SPATIAL ANALYSIS, THE WISCONSIN IDEA AND THE UW-SYSTEM.
The Use and Abuse of Dispersion Statistics

by
Frank E. Barmore
University of Wisconsin - La Crosse

Abstract

Dispersion statistics (the second moment of a distribution) from mathematical Physics are adapted to Geography for the description and analysis of distributions in a two-dimensional non-Euclidean space (the curved surface of the Earth). The University of Wisconsin System is used as an example. Also, it is again pointed out that the "ellipse of dispersion" is a misleading representation of dispersion statistics and its use should be discontinued.

I. Introduction

I have long been impressed with the power of the simplest statistical measures (population size, average and standard deviation) to effectively characterize the distribution of some property of a collection of things. These simple statistics for describing distributions are widely recognized, widely understood and, when properly used, possess significant descriptive and analytical power. These simple statistics have proven to be very useful in description and analysis in other disciplines. For example, the results of polls reported in the popular press often give the sample size, the average result and some measure of the reproducibility based on the standard deviation or its estimated value. In the study of dynamics of rigid bodies, these three measures of the spatial distribution of mass are equivalent to the total mass, the center of mass and the moment of inertia [see note 1]. No other spatial characteristics of the mass distribution are needed for the rigid body equations of motion. When describing the spatial distribution of electrical charge, the equivalent three simplest measures of the distribution -- the total charge, the dipole moment and the quadrupole moment -- are often capable of characterizing all the important features of the distribution's interaction with other collections of charge. And there are many other examples.

"The distinctively geographical question is '<u>why are spatial distributions structured the way they are?</u>' " (Abler, et al., 1971). Clearly, such a question can not be answered until the distributions of interest can be described. Geographical spatial description and analysis have become quite sophisticated and quite complex. In spite of this it is still often desirable to describe and make comparisons between spatial distributions in the simplest possible terms -- size, average location and dispersion. Why not use the simple, widely used and widely understood statistical moments of a distribution for describing spatial distributions?

The following discussion will: a) demonstrate the adaptation of the idea of location average and location dispersion to distributions in two (and higher) dimensional spaces; b) point out that the misconceived "ellipse of dispersion" is an inappropriate description of dispersion; c) describe the adaptation of appropriate measures of location and dispersion to non-Euclidean spaces (the curved surface of the earth) and d) as an example, use these measures to describe and comment on some features of the University of Wisconsin System.

II. Definitions

For describing spatial distributions of a collection of things scattered in one dimension, an appropriate minimal set of statistics would be the following: i) the size of the population of the things; ii) their average location; and iii) the standard deviation of their location. The location is usually given as some distance from an arbitrarily chosen marker. For distributions in two or three dimensions these concepts must be appropriately extended. In the case of the size of the population of things, there is no difficulty. The concept of average location is easily extended into two or three dimensions by simply taking the location of each member of the population as a <u>location vector</u>. These location vectors are vectors whose magnitudes and directions are taken as the distances and directions from an arbitrary marker of the various members of the population. Then the power and convenience of vector algebra can be used to calculate the average location or center. The result is a location that corresponds to the "center of gravity" or balance point of the distribution (Barmore, 1991). The extension of the concept of standard deviation into two or three dimensions is more complex and merits more discussion.

The standard deviation in one dimension is S, where

$$S^2 = \Sigma \ [d^2(i)] \ / \ n. \qquad \text{Eq. (1)}$$

The d(i)s are the distances <u>from the average location</u> of the various members of the distribution and n is the number of things in the distribution. If this definition is to be extended to two or three dimensions by replacing the distances, d(i), with location vectors, as is done in determining the average location, then it must be decided how the "square" of the location vector will be calculated. In order to square a vector, it must be "multiplied" by itself. There are three forms of vector multiplication: i) the scalar product (or "dot" product or inner product); ii) the vector product (or "cross" product or outer product); and iii) the tensor product (or matrix or dyadic product). [See note 2].

If the scalar product is chosen, the result is a scalar -- a single number. This single number corresponds to the unnamed index of dispersion described by Furfey (1927). It is equal to the <u>dynamical radius</u> defined by Stewart and Warntz (1958, p.182). It is equal to the <u>standard distance deviation</u> defined by Neft (1981, p.55). It is also the same as the <u>standard distance</u> and is equal to 0.707 (the square root of 1/2) times <u>DS</u>, the root-mean-square of the d(ij)s, where the d(ij)s are the distances between the various pairs of members of the distribution -- both of which are discussed by Kellerman (1981, p.15, 16). But, Furfey (1927, p.97-98), who introduced the idea, was of the opinion that there were better ways of representing areal distributions. Also, there is another single number that represents dispersion -- average population density. But it has also been found wanting (Day and Day, 1973). A single number is too simple a statistic for dispersion. A statistic is needed that is capable of carrying a richer array of information. The extension of S into two or three dimensions using the scalar product would, at best, be incomplete.

