Arrangements of Curves and Algebraic Surfaces

by

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To Coni and Max

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[1] CONICYT stands for Comisión Nacional de Investigación en Ciencia y Tecnología. This is the government institution in charge of sciences in my country Chile.

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CHAPTER I

Introduction

1.1 Complex projective surfaces.

Definition I.1. A variety is an integral separated scheme of finite type over $\mathbb{C}$. We call it a curve (a surface) if its dimension is one (two). A projective variety is a variety which has a closed embedding into $\mathbb{P}^n$ for some positive integer n.

The main objects of this work are complex smooth projective surfaces. These objects are studied by means of divisors lying on them. We write $D = \sum_{i=1}^{r} \nu_i D_i$ for a divisor D on a smooth projective surface X, where the D_i are projective curves on X and $\nu_i \in \mathbb{Z}$. An effective divisor has $\nu_i > 0$, and it describes a one dimensional closed subscheme in X. The line bundle defined by a divisor D is denoted by $\mathcal{O}_X(D)$, and the corresponding Picard group by $\mathrm{Pic}(X)$. If D and D' are linearly equivalent, we write $D \sim D'$. A divisor D is called nef if for every curve Γ, we have $\Gamma.D \geq 0$.

A distinguished divisor class in a smooth projective variety X is the canonical class, which is defined by any divisor K_X satisfying $\mathcal{O}_X(K_X) \simeq \Omega_X^{\dim(X)}$. The sheaf of differential i-forms on a smooth variety X is denoted by Ω_X^i. We often use the notation $\omega_X := \Omega_X^{\dim(X)}$.

For any two divisors D, D' in a smooth projective surface X, we write $D \equiv D'$ if they are numerically equivalent. The Neron-Severi group of X is denoted by

$NS(X) = \mathrm{Pic}(X)/\equiv$. Some numerical invariants of X are: $\rho(X) = \mathrm{rank}(NS(X))$ (Picard number), $p_g(X) = h^2(X, \mathcal{O}_X) = h^0(X, K_X)$ (genus), $q(X) = h^1(X, \mathcal{O}_X)$ (irregularity), $P_m(X) = h^0(X, mK_X)$, $m > 0$ (m-th plurigenus).

Any smooth projective surface X falls in one of the following classes.

$-\infty$) $P_m(X) = 0$ for all m; X is a ruled surface.

0) $P_m(X)$ are either 0 or 1 for all m; X is birational to either a $K3$ surface, or an Enriques surface, or an elliptic surface, or a bi-elliptic surface.

1) $P_m(X)$ grows linearly in $m >> 0$; then, X has an elliptic fibration.

2) $P_m(X)$ grows quadratically in $m >> 0$; X is called a surface of general type.

This is Enriques' classification for algebraic surfaces. The Kodaira dimension $\kappa(X)$ of X is the maximum dimension of the image of $|mK_X|$ for $m > 0$, or $-\infty$ if $|mK_X| = \emptyset$ for all $m > 0$. This explains the indices of the previous list. We are mainly interested on surfaces X having $\kappa(X) = 2$.

Important are the Chern classes of X, defined as $c_1(X) := c_1(T_X)$ and $c_2(X) := c_2(T_X)$, where $T_X = {\Omega^1_X}^\vee$ is the tangent bundle of X. These classes are related via the Noether's formula

$$12\chi(X, \mathcal{O}_X) = c_1^2(X) + c_2(X)$$

which is an instance of the Hirzebruch-Riemann-Roch theorem, so independent of the characteristic of the ground field [39, p. 433]. The invariant $\chi(X, \mathcal{O}_X) = 1 - q(X) + p_g(X)$ is called the Euler characteristic of X. For any canonical divisor K_X, we have the equality $K_X^2 = c_1(X).c_1(X) = c_1^2(X)$.

We now use the complex topology of X to define the Betti numbers

$$b_i(X) = \mathrm{rank} H_i(X, \mathbb{Z}) = \dim_{\mathbb{C}} H^i(X, \mathbb{C}).$$

Because of the Poincaré duality, they satisfy $b_i(X) = b_{4-i}(X)$. Hence, the topological Euler characteristic of X can be written as $e(X) = 2 - 2b_1(X) + b_2(X)$. We also have the Hodge decomposition $H^i(X, \mathbb{C}) = \sum_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X) := H^q(X, \Omega_X^p)$ and $h^{p,q}(X) = h^{q,p}(X)$. Thus we have $q(X) = h^0(X, \Omega_X^1)$, $b_1(X) = 2q(X)$, and $b_2(X) = 2p_g(X) + h^{1,1}(X)$. Finally, it is well-known that $c_2(X) = e(X)$ (Gauss-Bonnet formula [35, p. 435], or Hirzebruch-Riemann-Roch Theorem and Hodge decomposition).

More subtle topological invariants of X are the intersection form on $H^2(X, \mathbb{Z})$ and the topological fundamental group $\pi_1(X)$. Let b^+ and b^- be the numbers of positive and negative entries in the diagonalized version of the intersection form over $\mathbb{R}$ (so $b_2(X) = b^+ + b^-$). Its signature $\mathrm{sign}(X) = b^+ - b^-$ can be expressed as (see [35])

$$\mathrm{sign}(X) = 2 + 2p_g(X) - h^{1,1}(X) = \frac{1}{3}\big(c_1^2(X) - 2c_2(X)\big).$$

In particular, the Chern numbers of X are topological invariants.

However, contrary to the case of curves, the Chern numbers do not determinate the topology of X. For example, consider X in $\mathbb{P}^4$ defined by $\sum_{i=0}^4 x_i^{25} = \sum_{i=0}^4 x_i^{50} = 0$. By the Lefschetz theorem $\pi_1(X) = \{1\}$. There are two natural distinct free actions on X given by the groups $\mathbb{Z}/25\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$. Therefore, the two corresponding quotients are smooth projective surfaces with the same Chern numbers but different fundamental groups [1].

It is known that any simply connected compact Riemann surface is isomorphic to the complex projective line. In the case of smooth projective surfaces, simply connectedness occurs for all Kodaira dimensions. There are several effective ways to compute fundamental groups when $\kappa < 2$, but it becomes more difficult for surfaces of general type. We are going to describe in Section 1.3 one well-known general tool

[1] This example due to Tankeev.

for fibrations, which will be used later in this work. We will show how this tool fits very well in p-th root covers to produce, in particular, several simply connected surfaces of general type.

1.2 Moduli and geography.

A smooth projective surface is said to be minimal (or a minimal model) if it does not contain any (-1)-curves (i.e., a curve E with $E^2 = -1$ and $E \simeq \mathbb{P}^1$). It is a classical fact that any birational class of a non-ruled projective surface has a unique minimal model. Moreover, any smooth projective surface is obtained by performing a finite sequence of blow ups from a minimal model in its birational class. Throughout this section, X is a minimal smooth projective surface of general type.

After Riemann, the parameter spaces which classify objects have been called moduli spaces. For smooth projective curves, we have the famous moduli space of curves M_g [38], which is represented by a quasi-projective irreducible variety. As in the case of curves, to talk about moduli of surfaces of general type, we fix some numerical invariants, which are the ones preserved under flat deformations. Since K_X is nef and big, by the Riemann-Roch theorem and Kodaira vanishing theorem, we have that $P_n(X) = \frac{n(n-1)}{2}c_1^2(X) + \chi(X, \mathcal{O}_X)$ for $n \geq 2$. Hence, Chern numbers are natural candidates for such invariants.

In 1977, Gieseker [32] proved that the moduli space of minimal surfaces of general type with fixed Chern numbers (of the canonical model) exists as a quasi-projective variety. We denote it as $M_{c_1^2, c_2}$. This moduli space is much more complicated than M_g. For example, the complex topology of the surfaces it classifies may be different (as we saw in the previous section), any singularity of finite type over $\mathbb{Z}$ shows up (Vakil's Murphy's law) (for the case of curves, because of dimension, there is no

obstruction to deform so the only possible singularities in M_g are quotient singularities), and it is highly disconnected (even more, there are diffeomorphic surfaces in distinct connected components [60, 8]).

For any $g \geq 2$, it is easy to exhibit curves of genus g. For example, there are hyperelliptic curves. The equivalent question for the surface case is harder: for which integers c_1^2, c_2 does there exist a minimal smooth projective surface X with $c_1^2(X) = c_1^2$ and $c_2(X) = c_2$? This is the so-called "geography problem" [73].

Chern numbers have well-known classical constrains. For example, by Noether's formula, $c_1^2(X) + c_2(X) \equiv 0(mod\ 12)$. Moreover, we have the following classical inequalities:

$$c_1^2(X) > 0, \quad c_2(X) > 0 \quad \text{(Castelnuovo 1905)}$$
$$\tfrac{1}{5}(c_2(X) - 36) \leq c_1^2(X) \quad \text{(Max Noether 1875)}.$$

Less classical and more difficult is the so-called Miyaoka-Yau inequality

$$c_1^2(X) \leq 3c_2(X)$$

which was proved in 1977 independently by Miyaoka [63] (for any surface of general type) and Yau [94] (in any dimensions, assuming that Ω_X^n is ample). Surfaces for which equality holds are very special. Yau proved that $c_1^2(X) = 3c_2(X)$ if and only if X is a quotient of the unit complex ball; in particular, such surfaces are not simply connected [2]. Moreover, older results of Calabi and Vesentini [14] imply that these surfaces are rigid, that is, they do not deform.

How could we measure the difficulty of finding surfaces of general type with fixed Chern numbers? A tentative answer is the Chern numbers ratio $\frac{c_1^2}{c_2}$. Below we mention some facts in favor to this statement.

[2] Yau assumed ampleness for Ω_X^2, but Miyaoka proved in [64] that the assumption always holds when $c_1^2 = 3c_2$.

The so-called Severi conjecture was recently proved by Pardini [72] [3]. Let X be a minimal smooth projective surface. Assume that $q(X) > 0$. Then, the image of the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is either a smooth projective curve of genus $q(X)$ or a projective surface [9, pp. 61-65]. When $\alpha(X)$ is a surface, we say that X has maximal Albanese dimension. Using results about the slope of fibrations, Pardini proved that if X has maximal Albanese dimension, then $\frac{1}{2}c_2(X) \leq c_1^2(X)$. In another words, if $c_1^2(X) \leq \frac{1}{2}c_2(X)$ and $q(X) > 0$, then $\alpha : X \to \mathrm{Alb}(X)$ is a fibration over a smooth projective curve of genus $q(X)$.

In general, smooth projective surfaces with $c_1^2(X) \leq 2c_2(X)$ are easy to find; for example any complete intersection [73, pp. 203-107] satisfies this inequality. We actually have the following theorem [73, p. 209].

Theorem I.2. *(Persson, Xiao) For any pair of positive integers (c_1^2, c_2) satisfying $c_1^2 + c_2 \equiv 0 (mod\ 12)$ and $\frac{1}{5}\big(c_2 - 36\big) \leq c_1^2 \leq 2c_2$, there exist a minimal smooth projective surface X of general type with $c_1^2(X) = c_1^2$ and $c_2(X) = c_2$.*

The surfaces Persson and Xiao produced to prove this theorem were all genus two fibrations, and the technique they used was double covers. Xiao actually proved that any X admitting a genus two fibration has to satisfy $c_1^2(X) \leq 2c_2(X)$ [4].

We now move to the harder case $2c_2 \leq c_1^2 \leq 3c_2$. These are the surfaces with non-negative signature. Old Italians suspected that there were no surfaces in this range [73, p. 213]. The first opposite evidence was given around 1955 by Hirzebruch [41], followed by concrete examples due to Borel. In [41], Hirzebruch proved that if G is a subgroup of the automorphisms of a bounded symmetric domain D in $\mathbb{C}^2$ (so D is isomorphic to either $\mathbb{B}^2$ (the complex 2-ball) or $\mathbb{H}^1 \times \mathbb{H}^1$ (product of two

[3] In [8, pp. 9-12], there is a sketch of the history of this conjecture, including the various attempts towards its proof.

[4] Xiao wrote the book [91] about genus two fibrations.

complex disks)), acting properly discontinuously and freely, then the Chern numbers of D/G and the Chern numbers of the dual homogeneous complex manifold of D are proportional [5]. The proportionality factor is precisely the arithmetic genus of D/G. This is the two dimensional case of the Hirzebruch Proportionality Theorem. In particular, if $D = \mathbb{H}^1 \times \mathbb{H}^1$ then $c_1^2(D/G) = 2c_2(D/G)$; and if $D = \mathbb{B}^2$ then $c_1^2(D/G) = 3c_2(D/G)$. Notice that any product of two curves satisfies $c_1^2 = 2c_2$. As we pointed out before, Yau [94] proved the converse for ratio 3, that is, any surface of general type with $c_1^2(X) = 3c_2(X)$ is a ball quotient.

Which Chern ratios are realizable in the range $(2, 3)$? There are various examples of surfaces with Chern ratios in that interval. Very spectacular are the surfaces constructed from line arrangements in $\mathbb{P}^2$ due to Hirzebruch [42]. In that article, he finds surfaces satisfying $c_1^2 = 3c_2$ without the explicit construction of the group acting on the ball. There is also a density theorem due to Sommese [80].

Theorem I.3. *(Sommese) Any rational point in* $\left[\frac{1}{5}, 3\right]$ *occurs as a limit of Chern ratios of smooth projective surfaces of general type.*

The surfaces which occur in Sommese's theorem have all positive irregularity. They are simple base changes of a Hirzebruch's example of a ball quotient that has a fibration over a curves of genus 6. The most general theorem about geography of surfaces is the following [7, p. 291] [6].

Theorem I.4. *(Persson, Xiao, Chen) For any pair of positive integers* (c_1^2, c_2) *satisfying* $c_1^2 + c_2 \equiv 0 \ (mod\ 12)$*, the Noether and Miyaoka-Yau inequalities, there exist a minimal smooth projective surface* X *of general type with* $c_1^2(X) = c_1^2$ *and* $c_2(X) = c_2$*, except for the cases where* $c_1^2 - 3c_2 + 4k = 0$ *with* $0 \leq k \leq 347$*. In fact, for at most*

[5] The dual of $\mathbb{B}^2$ is $\mathbb{P}^2$, and the dual of $\mathbb{H}^1 \times \mathbb{H}^1$ is $\mathbb{P}^1 \times \mathbb{P}^1$

[6] Xiao and Chen took care of the harder case of positive signature.

finitely many exceptions on these lines all admissible pairs occur as Chern pairs.

Hence there are still gaps as we approach 3. Why is it hard to find surfaces of general type with Chern numbers ratio close to 3? A recent paper of Reider [75] gives the following conceptual reason.

Theorem I.5. *(Reider) Let X be a smooth projective surface with Ω_X^2 ample. Assume $\frac{8}{3}c_2(X) \leq c_1^2(X)$. Then, $h^1(X, T_X) \leq 9\big(3c_2(X) - c_1^2(X)\big)$.*

The geography problem becomes harder when we impose additional properties on surfaces. We are interested in simply connected surfaces. In this situation, we have to exclude the case $c_1^2 = 3c_2$ since they are ball quotients, and so π_1 is not trivial. Around 25 years ago, Bogomolov conjectured that a simply connected smooth projective surface of general type has negative signature [73, p. 216]. In 1984 Moishezon and Teicher [66] presented a construction of a simply connected surface of positive index. After that, several examples appeared, but no examples in the range

$$2.\overline{703}c_2 < c_1^2 < 3c_2$$

are known. In 1996 Persson, Peters and Xiao gave the best known results in this direction. In [74], they found simply connected projective surfaces with Chern numbers ratio as large as $2.\overline{703}$. If we consider symplectic 4-manifolds instead, there are simply connected examples with Chern numbers ratio arbitrarily close to 3 [4].

Question I.6. *Do there exist simply connected projective surfaces X of general type having Chern numbers ratio $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to 3?*

One of the main purposes of this work is to show a way to encode this kind of problems in the existence of certain arrangements of curves.

1.3 Fibrations.

Definition I.7. Let X be a smooth projective surface and let B be a smooth projective curve. A fibration is a surjective map $f : X \to B$ with connected fibers. The generic (smooth) fiber is denoted by F and its genus by g, which is the genus of the fibration. A fibration is said to be

- Smooth if all the fibers are smooth.
- Isotrivial if all smooth fibers are mutually isomorphic.
- Locally trivial if it is smooth and isotrivial.

A fibration of genus zero is a ruled surface; a fibration of genus one is an elliptic fibration, and so they are not of general type. For higher genus, we often have surfaces with Kodaira dimension two. We notice that any projective surface can be considered as a fibration over $\mathbb{P}^1$ after some blow ups, so the study of fibrations is important for birational invariants. The proposition below is a good example, and it is a well-known fact.

Proposition I.8. *The topological fundamental group of a smooth projective surface is a birational invariant.*

We notice that any birational class has a simply connected normal projective surface [37], and so the assumption of smoothness in Proposition I.8 is essential. At the end of this section, we will see how to compute the fundamental group of a fibration.

Definition I.9. A fibration is said to be relatively minimal if the fibers contain no (-1)-curves.

Proposition I.10. *(see [7, p. 112]) Let $f : X \to B$ be a fibration of genus g. If $g > 0$, then f factors through a unique smooth projective surface X', $f : X \to X' \to B$, such that $X' \to B$ is relatively minimal.*

Proposition I.11. *Let $f : X \to B$ be a relatively minimal fibration of genus g, and let E be a (-1)-curve such that $E.F > 2g - 2$. Then, X is a ruled surface. Hence, for example, a non-ruled relatively minimal elliptic fibration must have X minimal.*

Proof. If $g = 0$, then X is ruled. So, we assume $g > 0$. Let $\sigma : X \to X'$ be the blow-down of E, so X' is also a smooth projective surface. Let K_X and $K_{X'}$ be canonical divisors for X and X'. Let F be a smooth fibre of f and F' be an irreducible curve in X' such that $\sigma^*(F') = F + nE$ where $n = E.F$. Now, $F'.K_{X'} = \sigma^*(F').\sigma^*(K_{X'}) = (F + nE).(K_X - E) = 2g - 2 - n < 0$. Also, $p_a(F') > 0$. Therefore, after finite blow-downs, we arrive to a minimal model. Since X is non-ruled, the canonical class is nef. In the process, the images of F cannot be blown down and the intersection with the canonical class is always negative. So, X has to be ruled. □

Let $f : X \to B$ be a fibration. Let

$$\text{sing}(f) = \{b \in B : f^*(b)\text{is not a smooth reduced curve}\}.$$

Since we are working over $\mathbb{C}$, this is a finite set of points in B (maybe empty). Over the open set $f^{-1}(B \setminus \text{sing}(f))$, we have the exact sequence $0 \to f^*(\Omega^1_B) \to \Omega^1_X \to \Omega^1_{X/B} \to 0$, where $\Omega^1_{X/B} \simeq \omega_X \otimes f^*(\omega_B^\vee)$.

Definition I.12. The dualizing sheaf of f is the line bundle $\omega_{X/B} := \omega_X \otimes f^*(\omega_B^\vee)$.

Let F be any fiber of a fibration $f : X \to B$. By Lemma 1.2 in [91, p. 1], we have that $h^0(F, \mathcal{O}_F) = 1$, and so by Grauert's theorem [39, p. 288], $f_*\mathcal{O}_X = \mathcal{O}_B$. Since f is a flat map, the arithmetic genus of F is $g = 1 - h^0(F, \mathcal{O}_F) + h^1(F, \mathcal{O}_F)$, and so

$h^1(F, \mathcal{O}_F) = g$. Let ω_F^0 be the dualizing sheaf of F. In particular, $\omega_{X/B}|F \simeq \omega_F^0$. By Serre's duality, $h^0(F, \mathcal{O}_F) = h^1(F, \omega_F^0) = 1$ and $h^1(F, \mathcal{O}_F) = h^0(F, \omega_F^0) = g$. Hence, by Grauert's theorem again, the sheaf $f_*(\omega_{X/B})$ is locally free of rank g.

Proposition I.13. *(Arakelov) Let $f : X \to B$ be a relatively minimal fibration. Then, $\omega_{X/B}$ is nef. Moreover, $\omega_{X/B}.C = 0$ for a curve C iff C is a (-2)-curve (that is, $C \simeq \mathbb{P}^1$ and $C^2 = -2$).*

Proposition I.14. *(see [9]) Let $f : X \to B$ be a fibration and let F be a general fiber. Then,*

$$e(X) = e(B)e(F) + \sum_{b \in sign(f)} \big(e(f^*(b)) - e(F)\big).$$

Moreover, $e(f^*(b)) - e(F) \geq 0$ for any $b \in \operatorname{sing}(f)$.

From the Leray spectral sequence associated to f and $\mathcal{O}_X$ (see [7, p. 13]), we have

$$0 \to H^1(B, f_*\mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to H^0(B, R^1 f_* \mathcal{O}_X) \to H^2(B, f_*\mathcal{O}_X) = 0$$

and since $f_*\mathcal{O}_X = \mathcal{O}_B$, we have $h^0(B, R^1 f_* \mathcal{O}_X) = q(X) - b$. By duality, $f_*\omega_{X/B} = R^1 f_* {\mathcal{O}_X}^\vee$ and so $h^1(B, R^1 f_* \mathcal{O}_X) = h^0(B, f_*\omega_X) = p_g(X)$. Then, by the Riemann-Roch theorem,

$$\chi(B, R^1 f_* \mathcal{O}_X) = h^0(B, R^1 f_* \mathcal{O}_X) - h^1(B, R^1 f_* \mathcal{O}_X) = \deg(R^1 f_* \mathcal{O}_X) + g(b-1)$$

and so $\deg(f_*\omega_{X/B}) = \chi(X, \mathcal{O}_X) - (g-1)(b-1)$.

Proposition I.15. *(Fujita) deg$(f_*\omega_{X/B}) \geq 0$.*

Proof. We have $\omega_{X/B}.\omega_{X/B} = K_X^2 - 8(g-1)(b-1) \geq 0$ (by Prop. I.13) and $e(X) - e(F)e(B) \geq 0$ (by Prop. I.14). Also, we know that $\deg(f_*\omega_{X/B}) = \chi(X, \mathcal{O}_X) - (g-1)(b-1)$. So, we apply Noether's formula to obtain the result. □

This gives us the main invariants of a fibration: $\chi_f := \chi(X, \mathcal{O}_X) - (g-1)(b-1)$, $K_f^2 := K_X^2 - 8(g-1)(b-1)$, and $e_f := e(X) - 4(g-1)(b-1)$. The idea now is to obtain the analog of the geography for smooth projective surfaces for fibrations. The following is the main ingredient; it was proved by Xiao [92] [7].

Proposition I.16. *(Xiao) Let $f : X \to B$ a relatively minimal fibration of genus $g \geq 2$. Then, $K_f^2 \geq \frac{4(g-1)}{g}\chi_f$.*

Therefore, for a relatively minimal fibration of genus $g \geq 2$, we have $K_f^2 + e_f = 12\chi_f$, $K_f^2 \geq 0$, $e_f \geq 0$, $\chi_f \geq 0$, $\frac{4(g-1)}{g}\chi_f \leq K_f^2 \leq 12\chi_f$. The extremal cases are characterized as follows.

- (Arakelov) $K_f^2 = 0$ iff f is isotrivial.
- $e_f = 0$ iff f is smooth (i.e., a Kodaira fibration)
- (Konno [54, Prop. 2.6]) If f is not locally trivial and $\frac{4(g-1)}{g}\chi_f = K_f^2$, then all smooth fibers of f are hyperelliptic curves [8].

For a non-locally trivial relatively minimal fibration $f : X \to B$, Xiao defined the slope of f as

$$\lambda_f := \frac{K_f^2}{\chi_f}.$$

One of the main problems is to understand how properties of the general fiber affect the slope. Several results in this direction can be found in [3]; for a recent work see [20]. Some consequences of the study of fibrations are the proof of the Severi conjecture [72], and the computation of relevant intersection numbers for $\overline{M}_g$ (see [19, 83]).

[7] In [19], Cornalba and Harris proved the inequality for the special case of semi-stable fibrations, mainly oriented to curves in $\overline{M}_g$.

[8] This was conjectured by Xiao in [92].

As explained at the beginning of this section, the fundamental group of a fibration is important in the study of the fundamental group of any surface. Below we show a well-known basic tool to compute this group for a fibration, following Xiao [93, pp. 600-602].

Let $f : X \to B$ be a fibration. Let $F = f^{-1}(b)$ with $b \in B \setminus \text{sing}(f)$. Hence, the inclusion $F \hookrightarrow X$ induces a homomorphism $\pi_1(F) \to \pi_1(X)$. Let $\mathcal{V}_f$ be the image of this homomorphism. We call it the vertical part of $\pi_1(X)$.

Lemma I.17. *The vertical part $\mathcal{V}_f$ is a normal subgroup of $\pi_1(X)$, and is independent of the choice of F.*

The horizontal part of $\pi_1(X)$ is $\mathcal{H}_f := \pi_1(X)/\mathcal{V}_f$, and so we have

$$1 \to \mathcal{V}_f \to \pi_1(X) \to \mathcal{H}_f \to 1.$$

Let F be now any fiber of f, we write $F = f^*(b) = \sum_{i=1}^n \mu_i \Gamma_i$ for some positive integers μ_i. The multiplicity of F is $m = gcd(\mu_1, \ldots, \mu_n)$. We say that F is a multiple fiber of f if $m > 1$. Let $\{x_1, \ldots, x_s\}$ be the images of all the multiple fibers of f (it may be empty), and $\{m_1, \ldots, m_s\}$ the corresponding multiplicities. Let $B' = B \setminus \{x_1, \ldots, x_s\}$, and let γ_i a be small loop around x_i. Then, there are generators $\alpha_1, \ldots, \alpha_b, \beta_1, \ldots, \beta_b$ such that

$$\pi_1(B') \simeq \langle \alpha_1, \ldots, \alpha_b, \beta_1, \ldots, \beta_b, \gamma_1, \ldots, \gamma_s \ : \ \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_b\beta_b\alpha_b^{-1}\beta_b^{-1}\gamma_1 \cdots \gamma_s = 1 \rangle.$$

Lemma I.18. *The horizontal part $\mathcal{H}_f$ is the quotient of $\pi_1(B')$ by the normal subgroup generated by the conjugates of $\gamma_i^{m_i}$ for all i.*

Lemma I.19. *Let F be any fiber of f with multiplicity m. Then the image of $\pi_1(F)$ in $\pi_1(X)$ contains $\mathcal{V}_f$ as a normal subgroup, whose quotient group is cyclic of order m, which maps isomorphically onto the subgroup of $\mathcal{H}_f$ generated by the class of a*

small loop around the image of F in B. In particular, $\mathcal{V}_f$ is trivial if f has a simply connected fiber.

The immediate consequence of these lemmas is the following.

Corollary I.20. *If f has a section, then $1 \to \mathcal{V}_f \to \pi_1(X) \to \pi_1(B) \to 1$. Moreover, if f has a simply connected fiber, then $\pi_1(X) \simeq \pi_1(B)$.*

In [93], one can find general results about the fundamental group of elliptic and Hyperelliptic fibrations.

1.4 Logarithmic surfaces.

Definition I.21. Let X be a smooth complete variety, and let D be a simple normal crossings divisor on X (as defined in [55, p. 240], abbreviated SNC). A log variety is a smooth variety U of the form $U = X \setminus D$. We refer to U as the pair (X, D). We call it log curve (surface) if its dimension is one (two).

Any smooth non-complete variety X_0 is isomorphic to a log variety (see [48, Ch. 11], the main ingredient is Hironaka's resolution of singularities). We do not want to consider X_0 as a member of its birational class, but rather as a pair (X, D) such that $X_0 \simeq X \setminus D$. We modify the usual invariants of X_0 into the logarithmic invariants of the pair (X, D), which we will describe below. This will take X_0 out of its birational class, creating a whole new world of log varieties, in where the "class" of X_0 depends heavily on the "geometry" of D (what matters here is D or a divisor whose log resolution is D, and not its linear class). Iitaka, Sakai, Kobayashi, and Miyanishi among others have systematically studied log varieties, mainly in the case of surfaces (e.g., see [46], [76], [47], [51] and [62]).

Let (X, D) be a log variety. The analogue of a canonical divisor will be $K_X + D$.

It comes from the following modification on the sheaf of differentials of X, due to Deligne [21], which keeps track of D.

Definition I.22. [48, p. 321] Let X be a smooth complete variety, and let D be a SNC divisor on X. The sheaf of logarithmic differentials along D, denoted by $\Omega^1_X(\log D)$, is the $\mathcal{O}_X$-submodule of $\Omega^1_X \otimes \mathcal{O}_X(D)$ satisfying

(i) $\Omega^1_X(\log D)|_{X\setminus D} = \Omega^1_{X\setminus D}$.

(ii) At any closed point P of D,

$$\omega_p \in \Omega^1_X(\log D)_P \text{ iff } \omega_p = \sum_{i=1}^{s} a_i \tfrac{dz_i}{z_i} + \sum_{j=s+1}^{\dim(X)} b_j dz_j,$$

where $(z_1, \ldots, z_{\dim(X)})$ is a local system around P for X, and $\{z_1 \cdots z_s = 0\}$ defines D around P.

Hence $\Omega^1_X(\log D)$ is a locally free sheaf of X of rank $\dim(X)$. We define $\Omega^q_X(\log D) := \bigwedge^q \Omega^1_X(\log D)$ for any $0 \le q \le \dim(X)$ (being $\Omega^0_X(\log D) = \mathcal{O}_X$). Since

$$\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_s}{z_s} \wedge dz_{s+1} \wedge \cdots \wedge dz_{\dim(X)} = \frac{1}{z_1 \cdots z_s} dz_1 \wedge \cdots \wedge dz_{\dim(X)},$$

we have $\Omega^{\dim(X)}_X(\log D) \simeq \mathcal{O}_X(K_X + D)$.

Definition I.23. For any integer $m > 0$, the m-th logarithmic plurigenus of the log variety (X, D) is $\bar{P}_m(X, D) := h^0(X, m(K_X + D))$. When $m = 1$, we call it logarithmic genus of (X, D), denoted by $\bar{p}_g(X, D)$. The logarithmic Kodaira dimension of (X, D) is defined as the maximum dimension of the image of $|m(K_X + D)|$ for all $m > 0$, or $-\infty$ if $|m(K_X + D)| = \emptyset$ for all m. It is denoted by $\bar{\kappa}(X, D)$.

The following are two conceptually interesting theorems for log varieties.

Theorem I.24. *(see [47], [62, p. 60]) Let X_0 be an smooth non-complete variety. Let (X, D) be any log variety such that $X_0 \simeq X \setminus D$. Then, the mth logarithmic*

plurigenus of (X, D) is a well-defined invariant of X_0, i.e., for any other such pair (X', D') we have $\bar{P}_m(X, D) = \bar{P}_m(X', D')$. In particular, there is a well-defined logarithmic Kodaira dimension of X_0.

Theorem I.25. *(Deligne [22]) Let (X, D) be a log variety. Then, we have the logarithmic Hogde decomposition*

$$H^i(X \setminus D, \mathbb{C}) \simeq \bigoplus_{p+q=i} H^p(X, \Omega^q_X(\log D)).$$

Example I.26. (Logarithmic curves) Let (X, D) be a log curve, i.e., X is a smooth projective curve and $D = \sum_{i=1}^r P_i$ a finite sum of distinct points. Then, (X, D) is classified in the following table. We include the complete cases, when $D = \emptyset$.

$\overline{\kappa}(X, D)$	(X, D)	$\bar{p}_g(X, D)$
$-\infty$	$\mathbb{P}^1$, $\mathbb{A}^1$	0
0	elliptic curve, $\mathbb{A}^1 \setminus \{0\}$	1
1		≥ 1

For the case of curves, the log classification tells us about the uniformization of the corresponding log curve, being D the branch divisor for the uniformizing map.

Important invariants for log varieties are the following.

Definition I.27. The logarithmic Chern classes of the log variety (X, D) are defined as $\bar{c}_i(X, D) = c_i(\Omega^1_X(\log D)^\vee)$ for $1 \leq i \leq \dim(X)$.

Let (X, D) be a log surface. In particular, X is a smooth projective variety and $D = \sum_{i=1}^r D_i$, where D_i are smooth projective curves and D has at most nodes as singularities. We define the logarithmic irregularity of (X, D) as $\bar{q}(X, D) := h^0(X, \Omega^1_X(\log D))$. As in the projective case, we have the log Chern numbers

$$\bar{c}_1^2(X, D) = (K_X + D)^2 \text{ and } \bar{c}_2(X, D) = e(X) - e(D),$$

and the corresponding log Chern numbers ratio $\frac{\bar{c}_1^2(X,D)}{\bar{c}_2(X,D)}$, whenever $\bar{c}_2(X,D) \neq 0$. The second formula is derived from the Hirzebruch-Riemann-Roch theorem [46, p. 6].

Let us consider the corresponding geography problem for log surfaces. In [76], Sakai studied the logarithmic pluricanonical maps and corresponding logarithmic Kodaira dimensions for log surfaces. Moreover, he proved the analogue of the Miyaoka-Yau inequality when D is a semi-table curve (i.e., D is a SNC divisor and any $\mathbb{P}^1$ in D intersects the other components in more than one point).

Theorem I.28. *([76, Theorem 7.6]) Let (X,D) be a log surface with D semi-stable. Suppose $\bar{\kappa}(X,D) = 2$. Then, $\bar{c}_1^2(X,D) \leq 3\bar{c}_2(X,D)$.*

This theorem is proved using Miyaoka's proof of his inequality. The analytic Yau's point of view was used by Kobayashi. In [51], he proves the same inequality, under the assumptions $\bar{c}_1^2(X,D) > 0$ and $K_X + D$ nef. The additional point here is that, in this case, $\bar{c}_1^2(X,D) = 3\bar{c}_2(X,D)$ if and only if the universal covering of $X \setminus D$ is biholomorphic to the complex ball $\mathbb{B}^2$ [51, p. 46]. This is quite interesting, because it classifies divisors that produce open ball quotients. It turns out that they are very special. For concrete examples for which equality holds, see [76, p. 118].

We also have a density theorem due to Sommese [80, Remark 2.4], whose proof applies again the base change "trick" in Theorem I.3, considering D as a collection of fibers.

Theorem I.29. *(Sommese) Let $\mathcal{U}_t$ be the set of all log surfaces (X,D) such that D has t connected components and $K_X + D$ is ample. Then, the set of limits of the log Chern ratios $\frac{\bar{c}_1^2(X,D)}{\bar{c}_2(X,D)}$, where $(X,D) \in \mathcal{U}_t$, contains $[\frac{1}{5}, 3]$.*

1.5 Main results.

For surfaces, geography and logarithmic geography seem to behave in a similar way. One of the main results of this thesis is to give a concrete strong relation between them.

Theorem V.2. Let Z be a smooth projective surface over $\mathbb{C}$, and let $\mathcal{A}$ be a simple crossings divisible arrangement on Z (Definition V.1). Let $(Y, \mathcal{A}')$ be the log surface associated to $(Z, \mathcal{A})$ (see Section 2.1), and assume $e(Y) \neq e(\mathcal{A}')$. Then, there exist smooth projective surfaces X with $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')}$.

This result can be used to construct smooth projective surfaces with exotic properties from log surfaces, equivalently, from arrangements of curves. It turns out that log surfaces with interesting properties are somehow easier to find (e.g. we will see that line arrangements in $\mathbb{P}^2$ give many examples). Our method is based on the p-th root cover tool introduced by Esnault and Viehweg (see [29]). We first find Chern numbers in relation to log Chern numbers, Dedekind sums and continued fractions. Then, we exploit a large scale behavior of the Dedekind sums and continued fractions to find "good" weighted partitions of large prime numbers. These partitions, which come from what we call divisible arrangements, produce the surfaces X in Theorem V.2.

We also show that random choices of these weighted partitions are "good", with probability tending to 1 as p becomes arbitrarily large. An interesting phenomena is that random partitions are necessary for our constructions, if we want to approach to the log Chern numbers ratio of the corresponding arrangement. We put this in evidence by examples, using a computer program that calculates the exact values of the Chern numbers involved (see Section 5.3 for a sample).

The following corollary is a sort of uniformization for minimal surfaces of general type via Chern numbers ratio ≈ 2. The proof of this corollary uses indeed random partitions.

Corollary V.4. Let Z be a smooth minimal projective surface of general type over $\mathbb{C}$. Then, there exist smooth projective surfaces X, and generically finite maps $f : X \to Z$ of high degree, such that

(i) X is minimal of general type.

(ii) The Chern numbers ratio $\frac{c_1^2(X)}{c_2(X)}$ is arbitrarily close to 2.

(iii) $q(X) = q(Z)$.

One of the properties of the construction is that the geometry of $\mathcal{A}$ controls some invariants of the new surfaces X, for certain arrangements of curves. For example, we may control their irregularity (Kawamata-Viehweg vanishing theorem) and their topological fundamental group. In Section 5.3, we use our method to find simply connected surfaces X of general type with large $\frac{c_1^2(X)}{c_2(X)}$, coming from arbitrary line arrangements in $\mathbb{P}^2_{\mathbb{C}}$ (Proposition V.6). In particular, we produce simply connected surfaces X with $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{8}{3}$. They correspond to the dual Hesse arrangement. Furthermore, we prove in Proposition II.8 that this arrangement gives the largest possible value for the Chern numbers ratio of X (in Theorem V.2) among all line arrangements, and it is the only one with that property. The proof relies on the Hirzebruch's inequality for complex line arrangements [42, p. 140] [9].

Proposition II.8. Let $\mathcal{A}$ be an arrangement of d lines on $\mathbb{P}^2_{\mathbb{C}}$, and assume that no $d-1$ lines pass through a common point. Then, $\bar{c}_1^2(Y, \mathcal{A}') \leq \frac{8}{3}\bar{c}_2(Y, \mathcal{A}')$. Moreover, the

[9] We notice that this was found by Sommese in [80], without mentioning the dual Hesse arrangement as the only case for equality. He used the point of view of Hirzebruch [42].

equality holds if and only if $\mathcal{A}$ is (projectively equivalent to) the dual Hesse arrangement. In particular, the surfaces X corresponding to the dual Hesse arrangement have the best possible Chern numbers ratio (i.e. closest to 3) for line arrangements in $\mathbb{P}^2_{\mathbb{C}}$.

Is this bound $\frac{8}{3}$ a restriction for general divisible arrangements? In Section 7.5 we provide a short discussion around this issue. In positive characteristic, as one may expect, we have different restrictions for log Chern numbers (see Proposition VII.9).

By using some facts about algebraic surfaces, we have found formulas involving Dedekind sums and continued fractions [10]. Relations between them are well-documented (see for example [5], [96], [45], [34]). These objects, which have appeared repeatedly in geometry (see for example [44]), play a fundamental role in the construction of the surfaces X. In algebraic geometry, they naturally arise when considering the Riemann-Roch theorem and resolution of Hirzebruch-Jung singularities. The proofs of these relations are based on the Noether's formula, and a rationality criteria for smooth projective surfaces. Definitions and notations can be found in the Appendix.

Proposition IV.13 and Subsection 5.1.1. Let p be a prime number and q be an integer such that $0 < q < p$. Let $\frac{p}{q} = [e_1, e_2, \ldots, e_s]$. Then,

$$12s(q,p) - \sum_{i=1}^{s} e_i + 3s = \frac{q+q'}{p} \quad \text{and} \quad s(q,p) = s(q+1,p) + s(q'+1,p) + \frac{p-1}{4p}.$$

Since we are encoding the existence of smooth projective surfaces into the existence of certain pairs $(Z, \mathcal{A})$, it is important for us to know more about them. We study arrangements of curves $\mathcal{A}$ on a fixed surface Z. We first see that there are combinatorial restrictions for them to exist. For example, in $\mathbb{P}^2$ two lines intersect

[10] We notice that this formula was found by Holzapfel in [45, Lemma 2.3], using the original definition of Dedekind sums via Dedekind η-function.

at one point. I n general, it is often not difficult to satisfy these type of conditions (i.e., thinking combinatorially about $\mathcal{A}$), but it is hard to decide whether we can realize $\mathcal{A}$ in Z (i.e., to prove or disprove its existence). There are more constrains, as it was shown by Hirzebruch [42, p. 140]. His inequality is a reformulation of the Miyaoka-Yau inequality plus some results of Sakai [76]. The extra restrictions for existence depend on the field of definition of $\mathcal{A}$. For example, the Fano arrangement (seven lines with only triple points on $\mathbb{P}^2$) exists only in char 2, the Hesse arrangement exists over $\mathbb{C}$ but not over $\mathbb{R}$, Quaternion $(3,6)$-nets (see below) do not exist in char 2 but they do exist over $\mathbb{C}$.

Our second main result is to show a one-to-one correspondence which translates the question of existence of certain arrangements of curves into the question of existence of a single curve in projective space. We first study line arrangements in $\mathbb{P}^2$ via the moduli spaces of genus zero marked curves $\overline{M}_{0,d+1}$. These spaces have a wonderful description due to Kapranov [50] and [49]. Using Kapranov construction, we prove that an arrangement of d lines in $\mathbb{P}^2$ corresponds to one line in $\mathbb{P}^{d-2}$. The precise result is the following.

Proposition III.6. There is a one-to-one correspondence between pairs $(\mathcal{A}, P)$ up to isomorphism, where $\mathcal{A}$ is an arrangement of d lines in $\mathbb{P}^2$ and P is a point outside of $\mathcal{A}$, and lines in $\mathbb{P}^{d-2}$ outside of a certain fixed arrangement of hyperplanes $\mathscr{H}_d$. This correspondence is independent of the field of definition.

We also provide an elementary proof of Proposition III.6 in Section 3.2. Local properties of this correspondence and the particular construction of Kapranov, allow us to prove the following more general theorem.

Let $d \geq 3$ be an integer. Let C be a smooth projective curve and let $\mathcal{L}$ be a

line bundle on C with $\deg(\mathcal{L}) > 0$. Let $\mathscr{A}_d$ be the set of all isomorphism classes of arrangements $\mathcal{A}(C, \mathcal{L})$ which are primitive (Definition II.16) and simple crossings (i.e., any two curves in $\mathcal{A}(C, \mathcal{L})$ intersect transversally). On the other hand, let $\mathscr{B}_d$ be the set of irreducible projective curves B in $\mathbb{P}^{d-2}$ that are birational to C, locally factor in smooth branches which are transversal to the hyperplanes of $\mathscr{H}_d$, and satisfy that, if H is a hyperplane in $\mathbb{P}^{d-2}$ and $\nu : C \to B$ is the normalization of B, then $\mathcal{L} \simeq \mathcal{O}_C(\nu^*(H \cap B))$.

Theorem III.10. There is a one-to-one correspondence between $\mathscr{A}_d$ and $\mathscr{B}_d$.

One good thing about this correspondence is that it involves directly the spaces $\overline{M}_{0,n}$, giving a recipe to find lots of curves in $\overline{M}_{0,n}$ from arrangements $\mathcal{A}$. The construction of curves in these spaces is an important issue around Fulton's conjecture, in particular rigid curves. We intend to use it for that purpose in the future.

Coming back to lines in $\mathbb{P}^2$, this correspondence gives an effective way to prove or disprove the existence of line arrangements. Moreover, it allows us to have a concrete parameter space for line arrangements with fixed combinatorial data (given by the intersections of the lines) in the corresponding Grassmannian of lines. To eliminate the "artificial" point P in the pair $(\mathcal{A}, P)$, we take P in $\mathcal{A}$ and consider the new pair $(\mathcal{A}', P)$ with $\mathcal{A}'$ equal to $\mathcal{A}$ minus the the lines through P. Hence, the lines in $\mathbb{P}^{d'-2}$ (where $d' = |\mathcal{A}'|$) corresponding to $(\mathcal{A}', P)$ give us the parameter space for $\mathcal{A}$.

We use this correspondence to find new line arrangements, and in doing so, we classify $(3, q)$-nets for $2 \leq q \leq 6$. In general, (p, q)-nets are particular line arrangements (with long history [23]). They can be thought as the geometric structures of finite quasigroups. Recently, they have appeared in the work of Yuzvinsky (see [95]), where they play a special role in the study of the cohomology of local systems on

the complements of complex line arrangements (see Section 7.2). On the other hand, some of them are key examples in the construction of extremal surfaces X. These arrangements are in one-to-one correspondence with certain special pencils in $\mathbb{P}^2$. If we consider them without ordering their lines, $(3, q)$-nets are in bijection with the main classes of $q \times q$ Latin squares [23]. These main classes are known for $q < 11$.

Subsection 3.5.2. Only nine of the twelve main classes of 6×6 Latin squares are realized by $(3, 6)$-nets in $\mathbb{P}^2$ over $\mathbb{C}$. There exists an explicit parametrization of these nine cases, giving the equations of the lines. Among them, we have four three dimensional and five two dimensional families, some of them define nets only over $\mathbb{C}$, for others we have nets over $\mathbb{R}$, and even for one of them over $\mathbb{Q}$.

This brings a new phenomena for 3-nets, since for example all main classes are realizable for $2 \leq q \leq 5$. It was expected that their parameter spaces have the same dimension, but we found that this is not true for $q = 6$. In [81], it is noticed that with current methods, it is hard to decide which $(3, q)$-nets can be realized on $\mathbb{P}^2$. Our tool seems to organize much better the information to actually compute them. To show that our method does indeed work, we take the 8×8 main class corresponding to the Quaternion group, and we prove that the corresponding $(3, 8)$-nets exist and form a three dimensional family defined over $\mathbb{Q}$ (Subsection 3.5.4). The new cases corresponding to the symmetric and Quaternion groups show that it is also possible to obtain 3-nets from non-abelian groups (in [95], Yuzvinsky conjectured that 3-nets only existed for certain abelian groups). In this way, we left the following question open: *find a characterization for the main classes of Latin squares which realize 3-nets on* $\mathbb{P}^2_{\mathbb{C}}$.

The core of our work is in the articles [87] and [86]. We will not refer to them in

this thesis. Our purpose is to develop the ideas and proofs of these articles in more detail. We include in Chapter VI our first steps towards deformations of the surfaces X of Theorem V.2. Deformations may help to to understand potential restrictions to obtain surfaces with Chern numbers ratio close to the Miyaoka-Yau bound. In addition, Deformations may reveal properties for certain surfaces, such as minimality and rigidity (see Section 7.1).

