The geometry of birationally commutative graded domains

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2008

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To Staša, who made it possible

ACKNOWLEDGEMENTS

Many people have contributed to my education as a mathematician and to this thesis. To begin, I would like to thank the University of Michigan Mathematics Department, and particularly former graduate chair Juha Heinonen, for welcoming me twice as a graduate student. During the writing of this thesis I was supported by a National Science Foundation graduate fellowship and by NSF grants DMS-0555750 and DMS-0502170, and by a Rackham Pre-Doctoral Fellowship from the University of Michigan; I am grateful to these funders.

I would like to thank the members of my committee: Harm Derksen, Mircea Mustaţă, Karen Smith, and Jamie Tappenden. They have all been unfailingly generous with their time and attention, and have made this a stronger document.

In addition to those mentioned above, I have benefited greatly from discussions and classes with the following people: Michael Artin, Dave Anderson, Jason Bell, Ken Brown, Susan Colley, Hailong Dao, Carolyn Dean, Igor Dolgachev, Farkhod Eshmatov, Bill Fulton, Iain Gordon, Milena Hering, Gerardo Hernandez, Mel Hochster, Kyle Hofmann, Colin Ingalls, Brian Jurgelewicz, Gary Kennedy, Dennis Keeler, Ryan Kinser, Rob Lazarsfeld, Ezra Miller, Yogesh More, Gopal Prasad, Jeffrey Rauch, Dan Rogalski, Lance Small, Paul Smith, David Speyer, Alan Stapledon, Kevin Tucker, Giancarlo Urzua, Chelsea Walton, and James Zhang. I would particularly like to thank Brian Conrad for finding an error in an earlier version of Chapter V.

This thesis would not have been written without the love and support of my partner, Staśa Morgan-Appel.

I have been privileged to be a doctoral student of Toby Stafford's, and am deeply grateful to him for his knowledge, wisdom and patience.

TABLE OF CONTENTS

INDEX OF TERMINOLOGY

CHAPTER I

Introduction

1.1 Overview

Algebraic geometry is built upon the correspondence between algebraic objects such as rings and ideals and the geometric structures of curves, surfaces, and more general varieties and schemes. The rich and beautiful interplay between algebra and geometry is integral to modern expositions of the field, such as [Har77], and has been essential to most recent progress in both algebraic geometry and commutative algebra.

Remarkably, over the past fifteen years the classical correspondence between commutative rings and schemes has been extended to many noncommutative rings, at least in the graded setting. This is the new field of *noncommutative algebraic geometry.* The use of geometric techniques to study noncommutative graded rings is interesting in its own right, but has also had important applications to noncommutative algebra. These include Artin and Stafford's classification of *noncommutative projective curves* [AS95] and the use of geometric techniques to study and classify the noncommutative analogues of $\mathbb{P}^2$, including the well-known *3-dimensional Sklyanin algebras* [ATV90, ATV91, Ste96, Ste97].

One of the most important active research areas in noncommutative algebraic geometry is the *classification of noncommutative projective surfaces*: formally, these are noetherian finitely graded domains of Gelfand-Kirillov dimension 3. In this thesis, we make a significant contribution to this program by classifying all *birationally commutative* projective surfaces, completely solving the classification problem for one of what is conjectured to be only four birational types of noncommutative surface.

For these classification results, it was necessary to understand many graded rings that had not previously been studied. In particular, we investigate *geometric idealizers*: idealizer subrings of twisted homogeneous coordinate rings. These rings are defined by geometric data, and in order to understand them algebraically, new geometric techniques were needed. Ultimately, these led us to a generalization of the classical Kleiman-Bertini theorem, which in its earliest forms goes back to the 1880s.

We remark that in addition to the work in this thesis, the paper [Sie06] and the preprint [Sie07] were completed while the author was a Ph.D. student in the University of Michigan Mathematics Department.

In the remainder of the introduction, we give a more leisurely overview of the context and main results of this thesis. In Section 1.2 we describe the commutative setting and give a general discussion of noncommutative projective geometry. In Section 1.3, we present Artin and Stafford's results on noncommutative curves and summarize the current state of knowledge of noncommutative surfaces. In Section 1.4, we specifically discuss birationally commutative graded rings, and present our results on birationally commutative surfaces and on idealizers. In Section 1.5, we relate the geometry underlying idealizers to modern versions of the Kleiman-Bertini theorem, and give our generalization of this classical result. Finally, in Section 1.6, we summarize the overall plan of this thesis.

1.2 Commutative and noncommutative projective schemes

All rings in this introduction will be algebras over a fixed uncountable algebraically closed field $\Bbbk$, and all schemes will be of finite type over $\Bbbk$. We will also assume that our rings R are *finitely* $\mathbb{N}$*-graded*: that is, R is $\mathbb{N}$-graded and all R_n are finite-dimensional over $\Bbbk$.

Before describing the framework of noncommutative algebraic geometry, we review some important results from classical (commutative) algebraic geometry. For details, we refer readers to [Har77, Chapter II].

Let X be a projective variety, and let $\mathcal{L}$ be an invertible sheaf on X. The *section ring* of $\mathcal{L}$ is defined as

$$B(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(\mathcal{L}^{\otimes n}).$$

Multiplication on $B(X, \mathcal{L})$ is given by the maps

$$H^0(\mathcal{L}^{\otimes n}) \otimes H^0(\mathcal{L}^{\otimes m}) \to H^0(\mathcal{L}^{\otimes (m+n)}).$$

In general, determining the properties of section rings is extremely difficult — see the recent paper [BCHM06] on finite generation of the canonical ring for an example! However, when $\mathcal{L}$ is *ample* — recall this means that for any coherent sheaf $\mathcal{F}$ on X, for $n \gg 0$ the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections and has no higher cohomology — then a well-known result of Serre says that not only is $B(X, \mathcal{L})$ noetherian, but there is a functorial relationship that is essentially an equivalence between the categories of coherent sheaves on X and finitely generated graded $B(X, \mathcal{L})$-modules.

We introduce the notation we will need to describe the relevant categories. Given a projective scheme X, let $\mathcal{O}_X$-mod denote the category of coherent sheaves on X. The relevant module category is a bit more complicated. Let R be a finitely $\mathbb{N}$-graded $\Bbbk$-

algebra. The category gr-R is the category of noetherian $\mathbb{Z}$-graded right R-modules, with morphisms preserving degree. Inside gr-R, let tors-R be the full subcategory of finite-dimensional modules. We define qgr-R to be the quotient category

$$\text{qgr-}R = \text{gr-}R/\operatorname{tors-}R,$$

and define R-qgr to be the corresponding category on the left. (For more details on this construction, see Section 2.2.1.)

Serre's fundamental theorem states:

Theorem 1.2.1. (Serre's Theorem [Ser55, Chapter III.3, Propositions 5 and 6]) *Let X be a projective scheme and let $\mathcal{L}$ be an ample invertible sheaf on X. Then $B(X, \mathcal{L})$ is noetherian, and there is an equivalence of categories*

$$\mathcal{O}_X\text{-mod} \simeq \text{qgr-}B(X, \mathcal{L}).$$

Furthermore, if $X = \operatorname{Proj} R$, where R is a finitely graded commutative $\Bbbk$-algebra generated in degree 1, then the Serre twisting sheaf $\mathcal{O}(1)$ is ample, and R and $B(X, \mathcal{O}(1))$ are equal in large degree. Thus qgr-R *is equivalent to the category of coherent sheaves on* $\operatorname{Proj} R$*.*

In their seminal paper [AV90], Artin and Van den Bergh showed that Serre's theorem has a noncommutative version. This noncommutative Serre's theorem relies on the important construction of a *twisted homogeneous coordinate ring*, which we describe here. As before, we begin with a projective scheme X and an invertible sheaf $\mathcal{L}$ on X. We have one additional piece of data: an automorphism σ of X.

For ease of notation, if $\mathcal{F}$ is a quasicoherent sheaf on X, we will let $\mathcal{F}^\sigma = \sigma^*\mathcal{F}$, the pullback of $\mathcal{F}$ along σ. We form the *twisted tensor powers* $\mathcal{L}_n$ of $\mathcal{L}$, where we define

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}.$$

Then the *twisted homogeneous coordinate ring* of $\mathcal{L}$ is defined to be

$$B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(\mathcal{L}_n).$$

This is a ring, with multiplication given by

$$H^0(\mathcal{L}_n) \otimes H^0(\mathcal{L}_m) \xrightarrow{1 \otimes \sigma^n} H^0(\mathcal{L}_n) \otimes H^0((\mathcal{L}_m)^{\sigma^n}) \longrightarrow H^0(\mathcal{L}_{n+m}).$$

Example 1.2.2. Let $X = \mathbb{P}^1 = \mathbb{P}^1(\Bbbk)$, let $\mathcal{L} = \mathcal{O}(1)$, and define $\sigma \in \mathbb{P}GL_2$ by $\sigma([a : b]) = [a : a + b]$. We let σ act on a function f as $f^\sigma = f \circ \sigma$. Then the twisted homogeneous coordinate ring $B(\mathbb{P}^1, \mathcal{O}(1), \sigma)$ may be presented by generators and relations as

$$\Bbbk\{x, y\}/(xy - yx - x^2)$$

This ring is commonly referred to as the *Jordan (affine) plane* and is usually written $\Bbbk_J[x, y]$.

Remarkably, if the twisted tensor powers of $\mathcal{L}$ satisfy the appropriate ampleness property (the technical term is that $\mathcal{L}$ is σ-*ample*, defined precisely in Section 2.3), then a version of Serre's Theorem still holds.

Theorem 1.2.3. ([AV90, Theorem 1.3, Theorem 1.4], [Kee00, Theorem 1.2]) *Let X be a projective scheme, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Then $B(X, \mathcal{L}, \sigma)$ is left and right noetherian, and there are equivalences of categories*

$$\text{qgr-}B(X, \mathcal{L}, \sigma) \simeq \mathcal{O}_X\text{-mod} \simeq B(X, \mathcal{L}, \sigma)\text{-qgr}\,.$$

Motivated by Theorem 1.2.3, if R is a graded $\Bbbk$-algebra, Artin and Zhang [AZ94] defined the *right noncommutative projective scheme* associated to R to be the pair

$$\text{Proj-}R = (\text{qgr-}R, [R]),$$

where $[R]$ denotes the image of R in qgr-R. For example, one can easily deduce from Theorem 1.2.3 that

$$\text{qgr-}\Bbbk_J[x,y] \simeq \mathcal{O}_{\mathbb{P}^1}\text{-mod} \simeq \text{qgr-}\Bbbk[x,y]$$

and that

$$\text{Proj-}\Bbbk_J[x,y] = (\text{qgr-}\Bbbk_J[x,y], [\Bbbk_J[x,y]]) \cong (\mathcal{O}_{\mathbb{P}^1}\text{-mod}, \mathcal{O}_{\mathbb{P}^1}).$$

In general, the distinguished object $[R]$ is supposed to play the role of the structure sheaf of a projective scheme. Thus one defines cohomology functors $H^q(\text{Proj-}R, \_\_)$ on Proj-R as the right derived functors of $H^0(\text{Proj-}R, \_\_)$, where

$$H^0(\text{Proj-}R, \mathcal{M}) = \text{Hom}_{\text{qgr-}R}([R], \mathcal{M}),$$

for $\mathcal{M} \in$ qgr-R. The *(right) cohomological dimension* of Proj-R is the maximum q such that $H^q(\text{Proj-}R, \mathcal{M}) \neq 0$ for some $\mathcal{M}$. It is an important question, asked by Stafford and Van den Bergh in [SV01, page 194], whether all graded noetherian rings have finite left and right cohomological dimension. (Note that commutative graded noetherian rings have finite cohomological dimension by [Har77, Theorem III.2.7]. This and Theorem 1.2.3 imply that twisted homogeneous coordinate rings have finite left and right cohomological dimension.)

Although twisted homogeneous coordinate rings are noncommutative, in many important ways they behave remarkably like commutative rings. This is certainly not true for all noncommutative graded rings, and to give an indication of the issues that can arise, we give a second example, due to Stafford and Zhang.

Example 1.2.4. ([SZ94, Example 0.1]) Let $B = \Bbbk_J[x,y]$ be the Jordan plane defined in Example 1.2.2. We consider the subring

$$R = \Bbbk + yB$$

of B. Intuition from commutative algebra might lead us to expect that R behaves pathologically, since the similarly constructed commutative ring

$$R' = \Bbbk + y\Bbbk[x, y] \subset \Bbbk[x, y]$$

is certainly not noetherian. In fact, if $\Bbbk$ is countable, then R' has countably many elements and uncountably many ideals.

However, if $\operatorname{char} \Bbbk = 0$, then, perhaps surprisingly, it turns out that R is left and right noetherian. Many of the properties of R in characteristic 0 derive from the fact that it is an *idealizer* inside B: that is, R is the maximal subring of B in which the right ideal yB, generated by sections that vanish at the point $[1 : 0]$, becomes a two-sided ideal.

Example 1.2.4 is a special case of the following construction, which we study in detail in Chapter III.

Construction 1.2.5. Let Z be a closed subscheme of a variety X, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Inside the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$, let I be the right ideal generated by sections that vanish on Z. We define the ring

$$R(X, \mathcal{L}, \sigma, Z) = \{x \in B \mid xI \subseteq I\}$$

to be the *idealizer* $\mathbb{I}_B(I)$ *of* I *in* B.

In this notation, the ring of Example 1.2.4 becomes $R(\mathbb{P}^1, \mathcal{O}(1), \sigma, [1 : 0])$.

1.3 Noncommutative curves and surfaces

Many of the important techniques of commutative algebraic geometry were initially developed to study curves and surfaces. Likewise, studying low-dimensional rings has been important in noncommutative algebraic geometry. We note that there is a technical difficulty: we must define what we mean by the "dimension" of a graded ring. We will use the *Gelfand-Kirillov dimension* (GK-dimension), which we define precisely in Section 2.2.2. For now we will simply say that a graded ring R has *GK-dimension* d if $\dim R_n$ grows like n^{d-1}.

Here we outline what is known about graded domains of low GK-dimension; we refer the reader to the survey article [SV01] for a more in-depth discussion. If R is a finitely generated $\mathbb{k}$-algebra that is a domain of GK-dimension 1, then a well-known result of Small and Warfield [SW84] says that R is commutative. GK-dimension 2 is thus the first case of interest to us. A noetherian graded domain of GK-dimension 2 is known as a *noncommutative projective curve*. We have already seen two examples of noncommutative curves: Example 1.2.2, and more generally any twisted homogeneous coordinate ring of a projective curve, and Example 1.2.4. By a remarkable result due to Artin and Stafford, all noncommutative projective curves fall into one of these two types.

If R is a graded ring and $k \geq 1 \in \mathbb{Z}$, we define the *k'th Veronese of R* to be

$$R^{(k)} = \bigoplus_{n \in \mathbb{Z}} R_{kn}.$$

That is, $(R^{(k)})_n = R_{kn}$.

Theorem 1.3.1. ([AS95]) *Let R be a noetherian finitely $\mathbb{N}$-graded domain of GK-dimension 2. Then there is an integer $k \geq 1$ so that $R^{(k)}$ is either:*

(1) *a twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$ for some projective curve X, automorphism σ of X, and σ-ample invertible sheaf $\mathcal{L}$ on X; or*

(2) *an idealizer at points of infinite order inside the twisted homogeneous coordinate ring of a projective curve: that is, a ring similar to Example 1.2.4.*

Theorem 1.3.1 implies that noncommutative curves are closely related to commutative curves. Artin and Stafford in fact show that if R is a noncommutative projective curve, then (even in the idealizer case) qgr-$R \simeq \mathcal{O}_X$-mod for some projective curve X. However, even though the noncommutative projective scheme associated to a noncommutative curve is in fact commutative, the idealizers that occur are rings for which there is no clear commutative analogue.

We now turn to discussing *noncommutative projective surfaces*: noetherian finitely $\mathbb{N}$-graded domains of GK-dimension 3. Here the situation is significantly more complex. It is natural to attempt a classification by birational type, mimicking the Enriques classification of (commutative) projective surfaces. In the noncommutative setting, we will define this as follows. Let R be a graded domain of GK-dimension 3. We form the *graded quotient ring* of R by inverting all homogeneous elements to obtain a *graded division ring*, which by standard results must be a skew-Laurent ring of the form

$$D[z, z^{-1}; \sigma]$$

for some division ring D and automorphism σ of D. For more details on noncommutative localization, see Section 2.2.3.

By abuse of notation, we will refer to D as the *function field* of R, and will say that two noncommutative projective surfaces are *birationally equivalent* if their function fields are isomorphic.

Clearly there is a large class of noncommutative surfaces whose function fields are actually (commutative) fields; these include twisted homogeneous coordinate rings of projective surfaces as well as their idealizers and other subrings. Such rings are called *birationally commutative*. For now, we postpone discussing birationally commutative surfaces until Section 1.4, and consider other surfaces that do not fall into this class.

One important set of examples are the *Artin-Schelter regular rings of dimension 3*, known less formally as "noncommutative $\mathbb{P}^2$s."

Definition 1.3.2. A finitely $\mathbb{N}$-graded domain R is called *Artin-Schelter regular of dimension* d if R satisfies the following properties:

(1) R has global dimension d;

(2) R has finite GK-dimension;

(3) R is homologically well-behaved in the sense that R has left and right injective dimension d and

$$\operatorname{Ext}^i_R(\Bbbk, R) = \begin{cases} 0 & \text{if } i \neq d \\ \Bbbk & \text{if } i = d. \end{cases}$$

The idea behind this definition is that R is supposed to be a good analogue of a polynomial ring in d (weighted) variables. Condition (3), which is known as the *Artin-Schelter Gorenstein condition*, is included to rule out unpleasant examples like $\Bbbk\{x, y\}/(xy)$.

Artin, Tate, Van den Bergh, and Stephenson [ATV90, ATV91, Ste96, Ste97] have classified the Artin-Schelter regular rings of dimension 3. The most interesting examples are the *3-dimensional Sklyanin algebras*

$$\mathrm{Skl}_3(a, b, c) = \Bbbk\{x_0, x_1, x_2\}/(ax_ix_{i+1} + bx_{i+1}x_i + cx_{i+2}^2 : i = 1, 2, 3 \mod 3),$$

where $[a : b : c] \in \mathbb{P}^2 \smallsetminus \{\text{a finite set of degenerate points}\}$. Techniques from noncommutative algebraic geometry were central to this work. It turns out that a Sklyanin algebra $S = \mathrm{Skl}_3(a, b, c)$ contains a normal element g of degree 3, and that $S/(g)$ is isomorphic to the twisted homogeneous coordinate ring $B(E, \mathcal{L}, \sigma)$ of an elliptic curve E. Further, S is determined by the data $(E, \mathcal{L}, \sigma)$, and in fact by E and σ.

Thus we may write

$$\mathrm{Skl}_3(a, b, c) = \mathrm{Skl}_3(E, \sigma).$$

We note that the Hilbert series of $\mathrm{Skl}_3(E, \sigma)$ is $1/(1-t)^3$, and so it is plausible that S is an analogue of a polynomial ring in 3 variables.

There is one more birational class of noncommutative surfaces that is easy to write down: they are built from curves. For example, we may take a (skew) polynomial extension of a noncommutative projective curve. A simple example is the ring

$$R = \Bbbk_J[x, y][z]$$

where $\Bbbk_J[x, y]$ is the Jordan plane defined in Example 1.2.2. The function field of R is the full quotient division ring of $\Bbbk_J[x, y]$. As a variation, we may consider the (homogenized) ring of differential operators on an affine curve; for example, the *homogenized Weyl algebra*

$$H = \Bbbk\{x, y, h\}/(xy - yx - h^2, xh - hx, yh - hy).$$

The function field of H is the quotient division ring of the Weyl algebra, the ring of differential operators on the affine line. More generally still, we may consider the quotient division ring of any *Ore extension* (see Definition 2.2.2) $K[x; \sigma, \delta]$, where K is a field of transcendence degree 1. These Ore extensions are noncommutative polynomial rings in one variable.

Michael Artin made the bold conjecture in 1995 that up to birational equivalence, all noncommutative surfaces fall into this short list of examples. We present a slightly modified form of his conjecture.

Conjecture 1.3.3. ([Art95, Conjecture 4.1]) *If R is a noncommutative projective surface, then its function field is either:*

(1) *a field of transcendence degree 2 (birationally commutative);*

(2) *a division ring finite-dimensional over a central field of transcendence degree 2;*

(3) *the full quotient division ring of an Ore extension* $K[x; \sigma, \delta]$, *where* K *is a field of transcendence degree 1 (a "quantum ruled surface"); or*

(4) $D(E, \sigma)$, *the function field of the Sklyanin algebra* $\mathrm{Skl}_3(E, \sigma)$ *for some elliptic curve* E *and automorphism* σ *of* E *(a "quantum rational surface").*

Artin's conjecture is the most important open problem in noncommutative algebraic geometry. It was extremely provocative at the time that it was made, and remains so. It is also notable for its difficulty: in the 13 years since it was made, there has been no significant progress towards either a proof or a counterexample.

We do not attempt to do either in this thesis. Instead, we restrict our attention to case (1), and completely classify the graded domains in this birational equivalence class. We discuss this classification in the next section.

1.4 Birationally commutative graded rings

In this section, we discuss birationally commutative projective surfaces and make some comments on higher dimensional birationally commutative graded rings. We begin with an example, due to Rogalski:

Example 1.4.1. *([Rog04a, Definition 1.1])* Let $X = \mathbb{P}^2$, and let

$$\sigma = \begin{bmatrix} 1 & & \\ & p & \\ & & q \end{bmatrix} \in \mathbb{P}GL_3.$$

We will assume that $p, q \in \Bbbk^*$ are very general; it is enough to assume that the multiplicative subgroup of $\Bbbk$ generated by p, q, and 1 is isomorphic to $\mathbb{Z}^3$. Then one can easily check that

$$B = B(\mathbb{P}^2, \mathcal{O}(1), \sigma) \cong \Bbbk\{x, y, z\}/(xy - pyx, xz - qzx, yz - qp^{-1}zy).$$

Now consider the ring $S \subset B$, where

$$S = \Bbbk\langle x - y, y - z\rangle.$$

By [Rog04a, Theorem 1.2], S is left and right noetherian.

The ring S constructed in Example 1.4.1 is an example of a so-called *naïve blowup algebra*. Some special cases were studied in [Rog04a], and a more general construction was given in [KRS05] and subsequently generalized in [RS07]. Here we follow the exposition in [KRS05].

To construct a naïve blowup algebra, begin as usual with a projective variety X, an automorphism σ of X, and a σ-ample invertible sheaf $\mathcal{L}$ on X. Also choose a point $P \in X$ (or more generally, let P be a 0-dimensional subscheme of X). Let $\mathcal{I} = \mathcal{I}_P$ be the ideal sheaf of P. Then we may form a ring

$$S(X, \mathcal{L}, \sigma, P) = \bigoplus_{n \geq 0} H^0(\mathcal{I}\mathcal{I}^{\sigma} \cdots \mathcal{I}^{\sigma^{n-1}}\mathcal{L}_n), \tag{1.4.2}$$

which we refer to as a *naïve blowup of X at P*. The construction of $S(X, \mathcal{L}, \sigma, P)$ mimics the construction of a commutative blowup as a Rees ring: we are taking (sections of) higher and higher successive powers of the ideal defining P, using the multiplication on the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$.

Let P be a 0-dimensional subscheme of X. Then the properties of the naïve blowup $S(X, \mathcal{L}, \sigma, P)$ depend on the geometry of the orbits $\{\sigma^n(p)\}_{n \in \mathbb{Z}}$ for $p \in P$.

Definition 1.4.3. Let X be a projective variety, let σ be an automorphism of X, and let $p \in X$. We say the orbit $\{\sigma^n(p)\}$ is *critically dense* if it is infinite and any infinite subset is Zariski-dense in X.

Then we have:

Theorem 1.4.4. ([RS07, Theorem 1.1]) *Let X be a projective variety, σ an automorphism of X, and $\mathcal{L}$ a σ-ample invertible sheaf on X. Let P be a 0-dimensional closed subscheme of X. If the set $\{\sigma^n(p)\}$ is critically dense for all $p \in P$, then the ring $S(X, \mathcal{L}, \sigma, P)$ is noetherian.*

We remark that [KRS05, Theorem 4.1] is an earlier version of this result, with less general hypotheses. We also note that we prove the converse to Theorem 1.4.4 in this thesis; see Proposition 4.7.14.

Rogalski and Stafford [RS06] have recently classified all birationally commutative projective surfaces that are generated in degree 1; remarkably, twisted homogeneous coordinate rings and naïve blowups are the only two types of rings that occur.

Theorem 1.4.5. ([RS06, Theorem 1.1]) *Let R be a birationally commutative projective surface that is generated in degree 1. Then there is an integer $k \geq 1$ so that $R^{(k)}$ is either:*

(1) *the twisted homogeneous coordinate ring of a projective surface; or*

(2) *the naïve blowup of a projective surface at a 0-dimensional subscheme supported on points that move in critically dense orbits.*

We note that Rogalski and Stafford consider a slightly more general class of rings than our noncommutative projective surfaces. They study finitely $\mathbb{N}$-graded noetherian domains R whose graded quotient ring is of the form

$$K[z, z^{-1}; \sigma]$$

where $K = \Bbbk(X)$ is the function field of a projective surface X such that σ induces an automorphism of X. By [Rog07, Theorem 1.1], any such R has GK-dimension 3 or 5, and any birationally commutative domain of GK-dimension 3 that is generated in degree 1 is of the form considered by Rogalski and Stafford. See Section 4.1 for more discussion of the GK-dimension of noncommutative surfaces.

The hypothesis in Theorem 1.4.5 that R be generated in degree 1 seems overly restrictive; note that in contrast with the commutative case, there are many noncommutative noetherian graded rings that have no Veronese subring generated in degree 1. (For example, the idealizers in Construction 1.2.5 have this property.) We remove this restriction in Chapter IV, and make a complete classification of birationally commutative surfaces, using methods that are quite different from the proof of Theorem 1.4.5. Besides idealizers, naïve blowups, and twisted homogeneous coordinate rings, one new type of ring arises; we refer to these as *ADC rings*. They are similar to, but more general than, naïve blowups, and give rise to a new class of *maximal orders* — the noncommutative version of integrally closed rings. (The formal definition is given in Definition 4.1.6.)

We obtain:

Theorem 1.4.6. (Theorem 4.1.4) *Let R be a finitely $\mathbb{N}$-graded birationally commutative noetherian domain of GK-dimension 3. Then there is an integer $k \geq 1$ so that $R^{(k)}$ is either:*

(1) *the twisted homogeneous coordinate ring of a projective surface;*

(2) *a naïve blowup or ADC ring on a projective surface;*

$(1')$, $(2')$ *an idealizer inside a ring of type* (1) *or* (2) *respectively.*

By classifying all rings falling within case (1) of Conjecture 1.3.3, Theorem 1.4.6 shows that relatively mild assumptions on rings of GK-dimension 3 can have powerful consequences. Artin's original formulation of Conjecture 1.3.3 assumed much stronger technical conditions on the rings under study; it is quite interesting that these assumptions do not, in fact, seem to be necessary to understand birationally commutative surfaces. Furthermore, by enumerating the possible types of birationally commutative surfaces, Theorem 1.4.6 opens up new avenues of future research: understanding the rings given in cases (1)–($2'$) of Theorem 1.4.6 should give new insight into the possibilities for important concepts such as the Artin-Zhang χ conditions, which are defined in Section 2.4. We plan to explore this further in future work.

The main difficulty in proving both Theorem 1.4.6 and Theorem 1.4.5 is constructing the classical projective surface X that is associated to a given birationally commutative projective surface R. Rogalski and Stafford prove Theorem 1.4.5 through a delicate analysis of a certain class of modules, called *point modules* over R. This is quite difficult because for naïve blowups such modules are parameterized by an infinite series of projective schemes but not by any individual projective scheme; see [KRS05, Theorem 1.1]. In contrast, in the proof of Theorem 1.4.6 we construct the surface X much more directly, through a method of successive approximations of the "correct" surface. While there are technical issues involved in this proof, most of them are involved with showing that this method does, in fact, lead to an appropriate projective surface, and the actual construction is relatively straightforward. We comment also that comparing the methods of proof of Theorems 1.4.5 and 1.4.6 may be a fruitful direction for future research.

Another important component of the proof of Theorem 1.4.6 is understanding idealizers in twisted homogeneous coordinate rings, as in Construction 1.2.5. These form a large class of examples, are often noetherian, and are never generated in degree 1. Idealizers on curves are understood, thanks to the classification of noncommutative curves in [AS95]; Rogalski [Rog04b] has also investigated idealizers of the form

$$R(\mathbb{P}^d, \mathcal{O}(1), \sigma, P), \tag{1.4.7}$$

where $P = \{p\}$ is a point in $\mathbb{P}^d$, from an algebraic perspective. However, until now the properties of general geometric idealizers were not known. In Chapter III, we investigate idealizers inside twisted homogeneous coordinate rings of arbitrary dimension.

