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INTEGRAL REPRESENTATIONS OF SOLUTIONS OF THE HELMHOLTZ EQUATION WITH APPLICATION TO DIFFRACTION BY A STRIP

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Chapter 1

INTRODUCTION

Exact solutions of diffraction problems are rare and until recently there was only one method of solution applicable to more than one particular problem. By diffraction problem is meant the general problem of determining a solution of the Helmholtz equation which satisfies homogeneous boundary conditions, either Dirichlet or Neumann, has a prescribed character both at infinity and in the neighborhood of any edges, and which may be singular only at points corresponding to sources. The physical problem then is the determination of a spatial field when an obstacle is in the presence of a time harmonic source of wave motion. Although the term diffraction is usually applied only when the boundary, in the language of geometric optics, has an "illuminated" and a "shadow" region it will not prove inconvenient to include the limiting cases of no shadow, e.g. when the boundary is an infinite plane.

The classic technique produces solutions as infinite series of eigenfunctions and is limited to coordinate systems in which the Helmholtz equation is separable and boundaries which are level surfaces of these coordinate systems. Important as this method is, the usefulness of the solutions thus obtained almost invariably suffers because of the slow convergence of the series.

Exact solutions obtained in "closed form" are even rarer. What constitutes a closed form solution is subject to debate but it is generally agreed that infinite series do not qualify. When Sommerfeld (Ref. 25) introduced his many valued wave functions to solve the half plane problem in 1896, he thought this idea could be extended to solve other diffraction problems, notably that of the strip. However, all subsequent attempts at extension, except to wedges of which the half plane is a special case, have been unsuccessful. Thus the Sommerfeld approach remains a technique, indeed a remarkable and elegant one, for solving a particular problem rather than a general method of solution.

More recently, Wiener Hopf techniques have been successful in treating certain problems involving parallel half planes (see Bouwkamp, Ref. 3, for a discussion of Wiener Hopf techniques in diffraction theory). Exact solutions by this technique are, to date, limited to boundaries of infinite extent.

A simpler integral equation formulation than that previously used in the Wiener Hopf treatment has been given by Clemmow (Ref. 8) The scattered field is considered as a superposition of plane waves of complex angles of incidence and in this sense is the method comparable to Sommerfeld's technique. This approach does not eliminate the restriction to infinite boundaries for exact results.

In the present work, however, we shall concern ourselves with a method for obtaining exact solutions in closed (integral) form even when the boundary is finite. A general class of solutions of the Helmholtz equation is derived which resemble Clemmow's and Sommerfeld's functions in that they are superpositions of elementary solutions except that the superposition is accomplished in a manner such that the solutions assume particularly simple form on boundaries which are level surfaces of the coordinate system used. The usefulness of these solutions is demonstrated by employing them to construct an exact integral representation of the field diffracted by a strip.

This problem of diffraction by a strip has occupied a prominent place in the literature since the work of Lord Rayleigh (Ref. 21) who found an approximate solution for long wavelengths in 1897. Since that time exact solutions in the form of infinite series have been found by Schwarzschild (Ref. 23) who used the Sommerfeld half plane solution as a basis for calculating successive interactions between the two edges, and Sieger (Ref. 24) who found the solution in terms of Mathieu functions as suggested by Wien (Ref. 34). We shall not attempt to give a complete bibliography but refer the reader to the treatments of this problem given by Sommerfeld (Ref. 26) Baker and Copson (Ref. 1) and Bouwkamp (Ref. 4). Bouwkamp's exhaustive survey just cited covers the many attempts to find long wavelength approximations and summarizes the numerical results. Short wavelength approximations have also been sought and, again without claiming completeness, we call attention to the work of Clemmow (Ref. 9), Karp and Russek (Ref. 12), Levine (Ref. 14) and Millar (Ref. 18), whose treatment is based in part on the Schwarzschild approach of successive interaction, and Burger (Ref. 6) and Timman (Ref. 30) who applied techniques of supersonic airfoil theory to the hyperbolic (time dependent) wave equation.

The success of the present approach depends on the fact that the problems of diffraction by a half plane and a strip are not really independent and in fact the method consists, in part, of transforming the solution of one into the solution of the other.

This is accomplished by using the new solutions of the Helmholtz equation to derive an integral equation for the wave function satisfying a Dirichlet condition on a line segment. We write these integral equations for both the half plane and strip problems, one in parabolic coordinates and the other in elliptic coordinates, and then assume that the unknown functions in the integrand are related via the same transformation relating the two coordinate systems. Since the half plane problem has been solved, we are able to obtain an explicit representation of this function, hence, if the assumption is valid, we also obtain, using the strip integral equation as an integral representation, the solution of the strip problem. The validity of the assumption is established by demonstrating in detail that it does indeed produce the solution of the problem of diffraction by a strip.

As will be evident, the construction of the solutions depends vitally on the fact that we consider line sources rather than plane wave incidence.

The plane wave solutions may be obtained but will involve a rather complicated limiting process. Since it is almost standard procedure in diffraction theory to consider the plane wave case first, the fact that line sources are apparently more appropriate in this case may help to explain why the strip problem has resisted closed form solution for so long.

We shall confine our attention entirely to two dimensional problems which are particularly appropriate to the present approach. The possibility of extension to three dimensional problems is not to be excluded but will not be treated here.

Chapter 2

A CLASS OF SOLUTIONS OF THE HELMHOLTZ EQUATION

In this chapter we shall derive a class of solutions of Helmholtz' equation which provides a basis for all that follows. These solutions can be characterized as a non-trivial superposition of elementary solutions where, as will be seen, non-trivial denotes that we integrate elementary wave functions between variable end points.

1. Some Remarks on Superposition

Although plane waves of the form $e^{\pm ikx}$ and $e^{\pm iky}$ are the elementary solutions of the homogeneous Helmholtz equation (2.1.1), in rectangular coordinates,

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right\} \Phi = 0 \qquad (2.1.1)$$

we shall consider line sources, the elementary solutions of cylindrical coordinates.

Following convention, we will use three dimensional terminology to describe two dimensional problems ; thus we shall speak of line sources rather than two dimensional point sources, diffraction by a half plane rather than a half line, etc.

In cylindrical coordinates the non-homogeneous Helmholtz equation, for a source at $r = r_o$, $\theta = \theta_o$,

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + k^2 \right\} \Phi = -4\pi \delta(\vec{r} - \vec{r}_o), \qquad (2.1.2)$$

has solutions $\Phi = -i\pi H_o^{(2)}(kR)$, $i\pi H_o^{(1)}(kR)$, and $-\pi N_o(kR)$ where

$R = |\vec{r} - \vec{r}_o| = \sqrt{r^2 + r_o^2 - 2rr\cos(\theta - \theta_o)} = \sqrt{(x-x_o)^2 + (y-y_o)^2}$. In keeping with a suppressed time dependence of $e^{+i\omega t}$, $-i\pi H_o^{(2)}(kR)$ represents waves diverging from R = 0. Neglecting the constant factor we write this as

$$H_o^{(2)}(kR) = H_o^{(2)}\left(k\sqrt{(x-x_o)^2 + (y-y_o)^2}\right) = H_o^{(2)}\left(k\sqrt{\left[(x+iy) - (x_o+iy_o)\right]\left[(x-iy) - (x_o-iy_o)\right]}\right)$$

and observe that the function obtained by setting $y_o = 0$, replacing x_o by some complex α, multiplying by an arbitrary function of α, and integrating over α, viz.

$$\Phi = \int_c H_o^{(2)}\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)d\alpha, \qquad (2.1.3)$$

will still be a solution of the Helmholtz equation if the contour c is independent of x and y and $f(\alpha)$ is sufficiently well behaved. The points of the path of integration represent sources of strength $-4if(\alpha)$. Similarly, letting $x_o = 0$ and $y_o = i\alpha$ we obtain

$$\Psi = \int_c H_o^{(2)}\left(k\sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \qquad (2.1.4)$$

which also remains a solution.

In a sense we have reversed the usual superposition where the elementary solution of rectangular coordinates (plane wave) is written in cylindrical coordinates and integrated over complex angles of incidence of the plane wave, i.e.

$$\int_C f(\alpha)\, e^{ikr\cos(\theta-\alpha)}\, d\alpha \quad . \tag{2.1.5}$$

We have written the elementary solution of cylindrical coordinates and integrated over complex positions of the line source.

2. Derivation of Non-Trivial Solutions

At first this reversed superposition may appear unnecessarily complicated since now we must worry about the branch points of the integrands. However, by choosing a particular path we are led to a rather surprising and interesting result, namely: if $f(\alpha)$ is analytic in a simply connected region containing the path of integration and A is constant, then the expressions

$$\int_A^{x+iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \tag{2.2.1}$$

$$\int_A^{x-iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \tag{2.2.2}$$

$$\int_A^{-x-iy} J_o\left(k\sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \tag{2.2.3}$$

and

$$\int_A^{x-iy} J_o\left(k\sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \tag{2.2.4}$$

are all solutions of the homogeneous Helmholtz equation in this region. J_o denotes the ordinary Bessel function of order 0.

To see that this is true, first for (2.2.1) we proceed as follows. In formula (2.1.3), where the integrand has branch points at $\alpha = x \pm iy$, take the path of integration to be a loop enclosing the point $\alpha = x + iy$ as shown in figure 2.2.1.

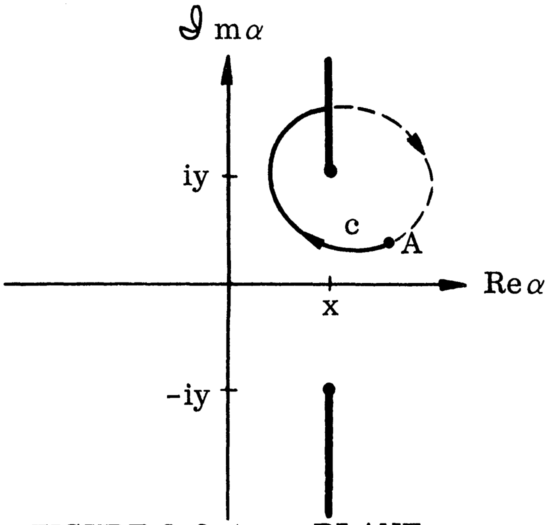


FIGURE 2.2.1: α-PLANE

Before discussing the branch cuts, let us note that, despite the square root of its argument, $J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right)$ is an analytic function of α in the entire finite α-plane, so we need only concern ourselves with the logarithmic branch points of the Hankel function. These can be separated out as follows:

$$H_o^{(2)}\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) = \frac{-2i}{\pi} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right)\log\left(\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right)$$

$$+ \quad F\left(\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) \qquad (2.2.5)$$

where F is an analytic function of α. This is evident on looking at the series representations. Using the shorthand $\rho = \sqrt{(x+iy-\alpha)(x-iy-\alpha)}$, formula (2.1.3) can be written as

$$\int_c H_o^{(2)}(k\rho)\, f(\alpha)\, d\alpha = \frac{-i}{\pi}\int_c J_o(k\rho) \log \rho^2 f(\alpha)\, d\alpha + \int_c F(\rho)\, f(\alpha)\, d\alpha \; . \qquad (2.2.6)$$

Since $f(\alpha)$ is assumed analytic throughout a region containing the contour c and $F(\rho)$ is also an analytic function of α, the contour shown in Figure 2.2.1 is really closed for the second term on the right hand side of (2.2.6) hence, by Cauchy's theorem, this term vanishes. We have written the logarithmic part as $\log \rho^2$ to eliminate, when defining the branch cuts, any complications due to the square root in the definition of ρ.

We choose as a branch cut the negative real axis in the ρ^2 plane. To see what this maps into in the α-plane, we write $\alpha = \xi + i\eta$ and examine

$$\begin{aligned} \rho^2 &= (x+iy-\alpha)(x-iy-\alpha) = (x+iy-\xi-i\eta)(x-iy-\xi-i\eta) \\ &= (x-\xi)^2 - \eta^2 + y^2 + 2i\eta(\xi - x). \end{aligned} \qquad (2.2.7)$$

The condition, ρ^2 real $\Longrightarrow 2\eta(\xi - x) = 0$ and $\rho^2 < 0 \Longrightarrow (x-\xi)^2 - \eta^2 + y^2 < 0$. If $\eta = 0$ the second of these is violated hence $\xi = x,\ \eta > |y|$ and $\xi = x,\ \eta < -|y|$ are the branch cuts in the α-plane, as shown in Figure 2.2.1.

To keep track of which sheet of the Riemann surface of $\log \rho^2$ we are dealing with, we employ the following notation: we define the $n^{\underline{th}}$ sheet to consist of all values of $\log \rho^2$ where $(2n-1)\pi < \arg \rho^2 < (2n+1)\pi$ and indicate this explicitly by writing $\log_n \rho^2$. Finally, by requiring that $\arg \log_o \rho^2 \Big|_{\alpha=0} = 0$, we remove all ambiguity from the definition. The values of $\log \rho^2$ on two successive sheets are related by

$$\log_{n+1}\rho^2 - \log_n \rho^2 = 2\pi i. \qquad (2.2.8)$$

Now we can make precise the meaning of the contour shown in Figure 2.2.1. Starting at the point $\alpha = A$, we choose the principal value of the logarithm, $\log_o \rho^2$, and let the function vary continuously along c. Thus the solid portion of c indicates points on the $o^{\underline{th}}$ sheet of the Riemann surface of the logarithm, but continuous variation across the branch cut takes us onto the $-1^{\underline{st}}$ sheet which is indicated by the dotted portion of c.

Since we have assumed $f(\alpha)$ to be analytic, however, we may deform the contour to that shown in Figure 2.2.2.

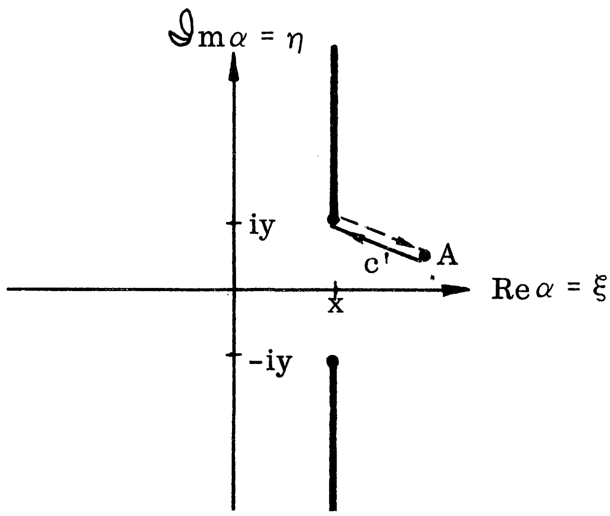


FIGURE 2.2.2: α-PLANE

The integral in (2.2.6) can now be written

$$\int_c H_o^{(2)}(k\rho)\, f(\alpha)\, d\alpha = \frac{-i}{\pi} \int_{c'} J_o(k\rho) \log \rho^2\, f(\alpha)\, d\alpha$$

$$= -\frac{i}{\pi} \int_A^{x+iy} J_o(k\rho)\, f(\alpha) (\log_o \rho^2 - \log_{-1} \rho^2)\, d\alpha\,. \tag{2.2.9}$$

Using relation (2.2.8), this becomes

$$\int_c H_0^{(2)}(k\rho)\, f(\alpha)\, d\alpha = 2\int_A^{x+iy} J_0(k\rho)\, f(\alpha)\, d\alpha . \tag{2.2.10}$$

Since the left hand side of (2.2.10) is a solution of Helmholtz' equation, the right hand side is also. Hence (2.2.1) is shown to be a solution of the Helmholtz equation. This same procedure can be readily used to establish that (2.2.2), (2.2.3), and (2.2.4) are also solutions and it would be needlessly repetitious to do this explicitly.

3. An Alternate Proof.

An alternate procedure, consisting of direct substitution in the differential equation, can be employed to establish that the expressions (2.2.1) - (2.2.4) are indeed solutions of Helmholtz' equation. Contrary to the method used above, this gives no hint as to how the relations were found but does have the advantage of being somewhat simpler. This is illustrated by demonstrating directly that (2.2.2) is a solution. Whereas before we needed $f(\alpha)$ to be analytic in order to deform contours, now the analyticity forms a sufficient condition to permit differentiation according to the usual rule.

Thus, keeping in mind the following easily verifiable relations,

$$\left.\begin{aligned} & J_0\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right)\Big|_{\alpha = x-iy} = 1 \\ & \frac{\partial}{\partial x} J_0\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right)\Big|_{\alpha = x-iy} = -i\frac{k^2 y}{2} \\ & \frac{\partial}{\partial y} J_0\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right)\Big|_{\alpha = x-iy} = -\frac{k^2 y}{2} \end{aligned}\right\} \tag{2.3.1}$$

we find that

$$\frac{\partial}{\partial x}\int_A^{x-iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\,d\alpha$$

$$= f(x-iy) + \int_A^{x-iy} \frac{\partial}{\partial x} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\,d\alpha \qquad (2.3.2)$$

and

$$\frac{\partial^2}{\partial x^2}\int_A^{x-iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\,d\alpha = \frac{\partial f(x-iy)}{\partial x} - \frac{ik^2y}{2} f(x-iy)$$

$$+ \int_A^{x-iy} \frac{\partial^2}{\partial x^2} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\,d\alpha\,. \qquad (2.3.3)$$

Similarly

$$\frac{\partial^2}{\partial y^2}\int_A^{x-iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\,d\alpha = -i\frac{\partial f}{\partial y}(x-iy) + i\frac{k^2y}{2} f(x-iy)$$

$$+ \int_A^{x-iy} \frac{\partial^2}{\partial y^2} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\,d\alpha\,. \qquad (2.3.4)$$

Since $-i\frac{\partial f(x-iy)}{\partial y} = -\frac{\partial f(x-iy)}{\partial x}$, upon substitution of (2.2.2) in the Helmholtz equation we obtain, with (2.3.3) and (2.3.4)

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right\} \int_A^{x-iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha$$

$$= \int_A^{x-iy} \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right\} J_o(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\,)\, f(\alpha)\, d\alpha \;=\; 0\,. \qquad (2.3.5)$$

Hence (2.2.2) is a solution of the Helmholtz equation and, of course, an almost identical procedure could be used to establish that the other relations (2.2.1), (2.2.3), and (2.2.4) are also solutions.

4. Some Properties of the Solutions

Upon subtracting (2.2.2) from (2.2.1), the constant end point of integration is eliminated and we obtain a function, $\phi(x, y)$, with some remarkable properties. Explicitly

$$\phi(x, y) = \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \quad . \qquad (2.4.1)$$

If the path of integration is entirely confined to a simply connected region, R, where $f(\alpha)$ is analytic, then with no other restrictions on $f(\alpha)$, $\phi(x, y)$ is a solution of the homogeneous Helmholtz equation and vanishes on the segment of the line $y = 0$ lying in R.

A similar expression is obtained from (2.2.3) and (2.2.4), viz.

$$\psi(x, y) = \int_{-x-iy}^{x-iy} J_0\left(k\sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) f(\alpha)\, d\alpha \tag{2.4.2}$$

where $\psi(x, y)$ vanishes on the appropriate segment of $x = 0$.

Further, the derivatives assume particularly simple form on these boundaries:

$$\left.\frac{\partial \phi(x, y)}{\partial x}\right|_{y=0} = 0 \quad , \qquad \left.\frac{\partial \phi(x, y)}{\partial y}\right|_{y=0} = 2if(x) \tag{2.4.3}$$

and

$$\left.\frac{\partial \psi(x, y)}{\partial x}\right|_{x=0} = 2f(-iy) \qquad \left.\frac{\partial \psi(x, y)}{\partial y}\right|_{x=0} = 0 \,. \tag{2.4.4}$$

As will be seen shortly, solutions of the form (2.4.1) and (2.4.2) can be constructed for the non-homogeneous Helmholtz equation by allowing $f(\alpha)$ to have singularities. We shall make use of these expressions in the following chapters to find integral representations of solutions of some boundary value problems, for which purpose these functions are obviously well suited.

5. A Limiting Case

To end this chapter we call attention to one of the most immediate consequences of the particular form of the solutions (2.4.1) and (2.4.2). With the simplest (a subjective but hopefully not an unreasonable judgment) non-trivial choice of $f(\alpha)$, namely $f(\alpha) \equiv 1$, (2.4.1) and (2.4.2) become the Helmholtz equation generalizations of the simplest non-trivial solutions of Laplace's equation, y and x.

Thus

$$\lim_{k \to 0} \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) d\alpha = \int_{x-iy}^{x+iy} d\alpha = 2iy \qquad (2.5.1)$$

and

$$\lim_{k \to 0} \int_{-x-iy}^{x-iy} J_o\left(k\sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) d\alpha = \int_{-x-iy}^{x-iy} d\alpha = 2x . \qquad (2.5.2)$$

The expressions for the wave functions can be simplified considerably. In the first case, with the substitution $\alpha = iy\cos\theta + x$, we find that

$$\int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) d\alpha = iy \int_{o}^{\pi} J_o(ky\sin\theta)\, \sin\theta\, d\theta. \qquad (2.5.3)$$

This last form can be integrated explicitly, (see reference 16), obtaining

$$\int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) = 2iy\ J_{1/2}(ky)\ 2^{-1/2}\ \frac{\Gamma(1/2)}{(ky)^{1/2}} \qquad (2.5.4)$$

$$= \frac{2i}{k} \sin ky . \qquad (2.5.5)$$

Similarly, the substitution $\alpha = x\cos\theta - iy$ in the second form enables us to write

$$\int_{-x-iy}^{x-iy} J_o\left(k\sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) d\alpha = \frac{2}{k} \sin kx. \qquad (2.5.6)$$

While these forms are, of course, among the most elementary wave functions it is noteworthy that when the expressions comparable to (2.4.1) and (2.4.2) are developed in other coordinates, (see for example the discussion of the elliptic coordinates in Section 5.3), it is possible to find wave equation generalizations of solutions of Laplace's equation that are not so well known.

Chapter 3

A REPRESENTATION THEOREM FOR WAVE FUNCTIONS SATISFYING DIRICHLET CONDITIONS ON A LINE SEGMENT

In this chapter, we use formula (2.4.1) as the basis for an integral representation theorem for certain solutions of the Helmholtz equation satisfying Dirichlet boundary conditions. This theorem is then employed to obtain integral representations of combinations of cylinder functions. Particular attention is devoted to the case of the line source, $H_0^{(2)}(kR)$.

1. The Representation Theorem

With the understanding that by an analytic function of the real variables x and y we mean that the function has a Taylor expansion in x and y but not necessarily in $z = x+iy$, the fundamental result of this section, the representation theorem, is formulated as follows.

Theorem: If $\phi(x, y)$ is an analytic solution of the Helmholtz equation in a simply connected region, $\mathcal{R}$, containing the line segment $y = 0$, $x_1 < x < x_2$, and $\phi(x, y) = 0$ on this segment, then, in this region, $\mathcal{R}$, $\phi(x, y)$ has the integral representation

$$\phi(x, y) = \frac{1}{2i} \int_{x-iy}^{x+iy} J_0\left(k \sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) \left.\frac{\partial \phi(\alpha, \nu)}{\partial \nu}\right|_{\nu=0} d\alpha . \tag{3.1.1}$$

The proof of this theorem, proving that (3.1.1) is not an equation but an identity, consists of showing that both sides of (3.1.1) have the same value on the line segment and that their normal derivatives are also equal on this segment. Then, by virtue of the Cauchy-Kowalewsky theorem which ensures that there cannot be more than one analytic solution of the Helmholtz equation in a neighborhood of a curve on which the function and its normal derivative are prescribed, the validity of (3.1.1) as an identity follows. Thus the left hand side of (3.1.1), $\phi(x, y)$, is given to be an analytic solution of the Helmholtz equation, vanishing on the line segment, and whose normal derivative on the segment is given by $\left.\frac{\partial\phi(x, y)}{\partial y}\right|_{y=0}$. The right hand side of (3.1.1) is an analytic solution of the Helmholtz equation since it is of the form (2.4.1); it obviously vanishes when $y = 0$; and its normal derivative at $y = 0$ (see (2.4.3)) is $\left.\frac{\partial\phi(x, \nu)}{\partial\nu}\right|_{\nu=0}$. Hence by the uniqueness cited above, (3.1.1) is established as an identity.

Sommerfeld, (Ref. 27), shows very clearly and constructively why uniqueness obtains when the solution of an elliptic equation is given, with its normal derivative, on a curve. Hadamard, (Ref. 11), discusses the more general results of Cauchy and Kowalewsky which guarantee existence as well as uniqueness. He also cites the work of Holmgren which indicates that the requirement of analyticity might be weakened to a condition of sufficient regularity, derivatives up to second order throughout the region, but we shall not consider this possible generalization at present.

Note that a completely analogous representation of wave functions vanishing on the line $x = 0$ can be obtained using the expression (2.4.2), namely:

If $\psi(x, y)$ is an analytic solution of the Helmholtz equation in a simply connected region, $\mathcal{R}$, containing the line segment $x = 0$, $y_1 \leqslant y \leqslant y_2$ and $\psi(x, y) = 0$ on this segment, then, in this region $\mathcal{R}$, $\psi(x, y)$ has the representation

$$\psi(x, y) = \frac{1}{2} \int_{-x-iy}^{x-iy} J_0\left(k \sqrt{(x+iy+\alpha)(x-iy-\alpha)}\right) \left.\frac{\partial \psi(\nu, i\alpha)}{\partial \nu}\right|_{\nu=0} d\alpha . \qquad (3.1.2)$$

The proof completely parallels that given above.

The theorem of this section can be considered in two different ways. On one hand it provides a method of obtaining integral representations of solutions of specific boundary value problems if the solutions are known. On the other hand, if the solution is not known, (3.1.1) provides us with an integral equation which it must satisfy. This integral equation is akin to that obtained through the use of Green's functions except that here we have a Volterra equation where the path of integration does not have such a ready interpretation as a physical boundary.

This integral representation can also be derived, as the Bessel function kernel suggests, by employing Riemann's method of integrating the linear second order hyperbolic differential equation (see Ref. 23). Introducing the characteristic coordinates

$$\xi = x + iy \quad , \qquad \eta = x - iy \qquad (3.1.3)$$

transforms the elliptic wave equation

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right\} \phi(x, y) = 0 \qquad (3.1.4)$$

into the hyperbolic equation

$$\left\{\frac{\partial^2}{\partial\xi\partial\eta} + \frac{k^2}{4}\right\} \phi\left[\xi, \eta\right] = 0. \tag{3.1.5}$$

Following Riemann, we express a solution, ϕ, of this hyperbolic equation in terms of its values on a curve in the $\xi\eta$-plane in a manner completely analogous to the use of Green's theorem in elliptic equations, where instead of Green's functions we employ the characteristic function or the Riemann-Green function

$$v = J_o\left(k\sqrt{(\xi-\xi_o)(\eta-\eta_o)}\right). \tag{3.1.6}$$

Requiring the solution $\phi\left[\xi, \eta\right]$ to vanish on the line $\xi = \eta$ (which corresponds to the boundary condition $\phi(x, 0) = 0$ in the xy-plane) and choosing the region of interest to be bounded by segments of this line and two characteristics, (see Figure 3.1.1) enables us to express the function at ξ_o, η_o in terms of its values on Γ, the segment of $\xi = \eta$, as follows

$$\phi\left[\xi_o, \eta_o\right] = \frac{1}{2}\int_{\Gamma}\left\{v\frac{\partial\phi}{\partial\eta}\cos(n,\xi) + v\frac{\partial\phi}{\partial\xi}\cos(n,\eta)\right\} ds \quad . \tag{3.1.7}$$

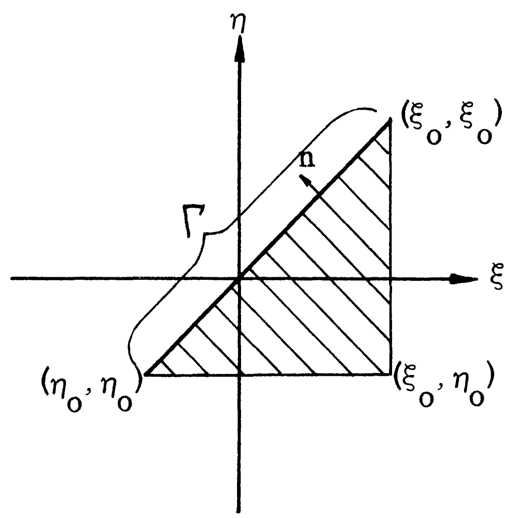


FIGURE 3.1.1: $\xi\eta$-PLANE

Since Γ is the segment of the line $\xi = \eta$ lying between $\xi = \xi_o$ and $\xi = \eta_o$, this can be simplified considerably. With the normal drawn outward, $\cos(n, \xi) = -\frac{1}{\sqrt{2}}$ and $\cos(n, \eta) = \frac{1}{\sqrt{2}}$, thus (3.1.7) becomes

$$\phi\left[\xi_o, \eta_o\right] = + \frac{1}{2\sqrt{2}} \int_\Gamma v \left\{ \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \eta} \right\} ds \quad . \tag{3.1.8}$$

The distance $\int_\Gamma ds$ must be positive if ξ and η are real thus we must define the line element along Γ as

$$ds = \sqrt{d\xi^2 + d\eta^2} = - \sqrt{2}\, d\xi \tag{3.1.9}$$

so that

$$\int_\Gamma ds = - \int_{\xi_o}^{\eta_o} \sqrt{2}\, d\xi = \xi_o - \eta_o > 0 \ . \tag{3.1.10}$$

If $\xi_o - \eta_o < 0$, that is if the point (ξ_o, η_o) were on the other side of the line $\xi = \eta$ from that shown in Figure 3.1.1 both the sense of Γ and the sign of the square root would alter. In either case (3.1.8) becomes, writing v explicitly,

$$\phi\left[\xi_o, \eta_o\right] = -\frac{1}{2} \int_{\xi_o}^{\eta_o} \left\{ J_o\left(k\sqrt{(\xi - \xi_o)(\eta - \eta_o)}\right) \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) \phi\left[\xi, \eta\right] \right\} \Bigg|_{\eta = \xi} d\xi \ . \tag{3.1.11}$$

Although it has been convenient to consider ξ, η and ξ_o, η_o as real, it is of course true that the function defined by (3.1.11) is still a solution of the hyperbolic equation (3.1.5) and vanishes when $\xi_o = \eta_o$ even if ξ, η, ξ_o, and η_o become complex. In particular, with the transformation (3.1.3) it is easily seen that

$$\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta} = \frac{1}{i}\frac{\partial}{\partial y} \tag{3.1.12}$$

and $\eta = \xi$ corresponds to $y = 0$. Thus (3.1.11) becomes

$$\phi(x_o, y_o) = \frac{1}{2i}\int_{x_o - iy_o}^{x_o + iy_o} J_o\left(k\sqrt{(x - x_o - iy_o)(x - x_o + iy_o)}\right) \frac{\partial\phi(x, y)}{\partial y}\bigg|_{y=0} dx . \tag{3.1.13}$$

Renaming the dummy variables appropriately and dropping the subscripts yields formula (3.1.1).