CHAPTER II

Arrangements of curves

In this Chapter, we define arrangements of curves, and we show various examples of them. We put emphasis on the realization of an incidence by means of an arrangement, and on formulas and restrictions for logarithmic Chern numbers. Sections 2.2 and 2.4 form the base for the one-to-one correspondence between arrangements and single curves, which will be developed and proved in Chapter III. In Section 2.3, we introduce very special line arrangements, which are called nets. We will use our one-to-one correspondence to classify, in Section 3.5, $(3,q)-$nets for $2 \leq q \leq 6$, and the Quaternion nets. Nets provide good examples for the realization problem, for non-trivial incidences (via Latin squares) and their corresponding parameter spaces, and for extreme logarithmic Chern numbers. Nets are also important to understand certain invariants of the fundamental group of the complement of complex line arrangements (this is explained in Section 7.2).

2.1 Definitions.

Definition II.1. Let Z be a smooth projective surface, and let $d \geq 3$ be a positive integer. An arrangement of curves $\mathcal{A}$ in Z is a collection of smooth projective curves $\{C_1, \ldots, C_d\}$ such that $\bigcap_{i=1}^{d} C_i = \emptyset$. An arrangement is said to be defined over a field $\mathbb{K}$ if all C_i are defined over $\mathbb{K}$. Two arrangements $\mathcal{A} = \{C_1, \ldots, C_d\}$ and $\mathcal{A}' =$

$\{C'_1, \ldots, C'_d\}$ are said to be isomorphic if there exists an automorphism $T : Z \to Z$ such that $T(C_i) = C'_i$ for all i.

We notice that with this definition of isomorphism, what matters is how the arrangement lies on Z, and also the order of its curves. We will consider $\mathcal{A}$ as the set $\{C_1, \ldots, C_d\}$, or as the divisor $C_1 + \ldots + C_d$, or as the curve $\bigcup_{i=1}^{d} C_i$. The most important arrangements for us are the following.

Definition II.2. An arrangement of curves $\mathcal{A}$ in Z is said to be simple crossings if any two curves of $\mathcal{A}$ intersect transversally. For $2 \leq k \leq d-1$, a k-point is a point in $\mathcal{A} = \bigcup_{i=1}^{d} C_i$ which belongs to exactly k curves. The number of k-points of $\mathcal{A}$ is denoted by t_k.

Definition II.3. Let e be a positive integer. An incidence of order (d, e) is a pair of sets $(\mathcal{A}, \mathcal{X})$ of cardinalities d and e respectively, such that for each element of $\mathcal{X}$ (points) we associate k elements of $\mathcal{A}$ (curves), for some $k \in \{2, 3, \ldots, d\}$. An incidence is denoted by $\mathcal{I}(d, e)$, or $\mathcal{I}$ when the pair (d, e) is understood or not relevant.

Classical examples of incidences are the so-called abstract (a_α, b_β)-configurations (see for example [24] or [36]). In [40], Hilbert encodes the incidence of certain (potential) line arrangements $\mathcal{A}$ in $\mathbb{P}^2$ by means of a matrix where columns are points in $\mathcal{A}$ and entries are labelled lines, indicating which lines are required to pass trough a common point. There are several ways to represent an incidence. For example, we will use Latin squares to encode incidences for nets. Of course, an incidence wants to model part of the intersections for an arrangement of curves. What happens is that we can often think of an arrangement abstractly, by only giving an incidence of d "pseudo" curves, and then we ask if the incidence can be realized as an arrangement

of curves in Z. We will consider only simple crossings arrangements as answers. In this way, we keep the incidence information simple enough.

Definition II.4. Let $\mathcal{I}$ be an incidence of order (d, e). We say that $\mathcal{I}$ is realizable in Z over $\mathbb{K}$ if there exists a simple crossings arrangement of d curves defined over a field $\mathbb{K}$ satisfying $\mathcal{I}$. The set of isomorphism classes of simple crossing arrangements of d curves over $\mathbb{K}$ satisfying $\mathcal{I}$ is denoted by $M(\mathcal{I}, \mathbb{K})$.

Notice that, according to our definition, an incidence does not determine all the intersections of the possible arrangement. This will be evident when we consider nets on $\mathbb{P}^2$.

Let $\mathcal{A} = \{C_1, \ldots, C_d\}$ be a simple crossings arrangement on Z. We now want to describe the open variety $Z \setminus \mathcal{A}$ from the log point of view. To this end, we consider the surface Y which is the blow-up at all the k-points of $\mathcal{A}$ with $k \geq 3$. Let $\sigma : Y \to Z$ be the corresponding birational map, and let $\mathcal{A}'$ be the reduced total transform of $\mathcal{A}$ under σ. Hence, it includes the exceptional divisors over the $(k \geq 3)$-points. Consider $\mathcal{A}'$ as an arrangement on Y. Then, $(Y, \mathcal{A}')$ is a log surface (Definition I.21). We refer to it as the associated pair to $(Z, \mathcal{A})$. We can easily compute the logarithmic Chern numbers of this pair with respect to $(Z, \mathcal{A})$:

$$\bar{c}_1^2(Y, \mathcal{A}') = c_1^2(Z) - \sum_{i=1}^{d} C_i^2 + \sum_{k \geq 2} (3k - 4)t_k + 4 \sum_{i=1}^{d} (g(C_i) - 1)$$

and

$$\bar{c}_2(Y, \mathcal{A}') = c_2(Z) + \sum_{k \geq 2} (k - 1)t_k + 2 \sum_{i=1}^{d} (g(C_i) - 1).$$

We will be interested in extremal arrangements, in the sense that we want the log Chern numbers ratio of $(Y, \mathcal{A}')$ be as close as possible to 3. Having this in mind, we

define for every simple crossings arrangement $\mathcal{A}$ having $\bar{c}_2(Y,\mathcal{A}') \neq 0$, the error of $(Z,\mathcal{A})$ as $E(Z,\mathcal{A}) = \frac{3\bar{c}_2(Y,\mathcal{A}')-\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')}$, and so

$$E(Z,\mathcal{A}) = \frac{3c_2(Z) - c_1^2(Z) + \sum_{i=1}^d C_i^2 + \sum_{k\geq 2} t_k + 2\sum_{i=1}^d (g(C_i)-1)}{c_2(Z) + \sum_{k\geq 2}(k-1)t_k + 2\sum_{i=1}^d (g(C_i)-1)}.$$

Remark II.5. We want to have this ratio close to 3 from below, in accordance to a log Miyaoka-Yau inequality. For example, it is well-known that a $K3$ surface can have at most 16 disjoint rational smooth curves, with equality if and only if it is a Kummer surface [69]. Let Z be any Kummer surface, and let $\mathcal{A}$ be the arrangement formed by the 16 disjoint (-2)-curves. Then, $\bar{c}_1^2 = -32$ and $\bar{c}_2 = -8$. Hence $\frac{\bar{c}_1^2}{\bar{c}_2} = 4$. In this way, we see that not any arrangement works for this Miyaoka-Yau point of view. However, in our constructions of surfaces of general type, we will use arrangements for which log Miyaoka-Yau inequality holds.

2.2 Line arrangements in $\mathbb{P}^2$.

In this section $Z = \mathbb{P}^2$, and $d \geq 3$. Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be an arrangement of lines in $\mathbb{P}^2$. It is a simple crossings arrangement. The study of line arrangements is an old subject (more than 100 years old) with a huge bibliography. For the importance it had back then, one can check [40].

An incidence $\mathcal{I}$ has a chance to be realized as a line arrangement if it does not violate the statement: two lines intersect at one point. In general this type of restriction comes from the Picard group of Z. For line arrangements, we also have the combinatorial fact

$$\frac{d(d-1)}{2} = \sum_{k\geq 2} \frac{k(k-1)}{2} t_k,$$

being the unique linear equation on t_k's that they satisfy. This is of course field independent.

More subtle restrictions come from the field of definition. The following is a non-trivial constraint for line arrangements defined over $\mathbb{C}$ due to Hirzebruch [42, p. 140][1]. If $t_{d-1} = 0$ ($t_d = 0$ is always assumed), then

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(k-4)t_k.$$

For example, it says that there are no complex line arrangements without 2- and 3- points. This inequality is also used to disprove the realization of certain incidences. A well-known example is the Fano arrangement which is defined only over fields of characteristic 2. It has 7 lines, $t_3 = 7$, and $t_k = 0$ otherwise. One easily checks that the inequality above is violated by the Fano arrangement.

Over the real numbers, any line arrangement must have a 2-point (Gallai 1933). Moreover, we have the harder lower bound $t_2 \geq \left[\frac{d}{2}\right]$ for any real line arrangement [42, p. 115]. This is no longer true over $\mathbb{C}$, as it is shown by the following examples.

Example II.6. (Fermat arrangements) The Fermat arrangement is defined by $(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0$. The name is because this arrangement is exactly the singular locus of the pencil $u(x^n - y^n) + t(y^n - z^n) = 0$, where all the non-singular members are isomorphic to Fermat curve $x^n + y^n + z^n = 0$. For $n = 1$ we have a triangle. For $n = 2$ we have the complete quadrilateral of 6 lines, which has $t_2 = 3$, $t_3 = 4$ and $t_k = 0$ otherwise. For $n = 3$, this is the dual Hesse arrangement having $t_3 = 12$ and $t_k = 0$ otherwise. For $n \geq 4$, we have $t_n = 3$, $t_3 = n^2$ and $t_k = 0$ otherwise. We see that for $n \geq 3$, a Fermat arrangement cannot be defined over $\mathbb{R}$.

Example II.7. (Klein arrangement) The simple group of order 168 acts on $\mathbb{P}^2$. It has 21 involutions, each leaving a line fixed. The arrangement of these 21 lines is called the Klein arrangement (Klein 1879). A nice description of it can be found in

[1] This inequality is actually due to Sakai and Hirzebruch. In [42], Hirzebruch found this inequality with $\frac{3}{4}$ replaced by 1. At the end of his paper, by using a result of Sakai, he was able to improve it.

[13]. It has $t_3 = 28$, $t_4 = 21$ and $t_k = 0$ otherwise.

We also have the following analogue of the Hirzebruch inequality due to Iitaka [46], which holds for arrangements strictly defined over $\mathbb{R}$, and is much easier to prove[2],

$$t_2 \geq 3 + \sum_{k\geq 4}(k-3)t_k.$$

Proof. Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be an arrangement of lines in $\mathbb{P}^2_{\mathbb{R}}$. We use that $\mathbb{P}^2_{\mathbb{R}}$ is a 2- dimensional real manifold, and any $\mathcal{A}$ defines a cell structure. Let p_k be the number of 2-cells bounded by k-gons. In this way, $p_2 = 0$, since we never consider the trivial arrangement. As in [42, p. 115], let f_0 be the number of vertices, f_1 the number of edges, and f_2 the number of 2-cells defined by the arrangement $\mathcal{A}$. Then, $f_0 = \sum_{k\geq 2} t_k$, $f_1 = \frac{1}{2}\sum_{k\geq 2} 2kt_k = \frac{1}{2}\sum_{k\geq 2} kp_k$, and $f_2 = \sum_{k\geq 2} p_k$. On the other hand,

$$f_0 - f_1 + f_2 = e(\mathbb{P}^2_{\mathbb{R}}) = 1.$$

We re-arrange the terms of the previous equation, to obtain

$$\sum_{k\geq 2}(k-3)p_k = -3 - \sum_{k\geq 2}(k-3)t_k.$$

Notice that the right-hand side is positive. The inequality follows. Notice that equality holds if and only if $p_k = 0$ for $k \geq 4$. In that case, the arrangement is called simplicial. □

The log Chern numbers and the error associated to $(\mathbb{P}^2, \mathcal{A})$ are

$$\bar{c}_1^2(Y, \mathcal{A}') = 9 - 5d + \textstyle\sum_{k\geq 2}(3k-4)t_k \quad \text{and} \quad \bar{c}_2(Y, \mathcal{A}') = 3 - 2d + \sum_{k\geq 2}(k-1)t_k$$

and

$$E(\mathbb{P}^2, \mathcal{A}) = \frac{\sum_{k\geq 2} t_k - d}{3 - 2d + \sum_{k\geq 2}(k-1)t_k}.$$

[2]Iitaka erroneously claimed that the inequality holds for any complex line arrangement. This error was noticed before [42, p. 136].

In [46], Iitaka studies various properties of the associated log surfaces. For example, he shows that almost all of these varieties are of log general type, that is, $\bar{\kappa}(Y, \mathcal{A}') = 2$.

The following inequality imposes a restriction to the log Chern numbers corresponding to line arrangements, and it shows which are the extremal cases. The proof relies on the Hirzebruch inequality.

Proposition II.8. *Let $\mathcal{A}$ be an arrangement of d lines in $\mathbb{P}^2$ over $\mathbb{C}$ ($t_d = 0$ as always). Then,*

$$\sum_{k\geq 2}(4-k)t_k \geq 3 + d$$

with equality if and only if $\mathcal{A}$ is isomorphic to the dual Hesse arrangement or $t_{d-1} = 1$ or $d = 3$. This inequality is equivalent to $\frac{\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')} \leq \frac{8}{3}$ for all allowed pairs $(\mathbb{P}^2, \mathcal{A})$, and so equality holds if and only if $\mathcal{A}$ is isomorphic to the dual Hesse arrangement.

Proof. First notice that when $d = 3$ or $t_{d-1} = 1$ (and so $t_2 = d-1$), we have equality. Therefore, we assume $t_d = t_{d-1} = 0$ and $d > 3$. Then, from [42] we have

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k\geq 5}(k-4)t_k$$

and so it is enough to prove $t_2 + \frac{1}{4}t_3 \geq 3$. The proof goes case by case. Suppose $t_2 + \frac{1}{4}t_3 < 3$. Its possible non-negative integer solutions are $(t_2, t_3) = (0, 0)$, $(1, n)$ for $n = 0, \ldots, 7$ and $(2, n)$ for $n = 0, \ldots, 3$. By using the combinatorial equality $\frac{d(d-1)}{2} = \sum_{k\geq 2}\frac{k(k-1)}{2}t_k$ and Hirzebruch inequality, we easily check that none of them are possible.

Assume we have equality, i.e., $\sum_{k\geq 2}(4-k)t_k = 3 + d$ and we do not have $d = 3$ or $t_{d-1} = 1$. Then, by Hirzebruch inequality, $t_2 + \frac{1}{4}t_3 = 3$. Its possible solutions are $(t_2, t_3) = (3, 0), (2, 4), (1, 8), (0, 12)$. Easily one checks that the first three are not

possible. So, $t_2 = 0$, $t_3 = 12$ and, by Hirzebruch inequality again, $d \leq 9$. By the combinatorial equality, $d = 9$ and $t_k = 0$ for all $k \neq 3$. Write $\mathcal{A} = \{L_1, L_2, \ldots, L_9\}$ such that $L_1 \cap L_2 \cap L_3$ is one of the twelve 3-points. Since over any line of $\mathcal{A}$ there are exactly four 3-points, there is a 3-point outside of $L_1 \cup L_2 \cup L_3$. Say $L_4 \cap L_5 \cap L_6$ is this point, then $L_7 \cap L_8 \cap L_9$ gives another 3-point. This gives a $(3,3)$-net with three special members $\{L_1, L_2, L_3\}$, $\{L_4, L_5, L_6\}$ and $\{L_7, L_8, L_9\}$. One can prove that this $(3,3)$-net is unique up to projective equivalence (see for example [87]). This is isomorphic to the dual Hesse arrangement. □

In [80, p. 220], Sommese proves almost the same statement in the spirit of the Hirzebruch's coverings [42]. Notice that there are several simplicial arrangements satisfying the Iitaka equality for real arrangements (see [42, pp. 116-118]), but the previous equality is satisfied by only one nontrivial arrangement, the dual Hesse arrangement.

Is the previous inequality a topological fact as the Iitaka inequality for real arrangements? In general, the question is: *are log Miyaoka-Yau inequalities topological facts of the underlying surface?*

2.3 (p,q)-nets.

We now introduce a special type of line arrangements in $\mathbb{P}^2$ which are called nets. Our references are [23], [81] and [95]. We start with the definition of a net taken from [81].

Definition II.9. Let $p \geq 3$ be an integer. A $\underline{p\text{-net}}$ in $\mathbb{P}^2$ is a $(p+1)$-tuple $(\mathcal{A}_1, ..., \mathcal{A}_p, \mathcal{X})$, where each $\mathcal{A}_i$ is a nonempty finite set of lines of $\mathbb{P}^2$ and $\mathcal{X}$ is a finite set of points of $\mathbb{P}^2$, satisfying the following conditions:

(1) The $\mathcal{A}_i$'s are pairwise disjoint.

(2) The intersection point of any line in $\mathcal{A}_i$ with any line in $\mathcal{A}_j$ belongs to $\mathcal{X}$ for $i \neq j$.

(3) Through every point in $\mathcal{X}$ there passes exactly one line of each $\mathcal{A}_i$.

One can prove that $|\mathcal{A}_i| = |\mathcal{A}_j|$ for every i, j and $|\mathcal{X}| = |\mathcal{A}_1|^2$ [81]. Let us denote $|\mathcal{A}_j|$ by q, this is the degree of the net. In classical notation, a p-net of degree q is an abstract (q_p^2, pq_q)-configuration. Following [81] and [95], we will use the notation (p, q)-net for a p-net of degree q. We label the lines of $\mathcal{A}_i$ by $\{L_{q(i-1)+j}\}_{j=1}^{q}$ for all i, and define the arrangement $\mathcal{A} = \{L_1, L_2, ..., L_{pq}\}$.

Example II.10. Any Fermat arrangement II.6 defines a $(3, n)$-net. We can take $\mathcal{A}_1 = \{x^n - y^n = 0\}$, $\mathcal{A}_2 = \{y^n - z^n = 0\}$ and $\mathcal{A}_3 = \{x^n - z^n = 0\}$, labelling the lines in some order. The dual of the dual Hesse arrangement, i.e. the Hesse arrangement, is a $(4, 3)$- net. See [1] for a concrete description of its lines. Each of the four sets of lines is a triangle, and so $d = 12$, $t_2 = 12$, $t_4 = 16$ and $t_k = 0$ otherwise.

Assume for now that the lines are defined over an algebraically closed field $\mathbb{K}$. A (p, q)-net $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_p, \mathcal{X})$ gives a pencil of curves of degree q with p distinguished members $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p\}$. Take any two sets of lines $\mathcal{A}_i$ and $\mathcal{A}_j$. Consider $\mathcal{A}_i$ and $\mathcal{A}_j$ as the equations which define them, that is, the multiplication of its lines. Let $\mathscr{P}$ be the pencil $u\mathcal{A}_i + t\mathcal{A}_j = 0$ in $\mathbb{P}^2$, where $[u, t] \in \mathbb{P}^1$. Let F be any curve of degree q passing through the q^2 points in $\mathcal{X}$. Take a point Q in $F \setminus \mathcal{X}$. Then, there exists $[u, t] \in \mathbb{P}^1$ such that $u\mathcal{A}_i(Q) + t\mathcal{A}_j(Q) = 0$. By Bezout's theorem, $F = au\mathcal{A}_i + at\mathcal{A}_j$ for some $a \in \mathbb{K}$. Therefore, every F of degree q containing $\mathcal{X}$ is part of the pencil $\mathscr{P}$, and in particular any two members of $\mathcal{A}$ give the same pencil $\mathscr{P}$. Moreover, if the characteristic of $\mathbb{K}$ is zero, the general member of this pencil is smooth. Hence, after we blow up the q^2 points in $\mathcal{X}$ we obtain a fibration of curves of genus $\frac{(q-1)(q-2)}{2}$

with p completely reducible fibers.

Nets in $\mathbb{P}^2$ defined over $\mathbb{C}$ have the following restriction [95]. The proof is a simple topological argument which uses the topological Euler characteristic of the fibration over $\mathbb{P}^1$ obtained by blowing up the q^2 points in $\mathcal{X}$.

Proposition II.11. *For an arbitrary (p,q)-net in $\mathbb{P}^2$ defined over $\mathbb{C}$, the only possible values for (p,q) are: $(p=3, q\geq 2)$, $(p=4, q\geq 3)$ and $(p=5, q\geq 6)$.*

A Latin square is a $q\times q$ table filled with q different symbols (in our case numbers from 1 to q) in such a way that each symbol occurs exactly once in each row and exactly once in each column [23]. They are the multiplication tables of finite quasigroups [3]. Let $\mathcal{A}=(\mathcal{A}_1,...,\mathcal{A}_p,\mathcal{X})$ be a (p,q)-net. The pair $(\mathcal{A},\mathcal{X})$ defines an incidence of degree (pq,q^2) (see Definition II.3). The set $\mathcal{X}$ and its properties fix the incidence which defines the net. This incidence is encoded in a collection of $q\times q$ Latin squares. More precisely, the q^2 p-points in $\mathcal{X}$ are determined by $(p-2)$ $q\times q$ Latin squares which form an orthogonal set, as explained in [23] or [81].

Although we have defined nets as arrangements of lines already on $\mathbb{P}^2$, we first think "combinatorially" about the (possible) (p,q)-net defined by this set of $(p-2)$ Latin squares, that is, we only consider the incidence defined by the orthogonal set of Latin squares. Then, we try to answer the question of realizability of this incidence as a (p,q)-net on $\mathbb{P}^2$ over some field.

In his Ph.D. thesis [82], Stipins proves that there are no $(4,d\geq 4)$-nets and $(5,d)$-nets over $\mathbb{C}$ [4]. His proof does not use the combinatorics given by the corresponding orthogonal set of Latin squares. In this way, by Proposition II.11, the only cases left over $\mathbb{C}$ are the 3-nets. In [95], it is proved that for every finite subgroup H of a

[3] A quasigroup is a set Q with a binary operation, such that for each $a,b\in Q$, there exist unique elements $x,y\in Q$ satisfying $ax=b$ and $ya=b$.

[4] Recently, I was told that there is a gap in his proof for $(4,q)$-nets with $q\geq 4$. We do not use this result in what follows.

smooth elliptic curve, there exists a 3-net in $\mathbb{P}^2$ over $\mathbb{C}$ corresponding to the Latin square of the multiplicative table of H. In the same paper, Yuzvinsky proves that there are no 3-nets associated to the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A classification of $(3, q)$-nets for $2 \leq q \leq 5$ can be found in [81]. The classification of $(3, 6)$-nets was unknown. We will classify $(3, q)$-nets for $2 \leq q \leq 6$, and the $(3, 8)$-nets corresponding to the multiplication table of the Quaternion group. In the next Chapter we will introduce a new method to deal with these kind of problems. Using this method, we achieve this classification.

Remark II.12. (Main classes of Latin squares) As we explained before, a $q \times q$ Latin square defines the set $\mathcal{X}$ for a $(3, q)$-net $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$. What if we are interested only in the realization of $\mathcal{A}$ in $\mathbb{P}^2$ as a curve, i.e., without labelling the lines? Then, we divide the set of all $q \times q$ Latin squares into main classes.

For a given $q \times q$ Latin square M corresponding to $\mathcal{A}$, by rearranging rows, columns and symbols of M, we obtain a new labelling for the lines in each $\mathcal{A}_i$. If we write M in its orthogonal array representation, $M = \{(r, c, s) : r = \text{row number}, \ c = \text{column number}, \ s = \text{symbol number}\}$, we can perform six operations on M, each of them a permutation of (r, c, s) which translates into relabelling the members $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$, and so we obtain the same arrangement on $\mathbb{P}^2$. We can partition the set of all $q \times q$ Latin squares in main classes (also called Species) in the following way: if M, N belong to the same class, then we can obtain N by applying several of the above operations to M. In what follows, we will consider one member of each main class, which is actually the multiplication table of a loop [5]. The following is a table for the number of main classes for small q.

[5] A loop is a quasigroup which has an identity element

q	1	2	3	4	5	6	7	8	9
# main classes	1	1	1	2	2	12	147	283,657	19,270,853,541

2.4 Arrangements of sections.

Let C be a smooth projective curve and let $\mathcal{E}$ be a normalized locally free sheaf of rank 2 on C, that is, $\mathcal{E}$ is a rank 2 locally free sheaf on C with the property that $H^0(\mathcal{E}) \neq 0$ but for all invertible sheaves $\mathcal{L}$ on C with $\deg(\mathcal{L}) < 0$, we have $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ [39, p. 372]. We consider the geometrically ruled surface $\pi : \mathbb{P}_C(\mathcal{E}) \to C$. As in [39, p. 373], we let $\mathfrak{e}$ be the divisor on C corresponding to the invertible sheaf $\bigwedge^2 \mathcal{E}$, so that the invariant e is $-\deg(\mathfrak{e})$. We fix a section C_0 of $\mathbb{P}_C(\mathcal{E})$ with $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(C_0) \simeq \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$, and so $C_0^2 = -e$.

Let $d \geq 3$ be an integer. Let $\mathcal{A} = \{S_1, S_2, ..., S_d\}$ be a set of d sections (as curves) of π. We will assume that $S_i \neq C_0$ for all i. By performing elementary transformations [6] on the points in $C_0 \cap \mathcal{A}$, we obtain another $\mathbb{P}_C(\mathcal{E}')$ and $\mathcal{A}'$ such that $S_i' \cap C_0' = \emptyset$ for all i. In particular there are two disjoint sections and so $\mathcal{E}'$ is decomposable. Again we normalize $\mathcal{E}'$ so that there is an invertible sheaf $\mathcal{L}$ on C with $\deg(\mathcal{L}) \geq 0$ such that $\mathcal{E}' \simeq \mathcal{O}_C \oplus \mathcal{L}^{-1}$. Hence, for every section $S_i' \in \mathcal{A}'$, we have $S_i' \sim C_0' + \pi^*(\mathcal{L})$. Therefore, we can always start with $\mathcal{A}$ on a decomposable geometrically ruled surface such that $S_i \in |C_0 + \pi^*(\mathcal{L})|$ for every $i \in \{1, 2, ..., d\}$. Assume this is the case. The following are two trivial situations we want to eliminate.

(1) (Base points) This means $\bigcap_{i=1}^d S_i \neq \emptyset$. Then, we perform elementary transformations on the points in $\bigcap_{i=1}^d S_i$, and we consider the new obvious arrangement $\mathcal{A}'$ on the corresponding new decomposable geometrically ruled surface.

(2) Assume $e = \deg(\mathcal{L}) = 0$. In this case $S_i.S_j = 0$ for all i, j. Since $d \geq 3$,

[6] An elementary transformation is the blow-up of a point in $\mathbb{P}_C(\mathcal{E})$ followed by the blow-down of the proper transform of the fiber containing that point.

we consider $\pi : \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \to C$ as a fibration of $(d+1)$-pointed smooth stable curves of genus zero and the corresponding commutative diagram.

$$\begin{array}{ccc} \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}) & \longrightarrow & \overline{M}_{0,d+2} \\ \pi \downarrow & & \downarrow \\ C & \longrightarrow & \overline{M}_{0,d+1} \end{array}$$

This implies that $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \simeq C \times \mathbb{P}^1$, that is, $\mathcal{L} \simeq \mathcal{O}_C$. Hence, $\mathcal{A}$ is a collection of fibers of the projection to $\mathbb{P}^1$.

If $\mathcal{A} \subseteq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})$ is such that (1) and (2) do not hold, then any elementary transformation on any point of the surface will give us back one of the above situations. We now introduce what seems to be the right definition for arrangements of sections on geometrically ruled surfaces.

Definition II.13. Let $d \geq 3$ be an integer. Let C be a smooth projective curve and $\mathcal{L}$ be an invertible sheaf on C of degree $e > 0$. An arrangement of sections $\mathcal{A} = \mathcal{A}(C, \mathcal{L})$ is a set of d sections $\{S_1, S_2, ..., S_d\}$ of $\pi : \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \to C$ such that $S_i \sim C_0 + \pi^*(\mathcal{L})$ for all $i \in \{1, 2, ..., d\}$ and $\bigcap_{i=1}^d S_i = \emptyset$.

Therefore, $\mathcal{A}$ is an arrangement on $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})$. We notice that $\bigcap_{i=1}^d S_i = \emptyset$ implies that $\mathcal{L}$ is base point free. To see this, take a point $c \in C$ and consider the corresponding fiber F_c. Since $\bigcap_{i=1}^d S_i = \emptyset$, there are two sections S_i, S_j such that $F_c \cap S_i \cap S_j = \emptyset$. Let $\sigma_j : C \to \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})$ be the map defining the section S_j. Then, $\mathcal{L} \simeq \sigma_j^*(\pi^*(\mathcal{L}) \otimes \mathcal{O}_{S_j}) \simeq \sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})}(C_0) \otimes \pi^*(\mathcal{L}) \otimes \mathcal{O}_{S_j}) \simeq \sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})}(S_i) \otimes \mathcal{O}_{S_j})$ and $\sigma_j^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})}(S_i) \otimes \mathcal{O}_{S_j})$ is given by an effective divisor on C not supported at c. This tell us that $\mathcal{L} \simeq \mathcal{O}_C(D)$ where D is a base point free effective divisor on C.

Definition II.14. Let $\mathcal{A}(C, \mathcal{L})$, $\mathcal{A}'(C', \mathcal{L}')$ be two arrangements of d sections. A

morphism between them is a finite map $f : C \to C'$ and a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}) & \xrightarrow{F} & \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}'^{-1}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ C & \xrightarrow{\quad f \quad} & C' \end{array}$$

such that $F(S_i) = S'_i$ for all $i \in \{1, 2, ..., d\}$. If F is an isomorphism, then the arrangements are said to be isomorphic.

Example II.15. An arrangement of d sections $\mathcal{A}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ is the same as a pair $(\mathcal{A}, P)$ with $\mathcal{A}$ an arrangement of d lines on $\mathbb{P}^2$ (as in Section 2.2) and P a point outside of $\mathcal{A}$. Given $\mathcal{A}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ on $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, we blow down the (-1)-curve C_0, and we obtain a pair $(\mathcal{A}, P)$. Conversely, given $(\mathcal{A}, P)$, we blow up P and obtain an arrangement $\mathcal{A}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.

Fix an arrangement of d sections $\mathcal{A} = \mathcal{A}(C, \mathcal{L})$. Let $f : C' \to C$ be a finite morphism between smooth projective curves. Consider the induced base change:

$$\begin{array}{ccc} \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}'^{-1}) & \xrightarrow{F} & \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ C' & \xrightarrow{\quad f \quad} & C \end{array}$$

Then, as already is shown in the diagram, we obtain a decomposable geometrically ruled surface $\pi' : \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}'^{-1}) \to C'$ together with an arrangement of d sections $\mathcal{A}'$ given by the pull back under F of the sections in $\mathcal{A}$. Notice that $C'^2_0 = -e \deg(f)$. This leads us to the following definition.

Definition II.16. An arrangement of d sections $\mathcal{A} = \mathcal{A}(C, \mathcal{L})$ is said to be primitive if whenever we have an arrangement $\mathcal{A}' = \mathcal{A}'(C', \mathcal{L}')$ and a morphism as in Definition II.14, F is an isomorphism.

Example II.17. Every $\mathcal{A}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ is clearly primitive. For another example, consider the configuration on $\mathbb{P}^2$ formed by one conic and three lines as in Figure

2.1. We blow up the point P (in that figure) and obtain $\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$. After that, we perform an elementary transformation on the node of the total transform of the tangent line at P. It is possible to check that the resulting arrangement of four sections in $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ is primitive.

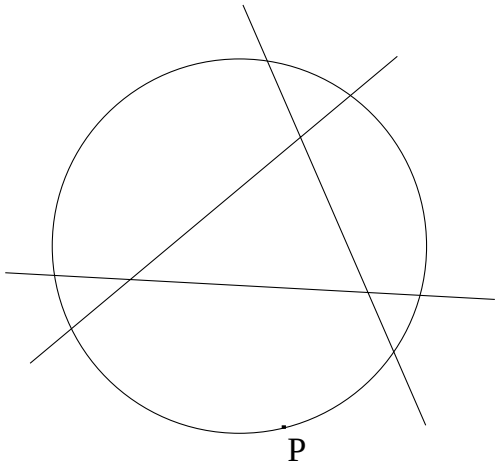


Figure 2.1: Configuration in example II.15.

Remark II.18. (Arrangements of sections in Hirzebruch surfaces) Consider the Hirzebruch surfaces $\mathbb{F}_e := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, with $e \geq 2$. Any arrangement of d sections can be seen as a collection of d curves in $\mathbb{F}_1$ by performing elementary transformations, so that the negative section C_0 goes to the (-1)-curve in $\mathbb{F}_1$. Notice there are several ways to perform these transformations. After doing that, we blow down this (-1)-curve to obtain an arrangement of d rational curves on $\mathbb{P}^2$.

Another way to induce an arrangement on $\mathbb{P}^2$ is the following. Let $\tau : \mathbb{F}_e \to \mathbb{P}^{e+1}$ be the map defined by the linear system $|S_{d+1} + \pi^*(\mathcal{O}_{\mathbb{P}^1}(e))|$. Then, it is an isomorphism outside of C_0 and $\tau(C_0)$ is a point. The image $\tau(\mathbb{F}_e)$ is a scroll in $\mathbb{P}^{e+1}$ swept by the lines passing through the point $\tau(C_0)$, and the normal rational curve $\tau(S_i)$ for any $i \neq d+1$. The sections $\{S_1, S_2, ..., S_d\}$ are all embedded into this scroll, and they are disjoint from the point $\tau(C_0)$. We can now choose a suitable point outside of $\tau(\mathbb{F}_e)$

to project this arrangement of d curves in the scroll to an arrangement of d rational nodal curves on $\mathbb{P}^2$. In general, this might also be done when $C \neq \mathbb{P}^1$ depending on what kind of line bundle $\mathcal{L}$ we are considering.

Let $\mathcal{A} = \{S_1, S_2, \ldots, S_d\}$ be a simple crossings arrangement of d sections in $Z = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})$. We now compute the log Chern numbers associated to $(Y, \mathcal{A}')$, using the formulas of Section 2.1. Notice that since $S_i \sim C_0 + \pi^*(\mathcal{L})$, we have $S_i^2 = e$. Also, we easily compute $c_1^2(Z) = 8(1-g(C))$ and $c_2(Z) = 4(1-g(C))$. Then,

$$\bar{c}_1^2(Y, \mathcal{A}') = 4(d-2)(g(C)-1) - de + \sum_{k\geq 2}(3k-4)t_k,$$

and

$$\bar{c}_2(Y, \mathcal{A}') = 2(d-2)(g-1) + \sum_{k\geq 2}(k-1)t_k.$$

Therefore,

$$E(\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1}), \mathcal{A}) = \frac{2(d-2)(g(C)-1) + de + \sum_{k\geq 2} t_k}{2(d-2)(g(C)-1) + \sum_{k\geq 2}(k-1)t_k}.$$

2.5 More examples of arrangements of curves.

2.5.1 Plane curves.

Let $\mathcal{A} = \{C_1, \ldots, C_d\}$ be an arrangement of d nonsingular plane curves in $\mathbb{P}^2$, such that any two intersect transversally and $\bigcap_{i=1}^d C_i = \emptyset$. Hence $\mathcal{A}$ is a simple crossings arrangement. Let n_a be the number of curves of degree a in $\mathcal{A}$. Then,

$$E(\mathbb{P}^2, \mathcal{A}) = \frac{\sum_{a\geq 1} a(2a-3)n_a + \sum_{k\geq 2} t_k}{\sum_{a\geq 1} a(a-3)n_a + \sum_{k\geq 2}(k-1)t_k + 3}.$$

Is it possible to improve $E(\mathbb{P}^2, \mathcal{A}) = \frac{1}{3}$ *(i.e., make it closer to zero) or*

$$\sum_{a\geq 1} a(5a-6)n_a + \sum_{k\geq 2}(4-k)t_k - 3 \geq 0$$

for any such arrangement? We saw that the last inequality is true for line arrangements. If $\mathcal{A}$ is composed by n_1 lines and n_2 conics, this potential inequality would read

$$8n_2 - n_1 + \sum_{k\geq 2}(4-k)t_k - 3 \geq 0.$$

2.5.2 Lines on hypersurfaces.

Our main reference here is [11]. Let Z be a smooth hypersurface in $\mathbb{P}^3$ of degree $n \geq 3$ (the quadric is treated in the next subsection). A line in Z is a line in $\mathbb{P}^3$ which also lives in Z. It is easy to prove that the set of all lines in Z form a finite set [7]. We consider the arrangement $\mathcal{A} = \{L_1, \ldots, L_d\}$ formed by all the lines in Z, subject to the condition $\bigcap_{i=1}^{d} L_i = \emptyset$. For $n = 3$, we always have lines, and the number is 27. For $n \geq 4$, we might not have any, so the surfaces we want to consider are special. Since we actually want extreme cases, with d large, they are very special. In [11], several examples are explicitly worked out. B. Segre proved in [78] that the maximum number of lines on a quartic is 64, and he gave the upper bound $(n-2)(11n-6)$ for the number of line on a smooth hypersurface of degree n. It remains an open question what is exactly the bound for $n \geq 5$.

Miyaoka [65] proved the following inequality for $\mathcal{A}$ (this is valid over $\mathbb{C}$),

$$nd - t_2 + \sum_{k\geq 3}(k-4)t_k \leq 2n(n-1)^2.$$

Any arrangement of lines in Z has simple crossings. One easily computes $L_i^2 = 2-n$, $c_1^2(Z) = n(n-4)^2$, and $\chi(Z, \mathcal{O}_Z) = \frac{1}{6}n(n^2-6n+11)$. Below we compute log Chern numbers for some interesting cases.

[7] Assume it is not finite. Divide the set of lines in Z in connected components. Since the Picard number of Z is finite, one of this components must have infinitely many lines. Take one line in this component, and consider the pencil associated to it. It has no fixed points, so it defines a fibration to $\mathbb{P}^1$. Infinitely many lines in a fiber is impossible, so there are infinitely many lines intersecting this fixed line. Then, Z has to be a ruled surface, contradicting $n > 2$.

Example II.19. (Fermat hypersurfaces) Let Z be the Fermat Hypersurface $x^n + y^n + w^n + z^n = 0$. This surface has exactly $3n^2$ lines. One can check that $t_2 = 3n^3$, $t_n = 6n$, and $t_k = 0$ otherwise. The log Chern numbers are $\bar{c}_1^2(n) = 2n(5n^2 - 4n - 4)$ and $\bar{c}_2(n) = 4n^2(n-1)$, and the error is $E(n) = \frac{n^2-2n+4}{2n(n-1)}$. Hence, $\frac{13}{28} \leq E(n) \leq \frac{7}{12}$, with equality on the left for $n = 7$ or 8, and on the right for $n = 3$.

Example II.20. (Cubics) Let Z be any smooth cubic, and let $\mathcal{A}$ be the arrangement formed by its 27 lines. We can only have 2-points, and 3-points (Classically called Eckardt points). For any cubic, we have $t_2 + 3t_3 = 135$ and so $\bar{c}_1^2 = 192 - t_3$, $\bar{c}_2 = 90 - t_3$ and $E(Z, \mathcal{A}) = \frac{2t_3-78}{t_3-90}$. Therefore, $\frac{7}{12} \leq E(Z, \mathcal{A}) \leq \frac{76}{89}$, with equality on the left when $t_3 = 18$ (i.e., only for the Fermat cubic) and on the right when $t_3 = 1$.

Example II.21. (Schur quartic) Let Z be the Schur quartic (F. Schur 1882)

$$x(x^3 - y^3) = w(w^3 - z^3).$$

It was studied by Schur in [77]. It achieves the maximum number of lines for a smooth quartic equal to 64. Let $\mathcal{A}$ be the arrangement formed by all the lines on Z. We use the general point of view of [11] (which is very helpful) to compute the numbers $t_4 = 8$, $t_3 = 64$, $t_2 = 192$ and $t_k = 0$ otherwise. Then, we have $E(Z, \mathcal{A}) = \frac{1}{3}$, or equivalently, $\bar{c}_1^2 = \frac{8}{3}\bar{c}_2$.

2.5.3 Platonic arrangements.

We denote the classes in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ by $\mathcal{O}(a, b)$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ be an arrangement of d curves in $Z = \mathbb{P}^1 \times \mathbb{P}^1$, such that $\mathcal{A}_1$ has d_1 curves in $|\mathcal{O}(1,1)|$, $\mathcal{A}_2$ has d_2 in $|\mathcal{O}(1,0)|$ and $\mathcal{A}_3$ has d_3 in $|\mathcal{O}(0,1)|$ (so $d = d_1 + d_2 + d_3$). Assume it is a simple crossings arrangement. Then, the error number is

$$E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{-2d_2 - 2d_3 + \sum_{k\geq 2} t_k + 4}{-2d_1 - 2d_2 - 2d_3 + \sum_{k\geq 2}(k-1)t_k + 4}.$$

Here we will consider some well-known arrangements coming from finite automorphism groups of $\mathbb{P}^1$. Let G be such group and g be an element of G. The arrangement $\mathcal{A}_1$ is defined by the orbit of the diagonal Δ of Z under the automorphisms $g : (x, y) \mapsto (x, g(y))$. Hence $\mathcal{A}_1$ is given by $\sum_{g\in G} g(\Delta)$. In particular, $d_1 = |G|$. The arrangement of fibers $\mathcal{A}_2$ and $\mathcal{A}_3$ will be either empty ($d_2 = 0$ and $d_3 = 0$) or formed all fibers F and S passing through the 2-points of $\mathcal{A}_1$. Hence, the arrangement $\mathcal{A}$ has simple crossings. Let A_n be the unique normal subgroup of index two in the symmetric group of n elements S_n. Then, we have

(A_4) : $d_1 = 12$, $d_2 = d_3 = 0$, $t_2 = 36$, $t_3 = 32$, $E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{9}{10} = \frac{1}{1.1\overline{1}}$.

(A'_4) : $d_1 = 12$, $d_2 = d_3 = 6$, $t_3 = 32$, $t_4 = 36$, $E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{3}{8} = \frac{1}{2.\overline{6}}$.

(S_4) : $d_1 = 24$, $d_2 = d_3 = 0$, $t_2 = 144$, $t_3 = 64$, $t_4 = 36$, $E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{31}{42} \approx \frac{1}{1.3548}$.

(S'_4) : $d_1 = 24$, $d_2 = d_3 = 12$, $t_3 = 64$, $t_4 = 180$, $E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{25}{72} = \frac{1}{2.88}$.

(A_5) : $d_1 = 60$, $d_2 = d_3 = 0$, $t_2 = 900$, $t_3 = 400$, $t_5 = 144$,
$E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{181}{270} \approx \frac{1}{1.4917}$.

(A'_5) : $d_1 = 60$, $d_2 = d_3 = 30$, $t_3 = 400$, $t_4 = 900$, $t_5 = 144$,
$E(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{A}) = \frac{83}{240} \approx \frac{1}{2.8915}$.

We do not consider the finite cyclic groups and Dihedral groups because the numbers are not higher than the ones above. We notice that these arrangements can be seen in $\mathbb{P}^2$ as simple crossings arrangements of conics and lines. They have large log Chern numbers ratio, but not higher than $\frac{8}{3}$.

2.5.4 Modular arrangements.

In [79], Shioda studies a very special class of elliptic fibrations, which he called elliptic modular surfaces. They are defined by certain subgroups of $\mathrm{SL}(2, \mathbb{Z})$. We

will work on the ones defined by the main congruence subgroups of level $n \geq 3$. In [6], there is a explicit description of these surfaces, and in particular the special cases $n = 3, 4, 5$. Let $X(n)$ be the elliptic modular surface, and let

$$\pi : X(n) \to B$$

be the corresponding elliptic fibration. The number of singular fibers is $\mu(n) = \frac{1}{2}n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right)$, where the product is over all primes dividing n. The images of these singular fibers are called cusps. All the singular fibers are of Kodaira type I_n, which means a cycle of n rational curves. This fibration admits n^2 sections, such that each component of each singular fiber intersects exactly n of them. The self-intersection of each of this sections is equal to $-\chi(X(n), \mathcal{O}_{X(n)}) = -\frac{1}{12}n\mu(n)$. The irregularity of $X(n)$ is the genus of B, and

$$g(B) = 1 + n\frac{\mu(n)}{12} - \frac{\mu(n)}{2}.$$

In addition, $c_1^2(X(n)) = 0$ and $c_2(X(n)) = n\mu(n)$.

Following [24, p. 433], we define the modular arrangement $\mathcal{A}$ as the set of n^2 sections mentioned above, and all the components of the singular fibers. This is a simple crossing arrangement (actually SNC) of $n^2 + n\mu(n)$ curves in $X(n)$. Hence, we have $t_2 = n\mu(n) + n^2\mu(n)$ and $t_k = 0$ otherwise. Therefore,

$$\bar{c}_1^2(n) = \frac{5}{12}n^3\mu(n) \qquad \bar{c}_2(n) = \frac{1}{6}n^3\mu(n) \qquad E(n) = \frac{1}{2}$$

and so the log Chern numbers ratio does not depends on n, being $\frac{\bar{c}_1^2(n)}{\bar{c}_2(n)} = 2.5$.

Let $\mathcal{A}$ be the arrangement formed only by the n^2 sections. Then, the numbers are

$$\bar{c}_1^2(n) = \frac{1}{12}n^2\mu(n)(5n-24) \qquad \bar{c}_2(n) = \frac{1}{6}n\mu(n)(n^2-6n+6) \qquad E(n) = \frac{(n-6)^2}{2(n^2-6n+6)}$$

and so for $n = 6$, we have that $E\big(X(6), \mathcal{A}\big) = 0$.

2.5.5 Hirzebruch elliptic arrangements.

The following example achieves $\frac{1}{3}$ as well. It is due to Hirzebruch [43].

Let $\zeta = e^{\frac{2\pi i}{3}}$ and T be the elliptic curve

$$T = \mathbb{C}/\big(\mathbb{Z} \oplus \mathbb{Z}\zeta\big).$$

Consider the abelian surface $Z = T \times T$ whose points are denoted by (z, w). Let $T_0 : \{w = 0\}$, $T_\infty : \{z = 0\}$, $T_1 : \{w = z\}$ and $T_\zeta : \{w = \zeta z\}$. These four curves only intersect at $(0, 0)$ because of the choice of ζ. Let U_n be the group of n-division points of Z,

$$U_n = \{(z, w) : \ (nz, nw) = (0, 0)\}.$$

It has order n^4. The group U_n acts on Z by translations. Each of the sets $U_n(T_0)$, $U_n(T_\infty)$, $U_n(T_1)$, $U_n(T_\zeta)$ consists of n^2 smooth disjoint elliptic curves. Let $\mathcal{A}_0$, $\mathcal{A}_\infty$, $\mathcal{A}_1$, $\mathcal{A}_\zeta$ be the corresponding arrangements. We define the arrangement $\mathcal{A}$ as $\mathcal{A}_0 \cup \mathcal{A}_\infty \cup \mathcal{A}_1 \cup \mathcal{A}_\zeta$.