If the vector product is chosen for the product of the d(i)s with themselves the result is always zero. This is not useful.

If the tensor product is chosen for the product of the d(i)s the summed result is a multiple component statistic . These components can be displayed in a variety of ways [again, refer to note 2]. I choose to simply arrange the components in a square array. Then,

$$\mathbf{S}^2 = \begin{matrix} s^2_{xx} & s^2_{xy} & s^2_{xz} \\ s^2_{yx} & s^2_{yy} & s^2_{yz} \\ s^2_{zx} & s^2_{zy} & s^2_{zz} \end{matrix} , \qquad \text{Eq. (2)}$$

for the case of extension of S into three dimensions. If the extension were into only two dimensions it would be a 2x2 array. The various components are given, for example, by

$$s^2_{xy} = \Sigma \; [\, d_x(i)\, d_y(i)\,] \,/\, n. \qquad \text{Eq. (3)}$$

The diagonal terms of $\mathbf{S}^2$ are the squares of the standard deviations in the directions of the three coordinate axes. The off-diagonal or cross-terms are invariant under an interchange of the subscripts so that, for example, $s^2_{yx} = s^2_{xy}$. The square of the standard distance (defined and used by others as noted above) is equal to the sum of the diagonal elements of the array. This sum is invariant under rotation of the coordinate system used in the computations. Hence, the standard distance does not depend on the orientation of the coordinate system chosen. Formal procedures exist for determining the square of the standard deviation in any chosen direction. More useful for our purposes here, is the possibility of finding a coordinate system, rotated relative to the initial one, for which the array appears in "reduced form."

In this reduced form all the off-diagonal terms are zero and the diagonal terms are: i) the value of the standard deviation squared in the direction for which it has its maximum possible value; ii) the value of the standard deviation squared in an orthogonal direction for which it has its minimum possible value and iii) the value of the standard deviation squared in a direction orthogonal to the previous two and which has a value between the maximum and minimum value. In the two-dimensional case there are two diagonal terms: i) and ii) above. Then one has:

$$\mathbf{S}^2 = \begin{matrix} s^2_{x'x'} & 0 & 0 \\ 0 & s^2_{y'y'} & 0 \\ 0 & 0 & s^2_{z'z'} \end{matrix} \qquad \text{Eq. (4)}$$

where the diagonal terms are the result of the calculations in directions parallel to the coordinate axes in the new (or rotated) coordinate system, indicated by the primes ('). The computation of these values and directions is an eigenvalue problem. The diagonal elements of $\mathbf{S}^2$ are the eigenvalues and are found by solving a single cubic equation (or quadratic equation for the two dimensional case). The orientation of the new or primed coordinate system is given by the eigenvectors which are found by solving a triplet of coupled linear equations (or a pair of coupled linear equations in the two dimensional case). The equations are simple and the procedure for solving them is straightforward (Band, 1959, Eqs. 5.18, 5.19).

In addition, as mentioned in note 2, the 3x3 array of Eqs. 2 and 4 can be represented geometrically by a closed surface in three dimensions. If the array is 2x2 then it can be represented by a closed figure in two dimensions. In either case the distance from the center of the figure to the boundary in any particular direction would equal the standard deviation in the same direction. But, this surface in three dimensions <u>is not an ellipsoid</u> and the two-dimensional figure <u>is not an ellipse</u>. This was pointed out by Furfey (1927, p.95) in response to Lefever's proposal of a "standard deviational ellipse" for measuring geographical concentration (Lefever, 1926). Later writers seem to have overlooked or misunderstood this portion of Furfey's paper. For example, Kellerman (1981, p.22) seems to believe that the two dimensional figure will fail to be an ellipse <u>only</u> in some special cases. <u>This is not so</u> and many of the statements by Kellerman about this nonexistent ellipse are incorrect! The only time when the surface or figure will be a simple one is the exceedingly unlikely case when the dispersion shows no directional variation whatsoever. Then the result is a sphere or a circle. An ellipsoid or ellipse does not represent the dispersion in any simple way and is, at best, misleading [see Note 3]. The use of the "ellipse or ellipsoid of dispersion" as usually defined should not continue.

More suitable and proper is a two or three dimensional <u>cross of dispersion</u>. The half-lengths of the arms of the cross are equal to the square root of the elements of the array in its reduced form (e.g., $s_{x'x'}$). The arms of the cross extend plus or minus one standard deviation from the center and the center has been taken as the average location. The cross is oriented so that its arms are in the directions for which the standard deviations are maximum and minimum. The standard distance is simply related to the cross, being the square root of the sum of the squares of the half-lengths of the arms.