Choosing Γ to be a segment of the line $\xi = -\eta$ rather than $\xi = \eta$ would lead to formula (3.1.2) by the same procedure.

2. Some Remarks on the Application of the Representation Theorem.

It must be pointed out that some caution should be exercised in the use of (3.1.1) as a representation of solutions of boundary value problems valid for all values of x and y.

First of all, it is often true that physically significant problems deal with functions that are not analytic at the boundary. As an example, consider the problem of finding the field, $\phi(x, y)$, of a single line source in the presence of a perfectly soft strip, a problem considered at some length in Chapter 5. An attempt to immediately write the solution of this problem in the form (3.1.1) for points in the region $\mathcal{R}$ (see Figure 3.2.1) might be unsuccessful because $\phi(x, y)$ is not analytic in y (when $x_1 \leqslant x \leqslant x_2$, $\frac{\partial\phi(x, y)}{\partial y}$ is not continuous at $y = 0$).

It is still possible to use the integral form (3.1.1) to represent the field in this case without relaxing the analyticity requirement by considering an intermediate problem, or more correctly, by considering an alternate but equivalent problem; i.e., the mixed boundary value problem depicted in Figure 3.2.2. The advantage in treating this problem is that the physical

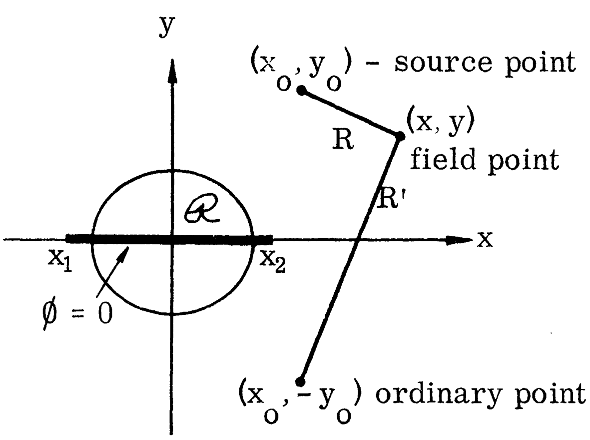


FIGURE 3.2.1: A LINE SOURCE IN THE PRESENCE OF A SOFT STRIP

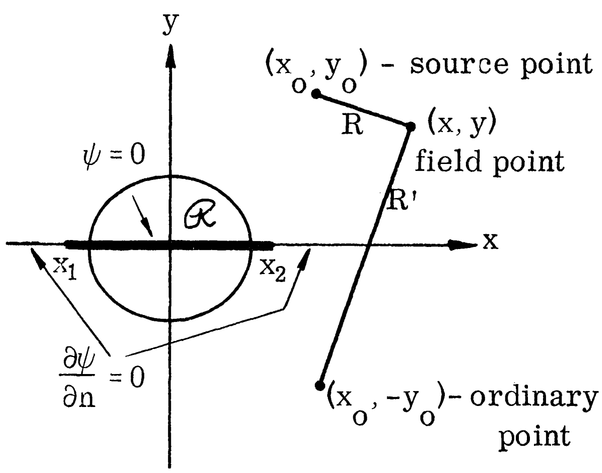


FIGURE 3.2.2: A LINE SOURCE IN THE PRESENCE OF A SOFT STRIP IN A RIGID SCREEN

problem is confined to the upper half plane and an analytic continuation of ψ to the lower half plane in the region $\mathcal{R}$ is possible, even if ψ has no physical meaning there. Hence ψ can be expressed in the integral form (3.1.1).

That the problems depicted in Figures 3.2.1 and 3.2.2 are really equivalent, that is, given the solution to one of them it is possible to construct the solution to the other, is easily seen. Following Bouwkamp, (Ref. 5) , the solution to the problem of Figure 3.2.1 can be written, save for a constant multiplicative factor that determines the strength of the source, as

$$\phi(x, y) = \begin{cases} H_o^{(2)}(kR) - H_o^{(2)}(kR') + \phi_D(x, y) & y \geq 0 \\ \phi_D(x, -y) & y \leq 0 \end{cases} \tag{3.2.1}$$

where $\phi_D(x, y)$ is finite for $-\infty < x < \infty$, $y > 0$ and vanishes for $x_1 < x < x_2$, $y = 0$. Moreover, $\frac{\partial \phi(x, y)}{\partial y}$ must be continuous at $y = 0$ when there is no physical barrier, i.e., when $x < x_1$, or $x > x_2$, hence, for these values of x,

$$\frac{\partial}{\partial y} \left[H_o^{(2)}(kR) - H_o^{(2)}(kR') \right] \Bigg|_{y=0} + \frac{\partial \phi_D(x, y)}{\partial y} \Bigg|_{y=0} = \frac{\partial \phi_D(x, -y)}{\partial y} \Bigg|_{y=0} . \tag{3.2.2}$$

Since, as is easily verified,

$$\frac{\partial H_o^{(2)}(kR)}{\partial y} \Bigg|_{y=0} = - \frac{\partial H_o^{(2)}(kR')}{\partial y} \Bigg|_{y=0} \tag{3.2.3}$$

it follows from (3.2.2) that

$$\frac{\partial \phi_D(x, y)}{\partial y} \Bigg|_{y=0} = - \frac{\partial H_o^{(2)}(kR)}{\partial y} \Bigg|_{y=0} , \quad \begin{matrix} x < x_1 \\ x > x_2 \end{matrix} . \tag{3.2.4}$$

With this relation it is clear that the function $\psi(x, y)$, defined as

$$\psi(x, y) = H_o^{(2)}(kR) - H_o^{(2)}(kR') + 2\phi_D(x, y), \quad \begin{matrix} -\infty < x < \infty \\ y > 0 \end{matrix} \tag{3.2.5}$$

is a solution of the problem depicted in Figure 3.2.2. Thus, knowing any one of the functions ϕ, ψ, or ϕ_D, it is possible to construct the other two. Furthermore, in the region $\mathcal{R}$, these functions can be expressed as integrals of the form (3.1.1)

Of course the next step is to attempt, by analytic continuation, to obtain an integral representation valid for the entire range of x and y. This step must also be taken with caution. The function $\phi(x, y)$ in the strip problem and $\psi(x, y)$ in the mixed problem both have non analytic behavior at three points of physical significance; $(x = x_o, y = y_o)$, $(x = x_1, y = 0)$, and $(x = x_2, y = 0)$. The first of these, the source point, can be eliminated by considering $\phi_D(x, y)$, the diffracted field. The other two remain however, so when extending the definition of the integral representation of $\phi_D(x, y)$ to the case when $x < x_1$, or $x > x_2$, the path of integration will have to vary as illustrated in Figure 3.2.3 in order that the continuation be analytic.

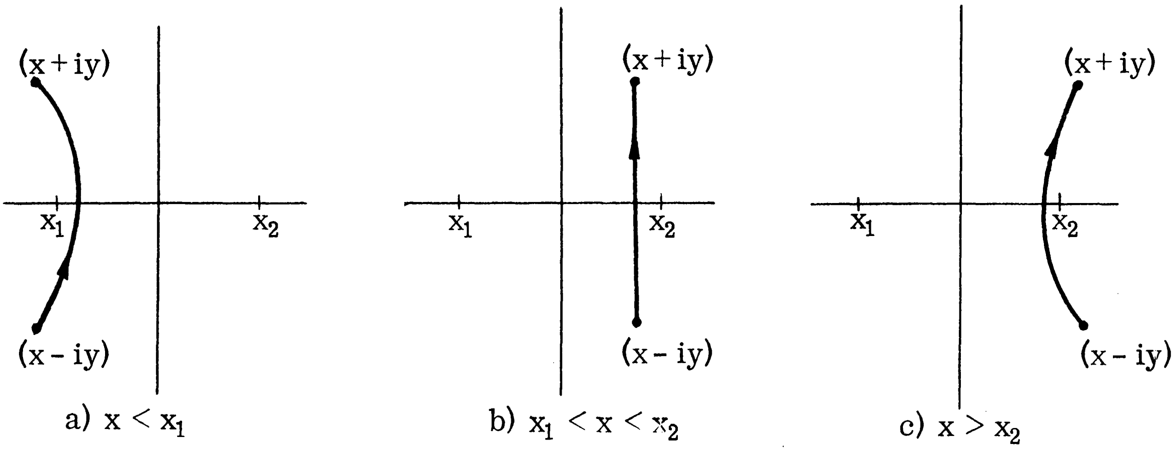


FIGURE 3.2.3: CONTOURS FOR THE INTEGRAL FORM OF $\phi_D(x, y)$

Thus it is seen how the integral (3.1.1), which apparently vanishes on the entire line $y = 0$, actually can represent a function which only vanishes on a segment of that line. It will prove more convenient, however, when considering the strip problem, to obtain a representation comparable to (3.1.1) in elliptic coordinates where the apparent behavior and the actual behavior are the same.

3. Integral Representation of a Class of Cylinder Functions

Having pointed out some of the hazards involved in using (3.1.1) to represent a function (solution of Helmholtz' equation) which vanishes on a line segment, we now consider the representation of functions for which the form (3.1.1) is ideally suited; that is, solutions of Helmholtz' equation which vanish on the entire line $y = 0$. A large class of such functions is known to be given by

$$\left.\begin{matrix} \cos\sqrt{k^2-\nu^2}\,x \\ \sin\sqrt{k^2-\nu^2}\,x \end{matrix}\right\} \sin \nu y. \tag{3.3.1}$$

In the sense that cylinder functions result from solving the two dimensional Helmholtz equation by separation of variables, leading to circular cylinder functions in polar (circular) coordinates, parabolic cylinder functions in parabolic coordinates, etc., the expressions (3.3.1) can be called rectangular cylinder functions. In this sense, "rectangular" coordinates are misnamed and would be more correctly called "right angle" coordinates. However, regardless of the possibly offensive nomenclature, the expressions (3.3.1) can be written, with the representation (3.1.1), as

$$\cos\sqrt{k^2-\nu^2}\,x \sin\nu y = \frac{\nu}{2i} \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) \cos\sqrt{k^2-\nu^2}\,\alpha\, d\alpha \tag{3.3.2}$$

and

$$\sin\sqrt{k^2-\nu^2}\,x \sin\nu y = \frac{\nu}{2i} \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) \sin\sqrt{k^2-\nu^2}\,\alpha\, d\alpha. \tag{3.3.3}$$

The representation of the sum of (3.3.2) and (3.3.3) has a particularly simple form, i.e.,

$$e^{i\sqrt{k^2-\nu^2}\,x} \sin\nu y = \frac{\nu}{2i} \int_{x-iy}^{x+iy} J_0\left(k\sqrt{(x-\alpha)^2+y^2}\right) e^{i\sqrt{k^2-\nu^2}\,\alpha}\, d\alpha \qquad (3.3.4)$$

which, with the substitution $\alpha = x + iy\cos\phi$, yields

$$\sin\nu y = \frac{\nu y}{2} \int_0^{\pi} J_0(ky\sin\phi)\, e^{-y\sqrt{k^2-\nu^2}\cos\phi} \sin\phi\, d\phi\,. \qquad (3.3.5)$$

This is a special case of a formula discovered by Gegenbauer (Ref. 31). We have already encountered formula (3.3.5) for the special case $\nu = k$ in section 2.5.

Another, perhaps more fruitful, application of the representation theorem involves the class of functions given by

$$Z_\nu(kR)\, e^{\pm i\nu\Theta} - Z_\nu(kR')\, e^{\mp i\nu\Theta'} \qquad (3.3.6)$$

where Z_ν represents any circular cylinder function of order ν, and (see Figure 3.3.1)

$$R^2 = (x-x_0)^2 + (y-y_0)^2 \qquad R'^2 = (x-x_0)^2 + (y+y_0)^2$$

$$\Theta = \tan^{-1}\frac{y-y_0}{x-x_0} \qquad \Theta' = \tan^{-1}\frac{y+y_0}{x-x_0} \qquad (3.3.7)$$

Clearly if $y = 0$, then $R = R'$ and $\Theta = -\Theta'$ hence the expression (3.3.6) vanishes. This expression is of course an analytic solution of the Helmholtz

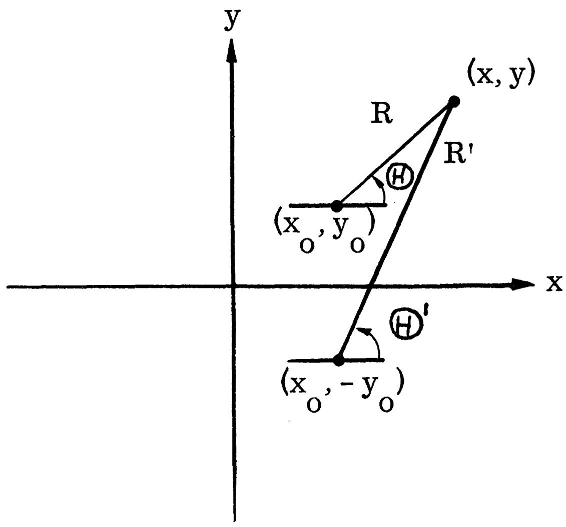


FIGURE 3.3.1: GEOMETRY OF R, Θ AND R', Θ'

equation in any simply connected region excluding the points $x = x_o$, $y = \pm y_o$. (Indeed if $Z_\nu \equiv J_{2n}$, $n = 0, \pm 1, \pm 2, \ldots,$ then it is analytic everywhere.) Then, if $y_o \neq 0$, it follows from the representation theorem that in a neighborhood of the line $y = 0$,

$$Z_\nu(kR)\, e^{+i\nu\Theta} - Z_\nu(kR')\, e^{-i\nu\Theta'} = \frac{1}{2i} \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2 + y^2}\right) f(x_o, y_o, \alpha)\, d\alpha \tag{3.3.8}$$

and

$$Z_\nu(kR)\, e^{-i\nu\Theta} - Z_\nu(kR')\, e^{+i\nu\Theta'} = \frac{1}{2i} \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2 + y^2}\right) g(x_o, y_o, \alpha)\, d\alpha \tag{3.3.9}$$

where

$$f(x_o, y_o, \alpha) = \frac{\partial}{\partial y} \left[Z_\nu(kR)\, e^{+i\nu \Theta} - Z_\nu(kR')\, e^{-i\nu \Theta'} \right] \Bigg|_{\substack{y=0 \\ x=\alpha}} \tag{3.3.10}$$

and

$$g(x_o, y_o, \alpha) = \frac{\partial}{\partial y} \left[Z_\nu(kR)\, e^{-i\nu \Theta} - Z_\nu(kR')\, e^{+i\nu \Theta'} \right] \Bigg|_{\substack{y=0 \\ x=\alpha}} \tag{3.3.11}$$

From the definition of Θ in (3.3.7) we find that

$$\cos \Theta = \frac{x - x_o}{\sqrt{(x - x_o)^2 + (y - y_o)^2}} \quad \text{and} \quad \sin \Theta = \frac{y - y_o}{\sqrt{(x - x_o)^2 + (y - y_o)^2}} \tag{3.3.12}$$

where, as Figure 3.3.1 makes clear, the positive square root is to be employed.

Thus

$$e^{i\Theta} = \cos \Theta + i \sin \Theta = \frac{x - x_o + i(y - y_o)}{\sqrt{(x - x_o)^2 + (y - y_o)^2}} \tag{3.3.13}$$

In exactly the same way we find that

$$e^{i\Theta'} = \frac{x - x_o + i(y + y_o)}{\sqrt{(x - x_o)^2 + (y + y_o)^2}} \tag{3.3.14}$$

With these expressions we may write $f(x_o, y_o, \alpha)$ and $g(x_o, y_o, \alpha)$ explicitly in terms of x_o, y_o, and α. Thus, carrying out the indicated differentiation, (3.3.10) becomes

$$f(x_o, y_o, \alpha) = \Bigg[k \frac{dZ_\nu(kR)}{d(kR)} e^{i\nu\Theta} \frac{\partial R}{\partial y} + i\nu Z_\nu(kR)\, e^{i\nu\Theta} \frac{\partial \Theta}{\partial y}$$

$$- k \frac{dZ_\nu(kR')}{d(kR')} e^{-i\nu\Theta'} \frac{\partial R'}{\partial y} + i\nu Z_\nu(kR')\, e^{-i\nu\Theta'} \frac{\partial \Theta'}{\partial y} \Bigg] \Bigg|_{\substack{y=0 \\ x=\alpha}} \tag{3.3.15}$$

Upon introducing the familiar recursion relations

$$\left.\begin{aligned} 2\,\frac{dZ_\nu(kR)}{d(kR)} &= Z_{\nu-1}(kR) - Z_{\nu+1}(kR) \\ \text{and} \qquad \frac{2\nu}{kR}\,Z_\nu(kR) &= Z_{\nu-1}(kR) + Z_{\nu+1}(kR) \end{aligned}\right\} \qquad (3.3.16)$$

we obtain

$$f(x_o, y_o, \alpha) = \frac{k}{2}\Bigg\{ \Big[Z_{\nu-1}(kR) - Z_{\nu+1}(kR)\Big] e^{i\nu\Theta}\,\frac{\partial R}{\partial y} + iR\Big[Z_{\nu-1}(kR) + Z_{\nu+1}(kR)\Big] e^{i\nu\Theta}\,\frac{\partial \Theta}{\partial y}$$

$$- \Big[Z_{\nu-1}(kR') - Z_{\nu+1}(kR')\Big] e^{-i\nu\Theta'}\,\frac{\partial R'}{\partial y} + iR'\Big[Z_{\nu-1}(kR') + Z_{\nu+1}(kR')\Big] e^{-i\nu\Theta'}\,\frac{\partial \Theta'}{\partial y}\Bigg\}\Bigg|_{\substack{y=o \\ x=\alpha}} \qquad (3.3.17)$$

It follows from their definition that

$$\left.\begin{aligned} R\Big|_{y=0} &= R'\Big|_{y=0} = \sqrt{(x-x_o)^2+y_o^2} \\ \frac{\partial R}{\partial y}\Big|_{y=0} &= -\frac{\partial R'}{\partial y}\Big|_{y=0} = -\frac{y_o}{\sqrt{(x-x_o)^2+y_o^2}} \\ \text{and} \qquad \frac{\partial \Theta}{\partial y}\Big|_{y=0} &= \frac{\partial \Theta'}{\partial y}\Big|_{y=0} = \frac{x-x_o}{\sqrt{(x-x_o)^2+y_o^2}} \end{aligned}\right\} \qquad (3.3.18)$$

Substituting these expressions, together with (3.3.13) and (3.3.14), in (3.3.17) we obtain, after simplification,

$$f(x_o, y_o, \alpha) = ik\, Z_{\nu-1}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)\left(\frac{\alpha-x_o-iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu-1}$$

$$+ ik\, Z_{\nu+1}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)\left(\frac{\alpha-x_o-iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu+1} \qquad (3.3.19)$$

A similar procedure yields

$$g(x_o, y_o, \alpha) = -ik\ Z_{\nu-1}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)\left(\frac{\alpha-x_o+iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu-1}$$

$$-ik\ Z_{\nu+1}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)\left(\frac{\alpha-x_o+iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu+1} \qquad (3.3.20)$$

If, in (3.3.19) and (3.3.20), $Z = J$ and $\nu = 2n+1$ then $f(x_o, y_o, \alpha)$ and $g(x_o, y_o, \alpha)$ have no singularities. In all other cases, however, they have branch points and possibly poles as well at $\alpha = x_o \pm iy_o$. In these cases we must specify the branch cuts and paths of integration in (3.3.8) and (3.3.9) in order for these representations to be meaningful. This is accomplished by employing the same branch cut convention adopted in Section 2.2. This ensures that when α is real the argument of the cylinder function is real and positive. We number the sheets of the Riemann surface of the logarithm exactly as in section 2.2 and then require that we remain on the 0^{th} sheet along the paths of integration in (3.3.8) and (3.3.9).

The branch of the factor $(\alpha - x_o \pm iy_o)^{\nu \pm 1}$ is determined by requiring consistency on both sides of (3.3.8) and (3.3.9). That is, in (3.3.8) the factors $e^{+i\nu\Theta}$ and $e^{-i\nu\Theta'}$ give rise (see (3.3.13) and (3.3.14)) to terms like $\left[x-x_o+i(y-y_o)\right]^{\nu}$ and $\left[x-x_o-i(y+y_o)\right]^{\nu}$ respectively, which are equal, when $y = 0$, to $\left[x-x_o-iy_o\right]^{\nu}$. We must then require that the factor $\left[\alpha-x_o-iy_o\right]^{\nu}$ in $f(x_o, y_o, \alpha)$ have exactly the same value when $\alpha = x$, then let it vary continuously as α becomes complex. Similarly, in (3.3.9) when $y = 0$ the factor $\left[x - x_o + iy_o\right]^{\nu}$ appears on the left and this determines the branch of $\left[\alpha - x_o + iy_o\right]^{\nu}$ in $g(x_o, y_o, \alpha)$. Figure 3.3.2 shows the branch cuts and a possible path of integration.

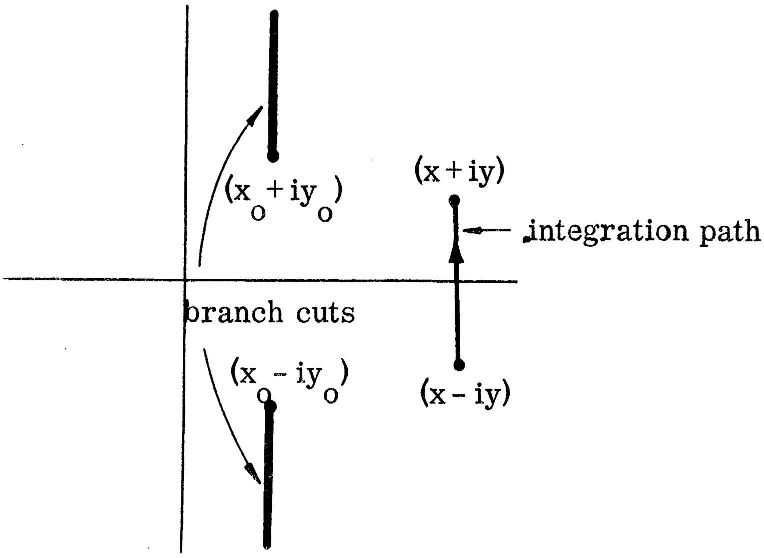


FIGURE 3.3.2: α-PLANE

While it is true that so far we have only established the validity of the representations in a neighborhood of $y = 0$ it is clear that by analytic continuation these representations remain valid throughout the cut xy-plane. The points $x = x_o$, $y = \pm y_o$ are singular and in general the functions $Z_\nu(kR)\, e^{\pm i\nu\Theta}$ and $Z_\nu(kR')\, e^{\pm i\nu\Theta'}$ are multiple valued thus we must cut the xy - plane as well as the α-plane. However, in the special case when $\nu = n$ $(n = 0, \pm 1, \pm 2, \ldots)$ the functions $Z_n(kR)\, e^{\pm in\Theta}$ and $Z_n(kR')\, e^{\pm in\Theta'}$ are single valued throughout the xy - plane hence in this case the integrals (3.3.8) and (3.3.9) must also be single valued, even though the integrand still has branch points. This means that for $x = x_o$, $y > y_o$, the path of integration can be either that shown in Figure 3.3.3a or 3.3.3b depending on whether $x \rightarrow x_o$ from right or left but both paths must yield the same value of the integral. As long as $y \neq y_o$ these integrals are defined and must be equal since they both represent the same function.

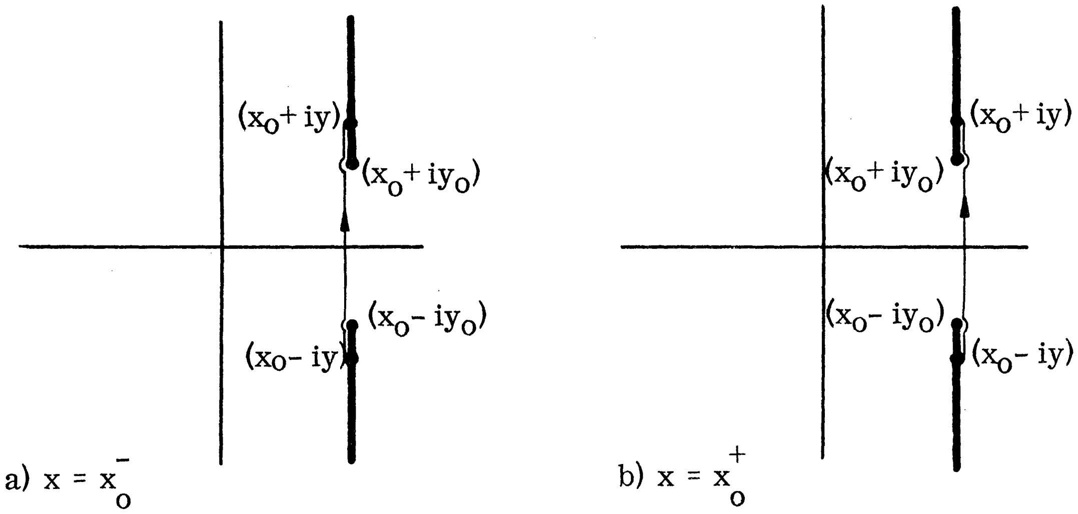


FIGURE 3.3.3: α-PLANE

The fact that (3.1.1) successfully represented a function with a singularity illustrates how (2.4.1) and (2.4.2) can be used to represent solutions of the inhomogeneous Helmholtz equation.

4. The Integral Representation of Line Sources

Of special interest is the particular case of (3.3.8) or (3.3.9) when Z_ν is chosen to be $H_o^{(2)}$, viz.:

$$H_o^{(2)}(kR) - H_o^{(2)}(kR') = \frac{k}{2} \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) \left[H_{-1}^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right) \left(\frac{\alpha - x_o - iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}} \right)^{-1} \right.$$

$$\left. + H_1^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right) \left(\frac{\alpha - x_o - iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}} \right)^{+1} \right] d\alpha \,. \qquad (3.4.1)$$

Further simplification yields

$$H_o^{(2)}(kR) - H_o^{(2)}(kR') = -iky_o \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) \frac{H_1^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)}{\sqrt{(\alpha-x_o)^2+y_o^2}} d\alpha \tag{3.4.2}$$

which can also be written

$$H_o^{(2)}(kR) - H_o^{(2)}(kR') = i \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) \frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right) d\alpha. \tag{3.4.3}$$

In addition to being of interest in itself, this last identity proves to be of considerable importance in the diffraction problems discussed in later chapters. It can be proven valid without making use of the representation theorem of this chapter and although this direct proof is somewhat tedious, formula (3.4.3) is felt to be of sufficient importance to warrant its inclusion here.