Rogalski proved that the properties of the idealizers (1.4.7) depend on the critical density of the orbit $\{\sigma^n(p)\}$. Notably, in order for R to be left noetherian, one needs that $\{\sigma^n(p)\}_{n\geq 0}$ is critically dense. Finding a condition on an arbitrary subscheme that will give rise to a noetherian idealizer is an important geometric question to be solved in generalizing Rogalski's results to arbitrary idealizers.

In Chapter III we answer this question. We define:

Definition 1.4.8. Let X be a projective variety and let $\sigma \in \operatorname{Aut} X$. Let $Z \subseteq X$ be a closed subscheme. The set $\{\sigma^n Z\}_{n\in\mathbb{Z}}$ is *critically transverse* in X if for all closed subschemes $Y \subseteq X$, for all but finitely many n we have $\mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) = 0$ for any $j \geq 1$.

We show that critical transversality of the set $\{\sigma^n Z\}$ controls the behavior of idealizers.

Theorem 1.4.9. (Theorem 3.1.6) *Let X be a projective scheme, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Z be a closed subscheme of X. For simplicity, assume that Z is reduced and irreducible and of infinite order under σ. (We treat the general case in the body of the thesis.) Let I be the right ideal of $B(X, \mathcal{L}, \sigma)$ generated by sections vanishing on Z. Let $R = R(X, \mathcal{L}, \sigma, Z)$, as in Construction 1.2.5.*

Then

$$R = \Bbbk + I,$$

and R is right noetherian if and only if the set $\{n \geq 0 \mid \sigma^n(p) \in Z\}$ is finite for any $p \in X$. If $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse, then R is left noetherian.

This generalizes the results in [SZ94, AS95, Rog04b] to arbitrary idealizers in twisted homogeneous coordinate rings.

As mentioned, the question of whether the left and right cohomological dimensions of a noetherian graded ring are finite is an important open problem in noncommutative geometry. Let R be one of the idealizers considered in Theorem 1.4.9. By a result of Rogalski [Rog04b, Proposition 3.5], the cohomological dimension of the left projective scheme associated to R is equal to $\dim X$. We study the cohomological dimension of the right projective scheme associated to R, and prove (Theorem 3.7.1) that if R is left noetherian, then $\operatorname{cd}(\text{Proj-}R)$ is finite, even if the global dimension of Proj-R is infinite. On the other hand, in Example 3.7.6 we give an example of a right but not left noetherian ring that has infinite right cohomological dimension.

1.5 Transversality

We have seen that the properties of many noncommutative rings defined by geometric data are controlled by the *critical transversality* of the underlying data. Here we discuss other concepts of transversality, and describe purely algebro-geometric results that relate these concepts. This is the subject of Chapter V of this thesis.

It is a fundamental principle of intersection theory that generic intersections are well-behaved. If the ambient variety is sufficiently nice, one expects two nonsingular subvarieties in general position to meet transversally, so a generic intersection should be nonsingular. Classically, these heuristics are made precise by the well-known

Kleiman-Bertini theorem, which goes back to 1882 [Ber82] in its earliest form.

Theorem 1.5.1. (Bertini, Kleiman [Har77, Theorem III.10.8]) *Assume* $\Bbbk$ *has characteristic 0. Let* X *be a variety (necessarily nonsingular) with a transitive left action of an algebraic group* G*. Let* Y *and* Z *be nonsingular closed subvarieties of* X*. Then there is a dense open subset* U *of* G *such that if* $g \in U$*, then* gZ *and* Y *intersect transversally. In particular, a general hyperplane section of a nonsingular projective variety is nonsingular.*

Recently, Miller and Speyer [MS06] generalized the Kleiman-Bertini theorem to apply to a more algebraic concept of a well-behaved intersection.

Definition 1.5.2. Let X be a scheme, and let Y and Z be closed subschemes of X. If $\mathcal{T}or_j^X(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for $j \geq 1$, we will say that Y and Z are *homologically transverse.*

Homological transversality has the following geometric meaning. If P is a component of $Y \cap Z$, then Serre's formula for the multiplicity of the intersection of Y and Z at P [Har77, p. 427] is:

$$i(Y, Z; P) = \sum_{j \geq 0} (-1)^j \operatorname{len}_P(\mathcal{T}or_j^X(\mathcal{O}_Z, \mathcal{O}_Y)),$$

where the length $\operatorname{len}_P(\_\_)$ is taken over the local ring at P. Thus if Y and Z are homologically transverse, their intersection multiplicity at P is simply the length of their scheme-theoretic intersection over the local ring at P.

We note that homological transversality does generalize classical transversality: if X, Y, and Z are nonsingular, Y and Z meet transversally, and $\operatorname{char} \Bbbk = 0$, then Y and Z are also homologically transverse.

Miller and Speyer's result is:

Theorem 1.5.3. [MS06] *Let X be a variety with a transitive left action of a smooth algebraic group G. Let Z and Y be closed subschemes of X. Then there is a dense Zariski open subset U of G such that, for all $g \in U$, the subschemes gZ and Y are homologically transverse.*

It is natural to ask what conditions on the action of G are necessary to conclude that homological transversality is generic in the sense of Theorem 5.1.1. In particular, the restriction to transitive actions is unfortunately strong, as it excludes important situations such as the torus action on $\mathbb{P}^n$. On the other hand, suppose that Z is the closure of a non-dense orbit. Then for all $g \in G$, we have

$$\mathcal{T}or_1^X(\mathcal{O}_{gZ}, \mathcal{O}_Z) = \mathcal{T}or_1^X(\mathcal{O}_Z, \mathcal{O}_Z) \neq 0,$$

and so the conclusion of Theorem 5.1.1 fails. Thus for non-transitive group actions some additional hypothesis is necessary.

In Chapter V, we show that there is a simple condition for homological transversality to be generic. We will state it here for algebraically closed fields, although in the text we make no assumptions on the ground field.

Theorem 1.5.4. (Theorem 5.1.2) *Let X be a variety with a left action of a smooth algebraic group G, and let Z be a closed subscheme of X. Then the following are equivalent:*

(1) *Z is homologically transverse to all G-orbit closures in X;*

(2) *For all closed subschemes Y of X, there is a Zariski open and dense subset U of G such that for all $g \in U$, the subscheme gZ is homologically transverse to Y.*

The investigations that led to Theorem 1.5.4 were motivated by the work in Chapter III. We have seen that if X is a projective variety, σ an automorphism of X, $\mathcal{L}$ a σ-ample invertible sheaf on X, and Z a closed subscheme of X, then the algebraic properties of the geometric idealizer ring $R(X, \mathcal{L}, \sigma, Z)$ in Construction 1.2.5 are controlled by the property that $\{\sigma^n Z\}$ is critically transverse: that is, for any closed subscheme Y and for any $j \geq 1$, the sheaves $\mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y)$ vanish for all but finitely many n. This certainly reminds one of generic transversality statements like the Kleiman-Bertini theorem or Theorem 1.5.3. Thus, in investigating critical transversality, one is naturally led to wonder what conditions on Z are necessary to conclude that some sort of generic transversality result holds for the translates of Z.

Using Theorem 1.5.4, we are able to answer this question, at least in many situations. We show:

Theorem 1.5.5. (Theorem 5.4.2) *Suppose that* $\operatorname{char} \Bbbk = 0$ *and that* σ *is an element of an algebraic group acting on* X*. Then the following are equivalent:*

(1) *Z is homologically transverse to all reduced σ-invariant subschemes of X;*

(2) *the set $\{\sigma^n Z\}$ is critically transverse.*

In particular, Theorem 1.5.5 implies that (in characteristic 0), if σ is a sufficiently general element of $\mathbb{P}GL_{d+1}$, then $\{\sigma^n Z\}$ is critically transverse for almost every $Z \subseteq \mathbb{P}^d$. (See Corollary 5.4.3 for a precise statement.)

1.6 Plan of this thesis

In this section we briefly discuss the plan of the rest of this thesis. In Chapter II, we review some of the ring theory, algebraic geometry, and noncommutative geometry that we will use. We give an overview of the current state of knowledge of noncommutative projective surfaces, and describe some of the techniques that we will use to prove the results in this thesis.

Chapter III is devoted to studying the geometric idealizers constructed in Construction 1.2.5. We determine many of the properties of geometric idealizers and show that they are controlled by the critical transversality of the underlying data; in particular, we prove Theorem 1.4.9. We also investigate when idealizers satisfy the Artin-Zhang χ *conditions* and are *strongly noetherian*, and study the cohomological dimension of idealizers.

Chapter IV is devoted to the proof of Theorem 1.4.6. We use many of the results from Chapter III in this proof.

Finally, in Chapter V, we investigate algebro-geometric questions related to homological transversality and critical transversality, and prove Theorem 1.5.4 and Theorem 1.5.5.

CHAPTER II

Background

2.1 Introduction

In this chapter, we lay out the fundamental notations and definitions that we will use in this thesis. In the first section, we collect some basic facts about graded rings, abelian categories, and Gelfand-Kirillov dimension. In the second section, we discuss *bimodule algebras*: a bimodule algebra, roughly speaking, is the noncommutative version of a sheaf of algebras. We also define and discuss σ-ampleness and outline the proof of Theorem 1.2.3, since it introduces techniques that we will use in the sequel. The third section is devoted to a discussion of two important technical conditions that we will see repeatedly in the rest of this thesis. Finally, we give a few results from classical algebraic geometry that we will need.

2.2 Basic definitions

In this section, we give basic definitions and notations that we use throughout this thesis.

2.2.1 Graded rings and abelian categories

We work throughout over a fixed field $\Bbbk$, which we assume to be algebraically closed unless otherwise stated. A $\Bbbk$-algebra R is called *graded* if it has a direct sum decomposition

$$R = \bigoplus_{n \in \mathbb{Z}} R_n$$

that satisfies $R_n R_m \subseteq R_{n+m}$ for all n, m. We adopt the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$, and we will say that R is $\mathbb{N}$-*graded* if $R_n = 0$ for $n < 0$. A graded $\Bbbk$-algebra R is called *connected graded* if $R_0 = \Bbbk$, and *connected* $\mathbb{N}$-*graded* if it is connected graded and $\mathbb{N}$-graded. A graded $\Bbbk$-algebra R is *finitely graded* if it is finitely generated as a $\Bbbk$-algebra and each R_n is finite-dimensional over $\Bbbk$, and *finitely* $\mathbb{N}$-*graded* if it is finitely graded and $\mathbb{N}$-graded. Note that a finitely graded domain is connected graded.

If R is a $\Bbbk$-algebra, we will denote the category of right, respectively left, R-modules by Mod-R, respectively R-Mod. If R is, in addition, graded, then by Gr-R we will denote the category of graded right R-modules: that is, modules M_R with a direct sum decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n,$$

satisfying $M_n R_m \subseteq M_{n+m}$. Morphisms in Gr-$R$ are module homomorphisms $\phi : M \to N$ such that $\phi(M_n) \subseteq N_n$ for all $n \in \mathbb{Z}$. We write

$$\hom_R(M, N) = \operatorname{Hom}_{\text{Gr-}R}(M, N),$$

and denote the derived functors of $\hom_R$ by ext^j_R. We similarly define the category R-Gr of graded left R-modules.

If $\mathcal{C}$ and $\mathcal{C}'$ are categories, then we will use the notation $\mathcal{C} \simeq \mathcal{C}'$ to mean that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent. We adopt the convention throughout that if Abc is the name of a category, then abc will denote the full subcategory of noetherian objects; that is, objects whose subobjects satisfy the ascending chain condition. Thus for a graded $\Bbbk$-algebra R, we also have categories R-gr, gr-R, mod-R, etc.

Let R be a graded $\Bbbk$-algebra. If M is a graded R-module and $n \in \mathbb{Z}$, we may define a new module $M[n]$ by shifting degrees: let

$$M[n] = \bigoplus_{i \in \mathbb{Z}} M[n]_i$$

and set $M[n]_i = M_{n+i}$. For all n, the functor

$$M \mapsto M[n]$$

is an autoequivalence of Gr-R and of R-Gr; we call these autoequivalences *shift functors*.

Let R be a graded $\Bbbk$-algebra, and let M and N be graded right R-modules. We define

$$\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{hom}_R(M, N[n]),$$

and write

$$\underline{\mathrm{Hom}}_R(M, N)_n = \mathrm{hom}_R(M, N[n]).$$

These are the maps ϕ such that $\phi(M_i) \subseteq N_{n+i}$, and we refer to them as *homomorphisms of degree n*. Similarly, we define, for any j,

$$\underline{\mathrm{Ext}}^j_R(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{ext}^j_R(M, N[n]).$$

If M is finitely generated, we may identify $\underline{\mathrm{Hom}}_R(M, N)$ with $\mathrm{Hom}_{\text{Mod-}R}(M, N)$.

If R is a graded $\Bbbk$-algebra and $k \neq 0 \in \mathbb{N}$, we denote the k'th *Veronese* of R by $R^{(k)}$, where

$$(R^{(k)})_n = R_{kn}.$$

If R is noetherian, so is $R^{(k)}$ for all $k \geq 1$. If $R^{(k)}$ is left (right) noetherian and R is a finitely generated left (right) $R^{(k)}$-module, then R is left (right) noetherian.

We briefly review the definition of a quotient category; we refer the reader to [Gab62] for a reference for the category theory used here. Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{A}$ of $\mathcal{C}$ that is closed under taking subobjects, quotients, and extensions is called a *Serre subcategory* or a *dense subcategory* of $\mathcal{C}$. If $\mathcal{A}$ is a Serre subcategory of $\mathcal{C}$, then we may form the *quotient category* $\mathcal{C}/\mathcal{A}$. The objects of $\mathcal{C}/\mathcal{A}$ are the same as the objects of $\mathcal{C}$. If M, N are two objects of $\mathcal{C}$, we define

$$\operatorname{Hom}_{\mathcal{C}/\mathcal{A}}(M, N) = \varinjlim \operatorname{Hom}_{\mathcal{C}}(M', N/N'),$$

where the direct limit is taken over all subobjects M' of M and N' of N such that M/M' and N' are both in $\mathcal{A}$. There is clearly a *quotient functor* $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{A}$: we define $\pi C = C$ for all $C \in \mathcal{C}$, and let $\pi(f : M \to N)$ be the image of f in the direct system that defines $\operatorname{Hom}_{\mathcal{C}/\mathcal{A}}(M, N)$.

We now specialize to the case that R is a finitely $\mathbb{N}$-graded $\Bbbk$-algebra and $\mathcal{C} =$ Gr-R. A graded right R-module M is called *right bounded* if $M_n = 0$ for all $n \gg 0$. We say that M is *torsion* if M is a direct limit of right bounded modules. Let Tors-R denote the full subcategory of Gr-R of torsion modules. We leave it to the reader to verify that Tors-R is a Serre subcategory of Gr-R. Thus we may form the quotient category

$$\text{Qgr-}R = \text{Gr-}R/\operatorname{Tors-}R.$$

We set qgr-$R =$ gr-$R/$ tors-R, where tors-$R =$ Tors-$R \cap$ gr-R. We note that the shift functors

$$M \to M[n]$$

descend to autoequivalences of Qgr-R and of qgr-R, and similarly on the left.

In fact, Tors-R is a *localizing subcategory* of Gr-R: this means that there is a *section functor* $\omega :$ Qgr-$R \to$ Gr-R such that $\pi\omega \cong \operatorname{Id}_{\text{Qgr-}R}$, where $\pi :$ Gr-$R \to$ Qgr-R is the quotient functor. If $M \in$ Gr-R is torsionfree, then $\omega\pi M$ is the largest essential extension M' of M such that M'/M is torsion.

Recall from Chapter I that if $B = B(X, \mathcal{L}, \sigma)$ is the twisted homogeneous coordinate ring of a σ-ample invertible sheaf $\mathcal{L}$ on the projective variety X, then by Theorem 1.2.3, we have that qgr-$B \simeq \mathcal{O}_X$-mod. Motivated by this, Artin and Zhang [AZ94] defined the *noncommutative projective scheme* associated to a graded $\Bbbk$-algebra R to be the pair

$$\text{Proj-}R = (\text{qgr-}R, \pi R).$$

The distinguished object πR plays the role of the structure sheaf of Proj-R. Now, if $\mathcal{F}$ is a quasicoherent sheaf on a projective variety X, then $H^0(X, \mathcal{F}) = \text{Hom}_X(\mathcal{O}_X, \mathcal{F})$. By analogy, we define cohomology functors on Proj-R by setting

$$H^i(\text{Proj-}R, \_\_) = \text{Ext}^i_{\text{Qgr-}R}(\pi R, \_\_).$$

We define the *right cohomological dimension of* R, or cd(Proj-R), to be

$$\max\{i \mid H^i(\text{Proj-}R, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \text{Qgr-}R\}.$$

We may of course mirror the constructions in the previous two paragraphs on the left; thus we also have R-Qgr, R-qgr, R-Proj, and the *left cohomological dimension of* R, or cd(R-Proj).

2.2.2 Gelfand-Kirillov dimension

One difficulty of noncommutative algebra is that the numerous equivalent notions of dimension for commutative rings diverge once one passes to the noncommutative realm. The dimension we will use in this thesis is the *Gelfand-Kirillov dimension* or *GK-dimension*. We will mention only a few properties here; for a general reference on GK-dimension, see [KL00].

Definition 2.2.1. Let R be a finitely generated $\Bbbk$-algebra, and let V be a finite-dimensional generating subspace for R that contains 1. The *GK-dimension of R* is

$$\begin{aligned}\operatorname{GKdim} R &= \inf\{\alpha \in \mathbb{R} \mid \dim_{\Bbbk}(V^n) \leq n^{\alpha} \text{ for all } n \gg 0\} \\ &= \limsup_{n\to\infty} \frac{\log \dim_{\Bbbk} V^n}{\log n}.\end{aligned}$$

One easily checks that this definition is independent of the generating subspace V.

For finitely generated commutative $\Bbbk$-algebras, the GK-dimension is equal to the Krull dimension. For noncommutative rings, GK-dimension can be quite badly behaved: in particular, it is not necessarily an integer. However, if R is a graded domain of GK-dimension ≤ 3, then $\operatorname{GKdim} R \in \{0, 1, 2, 3\}$ by results of Bergman [KL00, Theorem 2.5], Artin and Stafford [AS95], and Smoktunowicz [Smo06]. It is an open question whether there exist any domains with non-integer GK-dimension.

2.2.3 Noncommutative localization and skew polynomial rings

In the noncommutative setting, it is not always clear what one means by a "quotient ring." A general result due to Gabriel [Gab62, Théorème 1, p. 418] says that any noncommutative domain has what is known as a *maximal right quotient ring*; see [GW89, Chapter 4] for a construction. However, this ring is not necessarily a division ring! Furthermore, if R is badly behaved, then there may be many division rings sitting between R and its maximal quotient ring. For example, it appears that almost any division ring infinite-dimensional over its center K contains a free subalgebra $K\{x, y\}$ on two generators over K; cf. [ML83]. Thus one cannot form a "quotient division ring" of $K\{x, y\}$ in any canonical way.

Here we briefly review when noncommutative localization is possible.

Definition 2.2.2. Let R be a domain. A set $X \subset R$ of nonzero elements is a *right Ore set* if X is multiplicatively closed and if for all $x \in X$ and $r \in R$, we have

$$xR \cap rX \neq \emptyset.$$

If X is a right Ore set, then one can form a ring of fractions with denominators in X. That is, there is a unique way to form the localization

$$RX^{-1} = \{rx^{-1} \mid r \in R, x \in X\}$$

such that RX^{-1} is an overring of R with appropriate properties. If X is both a right and a left Ore set, then the rings RX^{-1} and $X^{-1}R$ are naturally isomorphic. In particular, any element of RX^{-1} may be written as both a right fraction rx^{-1} and a left fraction $y^{-1}s$ for some $r, s \in R$ and $x, y \in X$, and any finite set of elements of RX^{-1} has both a right and a left common denominator. (See [GW89, Chapter 9] for details.)

If R is a graded domain and is either noetherian or has finite GK-dimension, then the set

$$X = R \smallsetminus \{0\}$$

is automatically a right and left Ore set by Goldie's Theorem [GW89, Theorem 5.10] or by [KL00, Theorem 4.15]. Further, the set

$$Y = \left(\bigcup_{n \in \mathbb{Z}} R_n \right) \smallsetminus \{0\}$$

is also a right and left Ore set [GS00, Theorem 5a]. The ring RX^{-1} formed by inverting all nonzero elements is called the *quotient division ring of* R and written $Q(R)$. The ring RY^{-1} formed by inverting all nonzero homogeneous elements is known as the *graded quotient ring of* R and denoted $Q_{\mathrm{gr}}(R)$. Clearly $Q_{\mathrm{gr}}(R)$ is graded, and $Q_{\mathrm{gr}}(R)_0$ is a division ring. We will call $Q_{\mathrm{gr}}(R)_0$ the *function field* of

R; note that the function field of R need not be commutative! If z is any nonzero element of $Q_{\mathrm{gr}}(R)_1$ (in particular, we assume some such z exists), then it is not hard to see that the map

$$\sigma(x) = zxz^{-1}$$

defines an automorphism of $Q_{\mathrm{gr}}(R)_0$. In fact, by [NvO82, Corollary I.4.3], $Q_{\mathrm{gr}}(R)$ is isomorphic to the *skew-Laurent ring*

$$S = D[z, z^{-1}; \sigma],$$

where $D = Q_{\mathrm{gr}}(R)_0$ is the function field of R. Elements of S are are Laurent polynomials

$$\sum_{i=-\infty}^{\infty} d_i z^i,$$

with $d_i \in D$ and only finitely many d_i nonzero; multiplication is induced from the rule that if $d \in D$, then

$$zd = d^{\sigma} z.$$

We also mention the general construction of skew polynomial rings, which we use in the statement of Conjecture 1.3.3. Let K be a $\Bbbk$-algebra and let $\sigma \in \mathrm{Aut}_{\Bbbk}(K)$. Then we define a *$\sigma$-derivation* of K to be an additive map

$$\delta : K \to K$$

satisfying

$$\delta(rs) = \delta(r)\sigma(s) + r\delta(s)$$

for all $r, s \in K$. Given an automorphism σ of K and a σ-derivation δ of K, we define the *skew polynomial ring* or *Ore extension* $K[x; \sigma, \delta]$ to be the free K-module

$$\bigoplus_{n \geq 0} Kx^n,$$

with multiplication induced from the rule that

$$xr = \sigma(r)x + \delta(r)$$

for all $r \in K$. Details of skew-Laurent and skew polynomial rings may be found in [GW89, Chapter 1].

2.3 Bimodule algebras

In this section, we develop the notation to work with *bimodule algebras*. Most of the material in this section was developed in [Van96] and [AV90], and we refer the reader there for references. We will not work in full generality, however, and our presentation will follow that in [KRS05, Section 2].

We fix throughout this section a projective variety X; for us, a *variety* is an integral separated scheme of finite type over $\Bbbk$. We will denote the category of quasicoherent (respectively coherent) sheaves on X by $\mathcal{O}_X$-Mod (respectively $\mathcal{O}_X$-mod). If σ is an automorphism of X and $\mathcal{F}$ is a sheaf on X, recall the notation that $\mathcal{F}^\sigma = \sigma^*\mathcal{F}$. Thus σ acts on functions by sending f to $f^\sigma = f \circ \sigma$.

A bimodule algebra on X is, roughly speaking, a quasicoherent sheaf with a multiplicative structure. Before presenting an explicit definition, we give the fundamental example.

Example 2.3.1 (Twisted bimodule algebras)**.** Let σ be an automorphism of X and let $\mathcal{L}$ be an invertible sheaf on X. We define the *twisted bimodule algebra of* $\mathcal{L}$ to be

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}.$$

Let $\mathcal{L}_n$ be the nth twisted tensor power of $\mathcal{L}$; i.e., let

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}.$$

There is a natural map from $\mathcal{L}_n \otimes \mathcal{L}_m \to \mathcal{L}_{m+n} = \mathcal{L}_n \otimes (\mathcal{L}_m)^{\sigma^n}$, given by $1 \otimes \sigma^n$. Thus the multiplication on $\mathcal{B}$ is twisted by σ, in a sense that we will make precise in the following definitions.

Definition 2.3.2. An $\mathcal{O}_X$-*bimodule* is a quasicoherent $\mathcal{O}_{X\times X}$-module $\mathcal{F}$, such that for every coherent $\mathcal{F}' \subseteq \mathcal{F}$, we have that for $Z = \operatorname{Supp} \mathcal{F}'$, the projection maps $p_1, p_2 : Z \to X$ are both finite morphisms. The left and right $\mathcal{O}_X$-module structures associated to an $\mathcal{O}_X$-bimodule $\mathcal{F}$ are defined respectively as $(p_1)_*\mathcal{F}$ and $(p_2)_*\mathcal{F}$.

We note that by [Van96, Proposition 2.5], there is a tensor product operation on the category of bimodules that has the expected properties.

In general, operations with bimodules can be quite technical. However, all the bimodules that we consider will be constructed from bimodules of the following form:

Definition 2.3.3. Let $\sigma, \tau \in \operatorname{Aut}(X)$. Let (σ, τ) denote the map

$$\begin{aligned} X &\to X \times X \\ x &\mapsto (\sigma(x), \tau(x)). \end{aligned}$$

If $\mathcal{F}$ is a quasicoherent sheaf on X, we define the $\mathcal{O}_X$-bimodule ${}_\sigma\mathcal{F}_\tau$ to be

$${}_\sigma\mathcal{F}_\tau = (\sigma, \tau)_*\mathcal{F}.$$

If $\sigma = 1$ is the identity, we will often omit it; thus we write $\mathcal{F}_\tau$ for ${}_1\mathcal{F}_\tau$ and $\mathcal{F}$ for the $\mathcal{O}_X$-bimodule ${}_1\mathcal{F}_1 = \Delta_*\mathcal{F}$, where $\Delta : X \to X \times X$ is the diagonal.

We quote a lemma that shows how to work with bimodules of the form ${}_\sigma\mathcal{F}_\tau$, and, in particular, how to form their tensor product.

Lemma 2.3.4. ([KRS05, Lemma 2.3]) *Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, and let $\sigma, \tau \in \operatorname{Aut} X$.*

(1) ${}_{\tau}\mathcal{F}_{\sigma} \cong (\mathcal{F}^{\tau^{-1}})_{\sigma\tau^{-1}}$.

(2) $\mathcal{F}_{\sigma} \otimes \mathcal{G}_{\tau} \cong (\mathcal{F} \otimes \mathcal{G}^{\sigma})_{\tau\sigma}$.

(3) *In particular,* $\mathcal{L}_{\sigma}^{\otimes n} = (\mathcal{L}_n)_{\sigma^n}$. □

We will usually work with bimodules of the form $\mathcal{F}_{\tau}$. By Lemma 2.3.4(1), this is not a restriction. We make the notational convention that when we refer to an $\mathcal{O}_X$-bimodule simply as an $\mathcal{O}_X$-module, we are using the left-handed structure (for example, when we refer to the global sections or higher cohomology of an $\mathcal{O}_X$-bimodule).

Definition 2.3.5. Let X be a projective scheme and let $\sigma \in \operatorname{Aut} X$. An *$\mathcal{O}_X$-bimodule algebra*, or simply a *bimodule algebra*, $\mathcal{B}$ is an algebra object in the category of bimodules. That is, there is a unit map $1 : \mathcal{O}_X \to \mathcal{B}$ and a product map $\mu : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ that have the usual properties.

We follow [KRS05] and define

Definition 2.3.6. A bimodule algebra $\mathcal{B}$ is a *graded $(\mathcal{O}_X, \sigma)$-bimodule algebra* if:

(1) There are coherent sheaves $\mathcal{B}_n$ on X such that

$$\mathcal{B} = \bigoplus_{n \in \mathbb{Z}} {}_1(\mathcal{B}_n)_{\sigma^n};$$

(2) $\mathcal{B}_0 = \mathcal{O}_X$;

(3) the multiplication map μ is given by $\mathcal{O}_X$-module maps $\mathcal{B}_n \otimes \mathcal{B}_m^{\sigma^n} \to \mathcal{B}_{n+m}$, satisfying the obvious associativity conditions. Note that by Lemma 2.3.4(2),

$$(\mathcal{B}_n)_{\sigma^n} \otimes (\mathcal{B}_m)_{\sigma^m} = (\mathcal{B}_n \otimes \mathcal{B}_m^{\sigma^n})_{\sigma^{n+m}}.$$

Example 2.3.7. The twisted bimodule algebra $\mathcal{B}(X, \mathcal{L}, \sigma)$ from Example 2.3.1 is a graded $(\mathcal{O}_X, \sigma)$-bimodule algebra, with $\mathcal{B}_n = \mathcal{L}_n$ for all $n \geq 0$.

Definition 2.3.8. Let $\mathcal{B}$ be a graded $(\mathcal{O}_X, \sigma)$-bimodule algebra. A *right $\mathcal{B}$-module* $\mathcal{M}$ is a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ together with a right $\mathcal{O}_X$-module map μ :

$\mathcal{M} \otimes \mathcal{B} \to \mathcal{M}$ satisfying the usual axioms. We say that $\mathcal{M}$ is *graded* if there is a direct sum decomposition

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n)_{\sigma^n}$$

with multiplication giving a family of $\mathcal{O}_X$-module maps $\mathcal{M}_n \otimes \mathcal{B}_m^{\sigma^n} \to \mathcal{M}_{n+m}$, obeying the appropriate axioms.