The two-dimensional cross of dispersion is especially suitable for describing two-dimensional spatial distributions in Geography. It is rich in information, carrying with it <u>all</u>

the relevant information about the first and second moments of the distribution. It can be computed in a straightforward way using simple, elementary and non-iterative procedures. The computation is free of mathematical difficulties, provided only that the distributions are finite and dispersed in a finite space. It can be computed for continuous as well as discrete distributions. It can be displayed on a map at the same scale as the map. It is a visually effective and widely recognized symbol. Readers are accustomed to seeing statistical data represented as a point indicating the average and the "error bars" representing the standard deviation. The cross of dispersion is a natural extension of this symbol into two or three dimensions.

III. Statistics for Distributions in Non-Euclidean Spaces

Two limitations of the cross of dispersion must be considered. First, it carries information only about the first and second moments. If the skewness or higher moments of the distribution are important, then additional statistics must be calculated and methods of displaying the results devised. At least one effort has been made by Monmonier (1992). Second, the preceding discussion has assumed that space is Euclidean or "flat". But the surface of the Earth is not Euclidean or flat and as a result there are computational difficulties that must be dealt with if statistical concepts are to be extended to two-dimensional distributions on the Earth's surface.

Traditionally there have been two different ways of working on problems in non-Euclidean spaces. One way is to find a higher dimensional Euclidean space in which to embed the non-Euclidean space. Then the geometry is familiar and computations can proceed using familiar methods. Thus, one could embed the curved two-dimensional surface of the Earth in a Euclidean three-dimensional space (as indeed it is) and proceed.

The second possibility is to adapt and restrict the computation of statistics of distributions on the non-Euclidean surface of the Earth to the surface. We are largely confined to the Earth's surface and it is appropriate to adopt this provincial point of view when calculating statistics of surficial distributions. The key to the statistical computation on the Earth's non-Euclidean surface is the use of <u>location vectors</u> for specifying position. Two quantities remain well defined in non-Euclidean spaces: lengths of geodesics and the direction of geodesics at a given point. Therefore, a location vector of any particular location is a vector whose magnitude and direction are the length and direction of a geodesic (the arc length and direction of a great circle on a sphere) connecting the particular location and a reference location. The geodesics are curved but the location vectors are "straight". Thus, from the provincial or local point of view of someone at the reference point, the problem appears to be Euclidean -- one can proceed with the second moment computation as outlined in the preceding section. There is a long and honorable tradition in Geography of displaying the curved non-Euclidean surface of the Earth (and distributions on it) on a flat and, of necessity, distorted map. The use of location vectors for position is equivalent to working on an azimuthal-equidistant map centered at the reference location. The chosen reference point is the center (average location) of the distribution. While some distortion remains, distances and directions from the center are "true". This is all that is needed. Standard deviations are dispersions measured from the center.

Thus, the necessary techniques for calculating the simplest three moments of distributions in non-Euclidean spaces are in place. The zeroth moment is simply the population size. The population count is not changed or complicated by the non-Euclidean nature of the surface over which the population is distributed. The first moment is the center (average location) of the population. The computation of the center for distributions on the Earth's surface has been discussed previously (Barmore, 1991, 1992). The second moment (about the mean) is the standard deviation, suitably extended into a two-dimensional non-Euclidean space -- the surface of the Earth. The computation has been outlined above.

IV. The Wisconsin Idea and The University of Wisconsin System

The mission of the University of Wisconsin is often simply stated as serving the people of the state through its teaching, research and service. The University was unusual in its early history by working to serve <u>all</u> citizens of the state in the three areas mentioned. The University has done this by providing a wide range of instruction on and off the campus to a wide variety of citizens, doing research in areas with direct application to problems of the State and providing expert advice to citizens and agencies of the State. Universities were not always so conceived and the particular blend of ideas that drove and described these efforts has come to be called "The Wisconsin Idea." The meaning and the origin of the phrase, The Wisconsin Idea, and an associated phrase, "The Boundaries of the Campus are the boundaries of the State," are not precisely known (Stark, 1995 and note 4). In addition to the export and dispersion of the University activities from the campus to the population of the State, the University (the UW System) now consists of a variety of institutions whose campuses are dispersed about the State. Surely this dispersal is an important part of serving the people of the State and is an important part of The Wisconsin Idea. In what follows, this dispersal of the UW System will be used as an example. The dispersion statistics developed above will be used to describe and illustrate how well the UW System has developed this aspect of the Wisconsin Idea.

