ALGEBRAIC ASPECTS OF RATIOS

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To Douglas R. McManis, Editor of the *Geographical Review*,
whose cleverness with words has helped this author tame many a distorted thought!
Congratulations on your unparalleled service to the American Geographical Society
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A particularly exciting feature of the Internet permits documents stored on the World Wide Web to incorporate graphics, as well as text, on the same page. It is generally easy to cut and paste text from a word processor into the online file in which the home page is being created (into the html file). It is almost as easy to upload an image of a scanned photo (for example, saved in gif or jpg format) from a computer in one's home to that same online file (or to one linked to it). Indeed, the whole technical process is so easy, once one has a small set of keys that open merely a few doors, that documents can be created in advance of even simple knowledge about enduring concepts.

Thus, one might anticipate a host of web sites that do not take account of simple cartographic principles or of drafting lessons, let alone elements of written linguistic or mathematical style. From a more positive standpoint, however, one might see that the creation of web documents can serve as a favorable opportunity to cast new light on traditional material. Practice can motivate theory: in mathematics, geography, linguistics, or elsewhere.

Consider the simple task of creating web stationery incorporating an established logo. The American Geographical Society (New York) has given permission to experiment with its relatively symmetric logo that includes a sphere (Bird, M. L., 1996). When the logo is scanned at a relatively low resolution that even older or cheaper scanners can produce, and loaded as a scanned image onto a Web page, the result appears as it does below (Figure 1). The shape of the globe appears correct to the eye, but the image is too large to be viewed on one moderately sized computer screen. Clearly, this situation is unsatisfactory for general use: a large image requires not only too much visual space, but also too much computer space, and too much time to come up on the screen. The obvious answer is to scale the image down in size.