This arrangement is formed by $d = 4n^2$ elliptic curves. It can be checked that $t_4 = |U_n| = n^4$ and $t_k = 0$ for $k \neq 4$. The error number associated to this arrangement does not depend on n and is $E(T \times T, \mathcal{A}) = \frac{1}{3}$.

CHAPTER III

Arrangements as single curves and applications

In this Chapter, we translate the question of existence of an arrangement into the question of existence of a single curve in projective space. We construct a one-to-one correspondence between arrangements of d sections (Definition II.13) and certain curves in $\mathbb{P}^{d-2}$. As we already saw in Example II.15, these arrangements include line arrangements. We will use this correspondence to classify some nets in $\mathbb{P}^2$.

3.1 Moduli space of marked rational curves.

We denote the projective space of dimension n by $\mathbb{P}^n$ and a point in it by $[x_1 : ... : x_{n+1}] = [x_i]_{i=1}^{n+1}$. If $P_1, \ldots, P_r$ are r distinct points in $\mathbb{P}^n$, then $\langle P_1, \ldots, P_r \rangle$ is the projective linear space spanned by them. The points $P_1, \ldots, P_{n+2}$ in $\mathbb{P}^n$ are said to be in general position if no $n+1$ of them lie in a hyperplane.

We denote by $\overline{M}_{0,d+1}$ the moduli space of $(d+1)$-pointed stable curves of genus zero [50]. The boundary $\Delta := \overline{M}_{0,d+1} \setminus M_{0,d+1}$ of $\overline{M}_{0,d+1}$ is formed by the following divisors: for each subset $T \subset \{1, 2, ..., d+1\}$ with $|T| \geq 2$ and $|T^c| \geq 2$, we let $D^T \hookrightarrow \overline{M}_{0,d+1}$ be the divisor whose generic element is a curve with two components: the points marked by T in one, and the points marked by T^c on the other. We will assume $d+1 \in T$ so that there are no repetitions. These divisors are smooth and simple normal crossing.

We are interested in arrangements of d sections with simple crossings (Definition II.13). In this section we will explain how to obtain these arrangements from projective curves in $\mathbb{P}^{d-2}$ via moduli spaces of pointed stable curves of genus zero. The key ingredients are the construction of $\overline{M}_{0,d+1}$ via blow-ups of $\mathbb{P}^{d-2}$ and the description of $\overline{M}_{0,d+1}$ using Veronese curves, both due to Kapranov [50, 49].

Let $d \geq 3$ be an integer. It is well-known that $\overline{M}_{0,d+1}$ is a fine moduli space which is represented by a smooth projective variety of dimension $d-2$. For $i \in \{1, \ldots, d+2\}$, the i-th forgetful map $\pi_i : \overline{M}_{0,d+2} \to \overline{M}_{0,d+1}$, which forgets the i-th marked point, gives a universal family. The following are definitions and facts about these spaces, which can be found in [50] and [49].

Definition III.1. A Veronese curve is a rational normal curve of degree n in $\mathbb{P}^n$, $n \geq 2$, i.e., a curve projectively equivalent to $\mathbb{P}^1$ in its Veronese embedding.

It is a classical fact that any $d+1$ points in $\mathbb{P}^{d-2}$ in general position lie on a unique Veronese curve. The main theorem in [50] says that the set of Veronese curves in $\mathbb{P}^{d-2}$ and its closure are isomorphic to $M_{0,d}$ and $\overline{M}_{0,d}$ respectively.

Theorem III.2. *(Kapranov) Take d points $P_1, \ldots, P_d$ in projective space $\mathbb{P}^{d-2}$ which are in general position. Let $V_0(P_1, \ldots, P_d)$ be the space of all Veronese curves in $\mathbb{P}^{d-2}$ through P_i. Consider it as a subvariety in the Hilbert scheme $\mathscr{H}$ parametrizing all closed subschemes of $\mathbb{P}^{d-2}$. Then,*

(a) We have $V_0(P_1, \ldots, P_d) \cong M_{0,d}$.

(b) Let $V(P_1, \ldots, P_d)$ be the closure of $V_0(P_1, \ldots, P_d)$ in $\mathscr{H}$. Then $V(P_1, \ldots, P_d) \cong \overline{M}_{0,d}$. Moreover, the subschemes representing limit positions of curves from $V_0(P_1, \ldots, P_d)$ are, considered together with P_i, stable d-pointed curves of genus 0, which represent the corresponding points of $\overline{M}_{0,d}$.

(c) The analogs of statements (a) and (b) hold also for the Chow variety instead of the Hilbert scheme.

Theorem III.3. *(Kapranov, [49]) Choose d general points $P_1, \ldots, P_d$ in $\mathbb{P}^{d-2}$. The variety $\overline{M}_{0,d+1}$ can be obtained from $\mathbb{P}^{d-2}$ by a series of blowing ups of all the projective spaces spanned by P_i. The order of these blow-ups can be taken as follows:*

1. *Points $P_1, \ldots, P_{d-1}$ and all the projective subspaces spanned by them in order of the increasing dimension;*

2. *The point P_d, all the lines $\langle P_1, P_d\rangle$,...,$\langle P_{d-2}, P_d\rangle$ and subspaces spanned by them in order of the increasing dimension;*

3. *The line $\langle P_{d-1}, P_d\rangle$, the planes $\langle P_i, P_{d-1}, P_d\rangle$, $i \neq d-2$ and all subspaces spanned by them in order of the increasing dimension, etc, etc.*

Let us fix d points in general position in $\mathbb{P}^{d-2}$. We take $P_1 = [1 : 0 : \ldots : 0]$, $P_2 = [0 : 1 : 0 : \ldots : 0]$, ..., $P_{d-1} = [0 : \ldots : 0 : 1]$ and $P_d = [1 : 1 : \ldots : 1]$. Let

$$\Lambda_{i_1,\ldots,i_r} = \langle P_j : j \notin \{i_1, \ldots, i_r\}\rangle,$$

where $1 \leq r \leq d-1$ and $i_1, \ldots, i_r$ are distinct numbers, and let $\mathscr{H}_d$ be the set of the hyperplanes $\Lambda_{i,j}$. Hence, $\Lambda_{i,j} = \{[x_1 : \ldots : x_{d-1}] \in \mathbb{P}^{d-2} : x_i = x_j\}$ for $i, j \neq d$, $\Lambda_{i,d} = \{[x_1 : \ldots : x_{d-1}] \in \mathbb{P}^{d-2} : x_i = 0\}$ and

$$\mathscr{H}_d = \{[x_1 : \ldots : x_{d-1}] \in \mathbb{P}^{d-2} : \ x_1 x_2 \cdots x_{d-1} \prod_{i<j} (x_j - x_i) = 0\}.$$

Our goal is to build a simple crossings arrangement of d sections out of an irreducible projective curve $B \subseteq \mathbb{P}^{d-2}$. Because of our simple crossing requirement, this curve must satisfy some special properties.

Definition III.4. Let B be an irreducible projective curve in $\mathbb{P}^{d-2}$. The curve B is said to satisfy (*) if the following condition holds:

(*) For each $P \in B$, there is a local factorization of B formed by smooth branches, and each branch intersects each hyperplane $\Lambda_{i,j}$ transversally whenever $P \in B \cap \mathscr{H}_d$.

Hence B is not contained in $\mathscr{H}_d$ if it satisfies $(*)$. Fix a curve B satisfying $(*)$.

By Theorem III.3, there is a birational map $\psi_{d+1} : \overline{M}_{0,d+1} \to \mathbb{P}^{d-2}$ which is a composition of blow-ups along all linear projective spaces spanned by the points P_i, in a certain order. We have the following diagram of maps.

$$\begin{array}{ccc} \overline{M}_{0,d+2} & & \\ \downarrow{\scriptstyle \pi_{d+2}} & & \\ \overline{M}_{0,d+1} & \xrightarrow{\psi_{d+1}} & \mathbb{P}^{d-2} \supset B \end{array}$$

Let B' be the strict transform of the curve B under ψ_{d+1}. Then, by the property $(*)$ for B and the construction of ψ_{d+1} (Theorem III.3), the curve B' can only have local transversal intersections with each of the boundary divisors D^T, that is, for every point P of B', if $P \in D^T \cap B'$, then each local branch at P intersects D^T transversally.

Let B_0 be a local branch of B' at P such that $P = \Delta \cap B_0$. Then, since we are working with fine moduli spaces, we have the following unique commutative diagram.

$$\begin{array}{ccc} R_{B_0} & \xrightarrow{j} & \overline{M}_{0,d+2} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{d+2}} \\ B_0 & \xrightarrow{i} & \overline{M}_{0,d+1} \end{array}$$

In this diagram, R_{B_0} is the unique surface produced by the universal property of π_{d+2}, and so i and j are inclusions. The map $\pi : R_{B_0} \to B_0$ has one singular fiber which looks like the bold curve in Figure 3.1.

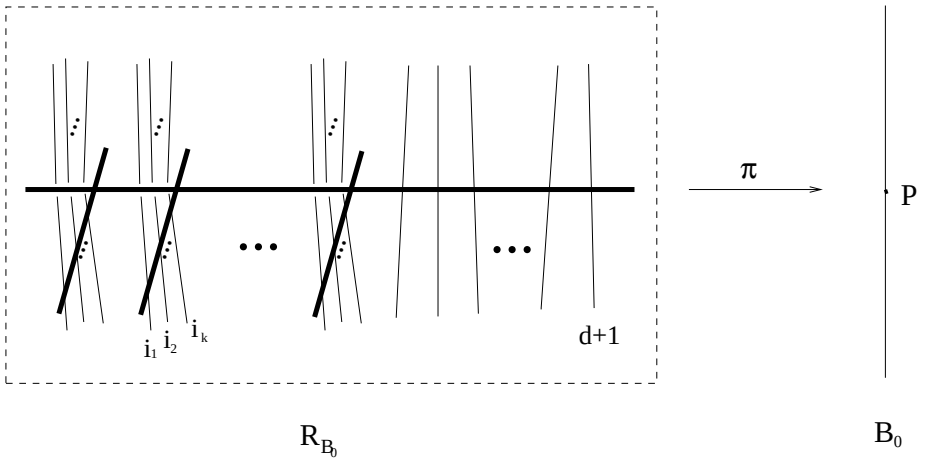


Figure 3.1: Singular fiber type.

Again, this is because B satisfies (*), and so locally intersects transversally each $\Lambda_{i,j}$. More precisely, let $Q = \psi_{d+1}(P) \in B \cap \mathscr{H}_d$ and let Λ_Q be the intersection of all the smallest $\Lambda_{i_1,i_2,\ldots,i_k}$ containing Q. This means, if $\Lambda_{i_1,i_2,\ldots,i_k}$ belongs to the intersection, then there is no $\Lambda_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{k+l}} \subset \Lambda_{i_1,i_2,\ldots,i_k}$ such that $Q \in \Lambda_{i_1,\ldots,i_k,i_{k+1},\ldots,i_{k+l}}$. We write

$$\Lambda_Q = \bigcap \Lambda_{i_1,i_2,\ldots,i_k}$$

where the intersection is taken over all these smallest linear spaces. Now, since B locally intersects every $\Lambda_{i,j}$ transversally, each $\Lambda_{i_1,i_2,\ldots,i_k}$ in the intersection Λ_Q corresponds to a component of the singular fiber of π, which does not intersect the $d+1$ section and intersects exactly the sections labelled by the set $\{i_1,\ldots,i_k\}$.

Also, since B_0 intersects transversally each component D^T of the boundary Δ, R_{B_0} is a smooth surface, and conversely. This assertion comes from the description of the versal deformation space of a stable curve [38, pp. 145-147].

In conclusion, for $T = \{1,2,\ldots,d+1\} \setminus \{i_1,i_2,\ldots,i_k\}$, the transversal intersection of B_0 with D^T is represented in R_{B_0} by the component of the singular fiber which intersects transversally the k sections of π labelled by the set $\{i_1,\ldots,i_k\}$, and the

intersection of this component with the singular fiber is a smooth point of R_{B_0}. The other components of the singular fiber give the transversal intersections of B_0 with another boundary divisors $D^{T'}$ at P.

Coming back to our curve B', we globally have the following commutative diagram.

$$\begin{array}{ccc} R_{B'} & \xrightarrow{j} & \overline{M}_{0,d+2} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{d+2}} \\ B' & \xrightarrow{i} & \overline{M}_{0,d+1} \end{array}$$

where $R_{B'}$ is a projective surface. We now produce another commutative diagram by considering the normalization of B', given by the map $\nu : \overline{B'} \to B'$.

$$\begin{array}{ccccc} R_{\overline{B'}} & \longrightarrow & R_{B'} & \xrightarrow{j} & \overline{M}_{0,d+2} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{d+2}} \\ \overline{B'} & \xrightarrow{\nu} & B' & \xrightarrow{i} & \overline{M}_{0,d+1} \end{array}$$

Because of our local description for B_0 above, we have that $R_{\overline{B'}}$ is an smooth projective surface, in particular it is a ruled surface over $\overline{B'}$. Let $R := R_{\overline{B'}}$ and $C := \overline{B'}$, so we have a ruled surface $\pi' : R \to C$ with distinguished $(d+1)$ sections $\{X_1, X_2, \ldots, X_{d+1}\}$. Now, we blow down all the (-1)-curves which are components of singular fibers which do not intersect X_{d+1} (it is easy to check that they are (-1)-curves). In this way, we arrive to a geometrically ruled surface $\mathbb{P}_C(\mathcal{E})$ over C, for some rank two locally free sheaf $\mathcal{E}$ on C. After applying an isomorphism over C of geometrically ruled surfaces, we can and do assume that $\mathcal{E}$ is normalized (as in Section 2.4). For each $i \in \{1, 2, \ldots, d+1\}$, let S_i be the sections in $\mathbb{P}_C(\mathcal{E})$ corresponding to the images of X_i under the composition of the blow-downs of (-1)-curves.

Proposition III.5. *Let us denote the previous map by $\pi : \mathbb{P}_C(\mathcal{E}) \to C$. Then, $\mathcal{E}$ is a decomposable vector bundle of the form $\mathcal{O}_C \oplus \mathcal{L}^{-1}$ with $\mathcal{L}$ invertible sheaf on*

C of degree $e = deg(B) > 0$. The section S_{d+1} is the unique curve on $\mathbb{P}_C(\mathcal{E})$ with negative self-intersection equal to $-e$. Moreover, for every $i \in \{1, 2, \ldots, d\}$ we have $S_i \sim S_{d+1} + \pi^*(\mathcal{L})$ and, if H is a hyperplane in $\mathbb{P}^{d-2}$ and $\overline{\nu} : C \to B' \to B$ is the corresponding normalization of B (as we did before), then $\mathcal{L} \simeq \mathcal{O}_C(\overline{\nu}^*(H \cap B))$. The set $\mathcal{A} = \{S_1, S_2, \ldots, S_d\}$ is a simple crossings arrangement of d sections.

Proof. First, since there are two disjoint sections, $\mathcal{E}$ is a decomposable vector bundle. Because we know explicitly the Picard group of $\mathbb{P}_C(\mathcal{E})$ [39, p. 370], for each $i \in \{1, 2, \ldots, d+1\}$ we can find a divisor D_i on C such that $S_i \sim C_0 + \pi^*(D_i)$, where C_0 is the section corresponding to $\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$ (and so $C_0^2 = -e$).

Now, $S_i.S_{d+1} = 0$ for all $i \neq d+1$, so $\deg(D_i) = e - \deg(D_{d+1})$ when $i \neq d+1$. Then, $S_i^2 = S_i.S_j$ for all $i, j \neq d+1$; and so $\deg(D_i) = \frac{1}{2}(e + S_i.S_j)$ and $\deg(D_{d+1}) = \frac{1}{2}(e - S_i.S_j)$ for $i, j \neq d+1$. But $S_{d+1}^2 = -e + 2\deg(D_{d+1})$ implies $S_{d+1}^2 = -S_i.S_j$ for $i, j \neq d+1$. Notice that $S_i.S_j > 0$ since we always have singular fibers (we always have $B \cap \mathscr{H}_d \neq \emptyset$). Hence, $S_{d+1}^2 < 0$.

We now suppose that $S_{d+1} \neq C_0$. Then, $S_{d+1}.C_0 \geq 0$ and so $-e + \deg(D_{d+1}) \geq 0$. But we have $0 > S_{d+1}^2 = -e + 2\deg(D_{d+1})$ and this implies $\deg(D_{d+1}) < 0$, and so $e < 0$. But for a decomposable normalized $\mathcal{E}$ we have $e \geq 0$ [39, p. 376]. Therefore, $S_{d+1} = C_0$.

The following is a known fact. Let $\Gamma \neq S_{d+1}$ be a curve in $\mathbb{P}_C(\mathcal{E})$ with $\Gamma^2 < 0$. Write $\Gamma \equiv aS_{d+1} + bF$, where F is the class of a fiber. Then, $\Gamma.F = a > 0$, $\Gamma^2 = -ea^2 + 2ab < 0$ and $\Gamma.S_{d+1} = -ea + b \geq 0$. Hence, we must have $b < 0$, and this contradicts the fact that $-e < 0$. Therefore, $\Gamma = S_{d+1}$ and so there is only one curve with negative self intersection.

Take $i \in \{1, 2, \ldots, d\}$. Let $\sigma_i : C \to \mathbb{P}_C(\mathcal{E})$ be the morphism defining the section S_i, i.e., $\sigma_i(C) = S_i$, and let $\mathcal{E} \to \mathcal{L}_i \to 0$ be the corresponding surjection of sheaves

on C [39, p. 370]. Then, $\mathcal{L}_i = \sigma_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(S_{d+1}) \otimes \mathcal{O}_{S_i})$, but $\sigma_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(S_{d+1}) \otimes \mathcal{O}_{S_i}) = \sigma_i^*(\mathcal{O}_{S_i}) = \mathcal{O}_C$ because $S_i.S_{d+1} = 0$. Therefore, $\mathcal{E} \simeq \mathcal{O}_C \oplus \mathcal{L}^{-1}$ with $\deg(\mathcal{L}) = e > 0$. Moreover, $S_i \sim S_{d+1} + \pi^*(\mathcal{L})$ for all $i \in \{1, 2, ..., d\}$.

Finally, by construction of this ruled surface, for any pair $i, j \in \{1, 2, \ldots, d\}$ with $i \neq j$, we have that $S_i.S_j$ is $B.\Lambda_{i,j}$, that is, $S_i.S_j = \deg(B)$. On the other hand, we proved that $S_i.S_j = e$, so $\deg(B) = e$. Moreover, it is not hard to see that, if H is a hyperplane in $\mathbb{P}^{d-2}$ and $\overline{\nu} : C \to B' \to B$ is the normalization as before, then $\mathcal{L} \simeq \mathcal{O}_C(\nu^*(H \cap B))$. □

3.2 Two proofs of the one-to-one correspondence for line arrangements.

We will be considering arrangements of d lines A together with a fixed point $P \in \mathbb{P}^2 \setminus \mathcal{A}$, denoted by $(\mathcal{A}, P)$. As always, we assume $t_d = 0$. If $(\mathcal{A}, P)$ and $(\mathcal{A}', P')$ are two such pairs, we say that they are isomorphic if there exists an automorphism T of $\mathbb{P}^2$ such that $T(L_i) = L_i'$ for every i and $T(P) = P'$. So, it is our former notion of isomorphism with the additional requirement that P goes to P'. Let $\mathscr{L}_d$ be the set of isomorphism classes of pairs $(\mathcal{A}, P)$. For example, the set $\mathscr{L}_3$ is clearly formed by one point, represented by the class of $(\{xyz = 0\}, [1 : 1 : 1])$.

We will prove in two different ways that $\mathscr{L}_d$ is the set of lines in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$. The first one is elemental, and it is inspired by a particular case of the so-called Gelfand-MacPherson correspondence [49] (although the idea is quite classical). The second proof is the way I saw the correspondence, and involves $\overline{M}_{0,d+1}$. It is important to understand how the correspondence works, explaining why we choose pairs, and to see how we can generalize it for arrangements of sections.

Proposition III.6. *There is a one-to-one correspondence between $\mathscr{L}_d$ and the set of lines in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$.*

Proof. (**I**) Let us fix a pair $(\mathcal{A}, P)$ where $\mathcal{A}$ is formed by linear polynomials

$$\{L_i(x,y,z)\}_{i=1}^{d}.$$

We consider a closed embedding $\iota_{(\mathcal{A},P)} : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{d-1}$ given by

$$[x:y:z] \mapsto \Big[\frac{L_1(x,y,z)}{L_1(P)} : \ldots : \frac{L_d(x,y,z)}{L_d(P)}\Big]$$

Then, $\iota_{(\mathcal{A},P)}(\mathbb{P}^2)$ is a projective plane, $\iota_{(\mathcal{A},P)}(P) = [1:1:\ldots:1]$ and $\iota_{(\mathcal{A},P)}(L_i) = \iota_{(\mathcal{A},P)}(\mathbb{P}^2) \cap \{y_i = 0\}$ for every $i \in \{1,2,\ldots,d\}$. Now, we consider the projection

$$\varrho : \mathbb{P}^{d-1} \setminus [1:1:\ldots:1] \to \mathbb{P}^{d-2}$$

defined by

$$[y_1 : y_2 : \ldots : y_d] \mapsto [ay_1 + b : ay_2 + b : \ldots : ay_{d-1} + b],$$

where $[a:b]$ is the unique point in $\mathbb{P}^1$ such that $ay_d + b = 0$. In this way, if

$$\Sigma_{i,j} = \{[y_1 : y_2 : \ldots : y_d] : \ y_i = y_j\},$$

we clearly see that $\varrho(\Sigma_{i,j}) = \Lambda_{i,j}$. Therefore, we have that $\varrho(\iota_{(\mathcal{A},P)}(\mathbb{P}^2))$ is a line in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$. To show the one-to-one correspondence, we need to prove that $(\mathcal{A}, P) \mapsto \varrho(\iota_{(\mathcal{A},P)}(\mathbb{P}^2))$ gives a well-defined bijection between $\mathscr{L}_d$ and the set of lines in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$. Clearly we have a bijection between projective planes in $\mathbb{P}^{d-1}$ passing through $[1:1:\ldots:1]$ and not contained in $\bigcup_{i,j} \Sigma_{i,j}$, and the set of lines in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$.

Let $T : \mathbb{P}^2 \to \mathbb{P}^2$ be an automorphism of $\mathbb{P}^2$. Suppose the arrangement $\mathcal{A}$ is defined by the linear polynomials $L_i(x,y,z) = a_{i,1}x + a_{i,2}y + a_{i,3}z$. Let $B = (b_{i,j})$ be the 3×3 invertible matrix corresponding to T^{-1}. Consider the pair $(\mathcal{A}', P')$ defined by $\mathcal{A}' = \{L_i' = T(L_i)\}_{i=1}^{d}$ and $P' = T(P)$. Then, the equations defining the lines L_i'

are $(\sum_{j=1}^{3} a_{i,j}b_{j,1})x + (\sum_{j=1}^{3} a_{i,j}b_{j,2})y + (\sum_{j=1}^{3} a_{i,j}b_{j,3})z = 0$. Hence, we obtain that $\iota_{(\mathcal{A},P)} = \iota_{(\mathcal{A}',P')} \circ T$, and so our map $(\mathcal{A}, P) \mapsto \varrho(\iota_{(\mathcal{A},P)}(\mathbb{P}^2))$ is well-defined on $\mathscr{L}_d$.

The map is clearly surjective, so we only need to prove injectivity. Let $\iota_{(\mathcal{A},P)}$ and $\iota_{(\mathcal{A}',P')}$ be the corresponding maps for the pairs $(\mathcal{A}, P)$ and $(\mathcal{A}', P')$ such that $\iota_{(\mathcal{A},P)}(\mathbb{P}^2) = \iota_{(\mathcal{A}',P')}(\mathbb{P}^2)$. Let $T = \iota^{-1}_{(\mathcal{A}',P')} \circ \iota_{(\mathcal{A},P)} : \mathbb{P}^2 \to \mathbb{P}^2$. Then, T is an automorphism of $\mathbb{P}^2$ such that $T(L_i) = L'_i$ for every i and $T(P) = P'$. Hence they are isomorphic, and so we have the one-to-one correspondence. □

Proof. (**II**) Let us fix a pair $(\mathcal{A}, P)$. Consider the genus zero fibration $\mathrm{Bl}_P(\mathbb{P}^2) \to C(= \mathbb{P}^1)$ given by the blow-up at P. It has singular fibers exactly at the k-points of $\mathcal{A}$. Consider the genus zero fibration $f : R \to C$, where R is the blow-up at all the k-points of $\mathcal{A}$ with $k \geq 3$. Then f is a family of $(d+1)$-marked stable curves of genus zero. The markings are given by the labelled lines of $\mathcal{A}$, which now are the sections $S_1, \ldots, S_d$ of f, and the (-1)-curve (section) S_{d+1} coming from the point P. It is a stable fibration because $t_d = 0$ for $\mathcal{A}$.

Therefore, we have the following unique commutative diagram.

$$\begin{array}{ccc} R & \xrightarrow{h} & \overline{M}_{0,d+2} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_{d+2}} \\ C & \xrightarrow{g} & \overline{M}_{0,d+1} \end{array}$$

Let B' be the image of g in $\overline{M}_{0,d+1}$. It is a projective curve, since f has singular and non-singular (stable) fibers, and so f is not isotrivial. Let us now consider the Kapranov map $\psi_{d+1} : \overline{M}_{0,d+1} \to \mathbb{P}^{d-2}$, and let $B = \psi_{d+1}(B')$. We saw in the previous section that B intersects all the hyperplanes $\Lambda_{i,j}$ transversally, and it has to intersect some of them, since f has singular fibers. Say B intersects $\Lambda_{i,j}$. This means that the lines L_i and L_j of $\mathcal{A}$ intersect in $\mathbb{P}^2$. But since they are lines, they can intersect only once. Therefore, $B.\Lambda_{i,j} = 1$ and so $\deg(B) = 1$, that is, B is a line in $\mathbb{P}^{d-2}$.

In particular, B' is a smooth rational curve. In Proposition III.5 we saw how to obtain a pair $(\mathcal{A}, P)$ from a line in $\mathbb{P}^{d-2}$ outside of $\mathscr{H}_d$. It is easy to check that the pair we obtain from this proposition is unique up to isomorphism of pairs. In this way, to finish the proof, we have to show that the map g in the diagram above is an inclusion.

Consider $g : C \to B'$ and assume $\deg(g) > 1$. We notice that g is totally ramified at the points corresponding to singular fibers, since again they come from intersections of lines in $\mathbb{P}^2$, and so all the singular fibers go to distinct points. Let $\operatorname{sing}(f)$ be the set of points in C corresponding to singular fibers of f. Then, since $t_d = 0$, we have $|\operatorname{sing}(f)| \geq 3$ (at least we have a triangle in $\mathcal{A}$). Now, by the Riemann-Hurwitz formula, we have

$$-2 = \deg(g)(-2) + (\deg(g) - 1)|\operatorname{sing}(f)| + \epsilon$$

where $\epsilon \geq 0$ stands for the contribution from ramification of f not in $\operatorname{sing}(f)$. But we re-write the equation as $0 = (\deg(g) - 1)(|\operatorname{sing}(f)| - 2) + \epsilon$, and since $\deg(g) > 1$ and $|\operatorname{sing}(f)| \geq 3$, this is a contradiction. Therefore, $\deg(g) = 1$ and we have proved the one-to-one correspondence. □

We notice that Proof I is field independent, and so this correspondence is true for arrangements defined over arbitrary fields. In Proof II we assume characteristic zero to apply the Riemann-Hurwitz formula.

Let $(\mathcal{A}, P)$ be a pair and L be the corresponding line in $\mathbb{P}^{d-2}$. Let λ be a line in $\mathbb{P}^2$ passing through P. We notice that λ corresponds to a point in L. Let $K(\lambda)$ be the set of k-points of $\mathcal{A}$ in λ, for all $1 < k < d$; it might be empty or consist of several points. We write

$$K(\lambda) = \{[[i_1, i_2, ..., i_{k_1}]], [[j_1, j_2, ..., j_{k_2}]], ...\}.$$

Example III.7. In Figure 3.2, we have the complete quadrilateral $\mathcal{A}$, formed by the set of lines $\{L_1, \ldots, L_6\}$, and a point P outside of $\mathcal{A}$. Through P we have the λ lines. In the figure, we have labelled two such lines: λ and λ'. Therefore, $K(\lambda) = \{[[3,6]], [[1,4]]\}$ and $K(\lambda') = \{[[1,2,3]]\}$.

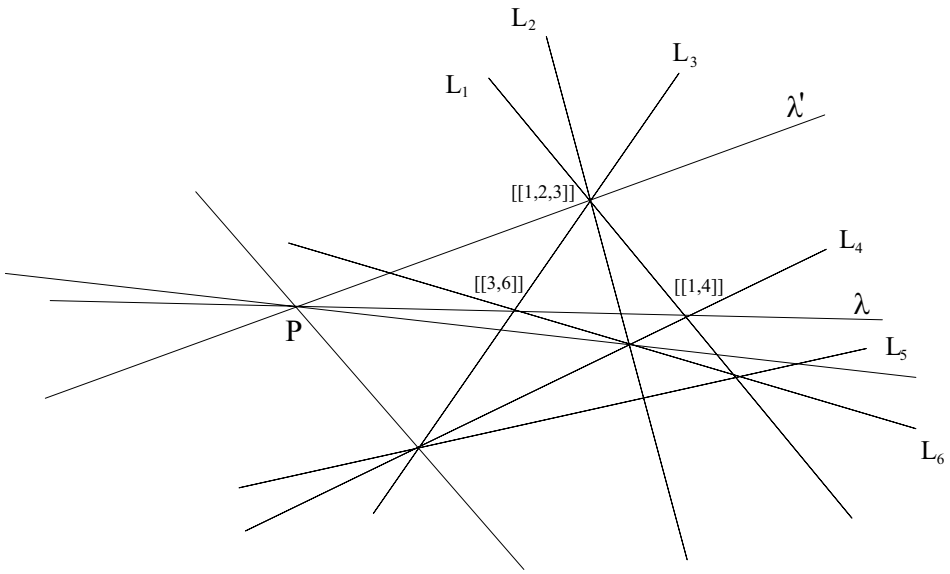


Figure 3.2: Some $K(\lambda)$ sets for (complete quadrilateral, P).

The set $K(\lambda)$ gives the following restrictions for the the point $[a_1 : a_2 : \ldots : a_{d-1}]$ in $L \subseteq \mathbb{P}^{d-2}$ corresponding to λ. For each k-point $[[i_1, i_2, ..., i_k]]$ of $\mathcal{A}$ in $K(\lambda)$,

- If for some j, $i_j = d$, then $a_{i_l} = 0$ for all $i_l \neq d$.
- Otherwise, $a_{i_1} = a_{i_2} = ... = a_{i_k} \neq 0$.

For $[[i_1, ..., i_{k_1}]]$, $[[j_1, .., j_{k_2}]]$ in $K(\lambda)$, we have that $a_{i_a} \neq a_{j_b}$, otherwise we would have a new (not considered before) k-point on λ. We will work out various examples at the end of this Chapter for the case of nets.

Example III.8. Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be an arrangement of d lines, and consider the pair $(\mathcal{A}, P)$ with P a general point in $\mathbb{P}^2 \setminus \mathcal{A}$. This means that for all lines

λ passing through P, the corresponding sets $K(\lambda)$ consist of one or zero elements. So, there is one line λ for each k-point $[[i_1, \ldots, i_k]]$ of $\mathcal{A}$, corresponding precisely to the point $L \cap \Lambda_{i_1,\ldots,i_k}$ (L being the line in $\mathbb{P}^{d-2}$ assigned to $(\mathcal{A}, P)$). For example, a 2-point $L_i \cap L_j$ corresponds to a proper intersection of L with the hyperplane $\Lambda_{i,j}$. Let $a = [a_i]_{i=1}^{d-1}$ and $b = [b_i]_{i=1}^{d-1}$ be distinct points in $\mathbb{P}^{d-2}$ so that the line L is given by $au + bt$ for $[u : t] \in \mathbb{P}^1$.

We now substitute L in the equation defining $\mathscr{H}_d$ to obtain the polynomial

$$p_L(u,t) = (a_1u + b_1t)(a_2u + b_2t)\cdots(a_{d-1}u + b_{d-1}t)\prod_{i<j}\big((a_j - a_i)u + (b_j - b_i)t\big).$$

This polynomial is not identically zero since L is outside of $\mathscr{H}_d$. We can and do assume that $b_i \neq 0$ for all i. Then, the number of simple roots of $p_L(1,t)$ is t_2, the number of double roots is t_3, and so on. The numbers t_k depend on the position of the complex numbers $\frac{a_i}{b_i}$ and $\frac{a_j - a_i}{b_j - b_i}$ on $\mathbb{C}$. The Hirzebruch inequality for t_k's gives a relation among the roots of $p_L(1,t)$.

Let $(\mathcal{A}, P)$ be a pair. The existence of this pair is equivalent to the existence of L in $\mathbb{P}^{d-2}$. If we are only interested in the line arrangement $\mathcal{A}$, the point P introduces unnecessary dimensions to realize $\mathcal{A}$. Instead, consider the pair $(\mathcal{A}', P')$ where $P' \in \mathcal{A}$ and the lines of $\mathcal{A}'$ are the lines in $\mathcal{A}$ not containing P'. Now, the line L' corresponding to this new pair $(\mathcal{A}', P')$ lies in $\mathbb{P}^{d'-2}$ (and $d' < d$). So we decrease dimensions, and L' still represents our arrangement $\mathcal{A}$, by keeping track of P'.

By taking P' as a k-point with k large, the previous observation will be important to simplify computations to prove or disprove the existence of $\mathcal{A}$. In addition, we find a moduli space for the combinatorial type of $\mathcal{A}$, forgetting the artificial point P. By combinatorial type we mean the information given by the intersection of its lines, that is, the "complete" incidence $\mathcal{I}$ which defines $\mathcal{A}$. This moduli space is what we

called $M(\mathcal{I}, \mathbb{K})$ (Definition II.4, $\mathbb{K}$ being the field of definition). We will show this parameter space through several examples.

Example III.9. Let $\mathcal{A} = \{L_1, L_2, \ldots, L_7\}$ be the Fano arrangement. The set of 7 triple points can be taken as $L_3 \cap L_4 \cap L_5$, $L_2 \cap L_5 \cap L_7$, $L_1 \cap L_5 \cap L_6$, $L_1 \cap L_2 \cap L_3$, $L_1 \cap L_4 \cap L_7$, $L_3 \cap L_6 \cap L_7$, and $L_2 \cap L_4 \cap L_6$. Take any point out of $\mathcal{A}$, and form the pair $(\mathcal{A}, P)$. Then, we have that the corresponding L lives in $\mathbb{P}^{7-2=5}$. In practices, it takes several computations to try to realize L.

Instead, consider $(\mathcal{A}', P')$ where $P' = L_1 \cap L_2 \cap L_3$ and $\mathcal{A}' = \{L_4, L_5, L_6, L_7\}$. Now, the lines L_1, L_2 and L_3 are points λ's on the line L', corresponding to $(\mathcal{A}', P')$. Rename the lines as $L_4 = L'_1$, $L_5 = L'_2$, $L_6 = L'_3$, and $L_7 = L'_4$. By our definition of $K(\lambda)$, we have $K(L_1) = \{[[1,4]], [[2,3]]\}$, $K(L_2) = \{[[1,3]], [[2,4]]\}$, and $K(L_3) = \{[[1,2]], [[3,4]]\}$. Therefore, according to our observation above, the corresponding points on L' are $[0:1:1]$, $[1:0:1]$, and $[1:1:0]$ respectively. However, this is possible if and only if the characteristic of the field is two (determinant $= 2$).

The only purpose of this example is to show how we easily reduced parameters by taking "P in $\mathcal{A}$". In general this strategy will be fruitful.

In the next two sections, we will be computing some special configurations by means of the line L corresponding to a pair $(\mathcal{A}, P)$. We make the following choices to write down the equations of the lines in $\mathcal{A}$:

- The point P will be always $[0:0:1]$.
- The arrangement $\mathcal{A}$ will be formed by $\{L_1, ..., L_d\}$ where L_i are the lines of $\mathcal{A}$ and also their linear polynomials $L_i = (a_i x + b_i y + z)$ for every $i \neq d$, and $L_d = (z)$.

With these assumptions, it is easy to check that the corresponding line L in $\mathbb{P}^{d-2}$

is $[a_i t + b_i u]_{i=1}^{d-1}$, where $[t : u] \in \mathbb{P}^1$.

3.3 General one-to-one correspondence.

In this section we prove the general correspondence for arrangements of sections. What mattered in Proof II is that we knew about the local intersections of B' with the boundary divisors of $\overline{M}_{0,d+1}$. Simple crossing arrangements are locally the same as line arrangements. We will also replace the "Riemann-Hurwitz argument" at the end of Proof II by imposing that they satisfy inclusion. This is the concept of primitive arrangements in Definition II.16, which trivially holds for line arrangements.

Let $d \geq 3$ be an integer. Let C be a smooth projective curve and $\mathcal{L}$ be a line bundle on C with $\deg(\mathcal{L}) = e > 0$. Let $\mathscr{A}_d$ be the set of all isomorphism classes of arrangements $\mathcal{A}(C, \mathcal{L})$ which are primitive and simple crossings.

On the other hand, let $\mathscr{B}_d$ be the set of irreducible projective curves B in $\mathbb{P}^{d-2}$ satisfying: (*) in Definition III.4, B birational to C and, if H is a hyperplane in $\mathbb{P}^{d-2}$ and $\nu : C \to B$ is the normalization of B, then $\mathcal{L} \simeq \mathcal{O}_C(\nu^*(H \cap B))$.

Theorem III.10. *There is a one-to-one correspondence between $\mathscr{A}_d$ and $\mathscr{B}_d$.*

Proof. Let $B \in \mathscr{B}_d$. Then, we use Proposition III.5 to obtain $\mathcal{A}(C, \mathcal{L}) \in \mathscr{A}_d$. Conversely, let $\mathcal{A} = \mathcal{A}(C, \mathcal{L}) \in \mathscr{A}_d$. Then, by blowing up all the k-points of $\mathcal{A}$ $(1 < k < d)$, we obtain a stable fibration of $(d+1)$-pointed curves of genus zero. Let us denote this fibration by $\rho : R \to C$. The $d+1$ distinguished sections of π are the strict transforms of the sections $\{S_1, \ldots, S_d\} = \mathcal{A}$ and the section C_0 in $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}^{-1})$. Hence, there is a unique commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R_{B'} & \hookrightarrow & \overline{M}_{0,d+2} \\ {\scriptstyle\rho}\downarrow & & \downarrow & & \downarrow{\scriptstyle\pi_{d+2}} \\ C & \longrightarrow & B' & \hookrightarrow & \overline{M}_{0,d+1} \end{array}$$

where B' and $R_{B'}$ are the images of C and R respectively under the unique maps to these fine moduli spaces. We notice that B' is a projective curve, $R_{B'}$ is a projective surface, and $B = \psi_{d+1}(B')$ satisfies (*) by the local description given in Section 3.1 (we recall that $\psi_{d+1} : \overline{M}_{0,d+1} \to \mathbb{P}^{d-2}$ is the composition of the blow-ups in Theorem III.3).

Let $\nu : \overline{B'} \to B'$ be the normalization of B'. Then again, by the local description of the family $R_{B'} \to B'$ and the universal property of these moduli spaces, we have the following commutative diagram

$$\begin{array}{ccccc} R_{\overline{B'}} & \xrightarrow{\nu'} & R_{B'} & \hookrightarrow & \overline{M}_{0,d+2} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_{d+2}} \\ \overline{B'} & \xrightarrow{\nu} & B' & \hookrightarrow & \overline{M}_{0,d+1} \end{array}$$

where ν' is the normalization map for $R_{B'}$. We notice that $R_{\overline{B'}}$ is a projective smooth surface. Then, this induces a unique commutative diagram.

$$\begin{array}{ccc} R & \xrightarrow{F} & R_{\overline{B'}} \\ {\scriptstyle \rho}\downarrow & & \downarrow \\ C & \xrightarrow{f} & \overline{B'} \end{array}$$

where f is a finite map, and F restricted to any fiber of ρ is an isomorphism sending the $d+1$ distinguished sections to $d+1$ sections. Therefore, we can blow down the (-1)-curves not intersecting the section $d+1$ on both surfaces, and so we arrive to a commutative diagram as in Definition II.14. But $\mathcal{A}(C, \mathcal{L})$ is a primitive arrangement, so F has to be an isomorphism. In particular, f is an isomorphism. Hence, this gives the construction in Section 3.1 starting with $B \subseteq \mathbb{P}^{d-2}$ satisfying (*); and so, by Proposition III.5, we finally obtain what we want for B. The one-to-one correspondence follows. □

Corollary III.11. *(Proposition III.6) There is a one-to-one correspondence between*

$\mathscr{L}_d$ and the set of lines in $\mathbb{P}^{d-2}$ not contained in $\mathscr{H}_d$.

Proof. A pair $(\mathcal{A}, P)$ (up to isomorphism) corresponds exactly to an arrangement $\mathcal{A}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ (up to isomorphism) by blowing up the point P. By Theorem III.10, these pairs are in one-to-one correspondence with curves B of degree one satisfying some properties. But then B is a line in $\mathbb{P}^{d-2}$, and the properties are reduced to: B is not contained in $\mathscr{H}_d$. □

3.4 Examples applying the one-to-one correspondence.

How do we construct arrangements using this correspondence? Let us consider Veronese curves in $\mathbb{P}^{d-2}$ (Definition III.1). The associated arrangements of d sections lie in Hirzebruch surfaces $\mathbb{F}_{d-2}$.

Let $d \geq 4$ and $0 \leq m \leq d$ be integers. A m-Veronese curve will be a Veronese curve in $\mathbb{P}^{d-2}$ passing through the points $P_1, P_2, \ldots, P_m$ (as we fixed in Section 3.1). A m-Veronese curve is said to be general if, apart from the intersections at the points $P_1, P_2, \ldots, P_m$, it intersects $\mathscr{H}_d$ transversally (proper points of $\Lambda_{i,j}$'s).

Proposition III.12. *There are general m-Veronese curves in $\mathbb{P}^{d-2}$.*

Proof. Consider the map $v : \mathbb{P}^1 \to \mathbb{P}^{d-2}$ given by

$$[u : t] \mapsto \Big[a_i b_i \prod_{j=0, j\neq i}^{m-1} (u - b_j t) \prod_{j=m, j\neq i}^{d-2} (c_{i,j} u - b_j t)\Big]_{i=0}^{d-2}.$$

For a general choice of the numbers a_i, b_i and $c_{i,j}$, this defines a m-Veronese curve in $\mathbb{P}^{d-2}$. We have $v([0 : 1]) = [a_i]_{i=0}^{d-2}$ and $v\big(\big[1 : \frac{1}{b_i}\big]\big) = P_{i+1}$ for $i \in \{0, 1, ..., m-1\}$. In order to have only transversal intersections out of $\{P_1, P_2, \ldots, P_m\}$, we choose a point outside of the locus described by a certain finite set of nonzero polynomials with variables a_i, b_i and $c_{i,j}$. These polynomials are the conditions to pass by specified points in $\mathscr{H}_d$ and with specified tangency. □

Let $B \subset \mathbb{P}^{d-2}$ be a general m-Veronese curve. Then, we have the following intersections in $\overline{M}_{0,d+1}$ of the corresponding B' (strict transform of B under ψ_{d+1}) and D^T's:

- If $T = \{i, d+1\}$ and $1 \leq i \leq m$, $B'.D^T = 1$.
- If $T^c = \{i, j\}$ and $i, j \in \{1, 2, ..., m\}$, $B'.D^T = d - m$.
- If $T^c = \{i, j\}$ and $i, j \in \{m+1, m+2, ..., d\}$, $B'.D^T = d - 2$.
- If $T^c = \{i, j\}$, $i \in \{1, 2, ..., m\}$ and $j \in \{m+1, m+2, ..., d\}$, $B'.D^T = d - m - 1$.
- $B'.D^T = 0$ otherwise.

Hence, $B'.\Delta = m + \frac{(d-m)}{2}\big(m(m-1) + (d-m-1)(d+2m-2)\big)$.

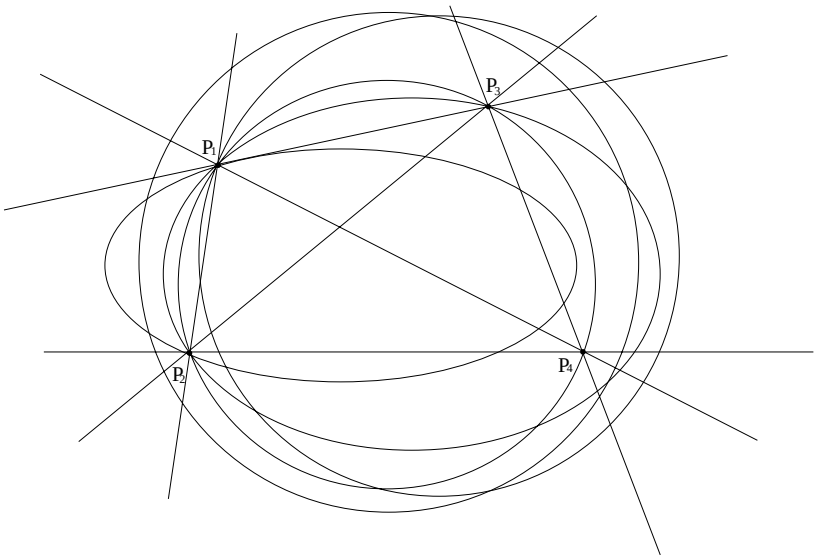


Figure 3.3: The five types of m-Veronese (conics) for $d = 4$.

A d-Veronese curve corresponds to a fiber of the map $\pi_{d+1} : \overline{M}_{0,d+1} \to \overline{M}_{0,d}$. The corresponding arrangement of d sections in $\mathbb{F}_{d-2}$ has $t_{d-1} = d$ and $t_k = 0$ for $k \neq d$. It can be seen in $\mathbb{P}^1 \times \mathbb{P}^1$ as d fibers, explaining its isotrivial nature.

In Figure 3.3, we show all the possible m-Veronese curves which satisfy the condition (*) in Definition III.4. The corresponding arrangements lie on $\mathbb{F}_2$ and can be seen as arrangements of conics in $\mathbb{P}^2$. For d higher, the possibilities for m-Veronese curves are richer, but it is hard to prove the existence. Although, we will show in the following sections an effective method to compute the arrangements corresponding to lines in $\mathbb{P}^{d-2}$.