We say that $\mathcal{M}$ is *coherent* if there is a coherent $\mathcal{O}_X$-module $\mathcal{M}'$ and a surjective map $\mathcal{M}' \otimes \mathcal{B} \to \mathcal{M}$ of ungraded $\mathcal{O}_X$-modules. We similarly define left $\mathcal{B}$-modules. The bimodule algebra $\mathcal{B}$ is *right (left) noetherian* if every right (left) ideal of $\mathcal{B}$ is coherent. By standard arguments, a graded $(\mathcal{O}_X, \sigma)$-bimodule algebra is right (left) noetherian if and only if every graded right (left) ideal is coherent.

If $\mathcal{B}$ is a graded $(\mathcal{O}_X, \sigma)$-bimodule algebra, we let Gr-$\mathcal{B}$ be the abelian category of graded right $\mathcal{B}$-modules, with morphisms those that preserve degree. A module $\mathcal{M} \in$ Gr-$\mathcal{B}$ is *bounded* if $\mathcal{M}_i = 0$ for all but finitely many i. We say that $\mathcal{M}$ is *torsion* if every coherent submodule of $\mathcal{M}$ is bounded. We denote the full subcategory of Gr-$\mathcal{B}$ of torsion modules by Tors-$\mathcal{B}$. This is a Serre subcategory, and as in the previous section, we define Qgr-$\mathcal{B}$ to be the quotient category Gr-$\mathcal{B}/$ Tors-$\mathcal{B}$. As before, let $\pi :$ Gr-$\mathcal{B} \to$ Qgr-$\mathcal{B}$ be the quotient functor. We will let gr-$\mathcal{B}$, qgr-$\mathcal{B}$, etc. be the full subcategories of noetherian objects, and we will similarly define $\mathcal{B}$-Gr, $\mathcal{B}$-qgr, etc.

Coherence for $\mathcal{B}$-modules should be viewed as analogous to finite generation, but it is unknown whether, for a general noetherian bimodule algebra, every submodule of a coherent module is coherent! Fortunately, in our situation the usual intuitions do hold. We restate [KRS05, Proposition 2.10] as:

Lemma 2.3.9. *Let $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{R}_n)_{\sigma^n}$ be a graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of a twisted bimodule algebra. Then $\mathcal{R}$ is right (left) noetherian if and only if all* *submodules of coherent right (left) $\mathcal{R}$-modules are coherent.*

Proof. ($\Leftarrow$) is clear. Conversely, suppose that $\mathcal{R}$ is right noetherian, so all right ideals of $\mathcal{R}$ are coherent. By [KRS05, Proposition 2.10], since the $\mathcal{R}_n$ are subsheaves of locally free sheaves, then all submodules of coherent right $\mathcal{R}$-modules are coherent. The left-handed result follows from symmetry. □

If $\mathcal{R}$ is a graded $(\mathcal{O}_X, \sigma)$ bimodule algebra, we may form its *section algebra*

$$H^0(X, \mathcal{R}) = H^0(\mathcal{R}) = \bigoplus_{n \geq 0} H^0(\mathcal{R}_n) = \bigoplus_{n \geq 0} H^0(X, \mathcal{R}_n).$$

(Throughout this thesis, we will omit the scheme X when taking global sections or cohomology unless the underlying scheme is not clear from context.)

Multiplication on $H^0(\mathcal{R})$ is induced from the multiplication map μ on $\mathcal{R}$; that is, from the maps

$$H^0(\mathcal{R}_n) \otimes H^0(\mathcal{R}_m) \to H^0(\mathcal{R}_n) \otimes H^0(\mathcal{R}_m^{\sigma^n}) \to H^0(\mathcal{R}_{n+m}).$$

Global sections give a functor from Gr-$\mathcal{R}$ to Gr-$H^0(\mathcal{R})$. If $\mathcal{M}$ is a graded right $\mathcal{R}$-module, define

$$H^0(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{M}_n).$$

This is a right $H^0(\mathcal{R})$-module in the obvious way.

If $R = H^0(\mathcal{R})$, and M is a graded right R-module, define $M \otimes_R \mathcal{R}$ to be the sheaf associated to the presheaf $V \mapsto M \otimes_R \mathcal{R}(V)$. This is a graded right $\mathcal{R}$-module, and the functor $\_\_ \otimes_R \mathcal{R} : \text{Gr-}R \to \text{Gr-}\mathcal{R}$ is a right adjoint to H^0.

Example 2.3.10. Let $\mathcal{B}(X, \mathcal{L}, \sigma)$ be a twisted bimodule algebra. Then the section algebra $H^0(\mathcal{B}(X, \mathcal{L}, \sigma))$ is the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$.

The fundamental result on when one can more closely relate Gr-$\mathcal{R}$ and Gr-R is due to Van den Bergh. We first give a definition:

Definition 2.3.11. Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ be a sequence of coherent sheaves on X. The sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is *right ample* if for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the following properties hold:

(i) $\mathcal{F} \otimes \mathcal{R}_n$ is globally generated for $n \gg 0$;

(ii) $H^q(\mathcal{F} \otimes \mathcal{R}_n) = 0$ for $n \gg 0$ and all $q \geq 1$.

The sequence $\{(\mathcal{R}_n)_{\sigma^n}\}$ is *left ample* if for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the following properties hold:

(i) $\mathcal{R}_n \otimes \mathcal{F}^{\sigma^n}$ is globally generated for $n \gg 0$;

(ii) $H^q(\mathcal{R}_n \otimes \mathcal{F}^{\sigma^n}) = 0$ for $n \gg 0$ and all $q \geq 1$.

We say that an invertible sheaf $\mathcal{L}$ is *σ-ample* if the $\mathcal{O}_X$-bimodules

$$\{(\mathcal{L}_n)_{\sigma^n}\} = \{\mathcal{L}_\sigma^{\otimes n}\}$$

form a right ample sequence. By [Kee00, Theorem 1.2], this is true if and only if the $\mathcal{O}_X$-bimodules $\{(\mathcal{L}_n)_{\sigma^n}\}$ form a left ample sequence.

The following result is a special case of a result due to Van den Bergh [Van96, Theorem 5.2], although we follow the presentation of [KRS05, Theorem 2.12]:

Theorem 2.3.12. (Van den Bergh) *Let X be a projective scheme and let σ be an automorphism of X. Let $\mathcal{R} = \bigoplus (\mathcal{R}_n)_{\sigma^n}$ be a right noetherian graded $(\mathcal{O}_X, \sigma)$-bimodule algebra, such that the bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ form a right ample sequence. Then $R = H^0(\mathcal{R})$ is also right noetherian, and the functors H^0 and $\_\_ \otimes_R \mathcal{R}$ induce an equivalence of categories*

$$\text{qgr-}\mathcal{R} \simeq \text{qgr-}R.$$

Theorem 1.2.3 follows easily from Theorem 2.3.12, and we give the proof here.

Proof of Theorem 1.2.3. Let $\mathcal{I} = \bigoplus (\mathcal{I}_n)_{\sigma^n}$ be a graded right ideal of $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$. The coherent $\mathcal{O}_X$-modules $\mathcal{I}_n \otimes (\mathcal{L}_n)^{-1}$ form an ascending chain of ideal sheaves on

X; thus they stabilize after some n_0, and we have a surjection $\mathcal{I}_{\leq n_0} \otimes \mathcal{B} \to \mathcal{I}$. Thus $\mathcal{B}$ is right noetherian. Arguing similarly, one sees that the functors

$$\mathcal{M} \mapsto \mathcal{M}_n \otimes (\mathcal{L}_n)^{-1} \text{ for } n \gg 0$$

and

$$\mathcal{F} \mapsto \mathcal{F} \otimes_X \mathcal{B}$$

give an equivalence of categories between $\mathcal{O}_X$-mod and qgr-$\mathcal{B}$. By assumption, $\mathcal{L}$ is σ-ample; thus by Theorem 2.3.12, the categories qgr-$\mathcal{B}$ and qgr-$B(X, \mathcal{L}, \sigma)$ are equivalent. Therefore $\mathcal{O}_X$-mod $\simeq$ qgr-$B(X, \mathcal{L}, \sigma)$. By symmetry, $\mathcal{O}_X$-mod $\simeq$ $B(X, \mathcal{L}, \sigma)$-qgr. $\square$

We introduce notation for the quasi-inverse functors between qgr-$\mathcal{B}$ and $\mathcal{O}_X$-mod. Define a functor

$$\Gamma_* : \mathcal{O}_X\text{-mod} \to \text{qgr-}B$$
$$\mathcal{F} \mapsto \bigoplus_{n \geq 0} H^0(\mathcal{F} \otimes \mathcal{L}_n).$$

The quasi-inverse of Γ_* is induced by a functor

$$\widetilde{\ } : \text{gr-}B \to \mathcal{O}_X\text{-mod}\,.$$

To define this functor, let $M \in$ gr-B. There is a unique coherent sheaf $\mathcal{F}$ such that $\mathcal{F} \otimes \mathcal{L}_n = (M \otimes_B \mathcal{B})_n$ for all $n \gg 0$. Define $\widetilde{M} = \mathcal{F}$. Note that if $\pi M = \pi N$ in qgr-R, then $\widetilde{M} = \widetilde{N}$.

Since right and left σ-ample invertible sheaves are the same, there is also an equivalence B-qgr $\simeq \mathcal{O}_X$-mod. The quasi-inverses between these two categories are defined by letting

$$\Gamma_* \mathcal{F} = \bigoplus_{n \geq 0} H^0(\mathcal{L}_n \otimes \mathcal{F}^{\sigma^n})$$

and letting $\widetilde{M}$ be the unique $\mathcal{F}$ such that $\mathcal{L}_n \otimes \mathcal{F}^{\sigma^n} = (\mathcal{B} \otimes_B M)_n$ for all $n \gg 0$.

We note that if $N \in \text{gr-}B$, then by [SV01, (3.1)], we have that

$$\widetilde{N[n]} \cong (\widetilde{N} \otimes \mathcal{L}_n)^{\sigma^{-n}} \tag{2.3.13}$$

for all $n \geq 0$.

We record here the observation that when working with bimodule algebras, we may in our setting suppose, without loss of generality, that we are working with sub-bimodule algebras of the twisted bimodule algebra $\mathcal{B}(X, \mathcal{O}_X, \sigma)$.

Lemma 2.3.14. *Let X be a projective scheme with automorphism σ, and let $\mathcal{L}$ be an invertible sheaf on X. Let*

$$\mathcal{R} = \bigoplus_{n \geq 0} (\mathcal{R}_n)_{\sigma^n}$$

be a graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of the twisted bimodule algebra $\mathcal{B}(X, \mathcal{L}, \sigma)$. Let $\mathcal{J}_n = \mathcal{R}_n \mathcal{L}_n^{-1}$ for $n \geq 0$.

(1) Let $\mathcal{S}$ be the graded $(\mathcal{O}_X, \sigma)$-bimodule algebra defined by

$$\mathcal{S} = \bigoplus_{n \geq 0} (\mathcal{S}_n)_{\sigma^n} = \bigoplus_{n \geq 0} (\mathcal{J}_n)_{\sigma^n}.$$

Then the categories gr-$\mathcal{R}$ *and* gr-$\mathcal{S}$ *are equivalent, and the categories* $\mathcal{S}$-gr *and* $\mathcal{R}$-gr *are equivalent.*

(2) Let $\mathcal{H}$ be an invertible sheaf on X and let $k \in \mathbb{Z}$. Then the functor $\mathcal{H}_{\sigma^k} \otimes \_\_$ that maps

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n)_{\sigma^n} \mapsto \bigoplus_{n \in \mathbb{Z}} (\mathcal{H} \otimes \mathcal{M}_n^{\sigma^k})_{\sigma^{k+n}} = \mathcal{H}_{\sigma^k} \otimes \mathcal{M}$$

is an autoequivalence of gr $\mathcal{R}$.

Proof. (1) By symmetry, it suffices to prove that gr-$\mathcal{R} \simeq$ gr-$\mathcal{S}$. For $n < 0$, define

$$\mathcal{L}_n = (\mathcal{L}^{-1})^{\sigma^n} \otimes (\mathcal{L}^{-1})^{\sigma^{n+1}} \otimes \cdots \otimes (\mathcal{L}^{-1})^{\sigma^{-1}}.$$

One easily verifies that

$$\mathcal{L}_m \otimes \mathcal{L}_n^{\sigma^m} \cong \mathcal{L}_{n+m}$$

for all $n, m \in \mathbb{Z}$. Define a functor $F : \text{gr-}\mathcal{R} \to \text{gr-}\mathcal{S}$ as follows: if

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n)_{\sigma^n}$$

is a graded right $\mathcal{R}$-module, define

$$F(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n \otimes (\mathcal{L}_n)^{-1})_{\sigma^n}.$$

The inverse functor $G : \text{gr-}\mathcal{S} \to \text{gr-}\mathcal{R}$ is defined as follows: if

$$\mathcal{N} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{N}_n)_{\sigma^n}$$

is a graded right $\mathcal{S}$-module, let

$$G(\mathcal{N}) = \bigoplus_{n \in \mathbb{Z}} (\mathcal{N}_n \otimes \mathcal{L}_n)_{\sigma^n}.$$

It is trivial that $GF \cong \mathrm{Id}_{\text{gr-}\mathcal{R}}$ and that $FG \cong \mathrm{Id}_{\text{gr-}\mathcal{S}}$.

(2) By Lemma 2.3.4(2), we have that

$$\big((\mathcal{H}^{\sigma^{-k}})^{-1}\big)_{\sigma^{-k}} \otimes \mathcal{H}_{\sigma^k} \cong {}_1\big((\mathcal{H}^{\sigma^{-k}})^{-1} \otimes \mathcal{H}^{\sigma^{-k}}\big)_1 \cong \mathcal{O}_X.$$

Thus the functor $((\mathcal{H}^{\sigma^{-k}})^{-1})_{\sigma^{-k}} \otimes \underline{\ \ }$ is a quasi-inverse to $\mathcal{H}_{\sigma^k} \otimes \underline{\ \ }$. $\square$

We also record here an elementary lemma on the two-sided ideals of twisted homogeneous coordinate rings; compare [AS95, Lemma 4.4].

Lemma 2.3.15. *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be an invertible sheaf on X. Let $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$ and let $\mathcal{J}$ be a two-sided ideal of $\mathcal{B}$. Then there is a σ-invariant ideal sheaf $\mathcal{I}$ so that $\mathcal{J}_n = \mathcal{I}\mathcal{L}_n$ for $n \gg 0$.*

Proof. By Lemma 2.3.14, without loss of generality we may let $\mathcal{L} = \mathcal{O}_X$. As $\mathcal{J}$ is a right ideal,

$$\mathcal{J}_{m+1} \supseteq \mathcal{J}_m \cdot \mathcal{B}_1 = \mathcal{J}_m$$

for all m. There is therefore some n so that $\mathcal{J}_m = \mathcal{J}_n$ for all $m \geq n$. Let $\mathcal{I} = \mathcal{J}_n$. As $\mathcal{J}$ is also a left ideal,

$$\mathcal{I} = \mathcal{J}_{n+1} \supseteq \mathcal{B}_1 \cdot \mathcal{J}_n^\sigma = \mathcal{J}_n^\sigma = \mathcal{I}^\sigma.$$

Thus $\mathcal{I} = \mathcal{I}^\sigma$. □

2.4 The χ conditions and the strong noetherian property

In this section we describe two properties, which, while technical, are needed to extend important techniques from commutative to noncommutative geometry. Because in their absence one's tools are relatively limited, it is important to understand when these properties hold.

We begin with the Artin-Zhang χ conditions.

Definition 2.4.1. Let R be a finitely $\mathbb{N}$-graded $\Bbbk$-algebra, and fix $j \in \mathbb{N}$. We say that R *satisfies right* χ_j if, for all $i \leq j$ and for all finitely generated graded right R-modules M, we have that

$$\dim_\Bbbk \underline{\operatorname{Ext}}^i_R(\Bbbk, M) < \infty.$$

We say that R *satisfies right* χ if R satisfies right χ_j for all $j \in \mathbb{N}$. We similarly define *left* χ_j and *left* χ; we say R *satisfies* χ if it satisfies left and right χ.

By [AZ94, Corollary 8.12], any commutative noetherian ring satisfies χ. It is an easy exercise to see that R satisfies right χ_0 if and only if R is right noetherian.

The most important of the χ conditions is χ_1. Artin and Zhang discovered that its presence allows one to reconstruct R from Proj-R. That is, we have:

Theorem 2.4.2. ([AZ94, Theorem 4.5]) *Let R be a finitely $\mathbb{N}$-graded $\Bbbk$-algebra, and let B be the $\mathbb{N}$-graded ring*

$$B = \underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}R}(\pi R, \pi R)_{n\geq 0} = \bigoplus_{n\geq 0} H^0(\mathrm{Proj}\text{-}R, \pi R[n])$$

If R satisfies right χ_1, then the canonical map $R \to B$ is an isomorphism in large degree. □

The higher conditions χ_j for $j > 1$ are less well understood. However, if a ring satisfies right or left χ, then it is well-behaved in some important ways. For example, by [AZ94, Theorem 7.4], R satisfies right χ if and only if the noncommutative version of Serre's finiteness theorem holds for Proj-R. That is, if R satisfies right χ, then for any $\mathcal{M} \in$ qgr-R, the cohomology $H^j(\mathrm{Proj}\text{-}R, \mathcal{M})$ is finite-dimensional for any $j \geq 0$, and for any $j \geq 1$,

$$H^j(\mathrm{Proj}\text{-}R, \mathcal{M}[n]) = 0$$

for $n \gg 0$.

The χ conditions are also needed in order to have a version of Serre duality for a noncommutative ring R. This is known as the existence of a *balanced dualizing complex* for R; see [Van97, Definition 6.2] for the precise definition. By results of Van den Bergh [Van97, Theorem 6.3] and Yekutieli and Zhang [YZ97, Theorem 4.2], R has a balanced dualizing complex if and only if R satisfies χ and both Proj-R and R-Proj have finite cohomological dimension.

The second technical condition we consider is the strong noetherian property.

Definition 2.4.3. We say that the $\Bbbk$-algebra R is *strongly right (left) noetherian* if, for any commutative noetherian $\Bbbk$-algebra C, the ring $R \otimes_{\Bbbk} C$ is right (left) noetherian.

The strong noetherian property is clearly related to questions of extending the base ring and working scheme-theoretically. All finitely generated commutative $\Bbbk$-algebras are strongly noetherian; however, the ring in Example 1.4.1 is an example of a finitely generated $\Bbbk$-algebra that is noetherian but not strongly noetherian on either side, by [Rog04a, Theorem 1.2].

The fundamental result about strongly noetherian rings is also due to Artin and Zhang. Before stating it, we define an important class of modules, which have the Hilbert series of a point in $\mathbb{P}^n$.

Definition 2.4.4. Let R be a connected $\mathbb{N}$-graded $\Bbbk$-algebra. A *(right or left) point module* is a cyclic graded (right or left) module M such that $\dim M_n = 1$ for all $n \geq 0$. A *(right or left) truncated point module of length d* is a module M such that $\dim M_n = 1$ for $0 \leq n \leq d$, and $M_n = 0$ otherwise.

Theorem 2.4.5. ([AZ01, Corollary E4.11]) *Let R be a connected $\mathbb{N}$-graded, strongly right noetherian $\Bbbk$-algebra. Then the right point modules over R are parameterized by a projective scheme.* □

We comment briefly on the construction of the point scheme for R. It is not hard to see that for any d, the right truncated point modules of length d are parameterized by a projective scheme, which we temporarily denote X_d. It turns out if R is strongly right noetherian, then there is some d such that the natural maps

$$X_{n+1} \to X_n$$

are isomorphisms for all $n \geq d$. The right point scheme for R is then isomorphic to this stable scheme X_d.

If we have a parameterization of the point modules for R, then understanding the point scheme can often provide crucial information about R. For example, consider the Sklyanin algebra $S = \mathrm{Skl}_3(E, \sigma)$ defined in Chapter I. The elliptic curve E turns out to be the point scheme for S, and this allows one to construct a map from S to a twisted homogeneous coordinate ring on E. On the other hand, we have seen that the ring of Example 1.4.1 is not strongly noetherian; by [KRS05, Theorem 1.1], the point modules over this ring are not parameterized by any scheme.

To end this section, we return to giving properties of twisted homogeneous coordinate rings. Intuition says that the χ conditions and the strong noetherian property should hold for "nice" rings, and in fact both hold for twisted homogeneous coordinate rings. We record this as

Theorem 2.4.6. *(Artin-Small-Zhang, Yekutieli, Van den Bergh) Let X be a projective variety, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let $B = B(X, \mathcal{L}, \sigma)$. Then B is strongly noetherian and satisfies χ.*

Proof. That twisted homogeneous coordinate rings are strongly noetherian is [ASZ99, Proposition 4.13]. By [Yek92, Theorem 7.3], B has a balanced dualizing complex. Then [Van97, Theorem 6.3] (or alternately, [YZ97, Theorem 4.2]) implies that B satisfies χ. $\square$

2.5 A few results from algebraic geometry

Our primary algebraic geometry reference is [Har77]. We include here a few results that we will use that are not included in that text.

For us, the term *divisor* means Cartier divisor. Recall that a (Cartier) divisor P on a projective variety X is *nef* if $P.C \geq 0$ for any curve C on X.

Theorem 2.5.1. (Fujita's Vanishing Theorem [Laz04, Theorem 1.4.35]) *Let X be a projective variety and let $\mathcal{F}$ be a coherent sheaf on X. Let N be an ample divisor on*

X. Then there is an integer m so that $H^i(\mathcal{F}(mN+P)) = 0$ for all nef divisors P and for all $i \geq 1$. □

We will also use the concept of *Castelnuovo-Mumford regularity.* Recall that if N is a very ample divisor on a projective variety X and $\mathcal{F}$ is a coherent sheaf on X, then $\mathcal{F}$ is *k-regular with respect to N* if, for all $i \geq 1$, we have that

$$H^i(\mathcal{F}((k-i)N)) = 0.$$

If $\mathcal{F}$ is k-regular with respect to N, it is $(k+n)$-regular for any $n \geq 0$, by [Laz04, Theorem 1.8.5(iii)]. The *regularity of $\mathcal{F}$* (with respect to N) is the minimal k such that $\mathcal{F}$ is k-regular with respect to N.

One of the most important applications of regularity is that it gives a criterion for a sheaf to be generated by its global sections.

Theorem 2.5.2. (Mumford's theorem [Laz04, Theorem 1.8.5(i)]) *Let X be a projective variety and let $\mathcal{F}$ be a coherent sheaf on X. Let N be a very ample divisor on X, and suppose that $\mathcal{F}$ is 0-regular with respect to N. Then $\mathcal{F}$ is globally generated.* □

We will need to use the Riemann-Roch theorem for singular curves, and we give that here as well. We denote linear equivalence of divisors by $\sim$.

Theorem 2.5.3. *Let X be a smooth surface, and let D be a reduced and irreducible curve on X. There are constants k and c, depending only on the isomorphism class of D, such that if C is any divisor on X with $C.D \geq k$, then*

(1) $H^1(\mathcal{O}_D(C)) = 0$;

(2) $\mathcal{O}_D(C)$ is globally generated;

(3) $h^0(\mathcal{O}_D(C)) = C.D + c$.

Proof. Because X is smooth, D is locally principal as a Cartier divisor, and so in particular is a local complete intersection.

Let N be a very ample Cartier divisor on D. Without loss of generality, by [Har77, Exercise IV.1.9(b)], we may assume that N is supported in D^{reg}. Let $b = \deg(N)$. Let ω be the Serre dualizing sheaf on D. By [Har77, Theorem III.7.11], ω is invertible; let K be a divisor with support in D^{reg} so that $\mathcal{O}_D(K) \cong \omega$. Let $w = \deg K$.

Suppose that C is a divisor on X such that $C.D \geq w + 1$. We claim that $H^1(\mathcal{O}_D(C)) = 0$. To see this, let H be a very ample divisor on X so that $H + C$ is also very ample. Applying Bertini's theorem, by [Har77, Lemma V.1.2], we may choose irreducible nonsingular curves $E \sim H$ and $E' \sim H + C$ such that both E and E' are nonsingular and meet D transversally (in particular, they do not meet the singular locus of D). Now, by Serre duality, we have that

$$h^1(\mathcal{O}_D(C)) = h^1(\mathcal{O}_D(E' - E)) = h^0(\mathcal{O}_D(K + E - E')).$$

This is 0, since $\deg_D(\mathcal{O}_D(K + E - E')) = w - D.C < 0$.

Now suppose that $C.D \geq w + b + 1$. By the above, $H^1(\mathcal{O}_D(C)) = 0$. Furthermore, $\deg_D(\mathcal{O}_D(C) - N) \geq w + 1$, and so $H^1(\mathcal{O}_D(C) - N) = 0$. Then $\mathcal{O}_D(C)$ is 0-regular with respect to N, and Theorem 2.5.2 implies that $\mathcal{O}_D(C)$ is globally generated.

(3) is [Har77, Exercise IV.1.9(a)] combined with (1). $\square$

CHAPTER III

Geometric idealizer rings

3.1 Introduction

In recent years, many examples have appeared of subrings of twisted homogeneous coordinate rings that have unusual and counter-intuitive properties. While these rings are often noetherian (indeed, generically so in many cases) and are birationally commutative by construction, subtler properties such as the χ conditions and the strong noetherian property can fail. Examples of such rings include the naïve blowup algebras defined in (1.4.2), first constructed by Keeler, Rogalski, and Stafford [KRS05], and the idealizers (1.4.7), studied by Rogalski in [Rog04b]. Since ideally a classification effort in noncommutative geometry would not depend on technical conditions, understanding when these kinds of examples occur is important. In particular, understanding a broad range of noetherian subrings of twisted homogeneous coordinate rings is necessary to fully classify noncommutative surfaces.

In this chapter we investigate one particular class of subrings of twisted homogeneous coordinate rings. We repeat Construction 1.2.5, with more detail.

Construction 3.1.1. Let X be a projective variety over an algebraically closed field $\Bbbk$, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Z be a closed subscheme of X. Following Example 2.3.10, form the twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$, and let I be the right ideal of B generated by sections that vanish on Z.

Our object of study is the ring

$$R = R(X, \mathcal{L}, \sigma, Z) = \mathbb{I}_B(I) = \{x \in B \mid xI \subseteq I\}.$$

By construction, R is the maximal subring of B in which the right ideal I becomes a two-sided ideal. We refer to R as a *geometric idealizer*, or more specifically, as the *(right) idealizer at Z inside B*.

The main goal of this chapter is to study the properties of geometric idealizers, and, in particular, to understand how these algebraic properties are controlled by the geometry of the defining data. At a basic level, we want to know when $R(X, \mathcal{L}, \sigma, Z)$ is noetherian. We also analyze when idealizers are strongly noetherian, satisfy various χ conditions, and have finite cohomological dimension. In general, the ways in which it is possible for these properties to fail are still poorly understood. Thus another goal of the work in this chapter is to gain more insight into these issues.

Our work generalizes work of Rogalski [Rog04b], who investigated idealizers at points in $\mathbb{P}^d$ using algebraic techniques. His work generalized earlier work of Stafford and Zhang [SZ94], who studied idealizers on $\mathbb{P}^1$. Rogalski worked in the more algebraic setting of *Zhang twists* of polynomial rings, and here we give the relevant definitions.

Definition 3.1.2. (cf. [Zha96]) Let $d \geq 1$ and let σ be an automorphism of $\mathbb{P}^d$. We also let σ denote the graded automorphism of $\mathbb{k}[x_0, \ldots, x_d]$ induced by σ; that is $x_j^\sigma = x_j \circ \sigma$. Then the *Zhang twist* of $\mathbb{k}[x_0, \ldots, x_d]$ by σ is written

$$\mathbb{k}[x_0, \ldots, x_d]^\sigma.$$

As graded vector spaces, $\Bbbk[x_0, \ldots, x_d]^\sigma$ and $\Bbbk[x_0, \ldots, x_d]$ are isomorphic. Let $\cdot$ denote the multiplication in $\Bbbk[x_0, \ldots, x_d]$. The multiplication $\star$ on $\Bbbk[x_0, \ldots, x_d]^\sigma$ is induced by the rule

$$x_i \star x_j = x_i \cdot x_j^\sigma.$$

Technically, the automorphism σ of $\Bbbk[x_0, \ldots, x_d]$ is defined up to a choice of multiplicative scalars. However, for any such choice of scalars, we obtain an isomorphic ring $\Bbbk[x_0, \ldots, x_d]^\sigma$. In fact,

$$\Bbbk[x_0, \ldots, x_d]^\sigma \cong B(\mathbb{P}^d, \mathcal{O}(1), \sigma).$$

We leave the verification to the reader.

