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FORCED CONVECTION FROM NONISOTHERMAL SURFACES

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SUMMARY

The convection of heat from surfaces at nonuniform temperatures is reviewed. Solutions for systems previously analyzed by others are collected and compared. A few new solutions are proposed. A method of treating heat fluxes with variable wall temperature by using the solutions for constant wall temperature is described.

SYMBOLS AND NOMENCLATURE

A	=	$k/\rho C_p g_c$
b	=	$(1/6)\ Pr\ f''\ (0)$
B	=	arbitrary constant
c	=	$(3/4)\ (1 + m)$
C	=	coefficients in Graetz and Poppendiek solutions
C_p	=	unit heat capacity at constant pressure, BTU/lb-°F
d	=	diameter of tube, ft
f	=	Hartree velocity function (Ref. 19)
g	=	integrating kernel, °F/(BTU/hr-ft)
g_c	=	32.2 ft/sec^2
h	=	integrating kernel, BTU/hr-ft^2-°F
k	=	thermal conductivity, BTU/hr-ft^2-(°F/ft)
k_e	=	"eddy conductivity", BTU/hr-ft^2-(°F/ft)
m	=	exponent for "wedge" flows
p	=	exponent in Poppendiek solution
P_r	=	$(3600\mu\ C_p g_c/R)$, Prandtl modulus
q	=	heat flux, BTU/hr-ft^2
Re_x	=	local Reynolds modulus,
S	=	diameter or width of channel or slot, ft
T	=	temperature, °F
T_∞	=	free-stream temperature, °F
T_s	=	slot temperature, °F
T_w	=	wall temperature, °F
T_o	=	initial stream temperature, °F
u	=	velocity parallel to surface, ft/hr
v	=	velocity perpendicular to surface, ft/hr
W	=	fluid weight rate, lbs/hr
x	=	distance along surface, ft
y	=	distance from surface, ft
z	=	dummy variable, ft
α	=	exponent in Graetz solution
β	=	$\pi R/2WC_p$ ft^{-1}
β_n	=	coefficient in Poppendiek solution
ξ	=	dummy variable, ft (unheated starting length)
θ	=	dimensionless temperature field, °F
ρ	=	fluid density, slugs/ft^3
μ	=	fluid viscosity, lb sec/ft^2
τ	=	wall shear stress, lb/ft^2

FORCED CONVECTION FROM NONISOTHERMAL SURFACES

INTRODUCTION

Rubesin[1,2] has outlined the problem of heat transfer from surfaces whose temperature varies in the direction of fluid flow. The analysis of heat flow for such cases shows that the heat fluxes which would be predicted when the surface temperature variation is not taken into account differ markedly from the case where the varying temperature effect is included. In some cases the actual heat flux is in the opposite direction to the prediction based on an isothermal surface.

In this paper the method of Rubesin will be reviewed and some new applications demonstrated. The reader who wishes to consider this problem further should also consult references 10 and 13, where numerical and graphical methods of analysis are given.

OTHER PAPERS ON THIS SUBJECT

Heat transfer from nonisothermal surfaces has in recent years attracted the attention of many workers. A number of exact solutions to the differential equations for conservation of momentum, mass, and energy have been given, as well as several suitable approximate solutions. One graphical method has been cited.

Table I is a tabulation of analytical solutions of the above type known to be available at the present time.

Experimental data to test the various analytical solutions have been almost completely lacking. The data of Scesa[12] were taken for the express purpose of testing the predictions and, as shown by Scesa[12] and Rubesin[2], the agreement is satisfactory. Sherwood and Maisel[18] give data

TABLE I

Author	Boundary-Layer Character	Flow Conditions Prescribed	Properties of the Fluid	Surface-Temperature Description	Reference
Rubesin (approximation)	Laminar	Flat plate, zero pressure gradient	Constant	Step function	1
Chapman-Rubesin (exact)	Laminar	Flat plate, zero pressure gradient	$\mu\rho$ = constant Pr = 0.72	Polynomial in x	4
Levy (exact	Laminar	Wedge flows $u1 = cx^m$	Constant $0.7 < Pr < 20$	Polynomial in x	5
Lighthill (approximation)	Laminar	Arbitrary, skin friction is taken as parameter	Constant	Arbitrary, solutions given in integral form	7
Graetz (exact)	Laminar	Flow in a tube, parabolic velocity distribution	Constant	step function	6
Lipkis (exact)	Laminar	Flow in a tube parabolic velocity distribution	Constant	Temperature a linear function of x	8
Seban (approximate)	Laminar	Arbitrary	Constant	Arbitrary	10
Leveque	Laminar	Flat plate, velocity profile in boundary layer taken as linear	Constant	Step function	9
Rubesin (approximation)	Turbulent	Flat plate, zero pressure gradient	Constant	Arbitrary, power function variation worked out as example	2
Sage, et al, (electrical analog)	Turbulent	Flat duct	Constant	Step function	11
Tribus (Graphical or numerical)	Turbulent	Round or flat duct	Constant	Arbitrary	11
Eckert	Laminar	Flat plate	Constant	Step function	15
Poppendiek	Turbulent	Entrance to a tube	Constant (liquid metals)	Step function	24
Poppendiek	Turbulent	Flat plate	Constant (liquid metals)	Step function	25
Yih-Cermak	Laminar	Round or flat duct	Constant	Arbitrary, opposite walls of duct not necessarily at same temperature	26

for the case of mass transfer.

Yih and Cermak[26] have presented a report which considers the laminar flow of a fluid in a pipe or duct with nonuniform wall temperature. Unfortunately, the report has not been widely circulated and the present author did not learn of its existence until after the completion of this paper.

THE GENERAL METHOD

Consider, for example, the form of the energy equation in two dimensions as used in boundary-layer calculations,

$$u\rho C_p \frac{\partial T}{\partial x} + v\rho C_p \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left\{ (k + k_e) \frac{\partial t}{\partial y} \right\} + \left(\mu \frac{\partial u}{\partial y} \right)^2, \qquad (1)$$

subject to the following restrictions:

1. The fluid properties ρ, C_p, k, μ are independent of temperature.
2. The velocity field (u, v) is known and independent of temperature.
3. The "eddy conductivity" is not a function of temperature.

The effect of the last term may be taken into account by adding a particular solution of the above equation to the solution of the following equation.

$$u\rho C_p \frac{\partial T}{\partial x} + v\rho C_p \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left\{ (k_e + k) \frac{\partial T}{\partial y} \right\} \qquad (1a)$$

The particular solution of most interest is known as the "adiabatic wall temperature" and is the temperature assumed by the wall when the heat flux from the wall to the fluid is everywhere zero. When a solution to Eq 1a is found, that is, a relation between $q(x)$ and $T_w(x)$, there must be added to $T_w(x)$ the adiabatic wall temperature $T_{ad}(x)$.

The important property of Eq 1a is its linearity with respect to temperature.

Consider now a function $\theta(\xi, x, y)$ which satisfies the above equation and a set of boundary conditions as follows:

$$
\begin{array}{llll}
\theta(\xi, x, 0) = 1 & x > \xi & & \\
\theta(\xi, x, 0) = 0 & x < \xi & & \text{if the flow is a} \\
\theta(\xi, x, \infty) = 0 & & & \text{"boundary-layer"} \\
\theta(\xi, \xi, y) = 0 & y > 0 & & \text{type}
\end{array}
$$

$$
\begin{array}{llll}
\theta(\xi, x, 0) = 1 & x > \xi & & \text{if the flow is of the} \\
\theta(\xi, x, 0) = 0 & x < \xi & & \text{"conduit" type with} \\
\theta_y(\ , x, r) = 0 & & & y = r \text{ a line of symmetry} \\
\theta(\ , \ , y) = 0 & y \ddagger 0 & & \text{for the flow and temperature}
\end{array}
$$

The above function may be used to construct the temperature field when the surface temperature varies in a stepwise fashion by considering the sum:

$$
T - T_\infty = \sum_{n = 1,2,3,\ldots} \left[T_w(\xi_n) - T_w(\xi_{n-1})\right] \theta(\xi_n, x, y) \qquad T_w(0^-) = T_\infty \tag{2}
$$

We may represent the above summation by an integral taken in the Stieltjes sense (see Appendix A).

$$
T - T_\infty = \int_{\xi = 0}^{x} \theta(\xi, x, y)\, dT_w(\xi), \qquad T_w(0^-) = T_\infty \tag{3}
$$

The above integral is the formal solution to the energy equation. It remains now to determine the form of the function $\theta(\xi, x, y)$ for some particular systems of practical interest. In every case the function $\theta(\xi, x, y)$ is the solution to the energy equation in the presence of a "step function" in temperature.

A number of "step-function" solutions exist. For example the Graetz[6] and Leveque[9] solutions are of this type. The solutions by Rubesin[1,2] are of such a nature. The analysis by Lighthill[7] is in the form above.

In general, it is not the temperature field but the heat flux at the wall which is sought. Thus we are usually interested not in $\theta(\xi, x, y)$ but in its derivative at $y = 0$, i.e.,

$$
q(x) = \left[-k \frac{\partial T(x,y)}{\partial y}\right]_{y = 0} = -k \int_{\xi = 0}^{x} \theta_y(\xi, x, 0)\, dT_w(\xi). \tag{4}
$$

As Rubesin has pointed out[2], -k times the normal gradient of θ at $y = 0$ is the usual definition of the local unit thermal conductance for the case where the wall temperature is a step function.

Defining

$$h(\xi, x) = -k\theta_y(\xi, x, 0),$$

we have:

$$q(x) = \int_{\xi=0}^{x} h(\xi, x)\, dT_W(\xi). \tag{5}$$

Table II gives a summary of the function $h(\xi, x)$ currently known for various systems.

FINDING THE WALL TEMPERATURE WHEN THE HEAT FLUX IS PRESCRIBED

The previous discussion has considered only cases where the wall temperature is prescribed and the heat flux to be found. In many cases of practical interest the heat flux is given and the temperature of the wall is to be found. The kernels must be modified for use with this second type of problem. Most of the kernels of Table II are of the form:

$$h(\xi, x) = f(x)\,(x^{\delta} - \xi^{\delta})^{-\alpha} \tag{6}$$

with

$$q(x) = \int_{\xi=0}^{\xi=x} h(\xi, x)\, dT_W(\xi). \tag{7}$$

Now multiply both sides by $(x^{\delta} - z^{\delta})^{\alpha-1}\, d(x^{\delta})$ and integrate from $x = 0$ to $x = z$.

$$\int_{x=0}^{z} \frac{q(x)}{f(x)} (x^{\delta} - z^{\delta})^{\alpha-1} d(x^{\delta}) = \int_{x=0}^{z} (x^{\delta} - z^{\delta})^{\alpha-1} d(x^{\delta}) \int_{\xi=0}^{x} (x - \xi^{\delta})^{-\alpha} dT_W(\xi). \tag{8}$$

Margenau and Murphy[17] discuss the above type of double integral; since x varies from 0 to z and ξ varies from 0 to x for every x, the result is equivalent to varying ξ from 0 to z and x from ξ to z for every value of ξ.