3.5 Applications to (p, q)-nets.

In this section we use the correspondence for pairs $(\mathcal{A}, P)$, considering the "trick" of forming a new pair $(\mathcal{A}', P')$ where P' belongs to $\mathcal{A}$, to find facts about 4-nets, to classify $(3, q)$-nets for $2 \leq q \leq 6$, and to find the Quaternion nets.

3.5.1 $(4, q)$-nets.

Example III.13. (Hesse arrangement) In this example we will use our method to reprove the existence of the Hesse configuration. This $(4, 3)$-net has nice applications in Algebraic geometry (see for example [42, 1]). We will obtain two Hesse configurations according to our definition of isomorphism, which keeps record of the labelling of the lines. Let us denote this net by $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$, with $\mathcal{A}_i = \{L_{3i-2}, L_{3i-1}, L_{3i}\}$. Without loss of generality, we assume that the combinatorics is given by the following set of orthogonal Latin squares.

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

These Latin squares give the intersections of $\mathcal{A}_3$ and $\mathcal{A}_4$ respectively with $\mathcal{A}_1$ (columns) and $\mathcal{A}_2$ (rows). For example, the left one tell us that L_2, L_6 and L_7 (values) have a common point of incidence. The right one says L_2, L_6 and L_{12} have

also non-empty intersection. Hence, $[[2,6,7,12]] \in \mathcal{X}$. In this way, we find $\mathcal{X}$, which is completely described in the following table.

	L_1	L_2	L_3
L_4	L_7	L_8	L_9
L_5	L_8	L_9	L_7
L_6	L_9	L_7	L_8

	L_1	L_2	L_3
L_4	L_{10}	L_{11}	L_{12}
L_5	L_{12}	L_{10}	L_{11}
L_6	L_{11}	L_{12}	L_{10}

We now consider a new arrangement of lines $\mathcal{A}' = \mathcal{A} \setminus \{L_3, L_4, L_9, L_{12}\}$ together with the point $P = [[3,4,9,12]]$. We rename the twelve lines in the following way: $\mathcal{A}' = \{L'_1 = L_1, L'_2 = L_2, L'_3 = L_5, L'_4 = L_6, L'_5 = L_7, L'_6 = L_8, L'_7 = L_{10}, L'_8 = L_{11}\}$ and the lines passing through P, $\alpha = L_3$, $\beta = L_4$, $\gamma = L_9$ and $\delta = L_{12}$. Our one-to-one correspondence tells us that the pair $(\mathcal{A}', P)$ corresponds to a unique line L' in $\mathbb{P}^6$, and it passes through these distinguished four points α, β, γ and δ (we abuse the notation, as we saw these lines correspond to points on L'). Then, $K(\alpha) = \{[[4,6,7]], [[3,5,8]]\}$, $K(\beta) = \{[[1,6,8]], [[2,5,7]]\}$, $K(\gamma) = \{[[1,3,7]], [[2,4,8]]\}$ and $K(\delta) = \{[[1,4,5]], [[2,3,6]]\}$. Hence, we write:

$$\alpha = [a_1 : a_2 : 0 : 1 : 0 : 1 : 1], \qquad \beta = [0 : 1 : a_3 : a_4 : 1 : 0 : 1]$$

$$\gamma = [1 : 0 : 1 : 0 : a_5 : a_6 : 1], \qquad \delta = [1 : a_7 : a_7 : 1 : 1 : a_7 : a_8]$$

for some numbers a_i (which have restrictions) and we take $L' : \alpha t + \beta u$, $[t : u] \in \mathbb{P}^1$. For some $[t : u]$, we have the equation $\alpha t + \beta u = \gamma$, and from this we obtain:

$$a_2 = 1 - a_1 \quad a_3 = \frac{a_1}{a_1 - 1} \quad a_4 = \frac{1}{1 - a_1} \quad a_5 = \frac{a_1 - 1}{a_1} \quad a_6 = \frac{1}{a_1}.$$

For another $[t : u]$, we have $\alpha t + \beta u = \delta$, and this gives $a_7^2 - a_7 + 1 = 0$ and $a_1 = \frac{1}{a_7}$. Therefore, our field will need to have roots for the equation $x^2 - x + 1$. For instance, over $\mathbb{C}$, they are $w = e^{\frac{\pi\sqrt{-1}}{3}}$ and its complex conjugate $\overline{w}$. The two lines for

the corresponding two Hesse configurations are: $[1-w : w : 0 : 1 : 0 : 1 : 1]t+[0 : w : w-1 : 1 : w : 0 : w]u$ and $[w-1 : 1 : 0 : w : 0 : w : w]t+[0 : 1 : 1-w : w : 1 : 0 : 1]u$. By our choices at the end of Section 3.2, we have that the lines α, β, γ and δ are given by $ux - ty = 0$, where $[t : u]$ are the corresponding points in $\mathbb{P}^1$ for each of them, as points of L'. We now evaluate to obtain:

$$\{L_1 = (x+rz), L_2 = (x+ry+z), L_3 = (y)\}\ \{L_4 = (x), L_5 = (y+\overline{r}z), L_6 = (x+\overline{r}y+z)\}$$

$$\{L_7 = (y+z), L_8 = (x+z), L_9 = (x-y)\}\ \{L_{10} = (x+y+z), L_{11} = (z), L_{12} = (x-ry)\}$$

where $r = w$ or $\overline{w}$.

Example III.14. (There are no $(4,4)$-nets in characteristic $\neq 2$) We again start by supposing their existence, let $\mathcal{A} = \{\mathcal{A}_i\}_{i=1}^4$ be such a net. Without loss of generality, we can assume that the orthogonal set of Latin squares is:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

Now we consider $(\mathcal{A}', P)$ given by the arrangement of twelve lines $\mathcal{A}' = \mathcal{A} \setminus \{L_4, L_5, L_{12}, L_{16}\}$ and the point $P = [[4, 5, 12, 16]]$. The lines of $\mathcal{A}'$ are $L'_1 = L_1$, $L'_2 = L_2$, $L'_3 = L_3$, $L'_4 = L_6$, $L'_5 = L_7$, $L'_6 = L_8$, $L'_7 = L_9$, $L'_8 = L_{10}$, $L'_9 = L_{11}$, $L'_{10} = L_{13}$, $L'_{11} = L_{14}$ and $L'_{12} = L_{15}$. The special lines are $\alpha = L_4$, $\beta = L_5$, $\gamma = L_{12}$ and $\delta = L_{16}$. Hence, we have that

$$\alpha = [a_1 : a_2 : a_3 : 1 : a_4 : 0 : 0 : a_4 : 1 : a_4 : 1],\ \beta = [1 : b_4 : 0 : b_1 : b_2 : b_3 : 1 : b_4 : 0 : 1 : b_4]$$

$$\gamma = [1 : 0 : c_4 : c_4 : 0 : 1 : c_1 : c_2 : c_3 : c_4 : 1],\ \delta = [d_1 : 1 : d_4 : 1 : d_1 : d_4, 1 : d_4 : d_1 : d_2 : d_3]$$

as points in L', where this is again the corresponding line for $(\mathcal{A}', P)$. Let L' be $\alpha t + \beta u$, where $[t : u] \in \mathbb{P}^1$. When we impose L' to pass through γ, we obtain:

$$a_1 = \frac{1-c_1}{c_3}\ \ a_2 = \frac{c_3-1}{c_3}\ \ a_3 = \frac{c_1+c_2+c_3-1}{c_3}\ \ a_4 = \frac{c_2+c_3-1}{c_3}\ \ c_4 = c_1+c_2+c_3-1$$

$$b_1 = \frac{c_1 + c_2 - 1}{c_1} \quad b_2 = \frac{1 - c_2 - c_3}{c_1} \quad b_3 = \frac{1}{c_1} \quad b_4 = \frac{1 - c_3}{c_1}.$$

Let $c_1 = a$, $c_2 = b$ and $c_3 = c$. When we impose to L' to pass through δ, we get $ad_4 = 1$ and $ad_1 + b = 1$ plus the following equations:

$$\textbf{(1)}\ d_1(1-c) = 1 - b - c, \qquad \textbf{(2)}\ d_1(1-b)(c-1) + d_1c(1-c) = (1-b)c,$$

$$\textbf{(3)}\ (1-b)(1 + d_4(b+c-1)) = d_4c, \qquad \textbf{(4)}\ c^2 = (1-b)(b+c-1)$$

among others. These equations will be enough to obtain a contradiction. By isolating d_1 in **(1)**, replacing it in **(2)** and using **(4)**, we get $c^3 = (1-b)^3$ which requires a 3-rd primitive root of 1. Say w is such, so $b = 1 - wc$. Then, by using **(3)**, we get $w^2(1+2c) = w - 1$. We now suppose that the characteristic of our field is not 2, and so $c = \frac{1}{w}$. Then, $b = 0$ which is a contradiction. Notice that there is no contradiction if the characteristic is equal to 2.

3.5.2 $(3, q)$-nets for $2 \leq q \leq 6$.

In this subsection we will be using again the trick of eliminating some lines passing by a k-point P of the arrangement, to consider a new arrangement $\mathcal{A}'$ together with this point P. First we will be working with $(3, q)$-nets, so P will be a 3-point of $\mathcal{X}$ (and so we eliminate three lines from $\mathcal{A}$). If the $(3, q)$-net is given by $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ such that $\mathcal{A}_i = \{L_{q(i-1)+j}\}_{j=1}^{q}$, then the new pair $(\mathcal{A}', P)$ will be given by $\{L'_1 = L_2, L'_2 = L_3, ..., L'_{q-1} = L_q, L'_q = L_{q+2}, L'_{q+1} = L_{q+3}, ..., L'_{2q-2} = L_{2q}, L'_{2q-1} = L_{2q+2}, L'_{2q} = L_{2q+3}, ..., L'_{3q-3} = L_{3q}\}$, $P = L_1 \cap L_{q+1} \cap L_{2q+1}$ and $\alpha = L_1$, $\beta = L_{q+1}$, $\gamma = L_{2q+1}$. The corresponding line for $(\mathcal{A}', P)$ will be $L' : \alpha t + \beta u$, $[t : u] \in \mathbb{P}^1$. We obtain $\mathcal{X}$ from a Latin square. Then, we fix a point P in $\mathcal{X}$, so the locus of the line L' is the moduli space of the $(3, d)$-nets (keeping the labelling) corresponding to that Latin square (or better its main class). We will give in each case equations for the lines of the nets depending on parameters coming from L'.

$(3,2)$-nets

Here we have one main class given by the multiplication table of $\mathbb{Z}/2\mathbb{Z}$: $\begin{array}{|cc|}\hline 1 & 2 \\ 2 & 1 \\ \hline\end{array}$. According to our set up, $(\mathcal{A}', P)$ is formed by an arrangement $\mathcal{A}'$ of three lines and $P = [[1,3,5]] \in \mathcal{X}$. The corresponding line L' is actually the whole space $\mathbb{P}^1$. This tells us that there is only one $(3,2)$-net up to isomorphism. The special points are $\alpha = [1:0]$, $\beta = [0:1]$ and $\gamma = [1:1]$. This $(3,2)$-net is represented by the singular members of the pencil $\lambda z(x-y) + \mu y(z-x) = 0$ on $\mathbb{P}^2$.

$(3,3)$-nets

Again, there is one main class given by the multiplication table of $\mathbb{Z}/3\mathbb{Z}$.

$$\begin{array}{|ccc|}\hline 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \\ \hline\end{array}$$

For $(\mathcal{A}', P)$ we have an arrangement of six lines $\mathcal{A}'$ and $P = [[1,4,7]] \in \mathcal{X}$, our line L' is in $\mathbb{P}^4$. The special points can be taken as $\alpha = [a_1 : a_2 : 1 : 0 : 1]$, $\beta = [1 : 0 : b_1 : b_2 : 1]$ and $\gamma = [1 : c_1 : c_1 : 1 : c_2]$. Then, for some $[t:u] \in \mathbb{P}^1$, we have $\alpha t + \beta u = \gamma$. Thus, if $a_2 = a$, $b_2 = b$ and $c_1 = c$, we have that $\alpha = \left[\frac{a(b-1)}{bc} : a : 1 : 0 : 1\right]$ and $\beta = \left[1 : 0 : \frac{bc(a-1)}{a} : b : 1\right]$. The rest of the points in $\mathcal{X}'$ (again, although $\mathcal{A}'$ is not a net, we think of $\mathcal{X}'$ as the set of 3-points in $\mathcal{A}'$ coming from $\mathcal{X}$) $[[1,3,6]]$ and $[[2,4,5]]$ give the same restriction $(a-1)(b-1) = 1$, i.e., $a = \frac{b}{b-1}$. Therefore, the line L' has two parameters of freedom and is given by $\left[\frac{1}{c} : \frac{b}{b-1} : 1 : 0 : 1\right]t + [1 : 0 : c : b : 1]u$ where c, b are numbers with some restrictions (for example, $c, b \neq 0$ or 1). Hence, we find that this family of $(3,3)$-nets can be represented by: $L_1 = (y)$, $L_2 = (\frac{1}{c}x + y + z)$, $L_3 = (\frac{b}{b-1}x + z)$, $L_4 = (x)$, $L_5 = (x + cy + z)$, $L_6 = (by + z)$, $L_7 = (x + c(1-b)y)$, $L_8 = (x + y + z)$ and $L_9 = (z)$.

$(3,4)$-nets

Here we have two main classes. We represent them by the following Latin squares.

$$M_1 = \left|\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array}\right| \qquad M_2 = \left|\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}\right|$$

They correspond to $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ respectively. We deal first with M_1. Then we have $\alpha = [a_1 : a_2 : a_3 : 1 : a_4 : 0 : 1 : a_4]$, $\beta = [1 : b_1 : 0 : b_2 : b_3 : b_4 : 1 : b_1]$ and $\gamma = [1 : c_1 : c_2 : c_2 : c_1 : 1 : c_3 : c_4]$. Let $a_3 = a$, $b_4 = b$ and $c_2 = c$. By imposing γ to L', we can find $a_1 = \frac{(-1+b)a}{bc}$, $a_2 = \frac{(-b_1+c_1b)a}{bc}$, $a_4 = \frac{(-b_1+c_4b)a}{bc}$, $b_2 = \frac{-(c-c_2a)b}{a}$, $b_3 = b_1 - c_4b + c_1b$ and $c_3 = \frac{1}{b} + \frac{c}{a}$. When we impose L' to pass through $[[1, 5, 9]]$, $[[2, 4, 9]]$ and $[[1, 4, 8]]$, we obtain equations to solve for c_4, c_1 and b_1 respectively. After that, the restrictions $[[2, 6, 7]]$, $[[3, 5, 7]]$ and $[[3, 6, 8]]$ are trivially satisfied. The line L' is parametrized by (a, b, c) in a open set of $\mathbb{A}^3$ and is given by: $a_1 = \frac{a(b-1)}{bc}$, $a_2 = \frac{ab}{abc+ab-a-bc}$, $a_3 = a$, $a_4 = \frac{a^2(b-1)}{abc+ab-a-bc}$, $b_1 = \frac{b^2(a-1)c}{abc+ab-a-bc}$, $b_2 = \frac{bc(a-1)}{a}$, $b_3 = \frac{abc}{abc+ab-a-bc}$ and $b_4 = b$.

Similarly, for M_2 we have $\alpha = [a_1 : a_2 : a_3 : 1 : a_4 : 0 : 1 : a_4]$, $\beta = [1 : b_1 : 0 : b_2 : b_3 : b_4 : 1 : b_1]$ and $\gamma = [1 : c_1 : c_2 : 1 : c_1 : c_2 : c_3 : c_4]$. Of course, the only change is γ. By doing similar computations, we have that L' is parametrized by (a, b, c) in a open set of $\mathbb{A}^3$ and is given by: $a_1 = \frac{(b-c)a}{bc}$, $a_2 = \frac{abc}{abc+ab-bc-ac}$, $a_3 = a$, $a_4 = \frac{a^2(b-c)}{abc+ab-bc-ac}$, $b_1 = \frac{b^2(a-c)}{abc+ab-bc-ac}$, $b_2 = \frac{b(a-c)}{ac}$, $b_3 = \frac{abc}{abc+ab-bc-ac}$ and $b_4 = b$.

Hence, the lines for the corresponding $(3, 4)$-nets for M_r can be represented by: $L_1 = (y)$, $L_2 = (a_1x + y + z)$, $L_3 = (a_2x + b_1y + z)$, $L_4 = (a_3x + z)$, $L_5 = (x)$, $L_6 = (x + b_2y + z)$, $L_7 = (a_4x + b_3y + z)$, $L_8 = (b_4y + z)$, $L_9 = (ax - bc^{2-r}y)$, $L_{10} = (x + y + z)$, $L_{11} = (a_4x + b_1y + z)$ and $L_{12} = (z)$. For example, if we evaluate the equations for the cyclic type M_1 at $a = \frac{1+i}{2}$, $b = \frac{1-i}{2}$ and $c = -i$ (where $i = \sqrt{-1}$), we obtain the very well known net: $\mathcal{A}_1 = \{y, (1+i)x+2y+2z, (1+i)x+y+2z, (1+i)x+2z\}$, $\mathcal{A}_2 = \{x, 2x + (1-i)y + 2z, x + (1-i)y + 2z, (1-i)y + 2z\}$ and $\mathcal{A}_3 = \{x + y, x + y + z, x + y + 2z, z\}$.

$(3,5)$-nets

Here we again have two main classes, we represent them by the following Latin squares.

$$M_1 = \begin{array}{|ccccc|}\hline 1&2&3&4&5\\ 2&3&4&5&1\\ 3&4&5&1&2\\ 4&5&1&2&3\\ 5&1&2&3&4\\ \hline\end{array} \qquad M_2 = \begin{array}{|ccccc|}\hline 1&2&3&4&5\\ 2&1&4&5&3\\ 3&5&1&2&4\\ 4&3&5&1&2\\ 5&4&2&3&1\\ \hline\end{array}$$

The Latin square M_1 corresponds to $\mathbb{Z}/5\mathbb{Z}$. As before, for M_1 and M_2 we have that $\alpha = [a_1 : a_2 : a_3 : a_4 : 1 : a_5 : a_6 : 0 : 1 : a_5 : a_6]$ and $\beta = [1 : b_1 : b_2 : 0 : b_3 : b_4 : b_5 : b_6 : 1 : b_1 : b_2]$, but for M_1, $\gamma = [1 : c_1 : c_2 : c_3 : c_3 : c_2 : c_1 : 1 : c_4 : c_5 : c_6]$, and for M_2, $\gamma = [1 : c_1 : c_2 : c_3 : 1 : c_1 : c_2 : c_3 : c_4 : c_5 : c_6]$.

In the case of M_1, after we impose γ to L', we use the conditions $[[2,3,5]]$, $[[4,8,11]]$, $[[2,6,12]]$, $[[2,8,9]]$ and $[[3,8,10]]$ to solve for b_2, c_6, c_5, b_1 and c_2 respectively. After that we have four parameters left: $a_4 = a$, $b_6 = b$, $c_3 = c$ and $c_1 = d$, and we obtain the following constrain for them:

$$b^2(a-1)(d-c)(c-ad)+b(-d^2a+dc+2d^2a^2-2da^2c-da+ca-dc^2+c^2da)$$
$$+ad(ca-da+1-c)=0.$$

Hence, the $(3,5)$-nets for M_1 are parametrized by an open set of the hypersurface in $\mathbb{A}^4$ defined by this equation. The values for the variables are:

$$a_1 = \frac{a(b-1)}{bc}, \quad a_2 = \frac{ab(d-1)}{a-ba+bc}, \quad a_3 = \frac{a(d-db+bc)}{c^2(a-1)b}, \quad a_4 = a,$$

$$a_5 = \frac{a^2(d-1)(d-db+bc)}{(a-ba+bc)(a-1)cd}, \quad a_6 = \frac{ad(b-1)}{bc}, \quad b_1 = \frac{b(da-adb+bc)}{a-ba+bc},$$

$$b_2 = \frac{d-db+bc}{c}, \quad b_3 = \frac{bc(a-1)}{a}, \quad b_4 = \frac{(da-adb+bc)(d-db+bc)a}{(a-ba+bc)(a-1)cd},$$

$$b_5 = d, \quad b_6 = b.$$

In the case of M_2, we obtain a three dimensional moduli space of $(3,5)$-nets as well. It is parametrized by (a,b,c) in an open set of $\mathbb{A}^3$ such that $a_4 = a$, $b_6 = b$ and $c_1 = c$, and:

$$a_1 = \frac{a^2(1-b)}{b(ab-a-b)}, \quad a_2 = c, \quad a_3 = \frac{(-a^2+a^2b+cba-ab-cb)b}{(ab+cb-a-b)(ab-a-b)}, \quad a_4 = a,$$

$$a_5 = \frac{(a^2-a^2b-cba+ab+cb)a}{(ab-a-b)^2}, \quad a_6 = \frac{c(b-1)a}{-a+ab+cb-b},$$

$$b_1 = \frac{cb^2(1-a)}{a(ab-a-b)}, \quad b_2 = \frac{(a-ab+b-c)b^2}{(-a+ab+cb-b)(ab-a-b)},$$

$$b_3 = \frac{b^2(1-a)}{a(ab-a-b)}, \quad b_4 = \frac{ab(ab-a-b+c)}{(ab-a-b)^2}, \quad b_5 = \frac{cb(a+b-ab)}{a(ab-a+bc-b)}, \quad b_6 = b.$$

To obtain the lines for the nets corresponding to M_r, we just evaluate and obtain: $L_1 = (y)$, $L_2 = (a_1x+y+z)$, $L_3 = (a_2x+b_1y+z)$, $L_4 = (a_3x+b_2y+z)$, $L_5 = (a_4x+z)$, $L_6 = (x)$, $L_7 = (x+b_3y+z)$, $L_8 = (a_5x+b_4y+z)$, $L_9 = (a_6x+b_5y+z)$, $L_{10} = (b_6y+z)$, $L_{11} = (ax - bc^{2-r}y)$, $L_{12} = (x+y+z)$, $L_{13} = (a_5x+b_1y+z)$, $L_{14} = (a_6x+b_2y+z)$ and $L_{15} = (z)$. These two 3 dimensional families of $(3,5)$-nets appear in [81]. We notice that both families of $(3,5)$-nets have members defined over $\mathbb{Q}$. For the case M_1, we can make b^2 disappear from the equation by declaring $c = ad$ (the relations $a = 1$ and $d = c$ are not allowed). Then, $b = \frac{2da-1-da^2}{2da-2da^2-1+a-d^2a+d^2a^2}$ and it can be checked that for suitable $a, d \in \mathbb{Z}$ the conditions for being $(3,5)$-net are satisfied.

$(3,6)$-nets

We have twelve main classes of Latin squares to check. The following is a list showing one member of each class. It was taken from [23, pp. 129-137].

$$M_1 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&3&4&5&6&1\\ 3&4&5&6&1&2\\ 4&5&6&1&2&3\\ 5&6&1&2&3&4\\ 6&1&2&3&4&5\\ \hline\end{array} \quad M_2 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&5&6&3&4\\ 3&6&1&5&4&2\\ 4&5&6&1&2&3\\ 5&4&2&3&6&1\\ 6&3&4&2&1&5\\ \hline\end{array} \quad M_3 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&3&1&5&6&4\\ 3&1&2&6&4&5\\ 4&6&5&2&1&3\\ 5&4&6&3&2&1\\ 6&5&4&1&3&2\\ \hline\end{array}$$

$$M_4 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&4&3&6&5\\ 3&4&5&6&1&2\\ 4&3&6&5&2&1\\ 5&6&1&2&4&3\\ 6&5&2&1&3&4\\ \hline\end{array} \quad M_5 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&4&3&6&5\\ 3&4&5&6&1&2\\ 4&3&6&5&2&1\\ 5&6&2&1&4&3\\ 6&5&1&2&3&4\\ \hline\end{array} \quad M_6 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&4&5&6&3\\ 3&6&2&1&4&5\\ 4&5&6&2&3&1\\ 5&3&1&6&2&4\\ 6&4&5&3&1&2\\ \hline\end{array}$$

$$M_7 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&4&3&6&5\\ 3&5&1&6&4&2\\ 4&6&5&1&2&3\\ 5&3&6&2&1&4\\ 6&4&2&5&3&1\\ \hline\end{array} \quad M_8 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&6&5&3&4\\ 3&6&1&2&4&5\\ 4&5&2&1&6&3\\ 5&3&4&6&1&2\\ 6&4&5&3&2&1\\ \hline\end{array} \quad M_9 = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&3&1&6&4&5\\ 3&1&2&5&6&4\\ 4&6&5&1&2&3\\ 5&4&6&2&3&1\\ 6&5&4&3&1&2\\ \hline\end{array}$$

$$M_{10} = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&6&5&4&3\\ 3&5&1&2&6&4\\ 4&6&2&1&3&5\\ 5&3&4&6&2&1\\ 6&4&5&3&1&2\\ \hline\end{array} \quad M_{11} = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&4&5&6&3\\ 3&4&2&6&1&5\\ 4&5&6&2&3&1\\ 5&6&1&3&2&4\\ 6&3&5&1&4&2\\ \hline\end{array} \quad M_{12} = \begin{array}{|cccccc|}\hline 1&2&3&4&5&6\\ 2&1&5&6&4&3\\ 3&5&4&2&6&1\\ 4&6&2&3&1&5\\ 5&4&6&1&3&2\\ 6&3&1&5&2&4\\ \hline\end{array}$$

The Latin squares M_1 and M_2 correspond to the multiplication table of $\mathbb{Z}/6\mathbb{Z}$ and S_3 respectively. The following will be the set up for the analysis of $(3,6)$-nets. We first fix one Latin square M from the list above. Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ be the corresponding (possible) $(3,6)$-net, where $\mathcal{A}_1 = \{L_1, ..., L_6\}$, $\mathcal{A}_2 = \{L_7, ..., L_{12}\}$ and $\mathcal{A}_3 = \{L_{13}, ..., L_{18}\}$. In analogy to what we did before, we consider a new arrangement $\mathcal{A}'$ together with a point P such that $\mathcal{A}' = \mathcal{A} \setminus \{L_1, L_7, L_{13}\}$ and $P = [[1, 7, 13]] \in \mathcal{X}$. We label the lines of $\mathcal{A}'$ from 1 to 15 following the order of $\mathcal{A}$, that is, $L'_1 = L_2$,..., $L'_5 = L_6$, $L'_6 = L_8$, etc, eliminating L_1, L_7 and L_{13}. Let L' be the line in $\mathbb{P}^{13}$ given by $(\mathcal{A}', P)$. The special lines (or points of L') $\alpha = L_1$, $\beta = L_7$ and $\gamma = L_{13}$ can be taken as $\alpha = [a_1 : a_2 : a_3 : a_4 : a_5 : 1 : a_6 : a_7 : a_8 : 0 : 1 : a_6 : a_7 : a_8]$, $\beta = [1 : b_1 : b_2 : b_3 : 0 : b_4 : b_5 : b_6 : b_7 : b_8 : 1 : b_1 : b_2 : b_3]$ and $\gamma = \gamma(c_1, c_2, ..., c_8)$ depending on M. Since there is $[t, u] \in \mathbb{P}^1$ satisfying $\alpha t + \beta u = \gamma$, we can and do write a_1, a_2, a_3, a_4, a_6, a_7, a_8, b_4, b_5, b_6 and c_5 depending on the resting variables. After that, we start to impose the points in $\mathcal{X}'$ which translates, as before, into 2×2 determinants equal to zero. At this stage we have 20 equations given by these determinants, and 12 variables. We choose appropriately from them to isolate

variables so that they appear linearly, i.e., with exponent equals to 1. In the way of solving these equations, we prove or disprove the existence of $\mathcal{A}$. In the case that this $(3,6)$-net exists, i.e. $\mathcal{A}$ is realizable on $\mathbb{P}^2$ over some field, the equations for its lines can be taken as: $L_1 = (y)$, $L_2 = (a_1x + y + z)$, $L_3 = (a_2x + b_1y + z)$, $L_4 = (a_3x+b_2y+z)$, $L_5 = (a_4x+b_3y+z)$, $L_6 = (a_5x+z)$, $L_7 = (x)$, $L_8 = (x+b_4y+z)$, $L_9 = (a_6x + b_5y + z)$, $L_{10} = (a_7x + b_6y + z)$, $L_{11} = (a_8x + b_7y + z)$, $L_{12} = (b_8y + z)$, $L_{13} = (ux - ty)$, $L_{14} = (x + y + z)$, $L_{15} = (a_6x + b_1y + z)$, $L_{16} = (a_7x + b_2y + z)$, $L_{17} = (a_8x + b_3y + z)$ and $L_{18} = (z)$, where $[t, u]$ satisfies $\alpha t + \beta u = \gamma$.

Now we apply this procedure case by case. We first give the result, after that we indicate the order we solve for the points in $\mathcal{X}'$, and then we give a moduli parametrization whenever the net exits. For simplicity, we will work always over the complex numbers $\mathbb{C}$. Sometimes we will omit the final expressions for the variables, although they all can explicitly be given.

$\underline{M_1}$: $(\mathbb{Z}/6\mathbb{Z})$ This gives a three dimensional moduli space. We also have that some of these nets can be defined over $\mathbb{R}$. We solve the determinants in the following order: $[[4, 6, 15]]$ solve for c_3, $[[5, 10, 14]]$ solve for c_8, $[[1, 9, 15]]$ solve for c_1, $[[5, 9, 13]]$ solve for c_7, $[[3, 10, 12]]$ solve for c_6, $[[2, 10, 11]]$ solve for b_3, $[[3, 9, 11]]$ solve for c_2 and $[[2, 8, 15]]$ solve for b_2. If $a_5 = a$, $b_1 = d$, $b_8 = b$ and $c_4 = c$, then they must satisfy:

$$c^2(-1+a)b^4(a^2 - a^2b + cab + ab - 2a + ca - bc) - b^2c(2c^2b^2 + 5cab + 4a^2b^2c - 4ca^2b - 2b^2a^3 - a^2c + 3a^2 - 2a^3 - 5a^2b + ca^3 + 2a^2b^2 - bc^2a + 4ba^3 - 4ac^2b^2 - 3ab^2c - a^3b^2c + c^2ba^2 + 2a^2b^2c^2)d + (bc + a - ab)(a^2b^2c^2 + c^2b^2 - 2ac^2b^2 + a^2b^2c - ab^2c + 2cab - ca^2b + a^2b^2 - 2a^2b + a^2)d^2 = 0.$$

So, the moduli space for these nets is an open set of this hypersurface.

$\underline{M_2}$: (S_3) This gives a three dimensional moduli space parametrized by an open

set of $\mathcal{A}^3$. It does not contains $(3,6)$-nets defined over $\mathbb{R}$. The reason is that we need the square root of -1 to define the nets. Moreover, all of them have extra 3-points, apart from the ones coming from $\mathcal{X}$. The order we take is: $[[5,10,14]]$ solve for c_8, $[[2,6,15]]$ solve for c_1, $[[1,10,13]]$ solve for c_7, $[[1,9,12]]$ solve for c_6, $[[2,10,11]]$ solve for b_1, $[[5,6,12]]$ solve for c_3, $[[1,8,15]]$ solve for b_2, $[[1,7,14]]$ solve for b_8 and $[[2,8,14]]$ solve for c_2. If $i=\sqrt{-1}$, $a_5=a$, $b_3=e$ and $c_4=c$, then the expressions for the variables are:

$$a_1=\tfrac{(1+i\sqrt{3})(2c+e-i\sqrt{3}e)a}{4ce},\ a_2=\tfrac{2ace}{2aec-ac-ce-ice\sqrt{3}+ae+iae\sqrt{3}+ica\sqrt{3}},$$
$$a_3=\tfrac{(-1+i\sqrt{3})(ae-iae\sqrt{3}-2ce+2ac)a}{2(2ae-2ce+2aec+ac+ica\sqrt{3})},\ a_4=a,\ a_5=a,$$
$$a_6=\tfrac{(-1+i\sqrt{3})(ae-iae\sqrt{3}-2ce+2ac)a}{2(2aec-ac-ce-ice\sqrt{3}+ae+iae\sqrt{3}+ica\sqrt{3})},\ a_7=\tfrac{(1+i\sqrt{3})(e-i\sqrt{3}e+2c)a^2}{2(2ae-2ce+2aec+ac+ica\sqrt{3})},\ a_8=\tfrac{(1+i\sqrt{3})a}{2},$$
$$b_1=\tfrac{(-1+i\sqrt{3})e^2(a-c)}{2aec-ac-ce-ice\sqrt{3}+ae+iae\sqrt{3}+ica\sqrt{3}},\ b_2=\tfrac{(1+i\sqrt{3})(-ce+ae+ac)e}{2ae-2ce+2aec+ac+ica\sqrt{3}},\ b_3=e,$$
$$b_4=\tfrac{(-1+i\sqrt{3})(a-c)e}{2ac},\ b_5=\tfrac{(1+i\sqrt{3})(-ce+ae+ac)e}{2aec-ac-ce-ice\sqrt{3}+ae+iae\sqrt{3}+ica\sqrt{3}},\ b_6=\tfrac{2aec}{2ae-2ce+2aec+ac+ica\sqrt{3}},$$
$$b_7=\tfrac{(1+i\sqrt{3})e}{2},\ b_8=\tfrac{(1+i\sqrt{3})e}{2}.$$

For instance, if we substitute $a=\frac{c+ic\sqrt{3}}{1+i\sqrt{3}-2c}$ and $e=\frac{c(1+i\sqrt{3})}{2(c-1)}$, we produce a one dimensional family of arrangements of 18 lines with $t_2=18$, $t_3=39$, $t_4=3$, $t_k=0$ otherwise.

$\underline{M_3}$: This gives a three dimensional moduli space which does not contains $(3,6)$-nets defined over $\mathbb{R}$. The reason again is that we need to have the square root of -1 to realize the nets. The order we take is: $[[5,10,11]]$ solve for b_8, $[[1,9,15]]$ solve for c_8, $[[5,9,12]]$ solve for c_6, $[[3,6,15]]$ solve for c_1, $[[1,10,13]]$ solve for c_7, $[[4,9,11]]$ solve for b_3, $[[1,6,12]]$ solve for b_1 and $[[3,10,12]]$ solve for b_2. If $a_5=a$, $c_3=d$, $c_2=e$ and $c_4=c$, then they must satisfy:

$$(e^2a^2+e^2-e^2a-2a^2de-de+d^2+3dea+d^2a^2-2d^2a)+(-ea-e+ad-d)c+c^2=0$$

and so its moduli space is an open set of this hypersurface. Moreover, by solving for

c, we have that: $c = \frac{1}{2}(ea + e - ad + d \pm \sqrt{-3}(a-1)(d-e))$. But, we cannot have $a = 1$ or $d = e$, so the square root of -1 is necessary.

$\underline{M_4}$: This case is not possible over $\mathbb{C}$. To achieve contradiction, we proceed as: $[[5, 10, 13]]$ solve for c_7, $[[3, 7, 15]]$ solve for c_6, $[[2, 8, 15]]$ solve for b_2, $[[4, 6, 15]]$ solve for c_3, $[[5, 6, 14]]$ solve for a_5, $[[1, 9, 15]]$ solve for c_8, $[[1, 10, 14]]$ solve for c_1, $[[3, 8, 14]]$ solve for c_2, $[[2, 10, 11]]$ solve for c_4 and $[[2, 6, 13]]$ solve for b_1. At this stage, we obtain several possibilities from the equation given by $[[2, 6, 13]]$, none of them possible (for example, $a_2 = a_6$).

$\underline{M_5}$: This case is not possible over $\mathbb{C}$. By solving $[[5, 10, 13]]$ for c_7 and then $[[3, 7, 15]]$ for c_6, we obtain $a_6 = a_7$ which is a contradiction.

$\underline{M_6}$: This gives a two dimensional moduli space, and so it is not always three dimensional. Some of these nets can be defined over $\mathbb{R}$. The order we take is: $[[5, 10, 11]]$ solve for a_5, $[[1, 9, 15]]$ solve for c_8, $[[3, 7, 15]]$ solve for c_6, $[[2, 6, 15]]$ solve for b_1, $[[5, 6, 13]]$ solve for c_7, $[[4, 9, 11]]$ solve for b_3, $[[2, 9, 13]]$ solve for c_1, $[[1, 10, 12]]$ solve for c_3 and $[[3, 9, 12]]$ solve for b_2. If $b_8 = b$, $c_2 = d$ and $c_4 = c$, then they must satisfy:

$$bc(1-c)(bc-c-b)+(bc^3+b^2-5bc^2+3bc-2b^2c+b^2c^2-c^3+2c^2)d+(-b+2bc-2c+c^2)d^2 = 0.$$

Thus, its moduli space is an open set of this hypersurface.

$\underline{M_7}$: This gives a two dimensional moduli space parametrized by an open set of $\mathbb{A}^2$. These nets can be defined over $\mathbb{Q}$. The order we solve is the following: $[[5, 6, 13]]$ solve for c_7, $[[3, 6, 15]]$ solve for b_2, $[[1, 9, 15]]$ solve for c_8, $[[5, 9, 12]]$ solve for c_6, $[[1, 10, 14]]$ solve for b_3, $[[3, 9, 11]]$ solve for b_8, $[[4, 8, 11]]$ solve for c_3, $[[4, 10, 13]]$ solve for c_2, $[[4, 7, 15]]$ solve for b_1 and $[[5, 7, 11]]$ solve for c_1. If $a_5 = a$ and $c_4 = c$, then the expressions for the variables are:

$$a_1 = \frac{(c^2-4c+2ac+4-2a)a}{c(a-2)(c-2)},\ a_2 = \frac{(c-1)(c-2)(a-2)a}{a^2c^2+a^2-2a^2c-2c^2a+5ac-2a+c^2-2c},\ a_3 = \frac{ac(a+c-2)}{-c^2-ac+c^2a+2c-2a+a^2},$$

$$a_4 = \frac{(a-2)(a-ac+c-2)a}{-c^2+c^2a-3ac+2c+a^2c+2a-a^2},\ a_5 = a,\ a_6 = \frac{(a+c-2)(-a+ac-c+2)a}{a^2c^2+a^2-2a^2c-2c^2a+5ac-2a+c^2-2c},$$

$$a_7 = \frac{(a-2)a^2(c-1)}{c^2+ac-c^2a-2c+2a-a^2},\ a_8 = \frac{a(c^2-4c+2ac+4-2a)}{-c^2+c^2a-3ac+2c+a^2c+2a-a^2},$$

$$b_1 = \frac{(a-1)(a-2)(c-2)^2c}{(a+c-2)(a^2c^2+a^2-2a^2c-2c^2a+5ac-2a+c^2-2c)},\ b_2 = \frac{(c-a)(a-2)(c-2)}{-c^2-ac+c^2a+2c-2a+a^2},$$

$$b_3 = \frac{(a-2)^2(c-1)a(c-2)}{(a+c-2)(c^2-c^2a+3ac-2c-a^2c-2a+a^2)},\ b_4 = \frac{(c-a)(a-2)(c-2)}{ac(a+c-2)},$$

$$b_5 = \frac{(c-1)(c-2)(a-2)a}{a^2c^2+a^2-2a^2c-2c^2a+5ac-2a+c^2-2c},\ b_6 = \frac{c(a-2)(c-2)a(a-1)}{(c^2+ac-c^2a-2c+2a-a^2)(a+c-2)},$$

$$b_7 = \frac{c(a-2)(c-2)}{-c^2+c^2a-3ac+2c+a^2c+2a-a^2},\ b_8 = \frac{(a-2)(c-2)}{2-a-c}.$$

$\underline{M_8}$: This gives a two dimensional moduli space. Some of these nets can be defined over $\mathbb{R}$. The order we solve is the following: $[[2,6,15]]$ solve for b_1, $[[1,10,13]]$ solve for c_7, $[[1,7,15]]$ solve for c_6, $[[5,7,14]]$ solve for c_8, $[[4,10,11]]$ solve for c_3, $[[5,6,13]]$ solve for b_2, $[[2,10,14]]$ solve for b_3, $[[5,9,11]]$ solve for a_5 and $[[3,7,11]]$ solve for c_1. If $b_8 = b$, $c_4 = c$ and $c_2 = e$, then they have to satisfy:

$$c^2(c-b)(4c^2-6cb-b^3+3b^2)+c(cb-2c+b)(6c^2-9cb-b^3+4b^2)e+(bc-b+c)(cb-2c+b)^2e^2 = 0.$$

Thus, its moduli space is an open set of this hypersurface. This family of nets is not possible in characteristic 2 because, for example, we have that $c_5 = \frac{2c}{b}$.

$\underline{M_9}$: This gives a three dimensional moduli space. Some of these nets can be defined over $\mathbb{R}$. The order we solve is the following: $[[5,10,11]]$ solve for a_5, $[[1,10,14]]$ solve for c_8, $[[4,7,15]]$ solve for c_6, $[[4,9,12]]$ solve for b_3, $[[1,8,15]]$ solve for c_7, $[[5,8,12]]$ solve for c_2, $[[5,6,14]]$ solve for c_1 and $[[3,6,15]]$ solve for b_2. If $b_1 = e$, $b_8 = b$, $c_4 = c$ and $c_3 = d$, then they have to satisfy:

$$(b^2c^2 + c^2 + bc - b^2c - 2bc^2) + (-2c + 2bc + ce - bec + e^2b - eb)d + (-e + 1)d^2 = 0.$$

Thus, its moduli space is an open set of this hypersurface.

$\underline{M_{10}}$: This gives a two dimensional moduli space. Some of these nets can be defined over $\mathbb{R}$. The order we solve is the following: $[[5,10,11]]$ solve for a_5, $[[1,7,15]]$ solve

for b_1, $[[1,10,12]]$ solve for c_6, $[[3,6,15]]$ solve for b_2, $[[5,6,13]]$ solve for c_7, $[[5,7,14]]$ solve for c_8, $[[4,8,15]]$ solve for b_3, $[[3,7,11]]$ solve for c_3 and $[[2,8,11]]$ solve for c_2. If $b_8 = b$, $c_4 = c$ and $c_1 = e$, then they have to satisfy:

$$ce(c-2e) + (2ce - c - e)(e-c)b + c(1-e)(e-c)b^2 = 0.$$

Thus, its moduli space is an open set of this hypersurface.

$\underline{M_{11}}$: This also gives a two dimensional moduli space. Some of these nets can be defined over $\mathbb{R}$. The order we solve is the following: $[[5,10,11]]$ solve for a_5, $[[1,9,15]]$ solve for c_8, $[[3,8,11]]$ solve for c_7, $[[3,7,15]]$ solve for c_6, $[[4,6,15]]$ solve for b_3, $[[2,8,15]]$ solve for b_1, $[[4,9,11]]$ solve for c_2, $[[5,7,14]]$ solve for c_3 and $[[1,8,14]]$ solve for c_1. For example, we have that c_7 has to be zero and so L_{13}, L_{16} and L_{18} have always a common point of incidence. If $b_2 = e$, $b_8 = b$ and $c_4 = c$, then they must satisfy:

$$c(b-1)(bc-b-c) + (b^2c - 2bc + c - b^2 + 2b)e - e^2 = 0.$$

Thus, its moduli space is an open set of this hypersurface.

$\underline{M_{12}}$: This case is not possible over $\mathbb{C}$. To achieve contradiction, we take: $[[2,9,15]]$ solve for c_8, $[[5,10,13]]$ solve for c_7, $[[3,6,15]]$ solve for b_2, $[[1,8,15]]$ solve for c_3, $[[1,10,12]]$ solve for c_6, $[[5,9,11]]$ solve for c_2 and $[[5,6,12]]$ solve for b_1. Then, the equation induced by $[[1,9,13]]$ gives six possibilities, all of them producing a contradiction.