We proceed by first substituting $\alpha = iy\cos t + x$ in the right hand side of (3.4.3) obtaining

$$i \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) \frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right) d\alpha$$

$$= -y \int_0^{\pi} J_o(ky\sin t) \frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(x_o-x-iy\cos t)^2+y_o^2}\right) \sin t\, dt. \tag{3.4.4}$$

We restrict y so that

$$(x-x_o)^2 + y_o^2 > y^2$$

which implies that

$$\left| (x-x_o)^2 + y_o^2 \right| > \left| iy\cos t \right|$$

and then make use of the addition theorem for the Hankel function (Ref. 32)

$$H_o^{(2)}\left(k\sqrt{r^2+\rho^2-2r\rho\cos\phi}\right) = \sum_{n=o}^{\infty} \epsilon_n J_n(k\rho) H_n^{(2)}(kr) \cos n\phi \tag{3.4.5}$$

where $\epsilon_o = 1$, $\epsilon_n = 2$ $(n = 1, 2, 3, \ldots)$, and $|r| > |\rho|$.

In our case we choose

$$\left.\begin{aligned} r &= \sqrt{(x-x_o)^2+y_o^2} \quad , \\ \rho &= iy\cos t \\ \text{and} \quad \cos\phi &= \frac{x_o - x}{\sqrt{(x_o-x)^2+y_o^2}} \end{aligned}\right\} \tag{3.4.6}$$

which implies

$$\begin{aligned} \cos n\phi &= \frac{1}{2}\left[\cos\phi + i\sin\phi\right]^n + \frac{1}{2}\left[\cos\phi - i\sin\phi\right]^n \\ &= \frac{1}{2}\left[\frac{x_o - x + iy_o}{\sqrt{(x_o-x)^2+y_o^2}}\right]^n + \frac{1}{2}\left[\frac{x_o - x - iy_o}{\sqrt{(x_o-x)^2+y_o^2}}\right]^n . \end{aligned} \tag{3.4.7}$$

Thus

$$H_o^{(2)}\left(k\sqrt{(x_o-x-iy\cos t)^2+y_o^2}\right) = \frac{1}{2}\sum_{n=o}^{\infty} \epsilon_n J_n(iky\cos t)\, H_n^{(2)}\left(k\sqrt{(x_o-x)^2+y_o^2}\right)\cdot$$

$$\left[\left(\frac{x_o-x+iy_o}{\sqrt{(x_o-x)^2+y_o^2}}\right)^n + \left(\frac{x_o-x-iy_o}{\sqrt{(x_o-x)^2+y_o^2}}\right)^n\right]. \tag{3.4.8}$$

To obtain a useful form of the derivative with respect to y_o, necessary in (3.4.4), we first examine the y_o dependent part of a general term of (3.4.8). With the substitution $r=\sqrt{(x-x_o)^2+y_o^2}$ $\left(\Rightarrow \frac{\partial r}{\partial y_o}=\frac{y_o}{r}\right)$ and the recursion formulas (3.3.16) we find that

$$\frac{\partial}{\partial y_o}\left\{H_n^{(2)}(kr)\left[\left(\frac{x_o-x+iy_o}{r}\right)^n + \left(\frac{x_o-x-iy_o}{r}\right)^n\right]\right\}$$

$$= \frac{ik}{2}\left\{H_{n-1}^{(2)}(kr)\left[\left(\frac{x_o-x+iy_o}{r}\right)^{n-1} - \left(\frac{x_o-x-iy_o}{r}\right)^{n-1}\right]\right.$$

$$\left. + H_{n+1}^{(2)}(kr)\left[\left(\frac{x_o-x+iy_o}{r}\right)^{n+1} - \left(\frac{x_o-x-iy_o}{r}\right)^{n+1}\right]\right\}. \tag{3.4.9}$$

Upon differentiating (3.4.8) with respect to y_o, utilizing (3.4.9), and adjusting the indices of summation so that $H_n^{(2)}$ occurs in the n^{th} term, we obtain

$$\frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(x_o-x-iy\cos t)^2+y_o^2}\right) = \frac{ik}{4}\,\epsilon_o J_o(iky\cos t)\, H_{-1}^{(2)}(kr)\left(\frac{r}{x_o-x+iy_0} - \frac{r}{x_o-x-iy_o}\right)$$

$$+ \frac{ik}{4}\sum_{n=1}^{\infty}\left[\epsilon_{n-1}J_{n-1}(iky\cos t) + \epsilon_{n+1}J_{n+1}(iky\cos t)\right] H_n^{(2)}(kr)\left[\left(\frac{x_o-x+iy_o}{r}\right)^n - \left(\frac{x_o-x-iy_o}{r}\right)^n\right]. \tag{3.4.10}$$

But $\dfrac{r}{x_o - x + iy_o} - \dfrac{r}{x_o - x - iy_o} = \dfrac{x_o - x - iy_o}{r} - \dfrac{x_o - x + iy_o}{r}$

thus, recalling the definition of ϵ_n together with the fact that $H_{-1}^{(2)} = -H_1^{(2)}$, (3.4.10) becomes

$$\frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(x_o - x - iy\cos t)^2 + y_o^2}\right)$$

$$= \frac{ik}{2}\sum_{n=1}^{\infty}\left[J_{n-1}(iky\cos t) + J_{n+1}(iky\cos t)\right] H_n^{(2)}(kr)\left[\left(\frac{x_o - x + iy_o}{r}\right)^n - \left(\frac{x_o - x - iy_o}{r}\right)^n\right]$$

(3.4.11)

which, again employing the recursion formula for the Bessel functions, becomes finally,

$$\frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(x_o - x - iy\cos t)^2 + y_o^2}\right)$$

$$= \sum_{n=1}^{\infty}\frac{n J_n(iky\cos t)}{y\cos t} H_n^{(2)}(kr)\left[\left(\frac{x_o - x + iy_o}{r}\right)^n - \left(\frac{x_o - x - iy_o}{r}\right)^n\right]. \qquad (3.4.12)$$

Substituting (3.4.12) in (3.4.4) we obtain, after interchanging summation and integration,

$$i\int_{x-iy}^{x+iy} J_o\left(k\sqrt{(\alpha - x)^2 + y^2}\right)\frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(\alpha - x_o)^2 + y_o^2}\right) d\alpha$$

$$= -\sum_{n=1}^{\infty} n\, H_n^{(2)}(kr)\left[\left(\frac{x_o - x + iy_o}{r}\right)^n - \left(\frac{x_o - x - iy_o}{r}\right)^n\right]\int_0^{\pi} J_o(ky\sin t)\, J_n(iky\cos t)\tan t\, dt\,.$$

(3.4.13)

However

$$\int_0^{\pi} J_0(ky\sin t)\,J_n(iky\cos t)\tan t\,dt = \left[1-(-1)^n\right]\int_0^{\pi/2} J_0(ky\sin t)\,J_0(iky\cos t)\tan t\,dt \tag{3.4.14}$$

and this last form has been explicitly integrated by Rutgers (Ref. 22) who found that

$$\int_0^{\pi/2} J_0(ky\sin t)\,J_n(iky\cos t)\tan t\,dt = \frac{i^n\Gamma(n/2)}{2\Gamma(n/2+1)}\,J_n(ky)$$

$$= \frac{i^n}{n}\,J_n(ky). \tag{3.4.15}^*$$

With this result, (3.4.13) can be written as

$$i\int_{x-iy}^{x+iy} J_0\left(k\sqrt{(\alpha-x)^2+y^2}\right)\frac{\partial}{\partial y_0}H_0^{(2)}\left(k\sqrt{(\alpha-x_0)^2+y_0^2}\right)d\alpha$$

$$= -\sum_{n=o}^{\infty} i^n\left[1-(-1)^n\right]J_n(ky)\,H_n^{(2)}(kr)\left[\left(\frac{x_0-x+iy_0}{r}\right)^n-\left(\frac{x_0-x-iy_0}{r}\right)^n\right] \tag{3.4.16}$$

which upon multiplying out the factors with n^{th} powers and reintroducing the ϵ notation $(\epsilon_n/2 = 1)$, which is possible since the $n = o$ term is zero, becomes

*Formula (3.4.15) appears in Watson (Ref. 33), but is in error by a factor of 2.

$$i \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(\alpha-x)^2+y^2}\right) \frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right) d\alpha$$

$$= \sum_{n=o}^{\infty} \epsilon_n J_n(ky)\, H_n^{(2)}(kr)\, \frac{1}{2}\left[\left(\frac{y_o - i(x_o - x)}{r}\right)^n + \left(\frac{y_o + i(x_o - x)}{r}\right)^n\right]$$

$$- \sum_{n=o}^{\infty} \epsilon_n J_n(ky)\, H_n^{(2)}(kr)\, \frac{1}{2}\left[\left(\frac{-y_o - i(x_o - x)}{r}\right)^n + \left(\frac{-y_o + i(x_o - x)}{r}\right)^n\right] \quad . \qquad (3.4.17)$$

Now we again make use of the addition theorem for which the earlier restriction $(x-x_o)^2+y_o^2 = r^2 > y^2$ still suffices. For the first sum on the right hand side of (3.4.17) we may write

$$\sum_{n=o}^{\infty} \epsilon_n J_n(ky)\, H_n^{(2)}(kr)\, \frac{1}{2}\left[\left(\frac{y_o - i(x_o - x)}{r}\right)^n + \left(\frac{y_o + i(x_o - x)}{r}\right)^n\right] = H_o^{(2)}\left(k\sqrt{y^2+r^2-2yr\cos\phi}\right) \qquad (3.4.18)$$

where

$$\cos\phi = \frac{1}{2}\left[\frac{y_o - i(x_o - x)}{r} + \frac{y_o + i(x_o - x)}{r}\right] = \frac{y_o}{r} \; .$$

Hence

$$H_o^{(2)}\left(k\sqrt{y^2+r^2-2yr\cos\phi}\right) = H_o^{(2)}\left(k\sqrt{y^2+r^2-2yy_o}\right) \qquad (3.4.19)$$

or, since $r^2 = (x-x_o)^2 + y_o^2$,

$$H_o^{(2)}\left(k\sqrt{y^2+r^2-2yr\cos\phi}\right) = H_o^{(2)}\left(k\sqrt{y^2+(x-x_o)^2+y_o^2-2yy_o}\right)$$

$$= H_o^{(2)}(kR) \quad . \qquad (3.4.20)$$

Similarly, the second sum on the right hand side of (3.4.17) is found to be $H_o^{(2)}(kR')$ consequently the relation we set out to prove, (3.4.3), is established. In order to use the addition theorem, it was necessary to restrict the values of x and y but this is clearly a restriction on the use of the addition theorem and not on the validity of (3.4.3) since the functions on both sides of (3.4.3) can be analytically continued throughout any simply connected region excluding the points x_o, $\pm y_o$.

In the ambiguous case when $x = x_o$ and $y > y_o$ two integration contours are possible, as shown in Figure 3.3.3. The two integrals thus formed must have the same value since they both represent the same function. That this is so may be directly demonstrated by showing that their difference vanishes, i.e., that

$$\int_c J_o\left(k\sqrt{(x_o-\alpha)^2+y^2}\right)\frac{\partial}{\partial y_o} H_o^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right) d\alpha = 0 \qquad (3.4.21)$$

where the contour, shown in Figure 3.4.1, is to lie entirely on the 0^{th} blade of the Riemann surface of the Hankel function.

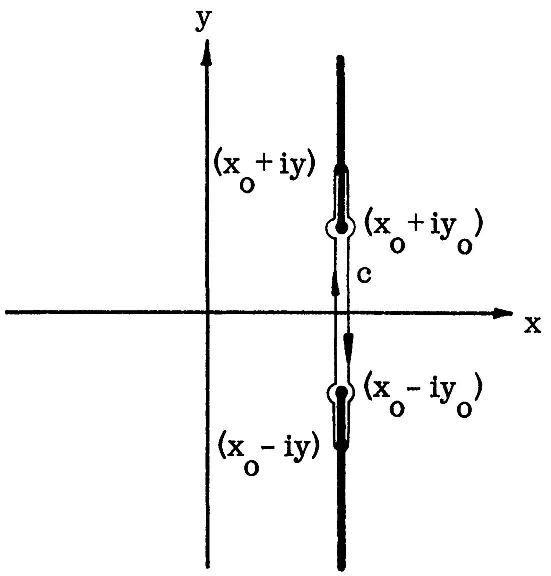


FIGURE 3.4.1: INTEGRATION CONTOUR

Rewriting the integrand with the help of (2.2.5) we may treat the singular part separately. Thus

$$\int_c J_o\left(k\sqrt{(x_o-\alpha)^2+y^2}\right)\frac{\partial}{\partial y_o}H_o^{(2)}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)d\alpha$$

$$= -\frac{i}{\pi}\int_c d\alpha\, J_o\left(k\sqrt{(x_o-\alpha)^2+y^2}\right)\left\{G(\alpha)+\frac{2y_o}{(\alpha-x_o-iy_o)(\alpha-x_o+iy_o)}\right.$$

$$\left.+\frac{\partial}{\partial y_o}J_o\left(k\sqrt{(x_o-\alpha)^2+y_o^2}\right)\log\left[(\alpha-x_o)^2+y_o^2\right]\right\} \qquad (3.4.22)$$

where $G(\alpha)$ is analytic. With Cauchy's theorem then

$$\int_c J_0\left(k\sqrt{(x_0-\alpha)^2+y^2}\right) G(\alpha)\, d\alpha = 0 \tag{3.4.23}$$

and, since the residues at the two poles are equal but opposite in sign,

$$\int_c \frac{J_0\left(k\sqrt{(x_0-\alpha)^2+y^2}\right) y_0\, d\alpha}{(\alpha-x_0-iy_0)(\alpha-x_0+iy_0)} = 0\ . \tag{3.4.24}$$

The logarithmic term is also easily evaluated. With the help of formula (2.2.8) we find that

$$\int_c J_0\left(k\sqrt{(x_0-\alpha)^2+y^2}\right) \frac{\partial}{\partial y_0} J_0\left(k\sqrt{(\alpha-x_0)^2+y_0^2}\right) \log\left[(\alpha-x_0)^2+y_0^2\right] d\alpha$$

$$= -2\pi i \int_{x_0+iy_0}^{x_0+iy} J_0\left(k\sqrt{(x_0-\alpha)^2+y^2}\right) \frac{\partial}{\partial y_0} J_0\left(k\sqrt{(\alpha-x_0)^2+y_0^2}\right) d\alpha$$

$$+ 2\pi i \int_{x_0-iy}^{x_0-iy_0} J_0\left(k\sqrt{(x_0-\alpha)^2+y^2}\right) \frac{\partial}{\partial y_0} J_0\left(k\sqrt{(\alpha-x_0)^2+y_0^2}\right) d\alpha\ . \tag{3.4.25}$$

The right hand side can be simplified, since it is independent of x_0, yielding

$$\text{R.H.S.} = -2\pi i \int_{iy_o}^{iy} J_o\left(k\sqrt{\alpha^2+y^2}\right) \frac{\partial}{\partial y_o} J_o\left(k\sqrt{\alpha^2+y_o^2}\right) d\alpha$$

$$+ 2\pi i \int_{-iy}^{-iy_o} J_o\left(k\sqrt{\alpha^2+y^2}\right) \frac{\partial}{\partial y_o} J_o\left(k\sqrt{\alpha^2+y_o^2}\right) d\alpha \qquad (3.4.26)$$

which is clearly seen to vanish on substitution of $-\alpha$ for α in one of the integrals but not in the other.

5. Asymptotic Behavior of the Line Source Representation

Formula (3.4.3) can be used to demonstrate that for large values of x_o and y_o, the circular cylinder function relations pass over into the rectangular cylinder function relations of section 3.3.

Consider first the case of large y_o, where

$$\left.\begin{aligned} R &= \sqrt{(x-x_o)^2+(y-y_o)^2} \sim y_o - y \\ R' &= \sqrt{(x-x_o)^2+(y+y_o)^2} \sim y_o + y \\ &\sqrt{(\alpha-x)^2+y_o^2} \sim y_o \ . \end{aligned}\right\} \qquad (3.5.1)$$

Retaining the y in the phase and neglecting it in the amplitude in the customary manner, we use Hankel's asymptotic form (Ref. 17) to obtain

$$H_o^{(2)}(kR) \sim \sqrt{\frac{2}{\pi y_o}}\; e^{-ik(y_o-y)+\frac{\pi i}{4}}$$

$$H_0^{(2)}(kR') \sim \sqrt{\frac{2}{\pi y_0}}\ e^{-ik(y_0+y)+\frac{\pi i}{4}}$$

and

$$H_0^{(2)}\left(k\sqrt{(\alpha-x_0)^2+y_0^2}\right) \sim \sqrt{\frac{2}{\pi y_0}}\ e^{-iky_0+\frac{i\pi}{4}} \tag{3.5.2}$$

Substituting (3.5.2) in (3.4.3) and retaining only the $\frac{1}{\sqrt{y_0}}$ term in the differentiation of $H_0^{(2)}\left(k\sqrt{(\alpha-x_0)^2+y_0^2}\right)$, we have, after cancelling common factors,

$$e^{iky} - e^{-iky} = k\int_{x-iy}^{x+iy} J_0\left(k\sqrt{(\alpha-x)^2+y^2}\right) d\alpha. \tag{3.5.3}$$

This expression has been previously encountered in section 2.5 where it appears as formula (2.5.5) and also as a special case of (3.3.5).

If instead of choosing y_0 to be large, we consider x_0 large, formula (3.4.3) becomes the trivial identity, 0 = 0. Of considerably more interest is the case when $r_0 = \sqrt{x_0^2+y_0^2}$ becomes large. Using polar coordinates (r_0, θ_0) for the source and rectangular coordinates (x, y) for the field point we have, for large r_0,

$$\left.\begin{aligned} R &= \sqrt{x^2+y^2+r_0^2-2xr_0\cos\theta_0-2yr_0\sin\theta_0} \sim r_0 - x\cos\theta_0 - y\sin\theta_0 \\ R' &= \sqrt{x^2+y^2+r_0^2-2xr_0\cos\theta_0+2yr_0\sin\theta_0} \sim r_0 - x\cos\theta_0 + y\sin\theta_0 \\ &\sqrt{\alpha^2+r_0^2-2\alpha r_0\cos\theta_0} \sim r_0 - \alpha\cos\theta_0 \end{aligned}\right\} \tag{3.5.4}$$

and for the Hankel functions

$$\left.\begin{aligned}
H_o^{(2)}(kR) &\sim \sqrt{\frac{2}{\pi r_o}}\; e^{-ik(r_o - x\cos\theta_o - y\sin\theta_o) + \frac{i\pi}{4}} \\
H_o^{(2)}(kR') &\sim \sqrt{\frac{2}{\pi r_o}}\; e^{-ik(r_o - x\cos\theta_o + y\sin\theta_o) + \frac{i\pi}{4}} \\
H_o^{(2)}\left(k\sqrt{(\alpha - x_o)^2 + y_o^2}\right) &\sim \sqrt{\frac{2}{\pi r_o}}\; e^{-ik(r_o - \alpha\cos\theta_o) + \frac{i\pi}{4}} .
\end{aligned}\right\} \qquad (3.5.5)$$

Substituting the expressions (3.5.5) in (3.4.3) and retaining only the $\frac{1}{\sqrt{r_o}}$ term in the derivative of $H_o^{(2)}\left(k\sqrt{(\alpha - x_o)^2 + y_o^2}\right)$ we have, after cancelling common factors,

$$e^{ikx\cos\theta_o}\left(e^{iky\sin\theta_o} - e^{-iky\sin\theta_o}\right) = k\sin\theta_o \int_{x-iy}^{x+iy} J_o\left(k\sqrt{(\alpha - x)^2 + y^2}\right) e^{ik\alpha\cos\theta_o}\, d\alpha \qquad (3.5.6)$$

With the substitution $\alpha = x + iy\cos t$ and the resulting simplification, (3.5.6) becomes

$$\sin(ky\sin\theta_o) = \frac{ky\sin\theta_o}{2} \int_0^{\pi} J_o(ky\sin t)\, e^{-ky\cos\theta_o\cos t} \sin t\, dt . \qquad (3.5.7)$$

Formula (3.5.7) is a disquised form of (3.3.5) but perhaps a more useful representation with application in the study of frequency modulation.

Although we have discussed the asymptotic results only for formula (3.4.3) which is a very special case of (3.3.8) and (3.3.9), there is not much to be gained in this regard from consideration of these more general cases.

Choosing $\nu \neq 0$ would not change the asymptotic forms, choosing $Z = J$ or N would lead to forms in which the dependence on the parameter which was taken to be large could not be factored, and choosing $Z = H_o^{(1)}$ rather that $H_o^{(2)}$ would lead to essentially the same results presented here.

6. Double Integral Representations

Of particular interest, from the point of view of subsequent applications, is the expression derivable from the formula obtained by adding (3.3.8) and (3.3.9), i.e.,

$$Z_\nu(kR)\cos\nu\Theta - Z_\nu(kR')\cos\nu\Theta' = \frac{1}{2i}\int_{x-iy}^{x+iy} J_o\left(k\sqrt{(\alpha-x)^2+y^2}\right) F(x_o, y_o, \alpha)\, d\alpha \tag{3.6.1}$$

where, adding (3.3.10) and (3.3.11),

$$F(x_o, y_o, \alpha) = \frac{\partial}{\partial y}\left[Z_\nu(kR)\cos\nu\Theta - Z_\nu(kR')\cos\nu\Theta'\right]\Bigg|_{\substack{y=o\\ x=\alpha}} . \tag{3.6.2}$$

With the aid of (3.3.19) and (3.3.20), (3.6.2) can be written as

$$F(x_o, y_o, \alpha) = \frac{ik}{2} Z_{\nu-1}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)\left[\left(\frac{\alpha-x_o-iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu-1} - \left(\frac{\alpha-x_o+iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu-1}\right]$$

$$+ \frac{ik}{2} Z_{\nu+1}\left(k\sqrt{(\alpha-x_o)^2+y_o^2}\right)\left[\left(\frac{\alpha-x_o-iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu+1} - \left(\frac{\alpha-x_o+iy_o}{\sqrt{(\alpha-x_o)^2+y_o^2}}\right)^{\nu+1}\right] . \tag{3.6.3}$$

With the expressions (3.3.13) and (3.3.14) which give $e^{i\Theta}$ and $e^{i\Theta'}$ explicitly in terms of x, y, x_o, and y_o it is a simple matter to show that

$$\cos\nu\Theta = \frac{1}{2}\left(\frac{x-x_o+i(y-y_o)}{\sqrt{(x-x_o)^2+(y-y_o)^2}}\right)^{\nu} + \frac{1}{2}\left(\frac{x-x_o-i(y-y_o)}{\sqrt{(x-x_o)^2+(y-y_o)^2}}\right)^{\nu} \tag{3.6.4}$$

and

$$\cos\nu\Theta' = \frac{1}{2}\left(\frac{x-x_o+i(y+y_o)}{\sqrt{(x-x_o)^2+(y+y_o)^2}}\right)^{\nu} + \frac{1}{2}\left(\frac{x-x_o-i(y+y_o)}{\sqrt{(x-x_o)^2+(y+y_o)^2}}\right)^{\nu} . \tag{3.6.5}$$

Thus exhibited explicitly it is clear that interchanging x with x_o and y with y_o in $\cos\nu\Theta$ and $\cos\nu\Theta$ is equivalent to multiplying by $(-1)^{\nu}$. Since this interchange leaves R and R' unaltered, we find that interchanging (x, y) with (x_o, y_o) in formula (3.6.1) yields, bringing the factor $(-1)^{\nu}$ to the right,

$$Z_{\nu}(kR)\cos\nu\Theta - Z_{\nu}(kR')\cos\nu\Theta' = \frac{(-1)^{\nu}}{2i}\int_{x_o-iy_o}^{x_o+iy_o} J_o\left(k\sqrt{(\beta-x_o)^2+y_o^2}\right) F(x,y,\beta)\, d\beta \tag{3.6.6}$$

where we have renamed the integration variable to avoid confusion in what follows. Substituting (3.6.6) in (3.6.2) we obtain

$$F(x_o, y_o, \alpha) = \frac{(-1)^{\nu}}{2i}\int_{x_o-iy_o}^{x_o+iy_o} J_o\left(k\sqrt{(x_o-\beta)^2+y_o^2}\right) \left.\frac{\partial}{\partial y}F(\alpha,y,\beta)\right|_{y=o} d\beta \tag{3.6.7}$$

where, using (3.6.3),

$$\left.\frac{\partial}{\partial y}F(\alpha,y,\beta)\right|_{y=o} = \frac{\partial}{\partial y}\left\{\frac{ik}{2}Z_{\nu-1}\left(k\sqrt{(\beta-\alpha)^2+y^2}\right)\left[\left(\frac{\beta-\alpha-iy}{\sqrt{(\beta-\alpha)^2+y^2}}\right)^{\nu-1} - \left(\frac{\beta-\alpha+iy}{\sqrt{(\beta-\alpha)^2+y^2}}\right)^{\nu-1}\right]\right.$$

$$\left.+\frac{ik}{2}Z_{\nu+1}\left(k\sqrt{(\beta-\alpha)^2+y^2}\right)\left[\left(\frac{\beta-\alpha-iy}{\sqrt{(\beta-\alpha)^2+y^2}}\right)^{\nu+1} - \left(\frac{\beta-\alpha+iy}{\sqrt{(\beta-\alpha)^2+y^2}}\right)^{\nu+1}\right]\right\}_{y=o} . \tag{3.6.8}$$

Upon performing the indicated differentiation and evaluation we obtain

$$\frac{\partial}{\partial y} F(\alpha, y, \beta)\bigg|_{y=0} = k\, Z_{\nu-1}\left(k\sqrt{(\beta-\alpha)^2}\right) \frac{(\nu-1)}{\sqrt{(\beta-\alpha)^2}} \left(\frac{\beta-\alpha}{\sqrt{(\beta-\alpha)^2}}\right)^{\nu-2}$$

$$+ k\, Z_{\nu+1}\left(k\sqrt{(\beta-\alpha)^2}\right) \frac{(\nu+1)}{\sqrt{(\beta-\alpha)^2}} \left(\frac{\beta-\alpha}{\sqrt{(\beta-\alpha)^2}}\right)^{\nu} \qquad (3.6.9)$$

We have written (3.6.9) in this purposefully complicated form so as to stress the importance of the choice of the sign of the square root. Actually, to be consistent with our previous convention, there is no choice left. We required that $\sqrt{(\alpha-x)^2+y^2}$ and $\sqrt{(\alpha-x_o)^2+y_o^2}$ (and hence $\sqrt{(\beta-x_o)^2+y_o^2}$) be real and positive when α (and β) were real. This implies that we must choose the sign of the square root so that when α and β are real, $\sqrt{(\alpha-\beta)^2}$ is also real and positive. Restricting the paths of integration to be straight lines connecting the end points, it is clear from (3.6.1) that the only real value α can assume is $\alpha = x$ and from (3.6.6) that the only real value β can assume is $\beta = x_o$ hence

$$\sqrt{(\beta-\alpha)^2} = \begin{cases} \beta-\alpha & x_o > x \\ \alpha-\beta & x_o < x\,. \end{cases} \qquad (3.6.10)$$

Formula (3.6.9) becomes, therefore

$$\frac{\partial}{\partial y} F(\alpha, y, \beta)\bigg|_{y=0} = \begin{cases} k(\nu-1)\,\dfrac{Z_{\nu-1}(k\beta-k\alpha)}{\beta-\alpha} + k(\nu+1)\,\dfrac{Z_{\nu+1}(k\beta-k\alpha)}{\beta-\alpha}\,, & x_o > x \\ \left\{ k(\nu-1)\,\dfrac{Z_{\nu-1}(k\alpha-k\beta)}{\alpha-\beta} + k(\nu+1)\,\dfrac{Z_{\nu+1}(k\alpha-k\beta)}{\alpha-\beta} \right\} (-1)^{\nu}\,, & x_o < x\,. \end{cases}$$

(3.6.11)

Substituting (3.6.11) in (3.6.7) yields

$$F(x_o, y_o, \alpha) = -\frac{k}{2i}\int_{x_o - iy_o}^{x_o + iy_o} d\beta \; \frac{J_o\left(k\sqrt{(\beta - x_o)^2 + y_o^2}\right)}{\beta - \alpha} \,.$$

$$\begin{cases} \left[(\nu-1)\, Z_{\nu-1}(k\beta - k\alpha) + (\nu+1)\, Z_{\nu+1}(k\beta - k\alpha)\right](-1)^{\nu+1}, & x_o > x \\ (\nu-1)\, Z_{\nu-1}(k\alpha - k\beta) + (\nu+1)\, Z_{\nu+1}(k\alpha - k\beta), & x_o < x \end{cases} \qquad (3.6.12)$$

and finally substituting (3.6.12) in (3.6.1) we obtain the general representation

$$Z_\nu(kR)\cos\nu\Theta - Z_\nu(kR')\cos\nu\Theta'$$

$$= \frac{k}{4}\int_{x-iy}^{x+iy} d\alpha \int_{x_o - iy_o}^{x_o + iy_o} d\beta \; J_o\left(k\sqrt{(\alpha - x)^2 + y^2}\right) \frac{J_o\left(k\sqrt{(\beta - x_o)^2 + y_o^2}\right)}{\beta - \alpha} \,.$$

$$\begin{cases} \left[(\nu-1)\, Z_{\nu-1}(k\beta - k\alpha) + (\nu+1)\, Z_{\nu+1}(k\beta - k\alpha)\right](-1)^{\nu+1}, & x_o > x \\ (\nu-1)\, Z_{\nu-1}(k\alpha - k\beta) + (\nu+1)\, Z_{\nu+1}(k\alpha - k\beta), & x_o < x\,. \end{cases} \qquad (3.6.13)$$

Choosing $Z_\nu = H_o^{(2)}$ in (3.6.13) yields the remarkable and useful representation

$$H_o^{(2)}(kR) - H_o^{(2)}(kR')$$

$$= -\frac{k}{2}\int_{x-iy}^{x+iy} d\alpha \int_{x_o - iy_o}^{x_o + iy_o} d\beta \; J_o\left(k\sqrt{(\alpha - x)^2 + y^2}\right) J_o\left(k\sqrt{(\beta - x_o)^2 + y_o^2}\right) \frac{H_1^{(2)}(\pm k\,[\beta - \alpha])}{\pm\,[\beta - \alpha]} \qquad (3.6.14)$$

where the positive sign must be employed when $x_o > x$ and the negative sign when $x_o < x$. The paths of integration and branch cuts are shown in Figure 3.6.1.