If the UW System and the State are to be compared, then each must be defined. I have arbitrarily taken the student population as the significant characteristic of the various components of the UW System. More specifically, the "fall head counts" of the student populations (UW System, 1994) was used to characterize each campus [see note 5]. The locations of the 13 two-year Center campuses, the 11 four-year Comprehensive University campuses and the two Doctoral University campuses were taken as the location of an arbitrarily chosen "central place" on each campus as shown on the 7.5 min. series, 1:24,000 scale topographical maps published by the U.S. Geological Survey. With equal arbitrariness, I have chosen to characterize the State two different ways -- by its population and by its area within its boundaries. The location of the various campuses of the UW System and the boundaries of the State are displayed on the map that is Figure 1. The display of the third distribution, the population of the State, is more difficult.

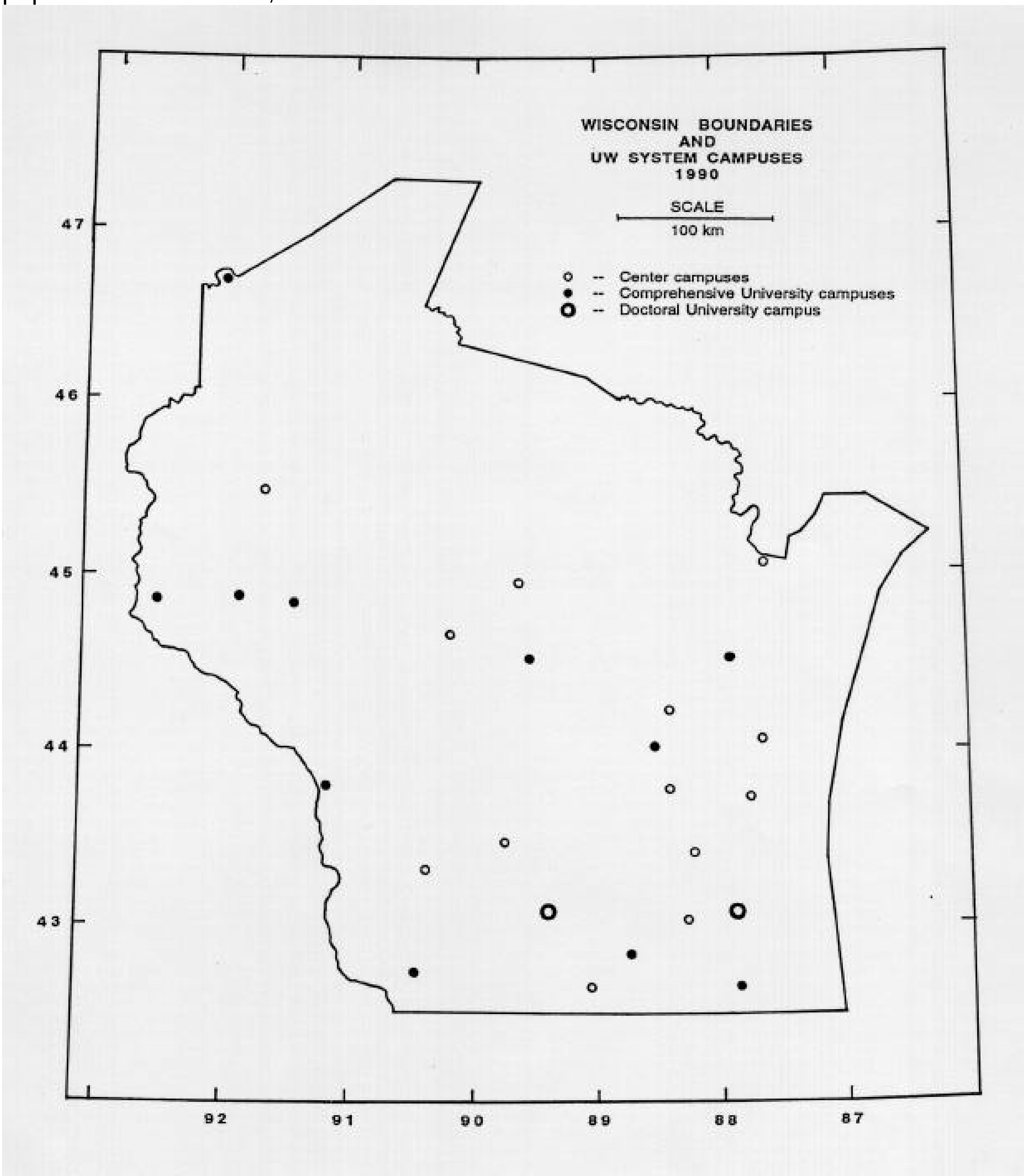

Figure 1. A map showing: a) The boundaries of Wisconsin (as distinct from the more familiar mix of some boundaries and some shore line). b) The 13 two-year Center campuses (small open circles), the 11 four-year Comprehensive University campuses (small solid circles) and the two Doctoral University campuses (larger open circles).

