# ON $L^1$ –CONVERGENCE OF MODIFIED SINE SUMS

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**Abstract**. In this paper a criterion for $L^1$ –convergence of a new modified sine sum with semi-convex coefficients is obtained. Also a necessary and sufficient condition for $L^1$ –convergence of the cosine series is deduced as a corollary.

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**1. Introduction.** Consider the cosine series

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

with partial sums defined by $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$

and

let $g(x) = \lim_{n\to\infty} S_n(x)$.

Concerning the $L^1$-convergence of cosine series (1. 1) Kolmogorov [5] proved the following theorem:

**Theorem A.** *If $\{a_n\}$ is a quasi-convex null sequence, then for the $L^1$-convergence of the cosine series* (1.1) *it is necessary and sufficient that* $\lim_{n\to\infty} a_n \log n = 0$.

The case in which the sequence $\{a_n\}$ is convex, of this theorem was established by Young [9]. That is why, sometimes, this Theorem A is known as Young-Kolmogorov Theorem.

**Definition[4].** A sequence $\{a_n\}$ is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and

(1.2) $$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \qquad (a_0 = 0)$$

where

$$\triangle^2 a_n = \triangle a_n - \triangle a_{n+1}$$

It may be remarked here that every quasi-convex null sequence is semi- convex.

Bala R. and Ram B. [1] have proved that Theorem A holds true for cosine series with semi-convex null coefficients in the following form:

**Theorem B.** *If $\{a_k\}$ is a semi–convex null sequence, then for the convergence of the cosine series* (1.1) *in the metric space L, it is necessary and sufficient that* $a_{k-1}\log k = o(1),\ k \to \infty$.

Garret and Stanojevic [2] have introduced modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n}\triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_j)\cos kx$$

Garret and Stanojevic [3], Ram [7] and Singh and Sharma [8] studied the $L^1$-convergence of this cosine sum under different sets of conditions on the coefficients $a_n$.

Later on, Kumari and Ram [6], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{j}\right)k\cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n}\triangle\left(\frac{a_j}{j}\right)k\sin kx$$

and have studied their $L^1$- convergence under the condition that the coefficients $a_n$ belong to different classes of sequences. Also they deduced some results about $L^1$-convergence of cosine and sine series as corollaries.

We introduce here new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx.$$

The aim of this paper is to study the $L^1$ –convergence of this modified sine sum with semi-convex coefficients and to obtain the above mentioned result of Bala R. and Ram B. as a corollary.

**2. Main Result.** The main result is the following theorem:

**Theorem 2.1.** *Let $\{a_n\}$ be the semi-convex null sequence, then $K_n(x)$ converges to $g(x)$ in $L^1$-norm.*

**Proof.** We have

$$\begin{aligned} S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n} a_k \cos kx 2\sin x \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n} a_k[\sin(k+1)x - \sin(k-1)x] \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}(a_{k-1} - a_{k+1})\sin kx + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x} \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_k + \Delta a_{k-1})\sin kx + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x} \end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned} S_n(x) &= \frac{1}{2\sin x}\left(\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k+1})\tilde{D}_k(x) + (a_n - a_{n+2})\tilde{D}_n(x)\right) \\ &\quad + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}. \end{aligned}$$

Thus

$$g(x) = \lim_{n\to\infty} S_n(x)$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{\infty}(\Delta^2 a_k + \Delta^2 \Delta a_{k-1})\tilde{D}_k(x)$$

Also

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\sin kx.$$

$$= \frac{1}{2\sin x}\left(\sum_{k=1}^{n}(a_{k-1} - a_{k+1})\sin kx - (a_n - a_{n+2})\tilde{D}_n(x)\right)$$

Applying Abel's transformation, we have

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta a_{k-1} - \Delta a_{k+1})\tilde{D}_k(x)$$

$$= \frac{1}{2\sin x}\sum_{k=1}^{n}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)$$

and

$$g(x) - K_n(x) = \frac{1}{2\sin x}\sum_{k=n+1}^{\infty}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)$$

$$= \lim_{m\to\infty}\left(\frac{1}{2\sin x}\sum_{k=n+1}^{m}(\Delta^2 a_k + \Delta^2 a_{k-1})\tilde{D}_k(x)\right)$$

Thus, we have

$$\int_{-\pi}^{\pi}|g(x) - K_n(x)|dx = O(\sum_{k=n+1}^{\infty} k|(\Delta^2 a_k + \Delta^2 a_{k-1})|)$$

$$= o(1), \text{by } (1.2).$$

This proves Theorem 2.1.

**Corollary.** *If $\{a_n\}$ be the semi-convex null sequence, then the necessary and sufficient condition for $L^1$-convergence of the cosine series (1.1) is* $\lim_{n\to\infty} a_n \log n = 0$.

**Proof.** We have

$$\|S_n(x) - g(x)\| \le \|S_n(x) - K_n(x)\| + \|K_n(x) - g(x)\|$$

$$= \|K_n(x) - g(x)\|$$

$$+ \left\|(a_n - a_{n+2})\frac{\tilde{D}_n(x)}{2\sin x} + a_{n+1}\frac{\sin nx}{2\sin x} + a_n\frac{\sin(n+1)x}{2\sin x}\right\|$$

Also

$$\left\| (a_n - a_{n+2}) \frac{D_n(x)}{2\sin x} + a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} \right\|$$

$$= \|K_n(x) - S_n(x)\|$$

$$\leq \|K_n(x) - g(x)\| + \|S_n(x) - g(x)\|,$$

and

$$|(a_n - a_{n+2})| = \left| \sum_{k=n}^{\infty} (\Delta a_k - \Delta a_{k+2}) \right|$$

$$= \left| \sum_{k=n+1}^{\infty} \frac{k}{k} (\Delta a_{k-1} - \Delta a_{k+1}) \right|$$

$$\leq \frac{1}{n} \left| \sum_{k=n+1}^{\infty} k(\Delta^2 a_k + \Delta^2 a_{k-1}) \right|$$

$$= o\left(\frac{1}{n}\right).$$

Since $\int_{-\pi}^{\pi} \frac{\tilde{D}_n(x)}{2\sin x}dx = O(n)$

Therefore

$$(a_n - a_{n+2}) \int_{-\pi}^{\pi} \frac{\tilde{D}_n(x)}{2\sin x}dx$$

$$= O((a_n - a_{n+2})n),$$

$$= o(1).$$

Moreover,

$$\int_{-\pi}^{\pi} \left| a_{n+1} \frac{\sin nx}{2\sin x} + a_n \frac{\sin(n+1)x}{2\sin x} \right| dx$$

$$\leq \int_{-\pi}^{\pi} a_n \left| \frac{\sin nx}{2\sin x} + \frac{\sin(n+1)x}{2\sin x} \right| dx$$

$$= a_n \int_{-\pi}^{\pi} |D_n(x)| dx$$

$$\sim (a_n \log n).$$

Since $\|K_n(x) - g(x)\| = o(1), \qquad (n \to \infty).$ by Theorem 2.1

Therefore it follows that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |g(x) - S_n(x)| dx = o(1),$$

if and only if $\lim_{n\to\infty} a_n \log n = 0$.

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