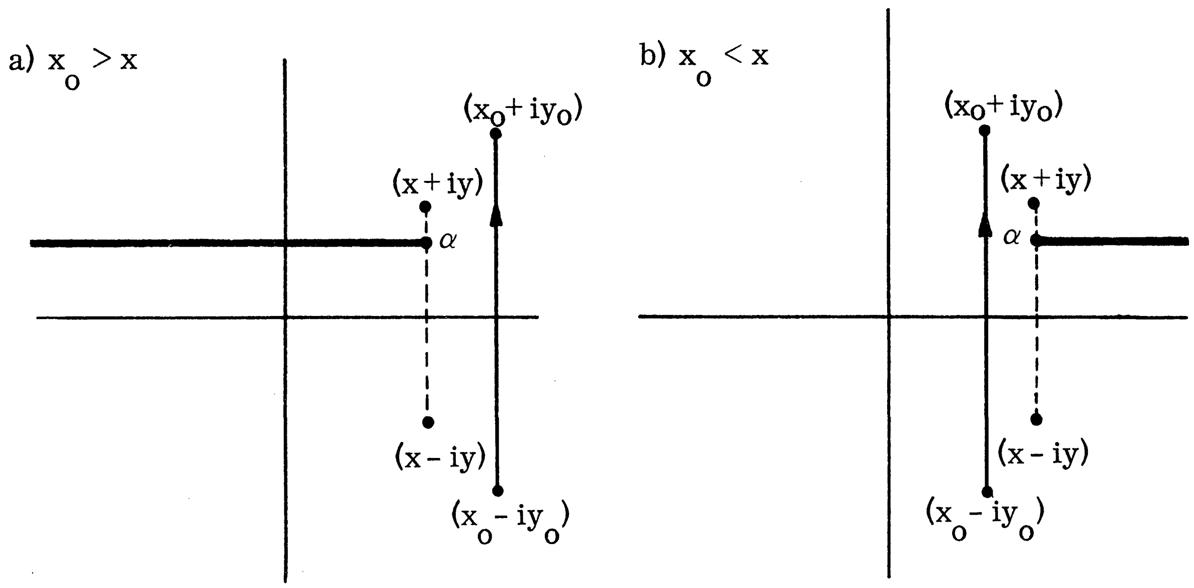


FIGURE 3.6.1: BRANCH CUTS IN THE β-PLANE FOR FORMULA (3.6.14)

The same figure can be used to represent the α-plane by interchanging α with β, x with x_o, and y with y_o.

As suggested by the representation of the two sources, $H_o^{(2)}(kR) - H_o^{(2)}(kR')$, we may now derive a general double integral representation of the analytic wave function, $G(x, y, x_o, y_o)$, which vanishes on a segment of the line $y = 0$ and, in addition, is symmetric in (x, y) and (x_o, y_o).

With (3.1.1) we write

$$G(x, y, x_o, y_o) = \frac{1}{2i} \int_{x-iy}^{x+iy} J_o\left(k \sqrt{(x-\alpha)^2 + y^2}\right) \frac{\partial}{\partial \nu} G(\alpha, \nu, x_o, y_o)\Big|_{\nu=o} \, d\alpha \qquad (3.6.15)$$

which, we have shown, is valid in a region containing $x = x_o$, $y = y_o$ even if this is a source as long as G is analytic everywhere else. Making use of the symmetry property,

$$G(x, y, x_o, y_o) = G(x_o, y_o, x, y) \quad , \tag{3.6.16}$$

we interchange x with x_o and y with y_o in (3.6.15) obtaining

$$G(x, y, x_o, y_o) = \frac{1}{2i} \int_{x_o - iy_o}^{x_o + iy_o} J_o\left(k\sqrt{(x_o - \beta)^2 + y_o^2}\right) \frac{\partial}{\partial \nu} G(\beta, \nu, x, y)\Big|_{\nu=o} d\beta \tag{3.6.17}$$

where we have renamed the variable of integration to avoid confusion in what follows. Differentiating (3.6.17) with respect to y, evaluating at y = 0, and replacing x by α, we obtain

$$\frac{\partial}{\partial y} G(\alpha, y, x_o, y_o)\Big|_{y=o} = \frac{1}{2i} \int_{x_o - iy_o}^{x_o + iy_o} J_o\left(k\sqrt{(x_o - \beta)^2 + y_o^2}\right) \frac{\partial^2}{\partial \nu \partial y} G(\beta, \nu, \alpha, y)\Big|_{\substack{\nu=o \\ y=o}} d\beta \, . \tag{3.6.18}$$

Changing the dummy variables appropriately and substituting (3.6.18) in (3.6.15), we obtain

$$G(x, y, x_o, y_o) = -\frac{1}{4} \int_{x-iy}^{x+iy} d\alpha \int_{x_o - iy_o}^{x_o + iy_o} d\beta \, J_o\left(k\sqrt{(x-\alpha)^2 + y^2}\right) J_o\left(k\sqrt{(x_o - \beta)^2 + y_o^2}\right) \frac{\partial^2}{\partial \nu \partial \mu} G(\beta, \nu, \alpha, \mu)\Big|_{\substack{\nu=o \\ \mu=o}} . \tag{3.6.19}$$

Again making use of the symmetry property, this can be written

$$G(x,y,x_o,y_o) = -\frac{1}{4}\int_{x-iy}^{x+iy} d\alpha \int_{x_o-iy_o}^{x_o+iy_o} d\beta\ J_o\left(k\sqrt{(x-\alpha)^2+y^2}\right) J_o\left(k\sqrt{(x_o-\beta)^2+y_o^2}\right) \frac{\partial^2}{\partial\mu\partial\nu} G(\alpha,\mu,\beta,\nu)\Big|_{\substack{\nu=o\\ \mu=o}} \tag{3.6.20}$$

where, as (3.6.14) bears witness, this interchange must be performed with considerable care in particular cases.

This representation and the corresponding forms in parabolic and elliptic coordinates prove to be of considerable value in solving the problem of diffraction by a strip.

Until now we have avoided the term "Green's function", although suggested it in the notation, because Green's functions are most often not analytic at the significant boundary (in this case the segment of the line $y = 0$ where $G = 0$) whereas our function must be analytic. In section 3.2 we showed that this is not an essential difficulty, and hence we shall in the future refer to (3.6.20) as the double integral representation for the Green's function for a line segment.

After having so thoroughly discussed the representation of known wave functions, it would seem proper to devote some consideration to the use of (3.1.1) or (3.6.20) as an integral equation for unknown wave functions. Unfortunately, the scattering problems in rectangular coordinates for which

(3.1.1) is appropriate are either so elementary in nature or so difficult that such consideration is fruitless. That is, wave functions satisfying a Dirichlet condition on the entire line $y = 0$ are easily found by the well known method of images (Ref. 19), whereas wave functions satisfying Dirichlet conditions on a part of the line $y = 0$, as indicated in section 3.2, will best be treated, at least initially, in more appropriate coordinates.

Chapter 4

DIFFRACTION BY A HALF PLANE

In this chapter we consider the canonical two dimensional diffraction problem, diffraction by a half plane. The solution is expressed in the integral form comparable to (3.6.19) in parabolic coordinates, the coordinates most appropriate for this approach to the problem. The usefulness of parabolic coordinates in connection with the half-plane problem has long been recognized, having been used by Lamb (Ref. 13) early in this century.

First the parabolic coordinates are introduced, results comparable to those of Chapters 2 and 3 are presented, and finally the solution is given.

The half plane problem, perhaps the most frequently solved problem in diffraction theory, is discussed, not as a model to test a method of solution, but because we assert (and prove in Chapter 5) that implicit in the solution of the half plane problem is the solution of the problem of diffraction by a strip and indeed it may, when interpreted correctly, yield the exact solution of a much wider class of two dimensional diffraction problems.

1. Important Relations in Parabolic Coordinates

We set

$$x + iy = (\xi + i\eta)^2 \qquad (4.1.1)$$

or

$$x = \xi^2 - \eta^2$$
$$y = 2\xi\eta \qquad (4.1.2)$$

where the level curves are shown in Figure 4.1.1.

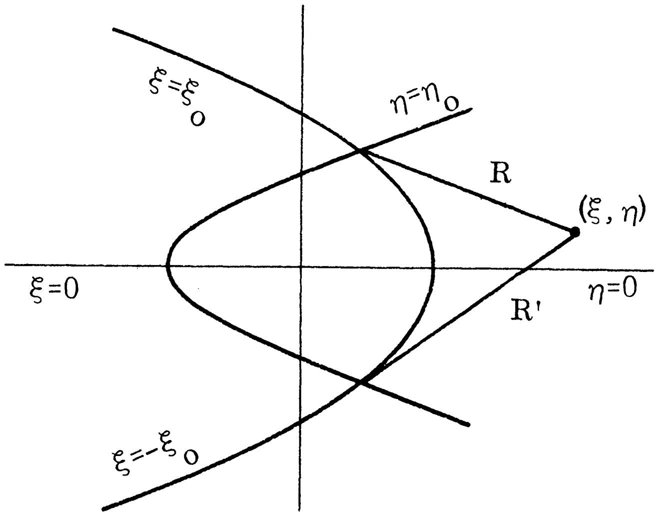


FIGURE 4.1.1: PARABOLIC COORDINATES

In the figure the range of ξ and η covering the entire xy - plane is taken as $-\infty < \xi < \infty$, $0 \leqslant \eta < \infty$. We could also describe the plane with $0 \leqslant \xi < \infty$, $-\infty < \eta < \infty$. The appropriate choice really depends on the problem being considered but for the present we shall restrict the discussion to the upper half plane, $\xi, \eta \geqslant 0$.

The distance R becomes

$$R = \sqrt{(x-x_o)^2+(y-y_o)^2} = \sqrt{\left[(x+iy)-(x_o+iy_o)\right]\left[(x-iy)-(x_o-iy_o)\right]}$$

$$= \sqrt{\left[(\xi+i\eta)^2-(\xi_o+i\eta_o)^2\right]\left[(\xi-i\eta)^2-(\xi_o-i\eta_o)^2\right]} \qquad (4.1.3)$$

and similarly

$$R' = \sqrt{\left[(\xi-i\eta)^2-(\xi_o+i\eta_o)^2\right]\left[(\xi+i\eta)^2-(\xi_o-i\eta_o)^2\right]} \quad . \qquad (4.1.4)$$

The Helmholtz equation is now

$$\left\{ \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + (2k)^2 (\xi^2+\eta^2) \right\} \phi = 0 \tag{4.1.5}$$

and the solutions which correspond to (2.4.1) and (2.4.2) are

$$\phi(\xi,\eta) = \int_{\xi - i\eta}^{\xi + i\eta} J(\xi,\eta,u)\, f(u)\, du \tag{4.1.6}$$

and

$$\psi(\xi,\eta) = \int_{-\xi + i\eta}^{\xi + i\eta} J(\xi,\eta,u)\, f(u)\, du \tag{4.1.7}$$

where we have used, and will continue to use, the short hand

$$J(\xi,\eta,u) = J_o\left(k \sqrt{\left[(\xi + i\eta)^2 - u^2\right] \left[(\xi - i\eta)^2 - u^2\right]} \right) . \tag{4.1.8}$$

Of course it must be verified that (4.1.6) and (4.1.7) are indeed solutions of the Helmholtz equation. While it is true that they are solutions if f(u) is analytic in a simply connected region containing the path of integration this unfortunately does not follow directly from the already proven results in rectangular coordinates. That is, under the conditions for which (2.4.1) and (2.4.2) were shown to be solutions, it is possible by a simple transformation to show that (4.1.6) and (4.1.7) are solutions for a wide class of functions f(u) but not all analytic functions since $u = 0$ is a singular point of the transformation. However, since the proof that (4.1.6) and (4.1.7) are solutions with any analytic function f(u) is completely analogous to that of section 2.2, it will not be reproduced here. Of course this can also be established by direct substitution in the Helmholtz equation but these tedious details will also be omitted.

2. Parabolic Form of the Representation Theorems and Applications

With (4.1.6) and (4.1.7) we can now establish representation theorems comparable to (3.1.1) and (3.1.2).

If $\phi(\xi, \eta)$ is an analytic solution of the Helmholtz equation in a simply connected region containing a segment of the line $\eta = 0$, and $\phi = 0$ on this segment then, in this region, ϕ has the integral representation

$$\phi(\xi, \eta) = \frac{1}{2i} \int_{\xi - i\eta}^{\xi + i\eta} J(\xi, \eta, u) \left. \frac{\partial}{\partial \nu} \phi(u, \nu) \right|_{\nu=0} du \,. \tag{4.2.1}$$

Similarly, if $\psi(\xi, \eta)$ is an analytic solution of the Helmholtz equation in a simply connected region containing a segment of the line $\xi = 0$, and $\psi = 0$ on this segment then, in this region, ψ has the representation

$$\psi(\xi, \eta) = \frac{1}{2} \int_{-\xi + i\eta}^{\xi + i\eta} J(\xi, \eta, u) \, \frac{\partial}{\partial \nu} \psi(\nu, -iu) \, du \,. \tag{4.2.2}$$

The proof of these forms of the representation theorem is exactly the same as presented in 3.1. It consists of demonstrating that the integral representation preserves the value of the function and its normal derivative on the line segment and since the function is given to be a solution and the integral has been shown to be a solution of the Helmholtz equation, the Cauchy-Kowalewsky theorem assures that they are identical.

We could, of course, employ (4.2.1) and (4.2.2) to find integral representations of products of parabolic cylinder functions which vanish when $\xi = 0$ or $\eta = 0$, corresponding to the representation of "rectangular" cylinder functions of section 3.3, but we shall confine our remarks to topics more intimately connected with the half plane problem.

The expression for the difference of two sources, $H_o^{(2)}(kR) - H_o^{(2)}(kR')$, which is of interest in this regard, can be derived using these representations. The two forms of the representation theorem produce two different expressions which can also be obtained by a transformation of our previous result. Replacing $x \pm iy$ with the parabolic equivalent $(\xi \pm i\eta)^2$ and substituting $\alpha = u^2$ in (3.4.2) we obtain two expressions, because of the sign ambiguity in u (i.e. $\alpha = u^2 = (-u)^2$), as follows:

$$H_o^{(2)}(kR) - H_o^{(2)}(kR') = -4ik\xi_o\eta_o \int_{\xi - i\eta}^{\xi + i\eta} \frac{J(\xi, \eta, u)\, H_1^{(2)}(\xi_o, \eta_o, u)\, udu}{\sqrt{\left[(\xi_o + i\eta_o)^2 - u^2\right]\left[(\xi_o - i\eta_o)^2 - u^2\right]}} \tag{4.2.3}$$

and

$$H_o^{(2)}(kR) - H_o^{(2)}(kR') = -4ik\xi_o\eta_o \int_{-\xi + i\eta}^{\xi + i\eta} \frac{J(\xi, \eta, u)\, H_1^{(2)}(\xi_o, \eta_o, u)\, udu}{\sqrt{\left[(\xi_o + i\eta_o)^2 - u^2\right]\left[(\xi_o - i\eta_o)^2 - u^2\right]}} \tag{4.2.4}$$

where the same abbreviation is used for $H_1^{(2)}$ as for J_o. Under this transformation, straight lines are mapped into hyperbolas and the branch cuts of the integrands lie along the curves

$$\mathscr{I}m \left[(\xi_o + i\eta_o)^2 - u^2\right] \left[(\xi_o - i\eta_o)^2 - u^2\right] = 0 \; . \qquad (4.2.5)$$

If we write $u = u_1 + iu_2$ then (4.2.5) implies that

$$u_1^2 - u_2^2 + \eta_o^2 - \xi_o^2 = 0 \qquad (4.2.6)$$

and the curves described by this equation are plotted in Figure 4.2.1 with the portion chosen as a branch cut indicated. Also shown are the values of $\arg \left[(\xi_o + i\eta_o)^2 - u^2\right]\left[(\xi_o - i\eta_o)^2 - u^2\right]$ along these curves on the n^{th} sheet of the Riemann surface of $\log \left[(\xi_o + i\eta_o)^2 - u^2\right] \left[(\xi_o - i\eta_o)^2 - u^2\right]$.

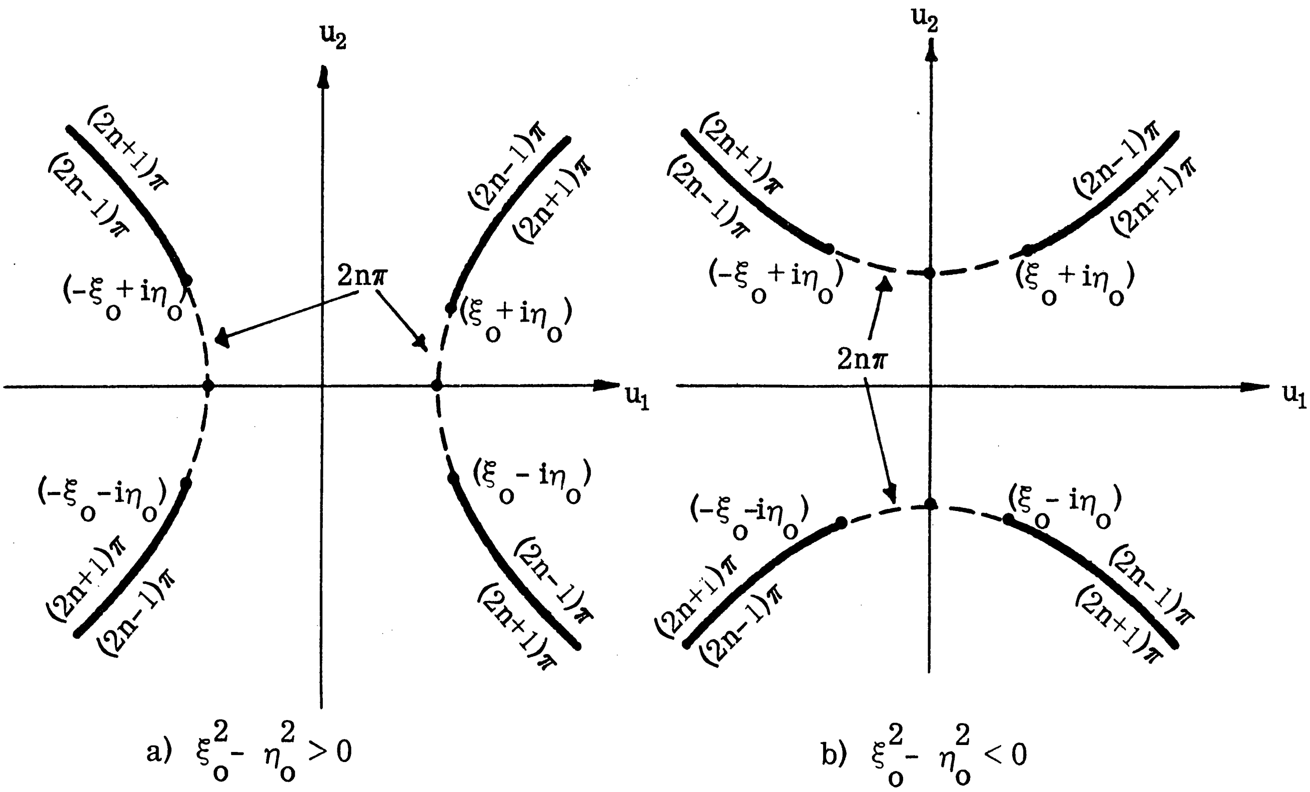
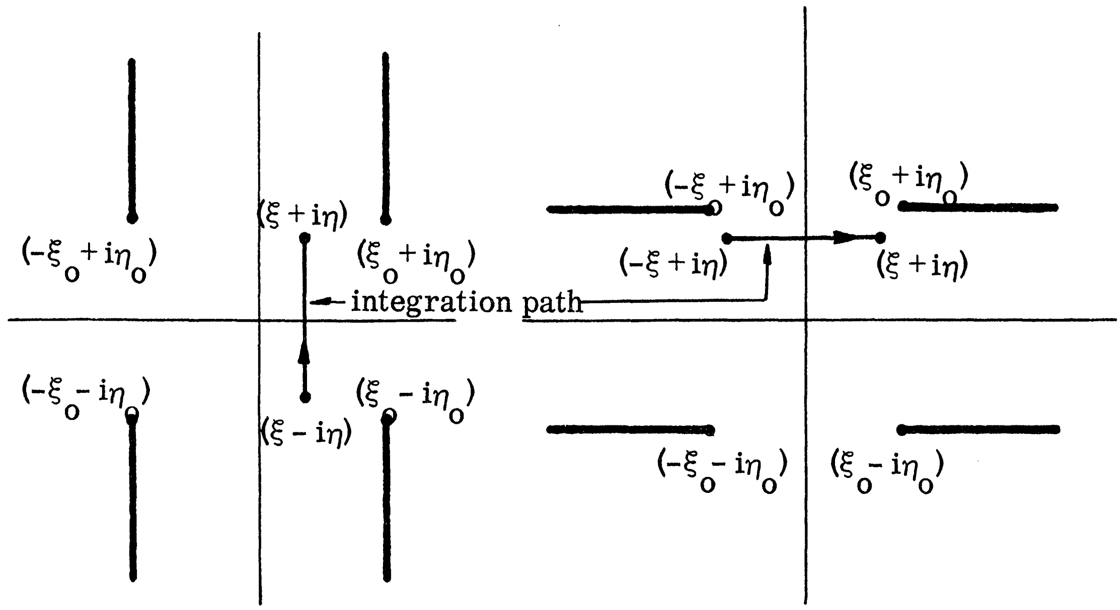


FIGURE 4.2.1: BRANCH CUTS IN THE u-PLANE

Since it is desirable to always be able to treat the path of integration as the straight line connecting the end points, we alter the cuts somewhat as shown in Figure 4.2.2. We still keep track of which sheet of the Riemann surface we are on by calling the n^{th} sheet that sheet where the argument of $\left[(\xi_o + iy_o)^2 - u^2\right]\left[(\xi_o - i\eta_o)^2 - u^2\right]$ along the dotted curves of Figure 4.2.1 is $2n\pi$ and then confine our attention to the 0^{th} sheet.



a) Branch Cuts for Formula (4.2.3) b) Branch Cuts for Formula (4.2.4)

FIGURE 4.2.2: ALTERED BRANCH CUTS IN u-PLANE

In the same way, the double integral expression (3.6.14) leads to two expressions in this parabolic form as follows:

$$H_0^{(2)}(kR) - H_0^{(2)}(kR')$$

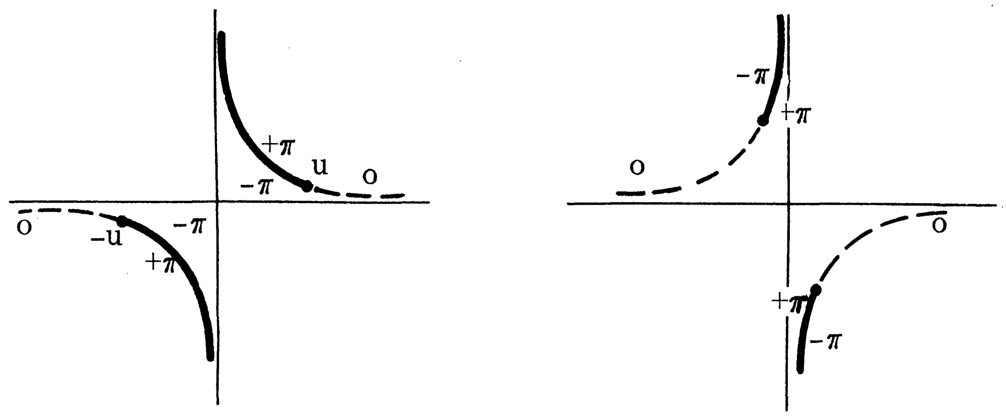
$$= -2k \int_{\xi - i\eta}^{\xi + i\eta} du \int_{\xi_o - i\eta_o}^{\xi_o + i\eta_o} dv \; J(\xi, \eta, u) J(\xi_o, \eta_o, v) \frac{H_1^{(2)}(\pm k [v^2 - u^2]) uv}{\pm (v^2 - u^2)} , \quad \begin{array}{l} + \text{ if } \xi_o > \xi \\ - \text{ if } \xi_o < \xi \end{array} \tag{4.2.7}$$

and

$$H_0^{(2)}(kR) - H_0^{(2)}(kR')$$

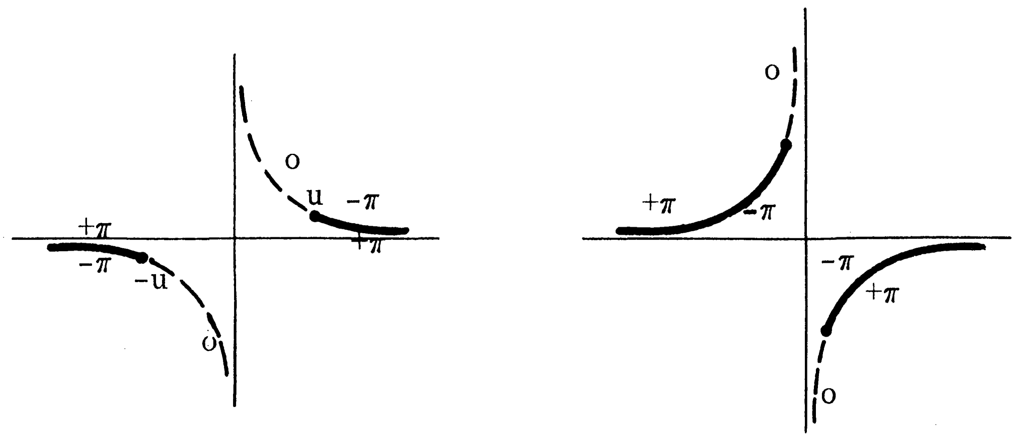
$$= -2k \int_{-\xi + i\eta}^{\xi + i\eta} du \int_{-\xi_o + i\eta_o}^{\xi_o + i\eta_o} dv \; J(\xi, \eta, u) J(\xi_o, \eta_o, v) \frac{H_1^{(2)}(\pm k [v^2 - u^2]) uv}{\pm (v^2 - u^2)} , \quad \begin{array}{l} + \text{ if } \eta > \eta_o \\ - \text{ if } \eta < \eta_o . \end{array} \tag{4.2.8}$$

In each case the branch cuts of the integrand are chosen to lie along the negative real axis of the argument of the Hankel function. In the v plane this criterion leads to the cuts shown in Figure 4.2.3. Note that in formula (4.2.7), $\mathrm{Re}\, u > 0$, $-\infty < \mathcal{I}m\, u < \infty$, and in formula (4.2.8), $-\infty < \mathrm{Re}\, u < \infty$, $\mathcal{I}m\, u > 0$, so while we may use the same figures for both formulas, they correspond to different u values as indicated. In the cases illustrated in Figures 4.2.3 b and d, the point u for formula (4.2.7) corresponds to the point $-u$ for formula (4.2.8). In all cases, the value of the phase of the argument of the Hankel function on the branch cut and its extension (the positive real axis of the argument) are indicated.



a) Formula (4.2.7): $\xi_o > \xi$, $\mathcal{I}m\, u > 0$
Formula (4.2.8): $\eta_o < \eta$, $\mathrm{Re}\, u > 0$

b) Formula (4.2.7): $\xi_o > \xi$, $\mathcal{I}m\, u < 0$
Formula (4.2.8): $\eta_o < \eta$, $\mathrm{Re}\, u < 0$



c) Formula (4.2.7): $\xi_o < \xi$, $\mathcal{I}m\, u > 0$
Formula (4.2.8): $\eta_o > \eta$, $\mathrm{Re}\, u > 0$

d) Formula (4.2.7): $\xi_o < \xi$, $\mathcal{I}m\, u < 0$
Formula (4.2.8): $\eta_o > \eta$, $\mathrm{Re}\, u < 0$

FIGURE 4.2.3: BRANCH CUTS IN THE v-PLANE

The same branch cut criterion in the u-plane leads to completely analogous results. (Indeed, the same figures can be employed with some changes in notation.)