3.5.3 The Quaternion nets.

We now do the analysis of $(3,8)$-nets corresponding to the multiplication table of the Quaternion group.

$$M = \begin{array}{|cccccccc|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 6 & 7 & 8 & 3 & 4 & 5 \\ 3 & 6 & 2 & 5 & 7 & 1 & 8 & 4 \\ 4 & 7 & 8 & 2 & 3 & 5 & 1 & 6 \\ 5 & 8 & 7 & 6 & 2 & 4 & 3 & 1 \\ 6 & 3 & 1 & 8 & 4 & 2 & 5 & 7 \\ 7 & 4 & 5 & 1 & 6 & 8 & 2 & 3 \\ 8 & 5 & 4 & 3 & 1 & 7 & 6 & 2 \\ \hline \end{array}$$

In this case, we have that there is a three dimensional moduli space for them, given by an open set of $\mathbb{A}^3$. Also, these 3-nets can be defined over $\mathbb{Q}$. This example shows again that a non-abelian group can also realize a 3-net on $\mathbb{P}^2$. The set up is analogous to what we have done before. In this case, $\mathcal{A}' = \mathcal{A} \setminus \{L_1, L_9, L_{17}\}$ and $P = [[1, 9, 17]]$. Our distinguished points on $L' \subseteq \mathbb{P}^{19}$ are: $\alpha = [a_1 : a_2 : a_3 : a_4 : a_5 : a_6 : a_7 : 1 : a_8 : a_9 : a_{10} : a_{11} : a_{12} : 0 : 1 : a_8 : a_9 : a_{10} : a_{11} : a_{12}]$, $\beta = [1 : b_1 : b_2 : b_3 : b_4 : b_5 : 0 : b_6 : b_7 : b_8 : b_9 : b_{10} : b_{11} : b_{12} : 1 : b_1 : b_2 : b_3 : b_4 : b_5]$ and $\gamma = [1 : c_1 : c_2 : c_3 : c_4 : c_5 : c_6 : 1 : c_4 : c_5 : c_6 : c_1 : c_2 : c_3 : c_7 : c_8 : c_9 : c_{10} : c_{11} : c_{12}]$. Let $[t : u] \in \mathbb{P}^1$ such that $\alpha t + \beta u = \gamma$. We isolate first a_1, a_2, a_3, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{11}, b_6, b_7, b_8, b_9, b_{10}, b_{11} and c_7 with respect to the other variables. The following is the order we solve some of the 2×2 determinants given by the 3-points in $\mathcal{X}'$: $[[1, 11, 21]]$ solve for c_{10}, $[[2, 10, 21]]$ solve for c_9, $[[3, 12, 21]]$ solve for c_{11}, $[[4, 8, 21]]$ solve for b_3, $[[5, 13, 21]]$ solve for c_8, $[[7, 14, 15]]$ solve for b_{12}, $[[5, 14, 20]]$ solve for c_4, $[[2, 14, 17]]$ solve for c_1, $[[4, 9, 20]]$ solve for c_2, $[[6, 13, 15]]$ solve for c_5, $[[3, 8, 20]]$ solve for b_5, $[[3, 10, 15]]$ solve for b_4, $[[3, 9, 18]]$ solve for c_6 and $[[3, 11, 19]]$ solve for b_1. Then, if we let $a_7 = a$, $b_2 = e$ and $c_3 = d$, the expressions for all the variables are:

$$a_1 = \tfrac{ad-a-d}{a-2}, \quad a_2 = \tfrac{2e^2d-2ed+ed^2-e^2d^2+(-2ed^2+e^2d^2+2e+6ed-3e^2d-4)a+(-4ed-e+4+ed^2+e^2d)a^2}{(ae-2a-2e+2)(ade-ed+d-a-da)},$$

$$a_3 = \tfrac{e(ade-ed+d-a-da)}{ae-2a-2e+2}, \quad a_4 = d,$$

$$a_5 = \tfrac{4d+2e^2d-6ed+e^2d^2-ed^2+(-2e^2d^2+2ed^2-8d-e^2d+10ed-2e)a+(4d+e+e^2d^2-ed^2-4ed)a^2}{(ae-2a-2e+2)(a+d-ad-de)},$$

$$a_6 = \tfrac{(a+d-ad-de)(ae+dae-4a-2e-ed+4)}{(ae-2a-2e+2)(ade-a-da-2ed+2+d)}, \quad a_7 = a, \quad a_8 = \tfrac{(a+d-da-2)e}{ae-2a-2e+2},$$

$$a_9 = \tfrac{2e^2d-2ed+ed^2-e^2d^2+(-2ed^2+e^2d^2+2e+6ed-3e^2d-4)a+(-4ed-e+4+ed^2+e^2d)a^2}{(a+d-ad-2)(ae-2a-2e+2)},$$

$$a_{10} = a + d - ad, \quad a_{11} = \frac{ade+ae-4a-2e-ed+4}{ae-2a-2e+2},$$

$$a_{12} = \frac{2e^2d+4d-6ed+e^2d^2-ed^2+(-2e^2d^2+2ed^2-8d-e^2d+10ed-2e)a+(4d+e+e^2d^2-ed^2-4ed)a^2}{(ade-a-da-2ed+d+2)(ae-2a-2e+2)},$$

$$b_1 = \frac{-2e+ed+ae-2a}{-ed+dae+d-a-da}, \quad b_2 = e, \quad b_3 = 2, \quad b_4 = \frac{ae-2a-2e-ed+4}{a+d-ad-de}, \quad b_5 = \frac{ade+ae-4a-2e-ed+4}{ade-a-da-2ed+d+2}, \quad b_6 = \frac{2}{d},$$

$$b_7 = \frac{e(ad-d+2-a)}{ad+de-a-d}, \quad b_8 = -\frac{-2e+ed+ae-2a}{a+d-ad-2}, \quad b_9 = 2 - a, \quad b_{10} = \frac{ae+dae-4a-ed-2e+4}{ade-ed+d-a-da},$$

$$b_{11} = \frac{ae-2a-2e+4-ed}{ade-a-da-2ed+d+2}, \quad b_{12} = \frac{a}{a-1},$$

and $[t : u] = [2 - a : d(a - 1)] \in \mathbb{P}^1$. Since $b_3 = 2$, these $(3, 8)$-nets are not possible in characteristic 2. The lines for these $(3, 8)$-nets can be written as: $L_1 = (y)$, $L_2 = (a_1x + y + z)$, $L_3 = (a_2x + b_1y + z)$, $L_4 = (a_3x + b_2y + z)$, $L_5 = (a_4x + b_3y + z)$, $L_6 = (a_5x + b_4y + z)$, $L_7 = (a_6x + b_5y + z)$, $L_8 = (a_7x + z)$, $L_9 = (x)$, $L_{10} = (x + b_6y + z)$, $L_{11} = (a_8x + b_7y + z)$, $L_{12} = (a_9x + b_8y + z)$, $L_{13} = (a_{10}x + b_9y + z)$, $L_{14} = (a_{11}x + b_{10}y + z)$, $L_{15} = (a_{12}x + b_{11}y + z)$, $L_{16} = (b_{12}y + z)$, $L_{17} = (ux - ty)$, $L_{18} = (x+y+z)$, $L_{19} = (a_8x+b_1y+z)$, $L_{20} = (a_9x+b_2y+z)$, $L_{21} = (a_{10}x+b_3y+z)$, $L_{22} = (a_{11}x + b_4y + z)$, $L_{23} = (a_{12}x + b_5y + z)$ and $L_{24} = (z)$.

3.5.4 Realizable Latin squares.

We have seen that in order to answer the realization question of the incidence of a $(3, q)$-net, we have to solve the equations modelled by $q \times q$ Latin squares representing main classes (Remark II.12). We have solved this for $2 \leq q \leq 6$ over $\mathbb{C}$, and for the quaternion nets. This give us an extra row in the following table.

q	1	2	3	4	5	6	7	8	9
# main classes	1	1	1	2	2	12	147	283,657	19,270,853,541
realizable over $\mathbb{C}$	1	1	1	2	2	9	≥ 1	≥ 2	≥ 1

We see that the case $q = 7$ has too many main classes to apply the proposed method. We may easily fill a little more this table, but our analysis for the case

$q = 6$ shows that the realization of a general $q \times q$ Latin square is quite subtle. We have different dimensions for parameter spaces, some of them defined strictly over $\mathbb{C}$, others with equations over $\mathbb{Q}$, and some of them do not even exist. The natural question to find a characterization for the main classes of Latin squares which realize 3-nets on $\mathbb{P}^2$ over $\mathbb{C}$ is very interesting, and may reveal some unknown combinatorial invariants of Latin squares.

CHAPTER IV

n-th root covers

Let us introduce some notation for this and the next chapters. If q is an integer with $0 < q < p$, we denote by q' the unique integer satisfying $0 < q' < p$ and $qq' \equiv 1 (\mathrm{mod}\ p)$. For positive integers $\{a_1, \ldots, a_r\}$, we denote by $(a_1, \ldots, a_r)$ their greatest common divisor. If $(a_1, \ldots, a_r) = 1$, we call them coprime. For a real number b, let $[b]$ be the integral part of b, i.e., $[b] \in \mathbb{Z}$ and $[b] \leq b < [b] + 1$. We use the notation $\mathcal{L}^n := \mathcal{L}^{\otimes n}$ for a line bundle $\mathcal{L}$.

4.1 General n-th root covers.

The n-th root cover tool, which we are going to describe in this section, was introduced by H. Esnault and E. Viehweg in [88] and [27]. Much more can be found in their paper [28] and in their book [29], where this tool is used for varieties over fields of arbitrary characteristic. For us, the ground field is always the field of complex numbers $\mathbb{C}$.

Let Y be a smooth projective variety. Let $D \neq 0$ be an effective divisor on Y with simple normal crossings (Definition I.21), and let $D = \sum_{i=1}^{r} \nu_i D_i$ be its decomposition into prime divisors. In particular, all the varieties D_i are non-singular. Assume that there exist a positive integer n and a line bundle $\mathcal{L}$ on Y satisfying

$$\mathcal{L}^n \simeq \mathcal{O}_Y(D).$$

Remark IV.1. We are not considering $D = 0$. This would correspond to étale coverings, and we are not interested in them.

We construct from the data $(Y, D, n, \mathcal{L})$ a new smooth variety $X = X(Y, D, n, \mathcal{L})$ which represents a "n-th root of D". Let s be a section of $\mathcal{O}_Y(D)$ with the divisor of zeros equal to D. The section s defines a structure of $\mathcal{O}_Y$-algebra on $\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}$ by means of the the induced injection $0 \to \mathcal{L}^{-n} \simeq \mathcal{O}_Y(-D) \to \mathcal{O}_Y$ (i.e., the multiplication $\mathcal{L}^{-i} \otimes \mathcal{L}^{-j} \to \mathcal{L}^{-i-j}$ is well defined). If we write

$$Y' := \mathrm{Spec}_Y\big(\bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}\big)$$

(as in [39, p. 128]), the first step in this construction is given by the induced map $f_1 : Y' \to Y$.

If some $\nu_i > 1$, the variety Y' might not be normal. The second step is the normalization $\overline{Y}$ of Y'. Let

$$f_2 : \overline{Y} \to Y$$

be the composition of f_1 with the normalization of Y'. Notice that $\overline{Y}$ is a projective variety as well.

Definition IV.2. As it is done in [88], we define the following line bundles on Y

$$\mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_Y\Big(-\sum_{j=1}^{r} \Big[\frac{\nu_j\ i}{n}\Big] D_j\Big)$$

for $i \in \{0, 1, ..., n-1\}$.

Proposition IV.3. *(see [88]) The variety $\overline{Y}$ has only rational singularities, f_2 is flat and*

$$f_{2*}\mathcal{O}_{\overline{Y}} = \bigoplus_{i=0}^{n-1} {\mathcal{L}^{(i)}}^{-1}.$$

Moreover, this is the decomposition of $f_{2*}\mathcal{O}_{\overline{Y}}$ into eigenspaces with respect to the action of $\mathbb{Z}/n\mathbb{Z}$.

The key point in the proof of Proposition IV.3 is the following lemma.

Proposition IV.4. *(see [88, p. 3]) Let C be a regular local ring, x a regular parameter and u a unit. Let $A = C[t]/(t^n - x^\nu u)$ (local first step) and B be the normalization of A (local second step). Then, B is generated as a C-module by $t^i x^{-[\frac{\nu i}{n}]}$ for $0 \leq i < n$.*

The map $f_2 : \overline{Y} \to Y$ is finite, $\overline{Y}$ is normal and Y is smooth. Hence, the ramification locus R of f_2 is a divisor on $\overline{Y}$. This divisor is defined by the zero section of the Jacobian of f_2. The branch divisor of f_2 is defined as $f_2(R)$ with the reduced scheme structure. It is contained in D.

The variety $\overline{Y}$ has an action of $\mathbb{Z}/n\mathbb{Z}$ such that the map $f_2 : \overline{Y} \to Y$ is the quotient map. The local picture of the action is the following. Take a point $P \in D_j$ such that it is smooth for D_{red} and ν_j is not congruent to zero module n. Let $d = (n, \nu_j)$, and write $n = dn'$ and $\nu_j = d\nu_j'$. Then, the fiber $(f_2)^{-1}(P)$ consists of d distinct smooth points of $\overline{Y}$. The action of $\mathbb{Z}/n\mathbb{Z}$ around any of these points looks like a linear action $A : \mathbb{C}^{\dim(Y)} \to \mathbb{C}^{\dim(Y)}$ having as eigenvalues $(\dim(Y) - 1)$ 1's and $e^{\frac{2\pi i q_j}{n'}}$ where $0 < q_j < n'$ is a uniquely determined number. It is sometimes called the rotation number. It can be checked that $q_j \nu_j' \equiv 1 (mod\ n')$. Therefore, ν_j' can be thought as the inverse mod n' of a rotation number of the local action of $\mathbb{Z}/n\mathbb{Z}$. The following proposition tell us how to modify the multiplicities ν_i of D to produce isomorphic varieties over Y.

Our multiplicities ν_i can always be taken in the range $0 \leq \nu_i < n$. If we change the multiplicities ν_i to μ_i such that $\mu_i \equiv \nu_i (mod\ n)$ and $0 \leq \mu_i < n$ for all i, then the corresponding varieties $\overline{Y}$ will be isomorphic over Y.

Proof. Let $D' = \sum_{i=1}^r \mu_i D_i$ where $0 \leq \mu_i < n$ and $\mu_i \equiv \nu_i (\text{mod } n)$. So, $\nu_i = \mu_i + c_i n$

for some integer $c_i \geq 0$. We define the line bundle $\mathcal{L}' = \mathcal{L} \otimes \mathcal{O}_Y(\sum_{i=1}^r -c_i D_i)$. Then, $\mathcal{L}'^n \simeq \mathcal{O}_Y(D')$. Moreover, the definition of $\mathcal{L}'$ induces isomorphisms between the algebras $\bigoplus_{i=0}^{n-1} {\mathcal{L}^{(i)}}^{-1}$ and $\bigoplus_{i=0}^{n-1} {\mathcal{L}'^{(i)}}^{-1}$, and so it gives an isomorphism of the corresponding varieties $\overline{Y}$ and $\overline{Y}'$ over Y. □

More generally, let b be a positive integer such that $(b, n) = 1$. Consider the effective simple normal crossings divisors on Y, $D = \sum_{i=1}^r \nu_i D_i$ and $D' = \sum_{i=1}^r \mu_i D_i$, such that $b\nu_i = \mu_i + c_i n$ and $0 \leq \mu_i < n$. Then again the corresponding varieties $\overline{Y}$ and $\overline{Y}'$ are isomorphic over Y. In this case, one takes $\mathcal{L}' = \mathcal{L}^b \otimes \mathcal{O}_Y(\sum_{i=1}^r -c_i D_i)$.

The third and the last step is to choose a <u>minimal</u> desingularization $f_3 : X \to \overline{Y}$ of $\overline{Y}$. Hence, our projective variety X depends not only on $(Y, D, n, \mathcal{L})$ but also on the chosen desingularization. However, we will be only working with curves and surfaces, and for that case X is uniquely determined by $(Y, D, n, \mathcal{L})$. Let

$$f : X \to Y$$

be the composition map, i.e., $f = f_2 \circ f_3$. We call it <u>n-th root cover</u> over Y along D (although it depends on $\mathcal{L}$ [1], this line bundle will be irrelevant for our numerical purposes).

In any case, we have the following general facts about X.

(1) Since the singularities of $\overline{Y}$ are rational (i.e., $R^b {f_3}_* \mathcal{O}_X = 0$ for all $b > 0$) and f_2 is affine, we have by Proposition IV.3

$$H^j(X, \mathcal{O}_X) \simeq \oplus_{i=0}^{n-1} H^j(Y, {\mathcal{L}^{(i)}}^{-1})$$

for every non negative integer j. In particular, $\chi(X, \mathcal{O}_X) = \sum_{i=0}^{n-1} \chi(Y, {\mathcal{L}^{(i)}}^{-1})$. A simple consequence is the following.

[1] Du Val discusses the significance of $\mathcal{L}$ in [25] for double covers.

Proposition IV.5. *X is connected if and only if $(\nu_1, ..., \nu_r, n) = 1$.*

Proof. We have that $h^0(X, \mathcal{O}_X) = \sum_{i=0}^{n-1} h^0(Y, {\mathcal{L}^{(i)}}^{-1}) = 1 + \sum_{i=1}^{n-1} h^0(Y, {\mathcal{L}^{(i)}}^{-1})$. If X is not connected, then $h^0(X, \mathcal{O}_X) \geq 2$ and so there is i such that $h^0(Y, {\mathcal{L}^{(i)}}^{-1}) > 0$. In particular, ${\mathcal{L}^{(i)}}^{-1} \simeq \mathcal{O}_Y(E)$ where E is an effective divisor. Then, by intersecting E with curves Γ_j such that $D_j.\Gamma_j > 0$, we have that $\left[\frac{\nu_j i}{n}\right] - \frac{\nu_j i}{n} = 0$ for all j, and so $i\nu_j \equiv 0 (mod\ n)$ for all j. This happens if and only if $(\nu_1, ..., \nu_r, n) \neq 1$. □

Assume X is connected. Since it is smooth, it must be irreducible. We notice that, by the proof of the previous proposition, if $(\nu_1, ..., \nu_r, n) = d > 1$, then $f : X \to Y$ is equal to d copies of the $\frac{n}{d}$-th root cover given by the data $(Y, \frac{1}{d}D, \frac{n}{d}, \mathcal{L})$. Notice that for $D = 0$, the variety X is connected if and only if for each integer $0 < r < n$, $\mathcal{L}^r$ is not isomorphic to $\mathcal{O}_Y$.

(2) If K_X and K_Y are canonical divisors for X and Y respectively, local computations give us the $\mathbb{Q}$-numerical equivalence

$$K_X \equiv f^*\Big(K_Y + \sum_{i=1}^{r}\Big(1 - \frac{(n, \nu_i)}{n}\Big)D_i\Big) + \Delta$$

where Δ is a $\mathbb{Q}$-divisor supported on the exceptional locus of the chosen desingularization.

4.2 n-th root covers for curves.

Let Y be a smooth projective curve of genus $g(Y)$ and $D = \sum_{i=1}^{r} \nu_i D_i$ be an effective divisor on Y. Here the D_i are closed points of Y. We assume that $0 < \nu_i < n$ for all i, and $(\nu_1, \ldots, \nu_r, n) = 1$. Notice that an invertible sheaf $\mathcal{L}$ on Y such that $\mathcal{L}^n \simeq \mathcal{O}_Y(\sum_{i=1}^{r} \nu_i D_i)$ exists if and only if $\sum_{i=1}^{r} \nu_i \equiv 0 (\text{mod } n)$.

In this way, we assume $\sum_{i=1}^{r} \nu_i \equiv 0 (\text{mod } n)$ and we fix $\mathcal{L}$ satisfying our condition. Then, the resulting $X = \overline{Y}$ is a smooth curve with a $\mathbb{Z}/n\mathbb{Z}$ action, having

$f : X \to Y$ as quotient map. By our previous facts, we have that $H^1(X, \mathcal{O}_X) \simeq \oplus_{i=0}^{n-1} H^1(Y, {\mathcal{L}^{(i)}}^{-1})$ and $\chi(X, \mathcal{O}_X) = \chi(Y, f_*\mathcal{O}_X)$. Then, by the Riemann-Roch theorem, $1 - g(X) = \deg(f_*\mathcal{O}_X) + n(1 - g(Y))$. According to the formula for $f_*\mathcal{O}_X$, we have

$$\deg(f_*\mathcal{O}_X) = -\sum_{i=1}^{n-1} \deg(\mathcal{L}^i) + \sum_{j=1}^{r}\sum_{i=1}^{n-1}\Big[\frac{\nu_j i}{n}\Big].$$

Also, $\deg(\mathcal{L}) = \frac{\deg(D)}{n}$. It is easy to check that $\sum_{i=1}^{n-1}[\frac{\nu i}{n}] = \frac{\nu(n-1)-n+(n,\nu)}{2}$ (although, a similar formula for $\sum_{i=1}^{n-1}[\frac{\nu i}{n}]^2$ seems hard to find). Hence, we have

$$\deg(f_*\mathcal{O}_X) = -\sum_{i=1}^{n-1} \frac{i}{n}\deg(D) + \sum_{j=1}^{r} \frac{\nu_j(n-1)-n+(n,\nu_j)}{2}$$

and so we recover the Riemann-Hurwitz formula

$$2g(X) - 2 = n(2g(Y) - 2) + \sum_{j=1}^{r}(n - (n, \nu_j)).$$

Let n be a prime number. We want to point out that the multiplicities ν_i play a role in the determination of the isomorphism class of X (in [85] we worked out these isomorphism classes for certain curves), but they do not play any role in the determination of its genus. This is highly not the case for algebraic surfaces, mainly because the D_i may intersect among each other. We will see that their numerical invariants are indeed affected by these multiplicities.

4.3 n-th root covers for surfaces.

Let Y be a smooth projective surface. Let D be an effective divisor on Y with simple normal crossings whose decomposition into prime divisors is $D = \sum_{i=1}^{r} \nu_i D_i$. In this way, D_i's are smooth projective curves of genus $g(D_i)$, and the singularities of D_{red} are at most nodes. Let n be a positive integer and $\mathcal{L}$ a line bundle on Y such that $\mathcal{L}^n \simeq \mathcal{O}_Y(D)$. We will assume that $0 < \nu_i < n$ for all i and $(\nu_1, \ldots, \nu_r, n) = 1$.

Let $f : X \to Y$ be the n-th root cover over Y along D. We remark that the smooth complex projective surface X is uniquely determined by $(Y, D, n, \mathcal{L})$.

By the facts in Section 4.2, the Riemann-Roch theorem and Serre's duality, we have

$$q(X) = q(Y) + \sum_{i=1}^{n-1} h^1(Y, {\mathcal{L}^{(i)}}^{-1}) \quad p_g(X) = p_g(Y) + \sum_{i=1}^{n-1} h^0(Y, \omega_Y \otimes \mathcal{L}^{(i)})$$

$$\chi(X, \mathcal{O}_X) = n\chi(Y, \mathcal{O}_Y) + \frac{1}{2}\sum_{i=1}^{n-1} \mathcal{L}^{(i)}.(\mathcal{L}^{(i)} \otimes \omega_Y).$$

We now develop a little more the formula for $\chi(X, \mathcal{O}_X)$. We have

$$\sum_{i=1}^{n-1} {\mathcal{L}^{(i)}}^2 = \sum_{i=1}^{n-1} \frac{i^2}{n^2} D^2 - \sum_{i=1}^{n-1} \frac{2i}{n}\Big(\sum_{j=1}^{r}\Big[\frac{\nu_j i}{n}\Big] D_j.D\Big) + \sum_{i=1}^{n-1}\Big(\sum_{j=1}^{r}\Big[\frac{\nu_j i}{n}\Big] D_j\Big)^2$$

and so

$$\sum_{i=1}^{n-1} {\mathcal{L}^{(i)}}^2 = \sum_{j=1}^{r} \frac{(n-(n,\nu_j))(2n-(n,\nu_j))}{6n} D_j^2 +$$

$$\sum_{j<k}\Big(\frac{\nu_j}{6n\nu_k}(n-(n,\nu_k))(2n-(n,\nu_k)) + \frac{\nu_k}{6n\nu_j}(n-(n,\nu_j))(2n-(n,\nu_j))$$

$$-\sum_{i=1}^{n-1}\Big[\frac{\nu_j i}{n}\Big]^2\frac{\nu_k}{\nu_j} - \sum_{i=1}^{n-1}\Big[\frac{\nu_k i}{n}\Big]^2\frac{\nu_j}{\nu_k} + 2\sum_{i=1}^{n-1}\Big[\frac{\nu_j i}{n}\Big]\Big[\frac{\nu_k i}{n}\Big]\Big) D_j.D_k$$

and

$$\sum_{i=1}^{n-1} \mathcal{L}^{(i)}.\omega_Y = \sum_{j=1}^{r}\sum_{i=1}^{n-1}\Big(\frac{\nu_j i}{n} - \Big[\frac{\nu_j i}{n}\Big]\Big) D_j.K_Y = \sum_{j=1}^{r} \frac{n-(n,\nu_j)}{2} D_j.K_Y.$$

For instance, when $D_i.D_j = 0$ for all i, j we have the following simple formulas for the main numerical invariants of X (resolution of singularities is not needed).

$$\chi(X, \mathcal{O}_X) = n\chi(Y, \mathcal{O}_Y) - \sum_{i=1}^{r} \frac{n^2-(n,\nu_j)^2}{12n} D_j^2 + \sum_{j=1}^{r} \frac{(n-(n,\nu_j))(g(D_j)-1)}{2}$$

$$K_X^2 = nK_Y^2 - \sum_{i=1}^{r} \frac{n^2-(n,\nu_j)^2}{n} D_j^2 + 4\sum_{j=1}^{r}(n-(n,\nu_j))(g(D_j)-1)$$

$$e(X) = ne(Y) + 2\sum_{j=1}^{r}(n-(n,\nu_j))(g(D_j)-1).$$

From now on, the number n will be a prime number, denoted by p.

Definition IV.6. Let p be a prime number and q an integer such that $0 < q < p$. As usual, the Dedekind sum associated to q, p is defined as

$$s(q,p) = \sum_{i=1}^{p-1} \Big(\Big(\frac{i}{p}\Big)\Big)\Big(\Big(\frac{iq}{p}\Big)\Big)$$

where $((x)) = x - [x] - \frac{1}{2}$ for any rational number x.

Connections between Dedekind sums and algebraic geometry appear in [44]. These sums naturally show up when considering the Riemann-Roch theorem. They will show the significance of the multiplicities ν_i.

Proposition IV.7. *Let p be a prime number. Let $(Y, D, p, \mathcal{L})$ be the data to construct the p-th root cover $f : X \to Y$. Assume $0 < \nu_i < p$ for all i. Then,*

$$\chi(X, \mathcal{O}_X) = p\chi(Y, \mathcal{O}_Y) - \frac{p^2-1}{12p}\sum_{i=1}^{r} D_i^2 - \frac{p-1}{4}e(D) + \sum_{i<j} s(\nu_i'\nu_j, p) D_i.D_j$$

where $e(D)$ is the topological Euler characteristic of the underlying complex topological space of D.

Proof. We temporarily define $S(a,b;p) = \sum_{i=1}^{p-1} \left[\frac{ai}{p}\right]\left[\frac{bi}{p}\right]$ for any integers a, b satisfying $0 < a, b < p$. Then, since $\sum_{i=1}^{p-1} \left(ai - \left[\frac{ai}{p}\right]p\right)^2 = \sum_{i=1}^{p-1} i^2 = \frac{p(p-1)(2p-1)}{6}$, one can check that $\sum_{i=1}^{p-1} i\left[\frac{ai}{p}\right] = \frac{1}{12a}(a^2-1)(p-1)(2p-1) - \frac{p}{2a}S(a,a;p)$.

Also, one can easily verify (see [44, p. 94]) that $s(a,p) = \frac{p-1}{6p}(2ap - a - \frac{3}{2}p) - \frac{1}{p}\sum_{i=1}^{p-1} i\left[\frac{ai}{p}\right]$ and so

$$s(a,p) = \frac{1}{12ap}(p-1)(2pa^2 - a^2 - 3ap + 2p - 1) - \frac{1}{2a}S(a,a;p).$$

On the other hand, we have

$$S(a,b;p) = s(a'b,p) - as(b,p) - bs(a,p) + \frac{p-1}{12p}\big(3p - 3pa - 3pb + 2ab(2p-1)\big).$$

Putting all together,

$$-\frac{a}{b}S(b,b;p) - \frac{b}{a}S(a,a;p) + 2S(a,b;p) = \frac{1-p}{6abp}\big(a^2(2p-1) + b^2(2p-1) - 3abp\big) + 2s(a'b;p).$$

We now replace these sums in the formula for $\sum_{i=1}^{p-1} \mathcal{L}^{(i)^2}$. Finally, we use again that $(\nu_i, p) = 1$ for all i and the adjunction formula to write down the wanted formula for $\chi(X, \mathcal{O}_X)$. The computation of the Euler characteristic $e(D)$ can be done using Lemma IV.9 below. □

We now work out the resolution of singularities $f_3 : X \to \overline{Y}$. Since we are assuming that D_{red} has only nodes as singularities, the singular points of $\overline{Y}$ are of Hirzebruch-Jung type, that is, locally isomorphic to the singularity of the normalization of $\mathrm{Spec}(\mathbb{C}[x, y, z]/(z^p - x^{\nu_i} y^{\nu_j}))$. These type of singularities have an explicit resolution (see for example [7]), which we explain now.

Let p be a prime number and q be a positive integer with $0 < q < p$. Let U be the normalization of $\mathrm{Spec}(\mathbb{C}[x, y, z]/(z^p - xy^{p-q}))$ and Q its singular point (a common notation for this singularity is $\frac{1}{p}(1, q)$). Let $\rho : V \to U$ be a minimal resolution of U. Then, $\rho^{-1}(Q)$ is composed of a chain of smooth rational curves $\{E_1, E_2, \ldots, E_s\}$ such that $E_j.E_{j+1} = 1$ for $j \in \{1, \ldots, s-1\}$ and no further intersections. The numbers s and $E_j^2 = -e_j$ are encoded in the negative-regular continued fraction

$$\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \frac{1}{e_s}}}$$

which comes from a recursion formula explained in the Appendix. We abbreviate this continued fraction by $\frac{p}{q} = [e_1, ..., e_s]$.

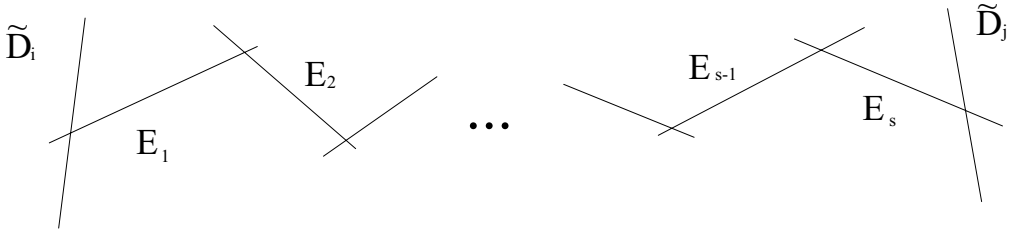


Figure 4.1: Resolution over a singular point of $D_i \cap D_j$.

Let $0 < \nu_i, \nu_j < p$ be the two multiplicities corresponding to D_i and D_j in D. Assume that these curves do intersect, and consider one of the points of intersection. Then, over this point (on $\overline{Y}$), we have an open neighborhood isomorphic to the normalization of $\mathrm{Spec}(\mathbb{C}[x,y,z]/(z^p - x^{\nu_i}y^{\nu_j}))$ (see [7, pp. 99-105]). Denote this open set by U'. The resolution $f_3 : X \to \overline{Y}$ looks locally like the resolution of U'. To resolve this open set, one proves that the normalization of the singularity in $\mathbb{C}[x,y,z]/(z^p - x^{\nu_i}y^{\nu_j})$ is isomorphic to the normalization of the singularity in $\mathbb{C}[x,y,z]/(z^p - xy^{p-q})$, where $0 < q < p$ is the unique integer satisfying $\nu_i q + \nu_j \equiv 0 (mod\ p)$. Then, we apply what we did previously.

Definition IV.8. For $0 < a, b < p$ (p prime as always), we define the length of a, b with respect to p, denoted by $l(a,b;p)$, as the number s in the continued fraction of $\frac{p}{q} = [e_1, \ldots, e_s]$, where $0 < q < p$ is the unique integer satisfying $aq + b \equiv 0 (mod\ p)$.

This number is symmetric with respect to (a,b), i.e., $l(a,b;p) = l(b,a;p)$ (see Appendix). Also, $l(a,b;p) = l(1,a'b;p)$. The different lengths we obtain in the p-th root cover $f : X \to Y$ appear in the Euler characteristic of X as we will see below. The following is a well-known topological lemma which will be used to prove Proposition IV.10.

Lemma IV.9. *Let B be a complex projective variety and $A \subseteq B$ a subvariety such that $B \setminus A$ is non-singular. Then, $e(B) = e(A) + e(B \setminus A)$.*

Proposition IV.10. *Let p be a prime number and $f : X \to Y$ be the p-th root cover with data $(Y, D = \sum_{i=1}^{r} \nu_i D_i, p, \mathcal{L})$. Assume $0 < \nu_i < p$ for all i. Then,*

$$e(X) = p\big(e(Y) - e(D)\big) + e(D) + \textstyle\sum_{i<j} l(\nu_i, \nu_j; p) D_i . D_j .$$

Proof. We consider X, Y, D and $R = f^{-1}(D)$ as topological spaces with the induced complex topology, and f as a continuous map. By Lemma IV.9, $e(X) = e(X \setminus R) +$

$e(R)$. Since $f|_{X\setminus R}$ is an étale cover of $Y \setminus D$, we have that $e(X \setminus R) = pe(Y \setminus D)$, and so $e(X \setminus R) = pe(Y) - pe(D)$. Let $\{R_\alpha\}_\alpha$ be the connected components of R. Since our minimal resolution gives us connected fibers, the R_α's are in one to one correspondence with the connected components of $f_2^{-1}(D)$, where $f_2 : \overline{Y} \to Y$. Let $\{f_2^{-1}(D)_\alpha\}_\alpha$ be the connected components of $f_2^{-1}(D)$ such that R_α corresponds to $f_2^{-1}(D)_\alpha$. Then, $e(R_\alpha) - e(f_2^{-1}(D)_\alpha)$ is the number of $\mathbb{P}^1$'s over the singular points of $f_2^{-1}(D)_\alpha$, that is, the corresponding lengths at the singular points. Therefore,

$$e(R) = \sum_\alpha R_\alpha = \sum_\alpha e(f_2^{-1}(D)_\alpha) + \sum_{i<j} l(\nu_i, \nu_j; p) D_i.D_j = e(f_2^{-1}(D)) + \sum_{i<j} l(\nu_i, \nu_j; p) D_i.D_j.$$

We now use $(\nu_i, p) = 1$ to say that $f_2^{-1}(D)$ is homeomorphic to D, in particular $e(f_2^{-1}(D)) = e(D)$. □

Example IV.11. When $\nu_i = 1$ for all i (i.e., D reduced divisor), the surface X has

$$K_X^2 = pK_Y^2 + \tfrac{(p-1)^2}{p} D^2 + 2(p-1) D.K_Y \quad e(X) = pe(Y) + (p-1)(D^2 + D.K_Y)$$
$$\chi(X, \mathcal{O}_X) = p\chi(Y, \mathcal{O}_Y) + \tfrac{(p-1)(2p-1)}{12p} D^2 + \tfrac{(p-1)}{4} D.K_Y$$

Remark IV.12. (**Irregularity**) Assume the hypothesis in Proposition IV.7. Moreover, assume that there is i such that $D_i^2 > 0$. Then, $q(X) = q(Y)$. This is a direct application of the Viehweg vanishing theorem in [88]. For simplicity, assume $i = 1$. We repeat the argument of multiplying by a unit as in Section 4.1. Let us multiply all the numbers ν_j by ν_1', and consider the new divisor $D' = \sum_{i=1}^r \upsilon_j D_j$ where $\nu_1' \nu_j = c_j p + \upsilon_j$, $0 < \upsilon_j < p$. Let $\overline{Y}$ and $\overline{Y}'$ be the normal varieties constructed in Section 4.1 from $(Y, D, p, \mathcal{L})$ and $(Y, D', p, \mathcal{L}')$ respectively, where $\mathcal{L}' = \mathcal{L}^{\nu_1'} \otimes \mathcal{O}_Y(-\sum_{i=1}^r c_j D_j)$. Then, $\overline{Y}$ and $\overline{Y}'$ are isomorphic over Y. Since the corresponding resolutions X and X' are minimal, and so unique, they are isomorphic as well. In particular, $q(X) = q(X')$. Notice that by construction $\upsilon_1 = 1$. Then, by the Viehweg vanishing theorem,

$H^1(Y, \mathcal{L}'^{(i)^{-1}}) = 0$ for all $i > 0$. But, $q(X') = q(Y) + \sum_{i=1}^{p-1} h^1(Y, \mathcal{L}'^{(i)^{-1}})$. Therefore, $q(X) = q(Y)$.

4.3.1 A formula for Dedekind sums.

As an application and to exemplify the p-th root method, we will now use the Noether formula to relate Dedekind sums and continued fractions. Relations between them are well-known for regular continued fractions [5] and for negative-regular continued fractions [96].

Let p be a prime number and q an integer with $0 < q < p$. Let $Y = \mathbb{P}^2$ and consider $r = q+1$ lines in $\mathbb{P}^2$ in general position. We denote this line arrangement as $\mathcal{A} = \{L_1, \ldots, L_r\}$. Then, $D = L_1 + \ldots + L_{r-1} + (p-q)L_r$ satisfies our requirements, and $\mathcal{O}_Y(D) \simeq \mathcal{O}_Y(p)$. Fix any such isomorphism and consider the p-th root cover $f : X \to Y$ along D. Then, we apply Propositions IV.7 and IV.10 (and $s(1,p) = \frac{(p-1)(p-2)}{12p}$) to compute

$$\chi(X, \mathcal{O}_X) = p - \frac{(p^2-1)r}{12p} - \frac{1}{8}(p-1)r(5-r) + \frac{1}{24p}(r-1)(r-2)(p-1)(p-2) + (r-1)s(p-q,p)$$

and

$$e(X) = 3p + (1-p)\frac{r(5-r)}{2} + \frac{(r-1)(r-2)}{2}(p-1) + (r-1)l(1, p-q; p).$$

The canonical divisor K_X is numerically equivalent to $f^*(-3L + \frac{p-1}{p}\sum_{i=1}^{r} L_i) + \sum \Delta_j$ where Δ_j are effective divisors on X supported on the exceptional locus coming from each node of D_{red}. Hence, Δ_j is zero whenever the node comes from $L_i \cap L_k$ with $i, k \neq r$. Otherwise, we write $\Delta_j = \sum_{i=1}^{l(1,q;p)} \alpha_i E_i$ with $E_i^2 = -e_i$, coming from $\frac{p}{q} = [e_1, \ldots, e_{l(1,p-q;p)}]$. Let $s = l(1, p-q; p)$. Then, $\Delta_j^2 = \sum_{i=1}^{s} \alpha_i(e_i - 2)$ (use adjunction formula for E_i) and

$$K_X^2 = \frac{1}{p}\big((p-1)r - 3p\big)^2 + (r-1)\sum_{i=1}^{s} \alpha_i(e_i - 2).$$

The numbers α_i can be found as follows. As always, let q' be the inverse of q module p, i.e., the unique integer $0 < q' < p$ such that $qq' \equiv 1(mod\ p)$. We know that $\frac{p}{q} = [e_1, \ldots, e_s]$ implies $\frac{p}{q'} = [e_s, \ldots, e_1]$ (see Appendix). Let $\{b_i\}_{i=-1}^{s}$ and $\{b'_i\}_{i=-1}^{s}$ be the associated sequences for q and q' respectively (definition of b_i's in the Appendix). Then, by using the adjunction formula for E_i and Cramer's rule (using the determinant formulas for b_i's and b'_i's in the Appendix), we have

$$\alpha_i = -1 + \frac{b_{i-1}}{p} + \frac{b'_{s-i}}{p}$$

for all $i \in \{1, 2, \ldots, s\}$. In Lemma .15, we compute $\sum_{i=1}^{s} \alpha_i(e_i - 2)$. Then, we substitute all the numbers in the Noether formula $12\chi(X, \mathcal{O}_X) = K_X^2 + e(X)$ to obtain

Proposition IV.13. *Let p be a prime number and q be an integer such that $0 < q < p$. Let $\frac{p}{q} = [e_1, e_2, \ldots, e_s]$. Then,*

$$12s(q,p) - \textstyle\sum_{i=1}^{s} e_i + 3s = \frac{q+q'}{p}.$$

We notice that this formula was found by Holzapfel in [45, Lemma 2.3], using the original definition of Dedekind sums via Dedekind η-function.

4.3.2 Pull-back of branch divisors.

Let $f : X \to Y$ be a p-th root cover, as in Theorem IV.7. We want to know the decomposition into prime divisors of $f^*(D_i)$, where the D_i's are the irreducible components of D.

To this end, we first do it locally. Let q be an integer with $0 < q < p$. Let $Y = \mathrm{Spec}\big(\mathbb{C}[x, y]\big)$ and $D = D_1 + (p-q)D_2$, where $D_1 = \{x = 0\}$ and $D_2 = \{y = 0\}$. Then, we have a p-th root cover as before $f : X \to Y$. This could be seen as a local picture of the p-th root cover over $\mathbb{P}^2$ along the divisor defined by $\{xy^{p-q}z^{q-1} = 0\}$.

Let $\frac{p}{q} = [e_1, \ldots, e_s]$. The resolution of the corresponding singularity is shown in Figure 4.1 (we take $i = 1$ and $j = 2$), where $\widetilde{D}_i$ are the reduced strict transforms of D_i under f. We remark that the self-intersection of E_i is $-e_i$.

We have $f^*(D_1) = p\widetilde{D}_1 + \sum_{i=1}^{s} a_i E_i$ and $f^*(D_2) = p\widetilde{D}_2 + \sum_{i=1}^{s} d_i E_i$, for some positive integers a_i, d_i. As it is done in [27, p. 481], these integers satisfy:

- $a_1 = q$, and if $\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_{i-1} - \frac{1}{c_i}}}}$, then $c_i a_i = a_{i-1}$.
- $d_0 = 0$, $d_1 = 1$, and $d_{i+1} = e_i d_i - d_{i-1}$.

Hence, it is easy to check that $a_i = b_{i-1}$ and $d_i = P_{i-1}$, as in the Appendix .1.

We now consider the global situation, where $f : X \to Y$ is a p-th root cover as in Theorem IV.7. As before, this p-th root cover is along $D = \sum_{i=1}^{r} \nu_i D_i$. Assume that D_i intersects only $D_{j_1}, \ldots, D_{j_t}$ in D. For each intersection, we have the above situation with $D_i = D_1$, up to multiplication by $\nu_i{}'$, and $D_{j_k} = D_2$. Using the previous notation, we take $q = p - \nu_{j_k}\nu_i'$. Let $E_{k,b}$ be the corresponding exceptional divisors from the Hirzebruch-Jung resolution at these points of intersection. Then,

$$f^*(D_i) = p\widetilde{D}_i + \sum_{b=1}^{t} \Big(\sum_{k=1}^{s_b} a_{k,b} E_{k,b} \Big) D_{j_b}.D_i.$$

We have $f^*(D_i).\big(\sum_{k=1}^{s_b} a_{k,b} E_{k,b} \big) = 0$, and so

$$0 = p(p - \nu_{j_b}\nu_i') + \big(\sum_{k=1}^{s_b} a_{k,b} E_{k,b} \big)^2 D_{j_b}.D_i.$$

Then, $pD_i^2 = f^*(D_i)^2 = p^2 \widetilde{D}_i^2 + 2p \sum_{b=1}^{t} (p - \nu_{j_b}\nu_i') D_{j_b}.D_i - p \sum_{b=1}^{t} (p - \nu_{j_b}\nu_i') D_{j_b}.D_i$. Therefore, the formula for the self-intersection is

$$\widetilde{D}_i^2 = \frac{1}{p} \Big(D_i^2 - \sum_{b=1}^{t} (p - \nu_{j_b}\nu_i') D_{j_b}.D_i \Big).$$

We always take $\nu_{j_b}\nu_i'$ in the interval $(0, p)$.

For us, it will be important to know the behavior of $\widetilde{D}_i^2$ when p is large. Assume that p tends to infinity and that the multiplicities are randomly chosen. Then, it is expected that the following inequality

$$-\sum_{b=1}^{t} D_{j_b}.D_i \leq \widetilde{D}_i^2 \leq -1$$

will hold with values tending to concentrate in the center of the interval

$$\Big[-\sum_{b=1}^{t} D_{j_b}.D_i, -1\Big],$$

that is, around $-\frac{1+\sum_{b=1}^{t} D_{j_b}.D_i}{2}$.

4.4 (-1)- and (-2)-curves on X.

Let $f : X \to Y$ be the p-th root cover over Y along $D = \sum_{i=1}^{r} \nu_i D_i$ in Theorem IV.7. By the fact **(2)** in Section 4.1, we have the $\mathbb{Q}$-numerical equivalence

$$K_X \equiv f^*\Big(K_Y + \frac{(p-1)}{p}\sum_{i=1}^{r} D_i\Big) + \Delta$$

where Δ is a $\mathbb{Q}$-divisor supported on the exceptional locus of the minimal desingularization of $\overline{Y}$. Let $\{P_1, \ldots, P_\delta\}$ be the set of nodes of D_{red}. Over each P_i, we have the exceptional divisor $\sum_{j=1}^{s_i} E_{j,i}$ given by the corresponding Hirzebruch-Jung resolution. Hence, there are $\alpha_{j,i} \in \mathbb{Q}$ such that

$$\Delta = \sum_{i=1}^{\delta}\Big(\sum_{j=1}^{s_i} \alpha_{j,i} E_{j,i}\Big).$$

The numbers $\alpha_{j,i}$ are known. They were computed in Subsection 4.3.1 over the point $L_1 \cap L_r$. If P_i is a point in $D_a \cap D_b$, then q in Subsection 4.3.1 is taken as $p - \nu_a \nu'_b$. In the Appendix, after Lemma .14, we give a formula for $\alpha_{i,j}$ as a function of p and q. As in the previous subsection, $\widetilde{D}_i$ denotes the strict transform of D_i.

Proposition IV.14. *The $\mathbb{Q}$-divisor $f^*\Big(\frac{(p-1)}{p}\sum_{i=1}^{r} D_i\Big) + \Delta$ is an effective $\mathbb{Z}$-divisor.*

Proof. Let P be a node of D_{red}. Assume P is a point in $D_i \cap D_j$, where q in Subsection 4.3.1 is taken as $p - \nu_j \nu_i'$. Assume that over P, the exceptional divisor has components $\{E_1, \ldots, E_s\}$, with $E_1 \cap \widetilde{D}_1 \neq \emptyset$. Then, locally over P, we have

$$f^*(D_i + D_j) = p\widetilde{D}_i + \sum_{k=1}^{s} a_k E_k + p\widetilde{D}_j + \sum_{k=1}^{s} d_k E_k$$

by Subsection 4.3.2. In the same subsection, we have $a_k = b_{k-1}$ and $d_k = P_{k-1}$ (using the notation in the Appendix). On the other hand, we know that $\Delta = \sum_{k=1}^{s} \alpha_k E_k$ over P. But in the Appendix we compute $\alpha_k = -1 + \frac{b_{k-1}}{p} + \frac{P_{k-1}}{p}$. Therefore, over P,

$$f^*\Big(\frac{p-1}{p} D_i + \frac{p-1}{p} D_j\Big) + \Delta = (p-1)\widetilde{D}_i + (p-1)\widetilde{D}_j + \sum_{k=1}^{s} \Big((p-1)(1+\alpha_k) + \alpha_k\Big) E_k$$

and $(p-1)(1+\alpha_k) + \alpha_k \geq 0$ by Lemma .16. Also, $p\alpha_k + (p-1) \in \mathbb{Z}$. □

An immediate corollary is the following.

Corollary IV.15. *Assume that K_Y is ample. Then, the curves $\widetilde{D}_i$ are the only possible (-1)-curves on X, and the curves $\widetilde{D}_i$ and $E_{j,i}$ are the only possible (-2)-curves on X.*

Proof. Let Γ be a smooth rational curve in X with $\Gamma^2 = -1$ or -2. Suppose Γ is not a component of $f^*(D)$. Then, $K_X.\Gamma = \Big(f^*\big(K_Y + \frac{(p-1)}{p} \sum_{i=1}^{r} D_i\big) + \Delta\Big).\Gamma > 0$ by Proposition IV.14 and K_Y ample. But $K_X.\Gamma \leq 0$ by adjunction. Therefore, Γ is a component of $f^*(D)$. □

4.4.1 Along line arrangements.

In this subsection, we detect (-1)- and (-2)-curves for p-th root coverings along line arrangements. Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be an arrangement of d lines in $\mathbb{P}^2$ (with $t_d = 0$). We use the notation in Section 2.2.

Let $\sigma : Y \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at all k-points with $k \geq 3$. Let p be a prime number, and let $\{\mu_i\}_{i=1}^d$ be a collection of d positive integers satisfying

$$\mu_1 + \mu_2 + \ldots + \mu_d = p.$$

Consider the divisor $D = \sigma^*(\sum_{i=1}^d \mu_i L_i)$, whose decomposition into prime divisors is written as $D = \sum_{i=1}^r \nu_i D_i$. Let the first d D_i be the strict transforms of L_i, and the rest the exceptional divisors of σ. Let $H = \sigma^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)$. Then, $pH \sim D$, and we construct the corresponding p-th root cover over Y along D, denoted by $f : X \to Y$.