Recall (Definition 1.4.3) that $\{\sigma^n(x)\}$ is *critically dense* if it is infinite and any infinite subset is Zariski dense in $\mathbb{P}^d$.

Theorem 3.1.3. (Rogalski) *Let σ be an automorphism of $\mathbb{P}^d$, and let*

$$B = k[x_0, x_1, \ldots, x_d]^\sigma \cong B(\mathbb{P}^d, \mathcal{O}(1), \sigma).$$

Let $p \in \mathbb{P}^d$, and let I be the right ideal of B of functions vanishing at p. Assume that x is of infinite order under σ, and let $R = \mathbb{I}_B(I) \cong R(\mathbb{P}^d, \mathcal{O}(1), \sigma, \{p\})$. Then

(1) *R is strongly right noetherian.*

(2) *R fails left χ_1.*

Further, if the set $\{\sigma^n(p)\}$ is critically dense, then:

(3) *R is left noetherian but not strongly left noetherian.*

(4) *R satisfies right χ_{d-1} but fails right χ_d.*

One interesting aspect of Rogalski's work is that the geometry driving the algebraic conclusions of (3) and (4) is rather subtle. Theorem 3.1.3 shows that right idealizers at points of infinite order are automatically right noetherian, but in order for them to be left noetherian, σ must move x significantly and in some sense uniformly around $\mathbb{P}^d$.

The aim of this chapter is to work out in detail the properties of $R(X, \mathcal{L}, \sigma, Z)$ for a general projective variety X and subschemes $Z \subseteq X$ of arbitrary dimension. In particular, we would like to understand if there is a higher-dimensional analogue of critical density that controls the behavior of more general idealizers than those studied in Theorem 3.1.3.

The answer is "yes." We define:

Definition 3.1.4. Let X be a projective variety and let $Z, Y \subseteq X$ be closed subschemes. We say that Z and Y are *homologically transverse* if

$$\mathcal{T}or_j^X(\mathcal{O}_Z, \mathcal{O}_Y) = 0$$

for all $j \geq 1$.

Definition 3.1.5. Let X be a projective variety and let $\sigma \in \operatorname{Aut} X$. Let $Z \subseteq X$ be a closed subscheme. The set $\{\sigma^n Z\}_{n \in \mathbb{Z}}$ is *critically transverse* in X if for all closed subschemes $Y \subseteq X$, the subschemes $\sigma^n(Z)$ and Y are homologically transverse for all but finitely many n.

In this chapter, we generalize Theorem 3.1.3 to arbitrary idealizers in twisted homogeneous coordinate rings. We show that critical transversality controls the behavior of these rings, and we prove:

Theorem 3.1.6. (Theorem 3.8.2) *Let X be a projective variety, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Form the ring $R(X, \mathcal{L}, \sigma, Z)$ as above. For simplicity, assume that Z is irreducible and of infinite order under σ. (We treat the general case in the body of the chapter).*

If for all $p \in X$, the set $\{n \geq 0 \mid \sigma^n(p) \in Z\}$ is finite, then:

(1) *R is strongly right noetherian.*

(2) *R fails left χ_1.*

If the set $\{\sigma^n Z\}_{n\in\mathbb{Z}}$ is critically transverse, then

(3) *R is left noetherian, but R is strongly left noetherian if and only if all components of Z have codimension 1.*

(4) *Let $d = \operatorname{codim} Z$. Then R fails right χ_d. If X and Z are smooth, then R satisfies right χ_{d-1}.*

Furthermore, if R is noetherian, then R has finite left and right cohomological dimension.

On the other hand, we give an example of a right but not left noetherian ring that has infinite right cohomological dimension, partially answering a question of Stafford and Van den Bergh [SV01, page 194].

In the remainder of the introduction, we explain the geometric meaning behind the technical-looking definition of critical transversality. We first explain the use of the term "transverse." Let Y and Z be closed subschemes of X, and recall [Har77, p. 427] Serre's definition of the intersection multiplicity of Y and Z along the proper component P of their intersection:

$$i(Y, Z; P) = \sum_{i\geq 0} (-1)^i \operatorname{len}_P(\mathcal{T}or_i^X(\mathcal{O}_Y, \mathcal{O}_Z)),$$

where $\operatorname{len}_P(\mathcal{F})$ is the length of $\mathcal{F}_P$ over the local ring $\mathcal{O}_{X,P}$.

Suppose that Y and Z are homologically transverse. Then their intersection multiplicity is given by the naïve formula that $i(Y, Z; P)$ is the length of the structure sheaf of their scheme-theoretic intersection over the local ring at P. We note that if $\operatorname{char} \Bbbk = 0$, X, Y, and Z are smooth, and Y and Z meet transversally, then Y and

Z are homologically transverse.

Another way of viewing the critical transversality of $\{\sigma^n(Z)\}$ is that for any Y, the general translate of Z is homologically transverse to Y. This sort of statement is clearly reminiscent of the Kleiman-Bertini theorem, and in fact the investigations in this chapter have led to a new, purely algebro-geometric, generalization of this classical result. Furthermore, as an application of our generalized Kleiman-Bertini theorem, we are able to obtain a simple criterion for the critical transversality of $\{\sigma^n(Z)\}$ in many cases. We discuss these results in Chapter V.

3.2 Right noetherian bimodule algebras

Let X, $\mathcal{L}$, σ, and Z be as in Construction 3.1.1, and let R be the geometric idealizer ring

$$R = R(X, \mathcal{L}, \sigma, Z).$$

The key technique in this chapter is to work, not with R, but with the corresponding bimodule algebra. To define this object, we first introduce some notation on operations with ideal sheaves. For any two ideal sheaves $\mathcal{K}$ and $\mathcal{J}$ on X, we define the *ideal quotient*

$$(\mathcal{J} : \mathcal{K})$$

to be the maximal coherent subsheaf $\mathcal{F}$ of $\mathcal{O}_X$ such that $\mathcal{K}\mathcal{F} \subseteq \mathcal{J}$.

Notation 3.2.1. Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be an invertible sheaf on X. Let Z be a closed subscheme of X and let $\mathcal{I} = \mathcal{I}_Z$ be its defining ideal. Following Example 2.3.1, let

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma),$$

and let $\mathcal{R}$ be the graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of $\mathcal{B}$ defined by

$$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} \big((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n\big)_{\sigma^n}.$$

It is straightforward to compute that $\mathcal{R}$ is the maximal sub-bimodule algebra of $\mathcal{B}$ such that $\mathcal{I}\mathcal{B}_+$ is a two-sided ideal of $\mathcal{R}$, and we will write

$$\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I}\mathcal{B}_+)$$

and speak of $\mathcal{R}$ as an *idealizer bimodule algebra* inside $\mathcal{B}$. As usual we write

$$\mathcal{R} = \bigoplus_{n \geq 0} (\mathcal{R}_n)_{\sigma^n},$$

so

$$\mathcal{R}_n = (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n.$$

We note here that

$$\mathcal{B}(X, \mathcal{L}, \sigma)^{(n)} \cong \mathcal{B}(X, \mathcal{L}_n, \sigma^n)$$

and that

$$\mathcal{R}(X, \mathcal{L}, \sigma, Z)^{(n)} \cong \mathcal{R}(X, \mathcal{L}_n, \sigma^n, Z).$$

In the next lemma, we show that $R(X, \mathcal{L}, \sigma, Z)$ is precisely the section ring of the bimodule algebra $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$.

Lemma 3.2.2. *Assume Notation 3.2.1, and let $R = R(X, \mathcal{L}, \sigma, Z)$ as in Construction 3.1.1. If $\mathcal{L}$ is σ-ample, then*

$$R = R(X, \mathcal{L}, \sigma, Z) = H^0(\mathcal{R}(X, \mathcal{L}, \sigma, Z)).$$

Proof. Let $I = \Gamma_*(\mathcal{I})$ be the right ideal of $B(X, \mathcal{L}, \sigma)$ generated by sections vanishing along Z. Suppose that $x \in R_n$, so $xI \subseteq I$. Since $\mathcal{L}$ is σ-ample, $\mathcal{I}\mathcal{L}_m$ is globally generated by $I_m = H^0(\mathcal{I}\mathcal{L}_m)$ for $m \gg 0$, and so for $m \gg 0$

$$x\mathcal{O}_X(\mathcal{I}\mathcal{L}^{\sigma^m})^{\sigma^n} = x\mathcal{O}_X(I_m\mathcal{O}_X)^{\sigma^n} \subseteq I_{m+n}\mathcal{O}_X = \mathcal{I}\mathcal{L}_{m+n}$$

for any n. Thus $x\mathcal{O}_X \subseteq (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n$ and $x \in H^0((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n) = H^0(\mathcal{R}_n)$.

For the other containment, suppose that $x \in H^0(\mathcal{R}_n)$. Then for any $m \geq 0$ we have

$$xI_m \subseteq H^0((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n) \cdot H^0((\mathcal{I}\mathcal{L}_m)^{\sigma^n}) \subseteq H^0(\mathcal{I}\mathcal{L}_{m+n}) = I_{n+m}.$$

Thus $x \in R_n$, and we have established the equality we seek. $\square$

In this section, we will determine when $\mathcal{R}$ is right noetherian; we will show that this is controlled by a straightforward geometric property of the motion of Z under σ. To analyze the bimodule algebra $\mathcal{R}$, we will need some basic lemmas. We first give an elementary result that allows us to pass from one noetherian idealizer bimodule algebra to a larger one.

Lemma 3.2.3. *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be an invertible sheaf on X. Let $\mathcal{B}$ be a graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of $\mathcal{B}(X, \mathcal{L}, \sigma)$, and let $\mathcal{R}$ and $\mathcal{R}'$ be graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebras of $\mathcal{B}$. Suppose that $\mathcal{R}$ is right noetherian and contains a nonzero graded right ideal of $\mathcal{B}$ and that there is some n_0 so that*

$$\mathcal{R}_{\geq n_0} \subseteq \mathcal{R}'_{\geq n_0}.$$

Then $\mathcal{R}'$ is right noetherian. If $\mathcal{R} \subseteq \mathcal{R}'$, then $\mathcal{R}'$ is a coherent right $\mathcal{R}$-module.

Proof. By Lemma 2.3.14, without loss of generality we may assume that $\mathcal{L} = \mathcal{O}_X$.

We note that $\mathcal{R}_{\geq n_0}$ also contains a nonzero graded right ideal of $\mathcal{B}$. Further, $\mathcal{R} \cap \mathcal{R}'$ is also right noetherian, as $(\mathcal{R} \cap \mathcal{R}')_{\geq n_0} = \mathcal{R}_{\geq n_0}$. Thus without loss of generality we may assume that $\mathcal{R} \subseteq \mathcal{R}'$.

Let $\mathcal{J}$ be a nonzero graded right ideal of $\mathcal{B}$ that is contained in $\mathcal{R}$; let m be such that $\mathcal{J}_m \neq 0$. Let $\mathcal{H}$ be an invertible ideal sheaf contained in $\mathcal{J}_m$. As $\mathcal{R}$ is right noetherian and $\mathcal{H}\mathcal{R}' \subseteq \mathcal{H}\mathcal{B} \subseteq \mathcal{R}$, we see that $\mathcal{H}\mathcal{R}'$ is a coherent right $\mathcal{R}$-module. Lemma 2.3.14 now implies that $\mathcal{R}'$ is a coherent right $\mathcal{R}$-module.

Any right ideal of $\mathcal{R}'$ is also a right $\mathcal{R}$-submodule, and so is coherent as an $\mathcal{R}$-module. It is thus also coherent as an $\mathcal{R}'$-module. Thus $\mathcal{R}'$ is right noetherian. $\square$

We will also use primary decomposition of ideal sheaves. We give the definitions here. Let $\mathcal{I}$ be a proper ideal sheaf on X. We will say that $\mathcal{I}$ is *prime* if it defines a reduced and irreducible subscheme of X. We say that $\mathcal{I}$ is $\mathcal{P}$*-primary* if there is a prime ideal sheaf $\mathcal{P}$ such that some $\mathcal{P}^n \subseteq \mathcal{I}$, and for all ideal sheaves $\mathcal{J}$ and $\mathcal{K}$ on X, if $\mathcal{J}\mathcal{K} \subseteq \mathcal{I}$ but $\mathcal{J} \not\subseteq \mathcal{P}$, then $\mathcal{K} \subseteq \mathcal{I}$.

Since primary decompositions localize, the theory of primary decomposition of ideals in a commutative ring translates straightforwardly to ideal sheaves on X. In particular, any ideal sheaf $\mathcal{I}$ has a *minimal primary decomposition*

$$\mathcal{I} = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_c,$$

where each $\mathcal{I}_i$ is $\mathcal{P}_i$-primary for some prime ideal sheaf $\mathcal{P}_i$, the $\mathcal{P}_i$ are all distinct, and $\mathcal{I}$ may not be written as an intersection with fewer terms. If $\mathcal{P}_i$ is a minimal prime over $\mathcal{I}$, then we will refer to $\mathcal{I}_i$ as a *minimal primary component* of $\mathcal{I}$. If $\mathcal{P}_i$ is not minimal over $\mathcal{I}$, we will refer to $\mathcal{I}_i$ as an *embedded primary component.* As is well-known, the primes $\mathcal{P}_i$ and the minimal primary components of $\mathcal{I}$ are uniquely determined by $\mathcal{I}$, while the embedded primary components are not necessarily unique.

Now let Z be a closed subscheme of X and let $\mathcal{I}$ be the ideal sheaf of Z. Let $\mathcal{I} = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_c$ be a minimal primary decomposition of $\mathcal{I}$. We will refer to the closed subschemes Z_i defined by the minimal primary components $\mathcal{I}_i$ of $\mathcal{I}$ as the *irreducible components* of Z. We will refer to the subschemes defined by embedded primary components as *embedded components* of Z. Together, the irreducible and embedded components make up the *primary components* of Z.

We record the following elementary lemmas for future use.

Lemma 3.2.4. *Let $\mathcal{I} = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_c$ be a primary decomposition of the ideal sheaf $\mathcal{I}$, where $\mathcal{I}_i$ is $\mathcal{Q}_i$-primary for some prime ideal sheaf $\mathcal{Q}_i$.*

(1) *If $\mathcal{K}$ and $\mathcal{J}$ are ideal sheaves so that $\mathcal{K} \not\subseteq \mathcal{Q}_i$ for some i, then*

$$\big(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})\big) \subseteq (\mathcal{I}_i : \mathcal{J}).$$

(2) *If $\mathcal{K}$ is not contained in any $\mathcal{Q}_i$, then $(\mathcal{I} : \mathcal{K}) = \mathcal{I}$.*

Proof. (1) We have

$$\big(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})\big)\mathcal{K}\mathcal{J} \subseteq \big(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})\big)(\mathcal{K} \cap \mathcal{J}) \subseteq \mathcal{I} \subseteq \mathcal{I}_i.$$

As $\mathcal{K} \not\subseteq \mathcal{Q}_i$, we have

$$\big(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})\big)\mathcal{J} \subseteq \mathcal{I}_i$$

and so

$$\big(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})\big) \subseteq (\mathcal{I}_i : \mathcal{J}).$$

(2) Applying (1) with $\mathcal{J} = \mathcal{O}_X$, we see that

$$(\mathcal{I} : \mathcal{K}) \subseteq \bigcap_{i=1}^{c} (\mathcal{I}_i : \mathcal{O}_X) = \bigcap_{i=1}^{c} \mathcal{I}_i = \mathcal{I}.$$

The other containment is automatic. □

Lemma 3.2.5. *Let $\mathcal{P}$ and $\mathcal{I}$ be ideal sheaves on the variety X, where $\mathcal{P}$ is prime and $\mathcal{I}$ is $\mathcal{P}$-primary. If $\mathcal{J}$ is an ideal sheaf on X that is not contained in $\mathcal{I}$, then $(\mathcal{I} : \mathcal{J})$ is also $\mathcal{P}$-primary.*

Proof. Since $\mathcal{J} \not\subseteq \mathcal{I}$, we have that $(\mathcal{I} : \mathcal{J}) \neq \mathcal{O}_X$. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are ideal sheaves with $\mathcal{F} \not\subseteq \mathcal{P}$ and $\mathcal{F}\mathcal{G} \subseteq (\mathcal{I} : \mathcal{J})$. Thus $\mathcal{F}\mathcal{G}\mathcal{J} \subseteq \mathcal{I}$, and since $\mathcal{I}$ is $\mathcal{P}$-primary, we have that $\mathcal{G}\mathcal{J} \subseteq \mathcal{I}$. This precisely says that $\mathcal{G} \subseteq (\mathcal{I} : \mathcal{J})$. Since for some m, we have $\mathcal{P}^m \subseteq \mathcal{I} \subseteq (\mathcal{I} : \mathcal{J})$, we see that $(\mathcal{I} : \mathcal{J})$ is $\mathcal{P}$-primary. □

We next translate some general results on idealizers to the context of bimodule algebras. We give these results in a slightly more general context than we are currently considering, to allow us to use them in Chapter IV.

The following result is originally due to Robson [Rob72, Proposition 2.3(i)], although we will follow Stafford's restatement of it.

Lemma 3.2.6. ([Sta85, Lemma 1.1]) *Let I be a right ideal of a right noetherian ring B, and let $R = \mathbb{I}_B(I)$. If B/I is a right noetherian R-module, then R is right noetherian.*

Our version of this is the following lemma.

Lemma 3.2.7. *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be an invertible sheaf on X. Let $\mathcal{B}$ be a right noetherian graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of the twisted bimodule algebra $\mathcal{B}(X, \mathcal{L}, \sigma)$, and let $\mathcal{I} = \bigoplus(\mathcal{I}_n)_{\sigma^n}$ be a nonzero graded right ideal of $\mathcal{B}$. Let $\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I})$. Then $\mathcal{B}/\mathcal{I}$ is a noetherian right $\mathcal{R}$-module if and only if $\mathcal{R}$ is right noetherian.*

Proof. The proof is a straightforward translation of Robson's proof into sheaf terminology. By Lemma 2.3.14, without loss of generality we may let $\mathcal{L} = \mathcal{O}_X$. Thus all $\mathcal{R}_n$ and all $\mathcal{I}_n$ are ideal sheaves on X.

By Lemma 3.2.3, if $\mathcal{R}$ is right noetherian, certainly $\mathcal{B}_{\mathcal{R}}$ and thus $(\mathcal{B}/\mathcal{I})_{\mathcal{R}}$ are also.

So suppose that $\mathcal{B}/\mathcal{I}$ is a noetherian right $\mathcal{R}$-module. Let $\mathcal{J}$ be a right ideal of $\mathcal{R}$; we will show that $\mathcal{J}$ is coherent. Because $\mathcal{B}$ is right noetherian, we may choose a coherent sheaf $\mathcal{J}' \subseteq \mathcal{J}$ such that $\mathcal{J}'\mathcal{B} = \mathcal{J}\mathcal{B}$. It suffices to show that $\mathcal{J}/\mathcal{J}'\mathcal{R}$ is a coherent right $\mathcal{R}$-module.

Now, $\mathcal{J}/\mathcal{J}'\mathcal{R}$ is a submodule of $(\mathcal{J}'\mathcal{B} \cap \mathcal{R})/\mathcal{J}'\mathcal{R}$. Further, it is killed by $\mathcal{I}$ and so is a subfactor of $\mathcal{J}' \otimes (\mathcal{B}/\mathcal{I})$. Since $\mathcal{B}/\mathcal{I}$ is a noetherian right $\mathcal{R}$-module, so is $\mathcal{J}' \otimes (\mathcal{B}/\mathcal{I})$. Thus the subfactor $\mathcal{J}/\mathcal{J}'\mathcal{R}$ is coherent. $\square$

The criterion in Lemma 3.2.6 can be hard to test. Stafford [Sta85, Lemma 1.2] gave a different criterion for an idealizer to be noetherian; it was later slightly strengthened by Rogalski [Rog04b, Proposition 2.1]. We give the following version, which is adequate for our needs.

Lemma 3.2.8. *Let B be a right noetherian domain, let I be a right ideal of B, and let $R = \mathbb{I}_B(I)$. Then the following are equivalent:*

(1) *R is right noetherian;*

(2) *B_R is finitely generated, and for all right ideals J of B such that $J \supseteq I$, we have that $\operatorname{Hom}_B(B/I, B/J)$ is a noetherian right R-module (or R/I-module).*

Proof. (2) $\Rightarrow$ (1) is [Sta85, Lemma 1.2]. For (1) $\Rightarrow$ (2), note that if R is noetherian, as B is a domain we have $B_R \hookrightarrow R_R$ and so B_R is finitely generated. The rest of the argument is [Rog04b, Proposition 2.1]. $\square$

Our version of this is the following lemma:

Lemma 3.2.9. *Let X be a projective variety, and let $\sigma \in \operatorname{Aut} X$. Let $\mathcal{B}$ be a right noetherian graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of the twisted bimodule algebra $\mathcal{B}(X, \mathcal{O}_X, \sigma)$, and let $\mathcal{I} = \bigoplus (\mathcal{I}_n)_{\sigma^n}$ be a nonzero graded right ideal of $\mathcal{B}$. Let $\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I})$. Suppose that for all graded right ideals $\mathcal{J} \supseteq \mathcal{I}$ of $\mathcal{B}$, we have that for $n \gg 0$,*

$$\mathcal{B}_n \cap \bigcap_{m \geq 0} (\mathcal{J}_{n+m} : \mathcal{I}_m^{\sigma^n}) = \mathcal{J}_n.$$

Then $\mathcal{R}$ is right noetherian.

Proof. We follow Stafford's proof of [Sta85, Lemma 1.2]. Assume that the hypotheses of the lemma hold; we claim that $\mathcal{B}/\mathcal{I}$ is a noetherian right $\mathcal{R}$-module.

Let $\mathcal{G}$ be a graded right $\mathcal{R}$-module with $\mathcal{I} \subseteq \mathcal{G} \subseteq \mathcal{B}$. We seek to prove that $\mathcal{G}/\mathcal{I}$ is coherent. Let $\mathcal{J}$ be the largest graded right ideal of $\mathcal{B}$ of the form $\mathcal{G}'\mathcal{I}$ for some coherent graded $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{G}$. ($\mathcal{J}$ exists because $\mathcal{B}$ is right noetherian.) By maximality of $\mathcal{J}$, we have $\mathcal{I} \subseteq \mathcal{J}$.

Using Zorn's lemma, let $\mathcal{C}$ be the maximal quasicoherent subsheaf of $\mathcal{B}$ such that $\mathcal{C}\mathcal{I} \subseteq \mathcal{J}$. Obviously, $\mathcal{C}$ is graded. Note that

$$\mathcal{C}_n = \mathcal{B}_n \cap \bigcap_{m \geq 0} (\mathcal{J}_{n+m} : \mathcal{I}_m^{\sigma^n}).$$

Since by assumption $\mathcal{C}\mathcal{R}\mathcal{I} \subseteq \mathcal{C}\mathcal{I} \subseteq \mathcal{J}$, we have that $\mathcal{C}\mathcal{R} \subseteq \mathcal{C}$ and $\mathcal{C}$ is a right $\mathcal{R}$-submodule of $\mathcal{B}$. Since $\mathcal{C}_n = \mathcal{J}_n$ for $n \gg 0$, the right $\mathcal{R}$-module $\mathcal{C}/\mathcal{J}$ is in fact a coherent $\mathcal{O}_X$-module.

We claim that $\mathcal{G} \subseteq \mathcal{C}$. Suppose not. We may choose a coherent graded $\mathcal{O}_X$-submodule $\mathcal{G}''$ of $\mathcal{G}$ such that $\mathcal{G}'' \not\subseteq \mathcal{C}$, and so $\mathcal{G}''\mathcal{I} \not\subseteq \mathcal{J}$. Then $(\mathcal{G}' + \mathcal{G}'')\mathcal{I} \supsetneqq \mathcal{J}$ by choice of $\mathcal{G}''$, contradicting the maximality of $\mathcal{J}$. Thus $\mathcal{G} \subseteq \mathcal{C}$.

Since $\mathcal{C}/\mathcal{J}$ is a coherent $\mathcal{O}_X$-module, so is the submodule $\mathcal{G}/\mathcal{J}$. Since $\mathcal{J}_{\mathcal{R}}$ is coherent and $\mathcal{G}/\mathcal{J}$ is a coherent $\mathcal{O}_X$-module, $\mathcal{G}_{\mathcal{R}}$ is coherent. Thus $\mathcal{G}/\mathcal{I}_{\mathcal{R}}$ is also coherent. Since $\mathcal{G}$ was arbitrary, we have shown that $\mathcal{B}/\mathcal{I}$ is a right noetherian $\mathcal{R}$-module. Applying Lemma 3.2.7, we obtain that $\mathcal{R}$ is a right noetherian bimodule algebra. □

One technical difficulty in studying the bimodule algebra $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is that if Z has multiple components, it may be difficult to compute $(\mathcal{I} : \mathcal{I}^{\sigma^n})$ and thus $\mathcal{R}$. However, if Z is irreducible, then computing $\mathcal{R}$ is straightforward.

Lemma 3.2.10. *Assume Notation 3.2.1. Suppose in addition that Z is irreducible and without embedded components. If Z^{red} has infinite order under σ, then $\mathcal{R} = \mathcal{O}_X \oplus \mathcal{I}\mathcal{B}_+$.*

Proof. Let $\mathcal{P}$ be the ideal sheaf of Z^{red}. For $n \geq 1$, clearly $\mathcal{I}^{\sigma^n} \not\subseteq \mathcal{P}$, since $\mathcal{I}^{\sigma^n}$ is $\mathcal{P}^{\sigma^n}$-primary and $\mathcal{P}^{\sigma^n} \neq \mathcal{P}$. The result follows from Lemma 3.2.4(2) and the identification $\mathcal{R}_n = (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n$. $\square$

We now give a geometric condition that is equivalent to $\mathcal{R}$ being right noetherian, at least in the setting that the components of Z are of infinite order under σ.

Definition 3.2.11. Let $x \in X$ and let σ be an automorphism of X. The *forward σ-orbit* or *forward orbit* of x is the set

$$\{\sigma^n(x) \mid n \geq 0\}.$$

If $Z \subset X$ is such that for any $x \in X$, the set

$$\{n \geq 0 \mid \sigma^n(x) \in Z\}$$

is finite, we say that Z *has finite intersection with forward orbits.* In particular, if Z has finite intersection with forward orbits, it contains no points of finite order under σ.

Lemma 3.2.12. *Assume Notation 3.2.1. Let*

$$\mathcal{I} = \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_c$$

be a minimal primary decomposition, where each $\mathcal{K}_i$ is $\mathcal{Q}_i$-primary for some prime ideal sheaf $\mathcal{Q}_i$. For $i = 1 \ldots c$, let Z_i be the primary component of Z corresponding to $\mathcal{K}_i$, and let

$$\mathcal{R}^i = \mathbb{I}_{\mathcal{B}}(\mathcal{K}_i\mathcal{B}_+) = \mathcal{R}(X, \mathcal{L}, \sigma, Z_i).$$

Suppose that for all $1 \leq i, j \leq c$ the set

$$\{m \geq 0 \mid \mathcal{K}_j^{\sigma^m} \subseteq \mathcal{Q}_i\}$$

is finite. (In particular, we assume that the $\mathcal{Q}_i$ are of infinite order under σ.) Then $\mathcal{R}_m = \mathcal{I}\mathcal{L}_m$ for $m \gg 0$. Further, the following are equivalent:

(1) *$\mathcal{R}$ is right noetherian;*

(2) *$\mathcal{R}^i$ is right noetherian for $i = 1 \ldots c$;*

(3) *Z has finite intersection with forward orbits;*

(4) *if $\mathcal{J}$ is an ideal sheaf on X such that $\mathcal{J} \supseteq \mathcal{I}$, then $(\mathcal{J} : \mathcal{I}^{\sigma^m}) = \mathcal{J}$ for $m \gg 0$;*

(5) *the bimodule algebra*

$$\mathcal{O}_X \oplus \mathcal{I}\mathcal{B}_+$$

is right noetherian.

We note that the assumptions of the lemma are satisfied if Z consists of one primary component such that Z^{red} is of infinite order under σ.

Proof. By Lemma 2.3.14, we may without loss of generality assume that $\mathcal{L} = \mathcal{O}_X$. By Lemma 3.2.4(2)

$$(\mathcal{I} : \mathcal{I}^{\sigma^m}) = \mathcal{I}$$

for $m \gg 0$. Thus $\mathcal{R}_m = \mathcal{I}$ for $m \gg 0$, as claimed. Note that this implies that (1) $\iff$ (5).

(1) $\Rightarrow$ (2). Fix i. By Lemma 3.2.10, $(\mathcal{R}^i)_m = \mathcal{K}_i$ for all $m \geq 1$. As $\mathcal{R}_m = \mathcal{I}$ for $m \gg 0$, there is some m_0 so that for $m \geq m_0$

$$\mathcal{R}(X, \mathcal{O}_X, \sigma, Z)_m = \mathcal{I} \subseteq \mathcal{K}_i = \mathcal{R}(X, \mathcal{L}, \sigma, Z_i)_m.$$

By Lemma 3.2.3, $\mathcal{R}^i$ is right noetherian.