3. Green's Function for the Half Plane

Corresponding to (3.6.20) we have the double integral representation for the Green's function which vanishes on a segment of the half line $\eta = 0$,

$$G(\xi, \eta, \xi_o, \eta_o)$$

$$= -\frac{1}{4} \int_{\xi - i\eta}^{\xi + i\eta} du \int_{\xi_o - i\eta_o}^{\xi_o + i\eta_o} dv \quad J(\xi, \eta, u)\, J(\xi_o, \eta_o, v) \left. \frac{\partial^2 G(u, \nu, v, \mu)}{\partial \nu \partial \mu} \right|_{\substack{\nu=0 \\ \mu=0}} . \tag{4.3.1}$$

This holds as long as $G(\xi, \eta, \xi_o, \eta_o)$ is analytic (save for sources) in a simply connected region containing the line segment. Clearly this representation will be most appropriate for the case when the segment on which G vanishes consists of the entire half line $\eta = 0$.

In order to have a physically significant problem and still have $G(\xi, \eta, \xi_o, \eta_o)$ analytic at $\eta = 0$ we must limit the problem so that the points in the neighborhood of one side of the line lie outside physical space. This of course can be accomplished in various ways. Perhaps the simplest is to impose another boundary condition when $\xi = 0$, i.e., on the complement of the half line $\eta = 0$, thus restricting physical space to the upper half plane. If we require that G also vanish when $\xi = 0$ we essentially will reproduce the problems for the entire line treated in the previous chapter.

However, if we require that the normal derivative vanish at $\xi = 0$, i.e. $\left. \frac{\partial G}{\partial \xi} \right|_{\xi=0} = 0$, we have a much more complicated problem. That the representation (4.3.1) is ideally suited to this problem is easily seen since it is only necessary to restrict G to be even in ξ in order for the integral expression to satisfy both the Helmholtz equation and the boundary conditions. Exactly the same procedure as discussed in section 3.2 shows that this mixed problem is entirely equivalent to the problems where either the function <u>or</u> its normal derivative vanish of the half plane, i.e., the classic half plane problems.

Baker and Copson (Ref. 2) give a very thorough, well referenced discussion of Sommerfeld's famous solution of this problem for plane wave incidence. Our concern is with line sources and the exact solution for this problem, finding the field, ϕ, of a line source in the presence of a perfectly soft screen (see Figure 4.3.1),was given by Carslaw (Ref. 7) in a form comparable to Sommerfeld's result. Macdonald (Ref. 15) simplified the result considerably and it is his form that we shall employ. Rewriting his result in parabolic coordinates, Macdonald found that the total field, $\phi(\xi, \eta, \xi_o, \eta_o)$, due to a line source, $-\pi i\, H_o^{(2)}(kR)$, could be expressed as

$$\phi(\xi, \eta, \xi_o, \eta_o) = \int_{-\infty + i\epsilon}^{Q_o} e^{-ikR \cosh Q}\, dQ - \int_{-\infty + i\epsilon}^{Q_1} e^{-ikR' \cosh Q}\, dQ \tag{4.3.2}$$

where

$$Q_o = \frac{1}{2} \log \frac{(\xi + \xi_o)^2 + (\eta + \eta_o)^2}{(\xi - \xi_o)^2 + (\eta - \eta_o)^2} \tag{4.3.3}$$

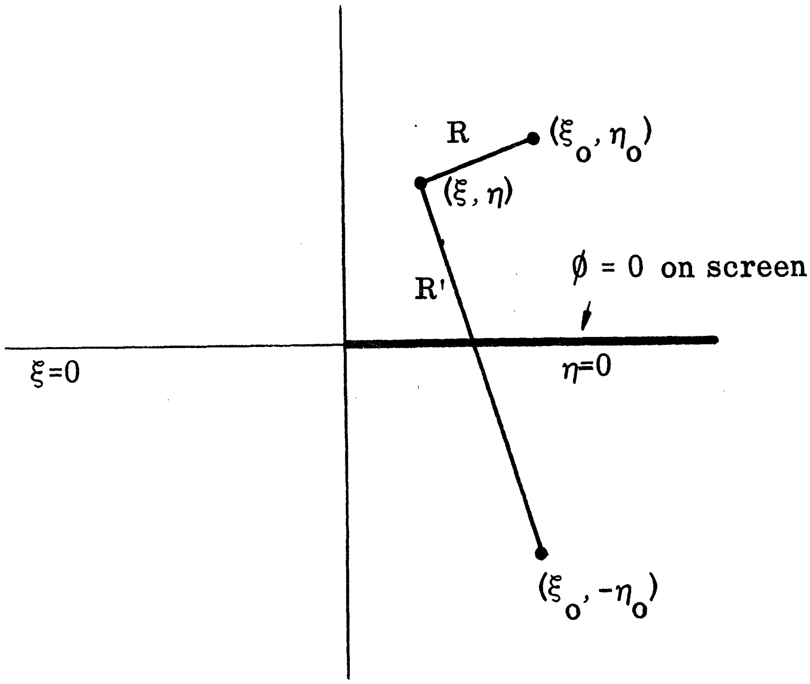


FIGURE 4.3.1: A LINE SOURCE
IN THE PRESENCE OF A SOFT HALF PLANE

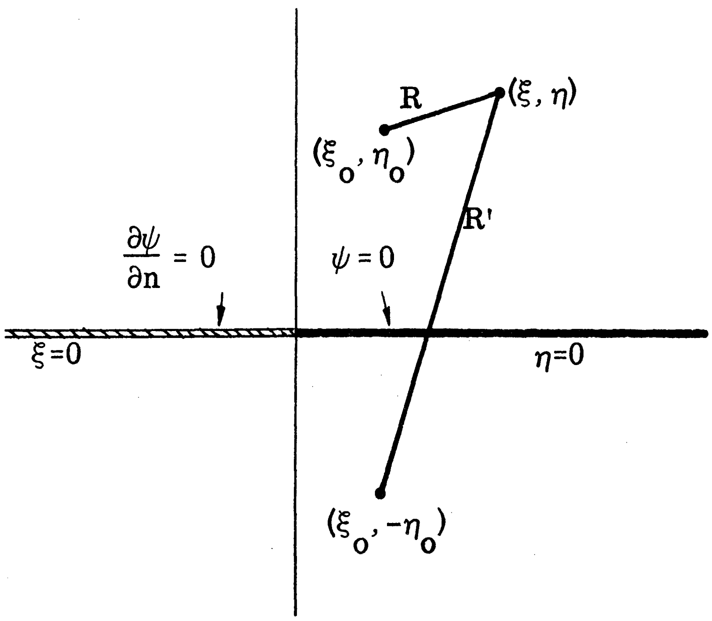


FIGURE 4.3.2: A LINE SOURCE
IN THE PRESENCE OF AN INFINITE SCREEN HALF SOFT; HALF RIGID

$$Q_1 = \frac{1}{2} \log \frac{(\xi+\xi_o)^2 + (\eta-\eta_o)^2}{(\xi-\xi_o)^2 + (\eta+\eta_o)^2} , \qquad (4.3.4)$$

R and R' are given by (4.1.3) and (4.1.4) respectively and the time dependence $e^{+i\omega t}$ is suppressed.

With the help of Sommerfeld's integral representation of the Hankel function (Ref. 29), this result has a ready interpretation as an "incomplete" Hankel function in the following sense. Sommerfeld represents the Hankel function as a complex integral where the contour goes from $-\infty i$ to $+\infty i$ in such a way that convergence is assured. By choosing only a part of this path we have an "incomplete" Hankel function in the same sense that Fresnel integrals are incomplete factorial functions.

Specifically, Sommerfeld's result can be given in the following form:

$$H_n^{(2)}(kR) = \frac{(i)^n}{\pi} \int_W \cos n\omega \, e^{-ikR\cos\omega} \, d\omega \qquad (4.3.5)$$

where the contour is shown in Figure 4.3.3.

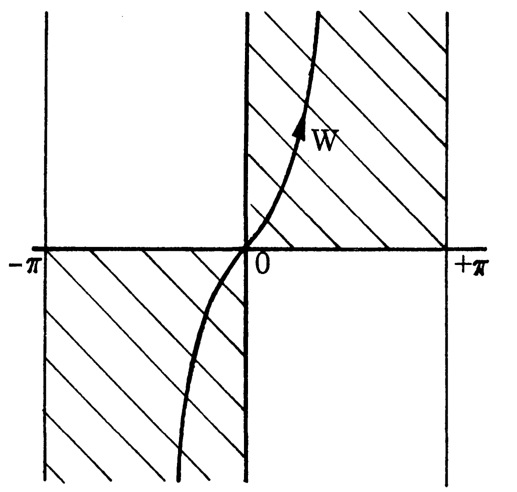


FIGURE 4.3.3: SOMMERFELD CONTOUR FOR $H_n^{(2)}(kR)$

Note that the contour must necessarily be confined to the shaded regions only in the neighborhood of infinity to ensure convergence.

Now with a slight transformation, $Q = i\omega$, (4.3.2) can be written as

$$\phi(\xi, \eta, \xi_o, \eta_o) = -i \int_{\omega_1 + \omega_3} e^{-ikR\cos\omega}\, d\omega + i \int_{\omega_1' + \omega_3'} e^{-ikR'\cos\omega}\, d\omega \qquad (4.3.6)$$

where the contours are shown in Figure 4.3.4.

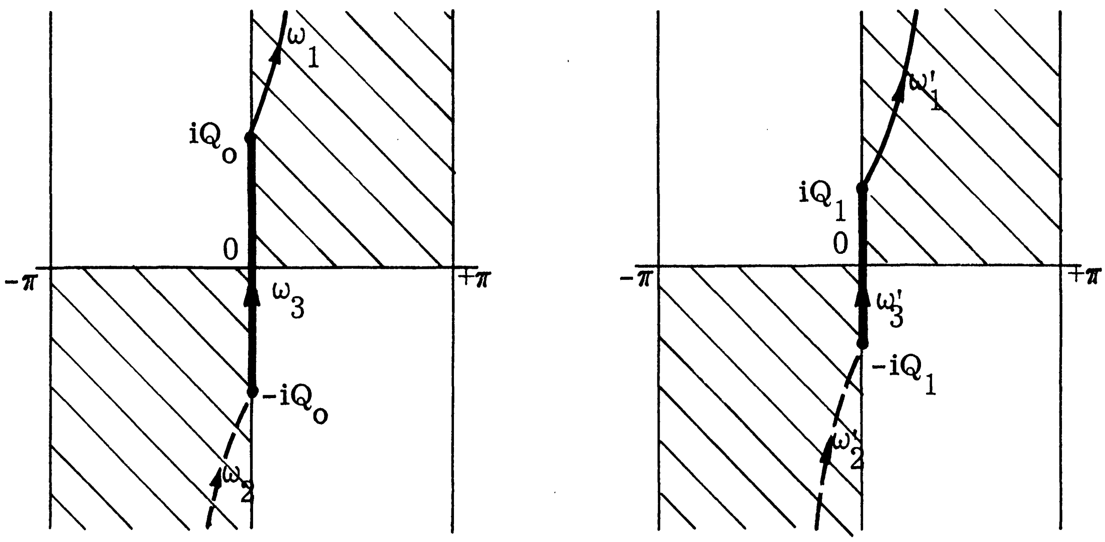


FIGURE 4.3.4: CONTOURS IN THE ω-PLANE

Comparison of Figures 4.3.3 and 4.3.4 shows that if we add on the dotted contours, ω_2 and ω_2', (4.3.6) becomes the integral representation of the geometric optics field,

$$-i\pi H_0^{(2)}(kR) + i\pi H_0^{(2)}(kR') = -i \int_{\omega_1+\omega_2+\omega_3} e^{-ikR\cos\omega} d\omega + i \int_{\omega_1'+\omega_2'+\omega_3'} e^{-ikR'\cos\omega} d\omega. \tag{4.3.7}$$

The usual Sommerfeld contour is deformed so that it is entirely contained within a shaded region (Figure 4.3.3) but this is only a convenience when considering the asymptotic behavior. In this instance it is convenient to have the contour symmetric about the origin passing through the points $\pm iQ_0$ or $\pm iQ_1$ which can lie anywhere on the imaginary axis. Thus it is seen that the diffraction phenomenon is described by the incomplete Hankel functions (4.3.6) and in the geometric optics limit, i.e., no diffraction, this expression becomes the complete Hankel functions, (4.3.7). From these integral forms for the total and geometric optics fields, it is clear that the diffracted field, ϕ_D, is given by

$$\phi_D = i \int_{\omega_2} e^{-ikR\cos\omega} d\omega - i \int_{\omega_2'} e^{-ikR'\cos\omega} d\omega. \tag{4.3.8}$$

As discussed in section 3.2, the total field, $\psi(\xi, \eta, \xi_0, \eta_0)$, for the mixed boundary value problem, Figure 4.3.2, with a source at ξ_0, η_0 is given by

$$\psi(\xi, \eta, \xi_0, \eta_0) = -\pi i \left[H_0^{(2)}(kR) - H_0^{(2)}(kR') \right] + 2\phi_D. \tag{4.3.9}$$

With the expressions (4.3.8) and (4.3.7), this becomes

$$\psi(\xi, \eta, \xi_0, \eta_0) = -i \int_{\omega_3} e^{-ikR\cos\omega} d\omega + i \int_{\omega_3'} e^{-ikR'\cos\omega} d\omega$$

or

$$\psi(\xi, \eta, \xi_o, \eta_o) = -i \int_{-iQ_o}^{iQ_o} e^{-ikR\cos\omega} d\omega + i \int_{-iQ_1}^{iQ_1} e^{-ikR'\cos\omega} d\omega. \tag{4.3.10}$$

To cast this in the integral form (4.3.1), we first must calculate

$\left.\frac{\partial^2 \psi(\xi, \eta, \xi_o, \eta_o)}{\partial\eta\partial\eta_o}\right|_{\substack{\eta=0\\ \eta_o=0}}$. Note that when $\eta = 0$ and $\eta_o = 0$, the distances R and R' are given by

$$R\Big|_{\substack{\eta=0\\ \eta_o=0}} = R'\Big|_{\substack{\eta=0\\ \eta_o=0}} = \sqrt{(\xi^2 - \xi_o^2)^2} \tag{4.3.11}$$

or, since distance is always ≥ 0,

$$R\Big|_{\substack{\eta=0\\ \eta_o=0}} = R'\Big|_{\substack{\eta=0\\ \eta_o=0}} = \pm (\xi^2 - \xi_o^2) \ , \quad \begin{matrix} + \text{ if } \xi > \xi_o \\ - \text{ if } \xi < \xi_o \end{matrix} . \tag{4.3.12}$$

With this sign choice thus dictated we find that

$$Q_o\Big|_{\substack{\eta=0\\ \eta_o=0}} = Q_1\Big|_{\substack{\eta=0\\ \eta_o=0}} = \log \frac{(\xi + \xi_o)^2}{\pm(\xi^2 - \xi_o^2)} \ , \quad \begin{matrix} + \text{ if } \xi > \xi_o \\ - \text{ if } \xi < \xi_o \end{matrix} \tag{4.3.13}$$

and from (4.3.10)

$$\left.\frac{\partial^2 \psi(\xi, \eta, \xi_o \eta_o)}{\partial\eta\partial\eta_o}\right|_{\substack{\eta=0\\ \eta_o=0}} = 8 \left\{ \frac{\xi^2 + \xi_o^2}{(\xi^2 - \xi_o^2)^2} e^{-ik(\xi^2 + \xi_o^2)} + \frac{2k\xi\xi_o}{\pm(\xi^2 - \xi_o^2)} \int_0^{i\log \frac{(\xi + \xi_o)^2}{\pm(\xi^2 - \xi_o^2)}} \cos\omega \, e^{-ik\left[\pm(\xi^2 - \xi_o^2)\right]\cos\omega} d\omega \right\}. \tag{4.3.14}$$

We now replace ξ and ξ_o by complex u and v respectively with the sign choice still

$$+ \text{ if } \xi = \text{Re}\, u > \xi_o = \text{Re}\, v$$

$$- \text{ if } \xi = \text{Re}\, u < \xi_o = \text{Re}\, v.$$

Then with (4.3.1) we have

$$\psi(\xi, \eta, \xi_o, \eta_o) = \int_{\xi - i\eta}^{\xi + i\eta} du \int_{\xi_o - i\eta_o}^{\xi_o + i\eta_o} dv \;\; J(\xi, \eta, u) J(\xi_o, \eta_o, v) \; F(u, v) \tag{4.3.15}$$

where

$$F(u, v) = \frac{-2(u^2+v^2)}{(u^2-v^2)^2} \; e^{-ik(u^2+v^2)} - \frac{4kuv}{\pm(u^2-v^2)} \int_0^{+i \log \frac{(u+v)^2}{\pm(u^2-v^2)}} \cos\omega \; e^{-ik\left[\pm(u^2-v^2)\right] \cos\omega} d\omega, \quad \begin{matrix} + \text{ if } \xi > \xi_o \\ - \text{ if } \xi < \xi_o \end{matrix} . \tag{4.3.16}$$

The paths of integration are straight lines in the u and v planes and the branch cuts are treated exactly as they were in the corresponding representation of the geometric optics field (4.2.7).

It would appear that this result represents the dubious accomplishment of changing the relatively simple result of Macdonald, (4.3.10), into more complicated form. However, while this is a valid objection in the half plane context, the form of (4.3.15) is such that, when properly interpreted, we are able to extrapolate to the problem of diffraction by a strip in the following way. For the moment consider the variables of integration, u and v as "quasi" * parabolic coordinate variables related to ordinary cylindrical coordinates, (ρ , γ), as follows

$$\begin{aligned} \rho \cos\gamma &= u^2 + v^2 \\ \rho \sin\gamma &= -2i\,uv. \end{aligned} \tag{4.3.17}$$

Then F(u,v) given by formula (4.3.16) becomes

$$F\left[\rho, \gamma\right] = -2\,\frac{\cos\gamma}{\rho}\, e^{-ik\rho\cos\gamma} + 2ik\sin\gamma \int_0^{\gamma} \cos\omega\, e^{-ik\rho\cos\omega}\, d\omega\, . \tag{4.3.18}$$

Now we assert that in a sense, the function $F[\rho, \gamma]$ describes the essence of the general phenomenon of diffraction and is only specifically related to the half plane problem through the specific integral (4.3.15) with the specific definition of ρ and γ (4.3.17). If we write the integral corresponding to (4.3.15) in elliptic coordinates and define ρ and γ in "quasi" elliptic coordinates, then this same $F[\rho, \gamma]$ yields the solution to the problem of diffraction by a strip.

In order to make more precise this admittedly vague assertion and demonstrate that it is true we must first phrase the strip problem in appropriate form and develop in elliptic coordinates the basic solutions and representation theorems corresponding to those already presented in rectangular and parabolic coordinates. This is done in the next chapter.

* If v were replaced by iv, then the definition (4.3.17) would correspond exactly to (4.1.2).

Chapter 5

DIFFRACTION BY A STRIP

An outstanding feature of the integral solutions of the Helmholtz equation presented in the preceding chapters is the ease with which boundary conditions are satisfied. The basic solution in rectangular coordinates, (2.4.1), is naturally suited to the problem of finding a wave function which vanishes on the entire line since, under very general limitations on the integrand, it already represents such a function. Similarly the basic solution in parabolic coordinates, (4.1.4), is ideally suited to the problem of finding wave functions which vanish on the half line.

In this chapter we present the corresponding solution in elliptic coordinates where the appropriate boundary value problem concerns the line segment or strip. Completely paralleling the presentation in parabolic coordinates of the previous chapter, we first develop the necessary machinery in elliptic coordinates and then use it to produce an exact solution to the problem of finding the field due to a line source in the presence of a perfectly soft strip. Since this problem is apparently of more than casual interest, the presentation will be slightly more detailed than that given in the half plane case.

1. Important Relations in Elliptic Coordinates

To begin with, we introduce the elliptic coordinates, express familiar quantities in them, and adopt a shorthand notation without which the analysis would be prohibitively cumbersome. Thus we set

$$x+iy = a \cosh(\mu + i\theta) \tag{5.1.1}$$

or

$$\begin{aligned} x &= a \cosh\mu \cos\theta \\ y &= a \sinh\mu \sin\theta \end{aligned} \tag{5.1.2}$$

where the level curves are shown in Figure 5.1.1.

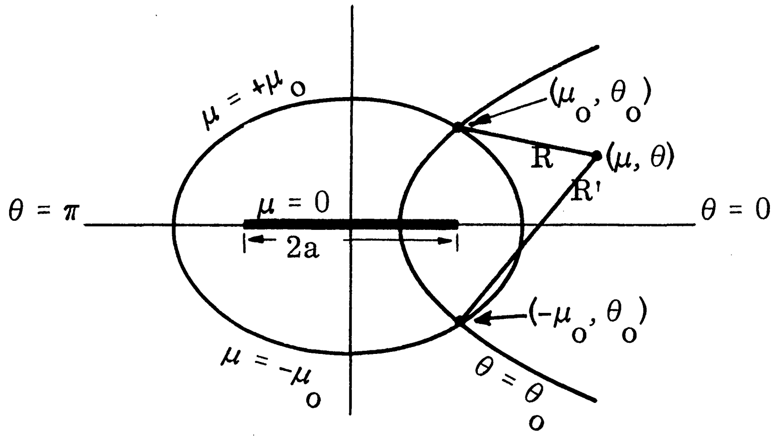


FIGURE 5.1.1: ELLIPTIC COORDINATES

As shown in the figure, the range of μ and θ covering the entire plane was taken as $-\infty < \mu < \infty$, $0 \leqslant \theta \leqslant \pi$. We could also describe the plane with $\mu \geqslant 0$, $-\pi \leqslant \theta \leqslant \pi$. The choice is important when discussing boundary value problems for the strip and slit but for now we shall restrict the discussion to the upper half plane, $\mu \geqslant 0$, $0 \leqslant \theta \leqslant \pi$.

The distance R becomes

$$R = \sqrt{(x-x_o)^2 + (y-y_o)^2} = \sqrt{\left[(x+iy)-(x_o+iy_o)\right]\left[(x-iy)-(x_o-iy_o)\right]}$$

$$= a\sqrt{\left[\cosh(\mu+i\theta)-\cosh(\mu_o+i\theta_o)\right]\left[\cosh(\mu-i\theta)-\cosh(\mu_o-i\theta_o)\right]} \tag{5.1.3}$$

and similarly

$$R' = a\sqrt{\left[\cosh(\mu + i\theta) - \cosh(\mu_o - i\theta_o)\right]\left[\cosh(\mu - i\theta) - \cosh(\mu_o + i\theta_o)\right]} . \qquad (5.1.4)$$

The Helmholtz equation is now

$$\left\{ \frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\theta^2} + (ka)^2 \left[\cosh^2\mu - \cos^2\theta\right] \right\} \phi = 0 \qquad (5.1.5)$$

and the solutions which correspond to (2.4.1) and (2.4.2) are

$$\phi(\mu, \theta) = \int_{-\mu + i\theta}^{\mu + i\theta} J\left[\mu, \theta, \alpha\right] f(\alpha)\, d\alpha \qquad (5.1.6)$$

and

$$\psi(\mu, \theta) = \int_{\mu - i\theta}^{\mu + i\theta} J\left[\mu, \theta, \alpha\right] f(\alpha)\, d\alpha \qquad (5.1.7)$$

where we have introduced and will continue to employ the shorthand

$$J\left[\mu, \theta, \alpha\right] = J_o\left(ka\sqrt{\left[\cosh(\mu + i\theta) - \cosh\alpha\right]\left[\cosh(\mu - i\theta) - \cosh\alpha\right]}\right). \qquad (5.1.8)$$

The square brackets are used to differentiate between this and the previously employed notation (4.1.8). Note that

$$J\left[\mu, \theta, \mu \pm i\theta\right] = J\left[\mu, \theta, \pm\mu + i\theta\right] = 1. \qquad (5.1.9)$$

2. Derivation of the Basic Solutions

Unfortunately, as in the parabolic case, it is not possible to establish with sufficient generality that $\phi(\mu, \theta)$ and $\psi(\mu, \theta)$ given by (5.1.6) and (5.1.7) are indeed solutions of the Helmholtz equation merely by a transformation of the already established solutions (2.4.1) and (2.4.2). This can be seen by writing (2.4.1) in elliptic coordinates obtaining

$$\phi[\mu, \theta] = \int_{a\cosh(\mu - i\theta)}^{a\cosh(\mu + i\theta)} J_o\left\langle k\sqrt{\left[a\cosh(\alpha + i\theta) - \alpha\right]\left[a\cosh(\mu - i\theta) - \alpha\right]}\right\rangle f(\alpha)\, d\alpha \tag{5.2.1}$$

where $f(\alpha)$ is analytic. With the substitution $\alpha = a\cosh\beta$ and the ambiguity thus introduced, i.e., $\cosh\beta = \cosh(-\beta)$, we obtain two expressions

$$\phi\left[\mu, \theta\right] = a \int_{-\mu + i\theta}^{\mu + i\theta} J\left[\mu, \theta, \beta\right]\ f(a\cosh\beta)\ \sinh\beta\ d\beta \tag{5.2.2}$$

and

$$\psi\left[\mu, \theta\right] = a \int_{\mu - i\theta}^{\mu + i\theta} J\left[\mu, \theta, \beta\right]\ f(a\cosh\beta)\ \sinh\beta\ d\beta\ . \tag{5.2.3}$$

Thus, while it is true that the expressions (5.2.2) and (5.2.3) are solutions of the Helmholtz equation under the condition that $f(a\cosh\beta)$ is an analytic function of $a\cosh\beta$, we wish to establish that this is true if $f(a\cosh\beta)\sinh\beta$ were replaced by a $g(\beta)$ where the only condition on $g(\beta)$ is analyticity in β. It is not true that requiring that $g(\beta)$ be analytic is sufficient to guarantee that $g(\beta)$ can always be written as $f(a\cosh\beta)\sinh\beta$ where $f(a\cosh\beta)$ is analytic. $g(\beta)$ then represents a wider class of functions than $f(a \cosh\beta)\sinh\beta$ and it is for this wider class that we wish to establish (5.1.6) and (5.1.7). Although this proof is quite similar to that of section 2.2, we present it here not only for the sake of completeness but because it gives an opportunity to discuss the branch cuts for the Hankel function in elliptic coordinates which will be necessary subsequently.

To this end, consider the following contour integral;

$$I = \int_c H_o^{(2)}\left(ka\sqrt{\left[\cosh(\mu+i\theta) - \cosh\alpha\right]\left[\cosh(\mu-i\theta) - \cosh\alpha\right]}\right) f(\alpha)\, d\alpha \qquad (5.2.4)$$

where $f(\alpha)$ is analytic in a simply connected region containing all points of the contour c and in order to specify the contour we must first discuss the branch points of the integrand.

Recall that the Hankel function $H_o^{(2)}(ka\rho)$ is an analytic function of the complex variable ρ in any simply connected region excluding $\rho = 0$ and has the form

$$H_o^{(2)}(ka\rho) = F_1(\rho^2)\log\rho^2 + F_2(\rho^2) \qquad (5.2.5)$$

where F_1 and F_2 are entire functions of the variable ρ^2. The point $\rho = 0$ is of course a logarithmic branch point while $\rho = \infty$ is an essential singularity. Clearly, in the finite plane, the singular behavior of the Hankel function is determined completely by $\log \rho^2$ and as we must consider the Hankel function in (5.2.4) as a function of the complex variable α we first investigate the behavior of the function

$$\log \rho^2 = \log \left[\cosh(\mu + i\theta) - \cosh \alpha\right] \left[\cosh(\mu - i\theta) - \cosh \alpha\right] . \qquad (5.2.6)$$

This function has the branch points

$$\alpha = \mu \pm i\theta + 2k\pi i$$

and

$$\alpha = -\mu \pm i\theta + 2k\pi i \;, \qquad k = 0, \pm 1, \pm 2, \ldots \; . \qquad (5.2.7)$$

In the ρ^2 plane, $\log \rho^2$ is single valued on a Riemann surface with a branch cut along the negative real axis. The zeroth blade is defined by $-\pi < \arg \rho^2 < \pi$ (see Figure 5.2.1) and every time a contour encircles the origin the amount $\pm 2\pi i$ is added to the function depending on the orientation of the contour. The Riemann surface has an infinity of blades and the argument of $\rho^2 (= \mathfrak{Im} \log \rho^2)$

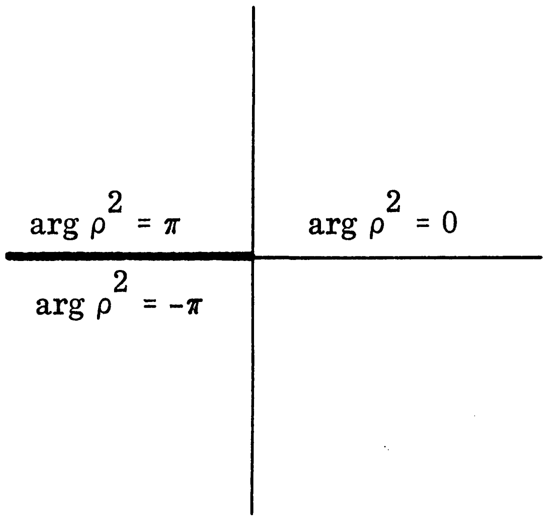


FIGURE 5.2.1: CUT ρ^2 PLANE

determines on which blade the function lies. By writing $\log_n \rho^2$ to denote the function on the nth blade of its Riemann surface, it is clear that the values of the function on successive blades are related by

$$\log_{n+1}\rho^2 - \log_n\rho^2 = 2\pi i. \tag{5.2.8}$$

The function

$$\rho^2 = \left[\cosh(\mu+i\theta) - \cosh\alpha\right]\left[\cosh(\mu-i\theta) - \cosh\alpha\right] \tag{5.2.9}$$

is periodic in α with period $2\pi i$ and in any strip of width $2\pi i$ four values of α correspond to each value of ρ^2. Thus in the strip $-\infty < \alpha_1 < \infty$, $-\pi < \alpha_2 < \pi$, where $\alpha = \alpha_1 + i\alpha_2$, there are four points, $\alpha = \pm\mu \pm i\theta$, which correspond to $\rho = 0$ and therefore the function $\log\rho^2$, (5.2.6), has, in this strip, logarithmic branch points at these four points.