Theorem IV.16. *Assume that $d \geq 6$, and that for any k-point of $\mathcal{A}$, $\frac{2}{3}d > k + 1$. We also exclude the case $d = 6$ with only nodes. Then, all the (-1)- and (-2)-curves of X are contained in $f^*(D)$.*

Proof. Let $F_{i,k}$ be the exceptional divisors of σ over the k-points of $\mathcal{A}$. Then, by Proposition IV.14,

$$K_X \equiv f^*\Big(-3H + \sum_{k\geq 3}\sum_{i=1}^{t_k} F_{i,k}\Big) + (p-1)\sum_{i=1}^{r}\widetilde{D}_i + \sum_{\text{nodes of D}}\sum_{i}\Big(p(\alpha_i+1)-1\Big)E_i.$$

Now, $dH = \sum_{i=1}^d D_i + \sum_{k\geq 3} k\big(\sum_{i=1}^{t_k} F_{i,k}\big)$, and so

$$K_X \equiv \frac{-3}{d} f^*\Big(\sum_{i=1}^{d} D_i + \sum_{k\geq 3} k\big(\sum_{i=1}^{t_k} F_{i,k}\big)\Big) + f^*\Big(\sum_{k\geq 3}\sum_{i=1}^{t_k} F_{i,k}\Big)$$
$$+(p-1)\sum_{i=1}^{r}\widetilde{D}_i + \sum_{\text{nodes of D}}\sum_{i}\Big(p(\alpha_i+1)-1\Big)E_i.$$

Now we analyze two cases. The first is $D_i \cap D_j$ for $i, j \leq d$ (i.e. a node of $\mathcal{A}$), and the second is $D_i \cap F_{j,k}$. We want to prove that K_X can be written as an effective $\mathbb{Q}$- divisor on $\widetilde{D}_i$'s and E_i's (with none of the coefficients zero). That would prove $\widetilde{D}_i$'s are the only potential (-1)-curves of X. For (-2)-curves, we notice that any such curve has to intersect $f^*(D)$, and so it has to be of the form $\widetilde{D}_i$ or E_i (since the intersection with K_X is zero).

For the first case, we look around $D_i \cap D_j$. Then,

$$-\frac{3}{d}f^*(D_i+D_j)+(p-1)\big(\widetilde{D}_i+\widetilde{D}_j\big)+\sum_{i=1}^{s}\Big(p(\alpha_i+1)-1\Big)E_i$$

$$\equiv \frac{1}{d}\big((d-3)p-d\big)\widetilde{D}_i+\frac{1}{d}\big((d-3)p-d\big)\widetilde{D}_j+\sum_{i=1}^{s}\Big(-\frac{3}{d}b_{i-1}-\frac{3}{d}P_{i-1}+b_{i-1}+P_{i-1}-1\Big)E_i.$$

But, $p \geq d > \frac{d}{3} \geq 2 \geq \frac{d}{d-3}$, and so $\frac{1}{d}\big((d-3)p-d\big) > 0$. Also, the integers b_{i-1} and P_{i-1} are greater or equal than one (see Appendix), and so

$$-\frac{3}{d}b_{i-1}-\frac{3}{d}P_{i-1}+b_{i-1}+P_{i-1}-1=\frac{(d-3)}{d}b_{i-1}+\frac{(d-3)}{d}P_{i-1}-1 \geq \frac{d-6}{d} \geq 0.$$

This could have been zero for $d=6$ with only nodes, but we are excluding that case.

For the second case, we look around $D_i \cap F_{j,k}$. Then,

$$-\frac{3}{d}f^*(D_i)-\frac{3}{d}kf^*(F_{j,k})+f^*(F_{j,k})+(p-1)\widetilde{D}_i+(p-1)\widetilde{F}_{j,k}+\sum_{i=1}^{s}\Big(p(\alpha_i+1)-1\Big)E_i$$

$$\equiv \frac{1}{d}\big((d-3)p-d\big)\widetilde{D}_i+\frac{1}{d}\big((2d-3k)p-d\big)\widetilde{F}_{j,k}+\sum_{i=1}^{s}\Big(-\frac{3}{d}b_{i-1}-\frac{3}{d}kP_{i-1}+P_{i-1}+P_{i-1}+b_{i-1}-1\Big)E_i.$$

But, $p \geq d > \frac{d}{3} > \frac{d}{2d-3k}$, and so $\frac{1}{d}\big((2d-3k)p-d\big) > 0$. Also,

$$-\frac{3}{d}b_{i-1}-\frac{3}{d}kP_{i-1}+P_{i-1}+P_{i-1}+b_{i-1}-1=\frac{d-3}{d}b_{i-1}+\frac{2d-3k}{d}P_{i-1}-1 \geq \frac{2d-3k-3}{d} > 0.$$

□

Remark IV.17. Many interesting arrangements satisfy the hypothesis of Theorem IV.16: any Fermat arrangement (Example II.6), any general arrangement with $d > 6$, the Hesse arrangement, the Klein arrangement (Example II.7), any $(3, q)$-net with $q \geq 3$.

As shown in Subsection 4.3.2 and confirmed in the samples of Section .2, we cannot expect minimality for X when D contains divisors D_i having at most 3 intersections with the rest. For example, any arrangement with 3-points will produce such situation (being D_i the exceptional curve over the 3-point).

CHAPTER V

Projective surfaces vs. logarithmic surfaces

In this section we show a strong relation between Chern and log Chern numbers for algebraic surfaces. The result is the following.

Let Z be a smooth projective surface. Let $\mathcal{A}$ be a simple crossings arrangement in Z, satisfying the **divisible** property (to be defined below). Let $(Y, \mathcal{A}')$ be the associated pair (Section 2.1). Then, there are sequences of smooth projective surfaces X having

$$\frac{c_1^2(X)}{c_2(X)} \text{ arbitrarily close to } \frac{\bar{c}_1^2(Y, \mathcal{A}')}{\bar{c}_2(Y, \mathcal{A}')}.$$

The construction is based on the p-th root cover tool developed in the previous chapter. The divisible property of $\mathcal{A}$ allows us to construct p-th root covers over Y along $\mathcal{A}'$ for arbitrarily large primes p. If $\mathcal{A}$ is divisible, there is a way to assign multiplicities to its curves so that the p-th root cover is granted. We do this by partitioning p in different weighted ways. The precise result is that, for large primes p, the Chern numbers of the new surfaces X are proportional to the log Chern numbers of the log surface $(Y, \mathcal{A}')$, with constant of proportionality equals to p. However, this does not work for any choice of multiplicities. We prove that this works for random partitions of p with probability tending to one, as p approaches infinity. An interesting phenomena is that random partitions are necessary for our constructions, if we

want to approach to the log Chern numbers ratio of the corresponding arrangement. We put this in evidence by examples, using a computer program which calculates the exact values of the Chern numbers involved (see table in Section 5.3).

In Propositions IV.7 and IV.10, we saw that the only, a priori, unmanageable terms in the formulas for $\chi(X, \mathcal{O}_X)$ and $c_2(X)$ are Dedekind sums and lengths of continued fractions. These arithmetic quantities are evaluated in pairs of the form $(\nu_i\nu'_j, p)$, where ν_i, ν_j are multiplicities of curves in $\mathcal{A}'$, coming from multiplicities of curves in $\mathcal{A}$. So, we have two problems: how these numbers $\nu_i\nu'_j$ behave module p after assigning multiplicities to $\mathcal{A}$, and how Dedekind sums and lengths of continued fractions behave with respect to them.

Girstmair has recently described some nice properties about the large scale behavior of Dedekind sums and regular continued fractions (see [33] and [34] respectively). By using these features together with known connections between negative-regular and regular continued fractions, we can prove similar properties for negative-regular continued fractions. The key large scale property is: for large values of p, and q outside of a "bad" subset of $\{0, 1, 2, \ldots, p-1\}$, these quantities evaluated at (q, p) are very small compared to p. Moreover, it can be proved that this "bad" set has measure tending to zero as p approaches infinity.

In Section 5.1 we explain how to find "good" multiplicities for $\mathcal{A}$, the ones that make $\nu_i\nu'_j$ to stay outside of the bad set for every i, j. We prove they exist for large p, and that a random choice of them works. In Section 5.2 we put all together to prove the main theorem which connects Chern and log Chern numbers. A consequence is that any minimal smooth projective surface of general type Z can be covered by minimal smooth projective surfaces of general type X such that $q(X) = q(Z)$, and $\frac{c_1^2(X)}{c_2(X)}$ is arbitrarily close to 2 (Corollary V.4).

In Section 5.3 we show how to use this theorem to find simply connected surfaces with large Chern numbers ratio. In that section, we also prove that all surfaces X coming from line arrangements are simply connected. The surfaces X are not necessarily minimal, but at least for certain abundant line arrangements we can control this issue. Finally, in Section 5.4 we give more examples, showing how to relax the divisible hypothesis on $\mathcal{A}$ to still be able to apply our main theorem.

5.1 Divisible arrangements.

Let Z be a smooth projective surface, and let $d \geq 3$ be an integer.

Definition V.1. Let $\mathcal{A} = \{C_1, C_2, \ldots, C_d\}$ be a simple crossings arrangement of d curves in Z (Definition II.2). We call it divisible if $\mathcal{A}$ splits into $v \geq 1$ arrangements of d_i curves $\mathcal{A}_i$ (so $d_i \geq 3$) satisfying:

1. A curve in $\mathcal{A}_i$ does not belong to $\mathcal{A}_j$ for all $j \neq i$.

2. For each $i \in \{1, \ldots, v\}$, there exists a line bundle $\mathcal{L}_i$ on Z such that for each C in $\mathcal{A}_i$, we have $\mathcal{O}_Z(C) \simeq \mathcal{L}_i^{u(C)}$ for some positive integer $u(C) > 0$.

Given a divisible arrangement, we can and do assume that for any fixed i, all the corresponding $u(C)$'s are coprime.

Any line arrangement $\mathcal{A} = \{L_1, \ldots, L_d\}$ in $\mathbb{P}^2$ is divisible (Section 2.2). We consider the data $v = 1$, $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^2}(1)$ and $u(L_i) = 1$ for all i. In fact, any simple crossings arrangement of smooth plane curves is divisible, but now the data is $v = 1$, $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^2}\big((\deg(C_1), \ldots, \deg(C_d))\big)$ and $u(C_i) = \frac{\deg(C_i)}{(\deg(C_1), \ldots, \deg(C_d))}$. For a $v > 1$ example, consider $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and the arrangement formed by finite horizontal and vertical fibers. Platonic arrangements in Subsection 2.5.3 are also divisible.

Let $(Z, \mathcal{A})$ a pair with $\mathcal{A}$ divisible, and let p be a prime number. Let $\mathcal{A}_i =$

$\{C_{1,i},\dots,C_{d_i,i}\}$ be the sub-arrangements of $\mathcal{A}$ in Definition V.1. Suppose that the linear system of Diophantine equations

$$\sum_{j=1}^{d_i} u(C_{j,i})x_{j,i} = p$$

has a positive integer solution $(x_{j,i})$ in $\{1,\dots,p-1\}^d$. This solution produces the multiplicities μ_i that we assign to the curves C_i in $\mathcal{A}$. If $\mathcal{L}' = \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_v$, then

$$\mathcal{L}'^p \simeq \mathcal{O}_Z\Big(\sum_{i=1}^{d} \mu_i C_i\Big).$$

Let $\sigma : Y \to Z$ be the composition of all the blow ups at the k-points of $\mathcal{A}$ with $k \geq 3$. Let $D = \sigma^*(\sum_{i=1}^d \mu_i C_i)$ so that $\mathcal{A}' = D_{\text{red}}$. Let $\mathcal{L} = \sigma^*(\mathcal{L}')$. Then, we have $\mathcal{L}^p \simeq \mathcal{O}_Y(D)$, and we can construct the p-th root cover over Y along $\mathcal{A}'$ as in Section 4.3. As before, it is denoted by $f : X \to Y$.

Let $D = \sum_{i=1}^r \nu_i D_i$ be the decomposition into prime divisors of D, such that D_i are the strict transforms of C_i for $i \in \{1,\dots,d\}$, and for $i > d$, D_i are the exceptional divisors over the k-points of $\mathcal{A}$ with $k \geq 3$ (so, $r = \sum_{k\geq 3} t_k + d$). Hence, $\nu_i = \mu_i$ when $i \in \{1,\dots,d\}$, and for $i > d$, ν_i is the sum of the multiplicities μ_i assigned to the k curves passing through the corresponding k-point. When $\nu_i \geq p$, we can and do reduce this multiplicity mod p.

Assume that $0 < \nu_i < p$ for all i. Then, the formulas in Propositions IV.7 and IV.10 take the following form

$$\chi(X,\mathcal{O}_X) = \tfrac{p}{12}\Big(12\chi(Z,\mathcal{O}_Z) - \textstyle\sum_{i=1}^d C_i^2 + \sum_{k\geq 2}(4k-5)t_k + 6\sum_{i=1}^d (g(C_i)-1)\Big) +$$

$$\frac{1}{4}\Big(\sum_{k\geq 3}(2-k)t_k - t_2 - 2\sum_{i=1}^{d}(g(C_i)-1)\Big) + \frac{1}{12p}\Big(\sum_{i=1}^{d} C_i^2 - \sum_{k\geq 3}(k+1)t_k\Big) + DS$$

where $DS = \sum_{i<j} s(\nu_i'\nu_j, p) D_i.D_j$, and

$$e(X) = p\Big(e(Z) + \sum_{k\geq 2}(k-1)t_k + 2\sum_{i=1}^{d}(g(C_i)-1)\Big) + \Big(\sum_{k\geq 3}(2-k)t_k - t_2 - 2\sum_{i=1}^{d}(g(C_i)-1)\Big) + LCF$$

where $LCF = \sum_{i<j} l(\nu_i, \nu_j; p) D_i.D_j$.

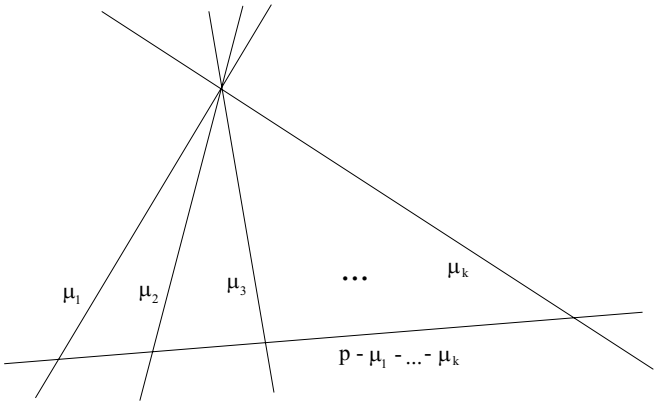
5.1.1 Example: Rational surfaces from p-th root covers.

Let $Z = \mathbb{P}^2$ and $\mathcal{A} = \{L_1, L_2, \ldots, L_{k+1}\}$ be an arrangement of $k+1$ lines such that $\bigcap_{i=1}^{k} L_i \neq \emptyset$ (and $\bigcap_{i=1}^{k+1} L_i = \emptyset$). Let $\{\mu_1, \ldots, \mu_k\}$ be an arbitrary collection of positive integers, and let p be a prime number greater than $\sum_{i=1}^{k} \mu_i + 1$. Then,

$$\mathcal{O}_Z(p) \simeq \mathcal{O}_Z\Big(\mu_1 L_1 + \mu_2 L_2 + \ldots + \mu_k L_k + \big(p - \sum_{i=1}^{k} \mu_i\big) L_{k+1}\Big)$$

and so we construct the corresponding p-th root cover $f : X \to Y$ as above (where Y is the blow up of Z at the k-point of $\mathcal{A}$). Then, X is a rational surface, because it has rational curves through every point. Let $0 < q < p$, and take $k = 2$, $\mu_1 = 1$ and $\mu_2 = q$. Since X is rational, we have $\chi(X, \mathcal{O}_X) = 1$, and so by our formulas

$$s(q, p) = s(q+1, p) + s(q'+1, p) + \frac{p-1}{4p}.$$



5.2 The theorem relating Chern and log Chern invariants.

Theorem V.2. *Let Z be a smooth projective surface, and let $\mathcal{A}$ be a simple crossings divisible arrangement on Z. Let $(Y, \mathcal{A}')$ be the corresponding associated pair, and*

assume $e(Y) \neq e(\mathcal{A}')$. *Then, there are smooth projective surfaces* X *having* $\frac{c_1^2(X)}{c_2(X)}$ *arbitrarily close to* $\frac{\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')}$.

Proof. Let $\mathcal{A}_i = \{C_{j,i}\}_{j=1}^{d_i}$ be the simple crossings sub-arrangements of $\mathcal{A}$ as in Definition V.1. The multiplicities μ_i we will assign to the curves in $\mathcal{A}$ come from a positive solution to the Diophantine linear system

$$(*) \quad \sum_{j=1}^{d_i} u(C_{j,i})x_{j,i} = p, \quad i = 1, 2, \ldots, v.$$

We will always assume that for a given i all $u(C_{j,i})$ are coprime (Definition V.1). To ensure the existence of solutions of **(*)**, we need p to be a large enough number. Then, it is well-known that the number of solutions is equal to (see [10] for example)

$$\prod_{i=1}^{v}\Big(\frac{p^{d_i-1}}{(d_i-1)!u(C_{1,i})u(C_{2,i})\cdots u(C_{d_i,i})} + O(p^{d_i-2})\Big).$$

We want to prove that when p approaches infinity, the majority of the solutions of **(*)** make the "error" numbers $\frac{DS}{p}$ and $\frac{LCF}{p}$ (in Propositions IV.7 and IV.10) tend to zero. In addition, we will prove that for a random choice of solutions, this is the case with probability tending to 1 as p approaches infinity. For this to happen, we use that outside of the bad set $\mathcal{F}$ (see Appendix Definition .17), whose size is $\approx \log(p)\sqrt{p}$, Dedekind sums (in DS) and length of continued fractions (in LCF) behave as $\sqrt{p}$.

We now prove that for a random choice of solutions $\{\mu_i\}_{i=1}^{d}$ of the system **(*)**, we do stay outside of this bad set. This means that the q's corresponding to the summands of the expressions DS and LCF do stay outside of $\mathcal{F}$. These summands are $s(\nu_i\nu'_j, p)D_i.D_j$ and $l(1, \nu_i\nu'_j; p)D_i.D_j$ respectively, and the corresponding common q is $p - \nu_i\nu'_j$. We re-write them using the multiplicities of $\mathcal{A}$, and so they are $p - \mu_i\mu'_j$ $(i \neq j)$ and $p - (\mu_{i_1} + \ldots + \mu_{i_k})\mu'_j$ (for any $1 < k < d$ and $j = i_l$ for some l).

We first consider $v = 1$, that is, one equation. Let us write it down as

$$Eq : t_1x_1 + \ldots + t_dx_d = p$$

where x_i are the variables (multiplicities) and $(t_1, \ldots, t_d) = 1$. As we said before, the number of positive solutions of Eq is $\frac{p^{d-1}}{(d-1)!t_1t_2\cdots t_d} + O(p^{d-2})$. In general, we denote the number of positive integer solutions of $x_1t_1 + \ldots + x_mt_m = a$ by $\alpha_m^{t_1,\ldots,t_m}(a)$. Let $b(x_ix'_j)$ be the set of solutions of Eq having $p - x_ix'_j$ in $\mathcal{F}$ for fixed i, j $(i \neq j)$; and similarly $b\big((x_{i_1}+\cdots+x_{i_k})x'_j\big)$ be the set of solutions of Eq having $p-(x_{i_1}+\ldots+x_{i_k})x'_j$ in $\mathcal{F}$ for fixed $i_1, \ldots, i_k, j$, having $1 < k < d$ and $j = i_l$ for some l. Then, we define the set of bad solutions $\mathcal{G}$ of Eq as the union of $b(x_ix'_j)$ and $b\big((x_{i_1} + \ldots + x_{i_k})x'_j\big)$ over all allowed indices. We want to bound the size of $\mathcal{G}$. We have the following two cases.

(1) Assume $i = 1$ and $j = 2$. Fix $0 < x_2 < p$. For each $0 < x_1 < p$, we have $\alpha_{d-2}^{t_3,\ldots,t_d}(p - x_1t_1 - x_2t_2)$ solutions of Eq. To each pair $(x_1, x_2) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, we associate the number $\alpha_{d-2}^{t_3,\ldots,t_d}(p - x_1t_1 - x_2t_2)$. The map $p - \bullet x'_2 : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is a bijection. To each new pair $(p - x_1x'_2, x_2)$ we associate the number $\alpha_{d-2}^{t_3,\ldots,t_d}(p - x_1t_1 - x_2t_2)$. Some pairs $(p - x_1x'_2, x_2)$ have $p - x_1x'_2 \in \mathcal{F}$, giving $\alpha_{d-2}^{t_3,\ldots,t_d}(p - x_1t_1 - x_2t_2)$ bad solutions to Eq. We know that for every x_2, there exists a positive number M such that $\alpha_{d-2}^{t_3,\ldots,t_d}(p - x_1t_1 - x_2t_2) < Mp^{d-3}$. Therefore, we have $|b(x_1x'_2)| < p \cdot |\mathcal{F}|Mp^{d-3}$, being the right-hand side the worse case scenario.

(2) Assume $j = 1$ and $i_1 = 1, \ldots, i_k = k$, $1 < k < d$ (there are not d-points in our arrangements by definition). Fix $0 < x_1 < p$. For each $x_2t_2 + \ldots + x_kt_k = p - x_1t_1$, we have $\alpha_{k-1}^{t_2,\ldots,t_k}(p - x_1t_1) < M(p - x_1t_1)^{k-2} < Mp^{k-2}$ solutions for some positive constant M. Also, for each pair $(x_1 + \ldots + x_k, x_1)$ we have $\alpha_{d-k}^{t_{k+1},\ldots,t_d}(p - x_1t_1 - \ldots - x_kt_k) < Np^{d-k-1}$ associated solutions of Eq, for some positive constant N. Since the map $p - \bullet x'_1 : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is a bijection, we associate to each new pair

$\big(p-(x_1+\ldots+x_k)x_1', x_1\big)$ the number of associated solutions of Eq, which is less than $Mp^{k-2}Np^{d-k-1}$. Hence, we obtain $|b\big((x_1+\ldots+x_k)x_1'\big)| < p\cdot|\mathcal{F}|Mp^{k-2}Np^{d-k-1}$, being the right-hand side the worse case scenario. We now prove that for a random choice of solutions $\{\mu_i\}_{i=1}^d$ of the system (*), we do stay outside of this bad set. This means that the entries of the expressions DS and LCF do stay outside of $\mathcal{F}$. These entries have the form $\nu_i\nu_j'$, where ν_i and ν_j are the multiplicities corresponding to D_i and D_j in D. We re-write them using the multiplicities of $\mathcal{A}$, and so they are $\mu_i\mu_j'$ $(i\neq j)$ and $(\mu_{i_1}+\ldots+\mu_{i_k})\mu_j'$ (for any $1<k<d$ and $j=i_l$ for some l).

Therefore, the number of bad solutions satisfies $|\mathcal{G}| < |\mathcal{F}|M_0p^{d-2}$, where M_0 is a positive number which depends on t_i's and d, and all possible combinations of pairs i,j and $i_1,\ldots,i_k,j$ as above. In particular M_0 does not depend on p. Now, by taking p large enough prime, we know that the exact number of solutions of Eq is $\frac{p^{d-1}}{(d-1)!t_1t_2\cdots t_d}+O(p^{d-2})$. On the other hand, by Theorem .18, we know that $|\mathcal{F}| < \sqrt{p}\big(\log(p)+2\log(2)\big)$ (we take $C=1$). Putting all together, we have proved the existence of non-bad solutions for large prime numbers p. In addition, we prove that for a random choice of solutions, the probability to be a bad solution is smaller than

$$\frac{\sqrt{p}\big(\log(p)+2\log(2)\big)M_0p^{d-2}}{\frac{p^{d-1}}{(d-1)!t_1t_2\cdots t_d}+O(p^{d-2})},$$

and so it approaches zero as p goes to infinity. In other words, for p large, a random choice of solutions has a great chance to be non-bad!

To prove the general case $v>1$, we go as before and prove that the bad solutions of (*) are $< |\mathcal{F}|M_0\sum_{i=1}^v(p^{d_i-2}\prod_{j\neq i}p^{d_j-1})$, where M_0 is a positive constant depending on t_i's and d, but not on p. Moreover, we know that the total number of solutions of (*) is $\prod_{i=1}^v\Big(\frac{p^{d_i-1}}{(d_i-1)!u(C_{1,i})u(C_{2,i})\cdots u(C_{d_i,i})}+O(p^{d_i-2})\Big)$. Therefore, we conclude the same for our linear system of Diophantine equations (*).

For each large enough prime number p, consider $\{\mu_i\}_{i=1}^d$ a non-bad solution of (*) (i.e., outside of $\mathcal{G}$). As in Section 4.3, we construct the corresponding p-th root cover $f : X \to Y$ branch along $\mathcal{A}'$. Then, by Propositions IV.7, IV.10, and the Noether's formula we have

$$\frac{c_1^2(X)}{p} = \Big(c_1^2(Z) - \sum_{i=1}^{d} C_i^2 + \sum_{k\geq 2}(3k-4)t_k + 4\sum_{i=1}^{d}(g(C_i)-1)\Big) +$$

$$\frac{1}{p}\Big(2\sum_{k\geq 3}(2-k)t_k - 2t_2 - 4\sum_{i=1}^{d}(g(C_i)-1)\Big) + \frac{1}{p^2}\Big(\sum_{i=1}^{d} C_i^2 - \sum_{k\geq 3}(k+1)t_k\Big) + 12\frac{DS}{p} - \frac{LCF}{p}$$

and

$$\frac{c_2(X)}{p} = \Big(c_2(Z) + \sum_{k\geq 2}(k-1)t_k + 2\sum_{i=1}^{d}(g(C_i)-1)\Big)$$
$$+\frac{1}{p}\Big(\sum_{k\geq 3}(2-k)t_k - t_2 - 2\sum_{i=1}^{d}(g(C_i)-1)\Big) + \frac{LCF}{p}$$

where

$$DS = \textstyle\sum_{i<j} s(\nu_i'\nu_j, p) D_i.D_j = -\sum_{i<j} s(p-\nu_i'\nu_j, p) D_i.D_j \text{ and}$$
$$LCF = \textstyle\sum_{i<j} l(1, \nu_i'\nu_j; p) D_i.D_j.$$

We know that the corresponding logarithmic Chern numbers are

$$\bar{c}_1^2(Y, \mathcal{A}') = \big(K_Y + \sum_{i=1}^{r} D_i\big)^2 = c_1^2(Z) - \sum_{i=1}^{d} C_i^2 + \sum_{k\geq 2}(3k-4)t_k + 4\sum_{i=1}^{d}(g(C_i)-1)$$

and

$$\bar{c}_2(Y, \mathcal{A}') = e(Y) - e(\mathcal{A}') = c_2(Z) + \sum_{k\geq 2}(k-1)t_k + 2\sum_{i=1}^{d}(g(C_i)-1).$$

By Theorems .18 and .20 (we take $C = 1$), we have

$$\Big|\frac{DS}{p}\Big| < \Big(\sum_{i<j} D_i.D_j\Big)\frac{(3\sqrt{p}+5)}{p} \quad \text{and} \quad \frac{LCF}{p} < \Big(\sum_{i<j} D_i.D_j\Big)\frac{(3\sqrt{p}+2)}{p}.$$

Since there are good solutions for arbitrary large p, we obtain that the corresponding surfaces X satisfy $c_1^2(X) \approx p\bar{c}_1^2(Y, \mathcal{A}')$ and $c_2(X) \approx p\bar{c}_2(Y, \mathcal{A}')$. Therefore,

if $e(Y) \neq e(\mathcal{A}')$ and p approaches infinity, there are smooth projective surfaces X having $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')}$. □

Remark V.3. (*Logarithmic Miyaoka-Yau inequality for divisible arrangements*) Let Z be a smooth projective surface and $\mathcal{A}$ be a divisible arrangement on Z, and let $(Y, \mathcal{A}')$ be the corresponding associated pair. Theorem V.2 proves the Miyaoka-Yau inequality for divisible arrangements, provided that $\bar{c}_1^2(Y, \mathcal{A}') + \bar{c}_2(Y, \mathcal{A}') > 0$. If this happens, then for large enough p, $\chi(X, \mathcal{O}_X)$ is a large positive number. In particular, by Enriques' classification of surfaces, X is not ruled, and so $c_1^2(X) \leq 3c_2(X)$. If in addition $\bar{c}_1^2(Y, \mathcal{A}') > 0$, then X is of general type, by the Enriques' classification.

We now substitute in this inequality the expressions for Chern numbers in the proof of Theorem V.2, and divide by p. Then, by making p tend to infinity, we obtain the inequality. Notice that this method can be used to disprove the divisibility property of an arrangement.

Corollary V.4. *Let Z be a smooth minimal projective surface of general type. Then, there exist smooth projective surfaces X, and generically finite maps $f : X \to Z$ of high degree, such that*

(i) X is minimal of general type.

(ii) The Chern numbers ratio $\frac{c_1^2(X)}{c_2(X)}$ is arbitrarily close to 2.

(iii) $q(X) = q(Z)$.

Proof. Since Z is projective, we have $Z \hookrightarrow \mathbb{P}^n$ for some n. For any integer $d \geq 4$, we consider a simple normal crossings arrangement $\mathcal{A} = \{H_1, \ldots, H_d\}$, where H_i are smooth hyperplane sections of Z. This is a divisible arrangement. Since Z is minimal of general type, we have that $5K_Z \sim C$, where C is a smooth projective

curve with $C^2 > 0$. This is because $|5K_Z|$ defines a birational map into its image, which is an isomorphism out of finitely many ADE configurations of (-2)-curves. We take C such that $\mathcal{A} \cup C$ has only nodes as singularities. Let p be a large prime number, and let $f : X \to Z$ be the p-th root cover associated to "random" partitions of p, $\nu_1 + \ldots + \nu_d = p$, as in Theorem V.2.

As in Section 4.1, we have the $\mathbb{Q}$-numerical relation $K_X \equiv f^*\big(K_Z + \frac{p-1}{p}\sum_{i=1}^d H_i\big) + \Delta$. Let Γ be a (-1)-curve of X. Then, $K_X.\Gamma = -1$. We know that $f^*(K_Z).\Gamma = f^*(\frac{1}{5}C).\Gamma \geq 0$. On the other hand, we have by Proposition IV.14 that

$$f^*\Big(\frac{(p-1)}{p}\sum_{i=1}^d H_i\Big) + \Delta$$

is an effective $\mathbb{Z}$-divisor. Therefore, the (-1)-curve Γ has to be a component of $f^*(D)$, where $D = \sum_{i=1}^d \nu_i H_i$. But all curves occurring on Hirzebruch-Jung resolutions have self-intersection ≤ -2, and so, for some i, we have $\Gamma = \widetilde{H}_i$, where $\widetilde{H}_i$ is the strict transform of H_i under f. But, as we will see, this is a contradiction because $\widetilde{H}_i^2 \leq -2$.

In Subsection 4.3.2, we computed the self-intersection of $\widetilde{H}_i$, which in this case is

$$\widetilde{H}_i^2 = \frac{1}{p}\Big(\deg(Z) - \sum_{j\neq i}(p - \nu_j\nu_i')\deg(Z)\Big).$$

An evident inequality is $\sum_{j\neq i}\nu_j\nu_i' \leq (d-2)p - 1$, since these numbers $\nu_j\nu_i'$ cannot sum the larger possible value $(d-1)p-1$. Also, since Z is not rational, we have that $\deg(Z) \geq 3$, and so $(d-2)p - 1 \leq \Big(d - 1 - \frac{2}{\deg(Z)}\Big)p - 1$. By rearranging the terms, we obtain the desired inequality.

The logarithmic Chern numbers of $(Z, \mathcal{A})$ are

$$\bar{c}_1^2(Z,\mathcal{A}) = \deg(Z)d^2 + \big(4g(H_1) - 4 - 2\deg(Z)\big)d + c_1^2(Z)$$

and

$$\bar{c}_2(Z,\mathcal{A}) = \tfrac{\deg(Z)}{2}d^2 + \big(2g(H_1) - 2 - \tfrac{\deg(Z)}{2}\big)d + c_2(Z).$$

We consider d large enough, so that $\bar{c}_1^2(Z,\mathcal{A}) + \bar{c}_2(Z,\mathcal{A}) > 0$. Let X be a surface produced by Theorem V.2 for $p >> 0$. Then, X is of general type. Moreover, the Chern numbers ratio $\frac{c_1^2(X)}{c_2(X)}$ is arbitrarily close to $\frac{\bar{c}_1^2(Z,\mathcal{A})}{\bar{c}_2(Z,\mathcal{A})}$. But when d is large, this log Chern numbers ratio tends to 2.

Finally, we notice that any curve in $\mathcal{A}$ is very ample. In particular, by Remark IV.12, we have $q(X) = q(Z)$ (this is exactly Viehweg vanishing theorem). □

Remark V.5. Corollary V.4 could be thought as a sort of "uniformization" via Chern numbers ratio ≈ 2. A minimal surface satisfying $c_1^2(X) = 2c_2(X)$ has signature zero, because the signature of a surface turns out to be $\frac{1}{3}(c_1^2(X) - 2c_2(X))$. The important geographical line $c_1^2 = 2c_2$ is the boundary between negative and positive signature. Notice that the surfaces X in Corollary V.4 have negative signature, i.e., their Chern numbers ratio tends to 2 from below.

One of the main properties of the construction is that the geometry of $\mathcal{A}$ can control $\pi_1(X)$, for certain arrangements of curves. This is developed by examples in the next section. In particular, as we have said before, our method could be used to attack the open problem of finding simply connected smooth projective surfaces over $\mathbb{C}$ with Chern numbers ratio in $(2.\overline{703}, 3)$.

5.3 Simply connected surfaces with large Chern numbers ratio.

In this section we show how to obtain simply connected surfaces with large Chern numbers ratio. Let $n \geq 2$ be an integer. Let $\mathcal{A}$ be the Fermat arrangement (Example II.6) defined by the equation

$$(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0.$$

This arrangement is formed by $3n$ lines, ad it is a divisible arrangement. To assign multiplicities as in the proof of Theorem V.2, we consider random partitions of

arbitrarily large prime numbers p:

$$\mu_1 + \mu_2 + \ldots + \mu_{3n} = p.$$

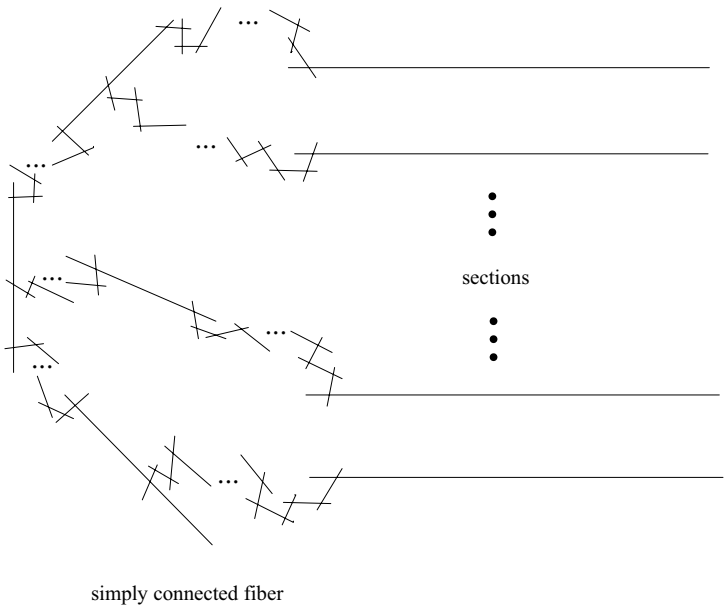


Figure 5.1: Fibration with sections and simply connected fiber.

Then, by Theorem V.2, there are smooth projective surfaces X with Chern numbers ratio $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to

$$\frac{\bar{c}_1^2(Y, \mathcal{A}')}{\bar{c}_2(Y, \mathcal{A}')} = \frac{5n^2 - 6n - 3}{2n^2 - 3n}.$$

Its maximum value is $\frac{8}{3}$, for $n = 3$, this is, the dual Hesse arrangement. Each of these arrangements corresponds exactly to union of the singular members in the Fermat pencil $u(x^n - y^n) + t(y^n - z^n) = 0$, $[u : t] \in \mathbb{P}^1$. If $(Y, \mathcal{A}')$ is the associated pair corresponding to $(\mathbb{P}^2, \mathcal{A})$, then Y is the blow up at the k-points of $\mathcal{A}$ with $k \geq 3$. In particular, Y resolves all the base points of the Fermat pencil, and so we have a fibration $g : Y \to \mathbb{P}^1$. This fibration has some sections and some singular fibers of g which are curves in $\mathcal{A}'$.

Then, the surfaces X in Theorem V.2 have an induced fibration $h : X \to \mathbb{P}^1$. Since the strict transform of any section of g in $\mathcal{A}'$ is a section of h, the map h does not have multiple fibers and all of its fibers are connected. The fibration g has three singular fibers, each of them simply connected. The inverse image of these fibers, under the p-th root covers, create simply connected singular fibers for the fibration h. If F is a singular fiber of g and $f : X \to Y$ is the p-th root cover, then $f^{-1}(F)$ is formed by the fixed components of F under f together with chains of $\mathbb{P}^1$'s coming from the Hirzebruch-Jung desingularization at the nodes of $\mathcal{A}'$. Hence, $f^{-1}(F)$ is simply connected. Therefore, by Corollary I.20, we have that $\pi(X) = \{0\}$. We can actually prove the following more general fact.

Proposition V.6. *All surfaces X coming from p-th root covers along line arrangements are simply connected.*

Proof. Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be an arbitrary arrangement of d lines in $\mathbb{P}^2$ with $t_d = 0$ as always. Let P be a point in L_1 which is smooth for $\mathcal{A}$. We consider the pencil $\alpha L_1 + \beta L = 0$, where $[\alpha : \beta] \in \mathbb{P}^1$, and L is a line in $\mathbb{P}^2$ passing through P.

To construct X in Theorem V.2, we considered the blow up $g : Y \to \mathbb{P}^2$ at the k-points of $\mathcal{A}$ with $k \geq 3$, and the divisor D in Y given by $D = g^*\big(\sum_{i=1}^{d} \mu_i L_i\big) \sim g^*\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)^p$, where $\mu_1 + \ldots + \mu_d = p$ is a partition of a prime number p. Let $f : X \to Y$ be the corresponding p-th root cover along D.

Let $\tau : Y' \to Y$ be the blow up at P of Y. Let E be the exceptional curve in Y' over P. Consider the divisor $D' = \tau^*(D)$. Then, we have a p-th root cover $f' : X' \to Y'$ along D', and so a birational map $\varsigma : X' \to X$ which is an isomorphism when restricted to $X' \setminus f'^{-1}(E)$. It is actually a birational morphism sending $f'^{-1}(E)$ to the point $f^{-1}(P)$. In particular, $\pi_1(X') \simeq \pi_1(X)$ (Proposition I.8). The self-intersection of the strict transform $\widetilde{E}$ of E under f' is -1, as computed in Subsection

4.3.2. Since in D', the strict transform of the line L_1 and E has the same multiplicity μ_1, the divisor $f'^*(E)$ consists of a chain of (-2)-curves and $\widetilde{E}$. Hence, $\varsigma : X' \to X$ blows down $\widetilde{E}$, and then each of the (-2)-curves in the chain.

To compute $\pi_1(X')$ we look at the fibration induced by the pencil $\alpha L_1 + \beta L = 0$. This fibration is over $\mathbb{P}^1$, and the exceptional curve E produces a section, mainly $\widetilde{E}$. Moreover, if L'_1 is the strict transform of L_1 in Y', then $f'^{-1}(L'_1)$ is simply connected, formed by $\widetilde{L}'_1 \simeq \mathbb{P}^1$ and chains of $\mathbb{P}^1$'s given by the Hirzebruch-Jung resolutions. Therefore, by Corollary I.20, X' is simply connected. □

The tables below give the actual values for the numerical invariants of simply connected surfaces X coming from the dual Hesse arrangement. This table was obtained by using a computer program, which was written for this purpose in C++. This program computes several invariants, and also self-intersections of the divisors involved (see Appendix for some examples).

Partition of $p = 61, 169$	$c_1^2(X)$	$c_2(X)$	$\frac{c_1^2(X)}{c_2(X)}$
1+2+3+4+5+6+7+8+61,133	1,441,949	733,435	1.9660
1+29+89+269+1,019+3,469+7,919+15,859+32,515	1,465,970	552,166	2.6549
6,790+6,791+6,792+6,793+6,794+6,795+6,796+6,797+6,821	1,464,209	633,619	2.3108
1+100+300+600+1,000+3,000+8,000+15,000+33,168	1,466,250	561,546	2.6110
1+30+90+270+1,020+3,470+7,920+15,860+32,508	1,465,778	553,594	2.6477
1+32+94+276+1,028+3,474+7,922+15,868+32,474	1,466,575	552,809	2.6529
1+1+1+1+1+1+1+1+6,1161	1,386,413	1,060,303	1.3075
1+1+89+89+1,019+3,469+7,919+15,859+32,723	1,465,370	553,402	2.6479

Table for the dual Hesse arrangement and $p = 61, 169$

Partition of p	$\frac{c_1^2(X)}{\chi(X,\mathcal{O}_X)}$	$\frac{c_1^2(X)}{c_2(X)}$
1+2+3+5+7+11+13+17+24=83	7.3319	1.5706
1+3+5+7+11+13+17+23+21=101	7.5035	1.6688
1+3+7+13+19+23+47+67+59=239	8.1248	2.0966
1+3+7+13+19+37+79+139+301=599	8.3908	2.3248
1+3+7+17+29+47+109+239+567=1,019	8.4088	2.3415
1+7+17+37+79+149+293+599+1,087=2,269	8.5866	2.5155
1+11+23+53+101+207+569+1,069+2,045=4,079	8.6462	2.5780
1+23+53+101+207+449+859+1,709+3,617=7,019	8.6853	2.6202
1+23+53+101+207+449+1,709+2,617+4,943=10,103	8.6954	2.6313
1+29+89+269+1019+3,469+7,919+15,859+32,515=61,169	8.7167	2.6549
1+101+207+569+1,069+10,037+22,441+44,729+66,623=145,777	8.7239	2.6629
1+619+1,249+2,459+5,009+10,037+32,323+68,209+110,421=230,327	8.7255	2.6647
1+929+1,889+3,769+6,983+15,013+32,323+87,443+163,751=312,101	8.7241	2.6632
1+929+1,889+3,769+6,983+15,013+45,259+90,749+172,397=336,989	8.7257	2.6649
1+929+1,889+3,769+6,983+15,013+45,259+90,749+187,637=352,229	8.7252	2.6644
1+1,709+3,539+7,639+15,629+31,649+62,219+150,559+271,165=544,109	8.7263	2.6656

Table for the dual Hesse arrangement and distinct primes p

By Proposition II.8, we know that our method cannot improve the bound $\frac{8}{3}$ when we use line arrangements. In other words, that proposition tells us (for our purposes):

The best (and unique) complex line arrangement is the dual Hesse arrangement.

Remark V.7. In the first table, we see that non-random looking partitions do not seem to approach to the corresponding log Chern numbers ratio. To be more precise, for example, for a general arrangement of d lines (i.e., only nodes as singularities), we

have log Chern numbers ratio tending to 2 as d approaches infinity. If we randomly choose the partition, we obtain surfaces with Chern numbers ratio tending to 2. If instead we choose multiplicities as in Subsection 4.3.1 (i.e., for a given p, all multiplicities equal to 1 except by one, which is $p - d + 1$), we obtain surfaces with Chern numbers ratio tending to 1.5 as d approaches infinity, whenever we take q $(= d - 1)$ in the good set (we can do it because the bad set has measure over p tending to 0). If, in the same case, we choose $q = p-1$, then the limit Chern numbers ratio is 1. Therefore, we put in evidence, by examples, that random multiplicities are necessary in our construction, if we want to approach to the log Chern numbers ratio of the corresponding arrangement.

5.4 More examples.

The following is a list of interesting examples of divisible arrangements, to which we apply Theorem V.2.

Plane curves.

Let $Z = \mathbb{P}^2$ and let $\mathcal{A}$ be a simple crossings arrangement of (smooth) plane curves $\{C_1, \ldots, C_d\}$. This is a divisible arrangement, and Theorem V.2 works through weighted random partitions

$$\sum_{i=1}^{d} \mu_i \frac{\deg(C_i)}{(\deg(C_1), \ldots, \deg(C_d))} = p.$$

For line arrangements, we have the bound given in Proposition II.8. However, we do not know to us what happens for arrangements of plane curves. These are very interesting, because they may belong to pencils of curves as in the last section. As in Subsection 2.5.1, let n_a be the number of curves of degree a in $\mathcal{A}$. Then, as we

computed before, the error number is

$$E(\mathbb{P}^2, \mathcal{A}) = \frac{\sum_{a\geq 1} a(2a-3)n_a + \sum_{k\geq 2} t_k}{\sum_{a\geq 1} a(a-3)n_a + \sum_{k\geq 2}(k-1)t_k + 3}.$$

This is the approximate distance from the Miyaoka-Yau bound of the surfaces X in Theorem V.2.

Horizontal and vertical curves in $\mathbb{P}^1 \times \mathbb{P}^1$

Let $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{A}$ be formed by two arrangements $\mathcal{A}_1 = \{F_1, \ldots, F_{d_1}\}$ with $F_i = \text{pt} \times \mathbb{P}^1$, and $\mathcal{A}_2 = \{G_1, \ldots, G_{d_2}\}$ with $G_i = \mathbb{P}^1 \times \text{pt}$. Assume $d_i > 2$. Then, this is a simple normal crossings arrangement and divisible. We partition p in two ways:

$$\sum_{i=1}^{d_1} \mu_i = p \qquad \sum_{i=d_1+1}^{d_2} \mu_i = p.$$

The error number associated to this arrangement is $E(Z, \mathcal{A}) = \frac{d_1 d_2 - 2d_1 - 2d_2 + 4}{d_1 d_2 - 2d_1 - 2d_2 + 4} = 1$. Hence, by Theorem V.2, we have sequences of surfaces X with Chern numbers ratio arbitrarily close to 2, independent of d_1 and d_2.