(2) ⇒ (3) Since Z is the set-theoretic union of finitely many irreducible components, it is enough to prove (3) in the case that $\mathcal{I}$ is itself primary; that is, in the case that $i = 1$. In this case, since $\mathcal{R} = \mathcal{R}^1$ is noetherian by assumption, by Lemma 3.2.3 $\mathcal{B}_{\mathcal{R}}$ is coherent.

Fix $x \in X$, and let $\mathcal{I}_x$ be its ideal sheaf. Let

$$\mathcal{M} = \bigoplus_{n \geq 0} (\mathcal{I}_x : \mathcal{I}^{\sigma^n})_{\sigma^n} \subseteq \mathcal{B} = \bigoplus_{n \geq 0} (\mathcal{O}_X)_{\sigma^n}.$$

Let $m \geq 1$ and $n \geq 0$. By Lemma 3.2.10, $\mathcal{R}_m = \mathcal{I}$. Therefore,

$$\mathcal{M}_n (\mathcal{R}_m)^{\sigma^n} = (\mathcal{I}_x : \mathcal{I}^{\sigma^n})\mathcal{I}^{\sigma^n} \subseteq \mathcal{I}_x \subseteq \mathcal{M}_{m+n},$$

and so $\mathcal{M}$ is a right $\mathcal{R}$-submodule of $\mathcal{B}$. It is therefore coherent, and so is the quotient $\mathcal{M}/\mathcal{I}_x\mathcal{B}$. Since $\mathcal{M} \cdot \mathcal{I}\mathcal{B}_+ \subseteq \mathcal{I}_x\mathcal{B}$, the $\mathcal{R}$-action on $\mathcal{M}/\mathcal{I}_x\mathcal{B}$ factors through $\mathcal{R}/\mathcal{I}\mathcal{B}_+ = \mathcal{O}_X$. In other words, $\mathcal{M}/\mathcal{I}_x\mathcal{B}$ is a noetherian and therefore coherent $\mathcal{O}_X$-module, and so the ideal sheaves $(\mathcal{I}_x : \mathcal{I}^{\sigma^n})$ and $\mathcal{I}_x$ are equal for $n \gg 0$. For fixed n, this is true if and only if $x \notin \sigma^{-n}Z$ or $\sigma^n(x) \notin Z$. Thus $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is finite.

(3) ⇒ (4). Let $\mathcal{P}$ be a nonzero prime ideal sheaf, defining a reduced and irreducible subscheme $W \subset X$. Since for any $m \in \mathbb{Z}$ we have that $\mathcal{I}^{\sigma^m} \subseteq \mathcal{P}$ if and only if $\sigma^m(W)$ is (set-theoretically) contained in Z, we see that the set

$$\{m \geq 0 \mid \mathcal{I}^{\sigma^m} \subseteq \mathcal{P}\}$$

is finite.

Now let $\mathcal{J} \supseteq \mathcal{I}$ be an ideal sheaf on X, and let $\mathcal{J} = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_e$ be a primary decomposition of $\mathcal{J}$, where $\mathcal{J}_i$ is $\mathcal{P}_i$-primary for a suitable prime ideal sheaf $\mathcal{P}_i$. For $m \gg 0$ and for $i = 1 \ldots e$, we have $\mathcal{I}^{\sigma^m} \not\subseteq \mathcal{P}_i$. Therefore by Lemma 3.2.4(2), $(\mathcal{J} : \mathcal{I}^{\sigma^m}) = \mathcal{J}$ for $m \gg 0$.

(4) $\Rightarrow$ (1). Suppose that for all $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{O}_X$, we have $(\mathcal{J} : \mathcal{I}^{\sigma^n}) = \mathcal{J}$ for $n \gg 0$. Let $\mathcal{F} \supseteq \mathcal{I}\mathcal{B}$ be a graded right ideal of $\mathcal{B}$, and for all $m \geq 0$ let

$$\mathcal{C}_m = \bigcap_{n \geq 0} (\mathcal{F}_{n+m} : \mathcal{I}^{\sigma^m}).$$

We saw in Section 2.3 that the categories qgr-$\mathcal{B}$ and $\mathcal{O}_X$-mod are equivalent, and that there is an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ such that, for some k, we have

$$\mathcal{F}_{\geq k} = \bigoplus_{m \geq k} (\mathcal{J})_{\sigma^m}.$$

By construction, $\mathcal{J} \supseteq \mathcal{I}$. For $m \geq k$, we have $\mathcal{C}_m = (\mathcal{J} : \mathcal{I}^{\sigma^m})$. This is equal to $\mathcal{J} = \mathcal{F}_m$ for $m \gg k$, and so the hypotheses of Lemma 3.2.9 hold. By Lemma 3.2.9, $\mathcal{R}$ is right noetherian. $\square$

We now give a general geometric criterion showing when an idealizer bimodule algebra is right noetherian.

Theorem 3.2.13. *Assume Notation 3.2.1. Let*

$$\mathcal{I} = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_c \cap \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_e \tag{3.2.14}$$

be a minimal primary decomposition of $\mathcal{I}$, where each $\mathcal{J}_i$ is $\mathcal{P}_i$-primary for some prime ideal sheaf $\mathcal{P}_i$ of finite order under σ, and each $\mathcal{K}_j$ is $\mathcal{Q}_j$-primary for some prime ideal sheaf $\mathcal{Q}_j$ of infinite order under σ. Let W be the closed subscheme of Z defined by the ideal sheaf $\mathcal{K} = \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_e$, and let $\mathcal{J} = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_c$. Then the following are equivalent:

(1) *$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is right noetherian;*

(2) *there is some n so that $\mathcal{J}^{\sigma^n} = \mathcal{J}$, and either $W = X$ or W has finite intersection with forward σ-orbits.*

Furthermore, if (2) and (1) hold, then $\mathcal{R}$ *is a finite module over* $\mathcal{R}^{(n)}$*, and there are a closed subscheme* W' *of* W*, with* $(W')^{\mathrm{red}} = W^{\mathrm{red}}$*, and an integer* n_0 *such that*

$$\mathcal{R}(X, \mathcal{L}_n, \sigma^n, Z)_{\geq n_0} = \mathcal{R}(X, \mathcal{L}_n, \sigma^n, W')_{\geq n_0}.$$

That is, any noetherian right idealizer is a finite module over a right idealizer at a subscheme without fixed components.

Proof. By Lemma 2.3.14, we may without loss of generality assume that $\mathcal{L} = \mathcal{O}_X$.

(1) $\Rightarrow$ (2). Suppose that $\mathcal{R}$ is right noetherian. We first show this implies that there is some n so that all $\mathcal{J}_i$ are fixed by σ^n. Suppose, in contrast, that for some i there is no n with $\mathcal{J}_i^{\sigma^n} = \mathcal{J}_i$. Since Veronese subrings of $\mathcal{R}$ are also right noetherian and $\mathcal{P}_i$ has finite order under σ, we may assume without loss of generality that $\mathcal{P}_i$ is fixed by σ.

Let $m \geq 1$. Since $(\mathcal{J}_i)^{\sigma^m} \neq \mathcal{J}_i$, by minimality of the primary decomposition (3.2.14), it is clear that $\mathcal{I}^{\sigma^m} \not\subseteq \mathcal{J}_i$. Thus by Lemma 3.2.5 $(\mathcal{J}_i : \mathcal{I}^{\sigma^m}) \neq \mathcal{O}_X$ is $\mathcal{P}_i$-primary. Therefore

$$\mathcal{R}_m = (\mathcal{I} : \mathcal{I}^{\sigma^m}) \subseteq (\mathcal{J}_i : \mathcal{I}^{\sigma^m}) \subseteq \mathcal{P}_i$$

for all $m \geq 1$.

Let $\mathcal{B} = \mathcal{B}(X, \mathcal{O}_X, \sigma)$. For any k, we have

$$(\mathcal{B}_{\leq k} \cdot \mathcal{R})_{k+1} = \sum_{j=0}^{k} (\mathcal{R}_{k+1-j})^{\sigma^j} \subseteq \mathcal{P}_i \neq \mathcal{O}_X = \mathcal{B}_{k+1}.$$

We see that $\mathcal{B}_{\mathcal{R}}$ is not finitely generated; by Lemma 3.2.3, this contradicts the assumption that $\mathcal{R}$ is right noetherian. Thus $\mathcal{J}_i$ is of finite order under σ.

As this holds for all i, there is some n so that $\mathcal{J}^{\sigma^n} = \mathcal{J}$. Suppose that $W \neq X$. Since W has finite intersection with forward σ-orbits if and only if W has finite intersection with forward σ^n-orbits, without loss of generality we may replace R by the Veronese $\mathcal{R}^{(n)}$ and assume that $\mathcal{J}$ is σ-invariant. Suppose that W has infinite intersection with some forward σ-orbit. We will derive a contradiction.

For $i = 1 \dots e$, let W_i be the primary component of Z corresponding to $\mathcal{K}_i$, and let Y_i be the subvariety corresponding to the prime ideal sheaf $\mathcal{Q}_i$. We claim that there is some i so that

(i) $Y_i \not\subseteq \sigma^{-m}(W)$ for $m \geq 1$;

(ii) for some $x \in X$, the set $\{m \geq 0 \mid \sigma^m(x) \in Y_i\}$ is infinite.

To see this, note that we may define a strict partial order $\prec$ on the set of the Y_i by defining

$$Y_i \prec Y_j \text{ if } Y_i \subseteq \sigma^{-m}(Y_j) \text{ for some } m \geq 1.$$

The order $\prec$ is strict because each Y_i has infinite order under σ. Now if (ii) holds for some Y_i, then (ii) holds for some Y_i that is maximal under $\prec$. But (i) holds for any such maximal Y_i, as the ideal sheaf of Y_i is prime.

Let i satisfy (i) and (ii). We thus have $\mathcal{K}^{\sigma^m} \not\subseteq \mathcal{Q}_i$ for any $m \geq 1$. As

$$\mathcal{I}^{\sigma^m} = \mathcal{K}^{\sigma^m} \cap \mathcal{J}^{\sigma^m} = \mathcal{K}^{\sigma^m} \cap \mathcal{J},$$

by Lemma 3.2.4(1) we have

$$\mathcal{R}_m = (\mathcal{I} : \mathcal{I}^{\sigma^m}) \subseteq (\mathcal{K}_i : \mathcal{J})$$

for all $m \geq 1$. By minimality of the primary decomposition (3.2.14) and Lemma 3.2.5, the ideal sheaf $(\mathcal{K}_i : \mathcal{J})$ is $\mathcal{Q}_i$-primary.

Let V be the closed subscheme of X defined by $(\mathcal{K}_i : \mathcal{J})$. By Lemma 3.2.10,

$$\mathcal{R}(X, \mathcal{O}_X, \sigma, V) = \mathcal{O}_X \oplus (\mathcal{K}_i : \mathcal{J})\mathcal{B}_+,$$

so

$$\mathcal{R}(X, \mathcal{O}_X, \sigma, Z) \subseteq \mathcal{R}(X, \mathcal{O}_X, \sigma, V).$$

Thus by Lemma 3.2.3, $\mathcal{R}(X, \mathcal{O}_X, \sigma, V)$ is right noetherian. But V also has infinite intersection with some forward σ-orbit. By Lemma 3.2.12, this is impossible.

Thus W has finite intersections with forward σ-orbits.

(2) $\Rightarrow$ (1). Suppose that (2) holds. We claim that

$$\mathcal{R}_m = (\mathcal{I} : \mathcal{J}^{\sigma^m}) \quad \text{for } m \gg 0. \tag{3.2.15}$$

If $W = X$ then $\mathcal{I} = \mathcal{J}$ and (3.2.15) holds for all m. If $W \neq X$ has finite intersection with forward σ-orbits, then for $m \gg 0$, $\mathcal{K}^{\sigma^m}$ is not contained in any minimal prime over $\mathcal{I}$. Thus by Lemma 3.2.4(1) we have that

$$(\mathcal{I} : \mathcal{I}^{\sigma^m}) \subseteq (\mathcal{I} : \mathcal{J}^{\sigma^m})$$

for $m \gg 0$. As the other containment is automatic, we see that (3.2.15) holds.

Now, if $n|m$ then

$$(\mathcal{I} : \mathcal{J}^{\sigma^m}) = (\mathcal{I} : \mathcal{J}) = (\mathcal{K} : \mathcal{J})$$

and so (3.2.15) implies in particular that $\mathcal{R}^{(n)}$ and $\mathcal{O}_X \oplus (\mathcal{K} : \mathcal{J})(\mathcal{B}^{(n)})_+$ are equal in large degree.

If $W = X$ then $(\mathcal{K} : \mathcal{J}) = \mathcal{O}_X$ and $\mathcal{R}^{(n)} = \mathcal{B}^{(n)}$. If W has finite intersection with forward σ-orbits, then note that $(\mathcal{K} : \mathcal{J})$ is the intersection of the $\mathcal{Q}_i$-primary ideal sheaves $(\mathcal{K}_i : \mathcal{J})$. Let W' be the closed subscheme defined by $(\mathcal{K} : \mathcal{J})$; then W' also has finite intersection with forward σ-orbits, and $(W')^{\mathrm{red}} = W^{\mathrm{red}}$. Thus we may apply Lemma 3.2.12 to $\mathcal{R}^{(n)}$ and we obtain that $\mathcal{R}^{(n)}$ is right noetherian. By Lemma 3.2.3, $\mathcal{B}^{(n)}$ is a coherent $\mathcal{R}^{(n)}$-module.

Thus in either case, $\mathcal{B}^{(n)}$ is a coherent ring $\mathcal{R}^{(n)}$-module. Therefore, for any m the right ideal

$$(\mathcal{I} : \mathcal{J}^{\sigma^m})\mathcal{B}^{(n)}$$

of $\mathcal{B}^{(n)}$ is a coherent $\mathcal{R}^{(n)}$-module. Applying (3.2.15) for $m = 0 \dots n-1$, we obtain that $\mathcal{R}$ is a finitely generated right $\mathcal{R}^{(n)}$-module and so $\mathcal{R}$ is right noetherian. $\square$

Example 3.2.16. We give an example illustrating what can go wrong when $\mathcal{J}^{\sigma^n}$ is never equal to $\mathcal{J}$. Let $X = \mathbb{P}^2$, let $\mathcal{L} = \mathcal{O}(1)$, and let

$$\sigma = \begin{bmatrix} 1 & & \\ & p & \\ & & q \end{bmatrix}$$

for some $p, q \in \mathbb{k}^*$ that are not roots of unity. Let $B = B(X, \mathcal{L}, \sigma)$. We saw in Definition 3.1.2 that B can be written as a Zhang twist $\mathbb{k}[x, y, z]^{\sigma}$.

Let $a = [0 : 0 : 1]$ and let $\mathcal{O} = \mathcal{O}_{X,a}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. As $\sigma(a) = a$, the automorphism σ acts on $\mathcal{O}$ via

$$\sigma(x) = x$$

$$\sigma(y) = py,$$

where (x, y) is an appropriate system of parameters for $\mathcal{O}$.

Let $\mathcal{I}$ be the ideal sheaf cosupported at a so that $\mathcal{I}_a = (x + y, \mathfrak{m}^2)$. Let $\mathcal{M}$ be the ideal sheaf of a. Then for any n we have that

$$(\mathcal{I}^{\sigma^n})_a = (x + p^n y, \mathfrak{m}^2).$$

We leave to the reader the computation that

$$(\mathcal{I} : \mathcal{I}^{\sigma^n}) = \mathcal{M}.$$

Thus, if Z is the subscheme defined by $\mathcal{I}$, we have that

$$R(X, \mathcal{L}, \sigma, Z) = \mathbb{k} + xB + yB.$$

This ring is not noetherian.

We end this section with a lemma giving conditions for $\mathcal{B}_{\mathcal{R}}$ to be coherent, even when $\mathcal{R}$ is not necessarily right noetherian.

Lemma 3.2.17. *Assume Notation 3.2.1. Let $Z_1, \ldots, Z_c$ be the primary components of Z. For $i = 1 \ldots c$, let $Y_i = Z_i^{\mathrm{red}}$.*

(1) *If for all i, $\{n \geq 0 \mid \sigma^n(Y_i) \subseteq Z\}$ is finite, then $\mathcal{R}_n = \mathcal{I}\mathcal{L}_n$ for $n \gg 0$.*

(2) *Assume (1) holds. Then $\mathcal{B}_{\mathcal{R}}$ is coherent if and only if Z contains no forward σ-orbits; that is, if and only if there is no point $x \in Z$ such that for all $n \geq 0$, we have $\sigma^n(x) \in Z$.*

(3) *If Z contains no σ-invariant subvarieties, then for all $n \geq 0$, $\mathcal{B}/(\mathcal{B} \cdot \mathcal{R}_{\geq n})$ is a coherent $\mathcal{O}_X$-module.*

Proof. (1) By Lemma 3.2.4(2),

$$\mathcal{R}_n = (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n = \mathcal{I}\mathcal{L}_n$$

for $n \gg 0$. Thus (1) holds.

(2) By (1), $\mathcal{B}$ is a coherent right $\mathcal{R}$-module if and only if $\mathcal{B}$ is a coherent right module over $\mathcal{S} = \mathcal{O}_X \oplus \mathcal{I}\mathcal{B}_+$. But this is true if and only if there is some k such that $\mathcal{I} + \mathcal{I}^{\sigma} + \cdots + \mathcal{I}^{\sigma^k} = \mathcal{O}_X$, i.e. if and only if $Z \cap \sigma^{-1}(Z) \cap \cdots \cap \sigma^{-k}(Z) = \emptyset$. This is equivalent to Z containing no forward σ-orbits.

(3) Certainly $\mathcal{B} \cdot \mathcal{R}_{\geq n}$ contains the two-sided ideal $\mathcal{B}\mathcal{I} \cdot \mathcal{B}_{\geq n}$ of $\mathcal{B}$. Since by assumption, $\mathcal{I}$ is contained in no nontrivial σ-invariant ideal sheaf, Lemma 2.3.15 implies that $\mathcal{B} \cdot \mathcal{R}_{\geq n}$ must contain $\mathcal{B}_{\geq m}$ for some m. $\square$

3.3 Left noetherian bimodule algebras

Since our ultimate goal is to understand noetherian idealizers, from now on we will assume the condition

Assumption-Notation 3.3.1. *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be an invertible sheaf on X. Let Z be a closed subscheme of X and let $\mathcal{I} = \mathcal{I}_Z$ be its defining ideal. Let*

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$$

and let

$$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} \big((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n\big)_{\sigma^n}.$$

For any associated prime $\mathcal{Q}$ of $\mathcal{I}$, we assume that the set $\{n \geq 0 \mid \mathcal{Q} \supseteq \mathcal{I}^{\sigma^n}\}$ is finite. By Lemma 3.2.17(1), this implies that $\mathcal{R}_n = \mathcal{I}\mathcal{L}_n$ for all $n \gg 0$.

By Theorem 3.2.13, any right noetherian bimodule algebra is, up to a finite extension, one whose defining data satisfies Assumption-Notation 3.3.1.

We now consider when the idealizer bimodule algebra $\mathcal{R}$ is left noetherian. We quote a result of Rogalski; we note that the original result was stated for left ideals of noetherian rings.

Proposition 3.3.2. ([Rog04b, Proposition 2.2]) *If $R = \mathbb{I}_B(I)$ for some right ideal I of a noetherian ring B, then R is left noetherian if and only if R/I is a left noetherian ring and for all left ideals J of B, the left R-module $\operatorname{Tor}_1^B(B/I, B/J)$ is noetherian.*

We note that if R/I is finite-dimensional, this result reduces to saying that R is left noetherian if and only if $\operatorname{Tor}_1^B(B/I, B/J)$ is a finite-dimensional vector space for all left ideals J of B.

We now prove a version of Proposition 3.3.2 for the bimodule algebra $\mathcal{R}$. Again, we give it in slightly more generality than we currently need.

Proposition 3.3.3. *Let $\mathcal{B}$ be a noetherian graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of $\mathcal{B}(X, \mathcal{L}, \sigma)$, and let $\mathcal{I} = \bigoplus(\mathcal{I}_n)_{\sigma^n}$ be a graded right ideal of $\mathcal{B}$. Let $\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I})$.*

Suppose that $\mathcal{R}_n = \mathcal{I}_n$ for all $n \gg 0$. Then $\mathcal{R}$ is left noetherian if and only if for all graded left ideals $\mathcal{J}$ of $\mathcal{B}$ we have

$$(\mathcal{I} \cap \mathcal{J})_n = (\mathcal{I}\mathcal{J})_n$$

for $n \gg 0$.

Proof. We follow Rogalski's proof of Proposition 3.3.2.

Since $(\mathcal{I} \cap \mathcal{J})/\mathcal{I}\mathcal{J}$ is a subfactor of ${}_{\mathcal{R}}\mathcal{R}$ that is killed by $\mathcal{I}$, if $\mathcal{R}$ is left noetherian then this is a coherent module over $\mathcal{R}/\mathcal{I}$ and so is certainly a coherent $\mathcal{O}_X$-module.

For the other direction, suppose that for all graded left ideals $\mathcal{J}$ of $\mathcal{B}$ we have that

$$(\mathcal{I} \cap \mathcal{J})_n = (\mathcal{I}\mathcal{J})_n$$

for $n \gg 0$. Let $\mathcal{K}$ be a graded left ideal of $\mathcal{R}$. Since $\mathcal{B}$ is noetherian, we may choose a graded coherent $\mathcal{O}_X$-submodule $\mathcal{K}'$ of $\mathcal{K}$ such that $\mathcal{B}\mathcal{K} = \mathcal{B}\mathcal{K}'$. Since $\mathcal{K}/\mathcal{R}\mathcal{K}'$ is a submodule of $(\mathcal{B}\mathcal{K}' \cap \mathcal{R})/\mathcal{R}\mathcal{K}'$, it is enough to show for any *coherent* graded left ideal $\mathcal{K}$ of $\mathcal{R}$, that $(\mathcal{B}\mathcal{K} \cap \mathcal{R})/\mathcal{K}$ is a noetherian left $\mathcal{R}$-module.

But now consider the exact sequences of left $\mathcal{R}$-modules

$$0 \to \frac{\mathcal{K}}{\mathcal{I}\mathcal{K}} \to \frac{\mathcal{B}\mathcal{K} \cap \mathcal{R}}{\mathcal{I}\mathcal{K}} \to \frac{\mathcal{B}\mathcal{K} \cap \mathcal{R}}{\mathcal{K}} \to 0 \tag{3.3.4}$$

and

$$0 \to \frac{\mathcal{B}\mathcal{K} \cap \mathcal{I}}{\mathcal{I}\mathcal{K}} \to \frac{\mathcal{B}\mathcal{K} \cap \mathcal{R}}{\mathcal{I}\mathcal{K}} \to \frac{\mathcal{B}\mathcal{K} \cap \mathcal{R}}{\mathcal{B}\mathcal{K} \cap \mathcal{I}} \to 0. \tag{3.3.5}$$

Since $(\mathcal{B}\mathcal{K} \cap \mathcal{R})/(\mathcal{B}\mathcal{K} \cap \mathcal{I})$ is a coherent $\mathcal{O}_X$-module, we see that $(\mathcal{B}\mathcal{K} \cap \mathcal{R})/\mathcal{K}$ is noetherian if $(\mathcal{B}\mathcal{K} \cap \mathcal{I})/\mathcal{I}\mathcal{K}$ is noetherian. Since $\mathcal{B}\mathcal{K}$ is a left ideal of $\mathcal{B}$, and $\mathcal{I}\mathcal{B}\mathcal{K} = \mathcal{I}\mathcal{K}$, we have by assumption that $(\mathcal{B}\mathcal{K} \cap \mathcal{I})/\mathcal{I}\mathcal{K}$ is a coherent $\mathcal{O}_X$-module. In particular, it is noetherian. Thus $\mathcal{R}$ is left noetherian. $\square$

Proposition 3.3.6. *Assume Assumption-Notation 3.3.1. Then $\mathcal{R}$ is left noetherian if and only if for all closed subschemes $Y \subseteq X$ the set*

$$\{n \geq 0 \,|\, \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

Proof. Let $\mathcal{J}$ be an ideal sheaf defining a closed subscheme Y of X. There are identifications of $\mathcal{O}_X$-modules

$$\frac{\mathcal{I}\mathcal{B} \cap \mathcal{B}\mathcal{J}}{\mathcal{I}\mathcal{B}\mathcal{J}} \cong \bigoplus_{n \geq 0} \frac{\mathcal{I} \cap \mathcal{J}^{\sigma^n}}{\mathcal{I}\mathcal{J}^{\sigma^n}} \otimes \mathcal{L}_n$$

$$\cong \bigoplus_{n \geq 0} \mathcal{T}or_1^X(\mathcal{O}_Z, \mathcal{O}_{\sigma^{-n} Y}) \otimes \mathcal{L}_n \cong \bigoplus_{n \geq 0} \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \otimes \mathcal{L}_n,$$

using [Wei94, Exercise 3.1.3] and the local property of $\mathcal{T}or$. As $\mathcal{R}/\mathcal{I}\mathcal{B}$ is a coherent $\mathcal{O}_X$-module, $(\mathcal{I}\mathcal{B} \cap \mathcal{B}\mathcal{J})/\mathcal{I}\mathcal{B}\mathcal{J}$ is a coherent left $\mathcal{R}$-module if and only if it is a coherent $\mathcal{O}_X$-module. This is true if and only if the set $\{n \geq 0 \,|\, \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$ is finite. $\square$

The vanishing of the sheaves $\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y)$ for large n is an important condition that in fact gives many further nice properties of $\mathcal{R}$. As remarked in the introduction, it is an analogue of critical density and can be viewed as a transversality property.

To begin, we define an algebraic generalization of classical transversality.

Definition 3.3.7. Let Y and Z be closed subschemes of X. We say that Y and Z are *homologically transverse* if

$$\mathcal{T}or_i^X(\mathcal{O}_Z, \mathcal{O}_Y) = 0$$

for all $i \geq 1$.

While this appears as an arcane algebraic condition, it does in fact have a geometric basis. As discussed in the introduction, Serre defined the intersection multiplicity of two closed subschemes Y and Z of X along the proper component P of their intersection by

$$i(Y,Z;P) = \sum_{i \geq 0} (-1)^i \operatorname{len}_P(\mathcal{T}or_i^X(\mathcal{O}_Y, \mathcal{O}_Z)).$$

The higher $\mathcal{T}or$ sheaves are needed to correct for possible mis-counting from the naïve attempt to define $i(Y,Z;P)$ as $\operatorname{len}_P(\mathcal{O}_Y \otimes \mathcal{O}_Z)$. [Har77, Appendix A, Example 1.1.1] gives an example where $\mathcal{T}or_1$ is needed to properly compute the intersection multiplicity.

We may think of the non-vanishing of $\mathcal{T}or_{\geq 1}^X(\mathcal{O}_Y, \mathcal{O}_Z)$ as indicating that Y and Z have an extremely non-transverse intersection (for example, the codimension of the intersection is smaller than $\operatorname{codim} Y + \operatorname{codim} Z$).

Definition 3.3.8. Let $A \subseteq \mathbb{Z}$ be infinite. We say that the set $\{\sigma^n(Z)\}_{n \in A}$ is *critically transverse* if for all closed subschemes Y of X, $\sigma^n(Z)$ and Y are homologically transverse for all but finitely many $n \in A$.

Critical transversality of $\{\sigma^n Z\}$ is a generic transversality property: for any closed subscheme Y, it implies that the general translate of Z is homologically transverse to Y.

In the remainder of this section, we prove some technical results on critical transversality. We first remark that although our definition of critical transversality looks stronger than the condition needed for $\mathcal{R}$ to be left noetherian, it is in fact equivalent.

The following lemma is due to Mel Hochster, and we thank him for allowing us to include it here. Recall that if $\mathcal{F}$ is a coherent sheaf on a projective variety X, we write $\mathrm{hd}_X(\mathcal{F})$ for the maximal length of a locally free resolution of $\mathcal{F}$; that is,

$$\mathrm{hd}_X(\mathcal{F}) = \sup_{x\in X}\{\mathrm{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x\}.$$

Lemma 3.3.9. (Hochster) *Suppose that Z is homologically transverse to all parts of the singular stratification of X. Then*

$$\mathrm{hd}_X(\mathcal{O}_Z) \leq \dim X.$$

Proof. Let $X = X^{(0)} \supset X^{(1)} \cdots \supset X^{(k)}$ be the singular stratification of X. By assumption, Z is homologically transverse to all $X^{(i)}$. By [Eis95, Corollary 19.5],

$$\mathrm{hd}_X(\mathcal{O}_Z) = \sup\{j \mid \text{for some closed point } x \in X, \mathcal{T}or_j^X(\mathcal{O}_Z, \Bbbk_x) \neq 0\}. \tag{3.3.10}$$

So fix $x \in X$, and let $\mathcal{O} = \mathcal{O}_{X,x}$. Let $F = \mathcal{O}_{Z,x}$, considered as an $\mathcal{O}$-module. Let i be such that $x \in X^{(i)} \smallsetminus X^{(i+1)}$. Let J be the ideal of $X^{(i)}$ in $\mathcal{O}$. By assumption on i, $\mathcal{O}/J$ is a regular local ring; in particular, $\mathrm{pd}_{\mathcal{O}/J} \Bbbk_x = \dim X^{(i)} \leq \dim X$.