The curves defined by the equation

$$\mathfrak{Im}\ \left[\cosh(\mu+i\theta) - \cosh\alpha\right]\left[\cosh(\mu-i\theta) - \cosh\alpha\right] = 0 \tag{5.2.10}$$

in the α-plane are mapped onto the real axis in the ρ^2-plane. These curves consist of the lines

$$\alpha_1 = 0\ , \qquad \alpha_2 = 0\ , \qquad \text{and} \qquad \alpha_2 = \pm\pi \tag{5.2.11}$$

and the curve

$$\cosh\mu\cos\theta = \cosh\alpha_1\cos\alpha_2\ . \tag{5.2.12}$$

It is easy to see that the straight lines (5.2.11) all map into portions of the positive real axis in the ρ^2 plane since

$$\left.\begin{aligned} \rho^2\Big|_{\alpha_1=0} &= \left|\cosh(\mu+i\theta) - \cos\alpha_2\right|^2 \geqslant 0 \\ \rho^2\Big|_{\alpha_2=0} &= \left|\cosh(\mu+i\theta) - \cosh\alpha_1\right|^2 \geqslant 0 \\ \text{and}\qquad \rho^2\Big|_{\alpha_2=\pm\pi} &= \left|\cosh(\mu+i\theta) + \cosh\alpha_1\right|^2 \geqslant 0\ . \end{aligned}\right\} \tag{5.2.13}$$

Since the four zeros of ρ^2 lie on the curve defined by (5.2.12), the image of the negative real axis in the ρ^2 plane lies on this curve (or curves) and extends from each of the zeros to infinity. These four curves are branch cuts for the Riemann surface in the α-plane since the negative real axis was chosen as the branch cut in the ρ^2 plane. These are shown in Figure 5.2.2 where the different shape of the curves (5.2.12) depending on the value of $\cosh\mu\cos\theta$ is illustrated. The value of $\mathcal{I}m \log_n \rho^2$ along these curves is noted. In all cases the lines $\alpha_2 = \pm\pi/2$ are asymptotes of the branch cuts. For each value of n, the function $\log_n \rho^2$, and therefore $H_o^{(2)}(ka\rho)$ is analytic in the cut α-plane.

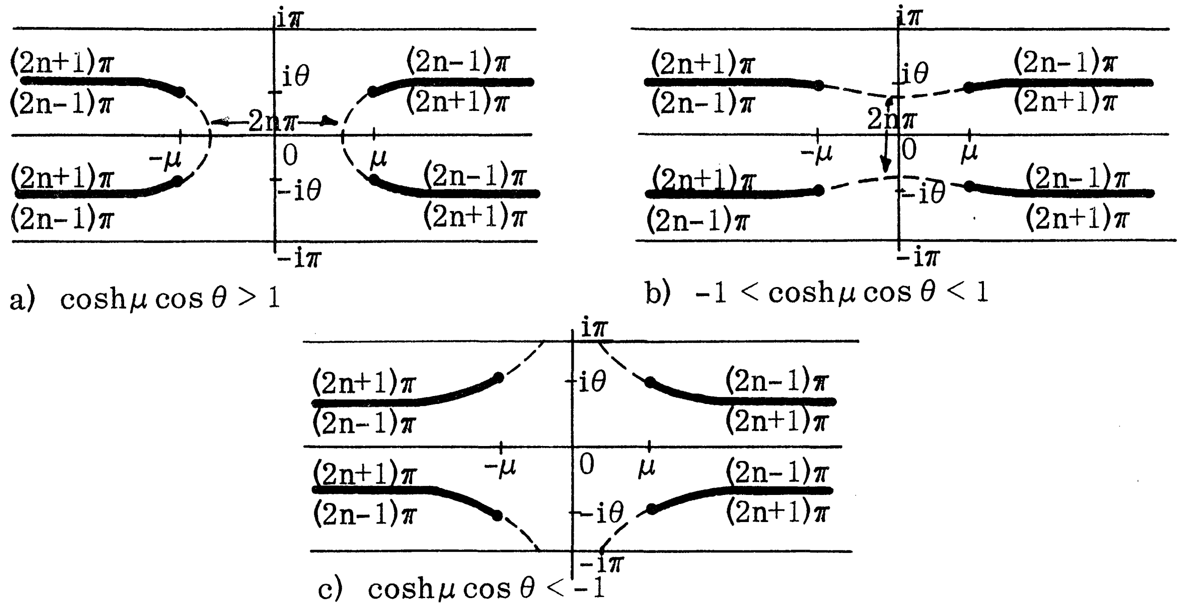


FIGURE 5.2.2: BRANCH CUTS IN THE α-PLANE

Returning to the integral (5.2.4), we designate the contour c as beginning at a constant point A on the zeroth blade, encircling the branch point $\alpha = \mu + i\theta$, and, since the integrand is to vary continuously across the branch cut, returning to the point A on the -1st blade. This is shown in

Figure 5.2.3 where the solid path lies on the 0^{th} blade and the dotted path on the -1^{st} blade.

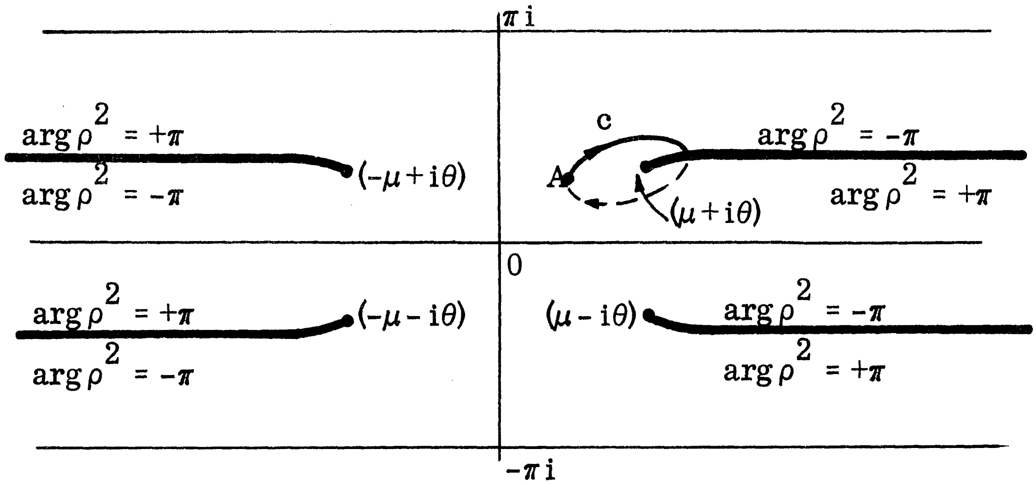


FIGURE 5.2.3: CONTOUR OF INTEGRATION IN α-PLANE

Since $f(\alpha)$ is an analytic function of α throughout a simply connected region containing the contour c, we may deform the contour to that shown in Figure 5.2.4.

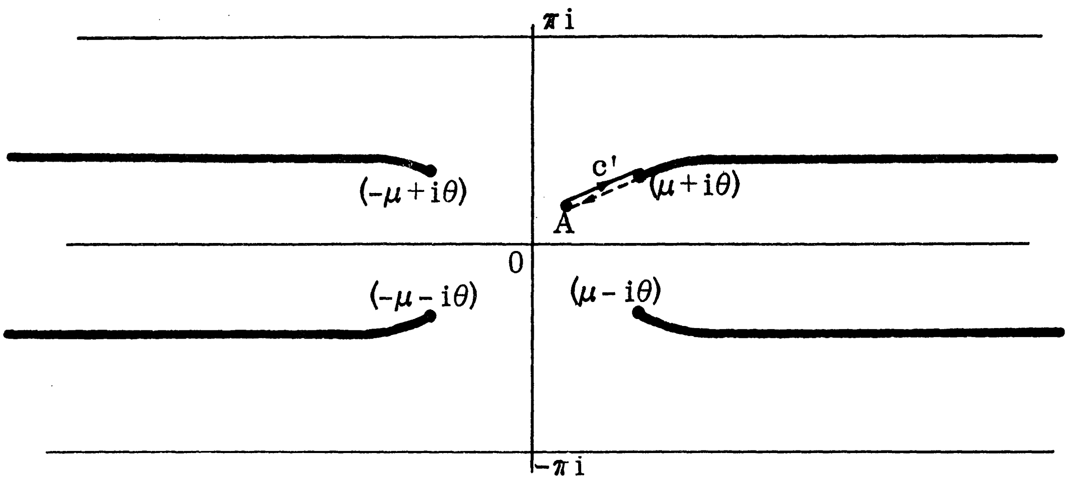


FIGURE 5.2.4: DEFORMED CONTOUR

With notation of (5.2.5) we see that

$$I = \int_c H_0^{(2)}(ka\rho)\, f(\alpha)\, d\alpha = \int_c \left[F_1(\rho^2)\log\rho^2 + F_2(\rho^2)\right] f(\alpha)\, d\alpha \tag{5.2.14}$$

or, since $F_2(\rho^2)f(\alpha)$ is analytic in α,

$$I = \int_c F_1(\rho^2) \log \rho^2 \; f(\alpha) \, d\alpha \quad . \tag{5.2.15}$$

With the deformed path, c', this becomes

$$I = \int_A^{\mu+i\theta} F_1(\rho^2) \, f(\alpha) \log_0 \rho^2 d\alpha + \int_{\mu+i\theta}^{A} F_1(\rho^2) \, f(\alpha) \log_{-1} \rho^2 \; d\alpha \tag{5.2.16}$$

which, with (5.2.8), is easily seen to be

$$I = 2\pi i \int_A^{\mu+i\theta} F_1(\rho^2) \, f(\alpha) \, d\alpha \quad . \tag{5.2.17}$$

Examination of the series representation of the Hankel function reveals that

$$F_1(\rho^2) = -\frac{i}{\pi} \; J\left[\mu, \theta, \alpha\right] \tag{5.2.18}$$

hence

$$I = 2 \int_A^{\mu+o\theta} J\left[\mu, \theta, \alpha\right] \, f(\alpha) \, d\alpha \, . \tag{5.2.19}$$

Since in its original form, I was a solution of the Helmholtz equation we have established that the integral (5.2.19) is a solution provided that $f(\alpha)$ is analytic in a neighborhood of the path of integration. In exactly the same way, and under the same conditions it can be shown that the expressions

$$\int_{A}^{\mu - i\theta} J\left[\mu, \theta, \alpha\right] f(\alpha)\, d\alpha$$

$$\int_{A}^{-\mu + i\theta} J\left[\mu, \theta, \alpha\right] f(\alpha)\, d\alpha \tag{5.2.20}$$

$$\int_{A}^{-\mu - i\theta} J\left[\mu, \theta, \alpha\right] f(\alpha)\, d\alpha$$

are also solutions of the Helmholtz equation. Since $\phi(\mu, \theta)$ and $\psi(\mu, \theta)$, (5.1.6) and (5.1.7), are linear combinations of these solutions, they are also solutions.

3. A Limiting Case: Laplace's Equation

Of some interest is the fact that if we choose $f(\alpha) = 1$ in (5.1.6) and (5.1.7) we obtain the wave functions

$$\phi_1(\mu, \theta) = \int_{-\mu + i\theta}^{\mu + i\theta} J\left[\mu, \theta, \alpha\right] \, d\alpha \tag{5.3.1}$$

and

$$\psi_1(\mu, \theta) = \int_{\mu - i\theta}^{\mu + i\theta} J\left[\mu, \theta, \alpha\right] \, d\alpha \,. \tag{5.3.2}$$

Since

$$\lim_{k \to 0} J\left[\mu, \theta, \alpha\right] = 1 \tag{5.3.3}$$

it is clear that

$$\lim_{k \to 0} \phi_1(\mu, \theta) = \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha = 2\mu \qquad (5.3.4)$$

and

$$\lim_{k \to 0} \psi_1(\mu, \theta) = \int_{\mu-i\theta}^{\mu+i\theta} d\alpha = 2i\theta \qquad (5.3.5)$$

hence $\phi_1(\mu, \theta)$ and $\psi_1(\mu, \theta)$ are the wave equation generalizations of the velocity potential and stream function respectively for an ideal fluid flowing through a slit of width 2a in a plane barrier (Ref. 20).

4. Elliptic Form of the Representation Theorems and Applications

Of importance is the fact that the fundamental solutions allow us to establish the elliptic form of the representation theorems as follows:

If $\phi(\mu, \theta)$ is an analytic solution of the Helmholtz equation in a simply connected region containing a segment of the line $\mu = 0$ and $\phi = 0$ on this segment then, in this region, ϕ has the integral representation

$$\phi(\mu, \theta) = \frac{1}{2} \cdot \int_{-\mu+i\theta}^{\mu+i\theta} J\left[\mu, \theta, \alpha\right] \left. \frac{\partial \phi}{\partial \nu}(\nu, -i\alpha) \right|_{\nu=0} d\alpha . \qquad (5.4.1)$$

Similarly, if $\psi(\mu, \theta)$ is an analytic solution of the Helmholtz equation in a simply connected region containing a segment of the line $\theta = 0$ and $\psi = 0$ on this segment, then, in this region, ψ has the integral representation

$$\psi(\mu, \theta) = \frac{1}{2i} \int_{\mu - i\theta}^{\mu + i\theta} J\left[\mu, \theta, \alpha\right] \left. \frac{\partial \psi(\alpha, \nu)}{\partial \nu} \right|_{\nu=0} d\alpha \, . \tag{5.4.2}$$

The proofs of these forms of the representation theorem are exactly the same as presented previously, consisting of demonstrating that the integral representation preserves the value of the function and its normal derivative on the line segment and since the function is given to be a solution and the integral has been shown to be a solution of the Helmholtz equation, the Cauchy-Kowalewsky theorem assures that they are identical.

As with the rectangular and parabolic versions of the representation theorem, (5.4.1) and (5.4.2) could be employed to find integral representations of products of appropriate cylinder functions, in this case elliptic cylinder or Mathieu functions. Here, in the integrand, special values of these Mathieu functions will appear. Since this line of investigation might prove too diverting from the main purpose of the present work, we shall confine our remarks to the representation of the line sources, $H_o^{(2)}(kR) - H_o^{(2)}(kR')$, in elliptic coordinates which is indeed pertinent to the strip problem. The same results obtained with the elliptic representation theorems can also be achieved by a transformation of the expressions already derived. Replacing $(x \pm iy)$ by the elliptic equivalent, $a\cosh(\mu \pm i\theta)$, and substituting $\alpha = a\cosh\alpha'$ in (3.6.14), we obtain two expressions due to the sign ambiguity in α' (i.e. $\alpha = a\cosh\alpha' = a\cosh(-\alpha')$). These are, dispensing with the primes,

$$H_o^{(2)}(kR) - H_o^{(2)}(kR')$$

$$= -ika\sinh\mu_o\sin\theta_o \int_{-\mu+i\theta}^{\mu+i\theta} \frac{J[\mu,\theta,\alpha]\; H_1^{(2)}[\mu_o,\theta_o,\alpha]\sinh\alpha\, d\alpha}{\sqrt{[\cosh(\mu_o+i\theta_o)-\cosh\alpha]\,[\cosh(\mu_o-i\theta_o)-\cosh\alpha]}} \tag{5.4.3}$$

and

$$H_o^{(2)}(kR) - H_o^{(2)}(kR')$$

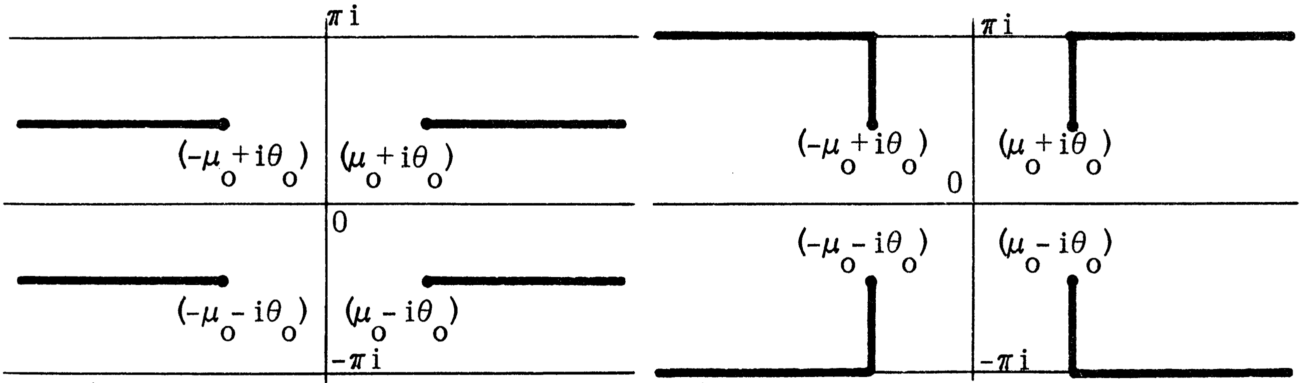
$$= -ika\sinh\mu_o\sin\theta_o \int_{\mu-i\theta}^{\mu+i\theta} \frac{J[\mu,\theta,\alpha]\; H_1^{(2)}[\mu_o,\theta_o,\alpha]\sinh\alpha\, d\alpha}{\sqrt{[\cosh(\mu_o+i\theta_o)-\cosh\alpha]\,[\cosh(\mu_o-i\theta_o)-\cosh\alpha]}} \tag{5.4.4}$$

where the same abbreviation is used for the argument of $H_1^{(2)}$ as for J_o. These expressions are exactly those obtained upon application of both elliptic forms of the representation theorem, (5.4.1) and (5.4.2).

The branch cuts of the integrands lie along the curves

$$\mathcal{I}m\;[\cosh(\mu_o+i\theta_o)-\cosh\alpha]\,[\cosh(\mu_o-i\theta_o)-\cosh\alpha] = 0 \tag{5.4.5}$$

and have been thoroughly discussed in section 5.2. With a slight change of notation, viz., $\mu \rightarrow \mu_o$, $\theta \rightarrow \theta_o$, Figure 5.2.2 illustrates the branch cuts. Since it is desirable to always be able to treat the path of integration as a straight line connecting the end points, lying entirely on one blade of the Riemann surface of the integrand, we alter the cuts somewhat as shown in Figure 5.4.1. The value of $\arg\left[\cosh(\mu_o + i\theta_o) - \cosh\alpha\right]\left[\cosh(\mu_o - i\theta_o) - \cosh\alpha\right]$



a) Branch Cuts for Formula 5.4.3 b) Branch Cuts for Formula 5.4.4

FIGURE 5.4.1: α-PLANE

on that portion of the curve $\cosh\alpha_1 \cos\alpha_2 = \cosh\mu_o \cos\theta_o$ where $\cos^2\alpha_2 < \cos^2\theta_o$ (i.e. along the dotted curves of Figure 5.2.2) is still utilized to number the sheets of the Riemann surface. The n^{th} sheet is that sheet where this argument is $2n\pi$. We then confine our attention to the 0^{th} sheet.

In the same way, the representation theorems or the double integral expression (3.6.14) lead to two expressions in this elliptic form as follows:

$$H_o^{(2)}(kR) - H_o^{(2)}(kR')$$

$$= -\frac{ka}{2} \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \; J\left[\mu, \theta, \alpha\right] J\left[\mu_o, \theta_o, \beta\right] \frac{H_1^{(2)}\left(\pm ka\left[\cosh\alpha - \cosh\beta\right]\right)}{\pm\left[\cosh\alpha - \cosh\beta\right]} \sinh\alpha \sinh\beta \qquad (5.4.6)$$

where we choose + if $\theta < \theta_o$ and - if $\theta > \theta_o$

and

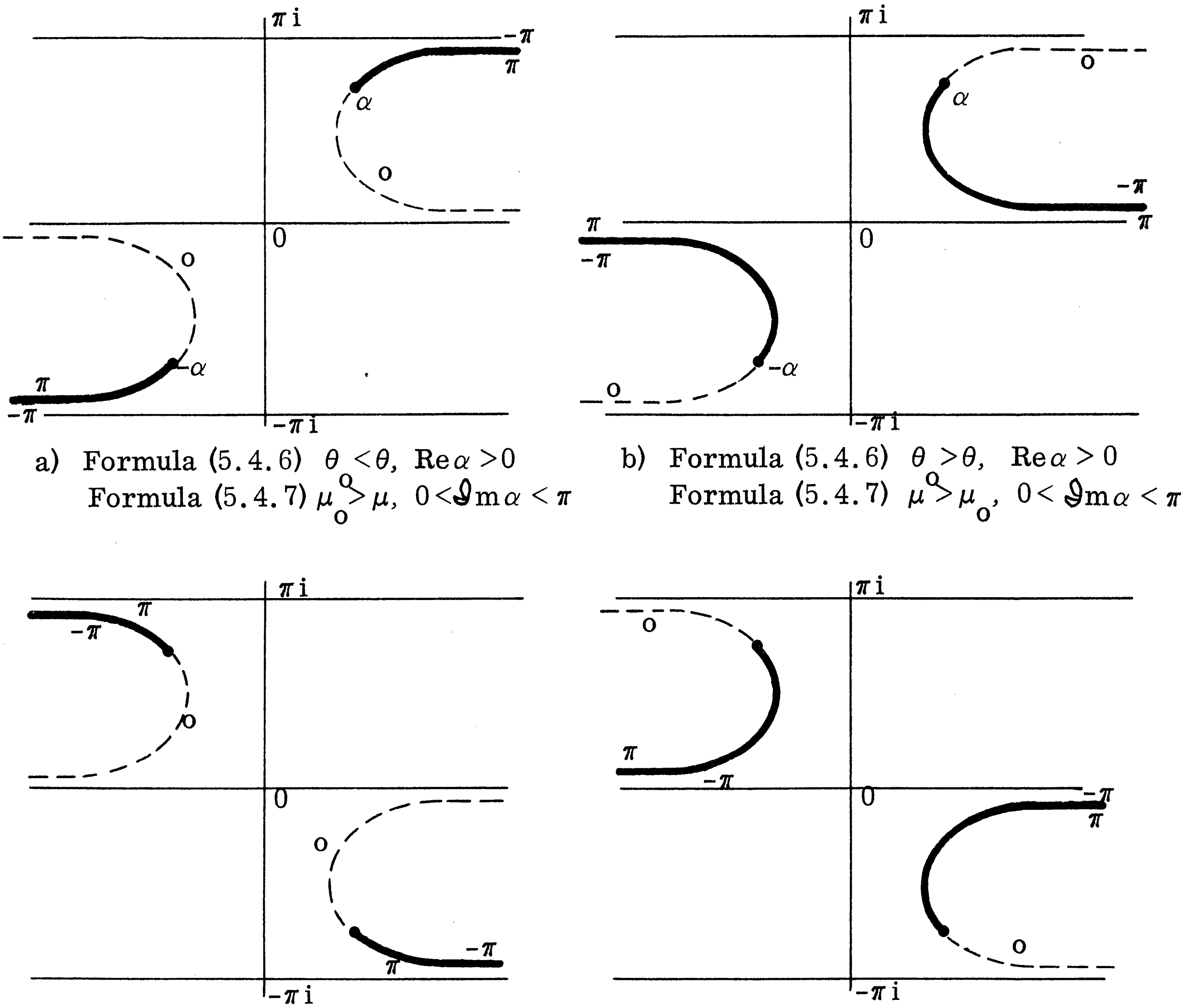
$$H_o^{(2)}(kR) - H_o^{(2)}(kR')$$

$$= -\frac{ka}{2} \int_{+\mu - i\theta}^{+\mu + i\theta} d\alpha \int_{\mu_o - i\theta_o}^{\mu_o + i\theta_o} d\beta \; J\left[\mu, \theta, \alpha\right] J\left[\mu_o, \theta_o, \beta\right] \frac{H_1^{(2)}(\pm ka \left[\cosh\alpha - \cosh\beta\right])}{\pm\left[\cosh\alpha - \cosh\beta\right]} \sinh\alpha \sinh\beta \qquad (5.4.7)$$

where we must use + if $\mu > \mu_o$ and - if $\mu < \mu_o$.

In each case the branch cuts of the integrand are chosen to lie along the negative real axis of the argument of the Hankel function. In the β-plane, this criterion leads to the cuts shown in Figure 5.4.2. Note that in formula (5.4.6), we can always restrict α so that $0 < \mathcal{I}m\,\alpha < \pi$ and in formula (5.4.7), $\mathrm{Re}\,\alpha > 0$ so while we may use the same figures for both formulas, they correspond to different α values as indicated. In the cases illustrated in Figures 5.4.2 c and d, the point α for formula (5.4.6) corresponds to the point $-\alpha$ for formula (5.4.7). The phase of the argument of the Hankel function on the branch cut and its extension, the positive real axis of the argument, or explicitly, the phase on the entire curve $\sinh\alpha_1 \sin\alpha_2 = \sinh\beta_1 \sin\beta_2$, is shown.

The same branch cut criterion in the α-plane leads to analogous results. Indeed, the same figures can be employed with some changes in notation. With these branch cuts we can deform the path of integration to be the straight line



a) Formula (5.4.6) $\theta_o < \theta$, Re$\alpha > 0$
Formula (5.4.7) $\mu_o > \mu$, $0 < \mathcal{I}m\,\alpha < \pi$

b) Formula (5.4.6) $\theta_o > \theta$, Re$\alpha > 0$
Formula (5.4.7) $\mu > \mu_o$, $0 < \mathcal{I}m\,\alpha < \pi$

c) Formula (5.4.6) $\theta_o < \theta$, Re$\alpha < 0$
Formula (5.4.7) $\mu_o > \mu$, $-\pi < \mathcal{I}m\,\alpha < 0$

d) Formula (5.4.6) $\theta_o > \theta$, Re$\alpha < 0$
Formula (5.4.7) $\mu > \mu_o$, $-\pi < \mathcal{I}m\,\alpha < 0$

FIGURE 5.4.2: BRANCH CUTS IN THE β-PLANE

connecting the end points in the case of formula (5.4.6) but this is not always true for formula (5.4.7). For example if $\mu_o > \mu$ and $\theta_o \to \pi$, the path might be as shown in Figure 5.4.3.

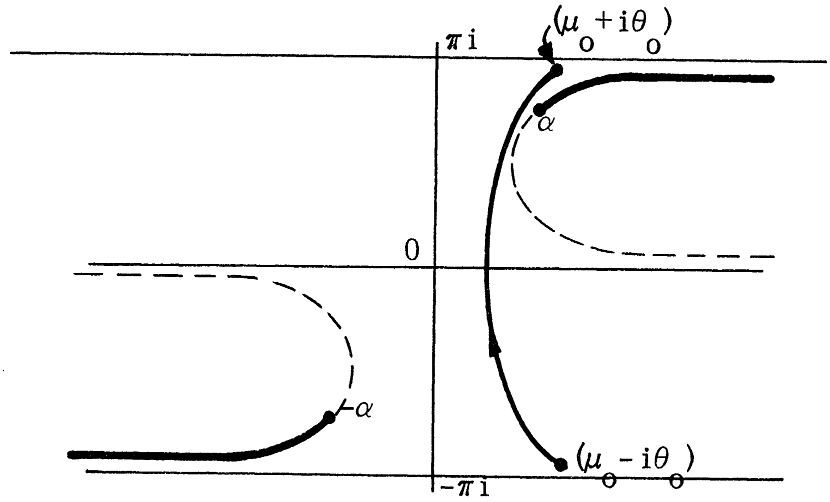


FIGURE 5.4.3: POSSIBLE β-CONTOUR - FORMULA (5.4.7)

To remedy this we alter the branch cuts somewhat for formula (5.4.7) as shown in Figure 5.4.4.