TABLE II

INTEGRATING KERNELS FOR NON-ISOTHERMAL CONVECTION

$$q(x)=\int_{\xi=0}^{x} h(\xi,x)\,dT_w(\xi) \qquad T_w(x)-T_\infty=\int_{\xi=0}^{x} g(\xi,x)\,q(\xi)\,d\xi$$

	AUTHOR	REF.	SYSTEM	METHOD OF SOLUTION	$h(\xi,x)$	$g(\xi,x)$
I	RUBESIN	1	Laminar flow over a flat plate. Zero pressure gradient. Fluid properties constant.	Velocity and temperature profiles postulated linear in y. Thermal boundary layer thickness proportional to momentum thickness. (Integral)	$\frac{0.304k}{x} Pr^{1/3} Re_x^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3}$	$\frac{1}{6(1/3)!(2/3)!}\frac{1}{0.304k} Pr^{-1/3} Re_x^{-1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3}$
II	ECKERT	15	Laminar flow over a flat plate. Zero pressure gradient. Fluid properties constant.	Velocity and temperature profiles postulated as a cubic in y. Integral method of solution.	$\frac{0.33k}{x} Pr^{1/3} Re_x^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3}$	$\frac{1}{6(1/3)!(2/3)!}\frac{1}{0.33k} Pr^{-1/3} Re_x^{-1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3}$
III	LEVEQUE	9	Laminar flow over a flat plate. Shear at the wall postulated constant. Fluid properties constant.	Velocity profile taken as u=cy, not dependent upon x. Term $v\partial T/\partial y$ thereby drops out of energy equation and resulting equation is solved.	$\frac{kPr^{1/3}}{3(1/3)!}(\rho/9\mu)^{1/3}\left[(du/dy)_{y=0}\right]^{1/3}(x-\xi)^{-1/3}$	$\frac{2}{3k(2/3)!}Pr^{-1/3}(\rho/9\mu)^{-1/3}\left[(du/dy)_{y=0}\right]^{-1/3}(x-\xi)^{-2/3}$
IV	LIGHTHILL	7	Laminar flow over a surface with known variation in surface shear stress. Constant fluid properties.	Velocity profile taken as $u=\frac{\tau(x)}{\mu}y$ where $\tau(x)$ = wall shear. Resulting differential equations solved.	$\frac{k}{(1/3)!}Pr^{1/3}(\rho/9\mu)^{1/3}\sqrt{\tau(x)}\left[\int_{\xi}^{x}\sqrt{\tau(z)}\,dz\right]^{-1/3}$	
V	BOND	16	"Wedge-flows". Fluid properties constant. (Velocity over wedge given by $u=c_1x^m$)	Velocity near wall taken as linear in y. Resulting differential equation solved for two cases: (a) Step function in temperature. (b) Step function in heat flux.	$\frac{(1+m)^{1/2}k}{\sqrt{2}\,(4/3)!\,x}b^{1/3}Re_x^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{c}\right]^{-1/3}$ $b=\frac{Pr}{6}f''(0) \qquad c=\frac{3}{4}(1+m)$ $f''(0)$ is a dimensionless velocity gradient tabulated as a function of m in reference 19.	$\frac{(4/3)\sqrt{2}}{(2/3)!(1+m)^{1/2}}\frac{x^{1/2}}{k}b^{-1/3}Re_x^{-1/2}c\xi^{-1/2}\left[\frac{x^c-\xi^c}{\xi^c}\right]^{-2/3}$
VI	MODIFIED LEVEQUE		Laminar flow over a surface. Fluid properties constant. $d\tau/dx$ small.	Velocity profile taken as $u=\frac{\tau(x)}{\mu}y$. $d\tau/dx$ small enough to make $v\partial T/\partial y$ negligible. $\tau(x)$ = wall shear.	$\frac{3k}{(1/3)!}Pr^{1/3}\left[9\rho/\mu^2\right]^{1/3}\left[\int_{\xi}^{x}\frac{dz}{\tau(z)}\right]^{-1/3}$	
VII	RUBESIN	2	Turbulent flow over a flat plate. Fluid properties constant.	Velocity and temperature profiles taken as following 1/7 power law. Integral method.	$\frac{0.0288k}{x}Pr^{1/3}Re_x^{0.8}\left[1-\left(\frac{\xi}{x}\right)^{39/40}\right]^{-7/39}$	$\frac{(28/195)\ Pr^{-1/3}}{(32/39)!(7/39)!\,0.0288k}Re_x^{-0.8}x^{0.8}\left(x^{\frac{39}{40}}-\xi^{\frac{39}{40}}\right)^{-}$ [illegible]
VIII	SEBAN	12	Turbulent flow over a flat plate. Constant fluid properties.	Velocity profile taken as 1/7 power of y. Temperature profile taken as linear in y near the wall, 1/7 power of y outside the laminar sub-layer.	$\frac{0.0289k}{x}Pr^{1/9}Re_x^{0.8}\left[1-\left(\frac{\xi}{x}\right)^{9/10}\right]^{-1/9}$	$\frac{(8/90)\ Pr^{-1/9}}{(8/9)!(1/9)!\,0.0289k}Re_x^{-0.8}x^{0.8}\left(x^{\frac{9}{10}}-\xi^{\frac{9}{10}}\right)^{-\frac{8}{9}}$
IX	MAISEL AND SHERWOOD	18	Experimental measurements on a mass transfer apparatus.	Empirical equation "best-fit" to data. (Translated to air heat transfer system by present author.)	$\frac{0.035k}{x}Re_x^{0.8}\left[1-\left(\frac{\xi}{x}\right)^{0.8}\right]^{-0.11}$	
X	GRAETZ	6	Laminar flow in a tube. Parabolic velocity distribution. Constant fluid properties.	Differential equation solved by separation of variables. Only first three eigen values known.	$\frac{4k}{d}\sum_{i=1,2,3}^{\infty}c_ie^{-\alpha_i\beta(x-\xi)}$ $\beta=\frac{\pi k}{2wc_p}$ i: 1, 2, 3 α: 7.317, 44.35, 106 c_i: 0.749, 0.539, 0.179	See Eq. 26
XI	POPPENDIEK	24	Turbulent flow of a liquid metal in a tube. Velocity profile established. Constant fluid properties.	Thermal conductivity of metal postulated large enough to render eddy heat transport negligible. Velocity profile follows. $u=B\left(\frac{y}{s}\right)^p$	$\frac{\frac{k}{s}\left[\frac{Re_sPr}{2(P+1)}\right]^{1/2(P+1)}}{\left(\frac{P+3}{P+2}\right)!}\sum_{n=1,2,3}^{\infty}c_n\beta_n^{\frac{1}{P+2}}e^{-\frac{\beta_n^2}{s}(x-\xi)}$ Constants c_n and β_n tabulated in reference 25.	
XII	POPPENDIEK MODIFICATION OF LEVEQUE	25	Turbulent flow of a liquid metal in a tube. Velocity profile established. Constant fluid properties.	Thermal conductivity of metal postulated large enough to render eddy heat transport negligible. Velocity profile follows. $u=B\left(\frac{y}{s}\right)^p$	$\frac{k}{\left(\frac{P+3}{P+2}\right)!}\left[\frac{B}{(P+2)^2s\rho_a}\right]^{\frac{1}{P+2}}(x-\xi)^{-\frac{1}{P+2}}$	$\frac{(P+3)(P+1)}{(P+2)^3k\left(\frac{P+1}{P+2}\right)!}\left[\frac{B}{(P+2)^2s\rho_a}\right]^{-\frac{1}{P+2}}(x-\xi)^{-\frac{P+1}{P+2}}$

Therefore,

$$\int_0^z \frac{q(x)}{f(x)} (x^\sigma - z^\sigma)^{\alpha - 1} d(x^\sigma) = \int_0^z dT_W(\xi) \int (x^\sigma - z^\sigma)^{\alpha - 1} (x^\sigma - \xi^\sigma)^{-\alpha} d(x^\delta) \tag{9}$$

Let $s = \dfrac{x^\sigma - \xi^\sigma}{z^\sigma - \xi^\sigma}$.

Hence

$$\int_\xi^z (x^\sigma - z^\sigma)^{\alpha - 1} (x^\sigma - \xi^\sigma)^{-\alpha} d(x^\sigma) = \int^1 (s-1)^{\alpha - 1} s^{-\alpha} d s \tag{10}$$

This latter integral is an Eulerian integral of the first kind and has the value

$$(-1)^{\alpha - 1} (-\alpha)!\ (\alpha - 1)! \tag{11}$$

$$T_W(x) - T_\infty = \int_{\xi=0}^{x} \frac{q(\xi)(x^\sigma - \xi^\sigma)^{\alpha - 1} \sigma \xi^{\sigma - 1} d\xi}{f(\xi)(-\alpha)!\ (\alpha - 1)!} \tag{12}$$

The above integral may be written in the form:

$$T_W(x) - T_\infty = \int_{\xi=0}^{x} q(\xi)\ g(\xi, x)\ d\xi \tag{13}$$

Table II gives values of $g(\xi, x)$ for some of the systems for which $h(\xi, x)$ is known. In the cases of the Lighthill solution and the Leveque modification (see Table II), when $(x)\gamma = Cx^{-n}$, as often occurs,

$$\int_\xi^x \sqrt{C}\, z^{\frac{-n}{2}}\, dz = \frac{\sqrt{C}}{1 - \frac{n}{2}} \left(x^{\frac{-n}{2}} - \xi^{\frac{-n}{2}}\right) \tag{14}$$

and

$$\int_\xi^x \frac{dz}{Cz^{-n}} = \frac{1}{C(n + 1)} \left(x^{n + 1} - \xi^{n + 1}\right). \tag{15}$$

Therefore, these solutions are of the form:

$$h(\xi, x) = \frac{-kC^{1/6} Pr^{1/3} (p/9\mu^2)^{1/3} x^{-n/2}}{(1/3)!\ (1 - n/2)^{-1/3}} \left(x^{-n/2} - \xi^{-n/2}\right)^{-1/3} \tag{16}$$

(Lighthill)

and

$$h(\xi, x) = \frac{3kC^{1/3}Pr^{1/3}(n+1)^{1/3}(9\rho/\mu^2)^{1/3}}{(1/3)!} (x^{n+1} - \xi^{n+1})^{-1/3} \qquad (17)$$

(Leveque Modification)

Most of the kernels of Table II have been published before. Two, however, are new and their derivation is presented in later sections of this paper. (The generalization of the Graetz solution was given in reference 26.)