When Furfey (1927) introduced what is now called "the standard distance" he was of the opinion that it was not suitable for graphical representation and that it would be "...better to use contour lines." in order to show how a distribution is dispersed. If the State's area and the State's population are to be compared to (with) the UW System dispersion then all three should be represented the same way. The choice of contour maps for all three representations results in peculiar maps. A contour map of the areal density of the areas of the State consists of a single contour -- the boundary of the State. A contour map of the areal density of students attending the various campuses would consist of a collection of a large number of nearly coincident contour lines tightly surrounding each campus location. If these two maps were combined it would appear much like Figure 1. A contour map of the general population density of the State would have a more familiar appearance but would be so different from the appearance of the other two maps that simple visual comparison would be difficult. Another possibility for displaying population density would be a "dot" map (where the number and size of the dots represent the population of places) such as those shown in <u>An Atlas of Wisconsin</u> (Collins, 1972). But, while these maps show the population distribution very effectively, it is still not possible to use it for simple visual comparison with the UW System distribution.

In contrast, the average location (center) and standard deviation (two-dimensional cross of dispersion) provide a uniform and consistent way of presenting distributions on a map no matter what the peculiarities of the distributions. I have calculated these statistics for the area of the State, the population of the State and the UW System. They are tabulated in Table 1 and displayed in Figure 2. The centers and crosses of dispersion were calculated as outlined in the previous sections. The computations were done assuming that all three distributions lay on the surface of a sphere whose area equals that of Clarke's (1866) ellipsoid. The data for computations involving the area of the State are the same as used previously in determining the geographic center of the State (Barmore, 1993). The data for computations involving the general population of the State are the locations and populations of the ca. 2000 sub-county and county units available from the U.S. Census Bureau (1994, 1995). The data sources for the UW System were previously given. The various population counts are for the year 1990.

As Figure 2 shows, the UW System, as I have defined it, is located and dispersed very much like the population of the State. Also, note that, although all three distributions show a strong northwest-southeast dispersion, both populations are well displaced from the location and dispersion of the area of the State. Finally, if the historical trends in these statistics [see the appendix] are reviewed, it is found that the location and dispersion of the UW System are approaching those of the population of the State. From these comparisons it seems that the UW System has been reasonably successful in serving the State by dispersing its facilities throughout the State. The dispersal of the UW System is well matched with the dispersal of the population it desires to serve.

Table 1.
Distribution Moments for 1990 for Wisconsin's Area and Population
and
The UW System Student Population.

	0th Moment	1st Moment		2nd Moment		
Distribution	Distribution Size	Center Location		Cross of Dispersion		
		Lat. deg.	Long. deg.	S(max) km	S(min) km	Azimuth deg.
Area of State (sq. km.)	169 609.8	44.6344	-89.7098	147.37	96.66	-39.31
Population	4 891 769	43.7280	-88.9829	128.60	69.49	-44.22
UW Students	159 979	43.6468	-89.4000	125.53	63.21	-48.34

Note: Azimuth is the direction of S(max). It is measured from the North and is taken as positive toward the East .