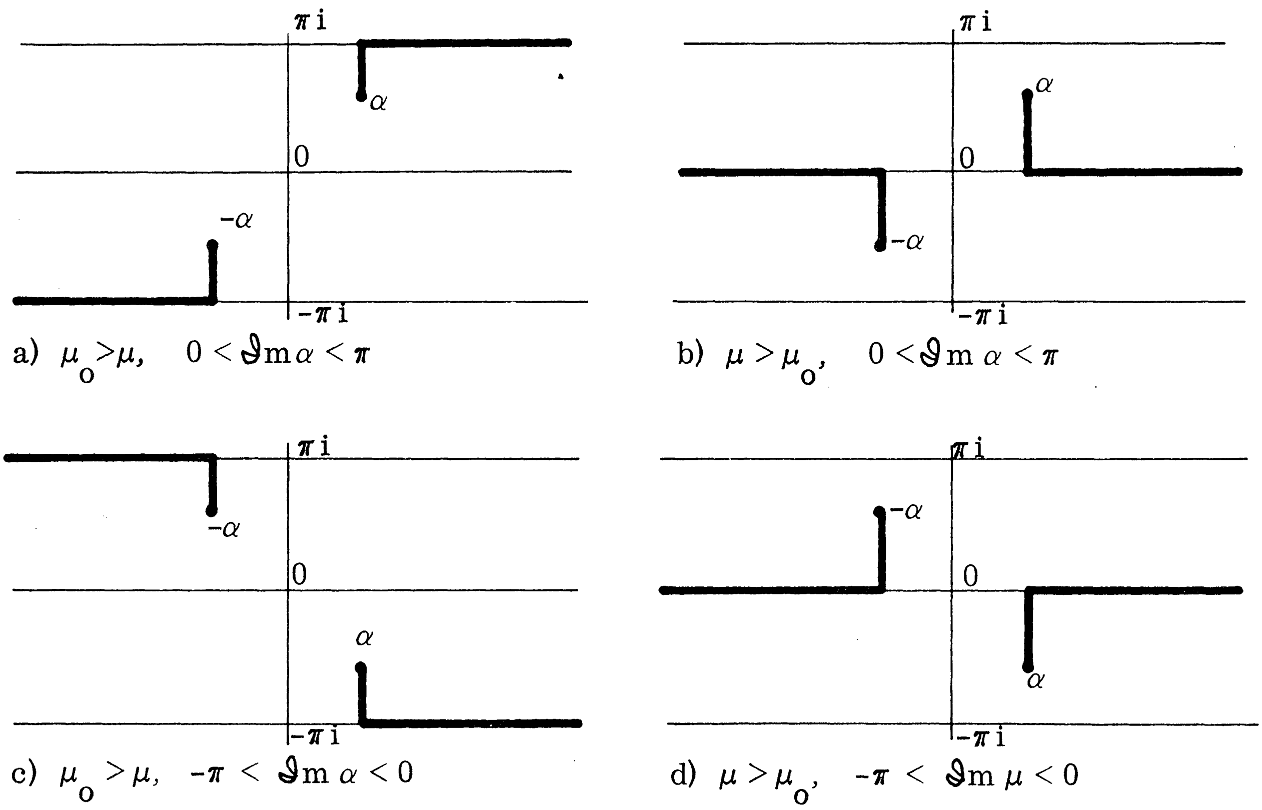


FIGURE 5.4.4: BRANCH CUTS IN THE β-PLANE FOR FORMULA (5.4.7)

5. Green's Function for the Strip

Corresponding to (3.6.20) we have the double integral representation for the Green's function which vanishes on a segment of the line $\mu = 0$,

$$G(\mu, \theta, \mu_o, \theta_o) = \frac{1}{4} \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \;\; J[\mu, \theta, \alpha]\, J[\mu_o, \theta_o, \beta] \frac{\partial^2}{\partial\sigma\partial s} G(\sigma, -i\alpha, s, -i\beta)\Big|_{\substack{\sigma=0\\ s=0}} . \quad (5.5.1)$$

Since the line $\mu = 0$ is itself a line segment in physical space, this expression, rather than the comparable representations in rectangular or parabolic coordinates, is clearly the most suitable form for representing the Green's function for a line segment. While it is true that the other forms, (3.6.20) and (4.3.1), can also represent such a Green's function, they are more naturally suited to the limiting cases when the line segment is infinite or semi-infinite in extent.

Since G must be analytic at $\mu = 0$, we limit the physical significance of G to the upper half plane by imposing the additional boundary condition that the normal derivative must vanish on the complement of the line segment $\mu = 0$, the two half lines $\theta = 0$ and $\theta = \pi$ (see Figure 5.5.1). On these lines, the

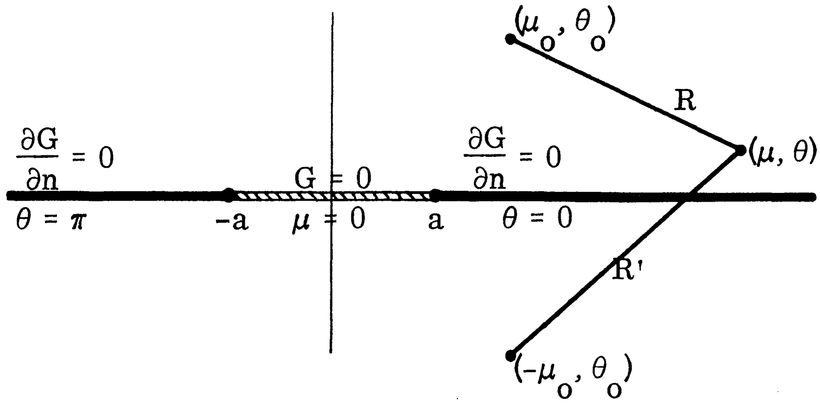


FIGURE 5.5.1: GEOMETRY FOR THE STRIP PROBLEM

normal derivative is proportional to $\frac{\partial}{\partial\theta}$. This problem, as shown in section 3.2, corresponds to the problems where <u>either</u> the function vanishes at $\mu = 0$ <u>or</u> the normal derivative vanishes at $\theta = 0$ and $\theta = \pi$.

We shall devote our attention to this problem. However, the complementary problem, where the function vanishes at $\theta = 0$ and $\theta = \pi$ and the normal derivative vanishes at $\mu = 0$, appears to present no new difficulties. It is expected that a completely analogous treatment, beginning with (5.1.7) which leads to the expression for the Green's function

$$G(\mu,\theta,\mu_o,\theta_o) = -\frac{1}{4}\int_{\mu-i\theta}^{\mu+i\theta} d\alpha \int_{\mu_o-i\theta_o}^{\mu_o+i\theta_o} d\beta \;\; J[\mu,\theta,\alpha]\; J[\mu_o,\theta_o,\beta]\frac{\partial^2}{\partial\sigma\partial s}G(\alpha,\sigma,\beta,s)\Big|_{\substack{\sigma=0\\ s=0}}, \tag{5.5.2}$$

would be successful.

If $\pi H_o^{(2)}(kR)\, e^{i(\omega t-\pi/2)}$ represents cylindrical waves* of length λ, frequency ω and velocity of propagation c diverging from a line (μ_o,θ_o) in the presence of a perfectly soft strip of width 2a set in a perfectly rigid screen, then the field at any point in space $(\mu \geqslant 0,\ 0 \leqslant \theta \leqslant \pi)$ can be expressed as

$$\phi(\mu,\theta,\mu_o,\theta_o)\; e^{i\omega t} \equiv \left\{\pi H_o^{(2)}(kR)\, e^{-\frac{\pi i}{2}} - \pi H_o^{(2)}(kR')\, e^{-\frac{\pi i}{2}} + \phi_D(\mu,\theta,\mu_o,\theta_o)\right\} e^{i\omega t} \tag{5.5.3}$$

where $k = \frac{2\pi}{\lambda}$ and

A. $(\nabla^2+k^2)\,\phi_D = 0, \qquad \mu > 0, \quad 0 < \theta < \pi$

B. $\phi(0,\theta,\mu_o,\theta_o) = 0, \quad \frac{\partial\phi}{\partial\theta}(\mu,\theta,\mu_o,\theta_o)\Big|_{\theta=0,\ \pi} = 0$

---

* This choice of source strength is consistent with that of Chapter 4. To obtain a "unit" source, we must multiply by $-1/4\pi$.

C. $\lim_{\mu \to \infty} \left(\frac{\partial \phi}{\partial \mu} + ika \cosh \mu \, \phi\right) = 0$

D. ϕ_D is everywhere finite.

We wish to find an explicit representation of ϕ, the total field, or equivalently, ϕ_D, the diffracted field. Introducing the diffracted field enables us to use the homogeneous Helmholtz equation, (5.1.5), and also enables us to employ the simple form of the edge condition, D. The boundary conditions, B, are more succinctly stated in terms of the total field while the radiation condition, C, applies equally well to ϕ_D. Implicit in condition A is the fact that ϕ_D must be at least twice differentiable everywhere except at the boundary.

From the representation (5.5.1), we know that the total field, $\phi(\mu, \theta, \mu_o, \theta_o)$, can be written in the form

$$\phi(\mu, \theta, \mu_o, \theta_o) = \int_{-\mu + i\theta}^{\mu + i\theta} d\alpha \int_{-\mu_o + i\theta_o}^{\mu_o + i\theta_o} d\beta \;\; J[\mu, \theta, \alpha] \; J[\mu_o, \theta_o, \beta] \; F(\alpha, \beta) \; . \tag{5.5.4}$$

Actually, (5.5.1) tells us more about $F(\alpha, \beta)$ but for the moment we will consider this expression for general $F(\alpha, \beta)$. It is clearly a solution of the Helmholtz equation (possibly the inhomogeneous equation depending on $F(\alpha, \beta)$) since it is of the form (5.1.6).

In order to satisfy the boundary conditions it is only necessary to require that $F(\alpha, \beta)$ be even and periodic in α. This follows since the function obviously vanishes when $\mu = 0$ and, upon differentiating (5.5.4), we find

$$\left.\frac{\partial \phi}{\partial \theta}\right|_{\theta=0} = i \int_{-\mu_o + i\theta_o}^{\mu_o + i\theta_o} d\beta \; J[\mu_o, \theta_o, \beta] \; \left\{ F(\mu, \beta) - F(-\mu, \beta) \right\} \tag{5.5.5}$$

and

$$\left.\frac{\partial \phi}{\partial \theta}\right|_{\theta=\pi} = i \int_{-\mu_o + i\theta_o}^{\mu_o + i\theta_o} d\beta \; J[\mu_o, \theta_o, \beta] \; \left\{ F(\mu + i\pi, \beta) - F(-\mu + i\pi, \beta) \right\} . \tag{5.5.6}$$

Clearly if $F(\mu, \beta) = F(-\mu, \beta)$, (5.5.5) vanishes and if $F(\mu + i\pi, \beta) = F(-\mu + i\pi, \beta)$, (5.5.6) vanishes.

Since the differential equation and the boundary conditions are satisfied for so general an $F(\alpha, \beta)$, the question remains as to how to completely specify $F(\alpha, \beta)$. A possible answer is to make use of all the information contained in (5.5.1) and try to solve this integral equation. The feasability of this procedure has not been seriously considered, however, since, as indicated at the close of chapter 4, the half plane solution provides us with the proper definition of $F(\alpha, \beta)$. That is, we use the same function $F[\rho, \gamma]$, (4.3.18), that provided the solution of the half plane problem except that now we must properly define ρ and γ in terms α and β, i.e. the "quasi" elliptic coordinates corresponding to the "quasi" parabolic coordinates of Chapter 4.

To do this we look to the corresponding expressions for the two Hankel functions. If we rewrite formula (4.2.7) using (4.3.17) to define ρ and γ, we obtain

$$H_0^{(2)}(kR) - H_0^{(2)}(kR') = -ik \int_{\xi - i\eta}^{\xi + i\eta} du \int_{\xi_0 - i\eta_0}^{\xi_0 + i\eta_0} dv \; J(\xi, \eta, u) \, J(\xi_0, \eta_0, v) \, H_1^{(2)}(k\rho)\sin\gamma. \qquad (5.5.7)$$

A comparable expression may be obtained from formula (5.4.6) if we define

$$\left.\begin{aligned} \rho\cos\gamma &= a(\cosh\alpha\cosh\beta - 1) \\ \rho\sin\gamma &= -i\,a\sinh\alpha\sinh\beta \end{aligned}\right\} . \qquad (5.5.8)$$

With this definition, (5.4.6) becomes

$$H_0^{(2)}(kR) - H_0^{(2)}(kR') = -\frac{ika}{2} \int_{-\mu + i\theta}^{\mu + i\theta} d\alpha \int_{-\mu_0 + i\theta_0}^{\mu_0 + i\theta_0} d\beta \; J[\mu, \theta, \alpha] \, J[\mu_0, \theta_0, \beta] \, H_1^{(2)}(k\rho)\sin\gamma. \qquad (5.5.9)$$

Now we maintain that, except for the factor $\frac{a}{2}$ which comparison of (5.5.7) and (5.5.9) shows is necessary to ensure that the source remains unchanged, the function $F[\rho, \gamma]$ given by (4.3.18) provides a solution of the strip problem. That is, the field ϕ is given by (5.5.4) where,

$$F(\alpha, \beta) = \frac{-a\cos\gamma}{\rho} \, e^{-ik\rho\cos\gamma} + ika\sin\gamma \int_0^{\gamma} \cos\omega \, e^{-ik\rho\cos\omega} \, d\omega \qquad (5.5.10)$$

and ρ and γ are defined by (5.5.8).

In this form it is not clear that $F(\alpha, \beta)$ has apparent singularities at $\alpha = \beta$. However, if we note that ρ and γ, written explicitly as functions of α and β, are

$$\left.\begin{aligned} \rho &= \pm(a\cosh\alpha - a\cosh\beta) \\ \gamma &= -i\log\frac{\cosh(\alpha+\beta) - 1}{\pm(\cosh\alpha - \cosh\beta)} \end{aligned}\right. , \quad \begin{aligned} &+ \text{ if } \theta < \theta_0 \\ &- \text{ if } \theta > \theta_0 \end{aligned} \qquad (5.5.11)$$

then the apparent singular behavior becomes more obvious. Hence the definition of ϕ is not complete until we specify the contours and branch cuts. This we do, in the by now familiar manner, by requiring that the contours in the α and β planes be deformable to straight lines connecting the end points and treating the branch cuts in exactly the same way as was done in the expression for the geometric optics field (5.4.6), as depicted in Figure 5.4.2.

Except for the brief mention of how the proper definition of ρ and γ in "quasi" elliptic coordinates was suggested by the comparable expressions for the two sources in elliptic and parabolic coordinates, we defer any further discussion of this point and shall occupy ourselves in the remainder of this chapter with demonstrating that the function ϕ defined above is indeed the solution we seek.

6. Analyticity of the Diffracted Field

It is a rather difficult task to show that the function ϕ defined above gives rise, through (5.5.3), to a diffracted field, ϕ_D, that is twice differentiable with respect to μ and θ throughout the region of physical significance, $\mu > 0$ and $0 < \theta < \pi$. Until we have shown that this is so, however, further discussion of other properties of the field would be suspect so we shall consider this point first. As will become evident, the same argument will suffice, not only to show that ϕ_D is twice differentiable, but that, in the range of μ and θ of interest which, it should be noted, excludes the boundaries and edges, ϕ_D is infinitely differentiable, i.e., ϕ_D is analytic in μ and θ.

First we explicitly exhibit the diffracted field. Utilizing the expressions for the geometric optics field and the total field, (5.5.3) yields

$$\phi_D(\mu, \theta, \mu_o, \theta_o) = \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \, J[\mu, \theta, \alpha] \, J[\mu_o, \theta_o, \beta] \; K(\alpha, \beta) \tag{5.6.1}$$

where $K(\alpha, \beta)$

$$= \frac{-\pi i k a}{2} \frac{\sinh\alpha \sinh\beta}{\pm[\cosh\alpha - \cosh\beta]} H_1^{(2)}(\pm ka[\cosh\alpha - \cosh\beta]) + \frac{1 - \cosh\alpha\cosh\beta}{(\cosh\alpha - \cosh\beta)^2} e^{+ika(1-\cosh\alpha\cosh\beta)}$$

$$+ \frac{ka \sinh\alpha \sinh\beta}{\pm[\cosh\alpha - \cosh\beta]} \int_0^{-i\log\frac{\cosh(\alpha+\beta)-1}{\pm[\cosh\alpha - \cosh\beta]}} \cos\omega \, e^{\pm ika(\cosh\beta - \cosh\alpha)\cos\omega} d\omega, \quad \begin{matrix} + \text{ if } \theta < \theta_o \\ - \text{ if } \theta > \theta_o \end{matrix} . \tag{5.6.2}$$

Since $J[\mu, \theta, \alpha]$ is analytic in μ, θ, and α, the kernel, $J[\mu, \theta, \alpha] \, J[\mu_o, \theta_o, \beta]$ offers no problem. Further, if $\theta \neq \theta_o$, the two contours of integration, which we have chosen as straight lines, have no common points, i.e. $\alpha \neq \pm\beta$. Hence $K(\alpha, \beta)$ is analytic in a neighborhood of the path of integration in both the α and β planes and, since the integral of an analytic function is an analytic function of its end points, considered separately, ϕ_D is analytic in μ and θ. However, when $\theta = \theta_o$, the two contours are coincident and, as (5.6.2) exhibits, $K(\alpha, \beta)$ presents three sorts of difficulty: poles, branch points and a sign change in its definition.

To show that the net effect of all this unpleasantness is nil and that in reality $K(\alpha, \beta)$ is analytic at $\alpha = \beta$ (and thus that ϕ_D is analytic when $\theta = \theta_o$) we shall separate $K(\alpha, \beta)$ into a component that is analytic and a component that is apparently singular and then show that the apparently singular part is really analytic. Note that by restricting θ_o so that $\theta_o \neq 0, \pi$, i.e., the source always lies off the screen, the paths of integration are such that we need only consider the singularity at $\alpha = +\beta$ since $\alpha \neq -\beta$ throughout the entire range of physical interest.

For convenience, we reintroduce the ρ, γ notation (5.5.11), obtaining

$$K(\alpha, \beta) = \frac{\pi ka}{2} \sin\gamma \, H_1^{(2)}(k\rho) - \frac{a\cos\gamma}{\rho} e^{-ik\rho\cos\omega} + ika\sin\gamma \int_o^{\gamma} \cos\omega \, e^{-ik\rho\cos\omega} \, d\omega \, . \tag{5.6.3}$$

To achieve the separation we seek, consider first the last term on the right hand side of (5.6.3). Expanding the exponential in the integrand in an infinite series and interchanging the order of summation and integration which is permissible, as long as $\alpha \neq \beta$, since the integral exists and the series is absolutely convergent, we find that

$$ika\sin\gamma \int_o^{\gamma} \cos\omega \, e^{-ik\rho\cos\omega} d\omega = ika\sin\gamma \sum_{n=o}^{\infty} \frac{(-ik\rho)^n}{n!} \int_o^{\gamma} \cos^{n+1}\omega \, d\omega \tag{5.6.4}$$

or, separating the even and odd powers,

$$ika \sin\gamma \int_0^{\gamma} \cos\omega \, e^{-ik\rho \cos\omega} d\omega$$

$$= ika \sin\gamma \sum_{n=0}^{\infty} \left\{ \frac{(-ik\rho)^{2n}}{(2n)!} \int_0^{\gamma} \cos^{2n+1}\omega \, d\omega + \frac{(-ik\rho)^{2n+1}}{(2n+1)!} \int_0^{\gamma} \cos^{2n+2}\omega \, d\omega \right\} . \quad (5.6.5)$$

Repeated integration by parts yields (see Ref. 10)

$$ika \sin\gamma \int_0^{\gamma} \cos\omega \, e^{-ik\rho \cos\omega} d\omega$$

$$= ika \sin\gamma \sum_{n=0}^{\infty} \left\{ \frac{(-ik\rho)^{2n}}{(2n)!} \sin\gamma \sum_{\nu=1}^{n+1} \frac{(2n;-2;\nu-1)}{(2n+1;-2;\nu)} \cos^{2n-2\nu+2}\gamma \right.$$

$$\left. + \frac{(-ik\rho)^{2n+1}}{(2n+1)!} \left[\sin\gamma \sum_{\nu=1}^{n+1} \frac{(2n+1;-2;\nu-1)}{(2n+2;-2;\nu)} \cos^{2n-2\nu+3}\gamma + \frac{(1;2;n+1)}{(2;2;n+1)} \gamma \right] \right\} \quad (5.6.6)$$

where

$$(p;r;s) = p(p+r)(p+2r) \; .... \; (p + [s-1] r), \quad s = 1, 2, 3, \ldots$$

$$= 1 \qquad , \quad s = 0 \; .$$

With this notational definition it is easy to show that

$$\frac{(1;2;n+1)}{(2;2;n+1)} = \frac{(2n+2)!}{2^{2n+2}[(n+1)!]^2}$$

hence (5. 6. 6) becomes

$$\text{ika} \sin\gamma \int_0^{\gamma} \cos\omega \, e^{-ik\rho\cos\omega} \, d\omega = \text{ika} \sin\gamma \sum_{n=o}^{\infty} \frac{\left(-\frac{ik\rho}{2}\right)^{2n+1}}{(n+1)!\, n!} \gamma$$

$$+ \text{ika} \sin^2\gamma \sum_{n=o}^{\infty} \sum_{\nu=1}^{n+1} \left\{ \frac{(2n+1;-2;\nu-1)(-ik\rho\cos\gamma)^{2n-2\nu+3}}{(2n+2;-2;\nu)\,(2n+1)!} \right.$$

$$\left. + \frac{(2n;-2;\nu-1)(-ik\rho\cos\gamma)^{2n-2\nu+2}}{(2n+1;-2;\nu)\,(2n)!} \right\} (-ik\rho)^{2\nu-2} . \quad (5.6.7)$$

The first term is of course a Bessel function and, since the general term in the double sum has the explicit α and β dependence

$$\sin^2\gamma(\rho\cos\gamma)^m \rho^{2\nu-2} = \frac{-\sinh^2\alpha \sinh^2\beta}{(\cosh\alpha - \cosh\beta)^2} (a\cosh\alpha\cosh\beta - a)^m (\cosh\alpha - \cosh\beta)^{2\nu-2}, \quad m \geqslant 0,$$

(5.6.8)

it is clear that only the $\nu = 1$ term is singular at $\alpha = \beta$, hence we may write

$$\text{ika} \sin\gamma \int_0^{\gamma} \cos\omega \, e^{-ik\rho\cos\omega} \, d\omega = \text{ka} \sin\gamma \, J_1(k\rho)\,\gamma + g_1$$

$$+ \text{ika} \sin^2\gamma \sum_{n=o}^{\infty} \left\{ \frac{(2n+1;-2;0)(-ik\rho\cos\gamma)^{2n+1}}{(2n+2;-2;1)(2n+1)!} + \frac{(2n;-2,0)(-ik\rho\cos\gamma)^{2n}}{(2n+1;-2;1)(2n)!} \right\}, \quad (5.6.9)$$

where g_1 is analytic at $\alpha = \beta$. This simplifies to

$$ika \sin\gamma \int_0^{\gamma} \cos\omega \, e^{-ik\rho\cos\omega} d\omega = ka \sin\gamma \, J_1(k\rho)\gamma + g_1 - \frac{a\sin^2\gamma}{\rho\cos\gamma} e^{-ik\rho\cos\gamma} + \frac{a\sin^2\gamma}{\rho\cos\gamma} . \tag{5.6.10}$$

Substituting this result in (5. 6. 3) we obtain

$$K(\alpha,\beta) = \frac{\pi ka\sin\gamma}{2} H_1^{(2)}(k\rho) + ka\sin\gamma J_1(k\rho)\gamma + g_1 - \frac{ae^{-ik\rho\cos\gamma}}{\rho\cos\gamma} + \frac{a\sin^2\gamma}{\rho\cos\gamma} . \tag{5.6.11}$$

The singular part of the Hankel function term can also be separated out as follows:

$$\sin\gamma \, H_1^{(2)}(k\rho) = \rho\sin\gamma \, g_2 - \frac{2i}{\pi} \sin\gamma \, J_1(k\rho) \log\rho + \frac{2i\sin\gamma}{\pi k\rho} \tag{5.6.12}$$

where g_2 is analytic in ρ^2 and hence in α and β. Substituting (5. 6. 12) in (5. 6. 11) we obtain

$$K(\alpha,\beta) = \frac{\pi ka}{2} \rho\sin\gamma \, g_2 + g_1 - \frac{ae^{-ik\rho\cos\gamma}}{\rho\cos\gamma} + \frac{ia\sin\gamma}{\rho\cos\gamma} e^{-i\gamma} + ka\sin\gamma \, J_1(k\rho)\left[\gamma - i\log\rho\right] . \tag{5.6.13}$$

If we now insert the explicit α and β dependence, given by (5. 5. 11), $K(\alpha,\beta)$ becomes

$$K(\alpha,\beta) = \frac{-\pi ka^2}{2} \sinh\alpha\sinh\beta \, g_2 + g_1 - \frac{e^{-ika(\cosh\alpha\cosh\beta - 1)}}{\cosh\alpha\cosh\beta - 1} + \frac{\sinh\alpha\sinh\beta}{\left[\cosh\alpha\cosh\beta - 1\right]\left[\cosh(\alpha+\beta) - 1\right]}$$

$$- \frac{ika\sinh\alpha\sinh\beta \, J_1(\pm k\left[\cosh\alpha\cosh\beta\right])}{\pm\left[\cosh\alpha - \cosh\beta\right]} \log\left[a\cosh(\alpha+\beta) - a\right] \tag{5.6.14}$$

which, since $\frac{J_1(k\rho)}{\rho}$ is analytic in ρ^2 and β has been bounded away from 0, is analytic at $\alpha = \beta$.

7. Properties of the Solution

With this sticky question of continuity resolved we now proceed to examine the other vital properties of the field. It is relatively simple to see that $\phi_D(\mu, \theta, \mu_o, \theta_o)$ given by (5.6.1) is a solution of the homogeneous Helmholtz equation since it belongs to the general class of such solutions, given by (5.1.6). Hence condition A of section 5.5 is satisfied. Further, the edge condition is also satisfied, since ϕ_D is expressed everywhere in the finite plane, including the boundary, as an integral over a finite path of a bounded function.

That the boundary conditions are satisfied is also demonstrable without undue difficulty. The function, ϕ, certainly vanishes when $\mu = 0$ since one of the contours is of zero length. The normal derivative conditions are only slightly more troublesome. From (5.5.5) we see that $\left.\frac{\partial \phi}{\partial \theta}\right|_{\theta=0} = 0$ if $F(\mu, \beta) = F(-\mu, \beta)$. To exhibit $F(\mu, \beta)$ and $F(-\mu, \beta)$ explicitly, we employ formula (5.5.10) for $F[\rho, \gamma]$ and use (5.5.11) to define ρ and γ in terms of α and β. We have chosen the + sign in these definitions since $\theta = 0$ and therefore is always less than θ_o. Hence

$$F(\mu, \beta) = \frac{1 - \cosh\mu \cosh\beta}{(\cosh\mu - \cosh\beta)^2}\, e^{ika(1 - \cosh\mu\cosh\beta)}$$

$$+ \frac{ka \sinh\mu \sinh\beta}{(\cosh\mu - \cosh\beta)} \int_0^{-i \log \frac{\cosh(\mu+\beta) - 1}{\cosh\mu - \cosh\beta}} \cos\omega\, e^{ika(\cosh\beta - \cosh\mu)\cos\omega}\, d\omega \qquad (5.7.1)$$

and

$$F(-\mu,\beta) = \frac{1-\cosh\mu\cosh\beta}{(\cosh\mu-\cosh\beta)^2}\, e^{ika(1-\cosh\mu\cosh\beta)}$$

$$-\frac{ka\sinh\mu\sinh\beta}{\cosh\mu-\cosh\beta}\int_0^{-i\log\frac{\cosh(\beta-\mu)-1}{\cosh\mu-\cosh\beta}} \cos\omega\, e^{ika(\cosh\beta-\cosh\mu)\cos\omega}\, d\omega. \tag{5.7.2}$$

But the identity

$$[\cosh(\beta-\mu)-1]\,[\cosh(\beta+\mu)-1] = (\cosh\mu-\cosh\beta)^2 \tag{5.7.3}$$

enables us to see that

$$\log\frac{\cosh(\beta-\mu)-1}{\cosh\mu-\cosh\beta} = -\log\frac{\cosh(\beta+\mu)-1}{\cosh\mu-\cosh\beta}\ . \tag{5.7.4}$$

With this relation, together with the fact that the integrand in (5.7.2) is an even function of ω, it follows that $F(\mu,\beta) = F(-\mu,\beta)$.