USE OF THE POPPENDIEK AND GRAETZ SOLUTIONS WHEN HEAT FLUX IS GIVEN*

The integrating kernels used in the Graetz or Poppendiek solutions are of the form:

$$h(\xi, x) = h(x - \xi) \qquad (18)$$

and the heat flux is given by:

$$q(x) = \int_0^x h(x - \xi) \frac{dT(\xi)}{d\xi} d\xi . \qquad (19)$$

Define now:

$$u(z) = \int_0^\infty e^{-zt} h(t)dt \quad v(z) = \int_0^\infty e^{-zt} \frac{dT}{dt} dt = -T(o) + z\int_0^\infty e^{-zt}T(t)dt. \qquad (20)$$

Then it follows[21]

$$u(z)\ v(z) = \int_0^\infty e^{-zt} q(t)\ dt \qquad (21)$$

Now if $q(x)$ and $h(x)$ are known, the temperature may be found by the Fourier-Mellin integral

*Yih and Cermak consider the case where the temperature outside a poorly insulated pipe is specified.[26]

$$T(x) = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{e^{zx}}{z} \left[v(z) - T(o)\right] dz \tag{22}$$

or

$$T_w(x) - T_w(o) = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{e^{zx}}{z} v(z)\, dz \ . \tag{23}$$

For the integrating kernels in question

$$h(t) = \sum_n C_n e^{-a_n t} \ . \tag{24}$$

Therefore,

$$u(z) = \sum_n C_n (A_n + z)^{-1} \tag{25}$$

and the formal solution for T(x) is

$$T(x) - T(o) = \frac{1}{2\pi i} \int_{C - i\infty}^{C + i\infty} \frac{e^{zx}}{z} \frac{\left\{\int_o^\infty e^{-zt} q(t)dt\right\}}{\sum_n \frac{C_n}{A_n + z}} dz \ . \tag{26}$$

The difficulty in the evaluation of the above integral lies in the infinitude of zeroes of

$$\sum_n C_n (A_n + z)^{-1} \tag{27}$$

An approximate solution may be obtained by taking only a few terms of the series. (In the case of the Graetz solution only the first three eigenvalues are known.) By way of example, the Graetz kernel is

$$h(x) = \frac{4k}{d} \left[0.749\, e^{-7.317\beta x} + 0.359 e^{-44.35\beta x} + 0.179 e^{-106\beta x} + \ldots\right] ; \tag{28}$$

hence, approximately

$$\sum \frac{C_n}{A_n + z} = \frac{0.749}{7.317\beta + z} + \frac{0.359}{44.35\beta + z} + \frac{0.179}{106\beta + z} \tag{29}$$

$$= \frac{(30.91\beta + z)(95.03\beta + z)}{(7.317\beta + z)(44{,}35\beta + z)(106\beta + z)} ,$$

and if $q(x)$ is a step function, $q(x) = \begin{cases} 0 & x < 0 \\ q & x > 0 \end{cases}$ the temperature is:

$$T(x) - T(0) = \frac{qd}{2\pi i 4k} \int_{C - i\infty}^{C + i\infty} \frac{e^{z} (7.317\beta + z)(44.35\beta + z)(106\beta + z)}{z \quad (30.91\beta + z)(95.03\beta + z)} dz. \tag{30}$$

The integral is evaluated to give:

$$T(x) - T(0) = \frac{qd}{4R} \left[11.7\beta x - 3.05 + 3.27e^{-30.91\beta x} + 2.51e^{-95.03\beta x} \right] \tag{31}$$

The above equation is in error at the origin (as in the Graetz solution, since only the first three terms of the infinite series are taken. However, the error disappears quickly since the exponentials decay rapidly.

FLOW IN A ROUND PIPE WITH PARABOLIC VELOCITY DISTRIBUTION ESTABLISHED

(Graetz Solution Modification[26])

The usual form for the Graetz solution is[6]:

$$q(x) = \frac{4k}{d} (t_0 - t_w) \sum_{i=1}^{\infty} C_i e^{-\alpha_i \beta x}, \tag{32}$$

where the C_i and α_i are given as:

i	=	1	2	3
C_i	=	0.749	0.539	0.179
α_i	=	7.3136	44.61	106

and $\beta = \dfrac{\pi k}{2WC_p}$.

In the Graetz solution, $\xi = 0$; hence the generalization is simple:

$$h(\xi, x) = \frac{4k}{d} \sum_{i} C_i e^{-\alpha_i \beta (x - \xi)} \tag{33}$$

and therefore, if the pipe has nonuniform wall temperature

$$q(x) = \frac{4k}{d} \sum_i C_i e^{\alpha_i \beta x} \int_0^x e^{\alpha_i \beta x} \, dT(\xi). \qquad (34)$$

Note that the conductance $h(\xi, x)$ is based on the inlet temperature of the fluid rather than on the mixed mean temperature at each axial position.

Certain wall temperature variations yield solutions readily, since the integrals are already known. For example, if the wall temperature variation is of the form: $T(x) = Ax^{n+1}$,

$$q(x) = (n+1)A \sum_i \frac{C_i}{\alpha_i \beta} (-x)^n \left[1 - \frac{n}{\alpha_i \beta x} \; \frac{n(n-1)}{\alpha_i^2 \beta^2 x^2} - \mp \; \frac{n}{\alpha_i^n \beta^n x^n} \right]. \qquad (35)$$

If the wall temperature is of the form: $T(x) = A \sin wx$,

$$q(x) = Aw \sum_i C_i \left(\frac{\alpha_i \beta \cos wx + w \sin wx}{\alpha_i^2 \beta^2 + w^2} - \frac{\alpha_i \beta}{\alpha_i^2 \beta^2 + w^2} \right) \qquad (36)$$

Yih and Cermak consider the case where the temperature is prescribed on the outside of an insulated pipe or plane duct[26].

MODIFICATION OF THE LEVEQUE SOLUTION

Consider the velocity profile $u = \tau(x)y/\mu$ and let $\tau(x)$ vary sufficiently slowly with x that $v \frac{\partial T}{\partial y} << u \frac{\partial T}{\partial x}$; i.e., if $u = \frac{\tau(x)}{\mu} y$,

then by continuity

$$v = \int_0^y \frac{\partial u}{\partial x} dy = \frac{\tau'(x)}{\mu} y^2. \quad \therefore \frac{\partial \tau}{\partial x} \text{ should be small.}$$

The energy equation is written:

$$u\frac{\partial T}{\partial x} = a\frac{\partial^2 T}{\partial y^2} \quad \text{or} \quad \frac{\tau(x)}{\mu} \frac{\partial T}{\partial x} = \frac{a}{y} \frac{\partial^2 T}{\partial y^2} \qquad (37)$$

Define

$$dS = \mu a \, dx/\gamma(x) \quad \text{or} \quad S = \int_0^x \frac{\mu a dx'}{\gamma(x')} \tag{38}$$

Then:

$$\frac{\partial T}{\partial S} = \frac{1}{y} \frac{\partial^2 T}{\partial y^2}$$

A solution to the above is:

$$\frac{T - T_w}{T_\infty - T_w} = C_1 \int_0^w e^{-\alpha^3} d\alpha \qquad \begin{array}{l} w > o \\ w = y(9S)^{-1/3}, \end{array} \tag{39}$$

satisfying the boundary conditions

$$\begin{array}{llll} T = T_w & \text{as} & y = o, \; x > o & \\ T \rightarrow T_\infty & \text{as} & y \rightarrow \infty, \; x > o & \text{or } x \rightarrow o, \; y > o. \end{array}$$

The heat flux at the surface $y = o$ is given by

$$-k\left(\frac{\partial T}{\partial y}\right)_{y = o} = kC_1 \, (T_w - T_\infty)(9S)^{-1/3} \, . \tag{40}$$

The constant C_1 is the reciprocal of

$$\int_0^\infty e^{-\alpha^3} \, d\alpha = 0.89297,$$

which satisfies the requirement that

$$T \rightarrow T_\infty \text{ as } y \rightarrow \infty.$$

The heat flux is given by:

$$q(s) = \frac{(T_w - T_\infty) \, k}{9^{1/3} \, (0.89297)} \; S^{-1/3} \, , \tag{41}$$

or

$$q(\xi, x) = \frac{(T_w - T_\infty) \, k \mu^{-1/3} a^{-1/3}}{9^{1/3} \, (0.89297)} \left\{ \int_\xi^x \frac{dx'}{\gamma(x')} \right\}^{-1/3} \tag{42}$$

The unit thermal conductance is thus

$$h(\xi, x) = \frac{k\mu^{-1/3}a^{-1/3}}{9^{1/3}\ (0.89297)} \left\{ \int_{\xi}^{x} \frac{dx'}{\tau(x')} \right\}^{-1/3} \quad (43)$$

For the case where the wall temperature varies with x, we have then

$$q(x) = \frac{k\mu^{-1/3}a^{-1/3}}{9^{1/3}\ (0.89297)} \int_{\xi = 0}^{x} \left\{ \int_{\xi}^{x} \frac{dx'}{\tau(x)} \right\}^{-1/3} dT(\xi) \quad (44)$$

$$a = \frac{k}{\rho C_p} .$$

The Leveque solution is useful as an approximation of the Graetz solution very close to the descontinuity in temperature. For example, near $x = \xi = 0$ a large number of terms should be taken to get accuracy in either the Graetz or Poppendiek series solutions. The Poppendiek generalization of the Leveque solution may be used in the thermal entrance region of a liquid metal system.

APPLICATION TO THE PROBLEM OF DISTRIBUTED HEAT SOURCES

The integrating kernels in the right-hand column of Table II permit the investigation of several problems of practical interest.

For example, consider the flat plate in laminar or turbulent flow with a heat source, q, which is uniform between points $x = a$ and $x = b$, and zero everywhere else.

Then for $x > b$ we have:

$$T(x) - T_\infty = \int_{\xi = a}^{\xi = b} g(\xi, x)\ q(\xi)\ d\xi \quad (45)$$

$$= q \int_{0}^{b} g(\xi, x)\ d\xi - q \int_{0}^{a} g(\xi, x)\ d\xi \quad (46)$$

Figs. 1 and 2 show the results of the integration for the flat plate in laminar and turbulent flow.