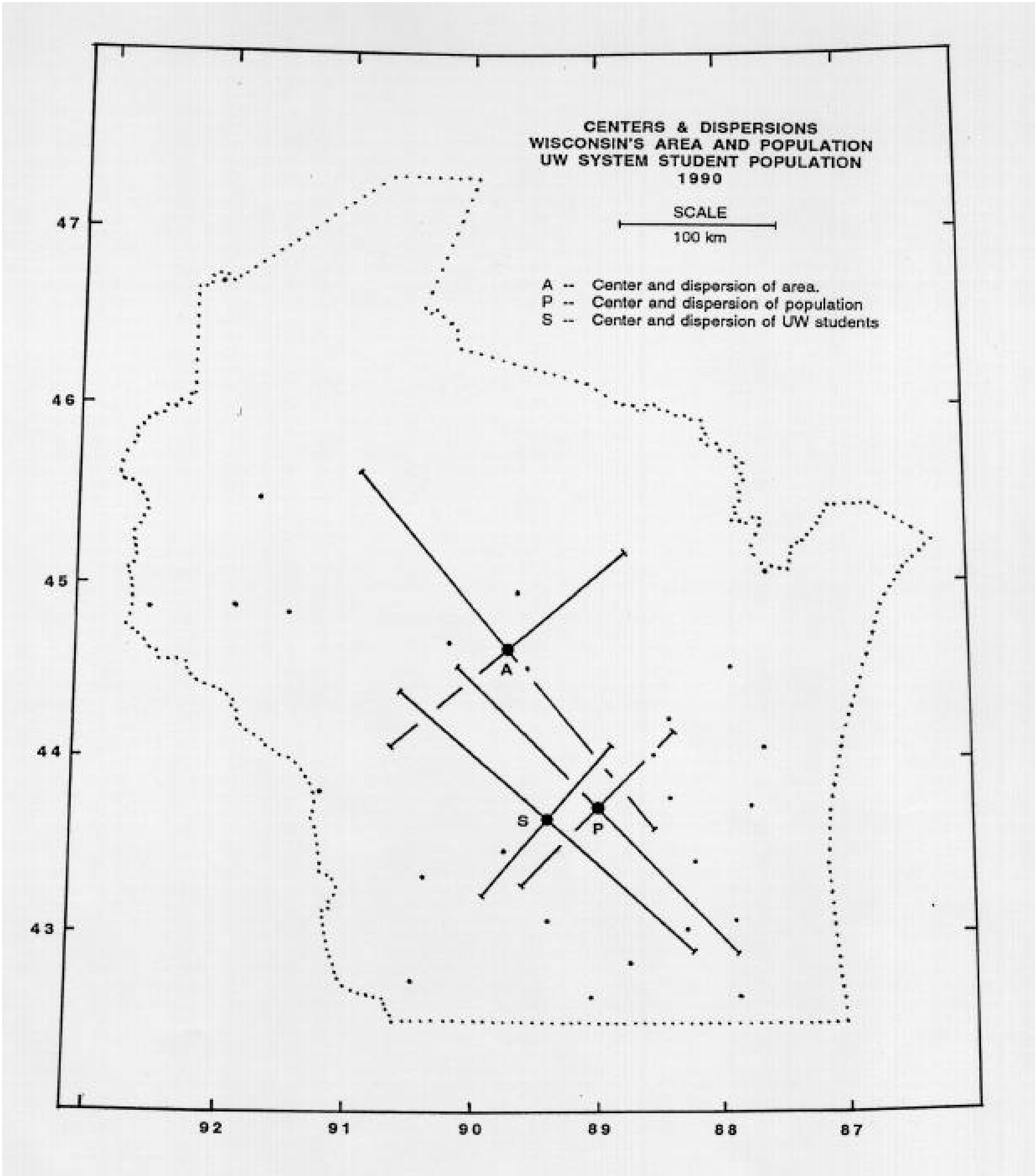

Figure 2. A map showing the location of the center (average location) and the cross of dispersion (two-dimensional standard deviation) of: a) the area of Wisconsin (labeled A); b) The 1990 population of Wisconsin (labeled P); c) The 1990 student population of the UW-System (labeled S).

V. Summary and Recommendations

The lowest three moments of a distribution (distribution size, average location and standard deviation of the location) are effective statistics for describing and comparing distributions. These three moments can be extended into two- and three-dimensional spaces. Their computation can be adapted to non-Euclidean spaces -- specifically the curved two-dimensional surface of the Earth. When extended into two-dimensional spaces this particular set of statistics is well suited to being displayed on a map. I have used these statistics to demonstrate that the UW System has fulfilled The Wisconsin Idea in one particular way. Finally, I have called attention to the misbegotten "dispersion ellipse," as usually defined, and recommend that its use not continue.

VI. Appendix.

Historical data are available that allow the statistics discussed in this work to be computed for somewhat more than a century into the past. The statistics are tabulated in Tables 2 and 3. While the quality of the data for 1990, 1980 and 1970 is quite good, the data used for earlier times are less reliable. I have not attempted to reconcile disparities in the data and I have made arbitrary, though reasonable, decisions in the face of historically shifting definitions of the various populations included. For example, I have arbitrarily included UW-Stout in the compilation in 1920, 1930 and 1950 even though it was not part of the system during these years. Because of the various limitations of the data, the statistics for the years prior to 1990 should be viewed as illustrative only.

VII. Notes.

Note 1. As it happens, the moment of inertia tensor is defined as the difference of two quantities [see Goldstein, Cha. 5], one of which is similar to the $\mathbf{S}^2$ tensor developed in this work. This is because the moment of inertia tensor is most useful if the distances are measured perpendicular to a specific direction. In contrast, in this work the measure wanted is to be based on distances measured parallel to a specific direction.