The change of rings theorem for Tor [Wei94, Theorem 5.6.6] gives a spectral sequence

$$\mathrm{Tor}_p^{\mathcal{O}/J}(\mathrm{Tor}_q^{\mathcal{O}}(F, \mathcal{O}/J), \Bbbk_x) \Rightarrow \mathrm{Tor}_{p+q}^{\mathcal{O}}(F, \Bbbk_x). \tag{3.3.11}$$

Now by assumption, Z is homologically transverse to $X^{(i)}$, and so (3.3.11) collapses for $q \neq 0$. We obtain

$$\mathrm{Tor}_p^{\mathcal{O}/J}(F \otimes_{\mathcal{O}} (\mathcal{O}/J), \Bbbk_x) \cong \mathrm{Tor}_p^{\mathcal{O}}(F, \Bbbk_x).$$

As $\mathcal{O}/J$ is a regular local ring of dimension no greater than $\dim X$, we have that $\mathrm{pd}_{\mathcal{O}/J} \Bbbk_x \leq \dim X$ and so $\mathrm{Tor}_p^{\mathcal{O}}(F, \Bbbk_x) = 0$ if $p > \dim X$. By (3.3.10), $\mathrm{hd}_X(\mathcal{O}_Z) \leq \dim X$. $\square$

Lemma 3.3.12. *Let $A \subseteq \mathbb{Z}$. The following are equivalent:*

(1) *For all closed subschemes Y of X, the set*

$$\{n \in A \mid \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

(2) *For all reduced and irreducible closed subschemes Y of X, the set*

$$\{n \in A \mid \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

(3) *For all closed subschemes Y of X, the set*

$$A'(Y) = \{n \in A \mid \ \sigma^n Z \text{ is not homologically transverse to } Y\ \}$$

is finite.

Proof. The implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are trivial. We prove (2) $\Rightarrow$ (3).

Assume (2). Without loss of generality we may assume that A is infinite. We first claim that for any coherent sheaf $\mathcal{F}$ and for any $j \geq 1$, the set

$$\{n \in A \mid \mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{F}) \neq 0\}$$

is finite. We induct on j. As any coherent sheaf on a projective variety has a finite filtration by products of invertible sheaves with structure sheaves of reduced and irreducible closed subvarieties, the claim is true for $j = 1$. Let $j > 1$ and fix a coherent sheaf $\mathcal{F}$. Because X is projective, it has enough locally frees, and there is an exact sequence

$$0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is locally free and $\mathcal{K}$ is also coherent. The long exact sequence in Tor implies that

$$\mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{F}) \cong \mathcal{T}or_{j-1}^X(\mathcal{O}_{\sigma^n Z}, \mathcal{K})$$

for any n. By induction, the right-hand side vanishes for all but finitely many $n \in A$.

The claim implies that Z is homologically transverse to any σ-invariant closed subscheme of X, and, in particular, that Z is homologically transverse to the singular stratification of X. By Lemma 3.3.9, we have $\mathrm{hd}_X(\mathcal{O}_Z) \leq \dim X$. Thus for a fixed Y,

$$A'(Y) = \{ n \in A \mid \ \mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0 \text{ for some } 1 \leq j \leq \dim X \ \}.$$

By the claim, this is finite. $\square$

Corollary 3.3.13. *Assume Assumption-Notation 3.3.1. Then the bimodule algebra $\mathcal{R}$ is left noetherian if and only if $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse.*

Proof. Combine Lemma 3.3.12 with Proposition 3.3.6. $\square$

We next verify that critical transversality generalizes critical density of the orbits of points. We first prove:

Lemma 3.3.14. *Let $W \subseteq V$ be closed subschemes of X. Then $\mathcal{T}or_1^X(\mathcal{O}_V, \mathcal{O}_W) \neq 0$.*

Proof. We work locally; let W' be an irreducible component of W, and let $P = (W')^{\mathrm{red}}$. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O} = \mathcal{O}_{X,P}$. Let J be the ideal of $\mathcal{O}$ defining V and let I be the $\mathfrak{m}$-primary ideal defining W locally at P. Then we have

$$\mathcal{T}or_1^X(\mathcal{O}_V, \mathcal{O}_W)_P = \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}/J, \mathcal{O}/I) \cong (J \cap I)/JI = J/JI,$$

as $J \subseteq I$. By Nakayama's Lemma, this is nonzero. $\square$

Corollary 3.3.15. *If Z is a 0-dimensional subscheme of X and $A \subseteq \mathbb{Z}$, then $\{\sigma^n(Z)\}_{n \in A}$ is critically transverse if and only if $\{\sigma^n(x)\}_{n \in A}$ is critically dense for all points x in the support of Z.*

Proof. Because $\mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y)$ is supported on $\sigma^n Z \cap Y$ for any j, critical density implies critical transversality. We prove that critical transversality implies critical density. By working locally, we may assume that Z is supported on a single point x. Suppose that critical density fails, so there is some infinite $A' \subseteq A$ and some reduced $W \subset X$ such that $\sigma^n(x) \in W$ for all $n \in A'$. Then there is some, not necessarily reduced, W' supported on W such that $\sigma^n(Z) \subseteq W'$ for all $n \in A'$. By Lemma 3.3.14, we have that $\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_{W'}) \neq 0$ for any $n \in A'$. Thus critical transversality also fails. $\square$

3.4 Ampleness

Our ultimate goal is to study, not the bimodule algebra $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$, but its section ring $R(X, \mathcal{L}, \sigma, Z)$. We have seen in Section 2.3 that, given appropriate ampleness of the graded pieces of a bimodule algebra, many properties descend from the bimodule algebra to its section ring. The goal of this section is to show that the sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is suitably ample.

We recall from Section 2.3 the definition of the properties we will need.

Definition 3.4.1. (Definition 2.3.11) Let $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ be a sequence of coherent sheaves on the projective variety X. The sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is *right ample* if for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the following properties hold:

(i) $\mathcal{F} \otimes \mathcal{R}_n$ is globally generated for $n \gg 0$;

(ii) $H^q(\mathcal{F} \otimes \mathcal{R}_n) = 0$ for $n \gg 0$ and all $q \geq 1$.

The sequence $\{(\mathcal{R}_n)_{\sigma^n}\}$ is *left ample* if for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the following properties hold:

(i) $\mathcal{R}_n \otimes \mathcal{F}^{\sigma^n}$ is globally generated for $n \gg 0$;

(ii) $H^q(\mathcal{R}_n \otimes \mathcal{F}^{\sigma^n}) = 0$ for $n \gg 0$ and all $q \geq 1$.

Recall also that if $\mathcal{L}$ is *σ-ample*, then $\{(\mathcal{L}_n)_{\sigma^n}\}$ is a left and right ample sequence.

Throughout this section we assume Assumption-Notation 3.3.1. Thus to prove that the sequence $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left or right ample, it suffices to prove that $\{(\mathcal{I}\otimes\mathcal{L}_n)_{\sigma^n}\}$ is left or right ample.

Given σ-ampleness of $\mathcal{L}$, right ampleness of $\{(\mathcal{R}_n)_{\sigma^n}\}$ is almost trivial; we record this in the next lemma.

Lemma 3.4.2. *Assume Assumption-Notation 3.3.1. Assume in addition that $\mathcal{L}$ is σ-ample. Then $\{(\mathcal{R}_n)_{\sigma^n}\}$ is right ample.*

Proof. From Assumption-Notation 3.3.1, we know that $\mathcal{R}_n = \mathcal{I}\mathcal{L}_n = \mathcal{I}\otimes\mathcal{L}_n$ for $n \gg 0$. Fix a coherent sheaf $\mathcal{F}$. Then for $n \gg 0$, $\mathcal{F}\otimes\mathcal{R}_n = \mathcal{F}\otimes\mathcal{I}\otimes\mathcal{L}_n$. By σ-ampleness of $\mathcal{L}$, for $n \gg 0$ this is globally generated and has no higher cohomology. $\square$

Left ampleness, however, is more subtle. In fact, we do not know when, in general, $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left ample. However, we will see that this does hold when $\mathcal{R}$ is left noetherian.

Lemma 3.4.3. *Let $\mathcal{L}$ be a σ-ample invertible sheaf.*

(1) If $\mathcal{M}$ and $\mathcal{N}$ are coherent sheaves on X, then there is an integer n_0 so $\mathcal{M}\otimes\mathcal{L}_n\otimes\mathcal{N}^{\sigma^n}$ is globally generated for all $n \geq n_0$.

(2) If $\mathcal{E}$ and $\mathcal{F}$ are invertible sheaves on X, there is an integer m_0 so that $\mathcal{E}\otimes\mathcal{L}_m\otimes\mathcal{F}^{\sigma^m}$ is ample for all $m \geq m_0$.

Proof. (1) Using the σ-ampleness of $\mathcal{L}$, take $i, j \gg 0$ so that $\mathcal{M}\otimes\mathcal{L}_i$ and $\mathcal{L}_j\otimes\mathcal{N}^{\sigma^j}$ are globally generated. Then $\mathcal{L}_j^{\sigma^i}\otimes\mathcal{N}^{\sigma^{i+j}}$ is also globally generated. Since the tensor product of globally generated sheaves is globally generated, $\mathcal{M}\otimes\mathcal{L}_{i+j}\otimes\mathcal{N}^{\sigma^{i+j}}$ is globally generated.

(2) In fact, we will show that $\mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m}$ is very ample for $m \gg 0$. Let $\mathcal{C}$ be an arbitrary very ample invertible sheaf. By (1) we may choose m_0 so that if $m \geq m_0$, the sheaf $\mathcal{K} = \mathcal{C}^{-1} \otimes \mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m}$ is globally generated. Since by [Har77, Exercise II.7.5(d)] the tensor product of a very ample invertible sheaf and a globally generated invertible sheaf is very ample, $\mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m} \cong \mathcal{C} \otimes \mathcal{K}$ is very ample. $\square$

Proposition 3.4.4. *If $\mathcal{L}$ is σ-ample and $\{\sigma^n(Z)\}_{n\geq 0}$ is critically transverse, then $\{(\mathcal{I} \otimes \mathcal{L}_n)_{\sigma^n}\}$ is a left ample sequence.*

Proof. Let $\mathcal{M}$ be an arbitrary coherent sheaf. By Lemma 3.4.3, we know that $\mathcal{I}\otimes\mathcal{L}_n\otimes\mathcal{M}^{\sigma^n}$ is globally generated for $n \gg 0$. We must establish that $H^j(\mathcal{I}\otimes\mathcal{L}_n\otimes\mathcal{M}^{\sigma^n}) = 0$ for all $j \geq 1$ and $n \gg 0$.

We know that $\mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{M}) = 0$ for all $n \gg 0$ and $j \geq 1$. Thus

$$\mathcal{T}or_j^X(\mathcal{I}, \mathcal{M}^{\sigma^n}) \cong \mathcal{T}or_j^X(\mathcal{I}^{\sigma^{-n}}, \mathcal{M})^{\sigma^n} = 0$$

for all $n \gg 0$ and $j \geq 1$.

First suppose that $\mathcal{M}$ is invertible. By Fujita's vanishing theorem, Theorem 2.5.1, choose an invertible sheaf $\mathcal{H}$ such that $H^i(\mathcal{I}\otimes\mathcal{H}\otimes\mathcal{F}) = 0$ for all $i \geq 1$ and any ample invertible sheaf $\mathcal{F}$. By Lemma 3.4.3(2), we may choose m_0 such that $\mathcal{H}^{-1}\otimes\mathcal{L}_m\otimes\mathcal{M}^{\sigma^m}$ is ample for all $m \geq m_0$. Then $\mathcal{I} \otimes \mathcal{L}_m \otimes \mathcal{M}^{\sigma^m} = \mathcal{I} \otimes \mathcal{H} \otimes \mathcal{H}^{-1} \otimes \mathcal{L}_m \otimes \mathcal{M}^{\sigma^m}$ and so its higher cohomology vanishes.

Now for general $\mathcal{M}$ let the cochain complex

$$\cdots \to \mathcal{P}^{-2} \to \mathcal{P}^{-1} \to \mathcal{P}^0 \to \mathcal{M} \to 0$$

be a (not necessarily finite!) projective resolution of $\mathcal{M}$. By tensoring on the left with $\mathcal{I} \otimes \mathcal{L}_n$, we obtain a complex $\mathcal{Q}^\bullet$, where $\mathcal{Q}^i = \mathcal{I} \otimes \mathcal{L}_n \otimes (\mathcal{P}^i)^{\sigma^n}$. The q-th cohomology of $\mathcal{Q}^\bullet$ is isomorphic to $\mathcal{T}or_{-q}^X(\mathcal{I}, \mathcal{M}^{\sigma^n}) \otimes \mathcal{L}_n$. Now, by [Wei94, 5.7.9], using a Cartan-Eilenberg resolution of $\mathcal{Q}^\bullet$ we obtain two spectral sequences

$$(3.4.5) \qquad {}^{I}E_1^{pq} = H^q(\mathcal{Q}^p)$$

and

$$(3.4.6) \qquad {}^{II}E_2^{pq} = H^p(\mathcal{T}or_{-q}(\mathcal{I}, \mathcal{M}^{\sigma^n}) \otimes \mathcal{L}_n).$$

Since X has finite cohomological dimension $d = \dim X$, these both converge to the hypercohomology groups $\mathbb{H}^{p+q}(\mathcal{Q}^\bullet)$.

Now, given $p + q = j \geq 1$, by critical transversality we may take $n \gg 0$ so that $\mathcal{T}or_{-q}(\mathcal{I}, \mathcal{M}^{\sigma^n}) = 0$ for all $j - d \leq q \leq -1$; thus (3.4.6) collapses and we obtain

$$\mathbb{H}^j(\mathcal{Q}^\bullet) = H^j(\mathcal{I} \otimes \mathcal{M}^{\sigma^n} \otimes \mathcal{L}_n).$$

On the other hand, since the sheaves $\mathcal{P}^i$ are locally free, applying the invertible case to each summand of $\mathcal{P}^i$ we may further increase n if necessary to obtain that

$$H^q(\mathcal{Q}^p) = H^q(\mathcal{I} \otimes \mathcal{L}_n \otimes (\mathcal{P}^p)^{\sigma^n}) = 0$$

for $d \geq q \geq 1$ and $1 - d \leq p \leq 0$. Thus if $j \geq 1$, (3.4.5) collapses to 0. Thus

$$H^j(\mathcal{I} \otimes \mathcal{L}_n \otimes \mathcal{M}^{\sigma^n}) = 0$$

for all $n \gg 0$ and $j \geq 1$. $\square$

3.5 Noetherian idealizer rings

We are now ready to begin translating our results on bimodules to results about geometric idealizer rings. We will work in the following setting:

Assumption-Notation 3.5.1. *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be an invertible sheaf on X, which we now assume to be σ-ample. Let Z be a* *closed subscheme of X and let $\mathcal{I} = \mathcal{I}_Z$ be its ideal sheaf. We continue to assume that for any associated prime $\mathcal{Q}$ of $\mathcal{I}$, the set $\{n \geq 0 \mid \mathcal{Q} \supseteq \mathcal{I}^{\sigma^n}\}$ is finite. Let*

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$$

and let

$$B = B(X, \mathcal{L}, \sigma).$$

Let

$$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} \big((\mathcal{I} : \mathcal{I}^{\sigma^n}) \mathcal{L}_n \big)_{\sigma^n}.$$

Let

$$I = \bigoplus_{n \geq 0} H^0(\mathcal{I}\mathcal{L}_n)$$

and let

$$R = R(X, \mathcal{L}, \sigma, Z) = \mathbb{I}_B(I)$$

as in Construction 3.1.1. By Lemma 3.2.2,

$$R = \bigoplus_{n \geq 0} H^0(\mathcal{R}(X, \mathcal{L}, \sigma, Z)_n).$$

Our assumptions imply that $\mathcal{R}_n = \mathcal{I}\mathcal{L}_n$ and $R_n = I_n$ for $n \gg 0$.

Assume Assumption-Notation 3.5.1. We next show that the right noetherian property for R, and in fact the strong right noetherian property, are equivalent to the simple geometric criterion from Theorem 3.2.13.

Proposition 3.5.2. *Assume Assumption-Notation 3.5.1. Then the following are equivalent:*

(1) *Z has finite intersection with forward σ-orbits;*

(2) *R is right noetherian;*

(3) *R is strongly right noetherian.*

Proof. (1) $\Rightarrow$ (3). By Theorem 3.2.13, if (1) holds then the bimodule algebra

$$\mathcal{R}(X, \mathcal{L}, \sigma, Z)$$

is right noetherian. Now let C be any commutative noetherian ring, and let

$$X_C = X \times \operatorname{Spec} C$$

and

$$Z_C = Z \times \operatorname{Spec} C \subseteq X_C.$$

Also define

$$B_C = B \otimes_{\Bbbk} C,$$

$$R_C = R \otimes_{\Bbbk} C,$$

and

$$I_C = I \otimes_{\Bbbk} C \cong I \otimes_B B_C.$$

It is clear that

$$R_C = \mathbb{I}_{B_C}(I_C)$$

and that R_C/I_C is a finitely generated C-module. Let $p : X_C \to X$ be projection onto the first factor.

The idea behind our proof is very simple: if Z has finite intersection with forward σ-orbits, then Z_C has finite intersection with forward $(\sigma \times 1)$-orbits, and so R_C should be noetherian by Theorem 3.2.13 and Theorem 2.3.12. However, neither of these were proved over an arbitrary base ring C; to work scheme-theoretically we instead follow the proof of [ASZ99, Proposition 4.13].

By [ASZ99, Proposition 4.13], B_C is noetherian. The proof of this proposition uses the fact that the shift functor in qgr-B_C satisfies the hypotheses of [AZ94, Theorem 4.5]. By [AZ94, Theorem 4.5], B_C satisfies right χ_1. In particular, for any graded right ideal J of B_C, the natural map

$$\underline{\mathrm{Hom}}_{B_C}(B_C/I_C, B_C/J) \to \underline{\mathrm{Hom}}_{\text{qgr-}B_C}(\pi(B_C/I_C), \pi(B_C/J)) \tag{3.5.3}$$

is an isomorphism in large degree, by [AZ94, Proposition 3.5].

As qgr-$B \simeq \mathcal{O}_X$-mod, it is clear that

$$\text{qgr-}B_C \simeq \mathcal{O}_{X_C}\text{-mod}\,. \tag{3.5.4}$$

We note that B_C/I_C corresponds to $\mathcal{O}_{Z_C}$ under this equivalence.

Let J be a graded right ideal of B_C containing I_C. We claim that

$$\underline{\mathrm{Hom}}_{B_C}(B_C/I_C, B_C/J)$$

is a finitely generated C-module. To see this, let $Y \subseteq Z_C$ be the closed subscheme of X_C such that B_C/J corresponds to $\mathcal{O}_Y$ under the equivalence (3.5.4). By (2.3.13), $(B_C/J)[n]$ corresponds to

$$(\mathcal{O}_Y \otimes p^*\mathcal{L}_n)^{(\sigma^{-n}\times 1)} \cong \mathcal{O}_{(\sigma^n\times 1)Y} \otimes p^*(\mathcal{L}_n^{\sigma^{-n}})$$

under (3.5.4). Thus

$$\underline{\mathrm{Hom}}_{\text{qgr-}B_C}(\pi(B_C/I_C), \pi(B_C/J))_{\geq 0} \cong \bigoplus_{n\geq 0} \mathrm{Hom}_{X_C}(\mathcal{O}_{Z_C}, \mathcal{O}_{(\sigma^n\times 1)Y} \otimes p^*(\mathcal{L}_n^{\sigma^{-n}})).$$

Now, Z_C has finite intersection with forward $(\sigma \times 1)$-orbits, and so for $n \gg 0$, no component of $(\sigma^n \times 1)Y$ is contained in Z_C. Thus

$$\mathrm{Hom}_{X_C}(\mathcal{O}_{Z_C}, \mathcal{O}_{(\sigma^n\times 1)Y} \otimes p^*(\mathcal{L}_n^{\sigma^{-n}})) = 0$$

for $n \gg 0$. As the map (3.5.3) is an isomorphism in large degree, we see that

$$\underline{\mathrm{Hom}}_{B_C}(B_C/I_C, B_C/J)_n = 0$$

for $n \gg 0$, and so

$$\underline{\mathrm{Hom}}_{B_C}(B_C/I_C, B_C/J)$$

is a finitely generated C-module, as claimed. As this is true for any graded $J \supseteq I_C$, by Lemma 3.2.8, R_C is right noetherian.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $x \in X$ and let J be the right ideal $\Gamma_*(\mathcal{I}_x)$ of B. As B and R are right noetherian, by Lemma 3.2.8,

$$\mathrm{Hom}_B(B/I, B/J) \cong \{r \in B \mid rI \subseteq J\}/J$$

is a noetherian right R/I-module. It is thus finite-dimensional, as R/I is finite-dimensional by assumption.

As $\mathcal{L}$ is σ-ample, $\mathcal{L}_n$ is globally generated for $n \gg 0$; in particular, $J_n \subsetneq B_n$ for $n \gg 0$. Now, suppose that

$$\{n \geq 0 \mid \sigma^n(x) \in Z\} = \{n \geq 0 \mid x \in \sigma^{-n}(Z)\}$$

is infinite. For any such n, we have that $B_n I \subseteq J$. Thus

$$\{r \in B \mid rI \subseteq J\}/J$$

is infinite-dimensional, giving a contradiction.

Thus $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is finite. $\square$

The left-hand side is very different. If $\mathcal{R}$ is left noetherian, then so is R; but R can only be strongly left noetherian if $\operatorname{codim} Z = 1$. In this case, R is both a left and a right idealizer, so the strong left noetherian property will follow from the left-handed version of Proposition 3.5.2.

Proposition 3.5.5. *Assume Assumption-Notation 3.5.1. If $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse, then $R = R(X, \mathcal{L}, \sigma, Z)$ is left noetherian.*

Proof. By Proposition 3.4.4 and Corollary 3.3.13, we have that $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is left noetherian and that $\{(\mathcal{R}_n)_{\sigma^n}\}$ is a left ample sequence. Thus by Theorem 2.3.12, the section ring $R(X, \mathcal{L}, \sigma, Z)$ is also left noetherian. $\square$

Unfortunately, we cannot prove the converse to Proposition 3.5.5 in full generality. We do give below several special cases where the converse does hold.

Proposition 3.5.6. *Assume Assumption-Notation 3.5.1. If* $\{\sigma^n(Z)\}_{n\geq 0}$ *is not critically transverse, and either*

(1) *there is some* σ*-invariant subscheme* Y *that is not homologically transverse to* Z*; or*

(2) $\operatorname{codim} Z = 1$*;*

then $R = R(X, \mathcal{L}, \sigma, Z)$ *is not left noetherian.*

Before giving the proof, we give a preliminary lemma.

Lemma 3.5.7. *Let* $X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \cdots$ *be the singular stratification of* X*. Suppose that* Z *is a subscheme of codimension 1 such that for all* j*,* $\mathcal{T}or_1^X(\mathcal{O}_Z, \mathcal{O}_{X^{(j)}}) = 0$*. Then* Z *is locally principal.*

Proof. Fix $x \in Z$; we will show that Z is locally principal at x. Let $\mathcal{O} = \mathcal{O}_{X,x}$.

Let j be maximal so that $x \in X^{(j)}$, and let J be the ideal of $X^{(j)}$ in $\mathcal{O}$. Let I be the defining ideal of Z in $\mathcal{O}$. By Lemma 3.3.14, $I \not\subseteq J$. Thus $(I+J)/J$ locally defines a hypersurface in $X^{(j)}$. Since $\mathcal{O}/J$ is a regular local ring, $(I+J)/J$ is principal in $\mathcal{O}/J$, and so there is $f \in I$ such that $(f) + J = I + J$.

By homological transversality, $I \cap J = IJ$. Thus

$$\frac{I}{(f)} \otimes_{\mathcal{O}} \frac{\mathcal{O}}{J} \cong \frac{I}{(f) + IJ} = \frac{I}{(f) + I \cap J}.$$

But

$$(f) + I \cap J = I \cap ((f) + J) = I \cap (I + J) = I.$$

Thus

$$\frac{I}{(f)} \otimes_{\mathcal{O}} \frac{\mathcal{O}}{J} \cong \frac{I}{I} = 0.$$

Let K be the residue field of $\mathcal{O}$. Since $I/(f) \otimes_{\mathcal{O}} (\mathcal{O}/J)$ surjects on to $(I/(f)) \otimes_{\mathcal{O}} K$ we see that $(I/(f)) \otimes_{\mathcal{O}} K = 0$. Nakayama's Lemma implies that $I = (f)$. □

Proof of Proposition 3.5.6. Suppose (1) holds. Let Y be a σ-invariant subscheme that is not homologically transverse to Z, and let $j \geq 1$ be such that

$$\mathcal{T}or_j^X(\mathcal{O}_Z, \mathcal{O}_Y) \neq 0.$$

Let $\mathcal{J} = \mathcal{I}_Y$, and let $J = \Gamma_*(\mathcal{J})$ be the right ideal of B generated by sections that vanish on Y. Since $\sigma Y = Y$, J is a two-sided ideal of B. We claim that $\operatorname{Tor}_j^B(B/I, B/J)_n \neq 0$ for $n \gg 0$.

Form a graded projective resolution

$$\cdots \to P^{-1} \to P^0 \to B/I \to 0$$

of B/I, where each P^i is a finitely generated graded free module. Thus for each $i \leq 0$, there is a finite multiset A_i of integers such that

$$P^i = \bigoplus_{a \in A_i} B[a].$$

Now, for each i let $\mathcal{P}^i = \widetilde{P^i}$. Since the functor $\widetilde{\ }$ is exact, the complex

$$\cdots \to \mathcal{P}^{-1} \to \mathcal{P}^0$$

is a resolution of $\mathcal{O}_Z = \widetilde{B/I}$. Furthermore, by the σ-invariance of Y and the σ-ampleness of $\mathcal{L}$, for $-j-d \leq i \leq -j+1$ and for $n \gg 0$, we have that

$$H^0(\mathcal{P}^i \otimes \mathcal{L}_n \otimes \mathcal{O}_Y) = \bigoplus_{a \in A_i} (B/J)_{n+a} = (P^i \otimes_B B/J)_n. \tag{3.5.8}$$

Fix n and let $\mathcal{Q}^\bullet = \mathcal{P}^\bullet \otimes \mathcal{L}_n \otimes \mathcal{O}_Y$. We will temporarily denote sheaf cohomology by $\check{H}^q$ to distinguish it from the cohomology H^p of a complex. As in Proposition 3.4.4, from a Cartan-Eilenberg resolution $\mathcal{C}^{\bullet,\bullet}$ of $\mathcal{Q}^\bullet$ we obtain two spectral sequences

$$ {}^{I}E^2_{pq} = H^p(\check{H}^q(\mathcal{Q}^\bullet)) \tag{3.5.9}$$

and

$$ {}^{II}E^2_{pq} = \check{H}^p(\mathcal{T}or^X_{-q}(\mathcal{O}_Z, \mathcal{O}_Y) \otimes \mathcal{L}_n), \tag{3.5.10}$$

both of which converge (since X has finite cohomological dimension) to the hypercohomology $\mathbb{H}^{p+q}(\mathcal{C}^{\bullet,\bullet})$.

By σ-ampleness of $\mathcal{L}$, by taking $n \gg 0$ we may assume that

$$\check{H}^p(\mathcal{T}or_{-q}(\mathcal{O}_Z, \mathcal{O}_Y) \otimes \mathcal{L}_n) = 0 \quad \text{for } p \geq 1 \text{ and } -j-d \leq q \leq -j-1$$

and that

$$\check{H}^q(\mathcal{Q}^p) = 0 \quad \text{for } q \geq 1 \text{ and } -j-d \leq p \leq -j-1.$$

Thus for $p+q = -j$, both (3.5.9) and (3.5.10) collapse, and we obtain that

$$\check{H}^0(\mathcal{T}or^X_j(\mathcal{O}_Z, \mathcal{O}_Y) \otimes \mathcal{L}_n) = H^{-j}(\check{H}^0(\mathcal{Q}^\bullet)). \tag{3.5.11}$$

Since $\mathcal{T}or^X_j(\mathcal{O}_Z, \mathcal{O}_Y) \neq 0$ and $\mathcal{L}$ is σ-ample, for $n \gg 0$ the left-hand side of (3.5.11) is nonzero; but (3.5.8) implies that for $n \gg 0$, the right-hand side is equal to

$$H^{-j}(P^\bullet \otimes_B B/J)_n = \operatorname{Tor}^B_j(B/I, B/J)_n.$$

Thus $\operatorname{Tor}^B_j(B/I, B/J)_n \neq 0$.