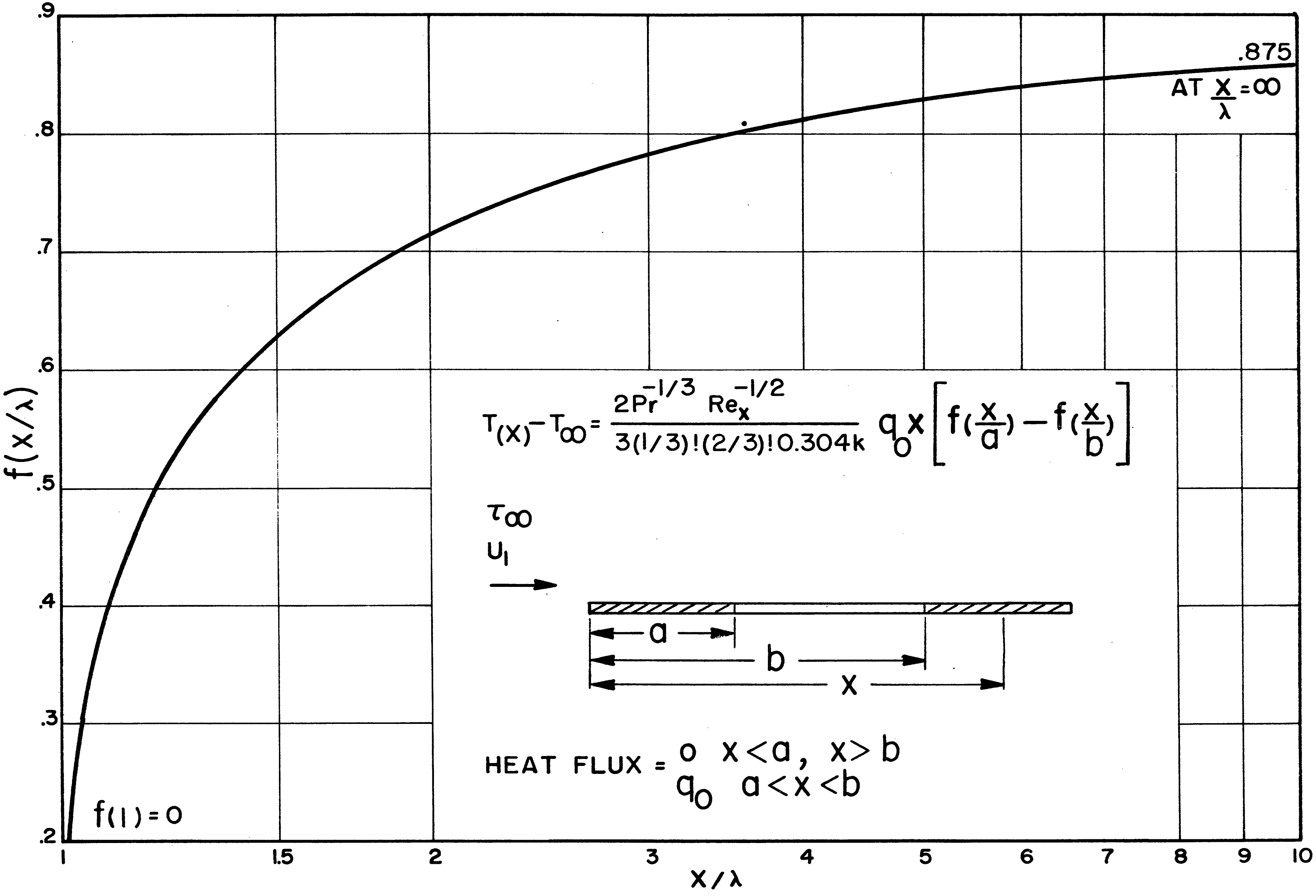

FIGURE 1. EFFECT OF A HEAT SOURCE ON A DOWNSTREAM TEMPERATURE. LAMINAR BOUNDARY LAYER.

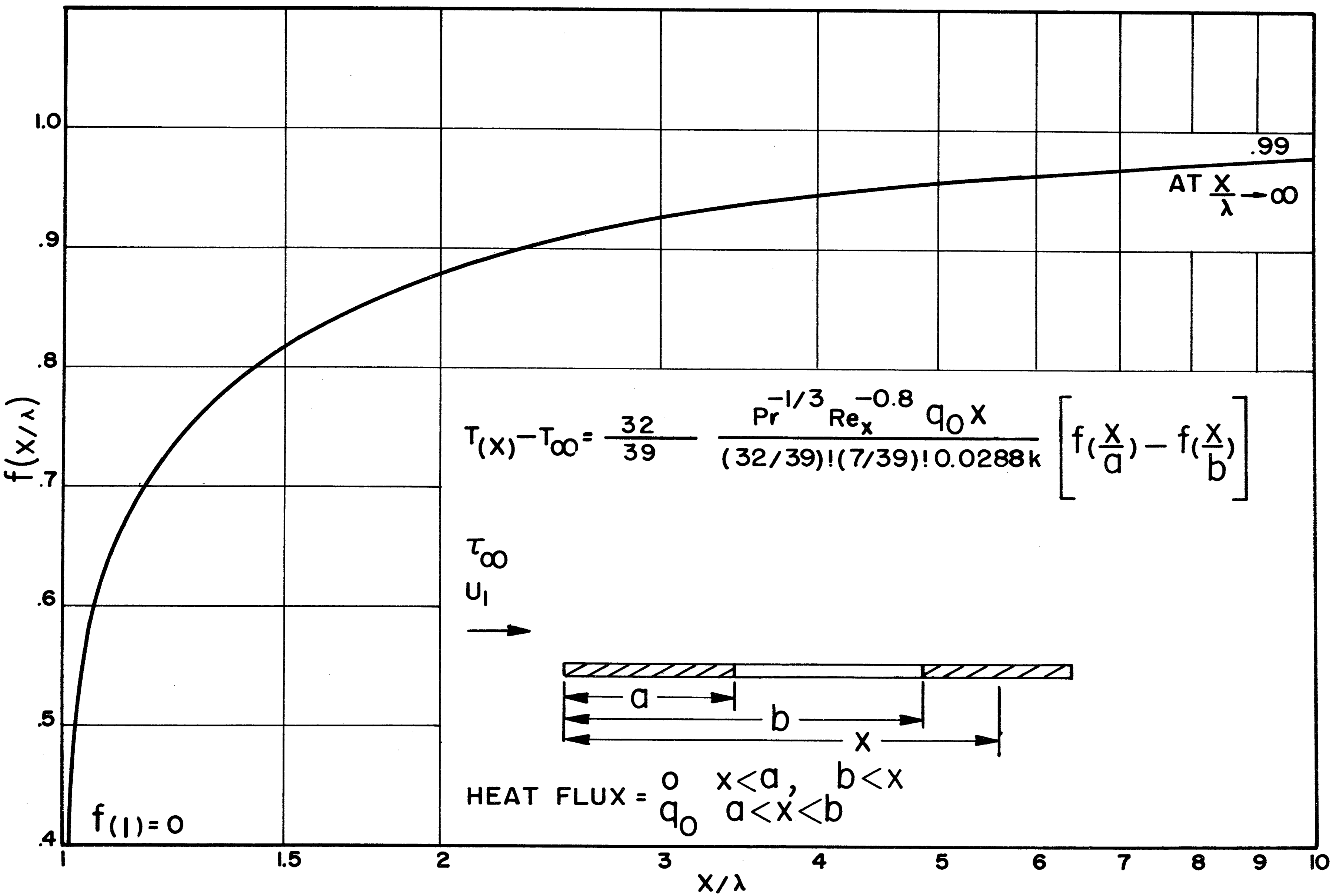

FIGURE 2. EFFECT OF A HEAT SOURCE ON A DOWNSTREAM TEMPERATURE. TURBULENT BOUNDARY LAYER.

As an example of the application of the above data, Figs. 3 and 4 show a comparison of temperature distributions when a uniform heat source is used and when a discontinuous heat source dissipating the same power is used. The discontinuous source is taken as twice the average power for $na < x < (n + 1)a$ when n is even and zero when n is odd (a = constant).

Another interesting case is that of the line source of heat, such as is approximated by a fine wire. For this case, the interval a b in the above integrals shrinks to zero, but the source intensity increases to keep $q d\xi = Q$, constant.

Therefore:

$$T(x) - T_\infty = \sum_n g(\xi_n, x)\, Q(\xi_n) \qquad \text{for } \xi_n > x,\ g(\xi_n, x) = 0. \tag{47}$$

Figs. 3 and 4 show the temperature distributions which result when the limiting case of the line source is approached.

COMPARISONS WITH EXPERIMENT

The integrating kernels proposed by Seban[12] and Rubesin[2] have been favorably compared to the data of Scesa [12,2]. In these cases step functions of temperature were imposed on a flat plate with a turbulent boundary layer parallel to an air stream.

Maisel and Sherwood[18] experimented with mass transfer and, as indicated in Table II, obtained results which were close to the prediction of Seban[12]. The data of Spielman and Jakob[22] for evaporation of water are also in good agreement with Seban's equation.

Flow In a Tube

The doctoral thesis of Kroll[23] yields data on heat transfer in a pipe with laminar flow and a variable wall temperature. Fig. 5 shows typical variations in q(x) and $T_w(x)$ measured by Kroll for three runs.

Fig. 6 shows the calculated value of q(x) using the data on $T_w(x)$ and graphically integrating Eq 48.