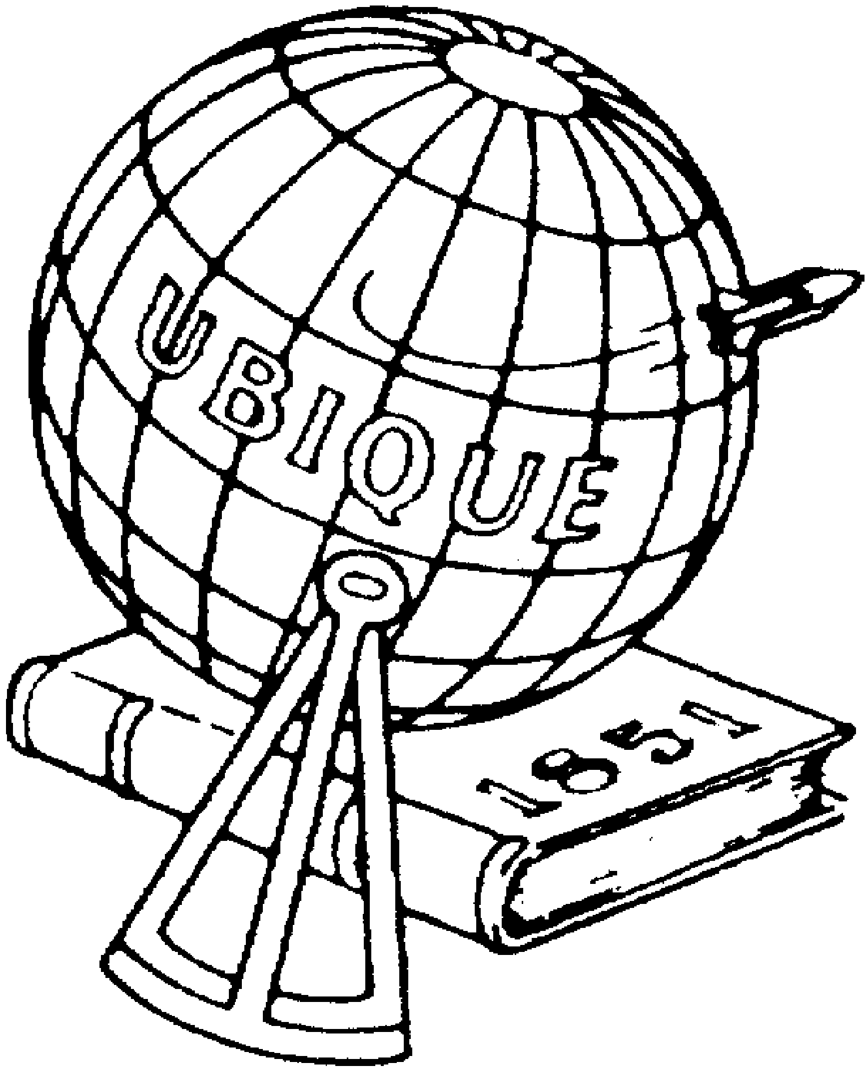

Figure 1.
Logo of the American Geographical Society, height 922 pixels, width 810 pixels.

With a rectangular image such scaling is of course a simple matter: divide the width in half (for example). To retain the same shape of image as the original, also divide the height in half (Figure 2). More generally, to retain shape when altering the scale of a two-dimensional rectangular shape, preserve the aspect ratio--the ratio of the horizontal dimension of an image to the vertical dimension of an image. Varying the aspect ratio of an image causes distortion, as in Figure 3, in which the width is divided by 2 and the height is divided by 3.

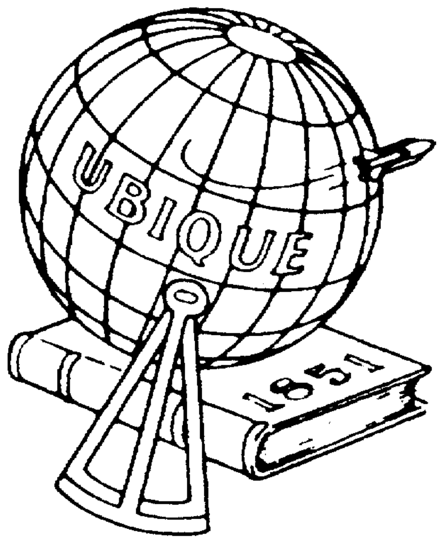

Figure 2.
Logo of the American Geographical Society, height of Figure 1 divided by 2 and width divided by 2. Correctly scaled image.

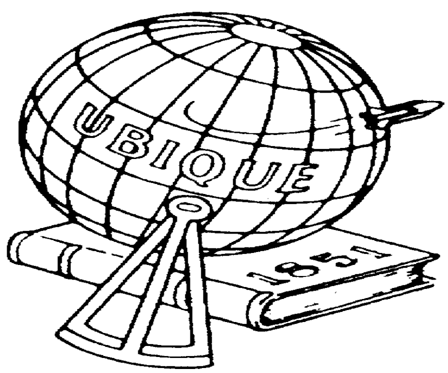

Figure 3.
Logo of the American Geographical Society, height of Figure 1 divided by 3 and width divided by 2. Distorted image.

PIXEL ALGEBRA

In the case of the AGS logo, the scanned image in Figure 1 (a .gif file) measured 810 pixels in width and 922 pixels in height. Pixels are the fundamental units in which these images are measured; fractions of pixels are not permitted. The algebra of pixels is an integer algebra that can be viewed to operate on a bounded, finite set of pixels (the dimensions of the cathode ray tube). In the image of the AGS logo, both dimensions are even numbers and so clearly both are divisible by 2. The aspect ratio, 810/922, can be preserved by taking 1/2 of each dimension; 810/922 = (405*2)/(461*2) = 405/461. Are both numerator and denominator divisible by 3?

In times when "flashy" applications relied more on elegant abstractions, the distributive law was often a source of the unusual or the "cute."

DISTRIBUTIVE LAW

For integers in a domain of two operations, + and *:

Suppose that a, b, and c are integers. Then a*(b+c)=a*b + a*c.