But if R is left noetherian, then, using Proposition 3.3.2 and a similar argument to that used in the proof of Lemma 3.3.12, for any finitely generated left B-module

M and for any $j \geq 1$, we must have that $\operatorname{Tor}_j^B(B/I, M)$ is torsion. Since we have shown this is false for $M = B/J$, R is not left noetherian.

Now suppose that (2) holds. Consider the singular stratification

$$X = X^{(0)} \supset X^{(1)} \supset \cdots$$

of X. If Z is not homologically transverse to some $X^{(i)}$, then by (1) R is not left noetherian. If Z is homologically transverse to all $X^{(i)}$, then by Lemma 3.5.7, Z is locally principal. By Lemma 3.3.12, there is some reduced and irreducible subscheme Y such that $\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0$ for infinitely many $n \geq 0$. But for the locally principal subvariety $\sigma^n Z$, $\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0$ if and only if $\sigma^n Z \supseteq Y$.

Thus $\sigma^n(Z) \supseteq Y$ for infinitely many $n \geq 0$. Let $\mathcal{J}$ be the ideal sheaf defining Y and let

$$A = \{n \geq 0 \mid Y \subseteq \sigma^n Z\} = \{n \geq 0 \mid \mathcal{J}^{\sigma^n} \supseteq \mathcal{I}\}.$$

Let $R' = \Bbbk \oplus H^0(\mathcal{I}\mathcal{B}_+)$. It is sufficient to show that R' is not left noetherian.

Let

$$J = \bigoplus_{n \geq 0} H^0((\mathcal{I} \cap \mathcal{J}^{\sigma^n})\mathcal{L}_n).$$

We will show that the left ideal J of R' is not finitely generated.

Fix an integer $k \geq 1$. By σ-ampleness of $\mathcal{L}$, we may choose $n > k$ such that $n \in A$ and $(\mathcal{I} \cap \mathcal{J}^{\sigma^n})\mathcal{L}_n = \mathcal{I}\mathcal{L}_n$ is globally generated. Then

$$(R' \cdot J_{\leq k})_n \subseteq H^0(\mathcal{I}\mathcal{J}^{\sigma^n}\mathcal{L}_n) \subsetneqq J_n$$

and ${}_{R'}J$ is not finitely generated. $\square$

Since the geometric condition required for a right idealizer to be left noetherian is fairly subtle, it is not surprising that right idealizers are almost never strongly left noetherian. To show this, we use the concept of *generic flatness*, as defined in

[ASZ99]. Let C be a commutative noetherian domain. We say that a C-module M is *generically flat* if there is some $f \neq 0 \in C$ such that M_f is flat over C_f. If R is a finitely generated commutative C-algebra, then by Grothendieck's generic freeness theorem [Gro65, Theorem 6.9.1], every finitely generated R-module is a generically flat C-module.

Artin, Small, and Zhang have generalized this result to strongly noetherian non-commutative rings. They prove:

Theorem 3.5.12. ([ASZ99, Theorem 0.1]) *Let R be a strongly noetherian algebra over an excellent Dedekind domain C. Then every finitely generated right R-module is generically flat over C.* $\square$

Lemma 3.5.13. *Assume Assumption-Notation 3.5.1. If Z' is a component of Z such that* $\operatorname{codim} Z' \geq 2$ *and such that $\bigcup_{m \geq 0} \sigma^m Z'$ is Zariski dense in X, then for every open affine $U \subseteq X$, the finitely generated left $R \otimes_k \mathcal{O}(U)$-module*

$$M = \bigoplus_{n \geq 0} \mathcal{R}(\sigma^{-n} U)$$

is not a generically flat $\mathcal{O}(U)$-module.

Proof. We first verify that M is a left R-module. By [AV90, Equation 2.5], the multiplication rule in $\mathcal{R}$ acts on sections via:

$$\mathcal{R}_n(V) \times \mathcal{R}_m(\sigma^n V) \to \mathcal{R}_{n+m}(V)$$

or, writing $V = \sigma^{-n-m} U$,

$$\mathcal{R}_n(\sigma^{-n-m} U) \times \mathcal{R}_m(\sigma^{-m} U) \to \mathcal{R}_{n+m}(\sigma^{-n-m} U).$$

Thus we have a map

$$R_n \times M_m = \mathcal{R}(X) \times \mathcal{R}_m(\sigma^{-m} U) \xrightarrow{\text{res}} \mathcal{R}_n(\sigma^{-n-m} U) \times \mathcal{R}_m(\sigma^{-m} U) \to M_{m+n}.$$

Verifying associativity is trivial, and so M is a left R-module.

Let $C = \mathcal{O}(U)$. By identifying C with C^{op}, consider the right action of C on M given by $g \star f = g \cdot f^{\sigma^n} = g \cdot (f \circ \sigma^n)$, where $g \in M_n$, $f \in C$. Note that $f \circ \sigma^n$ acts on $\sigma^{-n}U$ and so does act naturally on elements of M_n.

Now since for $n \gg 0$ the sheaves $\mathcal{I} \otimes \mathcal{L}_n$ are globally generated, the restriction map $R \to M$ is surjective in degree $\geq m$ for some m. But since $M_{<m}$ is a finitely generated C-module, therefore M is a finitely generated R_C module.

Now let f be an arbitrary element of C; let $M' = M_f$. Since the $\sigma^m(Z')$ are Zariski dense, there is some m such that $\sigma^m Z'$ meets U_f, say at a point p. But then $(M'_p)_m = (\mathcal{I}\mathcal{L}_m)_{\sigma^{-m}p}$, which is not flat over C_p, since codim $Z' \geq 2$. Thus M_f is not flat over C_f. $\square$

Corollary 3.5.14. *R is strongly left noetherian if and only if* $\operatorname{codim} Z = 1$ *and $\{\sigma^n Z\}_{n\geq 0}$ is critically transverse.*

Proof. If $\operatorname{codim} Z = 1$ and $\{\sigma^n Z\}_{n\geq 0}$ is critically transverse, then in particular Z is homologically transverse to the singular stratification of X and so by Lemma 3.5.7, Z is locally principal and $\mathcal{I} = \mathcal{I}_Z$ is invertible. Now, letting $\mathcal{L}' = \mathcal{I}\mathcal{L}(\mathcal{I}^{-1})^{\sigma}$, we have that $\mathcal{I}\mathcal{L}_n = (\mathcal{L}')_n \mathcal{I}^{\sigma^n}$. Since $\mathcal{L}'$ is clearly also σ-ample, we see that R is also the *left* idealizer at Z inside the twisted homogeneous coordinate ring $B(X, \mathcal{L}', \sigma)$. By assumption on critical transversality, we have in particular that for any $p \in X$, the set $\{n \leq 0 \,|\, \sigma^n(p) \in Z\}$ is finite. Thus by Proposition 3.5.2, R is strongly left noetherian.

If $\operatorname{codim} Z = 1$ and $\{\sigma^n Z\}_{n\geq 0}$ is not critically transverse, then by Proposition 3.5.6(2), R is not left noetherian so is certainly not strongly left noetherian.

If $\operatorname{codim} Z \neq 1$, fix an open affine $U \subseteq X$ such that $X \smallsetminus U$ has codimension 1. Let M be the module from Lemma 3.5.13. As M is not a generically flat left

$\mathcal{O}(U)$-module, by Theorem 3.5.12, $R \otimes \mathcal{O}(U)$ is not strongly left noetherian, so R is not strongly left noetherian. $\square$

3.6 The χ conditions for idealizers

In this section, we determine the homological properties of graded idealizers; specifically, we investigate the Artin-Zhang χ conditions, as defined in Section 2.4.

We first recall Rogalski's result that a right idealizer will fail χ_1 and all higher χ_j on the left.

Proposition 3.6.1. (Rogalski) *Assume Assumption-Notation 3.5.1. Then R fails left χ_1.*

Proof. This is proved in [Rog04b, Proposition 4.2]. To see it directly, note that changing R by a finite-dimensional vector space does not affect the χ conditions, so without loss of generality we have $R = \Bbbk + I$. Now B/R is infinite-dimensional and is killed on the left by I; thus we have an injection $B/R \hookrightarrow \operatorname{Ext}^1_R(\Bbbk, R)$ and we see that $\operatorname{Ext}^1_R(\Bbbk, R)$ is infinite-dimensional. $\square$

To analyze the right χ conditions, our key result is the following, due to Rogalski:

Proposition 3.6.2. ([Rog04b, Proposition 4.1]) *Let B be a noetherian ring that satisfies right χ. Let I be a a right ideal of B, and let $R = \mathbb{I}_B(I)$. Assume that B/I is infinite-dimensional, that B_R is finitely generated, and that R/I is finite-dimensional. Then R satisfies right χ_i for some $i \geq 0$ if and only if $\underline{\operatorname{Ext}}^j_B(B/I, M)$ is finite-dimensional for all $0 \leq j \leq i$ and all $M \in \text{gr-}B$.* $\square$

Rogalski proved that the right idealizer of a point in $\mathbb{P}^d$ satisfies right χ_{d-1} and fails right χ_d if the orbit of the point is critically dense. Here we extend Rogalski's result to higher-dimensional subvarieties.

Lemma 3.6.3. *Let X be a projective variety, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Z and Y be closed subschemes of X, and let $B = B(X, \mathcal{L}, \sigma)$. Let J be the right ideal of B consisting of sections vanishing along Y, and let I be the right ideal of B consisting of sections vanishing along Z. For $n \gg 0$, there is an isomorphism of k-vector spaces*

$$\underline{\operatorname{Ext}}^j_B(B/I, B/J)_n \cong \operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

Proof. There is a natural map from $\underline{\operatorname{Ext}}^j_B(B/I, B/J)$ to $\underline{\operatorname{Ext}}^j_{\text{Qgr-}B}(\pi(B/I), \pi(B/J))$. Since B satisfies χ by Theorem 2.4.6, this map has right bounded kernel and cokernel by [AZ94, Proposition 3.5]. Thus it suffices to show that for $n \gg 0$, we have

$$\underline{\operatorname{Ext}}^j_{\text{Qgr-}B}(\pi(B/I), \pi(B/J))_n \cong \operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

In fact, we show that we have this isomorphism for all n.

Using the equivalence between Qgr-B and $\mathcal{O}_X$-mod, we have that

$$\begin{aligned}\underline{\operatorname{Ext}}^j_{\text{Qgr-}B}(\pi(B/I), \pi(B/J))_n &\cong \operatorname{Ext}^j_{\text{Qgr-}B}(\pi(B/I), \pi((B/J)[n])) \\ &\cong \operatorname{Ext}^j_X(\widetilde{B/I}, \widetilde{(B/J)[n]}).\end{aligned}$$

Now, $\widetilde{B/I} = \mathcal{O}_Z$, and by (2.3.13),

$$\widetilde{(B/J)[n]} \cong (\mathcal{O}_Y \mathcal{L}_n)^{\sigma^{-n}} \cong \mathcal{O}_{\sigma^n Y} \mathcal{L}_n^{\sigma^{-n}}.$$

The result follows. □

We have seen that for R to be right Noetherian is relatively straightforward, but the left Noetherian property for R depends on the critical transversality of $\{\sigma^n Z\}$. It turns out that the right χ_j properties, for $j \geq 1$, also depend on the critical transversality of $\{\sigma^n Z\}$. In particular, we have:

Proposition 3.6.4. *Assume Assumption-Notation 3.5.1. Let k be the minimal codimension of an irreducible component of Z.*

(1) *If $\{\sigma^n Z\}_{n\leq 0}$ is critically transverse, and either*

(a) *X is nonsingular and Z is Gorenstein; or*

(b) *Z is 0-dimensional,*

then R satisfies right χ_{k-1} but fails right χ_k.

(2) *More generally, if Z contains an irreducible component of codimension k that is not contained in the singular locus of X, then R fails right χ_k. In particular, if R is left noetherian then R fails right χ_k.*

Proof. By Proposition 3.6.2, R satisfies right χ_i if and only if for all finitely generated M_B we have $\dim_k \underline{\operatorname{Ext}}^j_B(B/I, M) < \infty$ for all $j \leq i$. Furthermore, using the equivalence of categories between qgr-B and $\mathcal{O}_X$-mod, without loss of generality we may assume that $M = B/J$, where J is a right ideal of B consisting of sections vanishing along a reduced, irreducible subscheme Y of X.

Now by Lemma 3.6.3, for $n \gg 0$ we have isomorphisms $\underline{\operatorname{Ext}}^j_X(\pi B/I, \pi B/J)_n \cong \operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}})$. Thus we have:

$$(3.6.5)\qquad R \text{ satisfies right } \chi_i \iff \text{for all } Y \subseteq X,$$
$$\operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}) = 0 \text{ for all } j \leq i \text{ and } n \gg 0.$$

By [Gro57, Prop 4.2.1], for any coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ there is a spectral sequence

$$(3.6.6)\qquad H^p(\mathcal{E}xt^q_X(\mathcal{E}, \mathcal{F})) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathcal{E}, \mathcal{F}).$$

We consider the special case

$$(3.6.7)\qquad E^{pq} = H^p(\mathcal{E}xt^q_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}})) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

We first suppose that (1)(a) holds, and show that R satisfies right χ_{k-1}.

Fix a closed subscheme Y of X and consider the sheaf $\mathcal{E}xt^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y})$. This is supported on Z; we compute it by working locally at some closed point $x \in Z$. Gorenstein rings are Cohen-Macaulay and therefore locally equidimensional [Eis95, Corollary 18.11], so we may assume that Z is pure-dimensional of codimension $k' \geq k$. Let $J \subseteq \mathcal{O}$ be the ideal defining Z locally at x.

By [Eis95, Corollary 21.16], $\mathcal{O}/J$ has a self-dual free resolution as an $\mathcal{O}$-module

$$0 \to Q_{k'} \to \cdots \to Q_0 \to \mathcal{O}/J.$$

We write this resolution as $Q_\bullet \to \mathcal{O}/J$.

For a given n, let $K \subseteq \mathcal{O}$ be the ideal defining $\sigma^n Y$ at P. Let $M = \mathcal{O}/K$. Then we have isomorphisms of complexes

$$\operatorname{Hom}_{\mathcal{O}}(Q_\bullet, M) \cong \operatorname{Hom}_{\mathcal{O}}(Q_\bullet, \mathcal{O}) \otimes M \cong Q_\bullet \otimes M,$$

where the final isomorphism follows from the fact that $Q_\bullet$ is self-dual. The right-hand complex of this equation computes $\operatorname{Tor}^{\mathcal{O}}_{k'-j}(\mathcal{O}/J, M)$. Thus we obtain isomorphisms

$$\mathcal{E}xt^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) \cong \mathcal{T}or^X_{k'-j}(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) \cong \mathcal{T}or^X_{k'-j}(\mathcal{O}_{\sigma^{-n} Z}, \mathcal{O}_Y)^{\sigma^{-n}} \tag{3.6.8}$$

for all j.

We return to the Grothendieck spectral sequence (3.6.7). By [Har77, III.6.7], we have that

$$\mathcal{E}xt^q_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}) \cong \mathcal{E}xt^q_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) \otimes \mathcal{L}_n^{\sigma^{-n}}.$$

Using critical transversality and (3.6.8), choose n_0 such that $\mathcal{E}xt^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) = 0$ for all $n \geq n_0$ and $j < k \leq k'$. Then $E^{pq} = 0$ for $q < k$; so we see that if $p + q = j < k$, then (3.6.7) collapses to 0 and we have $\operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}) = 0$ for $n \gg 0$. By (3.6.5), R satisfies χ_{k-1}.

Let X^{sing} be the singular locus of X. We now suppose that (2) holds; that is, Z contains an irreducible component of codimension k that is not contained in X^{sing}. We show that in this situation, R fails right χ_k.

We consider the special case of (3.6.7) where $Y = X$:

$$H^p(\mathcal{E}xt^q_X(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}})) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}}). \tag{3.6.9}$$

Let $x \in Z$ be a nonsingular point of X such that the codimension of Z at x is k. Since X is nonsingular at x, by [BH93, Theorem 1.2.5]

$$\mathcal{E}xt^j_X(\mathcal{O}_Z, \mathcal{O}_X)_x = 0 \quad \text{ for } j < k \tag{3.6.10}$$

and

$$\mathcal{E}xt^k_X(\mathcal{O}_Z, \mathcal{O}_X)_x \neq 0. \tag{3.6.11}$$

Now (3.6.10) implies that for $p + q = k$, (3.6.9) collapses, and we obtain that

$$\operatorname{Ext}^k_X(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}}) \cong H^0(\mathcal{E}xt^k_X(\mathcal{O}_Z, \mathcal{O}_X) \otimes \mathcal{L}_n^{\sigma^{-n}}) \cong H^0((\mathcal{E}xt^k_X(\mathcal{O}_Z, \mathcal{O}_X)^{\sigma^n} \otimes \mathcal{L}_n).$$

This is nonzero for $n \gg 0$ by (3.6.11) and σ-ampleness of $\mathcal{L}$. Thus by (3.6.5), R fails right χ_k.

We have seen that if (2) holds, then R fails right χ_k. We note that if $\{\sigma^n Z\}_{n \leq 0}$ is critically transverse, then Z is homologically transverse to all σ-invariant subschemes, and certainly no component of Z is contained in X^{sing}. If R is left noetherian, then using Proposition 3.5.6 and Lemma 3.3.14, we again have that no component of Z is contained in the singular locus of X. Thus if (1)(a) or (1)(b) hold, or if R is left noetherian, then (2) holds and R fails right χ_k.

It remains to show that if (1)(b) holds, then R satisfies right χ_{k-1}. We have seen that X is nonsingular at all points of Z, and so (3.6.10) holds. Let $j \leq k - 1$. By

(3.6.9) we have that $\operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}}) = 0$. On the other hand, if $Y \subset X$ is a proper subvariety, then critical transversality of $\{\sigma^n Z\}_{n \leq 0}$ and Corollary 3.3.15 show that $\sigma^n Y$ and Z are disjoint for $n \gg 0$, and so certainly $\operatorname{Ext}^j_X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n) = 0$ for $n \gg 0$. By (3.6.5), R satisfies right χ_{k-1}. $\square$

3.7 Proj of graded idealizer rings and cohomological dimension

Assume Assumption-Notation 3.5.1. We are interested in understanding the cohomological dimension of the (right) noncommutative projective scheme associated to R, and here we briefly review the definitions.

Recall that Proj-R is defined as the pair (Qgr-$R, \pi R$). The cohomology groups on Proj-R are defined by setting

$$H^i(\text{Proj-}R, \mathcal{M}) = \operatorname{Ext}^i_{\text{Qgr-}R}(\pi R, \mathcal{M})$$

for any $\mathcal{M} \in$ Qgr-R. The *cohomological dimension* of Proj-R or the *right cohomological dimension of* R is

$$\max\{i \mid H^i(\text{Proj-}R, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \text{Qgr-}R\ \}.$$

If R is a finitely generated commutative graded $\Bbbk$-algebra, then its cohomological dimension is finite and in fact bounded by the dimension of $\operatorname{Proj} R$. The proofs of this are geometric, for example relying on Čech cohomology calculations, and do not generalize to the noncommutative situation. Stafford and Van den Bergh have asked [SV01, page 194] if every connected graded noetherian ring has finite left and right cohomological dimension.

In this section, we give a partial answer to Stafford and Van den Bergh's question. We prove:

Theorem 3.7.1. *Assume Assumption-Notation 3.5.1. If $R = R(X, \mathcal{L}, \sigma, Z)$ is noetherian, then R has finite left and right cohomological dimension.*

We also give an example of a right, but not left, noetherian ring with infinite right cohomological dimension. Amusingly, this ring has finite left cohomological dimension.

To begin, we review Rogalski's results on the cohomological dimension of idealizers.

Proposition 3.7.2. ([Rog04b, Lemma 3.2]) *Let B be a noetherian connected graded finitely $\mathbb{N}$-graded $\Bbbk$-algebra, and let I be a graded right ideal of B such that R/I is infinite-dimensional. Assume that B_R is finitely generated and R/I is finite-dimensional. Then there are isomorphisms of pairs*

$$R\text{-Proj} = (R\text{-Qgr}, \pi R) \cong (B\text{-Qgr}, \pi B) = B\text{-Proj} \tag{3.7.3}$$

and

$$\text{Proj-}R = (\text{Qgr-}R, \pi R) \cong (\text{Qgr-}B, \pi I). \tag{3.7.4}$$

□

Because of (3.7.3), it is clear that $\operatorname{cd}(R\text{-Proj}) = \operatorname{cd}(B\text{-Proj}) = \dim X$, and this was observed by Rogalski. We thus focus on calculating $\operatorname{cd}(\text{Proj-}R)$.

Lemma 3.7.5. *Assume Assumption-Notation 3.5.1. Then* $\operatorname{cd}(\text{Proj-}R)$ *is infinite if and only if* $\operatorname{hd}_X(\mathcal{O}_Z)$ *is infinite.*

Proof. Let $I = \Gamma_*(\mathcal{I}) \subseteq B$. Since $(\text{Qgr-}B, \pi I) \cong (\mathcal{O}_X\text{-Mod}, \mathcal{I})$, by (3.7.4) $\operatorname{cd}(\text{Proj-}R)$ is infinite if and only if for any $k \geq 0$, there is some quasi-coherent $\mathcal{F}$ such that $\operatorname{Ext}^k_X(\mathcal{I}, \mathcal{F}) \neq 0$.

Suppose that $\mathrm{hd}_X(\mathcal{O}_Z)$ and therefore $\mathrm{hd}_X(\mathcal{I})$ are infinite. Thus for any $k > 0$, there is some $\mathcal{G}$ such that $\mathcal{E}xt^k_X(\mathcal{I}, \mathcal{G}) \neq 0$. But let $\mathcal{O}(1)$ be any very ample invertible sheaf on X; by [Har77, III.6.9] we may choose n so that $\mathrm{Ext}^k_X(\mathcal{I}, \mathcal{G}(n)) = H^0(\mathcal{E}xt^k_X(\mathcal{I}, \mathcal{G}) \otimes \mathcal{O}(n)) \neq 0$. Thus $\mathrm{cd}(\text{Proj-}R) \geq k$ and since k was arbitrary, $\mathrm{cd}(\text{Proj-}R)$ is infinite.

Now suppose that $\mathrm{hd}_X(\mathcal{I})$ is finite, say equal to N, and let $\mathcal{G}$ be an arbitrary coherent sheaf. We apply (3.6.6) to obtain a spectral sequence

$$H^p(\mathcal{E}xt^q_X(\mathcal{I}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{I}, \mathcal{G}).$$

The left-hand side has nonzero terms only for $0 \leq p \leq \dim X$ and $0 \leq q \leq N$. Thus if $p+q$ is large (in particular $p+q > N+\dim X$), then all the groups on the left-hand side are 0, and so the right hand side is also 0. Thus $\mathrm{cd}(\text{Proj-}R) \leq N + \dim X$. $\square$

Proof of Theorem 3.7.1. If $R(X, \mathcal{L}, \sigma, Z)$ is left noetherian, then by Proposition 3.5.6, we have that $\{\sigma^n Z\}_{n \geq 0}$ is homologically transverse to all σ-invariant subvarieties of X, and in particular, to the singular stratification of X. Thus by Lemma 3.3.9, $\mathrm{hd}_X(\mathcal{O}_Z)$ is finite. By Lemma 3.7.5, $\mathrm{cd}(\text{Proj-}R)$ is finite. $\square$

We now give the promised example of a right noetherian ring with infinite right cohomological dimension.

Example 3.7.6. Assume that $\mathrm{char}\, \Bbbk = 0$. Let Y be the cuspidal cubic and let $X = Y \times \mathbb{P}^1$. Let $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ be the automorphism $\tau([x : y]) = [x + y : y]$, and let $\sigma = 1 \times \tau \in \mathrm{Aut}\, X$. Let P be the singular point of Y and let $Z = P \times [0 : 1] \in X$. Let $\mathcal{L}$ be any ample invertible sheaf on X, and let $R = R(X, \mathcal{L}, \sigma, Z)$. Since the numerical action of σ is trivial, by [Kee00, Theorem 1.2] $\mathcal{L}$ is σ-ample.

Now Z is certainly of infinite order under σ, and applying Proposition 3.5.2, we have that R is right noetherian. On the other hand, Z is contained in the singular locus of X, and so Proposition 3.5.6(1) and Lemma 3.3.14 imply that R is not left noetherian. Since X is not regular at Z, we have that $\mathrm{hd}_X(\mathcal{O}_Z)$ is infinite. Lemma 3.7.5 implies that $\mathrm{cd}\,\mathrm{Proj}\text{-}R = \infty$.

We note that Proposition 3.7.2 implies that the left cohomological dimension of R is 2.

Remark: Suppose that $R = R(X, \mathcal{L}, \sigma, Z)$ is a left noetherian idealizer. Together, Lemma 3.7.5 and Lemma 3.3.9 imply that the right cohomological dimension of R is bounded by $2 \dim X - 1$. We conjecture that in fact the left cohomological dimension of R is precisely $\dim X$. It is easy to see that $\mathrm{cd}(\mathrm{Proj}\text{-}R) \geq \dim X$.

3.8 Conclusion

Here we collect our results on geometric idealizers, and prove Theorem 3.1.6 and its promised generalization. Throughout, we make the following assumptions.

Assumptions 3.8.1. *Let X be a projective variety, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Z be a closed subscheme of X such that for any irreducible component Y of Z,*

$$\sigma^n(Y^{\mathrm{red}}) \not\subseteq Z$$

for $n \gg 0$.

Given this data, we let

$$R = R(X, \mathcal{L}, \sigma, Z).$$

Let $\mathcal{I} = \mathcal{I}_Z$ be the ideal sheaf of Z on X.

We note that since by Theorem 3.2.13 any noetherian right idealizer is up to a finite extension an idealizer at a scheme whose defining data satisfies Assumptions 3.8.1, these assumptions are not unduly restrictive.

We now summarize our results.

Theorem 3.8.2. *Assume Assumptions 3.8.1.*

(1) *R is right noetherian if and only if for any $x \in X$, the set $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is finite.*

(2) *If R is right noetherian, then R is strongly right noetherian.*

(3) *R fails left χ_1.*

(4) *If $\{\sigma^n(Z)\}_{n\geq 0}$ is critically transverse, then $\{(\mathcal{I}\mathcal{L}_n)_{\sigma^n}\}$ is a left and right ample sequence of bimodules, and R is left noetherian.*

(5) *R is strongly left noetherian if and only if $\operatorname{codim} Z = 1$ and $\{\sigma^n Z\}_{n\geq 0}$ is critically transverse.*

(6) *Let k be the minimal codimension of a component of Z. If $\{\sigma^n Z\}_{n\leq 0}$ is critically transverse and either $k = \dim X$ or X and Z are both smooth, then R satisfies right χ_{k-1}. If R is noetherian, then R fails right χ_k.*

(7) *If R is noetherian, then R has finite left and right cohomological dimension.*

We note that Theorem 3.1.6 is a special case of Theorem 3.8.2.

Proof. (1) and (2) are Proposition 3.5.2. (3) is Proposition 3.6.1. (4) is Lemma 3.4.2, Proposition 3.4.4 and Proposition 3.5.5. (5) is Corollary 3.5.14. (6) is a special case of Proposition 3.6.4, and (7) is Theorem 3.7.1. □

CHAPTER IV

Birationally commutative projective surfaces

4.1 Introduction

Artin and Stafford's classification [AS95] of *noncommutative projective curves*—finitely $\mathbb{N}$-graded domains of GK-dimension 2—was one of the early triumphs of noncommutative algebraic geometry. The classification of graded domains of GK-dimension 3, known as the problem of *classification of noncommutative projective surfaces*, is now the most important open problem in the field. It is much more difficult than the classification of curves: for example, while Artin and Stafford's work implies that all noncommutative curves are birationally commutative, the birational classification of surfaces is still unknown.

Artin's conjectured birational classification (Conjecture 1.3.3) says that a noncommutative projective surface is either birational to a quantum $\mathbb{P}^2$, birational to a quantum ruled surface, birationally commutative, or has a function field finite-dimensional over a field of transcendence degree 2. In this chapter, we classify birationally commutative surfaces, thus resolving one of the cases of Conjecture 1.3.3. We will always work over a fixed uncountable algebraically closed ground field, $\mathbb{k}$.

We formally define:

Definition 4.1.1. A finitely $\mathbb{N}$-graded domain R is a *birationally commutative pro-*

jective surface if

(1) R is noetherian of GK-dimension 3;

(2) the graded quotient ring of R is of the form $K[z, z^{-1}; \sigma]$ where K is a field of transcendence degree 2.

Some examples of birationally commutative projective surfaces are twisted homogeneous coordinate rings on projective surfaces, the naïve blowups defined in (1.4.2), and idealizer subrings of twisted homogeneous coordinate rings of surfaces, as studied in the last chapter. In addition, one expects that idealizers inside naïve blowups will provide examples of birationally commutative projective surfaces. Naturally, one asks if this is a complete enumeration of birationally commutative projective surfaces, and if and how one can construct the underlying geometric data of such a surface.