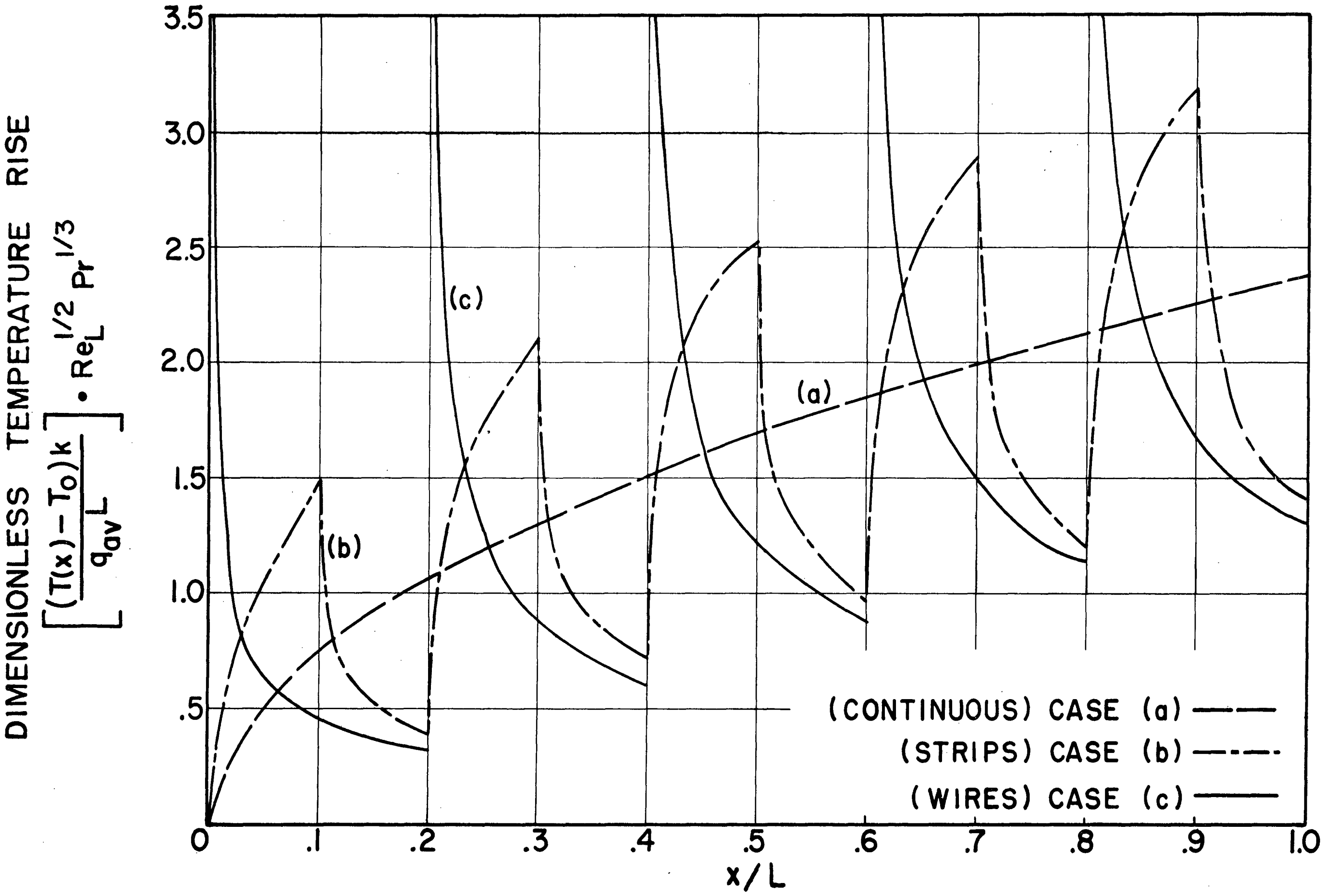

FIG. 3. EFFECT OF NON UNIFORM HEAT FLUX ON TEMPERATURE OF FLAT PLATE IN LAMINAR FLOW.

Effect of changing the distribution of a given heat flux to a length L of laminar boundary layer. In case (a), a constant heat flux is used. In case (b) the heat intensity is twice that of case (c), but the heat is applied to half the area, alternating with regions of zero heat flux. Case (c) considers line sources of heat such as fine wires at x/L = 0, 0.2, 0.6, and 0.8. In all cases $q_{av}L = \int_0^L q(x)dx$.

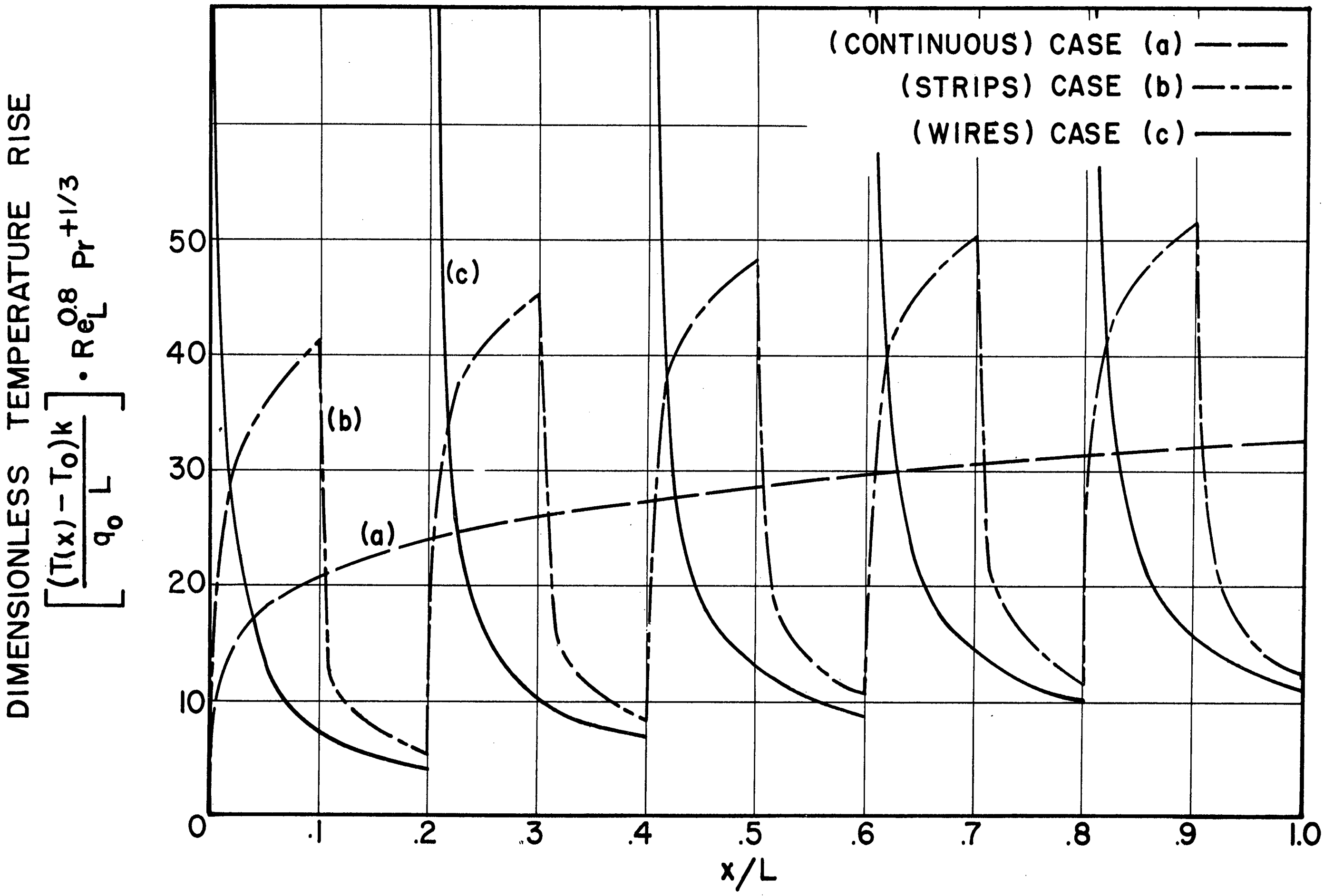

FIG. 4. EFFECT OF NON UNIFORM HEAT FLUX ON TEMPERATURE OF FLAT PLATE IN TURBULENT FLOW.

The effect of a nonuniform heat source on surface temperature for a turbulent boundary-layer. (See Fig. 3 for symbols.)

$$q(x) = \frac{4k}{d} \sum_{1,2,3} C_i e^{-\alpha_i \beta x} \int_0^x e^{\alpha_i \beta \xi} \, dT_w(\xi). \tag{48}$$

The points shown in Fig. 6 are measurements by Kroll.

Fig. 7 shows the computed local Nusselt modulus compared to the measured values. The computed values are taken from Eq 49:

$$\frac{hD}{k} = \frac{4}{T_w - T_g} \sum_{1,2,3} C_i e^{-\alpha_i \beta x} \int_{\xi=0}^{x} e^{\alpha_i \beta} \quad dT_w(\xi). \tag{49}$$

The measured values are taken from:

$$\frac{hD}{k} = \frac{q(x)\ D}{(T_w - T_g)\ k}, \tag{50}$$

where $q(x)$ and $(T_w - T_g)$ were obtained by Kroll.

The agreement in runs 46 and 22 is satisfactory.. The deviations in run 54 are attributable to the changed fluid properties caused by the large wall-temperature change. The better agreement in Fig. 7 is due to the fact that the "local" thermal conductivity was used in Eq 50 in reducing the experimental points to the nondimensional Nusselt modulus, thus in a measure compensating for the changed fluid properties.

BOUNDARY LAYERS PRODUCED BY JETS OF AIR DISCHARGED PARALLEL TO THE SURFACE

Wieghardt[27] has presented data for the temperature distribution downstream from a slot of hot air, as in Fig. 8.*

The results far downstream of the slot were represented by the empirical relation:

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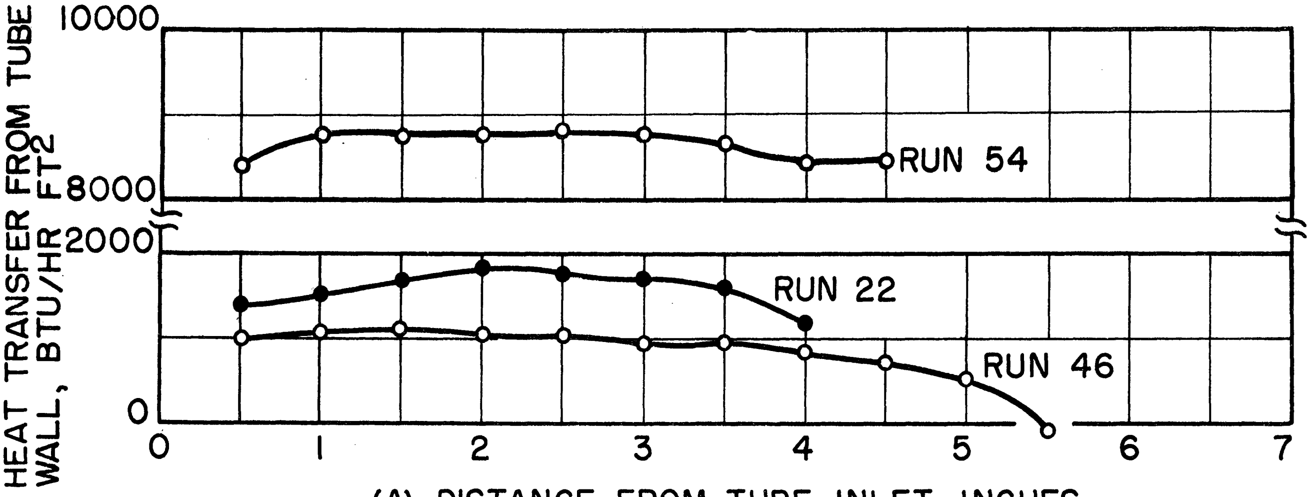
*The following comparison was suggested by E.R.G. Eckert.