One application of the distributive law yields the "trick" that to determine whether or not a number is divisible by three, add the digits which form it; repeat the process until a single digit is reached. If that single digit is divisible by 3, then the entire number is divisible by 3. In the case of the AGS logo dimensions 810 becomes 8+1+0=9 which is divisible by 3 so that 810 is also divisible by 3. The number 922 becomes 9+2+2=13 which becomes 1+3=4 which is not divisible by 3 so that 922 is not divisible by 3. Thus, 1/3 would not be a scaling factor for the scanned AGS logo that would preserve the aspect ratio: 3 is not a factor of both the numerator (810) and the denominator (922) of the aspect ratio of 810/922.

To see why this procedure is an application of the distributive law, work through 810 as an example, and note that numbers such as 9, 99, 999 are always divisible by 3 (conversation with W. C. Arlinghaus):

810 = 8*100 + 1*10 + 0*1 --- first use of distributive law
= 8*(99+1) + 1*(9+1) + 0*1 --- partitioning of powers of ten into convenient summands
= 8*99 + 8*1 + 1*9 + 1*1 + 0*1 --- second use of distributive law
= 8*99 + 1*9 + 8*1 + 1*1 + 0*1 --- commutative law of addition (a+b=b+a)
notice that the sum 8*99 + 1*9 is divisible by 3 because 9 and 99 are divisible by 3:
8*99 + 1*9 = (8*33 + 1*3)*3--third use of the distributive law, so now,
810 = (8*33 + 1*3)*3 + (8*1 + 1*1 + 0*1) and for the number 810 to be divisible by 3 (fourth use of the distributive law) all that is thus required is for the right summand in parentheses, 8*1 + 1*1 + 0*1 to be divisible by 3. Hence, the rule of three, for determining whether or not a given integer is divisible by 3 becomes clear.

A corresponding rule of nine is not difficult to understand, as are numerous other shortcuts for determining divisibility criteria. Clearly, one need test candidate divisors only up to the square root of the number in question. However, when one is faced with an image on the screen, it would be nice not to have taken the trouble (however little) of finding that 810 is divisible by 3, only to find that 922 is not. A far better approach is to rewrite each number using some systematic procedure and then compare a pair of expressions to determine, all at once, which numbers are divisors of BOTH 810 and 922. For this purpose, the Fundamental Theorem of Arithmetic is critical.

FUNDAMENTAL THEOREM OF ARITHMETIC

Any positive integer can be expressed uniquely as a product of powers of prime numbers (numbers with no integral divisors other than themselves and one).

Thus, in the AGS logo example,

810=2*405=2*5*81=2*3*3*3*3*5 (superscripts avoided by repetitive multiplication) and,

922=2*461.

The number 810 was easy to reduce to its unique factorization into powers of primes; one might not know whether or not 461 is a prime number or whether further reduction is required to achieve the prime power factorization of 922.

To this end, the Sieve of Eratosthenes (the same Librarian at Alexandria who measured the circumference of the Earth) works well. To use the sieve, simply test the prime numbers less than the square root of the number in question. The square root of 461 is about 21.47. So, the only primes that can possibly be factors are: 2, 3, 5, 7, 11, 13, 17, and 19. Clearly, 2, 3, and 5 are not factors of 461. A minute or two with a calculator shows that 7, 11, 13, 17, and 19 are also not factors of 461. Notice that these calculations need only be made once--when one has the unique factorization all divisors of both numbers are known from looking at the two factorizations, together. Thus:

810 = 2*3*3*3*3*5

922 = 2 * 461

and 2 is the only factor common to both numbers.

