORS IN COMPUTATIONS

After presentation of the new method of the present investigation, it is felt desirable to provide some material by which a comparison can be made between the new and the old methods. This material was provided by making arrangements for measuring the temperature at different heights.

A wire sounding device was set on the Huron Valley Golf Course, which is located at about one-third the distance between Station A and point C measured from Station A, and the temperatures at different heights were measured while the observations from Stations A and B were in progress. This procedure provided a way of determining the temperature and the lapse rate of temperature along the path of light. The figures which are given on pages 47-52 illustrate the results of the measured and the reduced values of the temperature and the lapse rate of temperature.

The pressure was measured at the observation station, and because the paths of light are very nearly horizontal, the pressure was assumed to be constant along the paths. The refraction angles $\mathcal{O}_a$ and $\mathcal{O}_b$ and the height $h_c$ are computed and compiled in Table 3 page 45.
### Table 3
MEASUREMENTS AND RESULTS OF COMPUTATIONS BY ADOPTING METEOROLOGICAL FACTORS

<table>
<thead>
<tr>
<th>Date</th>
<th>Station</th>
<th>Time</th>
<th>P (inches mercury)</th>
<th>T (°C)</th>
<th>$\frac{dT}{dh}$ (°C per ft.)</th>
<th>$k = 28000 \frac{P}{T^2} (0.004 + \frac{dT}{dh})$</th>
<th>Refraction angle</th>
<th>Vert. angle Correct. from Curvature &amp; Refraction</th>
<th>Computed $h_c$ (ft.)</th>
<th>Error in $h_c$ (ft.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>March 21, 1957</td>
<td>A</td>
<td>13.55</td>
<td>28.82</td>
<td>5.25</td>
<td>0.00204</td>
<td>0.0873</td>
<td>35.884&quot;</td>
<td>-00°-09'17.40</td>
<td>927.665</td>
<td>-0.887</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>12.45</td>
<td>28.88</td>
<td></td>
<td></td>
<td></td>
<td>27.433&quot;</td>
<td>-00°-08'20.19</td>
<td>928.203</td>
<td>-0.349</td>
</tr>
<tr>
<td>March 28, 1957</td>
<td>A</td>
<td>13.50</td>
<td>28.97</td>
<td>4.65</td>
<td>0.00506</td>
<td>0.0556</td>
<td>22.813'</td>
<td>-00°-09'07.76</td>
<td>929.485</td>
<td>+0.933</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>12.45</td>
<td>29.01</td>
<td></td>
<td></td>
<td></td>
<td>17.438'</td>
<td>-00°-08'15.17</td>
<td>929.800</td>
<td>+1.248</td>
</tr>
<tr>
<td>May 29, 1957</td>
<td>A</td>
<td>13.20</td>
<td>29.13</td>
<td>20.5</td>
<td>0.0057</td>
<td>.0445</td>
<td>18.290&quot;</td>
<td>-00°-09'03.52</td>
<td>930.265</td>
<td>+1.713</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>12.20</td>
<td>29.17</td>
<td></td>
<td></td>
<td></td>
<td>13.982&quot;</td>
<td>-00°-08'12.20</td>
<td>929.552</td>
<td>+1.00</td>
</tr>
</tbody>
</table>
\[ \frac{dT}{dh} = 0.00204 \, ^\circ C/Ft. \]

FIGURE 13
AVERAGE VALUES
TEMPERATURE VS. HEIGHT
HURON VALLEY GOLF COURSE
MARCH 21, 1957
FIGURE 14
TEMPERATURE VS. HEIGHT
HURON VALLEY GOLF COURSE
MARCH 28, 1957
$\frac{dT}{dh} = 0.00506 \, ^\circ C / Ft.$

**Figure 15**

AVERAGE VALUES

TEMPERATURE VS. HEIGHT

HURON VALLEY GOLF COURSE

MARCH 28, 1957
FIGURE 16
TEMPERATURE VS. HEIGHT
HURON VALLEY RIVER
MAY 29, 1957
$\frac{dT}{dh} = 0.0057 \degree C/ft.$

FIGURE 17
AVERAGE VALUES
TEMPERATURE VS. HEIGHT
HURON VALLEY GOLF COURSE
MAY 29, 1957
BIBLIOGRAPHY

BOOKS


PERIODICALS


Graff-Hunter, J. de

Gulatee, B. L.

McLod, A. R.

Morely, A. J.

Morely, A. J.

Morely, A. J.

Rainsford, H. F.