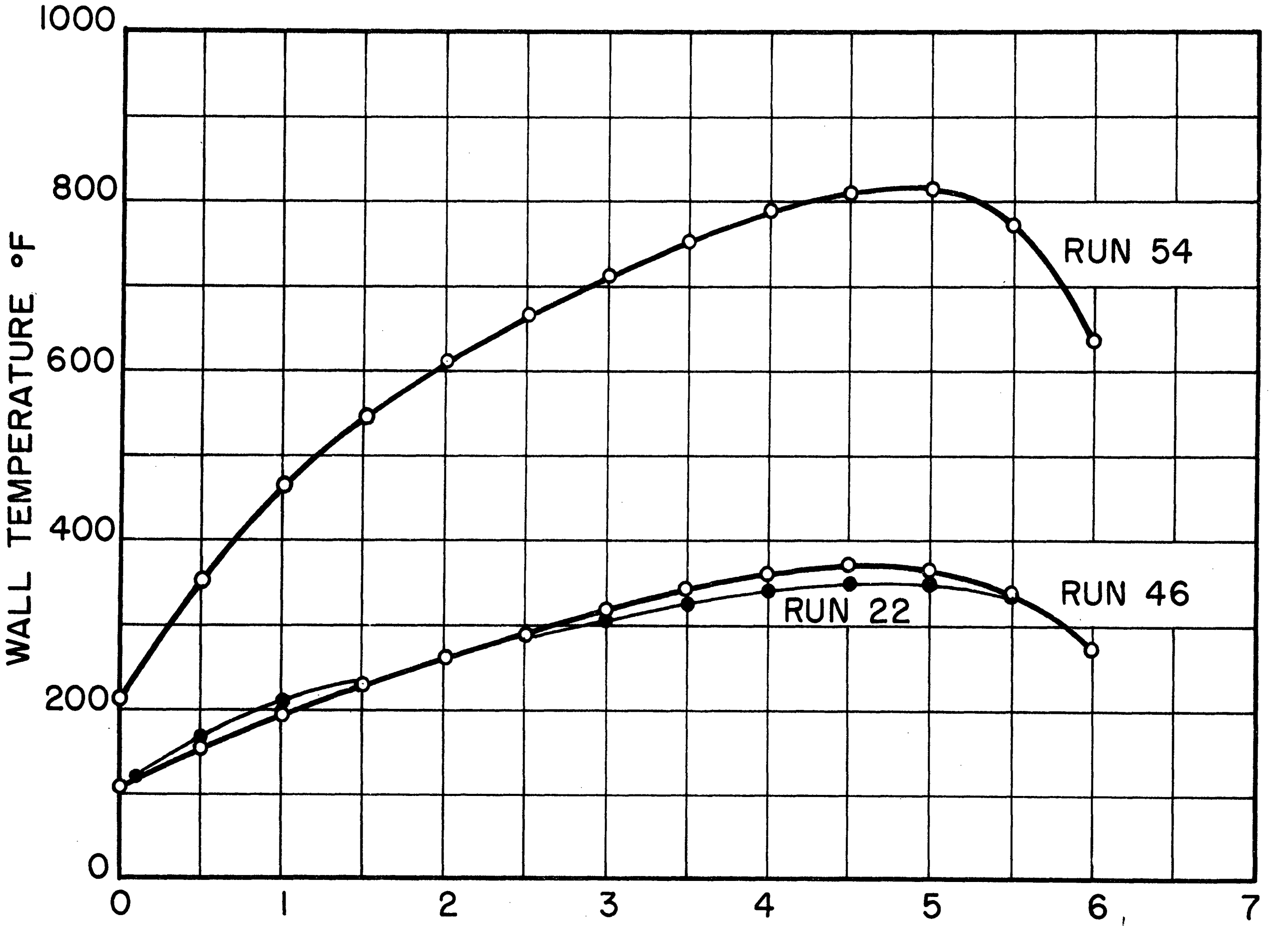


FIG. 5. (A) HEAT FLUX. (B) TEMPERATURE OF TUBE WALL. AT TUBE ENTRANCE, RUN 46, $Re_d = 489$, RUN 22, $Re_d = 1,390$, RUN 54, $Re_d = 2,850$.

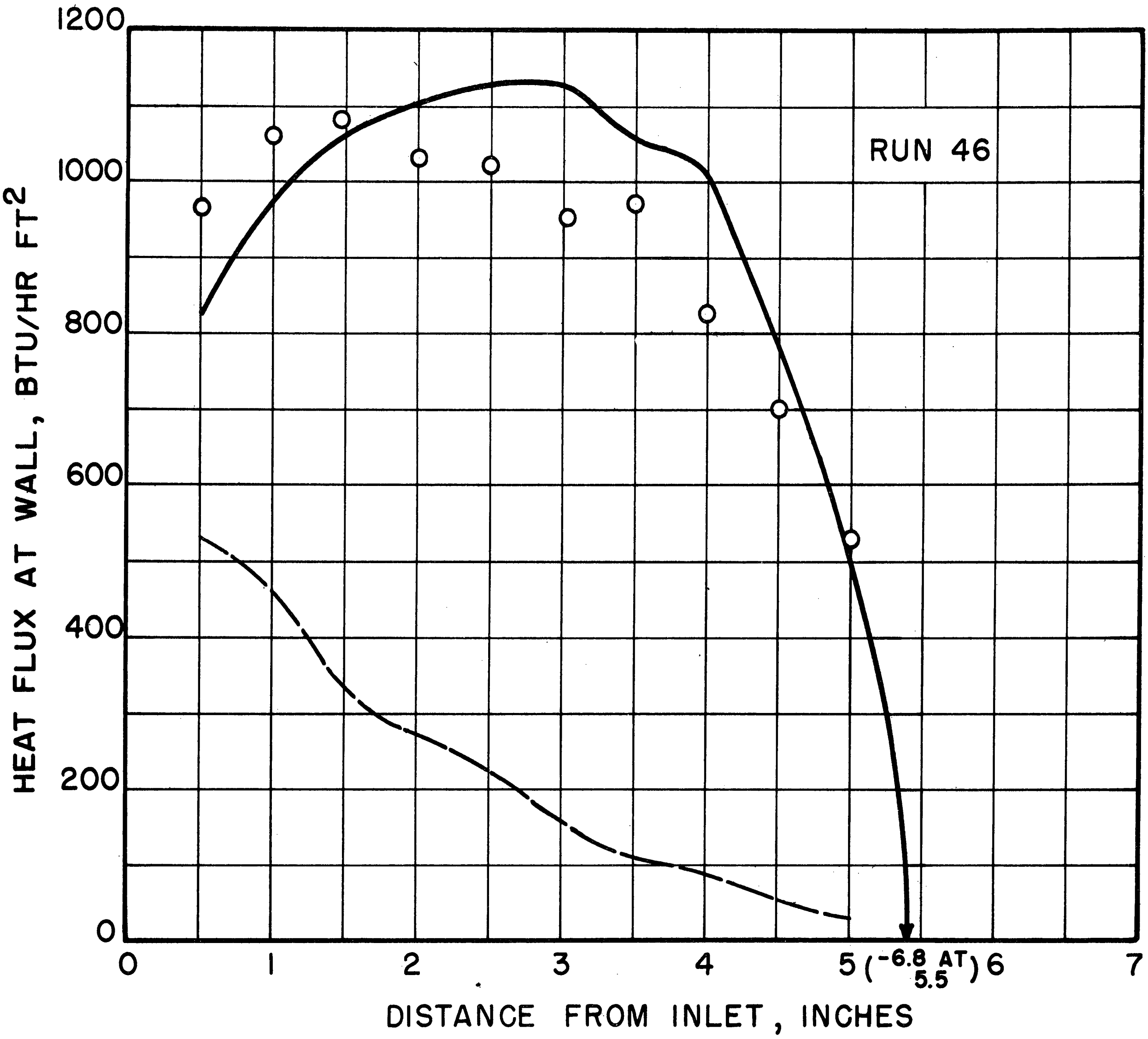


FIG. 6a. COMPARISON BETWEEN ANALYSIS AND EXPERIMENT. THE POINTS ARE FROM REFERENCE 23, THE SOLID LINE IS THE INTEGRATION OF EQUATION 47 AND THE DASHED LINE IS THE PREDICTION BASED UPON "ISOTHERMAL WALL" EQUATIONS.