PRESERVATION OF THE ASPECT RATIO

Because the only factor the two numbers 810 and 922 have in common is 2 it follows that the only scaling factor that can be used, THAT WILL PRESERVE THE ASPECT RATIO, is 1/2. Had 1/3 or some other ratio been employed, a distorted view of the AGS logo would have been the result. Figure 4 shows the result of using a scaling factor of 1/10, so that the image is 81 pixels wide and 92 pixels high (rounded off from 92.2 pixels). The image looks good, but in fact the aspect ratio was not preserved. Level of sensitivity to image distortion will vary with the individual; it is important to have absolute techniques that guarantee correct answers. Once one knows them, then one can choose when, and when not, to violate them.

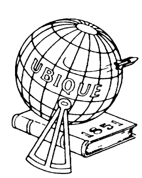

Figure 4.

Logo of the American Geographical Society, height and width divided by 10, aspect ratio not preserved. Distortion present but not evident.

Reduction of the AGS logo by 1/2, producing an image 1/4 of the original size, may still not be desirable. One can use the unique factorization to build what is desired. The value of 810 has a number of factors; thus, one might choose to rescan the image, holding the width at 810 pixels and shaving just a bit off the height (there appears to be room to do so in the background without touching the line drawing of the globe). Now, to create the possibility of various scaling factors, consider a tiny sliver removed to create a height of 920 = 2*2*2*5*23; there is an extra factor of 5 that 920 has in common with 810 that 922 did not, so all of 1/2, 1/5, and 1/10 (one over 2*5) are scaling factors that will preserve the aspect ratio. However, one might wish still more possible scaling factors; if a slightly larger sliver can be removed, so that the height of the scanned image is 900 pixels, with 900 = 2*2*3*3*5*5, then 810 and 900 have common prime power factors of 2, 3, 3*3, and 5, so that the set of scaling factors has been substantially expanded to include all of:

1/2, 1/3, 1/9, 1/5,	combination of four things one at a time--4!/(1!*3!)
1/6, 1/18, 1/10; 1/27, 1/15; 1/45	combination of four things two at a time--4!/(2!*2!)
1/54, 1/30, 1/90, 1/135	combination of four things three at a time--4!/(3!*1!)
1/270	combination of four things four at a time--4!/(4!*0!).

This set of values for the height, in pixels, should offer enough choices in various size ranges for the stationery to have a logo of reasonable size on it.

IMAGE SECURITY

In creating images, it is useful to have an aspect ratio with numerator and denominator with many common factors; greater flexibility in changing the image is a consequence. However, there may be situations, particularly on the Internet where downloading is just a right-click of the mouse away, in which one wishes to inhibit easy rescaling of an image. When that is the case, unique factorization via the Fundamental Theorem of Arithmetic again yields an answer: choose to scan the image so that the numerator and denominator of the aspect ratio are relatively prime--that is, so that numerator and denominator have no factors in common other than 1. In the case of the AGS logo, altering the aspect ratio from the original of 810/922 to 810/923 would serve the purpose: 923 is not divisible by any of 2, 3, or 5. Anyone downloading the AGS logo could not resize it without distortion: distortion, for once, becomes the mapmaker's friend.

FINALE

The abstract lessons learned in Modern Algebra are as important in the electronic world as they are elsewhere; indeed, the current technical realm therefore offers a refreshing host of new example to motivate theory, as suggested in this simple scaling example that drew on a variety of algebraic concepts including: the distributive law, the commutative law of addition, the Fundamental Theorem of Arithmetic and the needed associated concept of prime number, facts involving square roots and divisors, the Sieve of Eratosthenes, basic material on permutations and combinations together with the convention that 0!=1, and the idea of relatively prime numbers. The web is a rich source of example when one brings a rich source of abstract liberal arts training to it.

CITATIONS

Bird, M. L. Executive Director, American Geographical Society, May, 1996.

REFERENCES

For the reader wishing to learn more about modern algebra--not easy reading for most, but well worth looking at to gain some appreciation for the vastness of this field.

Birkhoff, G. and Mac Lane, S. A Survey of Modern Algebra. Macmillan,
Herstein, I., N. Topics in Algebra.
Mac Lane, S. and Birkhoff, G. Algebra.