Note 2. Vectors and tensors can be represented in a variety of ways. The components can be simply listed or arranged and displayed in some other suitable way. A vector can represented as a combination of its components with <u>unit vectors</u> (or pairs of unit vectors in a <u>dyadic</u> notation in the case of tensors). For vectors, the components can be formed into row or column matrices (and for tensor, the components can be represented by a two dimensional matrix of rows and columns). Vectors and tensors can even be represented as geometrical figures or surfaces in some suitable space as is mentioned in the main text of this work. The various representations of vectors and tensors and the algebra needed for working with them in various representations is available in an enormous number of texts. I have found the short summary in Band (1959, Chapter 1) particularly clear and concise.

Note 3. While the figure defined by the various components of the $\mathbf{S}^2$ tensor is not elliptical, it is possible to construct an ellipsoid or ellipse that is related to $\mathbf{S}^2$. Imagine a vector, **v**, whose magnitude can have various values when pointed in various directions. If this magnitude is chosen to vary so that

$$\mathbf{v}\cdot\mathbf{S}^2\cdot\mathbf{v} = 1 \qquad \text{Eq. (5)}$$

no matter what direction the vector, **v**, points, then the tip of the vector sweeps out an ellipsoid or ellipse as it is allowed to point in various directions. Thus, the ellipsoid or ellipse, "v", is defined in a way that makes it the reciprocal figure of the $\mathbf{S}^2$ tensor in the sense that when v^2 and $\mathbf{S}^2$ are "multiplied" in the appropriate way the result is unity. (The <u>inertial ellipsoid,</u> which is related to the moment of inertia tensor, is constructed in an analogous manner. [See Goldstein, sec. 5-4].) However, the elliptical figure "v" must be visualized in a space where the coordinates are measured in units of the reciprocal of length. How is such a figure to be displayed on a map? On a map of a physical space, it would be desirable to display statistics whose magnitudes are in units of length. Also, such a figure as "v" is counter-intuitive -- in directions for which it is small, "S" is large and vice versa. The characteristics of the elliptical "v" are the inverse of what is desired and "v" is only <u>related</u> (in a non-intuitive way) to $\mathbf{S}^2$. This is not suitable for a quantity that is to be displayed to scale on a map.

Note 4. I am unable to find any use of the phrase, "the Wisconsin Idea", prior to the publication of McCarthy's book of that title (McCarthy, 1912). Also, Jack Stark, author of "The Wisconsin Idea: The University's Service to the State" (Stark, 1995) has informed me (Stark, 1996) that neither he nor other scholars actively interested in the history of The Wisconsin Idea with whom he is in contact are aware of any use of the phrase prior to McCarthy's 1912 book.

Note 5. The UW System is characterized in this work as the "fall head count" for the various campuses for the years shown. The data used do not include the enormous enrollment in the UW-Extension of the UW System. It is somewhat ironic that the component of the system that has contributed so much to The Wisconsin Idea should be excluded. However, location data for the persons enrolled in the various Extension activities was not readily available.

Table 2.
Historical Distribution Moments for the UW System Student Population

	0th Moment	1st Moment		2nd Moment		
Year	Distribution Population	Center Location Lat. deg.	Center Location Long. deg.	Cross of Dispersion S(max) km	Cross of Dispersion S(min) km	Cross of Dispersion Azimuth deg.
1990	159 979	43.6468	-89.4000	125.53	63.21	-48.34
1980	155 499	43.6455	-89.4168	126.64	61.70	-48.84
1970	132 088	43.6537	-89.4017	126.75	63.05	-46.33
1960	44 587	43.524	-89.402	122.4	58.4	-43.3
1950	27 498	43.45	-89.49	116.4	51.9	-40.1
1930	16 455	43.52	-89.58	126.8	49.9	-37.1
1920	11 531	43.48	-89.57	115.8	46.6	-37.1
1910	7 167	43.54	-89.52	121.3	53.2	-31.1
1900	4 451	43.67	-89.45	131.3	55.1	-28.7
1890	2 043	43.37	-89.37	75.3	62.0	-14.5
1880	1 391	43.37	-89.35	74.9	61.1	+1.5
1870	846	42.95	-89.48	47.4	14.7	+82.0
1860	est. 237	43.07	-89.40	-.-	-.-	-.-
1850	est. 25	43.07	-89.40	-.-	-.-	-.-