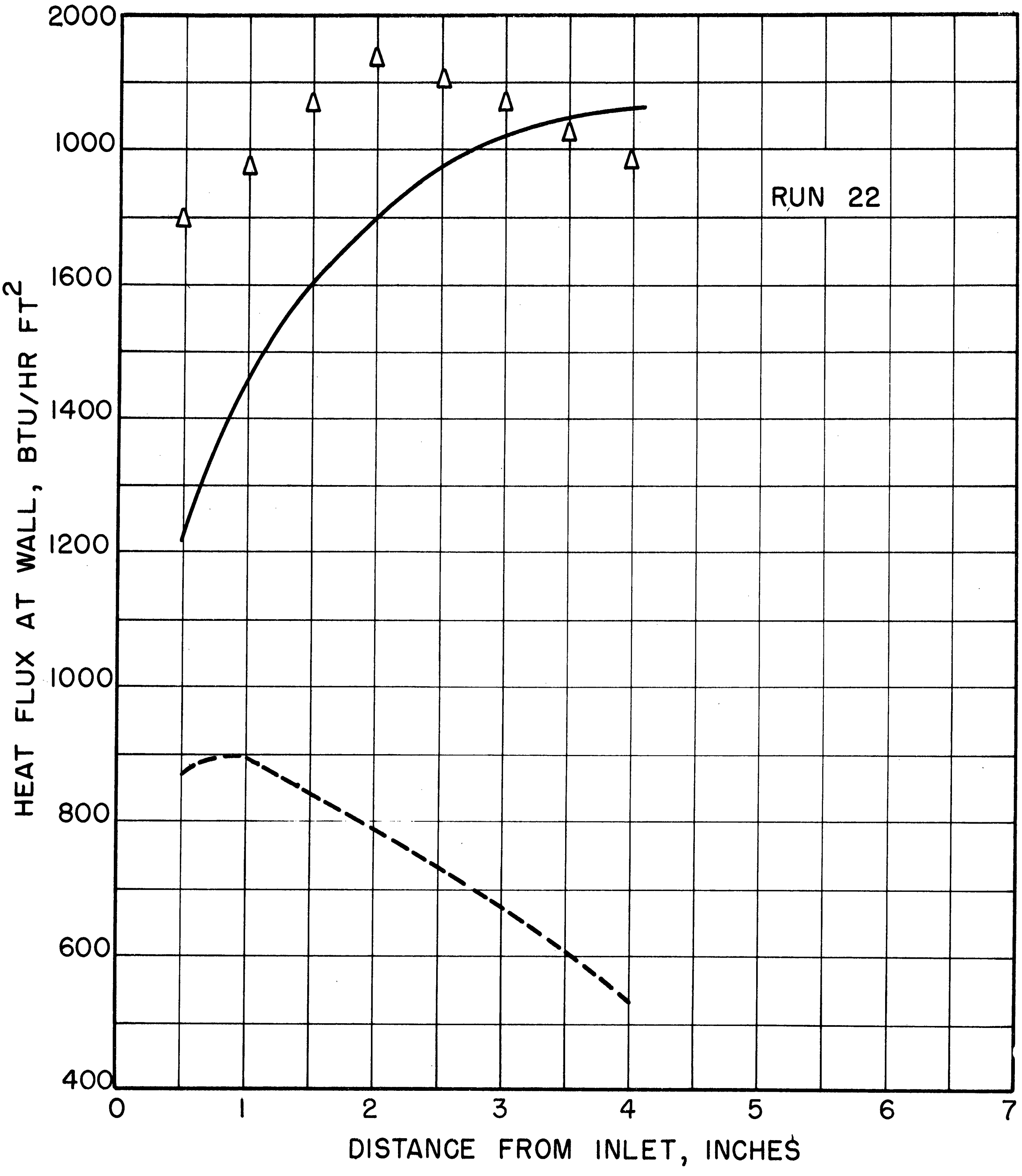


FIG. 6b. COMPARISON BETWEEN ANALYSIS AND EXPERIMENT. THE POINTS ARE FROM REFERENCE 23, THE SOLID LINE IS THE INTEGRATION OF EQUATION 47 AND THE DASHED LINE IS THE PREDICTION BASED UPON "ISOTHERMAL WALL" EQUATIONS.

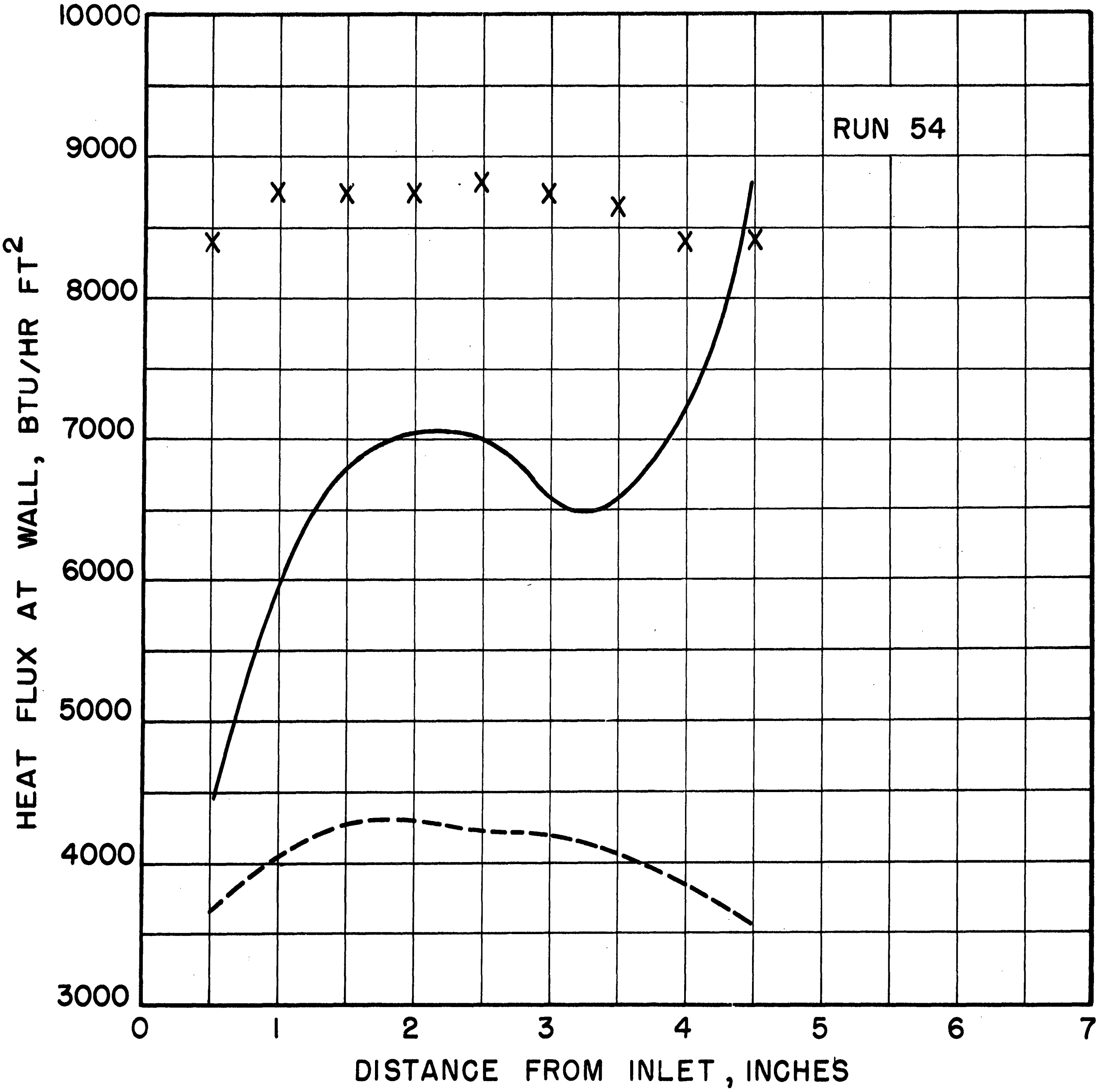


FIG. 6c. COMPARISON BETWEEN ANALYSIS AND EXPERIMENT. THE POINTS ARE FROM REFERENCE 23, THE SOLID LINE IS THE INTEGRATION OF EQUATION 47 AND THE DASHED LINE IS THE PREDICTION BASED UPON "ISOTHERMAL WALL" EQUATIONS.

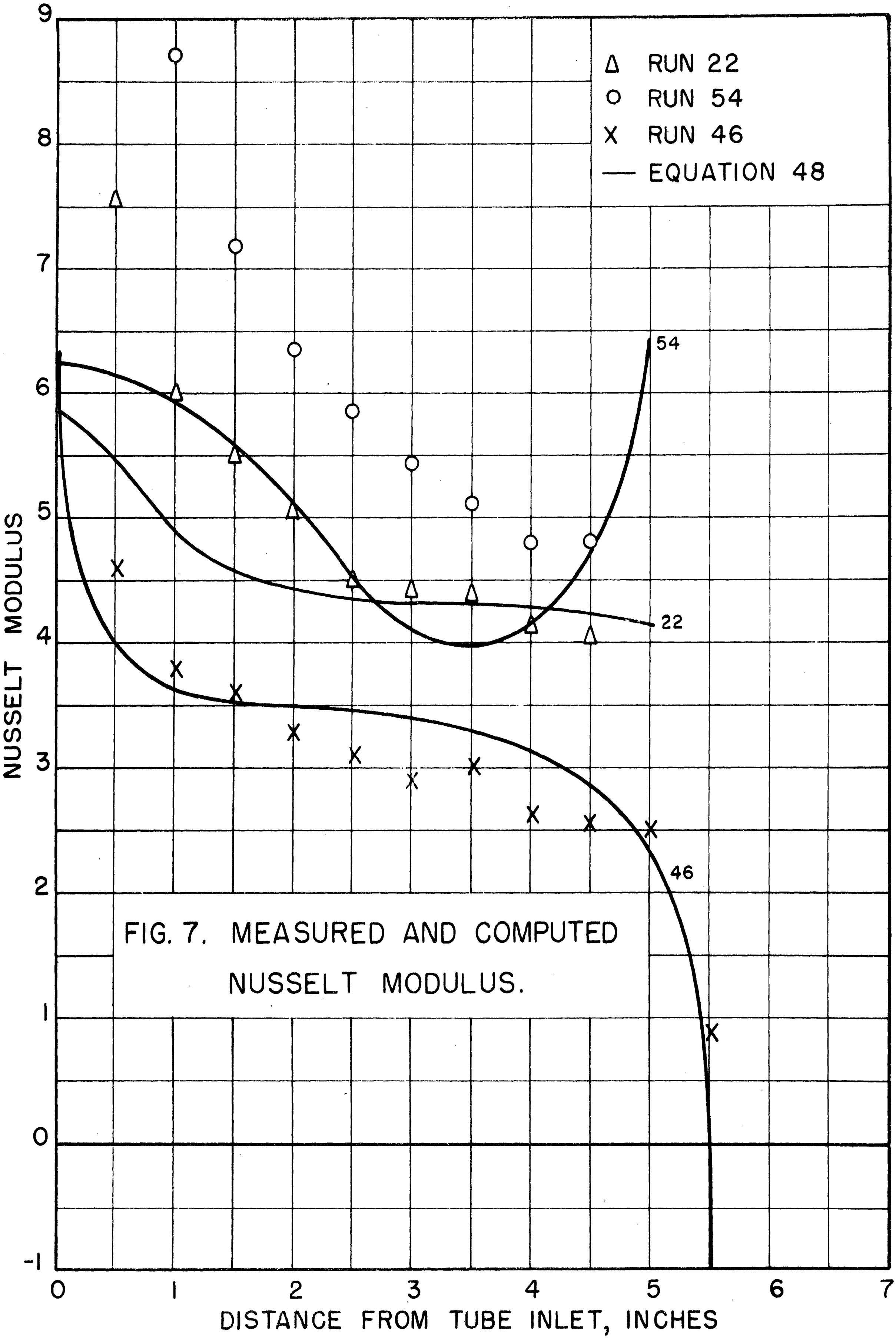


FIG. 7. MEASURED AND COMPUTED NUSSELT MODULUS.

$$\frac{T_w(x) - T}{T_s - T} = 21.8 \left(\frac{x\ U\ \rho}{S\ U_s \rho_s}\right)^{-0.8} \tag{51}$$

Except for the disturbance to the boundary later caused by the jet, the situation studied by Wieghardt is similar to the case of heat source of strength $U_s \rho_s S C_p \ (T_s - T_\infty)$ placed at the origin of the plate.