The same kind of procedure establishes the condition on the normal derivative when $\theta = \pi$, which vanishes if $F(\mu+i\pi,\beta) = F(-\mu+i\pi,\beta)$, see (5.5.6). Since $\theta = \pi$, we must chose the - sign in the definitions of ρ and γ and we find

$$F(\mu+i\pi,\beta) = \frac{\cosh\mu\cosh\beta+1}{(\cosh\mu+\cosh\beta)^2}\, e^{ika(\cosh\mu\cosh\beta+1)}$$

$$-\frac{ka\sinh\mu\sinh\beta}{\cosh\mu+\cosh\beta}\int_0^{-i\log\frac{-\cosh(\beta+\mu)-1}{\cosh\mu+\cosh\beta}} \cos\omega\, e^{-ika(\cosh\mu+\cosh\beta)\cos\omega}\, d\omega \tag{5.7.5}$$

and

$$F(-\mu+i\pi,\beta) = \frac{\cosh\mu\cosh\beta+1}{(\cosh\mu+\cosh\beta)^2}\, e^{ika(\cosh\mu\cosh\beta+1)}$$

$$+\frac{ka\sinh\mu\sinh\beta}{\cosh\mu+\cosh\beta}\int_0^{-i\log\frac{-\cosh(\beta-\mu)-1}{\cosh\mu+\cosh\beta}} \cos\omega\, e^{-ika(\cosh\mu+\cosh\beta)\cos\omega}\, d\omega . \tag{5.7.6}$$

As before, the identity

$$\left[\cosh(\mu+\beta)+1\right]\left[\cosh(\mu-\beta)+1\right] = (\cosh\mu+\cosh\beta)^2 \tag{5.7.7}$$

enables us to see that

$$\log\frac{-\cosh(\beta-\mu)-1}{\cosh\mu+\cosh\beta} = -\log\frac{-\cosh(\mu+\beta)-1}{\cosh\mu+\cosh\beta} \tag{5.7.8}$$

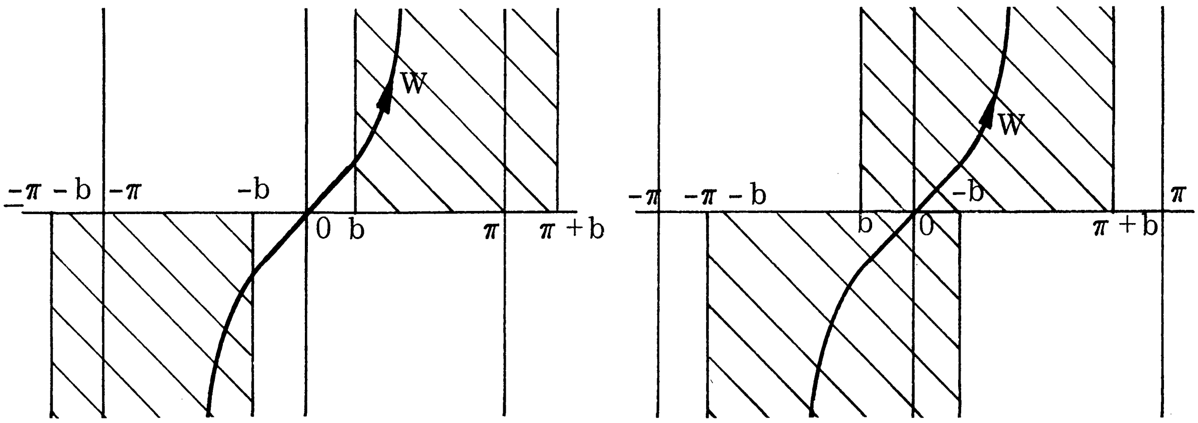
and this fact, together with the even-ness of the integrand, implies that $F(\mu+i\pi,\beta) = F(-\mu+i\pi,\beta)$.

In order to complete the proof that our function fulfills all the necessary conditions, A-D of section 5.5, we must discuss the behavior of the field at large distances from both the source and the strip. We shall bring this discussion to the point where compliance with the radiation condition appears assured but shall not present a complete asymptotic evaluation.

First note that the Sommerfeld representation for the Hankel function presented in Chapter 4, formula (4.3.5), is not limited to real values of the argument. In particular, if $\rho = |\rho|\, e^{ib}$, $-\pi < b < \pi$, we may write

$$H_1^{(2)}(k\rho) = \frac{i}{\pi} \int_W \cos\omega \, e^{-ik\rho\cos\omega} \, d\omega \tag{5.7.9}$$

where the contour is shown in Figure 5.7.1.



a) $0 < b = \arg\rho < \pi$ b) $-\pi < b = \arg\rho < 0$

FIGURE 5.7.1: SOMMERFELD CONTOURS

In order to assure that the integral converges, the contour must terminate in a shaded region, as can be seen from the following, if $\omega = \omega_1 + i\omega_2$, then

$$e^{-ik\rho\cos\omega} = e^{\frac{k|\rho|}{2}\left\{e^{\omega_2}\sin(b-\omega_1) + e^{-\omega_2}\sin(b+\omega_1) - ie^{\omega_2}\cos(b-\omega_1) - ie^{-\omega_2}\cos(b+\omega_1)\right\}} \, . \tag{5.7.10}$$

In this form we see that $e^{-ik\rho\cos\omega}$ vanishes when $\omega_2 \rightarrow +\infty$ if we require that $\sin(b-\omega_1) < 0$ or $b < \omega_1 < b+\pi$ and when $\omega_2 \rightarrow -\infty$ if we require that $\sin(b+\omega_1) < 0$ or $-\pi - b < \omega_1 < -b$. These regions of convergence are the shaded regions of Figure 5.7.1. The contour may be deformed as long as the tails lie in the shaded regions. In particular the contour may be chosen as symmetric about the origin and passing through the complex points γ and $-\gamma$ as shown in Figure 5.7.2. There is of course, a comparable picture

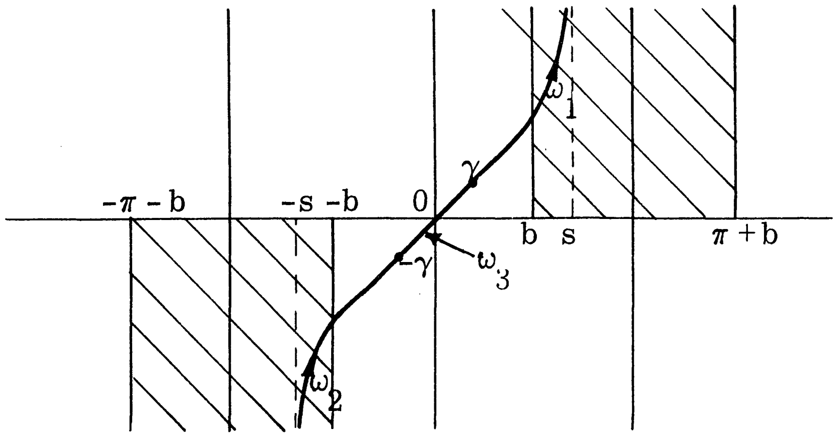


FIGURE 5.7.2: SPECIFIC CONTOUR

if $\arg \rho < 0$ and, despite the figure, the points $\pm\gamma$ may be either in or out of the shaded regions. The complete contour is $\omega_1 + \omega_2 + \omega_3$ but by dividing the contour in this way it is easily seen that

$$\int_{\omega_3} \cos\omega \; e^{-ik\rho\cos\omega} d\omega = -\, i\pi H_1^{(2)}(k\rho) - \int_{\omega_1+\omega_2} \cos\omega \; e^{-ik\rho\cos\omega} d\omega \; . \qquad (5.7.11)$$

But

$$\int_{\omega_1+\omega_2} \cos\omega \; e^{-ik\rho\cos\omega} d\omega = \int_{-i\infty - s}^{-\gamma} \cos\omega \; e^{-ik\rho\cos\omega} d\omega + \int_{\gamma}^{i\infty + s} \cos\omega \; e^{-ik\rho\cos\omega} d\omega \qquad (5.7.12)$$

and since the integrand vanishes at $\pm(i\infty + s)$, integration by parts yields

$$\int_{\omega_1+\omega_2} \cos\omega\, e^{-ik\rho\cos\omega}\, d\omega = \frac{-2\cos\gamma}{ik\rho\sin\gamma}\, e^{-ik\rho\cos\gamma} + \frac{1}{ik\rho}\int_{\omega_1+\omega_2} \frac{e^{-ik\rho\cos\omega}}{\sin^2\omega}\, d\omega\,. \tag{5.7.13}$$

If we substitute this result in (5.7.11) and make use of the fact that, because the integrand is even in ω,

$$\int_{\omega_3} \cos\omega\, e^{-ik\rho\cos\omega}\, d\omega = 2\int_{0}^{\gamma} \cos\omega\, e^{-ik\rho\cos\omega}\, d\omega \tag{5.7.14}$$

then we obtain

$$\int_{0}^{\gamma} \cos\omega\, e^{-ik\rho\cos\omega}\, d\omega = \frac{-i\pi}{2} H_1^{(2)}(k\rho) + \frac{\cos\gamma\, e^{-ik\rho\cos\gamma}}{ik\rho\sin\gamma} - \frac{1}{2ik\rho}\int_{\omega_1+\omega_2} \frac{e^{-ik\rho\cos\omega}}{\sin^2\omega}\, d\omega. \tag{5.7.15}$$

If we now express ρ and γ in terms of α and β using the definition (5.5.11) and substitute (5.7.15) in the expression for the total field given by formulas (5.5.4) and (5.5.10) we find that the field can be written as

$$\phi(\mu, \theta, \mu_o, \theta_o) = \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \;\; J[\mu, \theta, \alpha]\, J[\mu_o, \theta_o, \beta]\, \cdot$$

$$\left\{ \frac{\pi ka\sin\gamma}{2} H_1^{(2)}(k\rho) - \frac{a\sin\gamma}{2\rho}\int_{\omega_1+\omega_2} \frac{e^{-ik\rho\cos\omega}}{\sin^2\omega}\, d\omega \right\}. \tag{5.7.16}$$

With the use of formula (5.5.9), we could rewrite (5.7.16) as

$$\phi(\mu, \theta, \mu_o, \theta_o) = \pi i \left[H_o^{(2)}(kR) - H_o^{(2)}(kR') \right]$$

$$- \frac{a}{2} \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \int_{\omega_1+\omega_2} d\omega \quad J[\mu, \theta, \alpha] \quad J[\mu_o, \theta_o, \beta] \; \frac{\sin\gamma \, e^{-ik\rho\cos\omega}}{\rho \sin^2\omega} \tag{5.7.17}$$

but to say that the first term represents the geometric optics field and the second represents the diffracted field, tempting as it may seem, is not true. This is due to the fact that the first term in (5.7.17) differs in sign from the true geometric optics field and the second term, the integral, still has a singularity at the source, i.e. when $\mu = \mu_o$ and $\theta = \theta_o$. Thus, when discussing the diffracted field in section 5.6, we did not make use of (5.7.17). Our interest now, however, is in asymptotic behavior and in this regard it is quite informative to have an integral form with known asymptotic properties for comparison.

Specifically, if we integrate the Sommerfeld representation of the Hankel function by parts obtaining

$$H_1^{(2)}(k\rho) = \frac{1}{\pi k\rho} \int_{\omega_1+\omega_2+\omega_3} \frac{e^{-ik\rho\cos\omega}}{\sin^2\omega} \, d\omega \tag{5.7.18}$$

and substitute this in formula (5.7.16) we obtain

$$\phi(\mu,\theta,\mu_o,\theta_o) = \frac{a}{2}\int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \int_{\omega_1+\omega_2+\omega_3} d\omega \; J[\mu,\theta,\alpha]\, J[\mu_o,\theta_o,\beta]\, \frac{\sin\gamma\, e^{-ik\rho\cos\omega}}{\rho\sin^2\omega}$$

$$- \frac{a}{2}\int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \int_{\omega_1+\omega_2} d\omega \; J[\mu,\theta,\alpha]\, J[\mu_o,\theta_o,\beta]\, \frac{\sin\gamma\, e^{-ik\rho\cos\omega}}{\rho\sin^2\omega} \tag{5.7.19}$$

where we know that the first integral has the correct behavior as $\mu \to \infty$ since it is just a representation of the two Hankel functions. (Note that the ω contours must avoid $\omega = 0$.) The only difference in the two integrals is in the ω-contour; in one case it is "complete" and the other "incomplete". Of course this difference is not trivial since it accounts for the diffraction effect but it seems reasonable to assume that it would not drastically alter the asymptotic behavior since the integrands are exactly the same. Certainly by choosing the "incomplete" ω-contour no new saddle points are introduced hence it is expected that a direct demonstration that the first integral in (5.7.19) fulfills the radiation condition would suffice to show that the second does also.

8. Some Further Properties

We end this chapter with a brief discussion of some further properties of the field which the form of the solution makes easily demonstrable.

First we consider the behavior in the static, $k \to 0$, limit which can be found with ease. Since

$$\lim_{k \to 0} J[\mu, \theta, \alpha] = 1 \qquad (5.8.1)$$

the expression for the total field, (5.5.4) together with (5.5.10), becomes

$$\phi_s = \lim_{k \to 0} \phi(\mu, \theta, \mu_o, \theta_o) = - \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \; \frac{a \cos \gamma}{\rho} \qquad (5.8.2)$$

or, with (5.5.11), explicitly

$$\phi_s = \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \; \frac{1 - \cosh\alpha \cosh\beta}{(\cosh\alpha - \cosh\beta)^2} \; . \qquad (5.8.3)$$

Since

$$\frac{1 - \cosh\alpha \cosh\beta}{(\cosh\alpha - \cosh\beta)^2} = \frac{\partial^2}{\partial\alpha\partial\beta} \left\{ \frac{1 - e^{\alpha+\beta}}{1 - e^{-\alpha+\beta}} \right\} \qquad (5.8.4)$$

(5.8.3) can be integrated explicitly yielding

$$\phi_s = \log \frac{[\cosh(\mu+\mu_o) - \cos(\theta-\theta_o)]\,[\cosh(\mu+\mu_o) - \cos(\theta+\theta_o)]}{[\cosh(\mu-\mu_o) - \cos(\theta-\theta_o)]\,[\cosh(\mu-\mu_o) - \cos(\theta+\theta_o)]} \qquad (5.8.5)$$

which, as can be verified, is a solution of Laplace's equation, has a logarithmic singularity corresponding to a source at (μ_o, θ_o), and satisfies the boundary conditions $\phi_s\Big|_{\mu=0} = 0, \quad \dfrac{\partial\phi_s}{\partial\theta}\Big|_{\theta=0,\pi} = 0.$

Another feature of the form of ϕ or ϕ_D is that the symmetry in source and field points, the reciprocity relation, is readily apparent. With formulas (5.6.1) and (5.6.2) we see that the diffracted field can be written

$$\phi_D(\mu,\theta,\mu_o,\theta_o) = \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \; J[\mu,\theta,\alpha]\, J[\mu_o,\theta_o,\beta] \begin{cases} K_1(\alpha,\beta) \text{ if } \theta<\theta_o \\ K_2(\alpha,\beta) \text{ if } \theta>\theta_o \end{cases} \tag{5.8.6}$$

and further that

$$K_1(\alpha,\beta) = K_2(\beta,\alpha)$$

$$K_2(\alpha,\beta) = K_1(\beta,\alpha) \; . \tag{5.8.7}$$

Replacing μ with μ_o and θ with θ_o in (5.8.6) we obtain

$$\phi_D(\mu_o,\theta_o,\mu,\theta) = \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\alpha \int_{-\mu+i\theta}^{\mu+i\theta} d\beta \; J[\mu_o,\theta_o,\alpha]\, J[\mu,\theta,\beta] \begin{cases} K_1(\alpha,\beta) \text{ if } \theta_o<\theta \\ K_2(\alpha,\beta) \text{ if } \theta_o>\theta \end{cases} \tag{5.8.8}$$

Changing the order of integration, which is permissible since there are no singularities in the integrand, and renaming the integration variables yields

$$\phi_D(\mu_o,\theta_o,\mu,\theta) = \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \; J[\mu,\theta,\alpha]\, J[\mu_o,\theta_o,\beta] \begin{cases} K_1(\beta,\alpha) \text{ if } \theta_o<\theta \\ K_2(\beta,\alpha) \text{ if } \theta_o>\theta. \end{cases} \tag{5.8.9}$$

With (5.8.7) this is seen to be identical with (5.8.6) hence

$$\phi_D(\mu_o,\theta_o,\mu,\theta) = \phi_D(\mu,\theta,\mu_o,\theta_o). \tag{5.8.10}$$

Chapter 6

CONCLUSION

In this chapter we shall briefly review what has been presented and comment on some consequences of the results. A possible generalization of the work is also discussed.

In summary, we derived a class of solutions of the two dimensional Helmholtz equation which satisfied particularly simple boundary conditions. These were used to construct a double integral equation for the Green's function for a line segment. In particular, when the line segment was infinite we obtained a double integral representation of the difference of two Hankel functions (line sources), $H_o^{(2)}(kR) - H_o^{(2)}(kR')$. For the case when the line was semi-infinite we used the known result for the Green's function to obtain, via the integral equation, an integral representation for this same Green's function. We then asserted that this integral form, when properly interpreted, led to the corresponding form for the Green's function for a finite segment and this assertion was verified.

Essential to the argument is the assertion that the known solution of the half plane problem contains, in a sense, the solution of the strip problem. That this assertion proved valid, i.e., that the mathematical description of diffraction by one edge could be transformed so as to also describe diffraction by two edges, indicates that introducing a second diffracting edge is not an essential complication in the sense that scattering by a half plane is an essentially more complicated physical phenomenon than scattering by a full plane. Whether this means that there is no interaction between the diffracting edges or just that this interaction is describable in exactly the same way as is the half plane (single edge) phenomenon is still to be decided.

If we consider the manner in which we constructed the solution, however, a much more far reaching question is raised. Recall that we cast the solution of the half plane problem into a double integral form in which the dependence on the coordinates was contained in the limits of integration and the kernel. The other factor in the integrand was hypothesized to be the essence of the diffraction process, invariant under change of coordinate systems from parabolic to elliptic hence the corresponding diffraction problem in elliptic coordinates was essentially solved. Actually this "universal" diffraction kernel, to adopt a somewhat grandiose but perhaps not unwarranted nomenclature, was not completely invariant but by writing the corresponding expressions for $H_o^{(2)}(kR) - H_o^{(2)}(kR')$ in the two coordinate systems we were able to decide how it had to be altered. Now, of course, the question occurs as to whether it is possible to systemitize this procedure for transforming the solution of one diffraction problem into the solution for another.

Actually, we are able to induce a general expression which is valid at least for the three special cases we now have at our disposal, namely scattering by a strip, by a half plane, and by a full plane (a limiting case in which there is really no diffraction). Specifically, if we let $x+iy = f(u+iv)$ denote any of the three coordinate transformations

$$(1)\quad x+iy = u+iv$$

$$(2)\quad x+iy = (u+iv)^2$$

$$(3)\quad x+iy = a\cosh(u+iv)$$

then the functions

$$\phi_1(u,v,u_o,v_o) = \frac{i}{2}\int_{u-iv}^{u+iv} d\alpha \int_{u_o-iv_o}^{u_o+iv_o} d\beta \int_{-i\log\frac{f'(\alpha+\beta)}{\mp[f'(\alpha)-f'(\beta)]}}^{+i\log\frac{f'(\alpha+\beta)}{\mp[f'(\alpha)-f'(\beta)]}} d\omega \cdot$$

$$J\left\{u,\overset{+}{v},\alpha\right\} \; J\left\{u_o,\overset{+}{v}_o,\beta\right\} \; \frac{f'(\alpha)f'(\beta)\,e^{\pm ik[f(\beta)-f(\alpha)]\cos\omega}}{[f(\alpha)-f(\beta)]^2\sin^2\omega} \tag{6.1.1}$$

and

$$\phi_2(u,v,u_o,v_o) = \frac{i}{2}\int_{-u+iv}^{u+iv} d\alpha \int_{-u_o+iv_o}^{u_o+iv_o} d\beta \int_{-i\log\frac{f'(\alpha+\beta)}{\pm[f'(\alpha)-f'(\beta)]}}^{+i\log\frac{f'(\alpha+\beta)}{\pm[f'(\alpha)-f'(\beta)]}} d\omega \cdot$$

$$J\left\{u,\bar{v},\alpha\right\} \; J\left\{u_o,\bar{v}_o,\beta\right\} \; \frac{f'(\alpha)\,f'(\beta)e^{\pm ik[f(\beta)-f(\alpha)]\cos\omega}}{[f(\alpha)-f(\beta)]^2\sin^2\omega} \tag{6.1.2}$$

where

$$J\left\{u, \overset{+}{v}, \alpha\right\} = J_o\left(k\sqrt{[f(u+iv)-f(\alpha)]\,[f(u-iv)-f(\alpha)]}\right) \tag{6.1.3}$$

and

$$J\left\{u, \bar{v}, \alpha\right\} = J_o\left(k\sqrt{[f(u+iv)-f(\alpha)]\,[f(u-iv)-f(-\alpha)]}\right) \tag{6.1.4}$$

represent the total field due to a line source at u_o, v_o with the boundary conditions, in one case, that the function vanish on a line segment and the normal derivative vanish on its complement and in the other case that the normal derivative vanish on the segment and the function vanish on the complement. Since we have chosen the time dependence so that the Hankel function of the second kind represents the source, the sign ambiguity in these formulas is removed. In ϕ_1, the choice must be + if $u > u_o$ and - if $u < u_o$ while in ϕ_2 it must be + if $v < v_o$ and - if $v > v_o$. If we had chosen the opposite time dependence we would have to reverse the choices.

Thus when f is the identity transformation (1) u = x, v = y, and $f(\alpha) = \alpha$. Since the limits in the ω-integral become infinite the ω-contour must be the Sommerfeld contour if ϕ_1 is to be defined. Hence formula (6.1.1) becomes

$$\phi_1(x, y, x_o, y_o) = \frac{i}{2}\int_{x-iy}^{x+iy} d\alpha \int_{x_o-iy_o}^{x_o+iy_o} d\beta \int_{W} d\omega \quad \cdot$$

$$J_o\left(k\sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) J_o\left(k\sqrt{(x_o+iy_o-\beta)(x_o-iy_o-\beta)}\right) \frac{e^{\pm ik(\beta-\alpha)\cos\omega}}{(\alpha-\beta)^2\sin^2\omega} \tag{6.1.5}$$

where we use + if $x > x_o$ and - if $x < x_o$.

With the representation of the Hankel function, (5.7.18), this becomes

$$\phi_1(x, y, x_o, y_o) = \frac{ik\pi}{2} \int_{x-iy}^{x+iy} d\alpha \int_{x_o - iy_o}^{x_o + iy_o} d\beta \; \cdot$$

$$J_o\left(k \sqrt{(x+iy-\alpha)(x-iy-\alpha)}\right) J_o\left(k \sqrt{(x_o+iy_o-\beta)(x_o-iy_o-\beta)}\right) \frac{H_1^{(2)}(\pm k[\alpha-\beta])}{\pm(\alpha-\beta)} \; . \qquad (6.1.6)$$

This in turn, with formula (3.6.14), is seen to be the known result for scattering by a full plane

$$\phi_1(x, y, x_o, y_o) = -i\pi \left[H_o^{(2)}(kR) - H_o^{(2)}(kR')\right]. \qquad (6.1.7)$$

With the parabolic transformation (2), $f(\alpha) = \alpha^2$ and, using the more familiar notation $u = \xi$ and $v = \eta$, formula (6.1.1) becomes

$$\phi_1(\xi, \eta, \xi_o, \eta_o) = 2i \int_{\xi - i\eta}^{\xi + i\eta} d\alpha \int_{\xi_o - i\eta_o}^{\xi_o + i\eta_o} d\beta \int_{-i \log \frac{\alpha+\beta}{\pm(\alpha-\beta)}}^{+i \log \frac{\alpha+\beta}{\pm(\alpha-\beta)}} d\omega$$

$$J(\xi, \eta, \alpha) \; J(\xi_o, \eta_o, \beta) \; \frac{\alpha\beta \, e^{\pm ik(\beta^2 - \alpha^2)\cos\omega}}{(\alpha^2 - \beta^2)^2 \sin^2\omega} \qquad (6.1.8)$$

where we choose + if $\xi > \xi_o$ and - if $\xi < \xi_o$.

Integrating the ω-integral once by parts yields

$$\phi_1(\xi,\eta,\xi_o,\eta_o) = -2 \int_{\xi - i\eta}^{\xi + i\eta} d\alpha \int_{\xi_o - i\eta_o}^{\xi_o + i\eta_o} d\beta \quad J(\xi,\eta,\alpha)\, J(\xi_o,\eta_o,\beta) \quad \cdot$$

$$\left\{ \frac{(\alpha^2+\beta^2)\, e^{-ik(\alpha^2+\beta^2)}}{(\alpha^2-\beta^2)^2} + \frac{k\alpha\beta}{\pm(\alpha^2-\beta^2)} \int_{-i \log \frac{\alpha+\beta}{\pm(\alpha-\beta)}}^{i \log \frac{\alpha+\beta}{\pm(\alpha-\beta)}} \cos\omega\, e^{\pm ik(\beta^2-\alpha^2)\cos\omega} d\omega \right\} \tag{6.1.9}$$

which is identical with formula (4.3.15). This in turn was shown to be just another form for the known result for scattering by a half plane.

For the elliptic transformation (3), $f(\alpha) = a\cosh\alpha$ and, replacing u and v by the more familiar μ and θ, formula (6.1.2) becomes

$$\phi_2(\mu,\theta,\mu_o,\theta_o) = \frac{i}{2} \int_{-\mu+i\theta}^{\mu+i\theta} d\alpha \int_{-\mu_o+i\theta_o}^{\mu_o+i\theta_o} d\beta \int_{-i\log \frac{\sinh(\alpha+\beta)}{\pm(\sinh\alpha-\sinh\beta)}}^{+i\log \frac{\sinh(\alpha+\beta)}{\pm(\sinh\alpha-\sinh\beta)}} d\omega \quad \cdot$$

$$J[\mu,\theta,\alpha]\; J[\mu_o,\theta_o,\beta]\, \frac{\sinh\alpha\sinh\beta\, e^{\pm ik(\cosh\beta-\cosh\alpha)\cos\omega}}{(\cosh\alpha-\cosh\beta)^2\sin^2\omega} \tag{6.1.10}$$

where the + sign is to be employed if $\theta < \theta_o$ and the - sign if $\theta > \theta_o$. However, since

$$\frac{\sinh(\alpha+\beta)}{\sinh\alpha-\sinh\beta} = \frac{\cosh(\alpha+\beta)-1}{\cosh\alpha-\cosh\beta} \quad , \tag{6.1.11}$$

formula (6.1.10) is seen to be exactly the expression derived in Chapter 5 for the field scattered by a strip where, for this comparison, formula (5.7.19) is perhaps best employed.

Now, of course, having shown the similarity between these diffraction problems and, in fact, writing the solutions as special cases of a general form, it appears not unreasonable to conjecture that other two dimensional diffraction problems can also be solved using the "universal" diffraction kernel. Future research will determine whether we have indeed correctly induced the general solution for diffraction by bodies that can be conformally mapped onto a line segment. It would seem more than an idle hope, however, that even if formulas (6.1.1) and (6.1.2) do not constitute this general solution for an arbitrary analytic function f, either they will be significant for a more restricted class of functions f (certainly the three treated above) or by altering the form such a general result may be achieved. In any event there is reason to expect that the results of this work, in addition to presenting the long sought integral representation of the field diffracted by a strip, may actually provide a basis for the application of the theory of conformal mapping to the solution of two dimensional diffraction problems.

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SUMMARY

This work is concerned with the derivation and application of a useful class of solutions of the two dimensional Helmholtz equation. The utility is demonstrated by employing these solutions to find a closed (integral) form for the field diffracted by a strip.

Chapter 1 presents a general picture of what is to follow together with a short historical introduction to the strip problem. A brief discussion of exact methods of solving diffraction problems is given in which they are compared with the present approach.

In Chapter 2, a class of solutions of the Helmholtz equation is obtained in the form of an integral with a particular kernel and an arbitrary weighting function whose limits of integration are specified functions of the independent variables.

These solutions are utilized in Chapter 3 to establish an integral representation of those solutions of the Helmholtz equation which satisfy Dirichlet boundary conditions on a line segment. This representation theorem is then used to find new integral representations of combinations of cylinder functions with particular attention paid to the case of line sources. Finally a double integral representation of the Green's function for a line segment is obtained.

Chapter 4 is devoted to the particular case when the line segment is semi-infinite. Here the classical half plane result is recast into this double

integral form. In order to do this most conveniently the previous results are reformulated in parabolic coordinates. It is then hypothesized that the corresponding Green's function for the strip is essentially the equivalent form in elliptic coordinates.

In Chapter 5 this hypothesis is demonstrated to be correct and the Green's function for a strip is given explicitly in integral form.

Some of the consequences and possible extensions of the results are discussed in Chapter 6.