We notice that all the surfaces X have two induced isotrivial fibrations over $\mathbb{P}^1$ with general curves C_1 and C_2 respectively. Since we have some simply connected fibers in $\mathcal{A}$ which are fixed by the p-th root cover (take vertical or horizontal curves), the surfaces X are simply connected. In this way, the surfaces X may be seen as simply connected approximations of $C_1 \times C_2$. The latter surface satisfies $c_1^2 = 2c_2$, and it is not simply connected.

Line arrangements in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ having a k_1-point P and a k_2 point Q. Assume $k_1, k_2, d - k_1 - k_2 \geq 3$, and that the line L joining P and Q is not in $\mathcal{A}$ and intersects $\mathcal{A} \setminus \{P, Q\}$ transversally. We now blow up P and Q, and blow down the proper transform of L, to obtain $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the arrangement $\overline{\mathcal{A}}$ given by the lines

in $\mathcal{A}$. So, it is composed by $d-k_1-k_2$ curves in $\mathcal{O}(1,1)$, k_1 curves in $\mathcal{O}(0,1)$ and k_2 curves in $\mathcal{O}(1,0)$. This is a divisible arrangement. If t_k's are the data for $\mathcal{A}$, then the data for $\overline{\mathcal{A}}$ is $\bar{t}_{k_1} = t_{k_1} - 1$, $\bar{t}_{k_2} = t_{k_2} - 1$, $\bar{t}_{d-k_1-k_2} = t_{d-k_1-k_2} + 1$, and $\bar{t}_k = t_k$ otherwise. Then, for $\overline{\mathcal{A}}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ we have

$$\frac{\bar{c}_1^2(Y, \overline{\mathcal{A}}')}{\bar{c}_2(Y, \overline{\mathcal{A}}')} = \frac{-3d - 4k_1 - 4k_2 + \sum_{k\geq 2}(3k-4)t_k + 12}{-d - 2k_1 - 2k_2 + \sum_{k\geq 2}(k-1)t_k + 5}.$$

Notice that $\overline{\mathcal{A}}$ is not a "birational modification" of $\mathcal{A}$. These two arrangements are different from the log point of view. For example, let $\mathcal{A}$ be the Fermat arrangement of order $n > 2$ (see Example II.6). We have

$$\frac{\bar{c}_1^2(Y, \mathcal{A}')}{\bar{c}_2(Y, \mathcal{A}')} = \frac{5n^2 - 6n - 3}{2n^2 - 3n} \quad \text{and} \quad \frac{\bar{c}_1^2(Y, \overline{\mathcal{A}}')}{\bar{c}_2(Y, \overline{\mathcal{A}}')} = \frac{n(5n-8)}{2(n^2 - 2n + 1)}.$$

Hence, for example, the Dual Hesse arrangements has log Chern numbers ratio $\frac{8}{3} = 2.\bar{6}$ in $\mathbb{P}^2$, but in $\mathbb{P}^1 \times \mathbb{P}^1$ is $\frac{21}{8} = 2.625$. On the other hand, for $n = 4$ we have $\frac{53}{20} = 2.65$ for $\mathbb{P}^2$, but it is $\frac{8}{3}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. This is a new arrangement achieving the record, and because of that, we name it as 4-Fermat arrangement.

Platonic arrangements.

They were worked out in Subsection 2.5.3. The corresponding arrangements $\mathcal{A}$ are all divisible, with $v = 3$. By Theorem V.2, we have surfaces X with Chern numbers ratio arbitrarily close to the log Chern numbers ratio of these arrangements. These ratios are not greater than $\frac{8}{3}$ (they were computed in Subsection 2.5.3), but very close to it. The surfaces X are all simply connected by Corollary I.20.

Hirzebruch elliptic arrangements.

Hirzebruch elliptic arrangements achieve our bound $\frac{8}{3}$ as well, but the corresponding surfaces X are not simply connected. These examples were found by Hirzebruch in [43]. We worked them out in Subsection 2.5.5. This example shows how one can relax the hypothesis of being divisible, and still be able to use Theorem V.2.

Let $\zeta = e^{\frac{2\pi i}{3}}$ and T be the elliptic curve

$$T = \mathbb{C}/\big(\mathbb{Z} \oplus \mathbb{Z}\zeta\big).$$

Consider the abelian surface $Z = T \times T$ whose points are denoted by (z, w). Let $T_0 : \{w = 0\}$, $T_\infty : \{z = 0\}$, $T_1 : \{w = z\}$ and $T_\zeta : \{w = \zeta z\}$. These four curves only intersect at $(0, 0)$. Let U_n be the group of n-division points of Z, $U_n = \{(z, w) : \ (nz, nw) = (0, 0)\}$. This group has order n^4. The group U_n acts on Z by translations. Each of the sets $U_n(T_0)$, $U_n(T_\infty)$, $U_n(T_1)$, $U_n(T_\zeta)$ consists of n^2 smooth disjoint elliptic curves. Let $\mathcal{A}_0$, $\mathcal{A}_\infty$, $\mathcal{A}_1$, $\mathcal{A}_\zeta$ be the corresponding arrangements. We define the arrangement $\mathcal{A}$ by $\mathcal{A}_0 \cup \mathcal{A}_\infty \cup \mathcal{A}_1 \cup \mathcal{A}_\zeta$. Then, it is formed by $d = 4n^2$ elliptic curves. It can be checked that $t_4 = |U_n| = n^4$ and $t_k = 0$ for $k \neq 4$.

Consider $\mathcal{A}_0 = \{C_1, \ldots, C_{n^2}\}$. Then, by definition, $\mathcal{O}_Z(nC_i) \simeq \mathcal{O}_Z(nT_0)$ for all i. Hence, if $\sum_{i=1}^{n^2} \mu_i = p$, we have

$$\mathcal{O}_Z\Big(\sum_{i=1}^{n^2} n\mu_i C_i\Big) \simeq \mathcal{O}_Z(nT_0)^{\otimes p}.$$

Each T_i is a direct factor of Z, so we have the same situation for all T_i. We now look at four partitions of p which avoid bad multiplicities (as in Theorem V.2) in the 4-points. If $n \neq p$, the corresponding numbers module p are of type

$$n'\mu'_a(n\mu_a + n\mu_b + n\mu_c + n\mu_d) = \mu'_a(\mu_a + \mu_b + \mu_c + \mu_d),$$

so they are exactly as considered in Theorem V.2. Therefore, this theorem can also be applied for the pair $(Z, \mathcal{A})$. The error number associated to this arrangement does not depend on n and is our record $E(T \times T, \mathcal{A}) = \frac{1}{3}$ (see Subsection 2.5.5).

Smooth cubic surfaces.

In this example, we show how to apply Theorem V.2 to an arrangement that is not divisible a priori. Let Z be a cubic surface in $\mathbb{P}^3$, and let $\mathcal{A}$ be the arrangement

formed by the 27 lines on Z. In [39, Theorem 4.9], we have a way to write down the classes in $\mathrm{Pic}(Z)$ of these lines. The cubic surface Z is thought as the blow up of $\mathbb{P}^2$ at 6 points, say $P_1, \ldots, P_6$ (not all in a conic). Let E_i be the exceptional divisors of this blow up, and let H be the class of the pull back of a line in $\mathbb{P}^2$. We consider the lines as follows.

(a) E_i the exceptional divisor, $1 \leq i \leq 6$ (six of these),

(b) $F_{i,j} \sim H - E_i - E_j$ the strict transform of the line in $\mathbb{P}^2$ containing P_i and P_j, $1 \leq i < j \leq 6$ (fifteen of these),

(c) $G_i \sim 2H - \sum_{j \neq i} E_j$ the strict transform of the conic containing P_j for all $j \neq i$ (six of these).

We want to assign multiplicities to them such that the corresponding divisor is divisible by arbitrarily large primes p, and they are outside of the bad set. Let γ_i, $\alpha_{i,j}$ and β_i be the multiplicities for E_i, $F_{i,j}$ and G_i respectively. Let p be a prime number, and assume

$$\sum \alpha_{i,j} + 2\sum \beta_i = p \qquad \gamma_i = p + \sum_{j=1}^{6} \alpha_{i,j} + \sum_{j \neq i} \beta_j.$$

Then, the divisor $D' = \sum \alpha_{i,j} F_{i,j} + \sum \beta_i G_i + \sum \gamma_i E_i$ satisfies

$$D' \sim p\big(H + \sum F_{i,j} + \sum G_i + \sum E_i\big).$$

In order to have the result of Theorem V.2, we need to find solutions to $\sum \alpha_{i,j} + 2\sum \beta_i = p$ for arbitrarily large p, such that the corresponding numbers $\nu_i \nu'_j$ stay outside of the bad set $\mathcal{F}$. The multiplicities ν_i are $\alpha_{i,j}$, β_i, γ_i or three term sums of them, since $\mathcal{A}$ can have only 2- and 3- points. We can modify the argument in points **(1)** and **(2)** in the proof of Theorem V.2, to prove that for random solutions

of $\sum \alpha_{i,j} + 2\sum \beta_i = p$ the numbers $\nu_i \nu'_j$ do stay out of $\mathcal{F}$, so that we have surfaces X as in the conclusion of that theorem.

We computed in Subsection 2.5.2 the possible log Chern numbers for $\mathcal{A}$. The best possible one is given by the Fermat cubic surface, having Chern numbers ratio $\frac{7}{12} = 2.41\bar{6}$. In any case, if we consider a line in Z having a 3-point, and consider the corresponding pencil of conics, it can be checked (using the usual fibration's argument) that the surfaces X constructed out of $(Z, \mathcal{A})$ are simply connected.

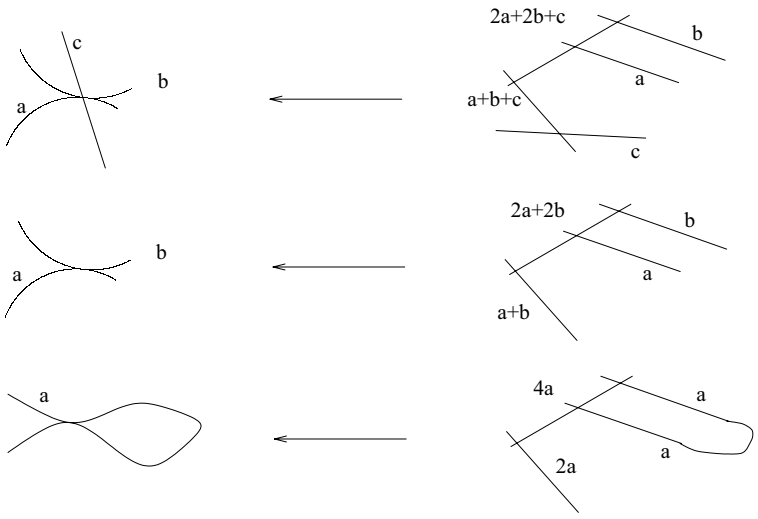


Figure 5.2: Some possible singularities for general arrangements.

Remark V.8. Some examples illustrate that there is a larger class of simple crossings arrangements for which Theorem V.2 holds. Our strategy to find surfaces with large Chern numbers ratio is: first we find a simple crossings arrangement and compute its log Chern numbers, if the numbers are good, we then prove that the arrangement is suitable to apply Theorem V.2. Divisible arrangements give us a large class for which the ideas in the proof go through. Not any arrangement is possible. For example, a p-th root cover along a divisor D with a disjoint primary component must have self-intersection divisible by p on that component. Hence, any arrangement which

has a disjoint component with non-zero self-intersection cannot be divisible.

A second remark is our method will be applicable when considering arrangements with worse singularities than simple crossings. In Figure 5.2, we illustrate three singularities, representing a singularity of our arrangement $\mathcal{A}$ and its log resolution (i.e., locally $\mathcal{A}'$). In the two last cases, we have a situation were the multiplicities cannot be taken randomly, since $4a(2a)' = 2$, $4a(a)' = 4$ and $2(a+b)(a+b)' = 2$, that is, these numbers do not vary and always belong to the bad set $\mathcal{F}$. Hence the errors from Dedekind sums and length of continued fractions count in the computation of the Chern invariants of X, for large prime numbers p. However, in the first case (a priori) we can assume random multiplicities by just taking a, b, c randomly.

CHAPTER VI

Deforming p-th root covers

In this Chapter, we develop tools to study deformations of the surfaces X coming from p-th root covers (Section 4.3). Our main goal is to compute the cohomology groups of T_X, the tangent sheaf of X, with respect to the p-th root data $(Y, D, \mathcal{L}, p)$. The study of deformations may help to understand potential restrictions to obtain surfaces with Chern numbers ratio arbitrarily close to the Miyaoka-Yau bound. Also, we want to obtain results about minimality and rigidity.

6.1 Deformations of surfaces of general type.

Let X be a normal projective surface over $\mathbb{C}$. An infinitesimal deformation of X is the existence of a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

where A is an Artin local $\mathbb{C}$-algebra, the map $X' \to \mathrm{Spec}(A)$ is flat, and

$$X \simeq \mathrm{Spec}(\mathbb{C}) \times_{\mathrm{Spec}(A)} X'.$$

When $A = \mathbb{C}[t]/(t^2)$ $(= \mathbb{C}[\mathbb{C}]$ in Olsson's notes [70]), we have a first order deformation. Let $\mathrm{Def}_X : Alg/\mathbb{C} \to \mathrm{Sets}$ be the functor of diagrams defined in [70,

p. 4]. The tangent space of the functor Def_X is defined as $\mathrm{Def}_X\big(\mathbb{C}[t]/(t^2)\big)$. It turns out that the tangent space has a structure of $\mathbb{C}$-module. In our case, it is actually a finite dimensional vector space over $\mathbb{C}$ [12, p. 74]. If X is smooth, then $\mathrm{Def}_X\big(\mathbb{C}[t]/(t^2)\big) = H^1(X, T_X)$ [70, Prop. 2.6].

Assume that X is of general type. Then, X has a finite number of automorphisms, and so $H^0(X, T_X) = 0$ (see for example [16, p. 98]). This fact implies the existence of a universal deformation $\pi : \mathcal{X} \to \mathrm{Def}(X)$ of X, where $\mathrm{Def}(X)$ is an analytic set with a distinguished point $0 \in \mathrm{Def}(X)$ such that $\pi^{-1}(0) = X$ [16, p. 97]. Let $S = \mathcal{O}_{\mathrm{Def}(X),0}$, and let $h_S = \mathrm{Hom}(S, \bullet)$ be the usual functor defined by the ring S. Then $h_S \simeq \mathrm{Def}_X$ [61, p. 25]. In particular, the tangent space of $\mathrm{Def}(X)$ at 0 is $\mathrm{Def}_X\big(\mathbb{C}[t]/(t^2)\big)$ [61, p. 23]. Again, if X is smooth, this tangent space is $H^1(X, T_X)$. In this case, the obstruction for the smoothness of $\mathrm{Def}(X)$ lies in $H^2(X, T_X)$. For example, if $H^2(X, T_X)$ vanishes, then $\mathrm{Def}(X)$ is smooth at 0. In general, we say that X has <u>obstructed deformations</u> if $\mathcal{O}_{\mathrm{Def}(X),0}$ is not regular.

We saw in Section 1.2 that there exists a quasi-projective variety representing the coarse moduli space of surfaces of general type. We denoted it by $M_{c_1^2,c_2}$, being c_1^2 and c_2 the fixed Chern numbers of the surfaces it classifies. Since to construct the moduli space we want ω_X to be ample, these surfaces are canonical models. The canonical model of a minimal smooth projective surface of general type X is the image of the map given by $|nK_X|$, for sufficiently large n. It turns out that if we fix $n \geq 5$, we can define the canonical model of X as $X_{\mathrm{can}} := |nK_X|(X)$, up to isomorphism. These are the objects parametrized by $M_{c_1^2,c_2}$. The normal projective surface X_{can} can only have rational double points as singularities (ADE singularities). This happens exactly when K_X is not ample, where X is the minimal model of the birational class of X_{can}.

Let X be a smooth surface of general type, and let X_{can} be its canonical model. Then, we have [61, p. 78]

$$\text{locally at } [X],\ M_{c_1^2,c_2} \text{ is analytically isomorphic to } \text{Def}(X_{\text{can}})/\operatorname{Aut}(X_{\text{can}}).$$

The number of moduli of X is defined as $M(X) := \dim \mathcal{O}_{\text{Def}(X_{\text{can}}),0}$, i.e., its Krull dimension.

What is the relation between deformations of X and deformations of X_{can}? Let X be a minimal smooth projective surface, and let $\rho : X \to X_{\text{can}}$ the minimal resolution of all the the rational double points of X_{can}. First, we have $\dim\text{Def}(X) = \dim\text{Def}(X_{\text{can}})$ [17, p. 299]. Let E be the exceptional locus of ρ, and let $H^1_E(X, T_X)$ be the local first cohomology group with coefficients in T_X supported on E (as in [12]). Then, by using results of Burns-Wahl and Pinkham, we have [17, p. 299] [1]

$$H^1(X, T_X) = H^1(X_{\text{can}}, T_{X_{\text{can}}}) \oplus H^1_E(X, T_X) \qquad H^2(X, T_X) = H^2(X_{\text{can}}, T_{X_{\text{can}}}).$$

Moreover, in [12, Prop.(1.10)], Burns and Wahl proved that $\dim H^1_E(X, T_X)$ is equal to the number of (-2)-curves in E. Hence, each rational curve produces a first order deformation of X. However, it might be misleading to find actual positive dimensional deformations. It seems unknown whether a (-2)-curve always produces a one dimensional deformation. It is known that the deformations coming from (-2)-curves may not be independent because, for example, we can have $H^1_E(X, T_X) > M(X)$. Examples are given in [12]. For the tangent space, we have the inequalities [90, Cor. (6.4)]

$$h^1(X, T_X) - \dim H^1_E(X, T_X) = h^1(X_{\text{can}}, T_{X_{\text{can}}}) \leq \dim\text{Def}_{X_{\text{can}}}\big(\mathbb{C}[t]/(t^2)\big) \leq h^1(X, T_X).$$

As explained in [16, p. 84], if X is a minimal smooth projective surface of general

[1] The tangent sheaf of X_{can} is defined as $T_{X_{\text{can}}} := \mathscr{H}om_{\mathcal{O}_{X_{\text{can}}}}(\Omega^1_{X_{\text{can}}}, \mathcal{O}_{X_{\text{can}}})$.

type, the number of moduli of X is coarsely bounded by the inequality

$$10\chi(X,\mathcal{O}_X) - 2c_1^2(X) \leq \dim\mathcal{O}_{\mathrm{Def}(X),0} = M(X) \leq h^1(X,T_X).$$

This is because $\mathrm{Def}(X)$ is a germ of analytic subset of $H^1(X,T_X)$ at 0 defined by $h^2(X,T_X)$ equations, plus the fact that $h^1(X,T_X) - h^2(X,T_X) = \chi(X,T_X) = 10\chi(X,\mathcal{O}_X) - 2c_1^2(X)$ by the Hirzebruch-Riemann-Roch theorem (and $H^0(X,T_X) = 0$). Notice that the left-hand side inequality does not give any information about the dimension of $\mathrm{Def}(X)$ when $5\chi(X,\mathcal{O}_X) < c_1^2(X)$, in particular when we are close to the Miyaoka-Yau bound. In this case, we have $h^2(X,T_X) \neq 0$, and so we cannot use the usual observation to try to prove smoothness for $\mathrm{Def}(X)$.

If we consider a non-minimal smooth projective surface of general type X, and X_0 is its minimal model, then $h^0(X,T_X) = 0$, $h^1(X,T_X) = h^1(X_0,T_{X_0}) + 2m$, where m is the number of blow downs to obtain X_0, and $h^2(X,T_X) = h^2(X_0,T_{X_0})$. If $\sigma : \widetilde{X} \to X$ is the blow up at a point P of X, then we have the short exact sequence [12, p. 72]

$$0 \to \sigma_* T_{\widetilde{X}} \to T_X \to N_P \to 0$$

where N_P is the normal bundle of P in X. We have $h^0(P,N_P) = 2$, and $h^1(P,N_P) = h^2(P,N_P) = 0$, and so the previous observation follows from the associated long exact sequence. In [7, p. 154], it is shown a way to blow down (-1)-curves in families, keeping the base fixed.

We finish with the key equation

$$h^1(X,T_X) = 10\chi(X,\mathcal{O}_X) - 2c_1^2(X) + h^0(X,\Omega_X^1 \otimes \Omega_X^2)$$

which is the Hirzebruch-Riemann-Roch theorem applied to T_X, and Serre's duality.

6.2 Some general formulas for n-th root covers.

Here we present some relevant facts about sheaves associated to n-th root covers. We will work on any dimension, as we did in Section 4.1. Let Y be a smooth projective variety, and let D be a SNC effective divisor with primary decomposition $D = \sum_{i=1}^{r} \nu_i D_i$. Assume that there exist a positive integer n and a line bundle $\mathcal{L}$ on Y satisfying

$$\mathcal{L}^n \simeq \mathcal{O}_Y(D).$$

Let $f : X \to Y$ be a n-th root cover associated to the data $(Y, D, n, \mathcal{L})$. Here we have chosen a minimal resolution of $\overline{Y}$ such that the divisor $f^*(D)_{\text{red}}$ has simple normal crossings. Let $\widetilde{D} = f^*(D) = \sum_{i=1}^{r'} \eta_i \widetilde{D}_i$. The main sheaves of these covers are the invertible sheaves

$$\mathcal{L}^{(i)} := \mathcal{L}^i \otimes \mathcal{O}_Y\Big(-\sum_{j=1}^{r} \Big[\frac{\nu_j\ i}{n}\Big] D_j\Big)$$

for $i \in \{0, 1, ..., n-1\}$. We start by rewording Proposition IV.3.

Proposition VI.1. *(see [88]) Let $f : X \to Y$ be the n-th root cover associated to $(Y, D, n, \mathcal{L})$. Then,*

$$f_*\mathcal{O}_X = \bigoplus_{i=0}^{n-1} {\mathcal{L}^{(i)}}^{-1} \quad \textit{and} \quad R^i f_*\mathcal{O}_X = 0 \quad \textit{for } i > 0.$$

Proposition VI.2. *Let $f : X \to Y$ be the n-th root cover associated to $(Y, D, n, \mathcal{L})$. Then,*

$$f_*\Omega^2_X = \bigoplus_{i=0}^{n-1} \left(\Omega^2_Y \otimes \mathcal{L}^{(i)}\right) \quad \textit{and} \quad R^i f_*\Omega^2_X = 0 \quad \textit{for } i > 0.$$

Proof. By [89, Lemma 2.3], we have that $f_*\Omega^2_X = \bigoplus_{i=0}^{n-1} \left(\Omega^2_Y \otimes \mathcal{L}^{(i)}\right)$. This follows from Hartshorne's book [39, Exercises 6.10, 7.2]. By [52, Theorem 2.1], we have $R^b f_*\Omega^2_X = 0$ for $b > 0$, since the dimension of a general fiber of f is zero. □

Let us now consider the logarithmic sheaves of differentials $\Omega^a_X(\log \widetilde{D})$ on X and $\Omega^a_Y(\log D)$ on X, as in Definition I.22. We remark that for these sheaves we are taking the reduced divisors of $\widetilde{D}$ and D. The following proposition is [29, Lemma 3.22].

Proposition VI.3. *Let $f : X \to Y$ be the n-th root cover associated to $(Y, D, n, \mathcal{L})$. Let a be an integer satisfying $0 \leq a \leq dim X$. Then,*

$$f_*\Omega^a_X(\log \widetilde{D}) = \bigoplus_{i=0}^{n-1} \left(\Omega^a_Y(\log D) \otimes \mathcal{L}^{(i)^{-1}}\right) \quad \textit{and} \quad R^i f_*\Omega^a_X(\log \widetilde{D}) = 0 \quad \textit{for } i > 0.$$

6.3 The case of surfaces.

We will use the same set up of the previous section, with the modifications $\dim Y = 2$ and $n = p$ prime number. We also assume $0 < \nu_i < p$. The smooth projective surface X is uniquely determined by $(Y, D, p, \mathcal{L})$. The following result does not require n to be a prime number.

Proposition VI.4. *Let $f : X \to Y$ be the n-th root cover associated to $(Y, D, n, \mathcal{L})$. Let $a \in \{0, 1, 2\}$. Then,*

$$f_*\left(\Omega^a_X(\log \widetilde{D}) \otimes \Omega^2_X\right) = \bigoplus_{i=0}^{n-1} \left(\Omega^a_Y(\log D) \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)}\right) \quad \textit{and} \quad R^i f_*\left(\Omega^a_X(\log \widetilde{D}) \otimes \Omega^2_X\right) = 0$$

for all $i > 0$.

Proof. The case $a = 0$ is Proposition VI.2. By [27, Corollaire 4.], we have

$$f^*\Omega^a_Y(\log D) = \Omega^a_X(\log \widetilde{D}).$$

Therefore, the proposition follows from the projection formula [39, Ch. III Exerc. 8.3] and Proposition VI.2. □

We will see that the sheaf $f_*(\Omega^1_X \otimes \Omega^2_X)$ is key to understand deformations of X. By the Leray spectral sequence, there is an exact sequence

$$0 \to H^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) \to H^1(X, \Omega^1_X \otimes \Omega^2_X) \to$$

$$H^0(Y, R^1 f_*(\Omega^1_X \otimes \Omega^2_X)) \to H^2(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) \to H^2(X, \Omega^1_X \otimes \Omega^2_X).$$

Assume that X is of general type (for example, this happens when X comes from Theorem V.2, and $\bar{c}_1^2(Y, D) + \bar{c}_2(Y, D) > 0$). Then, the last term vanishes because, by Serre's duality, $h^2(X, \Omega^1_X \otimes \Omega^2_X) = h^0(X, T_X) = 0$. It would be a great simplification to have $R^1 f_*(\Omega^1_X \otimes \Omega^2_X) = 0$, but the existence of (-2)-curves in the resolution shows that this is not true. The following proposition clarifies the behavior of $f_*(\Omega^1_X \otimes \Omega^2_X)$.

Theorem VI.5. *Let $f : X \to Y$ be the p-th root cover associated to $(Y, D, p, \mathcal{L})$. Then,*

$$f_*(\Omega^1_X \otimes \Omega^2_X) = \bigoplus_{i=0}^{p-1} \Omega^1_Y(\log D^{(i)}) \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)}, \quad R^1 f_*(\Omega^1_X \otimes \Omega^2_X) = H^1_E(X, T_X)^\vee$$

and $R^2 f_*(\Omega^1_X \otimes \Omega^2_X) = 0$, where $D^{(i)} := \sum_{i\nu_j \neq -1 \pmod p} D_j$ (in particular, $D^{(0)} = D$) and E is the exceptional divisor in the minimal resolution $X \to \overline{Y}$. Moreover, the dimension of $H^1_E(X, T_X)$ is equal to the number of (-2)-curves in E.

Proof. First, we have $R^2 f_*\big(\Omega^1_X \otimes \Omega^2_X\big) = 0$ because the dimension of the fibers of f is at most one, and so we apply [39, Corollary 11.2]. By [53, Prop. 11.6 (11.6.1)], we have $R^1 f_*\big(\Omega^1_X \otimes \Omega^2_X\big) = H^1_E(X, \Omega^2_X \otimes \big(\Omega^1_X \otimes \Omega^2_X\big)^\vee)^\vee \simeq H^1_E(X, T_X)^\vee$. On the other hand, it is a theorem of Wahl that $\dim H^1_E(X, T_X)$ is equal to the number of (-2)-curves in E [90, Theorem (6.1)]. This theorem is valid for any rational singularities in characteristic zero.

Now we compute $f_*(\Omega^1_X \otimes \Omega^2_X)$. First, we consider the residual exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log \widetilde{D}) \to \oplus_{i=1}^{r'} \mathcal{O}_{\widetilde{D}_i} \to 0.$$

Then, by Proposition VI.4, we have

$$f_*\big(\Omega^1_X \otimes \Omega^2_X\big) \hookrightarrow f_*\big(\Omega^1_X(\log \widetilde{D}) \otimes \Omega^2_X\big) = \bigoplus_{i=0}^{p-1} \big(\Omega^1_Y(\log D) \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)}\big).$$

We now locally compute, on the right-hand side, the sections that lift to sections of $\Omega^1_X \otimes \Omega^2_X$. We take a neighborhood of a point $P \in Y$ which is a node for D, say $P \in D_1 \cap D_2$. We consider the set up of Sub-section 4.3.2, and so let x, y be local coordinates around P such that $D_1 = \{x = 0\}$ and $D_2 = \{y = 0\}$. Let $\widetilde{D}_1$ and $\widetilde{D}_2$ be the strict transforms of D_1 and D_2 respectively, and let E_i be the components of the exceptional divisor over P. We will use the numbers a_i and d_i in Sub-section 4.3.2, taking $q = p - \nu_2\nu'_1$. We remark that our notation is $D = \sum_{i=1}^{r} \nu_i D_i$.

Let us take local coordinates around $Q = E_i \cap E_{i+1}$. For the purpose of having a notation that applies to all the cases, we define $E_0 = \widetilde{D}_1$ and $E_{s+1} = \widetilde{D}_2$. Let $\tilde{x}$ and $\tilde{y}$ be the local coordinates around Q such that $E_i = \{\tilde{x} = 0\}$ and $E_{i+1} = \{\tilde{y} = 0\}$. Then, we have that under f

$$x = u\tilde{x}^{a_i}\tilde{y}^{a_{i+1}} \quad \text{and} \quad y = v\tilde{x}^{d_i}\tilde{y}^{d_{i+1}},$$

where u, v are units. Therefore,

$$dx = u\tilde{y}^{a_{i+1}-1}\tilde{x}^{a_i}d\tilde{y} + u\tilde{y}^{a_{i+1}}\tilde{x}^{a_i-1}d\tilde{x} + \tilde{y}^{a_{i+1}}\tilde{x}^{a_i}du$$

and

$$dy = v\tilde{y}^{d_{i+1}-1}\tilde{x}^{d_i}d\tilde{y} + v\tilde{y}^{d_{i+1}}\tilde{x}^{d_i-1}d\tilde{x} + \tilde{y}^{d_{i+1}}\tilde{x}^{d_i}dv,$$

and so

$$\frac{dx}{x} = u\frac{d\tilde{x}}{\tilde{x}} + u\frac{d\tilde{y}}{\tilde{y}} + du$$

and

$$\frac{dy}{y} = v\frac{d\tilde{x}}{\tilde{x}} + v\frac{d\tilde{y}}{\tilde{y}} + dv.$$

On the other hand, we have that the line bundle $\mathcal{L}^{(}i)$ is locally generated by $t^{-i}x^{\left[\frac{i\nu_1}{p}\right]}y^{\left[\frac{i\nu_2}{p}\right]}$ where t^{-1} is a local generator for $\mathcal{L}$ such that locally satisfies $t^p = wx^{\nu_1}y^{\nu_2}$ (w a unit). All of this comes from $\mathcal{L}^p \simeq \mathcal{O}_Y(D)$.

We now look at the local sections of $\Omega^1_Y(\log D) \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)}$ using the previous parameters. When we go from x, y to $\tilde{x}, \tilde{y}$, we want these sections to be differential forms in $\Omega^1_X \otimes \Omega^2_X$, in particular with no poles. A simple computation shows that this requirement is equivalent to the inequality

$$-1 + a_i - 1 + d_i + \frac{a_i}{p}\Big(\Big[\frac{i\nu_1}{p}\Big]p - i\nu_1\Big) + \frac{d_i}{p}\Big(\Big[\frac{i\nu_2}{p}\Big]p - i\nu_2\Big) \geq 0,$$

and similarly for $i+1$. Assume that $\left[\frac{i\nu_j}{p}\right]p - i\nu_j \geq -(p-2)$ for $j = 1, 2$. This is equivalent to say that $i\nu_j$ is not -1 module p for $j = 1, 2$. Then, the inequality above follows from

$$(p-2) + \frac{-a_i(p-2)}{p} + \frac{-d_i(p-2)}{p} = (p-2)\Big(1 - \frac{a_i}{p} - \frac{d_i}{p}\Big) \geq 0,$$

which is always true since $p \geq 2$ and, by Lemma .16, $1 - \frac{a_i}{p} - \frac{d_i}{p} \geq 0$. In this way, if we assume $i\nu_j$ is not -1 module p for $j = 1, 2$, all the sections lift to $\Omega^1_X \otimes \Omega^2_X$.

If for some j we have $i\nu_j \equiv -1 (\mathrm{mod}\ p)$, then it is easy to check that the corresponding section will not lift. One can check it around a point $P \in D_j$ which is smooth for D_{red}. □

Remark VI.6. In [12, Corollary (1.3)], Burns and Wahl prove that

$$H^1_E(X, T_X) \hookrightarrow H^1(X, T_X),$$

using the long exact sequence for local cohomology. In the previous proof, we use that $H^0(Y, R^1 f_*(\Omega^1_X \otimes \Omega^2_X))^\vee = H^1_E(X, T_X)$. Is the dual of the inclusion map of

Burns and Wahl the corresponding map in the Leray spectral sequence above? That would induce a splitting, producing

$$H^1(X, T_X) \simeq H^1_E(X, T_X) \oplus H^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X))^\vee,$$

and $H^2(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) \hookrightarrow H^0(X, T_X)$.

For each i, consider the residual exact sequence for $D^{(i)}$

$$0 \to \Omega^1_Y \to \Omega^1_Y(\log D^{(i)}) \to \oplus_{i\nu_j \neq -1 (\mathrm{mod}\ p)} \mathcal{O}_{D_j} \to 0.$$

We tensor it by $\Omega^2_Y \otimes \mathcal{L}^{(i)}$, to obtain

$$(*) \quad 0 \to \Omega^1_Y \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)} \to \Omega^1_Y(\log D^{(i)}) \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)} \to \oplus_{i\nu_j \neq -1 (\mathrm{mod}\ p)} \Omega^2_Y \otimes \mathcal{L}^{(i)} \otimes \mathcal{O}_{D_j} \to 0.$$

The importance of this sequence relies on

- By Proposition VI.2, $f_*\big(f^*(\Omega^1_Y) \otimes \Omega^2_X\big) \simeq \bigoplus_{i=0}^{p-1} \Omega^1_Y \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)}$.
- By Theorem VI.5, $f_*(\Omega^1_X \otimes \Omega^2_X) = \bigoplus_{i=0}^{p-1} \Omega^1_Y(\log D^{(i)}) \otimes \Omega^2_Y \otimes \mathcal{L}^{(i)}$.

and so, it allows us to study the key cohomologies $H^1(X, f_*(\Omega^1_X \otimes \Omega^2_X))$ and $H^2(X, f_*(\Omega^1_X \otimes \Omega^2_X))$ through the cohomology of explicit sheaves on Y and on the curves D_i's, via the corresponding long exact sequence.

Remark VI.7. (*The cohomology groups of* $f_*\big(f^*(\Omega^1_Y) \otimes \Omega^2_X\big)$) First, we notice that by the projection formula and Proposition VI.2, $R^i f_*\big(f^*(\Omega^1_Y) \otimes \Omega^2_X\big) = 0$ for $i > 0$. Therefore, by [39, Ch. III Ex. 8.1], $H^j(X, f^*(\Omega^1_Y) \otimes \Omega^2_X) \simeq H^j(Y, f_*\big(f^*(\Omega^1_Y) \otimes \Omega^2_X\big))$. In addition, by Serre's duality, $H^{2-j}(X, f^*(\Omega^1_Y) \otimes \Omega^2_X) \simeq H^j(X, f^*T_Y)^\vee$. Finally, by the projection formula and Proposition VI.1, we have

$$H^j(X, f^*T_Y) \simeq \bigoplus_{i=0}^{p-1} H^j(Y, T_Y \otimes \mathcal{L}^{(i)^{-1}}).$$

Remark VI.8. (*The cohomology groups of* $\Omega_Y^2 \otimes \mathcal{L}^{(i)} \otimes \mathcal{O}_{D_j}$) First, by the adjunction formula,

$$\Omega_Y^2 \otimes \mathcal{L}^{(i)} \otimes \mathcal{O}_{D_j} \simeq \Omega_{D_j}^1 \otimes \mathcal{O}_Y(-D_j) \otimes \mathcal{L}^{(i)} \otimes \mathcal{O}_{D_j}.$$

The degree of $\mathcal{L}^{(i)}$ restricted to D_j is $D_j.\mathcal{L}^{(i)} = \sum_{b=1}^{t} \left(\frac{i\nu_{j_b}}{p} - \left[\frac{i\nu_{j_b}}{p}\right]\right) + \left(\frac{i\nu_j}{p} - \left[\frac{i\nu_j}{p}\right]\right) D_j^2$, where $D_{j_1}, \ldots, D_{j_t}$ are exactly the components of D that intersect D_j. Since in any case $0 < \frac{i\nu}{p} - \left[\frac{i\nu}{p}\right] < 1$, there are chances for $H^0(D_j, \Omega_Y^2 \otimes \mathcal{L}^{(i)} \otimes \mathcal{O}_{D_j})$ to be non-zero.

By Serre's duality, we have $h^1(D_j, \Omega_Y^2 \otimes \mathcal{L}^{(i)} \otimes \mathcal{O}_{D_j}) = h^0(D_j, \mathcal{O}_Y(D_j) \otimes \mathcal{L}^{(i)^{-1}} \otimes \mathcal{O}_{D_j})$. If $D_j^2 < 0$, we have $\deg_{D_j}\left(\mathcal{O}_Y(D_j) \otimes \mathcal{L}^{(i)^{-1}}\right) < 0$, and so it is expected to vanish in general.

The following long exact sequence was inspired by Catanese's article [15, p. 497]. We consider the sum of **(*)** for $i \in \{0, 1, \ldots, p-1\}$, dual cohomologies, and finally Serre's duality to obtain

(**)
$$\begin{aligned} 0 \to H^2(Y, f_*(\Omega_X^1 \otimes \Omega_X^2))^\vee \to H^0(X, f^*T_Y) \\ \to \bigoplus_{i=0}^{p-1} \bigoplus_{i\nu_j \neq -1 (\mathrm{mod}\ p)} H^0(D_j, \mathcal{O}_Y(D_j) \otimes \mathcal{L}^{(i)^{-1}} \otimes \mathcal{O}_{D_j}) \to H^1(Y, f_*(\Omega_X^1 \otimes \Omega_X^2))^\vee \\ \to H^1(X, f^*T_Y) \to \bigoplus_{i=0}^{p-1} \bigoplus_{i\nu_j \neq -1 (\mathrm{mod}\ p)} H^1(D_j, \mathcal{O}_Y(D_j) \otimes \mathcal{L}^{(i)^{-1}} \otimes \mathcal{O}_{D_j}) \\ \to H^2(X, T_X) \to H^2(X, f^*T_Y) \to 0. \end{aligned}$$

If the exceptional divisor of $f : X \to Y$ does not have (-2)-curves (e.g. when D_{red} is smooth), then $H^2(Y, f_*(\Omega_X^1 \otimes \Omega_X^2))^\vee \simeq H^0(T_X)$ and $H^1(Y, f_*(\Omega_X^1 \otimes \Omega_X^2))^\vee \simeq H^1(X, T_X)$ by Theorem VI.5. In this case, the above sequence is exactly a generalization of the sequence in [15, p. 497].

Are there p-th root covers over $\mathbb{P}^2$ *along (non-trivial) line arrangements with no* (-2)*-curves in their exceptional locus?* Negative-regular continued fractions $\frac{p}{q} =$

$[e_1, \ldots, e_s]$ with $e_i \neq 2$ for all i seem to be very scarce compared to p, and so such covers should be very special.

Example VI.9. (Deformation of a singular K3 surface) This example is to run the sequences above. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $D = \sum_{i=1}^{6} D_i$ with $D_i \sim \mathcal{O}_Y(1,0)$ for $i = 1,2,3$, and $D_i \sim \mathcal{O}_Y(0,1)$ for $i = 4,5,6$. We consider $\mathcal{L} = \mathcal{O}_Y(1,1)$ so that $\mathcal{L}^3 \simeq \mathcal{O}_Y(D)$, and the corresponding 3-th root cover $f : X \to Y$ along D. By using the formulas in Example IV.11, we have $\chi(X, O_X) = 2$ and $K_X^2 = 0$. Moreover, since X is simply connected, $q(X) = 0$. Also, by the formula in Section 4.1, $K_X \sim 0$. All in all, X is a K3 surface. We have an induced elliptic fibration $g : X \to \mathbb{P}^1$ which has 3 singular fibers of type IV^* (over each node of D, the Hirzebruch-Jung resolution produces two (-2)-curves). Hence, X is a singular K3 surface, i.e., it achieves the maximum Picard number 20 for a K3 surface.

We know the following numbers for any K3 surface [7, p. 311]: $h^0(T_X) = h^2(T_X) = 0$ and $h^1(T_X) = 20$. we want to see how these numbers fit in our sequences. First, we know that $T_Y = \mathcal{O}_Y(2,0) \oplus \mathcal{O}_Y(0,2)$, and so $T_Y \otimes {\mathcal{L}^{(1)}}^{-1} = T_Y \otimes \mathcal{L}^{-1} = \mathcal{O}_Y(1,-1) \oplus \mathcal{O}_Y(-1,1)$ and $T_Y \otimes {\mathcal{L}^{(2)}}^{-1} = T_Y \otimes \mathcal{L}^{-2} = \mathcal{O}_Y(0,-2) \oplus \mathcal{O}_Y(-2,0)$.

We compute $H^2(X, T_X)$ via the sequence **(**)**. We have for $i = 0,1$ that $H^1(D_j, \mathcal{O}_Y(D_j) \otimes {\mathcal{L}^{(i)}}^{-1} \otimes \mathcal{O}_{D_j}) = 0$ by Serre's duality and degrees, but for $i = 2$ the dimension of this cohomology group is 1. However, when $i = 2$ we have $i\nu_j \equiv -1 (\mathrm{mod}\ 3)$, and so this case does not appear in **(**)**. On the other hand, a straightforward computation shows $H^2(Y, T_Y \otimes {\mathcal{L}^{(i)}}^{-1}) = 0$, and so $H^2(X, f^*(T_Y)) = 0$. Therefore, $H^2(X, T_X) = 0$. Using Hirzebruch-Riemann-Roch and $H^0(X, T_X) = 0$, we obtain $H^1(X, T_X) = 20$.

The rest of the sequence gives us information about $H^a(Y, f_*(\Omega_X^1 \otimes \Omega_X^2))$ for

$a = 1, 2$. The group $H^0(D_j, \mathcal{O}_Y(D_j) \otimes {\mathcal{L}^{(i)}}^{-1} \otimes \mathcal{O}_{D_j})$ vanishes for $i = 1, 2$, and has dimension 6 when $i = 0$. Also, $H^1(Y, T_Y \otimes {\mathcal{L}^{(i)}}^{-1}) = 0$ for $i = 0, 1$ and it has dimension 2 for $i = 2$. We also have $H^0(Y, T_Y \otimes {\mathcal{L}^{(i)}}^{-1}) = 0$ for $i = 1, 2$ and it has dimension 6 when $i = 0$. This gives us that $h^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) = h^2(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) + 2$. If Remark VI.6 is true, then $h^2(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) = 0$ and $h^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) = 2$. Therefore, since $H^1_E(X, T_X)$ is the number of (-2)-curves in the exceptional locus (i.e., 18), we recover $H^1(X, T_X) = 2 + 18 = 20$.

CHAPTER VII

Further directions

7.1 Minimality and rigidity.

We use the notation of the previous Chapter. First, we would like Remark VI.6 to be true. Assume it holds, and that X is of general type. Then,

$$H^1(X,T_X) \simeq H^1_E(X,T_X) \oplus H^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X))^\vee \quad \text{and} \quad H^2(Y, f_*(\Omega^1_X \otimes \Omega^2_X)) = 0.$$

In this way, since $H^1_E(X,T_X)$ is known [90], we only need to work out $H^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X))^\vee$, and for that we may use the sequence

$$\begin{aligned}
\textbf{(**)}\ 0 \to H^0(X, f^*T_Y) &\\
\to \bigoplus_{i=0}^{p-1} \bigoplus_{i\nu_j \neq -1 (\mathrm{mod}\ p)} & H^0(D_j, \mathcal{O}_Y(D_j) \otimes {\mathcal{L}^{(i)}}^{-1} \otimes \mathcal{O}_{D_j}) \to H^1(Y, f_*(\Omega^1_X \otimes \Omega^2_X))^\vee \\
\to H^1(X, f^*T_Y) \to \bigoplus_{i=0}^{p-1} \bigoplus_{i\nu_j \neq -1 (\mathrm{mod}\ p)} & H^1(D_j, \mathcal{O}_Y(D_j) \otimes {\mathcal{L}^{(i)}}^{-1} \otimes \mathcal{O}_{D_j}) \\
\to H^2(X,T_X) \to H^2(X, f^*T_Y) \to 0. &
\end{aligned}$$

This would be a complete picture for deformations of p-th root covers for surfaces.

Question VII.1. *Let Y be a normal surface over $\mathbb{C}$ with only rational singularities, and let $\pi : X \to Y$ be the minimal resolution of Y. Let E be the exceptional divisor of π. In [12], Burns and Wahl considered the exact sequence of cohomologies*

$$\ldots \to H^0(X,T_X) \to H^0(X \backslash E, T_X) \to H^1_E(T_X) \xrightarrow{\alpha} H^1(X,T_X) \to H^1(X \backslash E, T_X) \to \ldots$$

to show that $H^1_E(T_X) \overset{\alpha}{\hookrightarrow} H^1(X, T_X)$. *This is proved by showing that the map* $H^0(X, T_X) \to H^0(X \setminus E, T_X)$ *is surjective.*

On the other hand, as we pointed out before, there is a Leray spectral sequence associated to π *and* $\Omega^1_X \otimes \Omega^2_X$, *which produces the exact sequence*

$$0 \to H^1(Y, \pi_*(\Omega^1_X \otimes \Omega^2_X)) \to H^1(X, \Omega^1_X \otimes \Omega^2_X) \overset{\beta}{\to} H^0(Y, R^1\pi_*(\Omega^1_X \otimes \Omega^2_X)) \to$$

$$H^2(Y, \pi_*(\Omega^1_X \otimes \Omega^2_X)) \to H^2(X, \Omega^1_X \otimes \Omega^2_X)$$

We know that $R^1\pi_*\big(\Omega^1_X \otimes \Omega^2_X\big) = H^1_E(X, T_X)^\vee$, *and by Serre's duality,* $H^1(X, \Omega^1_X \otimes \Omega^2_X) = H^1(X, T_X)^\vee$. *Is* β *the dual map of* α*?*

Another issues we would like to understand are minimality and (possible) rigidity of the surfaces X coming from rigid line arrangements. Let $\mathcal{A}$ be an arrangement of d lines on $\mathbb{P}^2_{\mathbb{C}}$, and let $(Y, \mathcal{A}')$ be the corresponding associated pair (end of Section 2.1). Let $g : Y \to \mathbb{P}^2_{\mathbb{C}}$ be the blow up at k-points of $\mathcal{A}$, with $k > 2$. We assume the rigidity condition:

$$D_i^2 < 0 \text{ for every } D_i \text{ in } \mathcal{A}'.$$

Let X be the limit random surfaces constructed in Theorem V.2, i.e., the multiplicities are randomly assigned and p is very large. Assume that these surfaces are of general type. Let X_0 be the minimal models of X. A proof of "quasi-minimality" of X, and some evidence about their rigidity, would follow from a positive answer to the question.