Using the integrating kernel on line VII of Table II and setting $q(\xi)\ d\xi = Q = U_s \rho_s\ S\ C_{ps}\ (T_s - T_\infty)$, $\xi = 0$ we have:

$$T_w(x) - T_\infty = \frac{28\ Pr^{-1/3}\ Re_x^{-0.8}\ U_s \rho_s\ S c\ \rho_s (T_s - T)}{195\ (32/39)!\ (0.0288k)} \tag{52}$$

A minor rearrangement yields

$$\frac{T_w(x) - T_\infty}{T_s - T_\infty} = \frac{28\ Pr\ \ Pr_s\ (k_s/k\ \ Re_s^{0.2}}{(32/39)!\ (7/39)!\ (0.0288k)} \left(\frac{x\ U\ \rho}{S\ U_s \rho s}\right)^{-0.8} \tag{53}$$

where

$$Re_s = \frac{U_s \rho_s S}{\mu_s}$$

For the range of temperatures used by Wieghardt

$$Pr = Pr_s = 0.72,\ (\mu/\mu_s)^{0.8}\ (k_s/k) \cong 1. \tag{54}$$

Thus

$$\frac{T_w(x) - T_\infty}{T_s - T_\infty} = 4.62\ Re_s^{0.2} \left(\frac{x\ U_\infty \rho_\infty}{S\ U_s \rho s}\right)^{-0.8} \tag{55}$$

The data given by Wieghardt did not include the actual temperature of the air during the tests, but if the laboratory air temperature is guessed to be 20°C, then the "slot" Reynolds modulus, Re_s, may be computed. The tests thus appear to have been run at values $3760 < Re_s < 12,630$. Eq 55 thus becomes

$$\frac{T_w(x) - T_\infty}{T_s - T_\infty} \begin{Bmatrix} 24 \\ \text{to} \\ 30 \end{Bmatrix} \left(\frac{x\ U_\infty \rho_\infty}{S\ U_s \rho_s}\right)^{-0.8} \tag{56}$$

Part of the discrepancy between Eqs 56 and 51 may be ascribed to the fact that the issuing jet has a unifrom temperature profile, whereas a turbulent boundary layer heated from one side has a profile similar to the

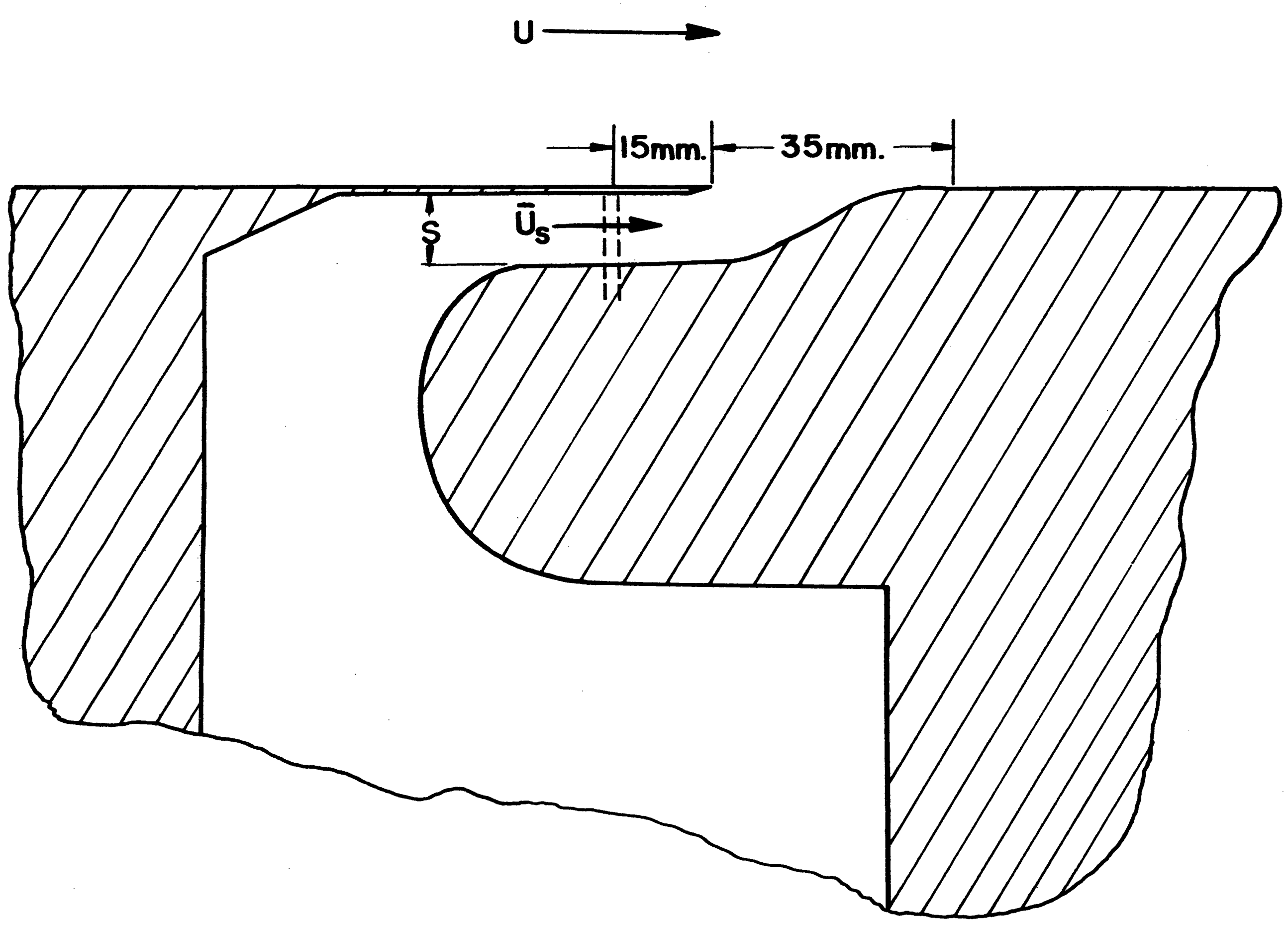


FIGURE 8. DISCHARGE SLOT USED BY WIEGHARDT IN TESTS OF REFERENCE 26.

1/7-power law. For the same enthalpy in the boundary layer, the wall temperature will be higher in the case of a heat source than for a jet.

FRICTIONAL HEATING

As mentioned in the beginning of this paper, the effects of viscous heating may be taken into account by adding the "adiabatic wall temperature" to the solutions obtained when viscous heating effects have been put equal to zero.

In the case of flow in the absence of a pressure gradient the adjustment is most simply accomplished by replacing the free-stream temperature by the adiabatic wall temperature.

When the adiabatic wall temperature varies, as it does over a wedge or where there is a pressure gradient, the effect of frictional heating may be introduced by writing $T_w(x) - T_{ad}(x)$ in place of $T_w(x)$ wherever it appears in the equations. For example, a nonisothermal surface in frictional flow has heat flux related to wall temperature by:

$$q(x) = \int_{\xi=0}^{x} h(\xi, x)\, d\left[T_w(\xi) - T_{ad}(\xi)\right] \tag{57}$$

If the fluid properties vary significantly, the above result is not valid.

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The author would like to express his appreciation to Professor Ruel V. Churchill for reviewing the paper and suggesting some of the methods of analysis. Professor James Neal Addoms of Massachusetts Institute of Technology read the manuscript and offered several helpful criticisms which were adopted. The author wishes to give specific thanks to both Professor Addoms and Dr. C. L. Kroll for permission to use data from an unpublished thesis.

APPENDIX A

The Stieltjes Integral

The more commonly known Riemann integral, which for most engineers is pictured as the area under a curve of the integrand plotted against the variable of integration, does not yield a value if the integrand is not "well behaved". The Stieltjes-type integral is so defined that for "well behaved" functions the result of the integration is the same as ordinary or Riemann integration. The Stieltjes integral does have an advantage, however, in providing (by definition) an integral in some cases where the Riemann integral is ambiguous. It is a shorthand notation for expressing a sum plus an integral.

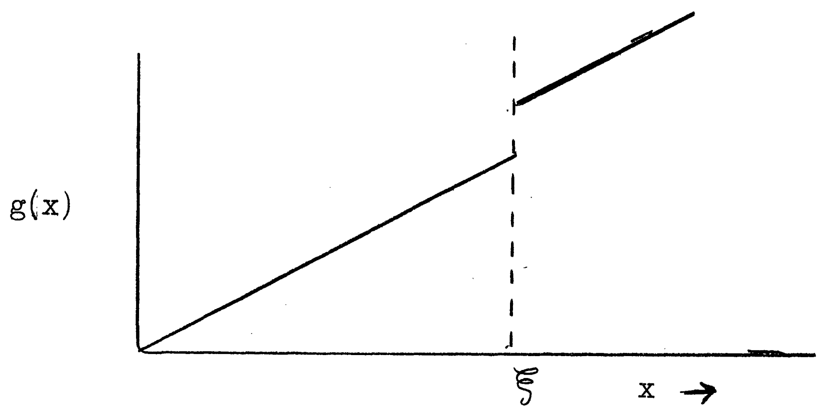
A discussion of Stieltjes integrals is contained in Reference 20. In this appendix we give only the interpretation necessary to the discussions of this paper.

Consider, for example the integral, I,

$$I = \int f(x) \frac{dg(x)}{dx} dx$$

So long as f(x) and dg(x)/dx are well behaved, no difficulty arises in finding the Riemann integral, I.

Now let $f(x) = X$ and suppose g(x) is given by the graph below



A discontinuity in g(x) occurs at ξ. Everywhere except at $x = \xi$, $dy(x)/dx = C_o$. If one attempts to form the Riemann integral for this case, the immediate vicinity of ξ does not yield an unambiguous "area under the curve" of $f(x)$ $[dg(x)/dx]$ plotted versus x.

The question is resolved as follows. Everywhere the integral is evaluated as an ordinary or Riemann integral. At $x = \xi$ we note $f(x) = f(\xi)$ and consider

$$\int_{\xi-\epsilon}^{\xi+\epsilon} f(x) \frac{dg(x)}{dx}\, dx = f(\xi) \int_{\xi-\epsilon}^{\xi+\epsilon} \frac{dg(x)}{dx}\, dx = f(\xi) \int_{\xi-\epsilon}^{\xi+\epsilon} dg$$

$$(\epsilon \to 0)$$

$$= f(\xi) \left[g(\xi^+) - g(\xi^-) \right]$$

Thus the Stieltjes integral is seen to be the ordinary Riemann integral plus contributions which occur whenever $g(x)$ has a discontinuity.

Therefore, we interpret the Stieltjes integral as:

$$\underset{\text{(Stieltjes)}}{\int_{\xi=0}^{x} h(\xi,x)\, d\, T_W(\xi)} = \underset{\text{(Riemann)}}{\int_{\xi=0}^{x} h(\xi,x) \frac{dT_W(\xi)}{d\xi}\, d\xi} + \sum_{\substack{\text{all jumps} \\ \text{in } T \text{ (at } \xi_i)}} h(\xi_i,x) \left[T(\xi^+_i) - T(\xi^-_i) \right]$$

i.e., a Riemann integral plus a summation.

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