Table 3.
Historical Distribution Moments for Wisconsin's Population

	0th Moment	1st Moment		2nd Moment		
Year	Distribution Population	Center Location		Cross of Dispersion		
		Lat. deg.	Long. deg.	S(max) km	S(min) km	Azimuth deg.
1990	4 891 769	43.7280	-88.9829	128.60	69.49	-44.22
1980	4 705 767	43.7263	-88.9753	129.28	69.41	-43.78
1970	4 417 731	43.6899	-88.9177	127.89	68.22	-43.78
1960	3 947 759	43.70	-89.92	130.9	68.7	-43.7
1950	3 434 620	43.79	-89.03	135.9	71.3	-43.8
1930	2 941 006	43.85	-89.11	140.2	72.8	-43.9
1920	2 632 067	43.93	-89.22	143.4	75.6	-43.5
1910	2 333 855	43.92	-89.26	141.3	77.8	-44.2
1900	2 068 994	43.94	-89.32	137.5	79.7	-46.1
1890	1 686 871	43.86	-89.31	129.6	79.7	-49.8
1880	1 315 497	43.71	-89.28	116.9	74.4	-61.1
1870	1 054 770	43.56	-89.19	104.2	72.2	-66.3
1860	775 881	43.43	-89.06	91.0	69.0	-70.8
1850	305 595	43.17	-88.82	70.3	56.7	+77.8

VIII. References.

Abler, Ronald et al. 1971. Spatial Organization: The Geographer's View of the World. Englewood Cliffs NJ, Prentice-Hall, Inc. p. 56.

Band, Wm. 1959. Introduction to Mathematical Physics. Princeton NJ, D. Van Nostrand.

Barmore, Frank E. 1991. "Where Are We? Comments on the Concept of the 'Center of Population'" The Wisconsin Geographer. Vol. 7, 40-50. [Reprinted (with the example data set used and with several corrections) in Solstice: An Electronic Journal of Geography and Mathematics, Vol. III, No. 2, pp. 22-38. Winter 1992. Ann Arbor MI, Inst. of Mathematical Geography].

Barmore, Frank E. 1992. "The Earth Isn't Flat. And It Isn't Round Either! Some Significant and Little Known Effects of the Earth's Ellipsoidal Shape." The Wisconsin Geographer, Vol. 8, pp. 1-9. [Reprinted in Solstice: An Electronic Journal of Geography and Mathematics, Vol. IV, no. 1, pp. 26-38. Summer 1993. Ann Arbor MI, Inst. of Mathematical Geography]

Barmore, Frank E. 1993. "Center Here; Center There; Center, Center Everywhere ! The Geographic Center of Wisconsin and the U.S.A.: Concepts, Comments and Misconceptions." The Wisconsin Geographer, Vol. 9, pp. 8-21. [Reprinted in Solstice: An Electronic Journal of Geography and Mathematics, Vol. V, no. 1, Summer 1994 Ann Arbor MI, Inst. of Mathematical Geography]

Collins, Charles W. 1972. An Atlas of Wisconsin, 2nd Ed. Madison WI, American Printing & Publishing Inc. p. 12, 13.

Day, A. T. and L. H. Day. 1973. "Cross-National Comparison of Population Density: Many uses of the concept of human population density generate more heat than light." Science. vol.181. pp. 1016-1023.

Furfey, Paul Hanly. 1927. "A Note on Lefever's 'Standard Deviational Ellipse'." American Journal of Sociology. vol. 33. pp. 94-98.

Goldstein, H. 1959. Classical Mechanics, 6th printing. Reading MA, Addison-Wesley.

Kellerman, Aharon. 1981(?). Centrographic Measures in Geography. Norwich UK, Geo Abstracts. No. 32 in the CAT MOG series, Concepts and techniques in Modern Geography.

Lefever, D. W. 1926. "Measuring Geographic Concentration by Means of the Standard Deviational Ellipse." American Journal of Sociology. vol.32. pp. 88-94.

McCarthy, Charles. 1912. The Wisconsin Idea. New York, Macmillan.

Monmonier, Mark. 1992. "Directional Profiles and Rose Diagrams to Complement Centrographic Cartography." Journal of the Pennsylvania Academy of Science. vol.66, no. 1. pp.29-34.

Neft, David. 1966. Statistical Analysis for Areal Distributions. Philadelphia PA, Regional Science Research Inst. Monograph Series No. 2.

Stark, Jack. 1995. "The Wisconsin Idea: The University's Service to the State" in the State of Wisconsin 1995-1996 Blue Book. Madison WI, [Wis.] Dept. of Administration. p. 99-179.

Staark, Jack. 1996. Letter to the author, October 7, 1996.

Stewart, J. Q. and Wm. Warntz. 1958. "Macrogeography and Social Science." The Geographical Review. vol. 48, no. 2, pp. 167-184.

U.S. Census Bureau. 1994. 1990 Census of Population and Housing: Geographical Identification Code Scheme [1990 Census, CD-ROM; CD-GICS issued Feb. 1994]. Washington DC, Bureau of the Census. Sup. of Doc. No. C3.275:G29.

U.S. Census Bureau. 1995. Unpublished data supplied by the Geography Division, Bureau of the Census.

University of Wisconsin System. 1994? 1994-95 Fact Book. Madison WI, UW System.