Question VII.2. *Is* $H^1(Y, \Omega^1_Y(\log D)^\vee \otimes \mathcal{L}^{(i)^{-1}}) = 0$ *for almost all* $i \in \{0, 1, \ldots, p-1\}$*? For almost all* i *means*

$$\frac{1}{p} h^1(Y, \Omega^1_Y(\log D)^\vee \otimes \mathcal{L}^{(i)^{-1}}) \to 0$$

as p *tends to infinity.*

A positive answer to this Question 7.1 would follow from the vanishing of $h^1(Y, T_Y \otimes \mathcal{L}^{(i)^{-1}})$ for almost all i. However, one can show that this is not true for rigid line arrangements.

Assuming that Question is true, one can prove that the surfaces X are "quasi-minimal". This means the following. Let $b(X)$ be the number of (-1)-curves one blows down to obtain X_0. Then, X is called quasi-minimal if $\frac{b(X)}{p} \to 0$ when $p \to \infty$. If the surfaces X are quasi-minimal, then $\frac{c_1^2(X_0)}{c_2(X_0)}$ is arbitrarily close to $\frac{\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')}$. In this way, we do not improve our records for Chern ratios by considering the minimal models of X. A direct proof of quasi-minimality may be possible by using the tools in Subsection 4.4.1, but so far seems too involved.

Also, the positivity of Question implies that we have a big difference between the first order deformation space of X and the number of equations defining $\mathrm{Def}(X)$, because $0 < h^1(X, T_X) <<< h^2(X, T_X)$. This may indicate rigidity. However, for general arrangements with random multiplicities, one can prove that $0 < h^1(X, T_X) <<< h^2(X, T_X)$ is true, but of course one can deform a general arrangement, and so obtain several (possible) non-trivial deformations of X. We want to remark that the existence of (-2)-curves always produce first order deformations [12], but it seems unknown if they induce one parameter deformations. Moreover, it seems unknown whether there exists a rigid surface with (-2)-curves.

7.2 3-nets and characteristic varieties.

In Section 3.5.2 we classified $(3, q)$-nets for $2 \leq q \leq 6$, being the new case $q = 6$. We saw in Chapter III that main classes of $q \times q$ Latin squares are in bijection with "combinatorial" $(3, q)$-nets [23]. The problem is whether these main classes are realizable as $(3, q)$-nets in $\mathbb{P}^2_{\mathbb{C}}$. For $q = 6$, we obtained that only nine of the twelve main

classes are realizable, and we found that these nine classes have distinguished properties among each other. There are $(3, q)$-nets for abelian and non-abelian groups, and also for Latin squares not coming from groups. Their moduli have different dimensions, and some of them may be defined strictly over $\mathbb{C}$ or $\mathbb{R}$. Also, they can be defined over $\mathbb{Q}$ (for an interesting example, see Quaternion nets in Subsection 3.5.4).

Question VII.3. *Is there a combinatorial characterization of the main classes of $q \times q$ Latin squares realizing $(3, q)$-nets in $\mathbb{P}^2_{\mathbb{C}}$? (see Subsection 3.5.4)*

One motivation to classify nets comes from topological invariants of the complement of line arrangements. Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be a line arrangement in $\mathbb{P}^2_{\mathbb{C}}$. An important and difficult problem is to compute $\pi_1 = \pi_1(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$. Some of the well-studied invariants of π_1 are the so called n-th characteristic varieties [57, 58], which we denote by $V_n(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$. These subvarieties of $\mathbb{C}^{*d-1} = \mathrm{Hom}\big(\pi_1/[\pi_1, \pi_1], \mathbb{C}^*\big)$ are unions of translated subtori. Their definition can be found in [58].

The relation with nets is via positive dimensional connected components of $V_n(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$, which contain the identity. In connection with Characteristic varieties, we have the n-th resonance varieties of $\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A}$ (for the definition, see for example [95, Section 2]), which we denote by $R_n(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$. A key fact is the following [59, 95].

Theorem VII.4. *For every positive dimensional component V of $V_n(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$ containing the identity, the tangent space of V at the identity is a component of $R_n(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$.*

It is known that every irreducible component of $R_n(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$ is defined by a subarrangement $\mathscr{B} \subseteq \mathcal{A}$ and a set of k-points $\mathcal{X}$ of $\mathscr{B}$ (see [59, 95, 30]). The set $\mathcal{X}$ induces a partition of $\mathscr{B}$ into $n+2$ subarrangements, and every point in $\mathcal{X}$ has multiplicity $\geq n+2$. It turns out that they are exactly the (n, q)-multinets defined in

[30], where (n, q)-nets (Section 3.5) are particular cases. More precisely, in [30], it is proved the following.

Theorem VII.5. *There is a one to one correspondence between the components of* $R_1(\mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{A})$ *and multinets contained in* $\mathcal{A}$.

Therefore, (n, q)-multinets are important for the classification of the positive dimensional Characteristic varieties of complex line arrangements. We notice that trivial $(n, 1)$-multinets almost always show up as components in Theorem VII.5, for every $n \geq 3$. However, if $q > 1$, we have the restriction $n \leq 5$ [59, 95, 30]. In [30, Remark 4.11], it is pointed out that at the combinatorial level, every multinet can be obtained from a net by gluing some points and lines. But it is not known if the resulting combinatorial net is indeed realizable in $\mathbb{P}^2_{\mathbb{C}}$. Falk and Yuzvinsky conjecture that every multinet can be obtained by a deformation of a net. If true, we need to classify nets and their degenerations.

In [82], it is proved that for a (n, q)-net, n must be equal to 3 or 4. The only 4-net known is the Hesse arrangement, and it is believed there are no more 4-nets. This gives the motivation to classify 3-nets. Here we do not only have a realization problem, but also a combinatorial one given by the classification of the main classes of $q \times q$ Latin squares when q is large.

The combinatorial $(4, q)$-nets have to do with pairs of orthogonal $q \times q$ Latin squares. These pairs exist in general, but the realization may not. An illustrative example is given by $(4, 4)$-nets. Although this case is combinatorially possible, there are no $(4, 4)$-nets over $\mathbb{C}$ (see for example Section 3.5.1).

Question VII.6. *Are there* 4*-nets apart from the Hesse arrangement?*

In [82], the proof of the non-existence of 5-nets (and the attempt for 4-nets) did

not use the strong combinatorial restrictions imposed by Latin squares. For 3-nets, it is key to know the combinatorics given by the corresponding Latin square.

7.3 p-th root covers over algebraically closed fields.

Let $\mathbb{K}$ be an algebraically closed field. Let p be a prime number such that $p \neq \text{Char}(\mathbb{K})$. In this section, We will show how to obtain the analog of Theorem V.2 for this more general setting.

In [29, p. 23-27], Esnault and Viehweg work out p-root covers for arbitrary algebraically closed fields $\mathbb{K}$, under the condition $p \neq \text{Char}(\mathbb{K})$. We will use the notation in Section 4.1, we will restrict to surfaces. Let $(Y, p, D = \sum_{j=1}^{r} \nu_j D_j, \mathcal{L})$ be the data for the corresponding p-th root cover. We always assume $0 < \nu_i < p$ for all i. Then, we have the chain of maps

$$\overline{Y} = \text{Spec}_Y\Big(\bigoplus_{i=0}^{p-1} {\mathcal{L}^{(i)}}^{-1}\Big) \to Y' = \text{Spec}_Y\Big(\bigoplus_{i=0}^{p-1} \mathcal{L}^{-i}\Big) \to Y$$

where the key part is the computation of the normalization [29]. As before, the line bundles $\mathcal{L}^{(i)}$ on Y are defined as

$$\mathcal{L}^i \otimes \mathcal{O}_Y\Big(-\sum_{j=1}^{r}\Big[\frac{\nu_j\ i}{n}\Big] D_j\Big)$$

for $i \in \{0, 1, ..., p-1\}$.

The construction shows that $\overline{Y}$ has only singularities of the type

$$T(p, \nu_i, \nu_j) := \overline{\text{Spec}\Big(\mathbb{K}[x, y, z]/(z^p - x^{\nu_i} y^{\nu_j})\Big)}$$

over all the nodes of D. The varieties $T(p, \nu_i, \nu_j)$ are affine toric surfaces. They correspond to pointed cones in a two dimensional lattice N. If q is the positive integer satisfying $\nu_i q + \nu_j \equiv 0 \text{mod}(p)$ and $0 < q < p$, then $T(p, \nu_i, \nu_j)$ is isomorphic to the affine toric variety defined by $(0, 1)$ and $(p, -q)$ [67, Ch.5]. The singularity

of $T(p, \nu_i, \nu_j)$ can be resolved by toric methods [67, Ch.5 p.5-8], obtaining the same situation as in the complex case. That is, the singularity is resolved by a chain of $\mathbb{P}^1$'s whose number and self-intersections are encoded in the negative-regular continued fraction $\frac{p}{q} = [e_1, \ldots, e_s]$.

The singularities of $\overline{Y}$ are rational, and the minimal toric resolutions produce the smooth projective surface X. To see this, let $\mathcal{Z}$ be the fundamental cycle of the singularity of $T(p, \nu_i, \nu_j)$ as defined in [2]. By definition, we have $\mathcal{Z} = \sum_{i=1}^{s} E_i$, where E_i's are the corresponding exceptional curves. In [2], it is proved that a normal singularity is rational if and only if $p_a(\mathcal{Z}) = 0$ (arithmetic genus of $\mathcal{Z}$ is zero). But $p_a(\mathcal{Z}) = p_a(\overline{\mathcal{Z}}) + s - 1$ [39, p. 298, Ex. 1.8(a)], and $p_a(\overline{\mathcal{Z}}) = 1 - s$, so the singularity is rational. As before, let us denote the composition of all maps by $f : X \to Y$.

We will now compute all the relevant numerical invariants of X, showing that we have the same results as for $\mathbb{C}$.

Euler characteristic: Since the singularities are rational, we have

$$\chi(X, \mathcal{O}_X) = p\chi(Y, \mathcal{O}_Y) + \frac{1}{2}\sum_{i=1}^{p-1} \mathcal{L}^{(i)}.(\mathcal{L}^{(i)} \otimes \omega_Y),$$

and so we can modify this formula as before to obtain the one in Proposition IV.7, which involves Dedekind sums (we change $e(D)$ by the corresponding combinatorial number).

First Chern number: As before, if K_X and K_Y are canonical divisors for X and Y respectively, local computations (which ese the fact $p \neq \mathrm{Char}(p)$) give us the $\mathbb{Q}$-numerical equivalence

$$K_X \equiv f^*\Big(K_Y + \Big(\frac{p-1}{p}\Big)\sum_{j=1}^{r} D_j\Big) + \Delta$$

where Δ is a $\mathbb{Q}$-divisor supported on the exceptional locus. The number of divisors and their self-intersections are the same as before, and so $c_1^2(X)$ is the number com-

puted when $\mathbb{K}$ was $\mathbb{C}$. Notice that this number includes the sums of e_i's over the nodes [1].

Second Chern number: For $\mathbb{K} = \mathbb{C}$, this was $e(X)$, the topological Euler characteristic of X. In general, $c_2(X)$ can be computed using the Hirzebruch-Riemann-Roch theorem [39, p. 432], in the form of Noether's formula. Hence, $c_2(X)$ can be expressed as in Proposition IV.10, changing Euler numbers by the corresponding combinatorial numbers.

Now, we consider divisible arrangements of d curves $\mathcal{A}$ on a smooth projective surface Z over $\mathbb{K}$. Let $g : Y \to Z$ be the corresponding SNC resolution of $\mathcal{A}$, and consider the log surface $(Y, \mathcal{A}')$ as in Section 5.1. The logarithmic Chern classes of $(Y, \mathcal{A}')$ are defined as $\bar{c}_i(Y, \mathcal{A}') := c_i(\Omega^1_Y(\log D)^\vee)$ for $i = 1, 2$.

First log Chern number: This is computed as before, being $\bar{c}_1^2(Y, \mathcal{A}') = (c_1(Y) + D_{\mathrm{red}})^2$. So, it can be written in combinatorial terms as in Section 2.1,

$$\bar{c}_1^2(Y, \mathcal{A}') = c_1^2(Z) - \sum_{i=1}^{d} C_i^2 + \sum_{k\geq 2}(3k-4)t_k + 4\sum_{i=1}^{d}(g(C_i)-1).$$

Second log Chern number: For $\mathbb{K} = \mathbb{C}$, one can prove $\bar{c}_2(Y, \mathcal{A}') = e(Y) - e(\mathcal{A}')$ by Hirzebruch-Riemann-Roch theorem (and Hodge decomposition to compute $c_2(Y) = e(Y)$, for example). We want to show that this is again the number when transformed to its combinatorial form, i.e.,

$$\bar{c}_2(Y, \mathcal{A}') = c_2(Z) + \sum_{k\geq 2}(k-1)t_k + 2\sum_{i=1}^{d}(g(C_i)-1).$$

As before, the Hirzebruch-Riemann-Roch theorem is valid over $\mathbb{K}$, and for the vector bundle $\Omega^1_Y(\log D)^\vee$ it reads

$$\chi(Y, \Omega^1_Y(\log D)^\vee) = \deg\Big(ch(\Omega^1_Y(\log D)^\vee).td(T_Y)\Big).$$

[1] The formula in Proposition IV.13 shows a correspondence between χ, c_1^2, and c_2 and Dedekind sums, sums of e_i's, and length of continued fractions respectively.

The left hand side is $\sum_{j=1}^{r}\chi(D_j,\Omega_Y^2\otimes\mathcal{O}_{D_j})+\chi(Y,T_Y)$ by the "canonical" log sequence. So, by applying Riemann-Roch theorem twice, we can compute $\chi(Y,\Omega_Y^1(\log D)^\vee)$ in terms of intersection numbers and Chern numbers. Since we have the same participants as for $\mathbb{C}$, the result follows.

Theorem VII.7. *Let $\mathbb{K}$ be an algebraically closed field. Let Z be a smooth projective surface over $\mathbb{K}$, and $\mathcal{A}$ be a divisible arrangement on Z. Let $(Y,\mathcal{A}')$ be the corresponding associated pair, and assume $\bar{c}_2(Y,\mathcal{A}')\neq 0$. Then, there are smooth projective surfaces X having $\frac{c_1^2(X)}{c_2(X)}$ arbitrarily close to $\frac{\bar{c}_1^2(Y,\mathcal{A}')}{\bar{c}_2(Y,\mathcal{A}')}$.*

Proof. We consider primes $p >$Char($\mathbb{K}$), and we use the exact same proof as for Theorem V.2. □

Positive characteristic brings more geometric possibilities for arrangements. It is immediately clear when we consider line arrangements in $\mathbb{P}^2_{\mathbb{K}}$.

Example VII.8. (Projective plane arrangements) Let $\mathbb{K}$ be an algebraically closed field of characteristic $n>0$. In $\mathbb{P}^2_{\mathbb{K}}$, we have n^2+n+1 $\mathbb{F}_n$-valued points, and there are n^2+n+1 lines such that through each of these points passes exactly $n+1$ of these lines, and each of these line contains exactly $n+1$ of these points [24, p. 426]. These lines define an arrangement of $d=n^2+n+1$ lines $\mathcal{A}$, we call them projective plane arrangements. When $n=2$, this is the Fano arrangement. We have that $t_{n+1}=n^2+n+1$ and $t_k=0$ otherwise (by the combinatorial equality), and the log Chern numbers are

$$\bar{c}_1^2(Y,\mathcal{A}')=3(n+1)(n-1)^2 \quad \text{and} \quad \bar{c}_2(Y,\mathcal{A}')=(n+1)(n-1)^2,$$

and so $\bar{c}_1^2=3\bar{c}_2$ for every n.

By Theorem VII.7, there are smooth projective surfaces X with $\frac{c_1^2(X)}{c_2(X)}\to 3$. These surfaces are of general type because $\bar{c}_1^2(Y,\mathcal{A}'),\bar{c}_2(Y,\mathcal{A}')>0$, and there is a classifica-

tion for surfaces in positive characteristic [9, p. 119-120]. Hence, these are surfaces arbitrarily close to the Miyaoka-Yau line. However, it is known there are surfaces with $c_1^2 > 3c_2$ (for example, see [26]).

This example leads us to wonder: is there a line arrangement in $\mathbb{P}^2_{\mathbb{K}}$ with $\bar{c}_1^2 > 3\bar{c}_2$? The answer is no.

Proposition VII.9. *Let $\mathbb{K}$ be an algebraically closed field, and let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2_{\mathbb{K}}$ with $t_d = 0$.*

1. *If Char$(\mathbb{K}) > 0$, then $\bar{c}_1^2 \leq 3\bar{c}_2$. This inequality is sharp because of the projective plane arrangements.*

2. *If Char$(\mathbb{K}) = 0$, then $\bar{c}_1^2 \leq \frac{8}{3}\bar{c}_2$. Equality holds if and only if $\mathcal{A}$ is a triangle, or $t_{d-1} = 1$ or $\mathcal{A}$ is the dual Hesse arrangement.*

Proof. Part 2. follows from Lefschetz's Principle [56] and Proposition II.8. We remark that this proposition involves Hirzebruch's inequality which relies on Miyaoka-Yau inequality. The inequality for positive characteristic is much more simple, as we now show.

We first notice that $\bar{c}_1^2 \leq 3\bar{c}_2$ is equivalent to $\sum_{k\geq 2} t_k \geq d$, because of our formulas in Section 2.2. Let $\sigma : \mathrm{Bl}_{k-\mathrm{pts}}(\mathbb{P}^2_{\mathbb{K}}) \to \mathbb{P}^2_{\mathbb{K}}$ be the blow up of $\mathbb{P}^2_{\mathbb{K}}$ at all the k-points of $\mathcal{A}$ (2-points included). Then, $\mathrm{Pic}(\mathrm{Bl}_{k-\mathrm{pts}}(\mathbb{P}^2_{\mathbb{K}})) \otimes \mathbb{Q}$ has dimension $1 + \Sigma_{k\geq 2} t_k$ [39, Ch. V.3]. Assume $\sum_{k\geq 2} t_k < d$.

Let $\{L_1, \ldots, L_d\}$ be the proper transforms of the lines in $\mathcal{A}$, and let H be the class of the pull-back of a general line. Since $t_d = 0$, we have $L_i^2 \leq -1$ for all i. Also, for $i \neq j$, $L_i.L_j = 0$. Therefore, they are linearly independent in $\mathrm{Pic}(\mathrm{Bl}_{k-\mathrm{pts}}(\mathbb{P}^2_{\mathbb{K}})) \otimes \mathbb{Q}$, and since $\sum_{k\geq 2} t_k < d$, they form a base (so $d = 1 + \Sigma_{k\geq 2} t_k$). In this way, there exist

$\alpha_i \in \mathbb{Q}$ such that

$$H = \Sigma_{i=1}^{d} \alpha_i L_i.$$

We have $L_i.H = 1 = \alpha_i L_i^2$ and $H.H = 1 = \Sigma_{i=1}^{d} \alpha_i$, and so $1 = \Sigma_{i=1}^{d} \frac{1}{L_i^2}$. But $L_i^2 \leq -1$, producing a contradiction. □

I think the projective plane arrangements are exactly the ones for which equality holds.

7.4 Coverings and geometric normalizations.

The leading idea is how we could modify coverings, so that we can apply similar large scale techniques, to find simply connected surfaces with large Chern numbers ratio. Hopefully, we want higher than $2.\overline{703}$. We start with a covering data on a smooth projective surface Y, construct an algebra representing the covering we want to perform, and compute normalization and resolution of singularities. That is what we have for p-th root covers. Below we discuss two situations which are manageable thanks to some known works.

Abelian covers: In [71], Pardini developed a general theory for abelian covers of algebraic varieties. Let Y be a smooth complete variety over an algebraically closed field $\mathbb{K}$, and let G be a finite abelian group. An abelian cover of Y with group G is a finite map $\pi : X \to Y$, where X is a normal variety, together with a faithful action of G on X such that π exhibits Y as the quotient of X via G. The main theorem in [71] says that abelian covers are determined, up to isomorphisms of G-covers, by certain building data L_χ (line bundles), $D_{H,\varphi}$ (divisors). Actual explicit constructions from this building data are a little difficult to carry out, because of the number of relations required and the general language used. It is possible to make it simpler by using a direct and more geometric method via p-th root covers,

following Esnault and Viehweg. This has been worked out in [31], we now explain that approach.

Let Y be a smooth projective surface over $\mathbb{K}$, and let $G \simeq \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_m\mathbb{Z}$. Then, the abelian cover data can be express as

$$\mathcal{L}_i^{n_i} \simeq \mathcal{O}_Y(D_i), \qquad i \in \{1, 2, \ldots, m\}$$

where L_i are line bundles and D_i are effective divisors on Y. If $\pi : X \to Y$ is the corresponding abelian cover, then

$$\pi_* \mathcal{O}_X = \bigoplus_{g \in G} {\mathcal{L}_g}^{-1}$$

where

$$\mathcal{L}_g := \mathcal{L}_1^{g_1} \otimes \mathcal{O}_Y\Big(- \Big[\frac{g_1}{n_1} D_1\Big]\Big) \otimes \cdots \otimes \mathcal{L}_m^{g_m} \otimes \mathcal{O}_Y\Big(- \Big[\frac{g_m}{n_m} D_m\Big]\Big),$$

$g = (g_1, \ldots, g_m) \in \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_m\mathbb{Z}$, and $\left[\frac{g_i}{n_i} D_i\right]$ is the sum of the integral parts of each primary divisor summand. Hence, the $m = 1$ case is the n_1-root cover associated to $(Y, D_1, n_1, \mathcal{L}_1)$ in Section 4.1. The key line bundles are the $\mathcal{L}_g$'s.

Let $\pi : X \to Y$ be an abelian cover. Assume that $\big(D_1 + \ldots + D_m\big)_{\text{red}}$ is a simple normal crossings divisor. Then, the singularities of X are of Hirzebruch-Jung type [71, Prop. 3.3], in particular rational singularities. This allows us to easily compute χ of the resolution of X. Also, it gives hopes on controlling its fundamental group. However, if G is not cyclic, the cover is not totally branched along any component of the D_i's, and it usually modifies their genus. This changes the situation we had before to prove simply connectedness. It seems that with non-cyclic covers, one almost never obtains simply connected surfaces. This is an example.

Example VII.10. (Hirzebruch's construction) In [42], Hirzebruch constructed very interesting surfaces, such as ball quotients, by using abelian covers of a certain kind.

Given an arrangement of lines $\mathcal{A} = \{L_1, \ldots, L_d\}$ in $\mathbb{P}^2_{\mathbb{C}}$ with $t_d = 0$, he considered abelian covers of $\mathbb{P}^2_{\mathbb{C}}$ with group $\mathbb{Z}/n\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n\mathbb{Z}$ ($d-1$ times), given by the data

$$\mathcal{O}_{\mathbb{P}^2}(1)^n = \mathcal{O}_{\mathbb{P}^2}(L_i + (n-1)L_d)$$

for $i \in \{1, 2, \ldots, d-1\}$. If $n > 2$ and there is a k-point P in $\mathcal{A}$ with $k \geq 3$, then the corresponding new smooth projective surface X has $q(X) > 0$, and so it is not simply connected. To see this, we notice that the singularity over P is resolved by a smooth projective curve C whose topological Euler characteristic is $n^{k-2}(n(2-k)+k) \leq 0$ [42, p. 122], and so $g(C) > 0$. This curve induces a fibration on X over C via the trivial pencil of lines through P. In particular, $q(X) \geq g(C) > 0$.

In that paper, he produced three surfaces of general type satisfying $c_1^2 = 3c_2$. The corresponding arrangements were the complete quadrilateral ($n = 5$), Hesse arrangement ($n = 3$), and dual Hesse arrangement ($n = 5$).

Bring-Jerrard covers: Cyclic covers are made out of gluing local data of the form $z^n + a = 0$, plus normalization. If instead we considered local data of the form $z^n + az + b = 0$, we obtain the so-called Bring-Jerrad covers. They have been recently studied by Tan and Zhang in [84]. They described their normalization from the cover data, generalizing the situation of cyclic covers. A good property of these coverings is that we often have total ramification along certain divisors. Also, we can consider extremal cases as n tends to infinity. Non-abelian coverings have been scarcely used to produce exotic surfaces. We would like to exploit Bring-Jarred covers in the spirit of p-root covers, to see if they bring something new to our search of simply connected surfaces with large Chern numbers ratio.

7.5 Upper bounds for log Chern ratios of divisible arrangements.

In this thesis, we have found the record number $\frac{8}{3}$ for log Chern ratios of divisible arrangements. In principle, we know that $\bar{c}_1^2 \leq 3\bar{c}_2$ holds for divisible arrangements (see Remark V.3), and so this record could be improved. The Log Chern numbers ratio $\frac{8}{3}$ has been achieved by rather elegant arrangements: dual Hesse arrangement (Example II.6), 4-Fermat arrangement (Section 5.4), Hirzebruch's elliptic arrangements (Subsection 2.5.5), and the 64 lines on the Schur quartic (Example II.21). We saw that for line arrangements in $\mathbb{P}^2_{\mathbb{C}}$, this number is actually an upper bound, and it is only achieved by the dual Hesse arrangement.

Question VII.11. *Is $\frac{8}{3}$ an upper bound for log Chern ratios of divisible arrangements?*

We remark that, for arbitrary arrangements, the set of limits of log Chern ratios contains $[\frac{1}{5}, 3]$ (Theorem I.29), and so it seems unlikely a positive answer to Question VII.11[2]. Curiously, the Chern numbers ratio $\frac{8}{3}$ has recently appeared as an upper bound for double Kodaira fibrations [18], and as a sufficient condition (assuming ampleness of the canonical class) to have small first order deformation space [75] (see Theorem I.5).

In any how, the open problem of simply connected surfaces with Chern numbers ratio higher than $2.\overline{703}$ together with our method gives a good excuse to search for Chern-beautiful arrangements. A starting point is to explore more plane arrangements, in particular conic-line (simple crossings) arrangements in $\mathbb{P}^2_{\mathbb{C}}$. Conic-line arrangements would produce simply connected surfaces, in resemblance with line arrangements. We have examples of conic-line arrangements with large log Chern

[2] Although, the arrangements Sommese used to prove this fact were rather special, collections of fibers of a fibration.

numbers ratio, but not higher than $\frac{8}{3}$ (see Subsection 2.5.3).

Question VII.12. *Consider a simple crossings arrangement of n_2 conics and n_1 lines in $\mathbb{P}^2_{\mathbb{C}}$. Is it true that $8n_2 - n_1 + \sum_{k\geq 2}(4-k)t_k - 3 \geq 0$.?*

Another interesting issue is the possible topological nature of log Miyaoka-Yau inequalities for divisible arrangements. In Section 2.2, we saw that for real line arrangements in $\mathbb{P}^2_{\mathbb{R}}$, one easily obtains the log Miyaoka-Yau inequality $\bar{c}_1^2 \leq 2.5\bar{c}_2$ from the induced cell decomposition of $\mathbb{P}^2_{\mathbb{R}}$. Equality is achieved by simplicial arrangements, this is, arrangements which produce only triangles in the corresponding cell decomposition.

Question VII.13. *For the case of line arrangements over $\mathbb{C}$, is there a topological proof for the inequality $\bar{c}_1^2 \leq \frac{8}{3}\bar{c}_2$? Are there topological reasons for the dual Hesse arrangement to be the unique non-trivial arrangement satisfying equality?*

Let Z be a smooth projective surface over $\mathbb{C}$. As we said above, for divisible arrangements on Z we have $\bar{c}_1^2 \leq 3\bar{c}_2$, but this may be too coarse. What are the sharp log Miyaoka-Yau inequalities for divisible arrangements on Z? This question is interesting even if we restrict to certain divisible arrangements. For example, divisible arrangements $\mathcal{A}$ whose members belong to the same line bundle $\mathcal{L}$, that is, for every $C \in \mathcal{A}$, $\mathcal{O}_Z(C) \simeq \mathcal{L}$. What are the sharp log Miyaoka-Yau inequalities for these arrangements when we fix $\mathcal{L}$? The example is line arrangements on $\mathbb{P}^2_{\mathbb{C}}$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$ and the sharp inequality is $\bar{c}_1^2 \leq \frac{8}{3}\bar{c}_2$.

APPENDICES

.1 Dedekind sums and continued fractions.

Most of the material in this appendix can be found in several places. Let p be a prime number and q be an integer satisfying $0 < q < p$. As in definition IV.6, we write the Dedekind sum associated to q, p as

$$s(q,p) = \sum_{i=1}^{p-1} \Big(\Big(\frac{i}{p}\Big)\Big)\Big(\Big(\frac{iq}{p}\Big)\Big)$$

where $((x)) = x - [x] - \frac{1}{2}$ for any rational number x. On the other hand, we have the (negative-regular) continued fraction

$$\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \frac{1}{e_s}}}$$

which we abbreviate as $\frac{p}{q} = [e_1, ..., e_s]$. We denote its length s by $l(1, p-q; p)$, following our notation (Definition IV.8). This continued fraction is defined by the following recursion formula: let $b_{-1} = p$ and $b_0 = q$, and define e_i and b_i by means of the equation $b_{i-2} = b_{i-1}e_i - b_i$ with $0 \leq b_i < b_{i-1}$ $i \in \{1, 2, \ldots, s\}$. In this way,

$$b_s = 0 < b_{s-1} = 1 < b_{s-2} < \ldots < b_1 < b_0 = q < b_{-1} = p.$$

In particular, $l(1, p-q; p) < p$. By induction one can prove that for every $i \in \{1, 2, \ldots, s\}$, $b_{i-2} = (-1)^{s+1-i} \det(M_i)$ where M_i is the matrix

$$\begin{pmatrix} -e_i & 1 & 0 & 0 & \ldots & 0 \\ 1 & -e_{i+1} & 1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \ldots & 0 & 1 & -e_{s-1} & 1 \\ 0 & \ldots & 0 & 0 & 1 & -e_s \end{pmatrix}$$

Hence, $s = p - 1$ if and only if $e_i = 2$ for all i.

Another well-known way to look at this continued fraction is the following. Let $\frac{p}{q} = [e_1, ..., e_s]$ and define the matrix

$$A(e_1, e_2, ..., e_s) = \begin{pmatrix} e_s & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e_{s-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} e_1 & 1 \\ -1 & 0 \end{pmatrix}$$

and the recurrences $P_{-1} = 0$, $P_0 = 1$, $P_{i+1} = e_{i+1}P_i - P_{i-1}$; $Q_{-1} = -1$, $Q_0 = 0$, $Q_{i+1} = e_{i+1}Q_i - Q_{i-1}$. Then, by induction again, one can prove that $\frac{P_i}{Q_i} = [e_1, e_2, ..., e_i]$ and

$$A(e_1, ..., e_i) = \begin{pmatrix} P_i & Q_i \\ -P_{i-1} & -Q_{i-1} \end{pmatrix}$$

for all $i \in \{1, 2, ..., s\}$. The following lemma can be proved using that $\det\big(A(e_1, ..., e_i)\big) = 1$.

Lemma .14. *Let p be a prime number and q be an integer such that $0 < q < p$. Let q' be the integer satisfying $0 < q' < p$ and $qq' \equiv 1 (mod\ p)$. Then, $\frac{p}{q} = [e_1, ..., e_s]$ implies $\frac{p}{q'} = [e_s, ..., e_1]$.*

We now express the number $\alpha_i = -1 + \frac{b_{i-1}}{p} + \frac{b'_{s-i}}{p}$ in terms of P_i's and Q_i's, to finally prove Proposition IV.13. Since

$$A(e_1, ..., e_s) = \begin{pmatrix} b_{i-1} & b_i \\ x & y \end{pmatrix} \begin{pmatrix} e_i & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} P_{i-1} & Q_{i-1} \\ -P_{i-2} & -Q_{i-2} \end{pmatrix}$$

we have $b_{i-1} = qP_{i-1} - pQ_{i-1}$ and $b'_{s-i} = P_{i-1}$.

Lemma .15. $\sum_{i=1}^{s} \alpha_i(2 - e_i) = \sum_{i=1}^{s}(e_i - 2) + \frac{q+q'}{p} - 2\frac{p-1}{p}$.

Proof. $\sum_{i=1}^{s} \alpha_i(2 - e_i) = \sum_{i=1}^{s}(e_i - 2) + \frac{1}{p}\sum_{i=1}^{s} \left((q+1)P_{i-1} - pQ_{i-1}\right)(2 - e_i)$. By definition, $e_iP_{i-1} = P_i + P_{i-2}$ and $e_iQ_{i-1} = Q_i + Q_{i-2}$, so

$$\sum_{i=1}^{s} \left((q+1)P_{i-1} - pQ_{i-1}\right)(2-e_i) = (q+1)\sum_{i=1}^{s}(2P_{i-1} - P_i - P_{i-2}) - p\sum_{i=1}^{s}(2Q_{i-1} - Q_i - Q_{i-2}).$$

$\sum_{i=1}^{s}(2P_{i-1}-P_i-P_{i-2}) = 1+P_{s-1}-p$ and $\sum_{i=1}^{s}(2Q_{i-1}-Q_i-Q_{i-2}) = -1+Q_{s-1}-q$, so

$$\sum_{i=1}^{s}\left((q+1)P_{i-1}-pQ_{i-1}\right)(2-e_i) = q+P_{s-1}+2-2p$$

since $qP_{s-1}-pQ_{s-1}=1$. □

Lemma .16. $0 \leq -\alpha_i \leq \frac{p-2}{p}$.

Proof. The statement $-\alpha_i \leq \frac{p-2}{p}$ is clear. For the left hand side, we need to prove $1-\frac{qP_{i-1}-pQ_{i-1}}{p}-\frac{P_{i-1}}{p} \geq 0$. This is equivalent to $\frac{p}{q}-\frac{Q_{i-1}}{P_{i-1}} \leq \frac{1}{P_i}-\frac{1}{p}$. For $i=s$, we have $P_{s-1}q-Q_{s-1}p=1$, and we know that for every j, $P_{j+1} \geq P_j+1$. So, we prove it by induction on i. Since $\frac{Q_i}{P_i}-\frac{Q_{i-1}}{P_{i-1}} = \frac{1}{P_iP_{i-1}}$, we have

$$\frac{q}{p}-\frac{Q_{i-1}}{P_{i-1}} = \frac{q}{p}-\frac{Q_i}{P_i}+\frac{1}{P_iP_{i-1}} \leq \frac{1}{P_i}-\frac{1}{p}+\frac{1}{P_iP_{i-1}} \leq \frac{1}{P_{i-1}}-\frac{1}{p}$$

by the previous remark. □

We are now going to describe the behavior of Dedekind sums and lengths of continued fractions when p is large and q does not belong to a certain bad set. All of what follows relies on the work of Girstmair (see [33] and [34]).

Definition .17. (from [33]) A Farey point (F-point) is a rational number of the form $p\cdot\frac{c}{d}$, $1 \leq d \leq \sqrt{p}$, $0 \leq c \leq d$, $(c,d)=1$. Fix an arbitrary constant $C>0$. The interval

$$I_{\frac{c}{d}} = \{x:\ 0 \leq x \leq p,\ \left|x-p\cdot\frac{c}{d}\right| \leq C\frac{\sqrt{p}}{d^2}\}$$

is called the F-neighbourhood of the point $p\cdot\frac{c}{d}$. We write $\mathcal{F}_d = \bigcup_{c\in\mathcal{C}} I_{\frac{c}{d}}$ for the union of all neighbourhoods belonging to F-points of a fixed d, where $\mathcal{C} = \{c:\ 0 \leq c \leq d\ \&\ (c,d)=1\}$. The bad set $\mathcal{F}$ is defined as

$$\mathcal{F} = \bigcup_{1\leq d\leq\sqrt{p}} \mathcal{F}_d.$$

The integers q, $0 \leq q < p$, lying in $\mathcal{F}$ are called F-neighbours. Otherwise, q is called an ordinary integer.

The following two theorems are stated and proved in [33].

Theorem .18. *Let $p \geq 17$ and q be an ordinary integer. Then, $|s(q,p)| \leq \big(2 + \frac{1}{C}\big)\sqrt{p} + 5$.*

The previous theorem is false for non-ordinary integers. For example, if $\frac{p+1}{m}$ is an integer, then $s(\frac{p+1}{m}, p) = \frac{1}{12mp}\big(p^2 + (m^2 - 6m + 2)p + m^2 + 1\big)$.

Theorem .19. *For each $p \geq 17$ the number of F-neighbours is $\leq C\sqrt{p}\big(\log(p) + 2\log(2)\big)$.*

A similar statement is true for the lengths of negative-regular continued fractions.

Theorem .20. *Let q be an ordinary integer and $\frac{p}{q} = [e_1, e_2, \ldots, e_s]$ be the corresponding continued fraction. Then, $s = l(1, p-q; p) \leq \big(2 + \frac{1}{C}\big)\sqrt{p} + 2$.*

Proof. For any integers $0 < n < m$ with $(n, m) = 1$ consider the regular continued fraction

$$\frac{n}{m} = f_1 + \cfrac{1}{f_2 + \cfrac{1}{\ddots_{+\frac{1}{f_r}}}}$$

and let us denote $\sum_{i=1}^{r} f_i$ by $t(n, m)$. Also, we write $\frac{n}{m} = [1, a_2, \ldots, a_{l'(n,m)}]$ for its negative-regular continued fraction. Observe that $\frac{p}{q} - \big[\frac{p}{q}\big] = \frac{x}{q} = [1, e_2, \ldots, e_s]$. By [68] corollary (iv), we have

$$t(q,p) = \left[\frac{p}{q}\right] + t(x,q) = \left[\frac{p}{q}\right] + l'(x,q) + l'(q-x,q) > l'(x,q) = s.$$

Now, by Proposition 3 in [34], we know that $t(q,p) \leq \big(2 + \frac{1}{C}\big)\sqrt{p} + 2$ whenever q is an ordinary integer. □

.2 Samples from the Fermat program.

Below we show some samples taken from the fermat program, which computes the exact invariants of p-th root covers over $\mathbb{P}^2$ along Fermat arrangements (Example II.6). This program was written using TURBO C++.

(1) pth root covers along q Fermat arrangements.

Enter q $(1 < q < 7)$: 3

Enter a prime number p : 1019

The multiplicities of the sections are integers $0 < v_i < 1019$ such

that $v_1 + ... + v_9 = 0 \pmod{1019}$.

Enter the multiplicities for the sections:

$v_1 = 1$

$v_2 = 3$

$v_3 = 7$

$v_4 = 17$

$v_5 = 29$

$v_6 = 47$

$v_7 = 109$

$v_8 = 239$

$v_9 = 567$

$\chi(X) = 2857 = 2803.746811 + 53.253189$

$c_1^2(X) = 24024$

$c_2(X) = 10260 = 9177 + 1083$

$e(D) = 6$

The ratio $c_1^2(X)/\chi(X)$ is 8.40882.

The Chern numbers ratio $c_1^2(X)/c_2(X)$ is 2.34152.

Self-intersections of pull-backs are:

$\widetilde{D}_1^2 = -3$ $\widetilde{D}_2^2 = -2$ $\widetilde{D}_3^2 = -2$ $\widetilde{D}_4^2 = -2$

$\widetilde{D}_5^2 = -2$ $\widetilde{D}_6^2 = -2$ $\widetilde{D}_7^2 = -2$ $\widetilde{D}_8^2 = -2$

$\widetilde{D}_9^2 = -3$

$\widetilde{D}_{10}^2 = -2$ $\widetilde{D}_{11}^2 = -1$ $\widetilde{D}_{12}^2 = -1$ $\widetilde{D}_{13}^2 = -2$

$\widetilde{D}_{14}^2 = -1$ $\widetilde{D}_{15}^2 = -2$ $\widetilde{D}_{16}^2 = -2$ $\widetilde{D}_{17}^2 = -2$

$\widetilde{D}_{18}^2 = -1$

$\widetilde{D}_{19}^2 = -1$ $\widetilde{D}_{20}^2 = -1$ $\widetilde{D}_{21}^2 = -1$

(2) pth root covers along q Fermat arrangements.

Enter q $(1 < q < 7)$: 3

Enter a prime number p : 145777

The multiplicities of the sections are integers $0 < v_i < 145777$ such that $v_1 + ... + v_9 = 0 \pmod{145777}$.

Enter the multiplicities for the sections:

$v_1 = 1$

$v_2 = 101$

$v_3 = 207$

$v_4 = 569$

$v_5 = 1069$

$v_6 = 10037$

$v_7 = 22441$

$v_8 = 44729$

$v_9 = 66623$

$\chi(X) = 400908 = 400888.249978 + 19.750032$

$c_1^2(X) = 3497491$

$c_2(X) = 1313405 = 1311999 + 1406$

$e(D) = 6$

The ratio $c_1^2(X)/\chi(X)$ is 8.723924.

The Chern numbers ratio $c_1^2(X)/c_2(X)$ is 2.662919.

Self-intersections of pull-backs are:

$\widetilde{D}_1^2 = -3$ $\widetilde{D}_2^2 = -2$ $\widetilde{D}_3^2 = -1$ $\widetilde{D}_4^2 = -1$

$\widetilde{D}_5^2 = -2$ $\widetilde{D}_6^2 = -2$ $\widetilde{D}_7^2 = -2$ $\widetilde{D}_8^2 = -3$

$\widetilde{D}_9^2 = -2$

$\widetilde{D}^2_{10} = -2$ $\widetilde{D}^2_{11} = -2$ $\widetilde{D}^2_{12} = -1$ $\widetilde{D}^2_{13} = -1$

$\widetilde{D}^2_{14} = -2$ $\widetilde{D}^2_{15} = -2$ $\widetilde{D}^2_{16} = -2$ $\widetilde{D}^2_{17} = -3$

$\widetilde{D}^2_{18} = -1$

$\widetilde{D}^2_{19} = -2$ $\widetilde{D}^2_{20} = -1$ $\widetilde{D}^2_{21} = -1$

(3) pth root covers along q Fermat arrangements.

Enter q $(1 < q < 7)$: 6

Enter a prime number p : 11239

The multiplicities of the sections are integers $0 < v_i < 11239$ such that $v_1 + ... + v_{18} = 0$ (mod 11239).

Enter the multiplicities for the sections:

$v_1 = 1$

$v_2 = 13$

$v_3 = 17$

$v_4 = 23$

$v_5 = 29$

$v_6 = 37$

$v_7 = 53$

$v_8 = 79$

$v_9 = 89$

$v_{10} = 139$

$v_{11} = 157$

$v_{12} = 317$

$v_{13} = 439$

$v_{14} = 641$

$v_{15} = 919$

$v_{16} = 1223$

$v_{17} = 2689$

$v_{18} = 4374$

$\chi(X) = 182792 = 182630.74891 + 161.25109$

$c_1^2(X) = 1580908$

$c_2(X) = 612596 = 606894 + 5702$

$e(D) = -12$

The ratio $c_1^2(X)/\chi(X)$ is 8.648672.

The Chern numbers ratio $c_1^2(X)/c_2(X)$ is 2.58067.

Self-intersections of pull-backs are:

$\widetilde{D}_1^2 = -6$ $\widetilde{D}_2^2 = -3$ $\widetilde{D}_3^2 = -3$ $\widetilde{D}_4^2 = -5$

$\widetilde{D}_5^2 = -2$ $\widetilde{D}_6^2 = -5$ $\widetilde{D}_7^2 = -2$ $\widetilde{D}_8^2 = -3$

$\widetilde{D}_9^2 = -3$ $\widetilde{D}_{10}^2 = -5$ $\widetilde{D}_{11}^2 = -4$ $\widetilde{D}_{12}^2 = -2$

$\widetilde{D}_{13}^2 = -3$ $\widetilde{D}_{14}^2 = -5$ $\widetilde{D}_{15}^2 = -4$ $\widetilde{D}_{16}^2 = -5$

$\widetilde{D}_{17}^2 = -4$ $\widetilde{D}_{18}^2 = -3$

$\widetilde{D}_{19}^2 = -1$ $\widetilde{D}_{20}^2 = -2$ $\widetilde{D}_{21}^2 = -2$ $\widetilde{D}_{22}^2 = -2$

$$\widetilde{D}^2_{23} = -1 \ \widetilde{D}^2_{24} = -1 \ \widetilde{D}^2_{25} = -2 \ \widetilde{D}^2_{26} = -2$$

$$\widetilde{D}^2_{27} = -2 \ \widetilde{D}^2_{28} = -1 \ \widetilde{D}^2_{29} = -1 \ \widetilde{D}^2_{30} = -2$$

$$\widetilde{D}^2_{31} = -1 \ \widetilde{D}^2_{32} = -2 \ \widetilde{D}^2_{33} = -1 \ \widetilde{D}^2_{34} = -1$$

$$\widetilde{D}^2_{35} = -2 \ \widetilde{D}^2_{36} = -2 \ \widetilde{D}^2_{37} = -1 \ \widetilde{D}^2_{38} = -2$$

$$\widetilde{D}^2_{39} = -1 \ \widetilde{D}^2_{40} = -1 \ \widetilde{D}^2_{41} = -1 \ \widetilde{D}^2_{42} = -2$$

$$\widetilde{D}^2_{43} = -1 \ \widetilde{D}^2_{44} = -1 \ \widetilde{D}^2_{45} = -2 \ \widetilde{D}^2_{46} = -1$$

$$\widetilde{D}^2_{47} = -1 \ \widetilde{D}^2_{48} = -1 \ \widetilde{D}^2_{49} = -1 \ \widetilde{D}^2_{50} = -1$$

$$\widetilde{D}^2_{51} = -2 \ \widetilde{D}^2_{52} = -2 \ \widetilde{D}^2_{53} = -1 \ \widetilde{D}^2_{54} = -2$$

$$\widetilde{D}^2_{55} = -2 \ \widetilde{D}^2_{56} = -2 \ \widetilde{D}^2_{57} = -3$$

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