In this chapter, we give a complete classification of birationally commutative projective surfaces. We show that there is one new class of such surfaces; we refer to these as *ADC rings*. ADC rings have similar properties to naïve blowups, although they are never generated in degree 1. We then show that (up to a Veronese, as usual) a birationally commutative projective surface is either:

- a twisted homogeneous coordinate ring;
- an ADC ring;
- or an idealizer in one of the above.

Further, we obtain strong constraints on the geometry of the defining data.

We make a remark on the GK-dimension of noncommutative surfaces. Artin and Van den Bergh showed [AV90, Theorem 1.7(iii)] that the GK-dimension of the twisted homogeneous coordinate ring of a projective surface is either 3 or 5 and may attain either value; see [AV90, Example 5.18] for an example of a surface whose twisted homogeneous coordinate ring has GK-dimension 5. Surely such twisted homogeneous coordinate rings should be considered noncommutative surfaces! Thus, although the requirement that a noncommutative surface have GK-dimension 3 seems natural, it does impose some restrictions. As yet, we have not been able to extend our results on birationally commutative surfaces to include the GK-dimension 5 case.

Let us describe the geometric data defining a birationally commutative surface in more detail.

Definition 4.1.2. The tuple $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is *surface data* if:

- X is a projective surface;
- σ is an automorphism of X;
- $\mathcal{L}$ is an invertible sheaf on X;
- $\mathcal{D}$ is the ideal sheaf of a 0-dimensional subscheme of X such that all points in the cosupport of $\mathcal{D}$ have distinct infinite σ-orbits;
- $\mathcal{A}$ and $\mathcal{C}$ are ideal sheaves on X such that $\mathcal{AC} \subseteq \mathcal{D}$ and such that the pair $(\mathcal{A}, \mathcal{C})$ is maximal with respect to this property (in particular, $\mathcal{D} \subseteq \mathcal{A} \cap \mathcal{C}$ and so $\mathcal{A}$ and $\mathcal{C}$ are cofinite);
- Ω is a curve on X; and
- Λ and Λ' are 0-dimensional subschemes of X supported on points of infinite order.

Given surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$, we define a graded $(\mathcal{O}_X, \sigma)$-bimodule algebra

$$\mathcal{T} = \mathcal{T}(\mathbb{D}) = \bigoplus_{n \geq 0} (\mathcal{T}_n)_{\sigma^n},$$

where $\mathcal{T}_0 = \mathcal{O}_X$ and

$$\mathcal{T}_n = (\mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n} \cap \mathcal{I}_{\Omega}\mathcal{I}_{\Lambda}\mathcal{I}_{\Lambda'}^{\sigma^n})\mathcal{L}_n$$

for $n \geq 1$, and a $\Bbbk$-algebra

$$T(\mathbb{D}) = H^0(\mathcal{T}(\mathbb{D})) = \bigoplus_{n\geq 0} H^0(\mathcal{T}_n).$$

Definition 4.1.3. The surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is *transverse* if:

- σ is numerically trivial;
- $\mathcal{L}$ is ample and σ-ample;
- all points in the cosupport of $\mathcal{D}$ have critically dense σ-orbits;
- $\{\sigma^n\Omega\}_{n\in\mathbb{Z}}$ is critically transverse; and
- both $\{\sigma^n\Lambda\}_{n\geq 0}$ and $\{\sigma^n\Lambda'\}_{n\leq 0}$ are critically transverse.

The main theorem of this chapter is:

Theorem 4.1.4. *Let R be a finitely $\mathbb{N}$-graded domain. If R is a birationally commutative projective surface, then there is transverse surface data*

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

so that some Veronese of R satisfies

$$R^{(k)} = T(\mathbb{D}).$$

Further, if the surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is transverse, then $T(\mathbb{D})$ is a birationally commutative projective surface.

Theorem 4.1.4 is an extension of Rogalski and Stafford's recent classification of birationally commutative projective surfaces that are generated in degree 1. Their result is:

Theorem 4.1.5. ([RS06, Theorem 1.1]) *Let R be a birationally commutative surface that is generated in degree 1. Then there are a projective surface X, an automorphism σ of X, a σ-ample invertible sheaf $\mathcal{L}$ on X, and a 0-dimensional subscheme Z of X, supported on points with critically dense orbits, so that for some $k \geq 1$, $R^{(k)} = S(X, \mathcal{L}, \sigma, Z)$. (If Z is nonempty, this is a naïve blowup; the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$ corresponds to $Z = \emptyset$.)* $\square$

We remark that Rogalski and Stafford work slightly more generally than we do, in that their rings may have GK-dimension 3 or 5. That is, they study finitely $\mathbb{N}$-graded noetherian domains R whose graded quotient ring is of the form

$$K[z, z^{-1}; \sigma]$$

where $K = \Bbbk(X)$ is the function field of a projective surface X such that σ induces an automorphism of X. By [Rog07, Theorem 1.1], any such R has GK-dimension 3 or 5, and any birationally commutative projective surface in the sense of Definition 4.1.1 that is generated in degree 1 is of the form studied by Rogalski and Stafford.

Let $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ be transverse surface data, and let $T = T(\mathbb{D})$. We comment on the various roles played by the pieces of $\mathbb{D}$ in the behavior of T.

The data Ω, Λ, and Λ' correspond to idealizing. That is, let

$$\mathbb{E} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \emptyset, \emptyset, \emptyset),$$

and let $S = T(\mathbb{E})$. Then

$$S_n = H^0(\mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n} \mathcal{L}_n)$$

for $n \geq 1$. We see that $T \subseteq S$; one may easily show that (in sufficiently large degree) T is a right idealizer inside a left idealizer inside S. In particular, if $\mathcal{A} = \mathcal{D} = \mathcal{C} = \mathcal{O}_X$, then $T(\mathbb{D})$ is a right idealizer inside a left idealizer inside the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$.

We studied the process of idealizing, at least in twisted homogeneous coordinate rings, in detail in Chapter III. We make a few comments now on the data defining S.

Definition 4.1.6. The tuple $(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ is *ADC data* if:

- X is a projective surface;
- σ is an automorphism of X;
- $\mathcal{L}$ is a σ-ample invertible sheaf on X;
- $\mathcal{D}$ is the ideal sheaf of a 0-dimensional subscheme of X such that all points in the cosupport of $\mathcal{D}$ have distinct critically dense σ-orbits;
- $\mathcal{A}$ and $\mathcal{C}$ are ideal sheaves on X such that $\mathcal{A}\mathcal{C} \subseteq \mathcal{D}$, and so that the pair $(\mathcal{A}, \mathcal{C})$ is maximal with respect to this property.

Given ADC data $(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$, we define the *ADC bimodule algebra*

$$\mathcal{S} = \mathcal{S}(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$$

to be the graded $(\mathcal{O}_X, \sigma)$-bimodule algebra

$$\mathcal{S} = \bigoplus_{n \geq 0} (\mathcal{S}_n)_{\sigma^n},$$

where $\mathcal{S}_0 = \mathcal{O}_X$ and

$$\mathcal{S}_n = \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n} \mathcal{L}_n$$

for $n \geq 1$. We define the *ADC ring* $S = S(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ as

$$S = H^0(\mathcal{S}) = \bigoplus_{n \geq 0} H^0(\mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n} \mathcal{L}_n).$$

Note that a naïve blowup is a special case of an ADC ring: if $\mathcal{A} = \mathcal{D}$ and $\mathcal{C} = \mathcal{O}_X$, then $S = S(X, \mathcal{L}, \sigma, \mathcal{D}, \mathcal{D}, \mathcal{O}_X)$ satisfies

$$S_n = H^0(\mathcal{D}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{L}_n)$$

and so S is a naïve blowup. More generally, if $\mathcal{A}\mathcal{C} = \mathcal{D}$, then S is a naïve blowup at the subscheme defined by $\mathcal{A}\mathcal{C}^{\sigma}$.

Example 4.1.7. To see that ADC rings are not idealizers inside naïve blowups, let X be a projective surface, let $\sigma \in \operatorname{Aut} X$, and let $p \in X$ be a (nonsingular) point with a critically dense orbit. Let $x, y \in \mathcal{O}_{X,p}$ be local coordinates at p. Let $\mathcal{A} = \mathcal{C}$ be the ideal sheaf cosupported at p so that

$$\mathcal{A}_p = \mathcal{C}_p = (x, y)\mathcal{O}_{X,p},$$

and let $\mathcal{D}$ be the ideal sheaf cosupported at p so that

$$\mathcal{D}_p = (x, y^2)\mathcal{O}_{X,p}.$$

We have $(\mathcal{A}\mathcal{C})_p = (x^2, xy, y^2)\mathcal{O}_{X,p} \subseteq \mathcal{D}_p$. Thus $\mathcal{A}\mathcal{C} \subseteq \mathcal{D}$, and clearly $(\mathcal{A}, \mathcal{C})$ is maximal with respect to this inclusion. Thus if $\mathcal{L}$ is a σ-ample invertible sheaf on X, the tuple $(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ is ADC data.

The ring $S = S(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ is not an idealizer. In fact, one can show that S is a *maximal order* — roughly speaking, the noncommutative equivalent of an integrally closed ring — although we do not do so in this thesis. One can also show that no Veronese subring of S is generated in degree 1.

The techniques used in the proofs of Theorem 4.1.4 and Theorem 4.1.5 are quite different. In order to construct the scheme X on which R lives, Rogalski and Stafford study the space of point modules (see Definition 2.4.4) over R. Since the point modules over a naïve blowup are not parameterized by any scheme, the arguments involving this space are quite subtle and technical.

In contrast, we are able to construct the data $(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ associated to R much more directly. We work via a method of successive approximations: we first construct a twisted homogeneous coordinate ring B that contains R, and then gradually modify the defining data for B to approach R more and more closely. Our philosophy is thus relatively straightforward, although showing that our methods do eventually converge to R is fairly involved.

4.2 Properties of rings defined by transverse data

We begin with the easy direction of Theorem 4.1.4. Suppose that the surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is transverse. Let $\mathcal{T} = \mathcal{T}(\mathbb{D})$ and let $T = T(\mathbb{D})$. In this section, we show that both $\mathcal{T}$ and T are noetherian, and study some of their properties.

Let $K = \Bbbk(X)$. The automorphism σ of X induces a $\Bbbk$-automorphism of K, which we also denote by σ. As a matter of notation, we will write $B(X, \mathcal{L}, \sigma)$ and all of its graded subrings as subrings of $K[z, z^{-1}; \sigma]$. That is, let $\mathcal{S}$ be any graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of $B(X, \mathcal{L}, \sigma)$. We write

$$H^0(\mathcal{S}) = \bigoplus_{n \geq 0} H^0(\mathcal{S}_n) z^n,$$

where z is a formal parameter. If S is any graded subring of

$$B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(\mathcal{L}_n) z^n,$$

we let

$$\overline{S}_n = S_n z^{-n} \subseteq K,$$

so

$$S = \bigoplus_{n \geq 0} \overline{S}_n z^n.$$

If I is a graded right or left ideal of S we will write

$$I = \bigoplus_{n \geq 0} I_n = \bigoplus_{n \geq 0} \overline{I}_n z^n.$$

In particular, for the rest of the chapter we will use the notation that if

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda'),$$

then

$$T = T(\mathbb{D})$$

is defined by

$$T_n = H^0(\mathcal{T}_n) z^n = H^0((\mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n} \cap \mathcal{I}_\Omega \mathcal{I}_\Lambda \mathcal{I}_{\Lambda'}^{\sigma^n}) \mathcal{L}_n) z^n. \tag{4.2.1}$$

Note that multiplication on T is now induced from its inclusion in $K[z, z^{-1}; \sigma]$.

We begin by showing that the sequence of bimodules $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample. We will use a result of Rogalski and Stafford that relates the ampleness of a sequence of bimodules of the form $\{(\mathcal{R}_n)_{\sigma^n}\}$ to the Castelnuovo-Mumford regularity of the sheaves $\mathcal{R}_n$.

Lemma 4.2.2. ([RS07, Corollary 3.14]) *Let X be a projective scheme with very ample invertible sheaf $\mathcal{N}$. Let $\mathcal{F}_n$ be a sequence of coherent sheaves on X such that for each n, the closed set where $\mathcal{F}_n$ is not locally free has dimension at most 2. Then $\{(\mathcal{F}_n)_{\sigma^n}\}$ is a right ample sequence if and only if*

$$\lim_{n \to \infty} \operatorname{reg}_{\mathcal{N}} \mathcal{F}_n = -\infty,$$

and $\{(\mathcal{F}_n)_{\sigma^n}\}$ *is a left ample sequence if and only if*

$$\lim_{n\to\infty} \operatorname{reg}_{\mathcal{N}^{\sigma^n}} \mathcal{F}_n = -\infty.$$

Proof. The right ampleness statement is a restatement of [RS07, Corollary 3.14]. The left ampleness statement follows by symmetry. □

We will also use the following result of Dennis Keeler:

Lemma 4.2.3. ([Kee06, Proposition 2.8]) *Let* X *be a projective scheme with very ample invertible sheaf* $\mathcal{N}$. *Then there is a constant* C, *depending only on* X *and* $\mathcal{N}$, *so that for any pair* $\mathcal{F}, \mathcal{G}$ *of coherent sheaves such that the dimension of the closed set where both* $\mathcal{F}$ *and* $\mathcal{G}$ *are not locally free is less than or equal to 2, we have that*

$$\operatorname{reg}_{\mathcal{N}} \mathcal{F} \otimes \mathcal{G} \leq \operatorname{reg}_{\mathcal{N}} \mathcal{F} + \operatorname{reg}_{\mathcal{N}} \mathcal{G} + C.$$

□

We will also frequently use the following easy observation about cohomology vanishing.

Lemma 4.2.4. *Let* X *be a projective scheme and suppose that*

$$0 \to \mathcal{K} \to \mathcal{M} \xrightarrow{\theta} \mathcal{N} \to \mathcal{K}' \to 0$$

is an exact sequence of coherent sheaves on X, *where* $\mathcal{K}$ *and* $\mathcal{K}'$ *are supported on subschemes of dimension 0. Further suppose that* $H^i(\mathcal{M}) = 0$ *for all* $i \geq 1$. *Then* $H^i(\mathcal{N}) = 0$ *for all* $i \geq 1$.

Proof. Note that $H^i(\mathcal{K}) = H^i(\mathcal{K}') = 0$ for all $i \geq 1$. Let $\mathcal{M}' = \operatorname{Im} \theta$. From the long exact cohomology sequence, we deduce that $H^i(\mathcal{M}') = 0$ for all $i \geq 1$. This implies that $H^i(\mathcal{N}) = 0$ for all $i \geq 1$. □

We will show that the sequence of bimodules $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample under slightly less restrictive assumptions on the defining data than transversality. We first assume that Λ and Λ' are empty.

Lemma 4.2.5. *Let X be a projective surface, let $\sigma \in \operatorname{Aut} X$, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Ω be a curve on X so that $\{\sigma^n \Omega\}$ is critically transverse.*

(1) *Let $\mathcal{E}$ be an ideal sheaf on X that defines a 0-dimensional subscheme supported on dense orbits. Then the sequence of bimodules*

$$\{\big((\mathcal{I}_\Omega \cap \mathcal{E}\mathcal{E}^\sigma \cdots \mathcal{E}^{\sigma^{n-1}})\mathcal{L}_n\big)_{\sigma^n}\}$$

is left and right ample.

(2) *In addition, let $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{C}$ be ideal shaves on X such that the tuple*

$$\mathbb{E} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \emptyset, \emptyset)$$

is surface data. Suppose also that the orbits of all points in the cosupport of $\mathcal{D}$ are dense. Let $\mathcal{T} = \mathcal{T}(\mathbb{E})$. Then the sequence of bimodules $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample.

Proof. (1) For all $n \geq 1$, let

$$\mathcal{J}_n = \mathcal{I}_\Omega \cap \mathcal{E}\mathcal{E}^\sigma \cdots \mathcal{E}^{\sigma^{n-1}}.$$

We will show that the sequence $\{(\mathcal{J}_n\mathcal{L}_n)_{\sigma^n}\}$ is left and right ample.

We first assume in addition that $\mathcal{L}$ is ample. By [AV90, Theorem 1.7], $\mathcal{L}$ is then also σ^2-ample. Note that all points in the cosupport of $\mathcal{E}\mathcal{E}^\sigma$ have dense σ^2-orbits. Let

$$\begin{aligned}\mathcal{F}_n = (\mathcal{E}\mathcal{E}^\sigma)(\mathcal{E}\mathcal{E}^\sigma)^{\sigma^2} \cdots (\mathcal{E}\mathcal{E}^\sigma)^{\sigma^{2n-2}} \mathcal{L} \otimes \mathcal{L}^{\sigma^2} \otimes \cdots \otimes \mathcal{L}^{\sigma^{2n-2}} \\ = \mathcal{E}\mathcal{E}^\sigma \cdots \mathcal{E}^{\sigma^{2n-1}} \mathcal{L} \otimes \mathcal{L}^{\sigma^2} \otimes \cdots \otimes \mathcal{L}^{\sigma^{2n-2}}.\end{aligned}$$

By [RS07, Theorem 3.1], the sequences $\{(\mathcal{F}_n)_{\sigma^{2n}}\}$ and $\{(\mathcal{F}_{n+1})_{\sigma^{2n+1}}\}$ are left and right ample.

Now let

$$\mathcal{G}_n = \mathcal{I}_\Omega \mathcal{L}^\sigma \mathcal{L}^{\sigma^3} \cdots \mathcal{L}^{\sigma^{2n-1}}.$$

By Proposition 3.4.4, the sequences $\{(\mathcal{G}_n)_{\sigma^{2n}}\}$ and $\{(\mathcal{G}_n)_{\sigma^{2n+1}}\}$ are left and right ample. By Lemma 4.2.3 and Lemma 4.2.2, the sequences

$$\{(\mathcal{F}_n \otimes \mathcal{G}_n)_{\sigma^{2n}}\}$$

and

$$\{(\mathcal{F}_{n+1} \otimes \mathcal{G}_n)_{\sigma^{2n+1}}\}$$

are left and right ample.

Let $\mathcal{M}$ be any coherent sheaf on X. For any $n \geq 0$, there is an exact sequence

$$0 \to \mathcal{H}_n \to \mathcal{F}_n \otimes \mathcal{G}_n \otimes \mathcal{M}^{\sigma^{2n}} \to \mathcal{J}_{2n}\mathcal{L}_{2n} \otimes \mathcal{M}^{\sigma^{2n}} \to \mathcal{K}_n \to 0$$

where both $\mathcal{H}_n$ and $\mathcal{K}_n$ are supported on dimension 0 subschemes of X. Since $H^i(\mathcal{F}_n \otimes \mathcal{G}_n \otimes \mathcal{M}^{\sigma^{2n}}) = 0$ for $i \geq 1$ and $n \gg 0$, Lemma 4.2.4 implies that

$$H^i(\mathcal{J}_{2n}\mathcal{L}_{2n} \otimes \mathcal{M}^{\sigma^{2n}}) = 0$$

for $i \geq 1$ and $n \gg 0$. Thus $\{(\mathcal{J}_{2n}\mathcal{L}_{2n})_{\sigma^{2n}}\}$ is a left ample sequence; the argument that it is right ample is similar. Likewise, from the maps

$$\mathcal{F}_{n+1} \otimes \mathcal{G}_n \to \mathcal{J}_{2n+1}\mathcal{L}_{2n+1}$$

we obtain that $\{(\mathcal{J}_{2n+1}\mathcal{L}_{2n+1})_{\sigma^{2n+1}}\}$ is left and right ample. Thus

$$\{(\mathcal{J}_n\mathcal{L}_n)_{\sigma^n}\}$$

is left and right ample.

Now consider the general case. By [AV90, Theorem 1.7], there is some $k \geq 1$ so that $\mathcal{L}_k$ is ample. Let $\mathcal{E}' = \mathcal{E}\mathcal{E}^{\sigma} \cdots \mathcal{E}^{\sigma^{k-1}}$. We have seen that the sequence of bimodules

$$\{((\mathcal{I}_\Omega \cap \mathcal{E}'(\mathcal{E}')^{\sigma^k} \cdots (\mathcal{E}')^{\sigma^{k(n-1)}})\mathcal{L}_{kn})_{\sigma^{kn}}\} = \{(\mathcal{J}_{kn}\mathcal{L}_{kn})_{\sigma^{kn}}\}$$

is left and right ample. Lemma 4.2.2 implies that for any $0 \leq i \leq k-1$, the sequence

$$\{(\mathcal{J}_{kn}\mathcal{L}_{kn-i})_{\sigma^{kn-i}}\}$$

is left and right ample.

Fix $0 \leq i \leq k-1$. We have $\mathcal{J}_{kn} \subseteq \mathcal{J}_{kn-i}$ for all $n \geq 1$. For any coherent $\mathcal{M}$ on X the kernel and cokernel of

$$\mathcal{M} \otimes \mathcal{J}_{kn}\mathcal{L}_{kn-i} \to \mathcal{M} \otimes \mathcal{J}_{kn-i}\mathcal{L}_{kn-i}$$

are supported on sets of dimension 0. Thus by Lemma 4.2.4 the sequence

$$\{(\mathcal{J}_{kn-i}\mathcal{L}_{kn-i})_{\sigma^{kn-i}}\}$$

is left and right ample for all $0 \leq i \leq k-1$. Thus

$$\{(\mathcal{J}_n\mathcal{L}_n)_{\sigma^n}\}$$

is a left and right ample sequence, as claimed.

(2) Let $\mathcal{E} = \mathcal{D}\mathcal{D}^{\sigma}$, so

$$\mathcal{E}\mathcal{E}^{\sigma} \cdots \mathcal{E}^{\sigma^{n-1}} \subseteq \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n}.$$

The cokernel of this inclusion is supported on a set of dimension 0. By (1) the sequence

$$\{\big((\mathcal{I}_\Omega \cap \mathcal{E}\mathcal{E}^{\sigma} \cdots \mathcal{E}^{\sigma^{n-1}})\mathcal{L}_n\big)_{\sigma^n}\}$$

is left and right ample. An argument similar to those above shows that

$$\{(\mathcal{T}_n)_{\sigma^n}\}$$

is left and right ample. $\square$

We thank Dennis Keeler for assistance with the following argument.

Lemma 4.2.6. *Let X be a projective surface and let σ be a numerically trivial automorphism of X. Let $\mathcal{L}$ be an invertible sheaf on X. Suppose that there are sheaves $\mathcal{R}_n \subseteq \mathcal{L}_n$ so that the sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is a left and right ample sequence. Let $\mathcal{I}$ be an ideal sheaf that is locally free except on a set of dimension ≤ 0. Then both $\{(\mathcal{I}\mathcal{L}_n \cap \mathcal{R}_n)_{\sigma^n}\}$ and $\{(\mathcal{I}\mathcal{R}_n)_{\sigma^n}\}$ are left and right ample sequences of bimodules.*

Proof. Fix a very ample invertible sheaf $\mathcal{N}$ on X. We first show that $\{(\mathcal{I} \otimes \mathcal{R}_n)_{\sigma^n}\}$ is left and right ample. Right ampleness is immediate. For left ampleness, by Lemma 4.2.2, it is sufficient to show that

$$\lim_{n\to\infty} \operatorname{reg}_{\mathcal{N}^{\sigma^n}} \mathcal{I} \otimes \mathcal{R}_n = -\infty.$$

By Fujita's Vanishing Theorem 2.5.1, we may choose m such that for any nef invertible sheaf $\mathcal{F}$, we have that $H^i(\mathcal{I} \otimes \mathcal{N}^{\otimes m} \otimes \mathcal{F}) = 0$ for all $i \geq 1$. As σ is numerically trivial, for any k the invertible sheaf $(\mathcal{N}^{-1} \otimes \mathcal{N}^{\sigma^k})^{\otimes m}$ is nef. Thus for any k and for any $i \geq 1$ we have that

$$H^i(\mathcal{I} \otimes (\mathcal{N}^{\sigma^k})^{\otimes m}) = H^i(\mathcal{I} \otimes \mathcal{N}^{\otimes m} \otimes (\mathcal{N}^{-1} \otimes \mathcal{N}^{\sigma^k})^{\otimes m}) = 0,$$

and so $\mathcal{I}$ is $(m+2)$-regular with respect to any $\mathcal{N}^{\sigma^k}$. Now by Lemma 4.2.3, we have that

$$\lim_{n\to\infty} \operatorname{reg}_{\mathcal{N}^{\sigma^n}} \mathcal{I} \otimes \mathcal{R}_n = -\infty.$$

Thus $\{(\mathcal{I} \otimes \mathcal{R}_n)_{\sigma^n}\}$ is left ample.

Now for any coherent $\mathcal{M}$, since the maps

$$\mathcal{I} \otimes \mathcal{R}_n \otimes \mathcal{M} \to (\mathcal{I}\mathcal{L}_n \cap \mathcal{R}_n) \otimes \mathcal{M}$$

and

$$\mathcal{I} \otimes \mathcal{R}_n \otimes \mathcal{M} \to \mathcal{I}\mathcal{R}_n \otimes \mathcal{M}$$

have kernel and cokernel supported on sets of dimension 0, by Lemma 4.2.4 we obtain that $\{(\mathcal{I}\mathcal{L}_n \cap \mathcal{R}_n)_{\sigma^n}\}$ and $\{(\mathcal{I}\mathcal{R}_n)_{\sigma^n}\}$ are left and right ample sequences. $\square$

Lemma 4.2.7. *Suppose that $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is surface data so that $\mathcal{L}$ is σ-ample, all points in the cosupport of $\mathcal{D}$ have dense orbits, and either*

(1) *$\{\sigma^n\Omega\}_{n\in\mathbb{Z}}$, $\{\sigma^n\Lambda\}_{n\geq 0}$, and $\{\sigma^n\Lambda'\}_{n\leq 0}$ are critically transverse; or*

(2) *σ is numerically trivial and Ω does not contain any 1-dimensional component of the singular locus of X.*

Let $\mathcal{T} = \mathcal{T}(\mathbb{D})$. Then the sequence of bimodules $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample.

Proof. In case (1), certainly the orbits of all points in Λ and Λ' are Zariski-dense. Thus there is an ideal sheaf $\mathcal{E}$ on X, supported on points with dense orbits, so that for all $n \geq 1$ we have

$$\mathcal{I}_\Omega \cap \mathcal{E}\mathcal{E}^\sigma \cdots \mathcal{E}^{\sigma^{n-1}} \subseteq \mathcal{I}_\Omega \mathcal{I}_\Lambda \mathcal{I}_{\Lambda'}^{\sigma^n} \cap \mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n} = \mathcal{T}_n\mathcal{L}_n^{-1}. \tag{4.2.8}$$

Let

$$\mathcal{M}_n = (\mathcal{I}_\Omega \cap \mathcal{E}\mathcal{E}^\sigma \cdots \mathcal{E}^{\sigma^{n-1}})\mathcal{L}_n.$$

By Lemma 4.2.5 the sequence of bimodules $\{(\mathcal{M}_n)_{\sigma^n}\}$ is left and right ample. Since the cokernel of the inclusion (4.2.8) is supported on a 0-dimensional scheme, $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample by Lemma 4.2.4.

In case (2), our assumption on Ω implies that $\mathcal{I}_\Omega$ is locally free except possibly on the 0-dimensional set where Ω meets the singular locus of X. Thus

$$\{(\mathcal{T}_n)_{\sigma^n}\}$$

is left and right ample by repeated applications of Lemma 4.2.6. $\square$

We will now prove that if the surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is transverse, then both the bimodule algebra $\mathcal{T}(\mathbb{D})$ and the $\Bbbk$-algebra $T(\mathbb{D})$ are left and right noetherian. As mentioned, the data Ω, Λ, and Λ' correspond to idealizing. We first assume that no idealizing is taking place, and show that ADC bimodule algebras are noetherian. To do this, we explicitly construct generators for graded right and left ideals.

Proposition 4.2.9. *Suppose that the tuple $(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ is ADC data, and let $\mathcal{S} = \mathcal{S}(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$. Let $\mathcal{J} = \bigoplus (\mathcal{J}_n)_{\sigma^n}$ be a graded right ideal of $\mathcal{S}$. Then there are an ideal sheaf $\mathcal{J}'$ on X and an integer $m \geq 0$ such that for $n > m$,*

$$\mathcal{J}_n = (\mathcal{J}' \mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n}) \mathcal{L}_n.$$

Further, for $n \geq m$, $\mathcal{J}'$ and $\mathcal{D}^{\sigma^n}$ are comaximal.

Likewise, let $\mathcal{K}$ be a graded left ideal of $\mathcal{S}$. Then there are an ideal sheaf $\mathcal{K}'$ on X and an integer $m' \geq 0$ such that for $n > m'$,

$$\mathcal{K}_n = (\mathcal{A} \mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-m'}} (\mathcal{K}')^{\sigma^n}) \mathcal{L}_n.$$

Further, for $j \leq -m'$, $\mathcal{K}'$ and $\mathcal{D}^{\sigma^j}$ are comaximal.

Proof. Let Z be the cosupport of $\mathcal{D}$; note that our assumptions imply that $\{\sigma^n Z\}$ is critically transverse. By Lemma 2.3.14, without loss of generality we may assume that $\mathcal{L} = \mathcal{O}_X$.

By symmetry, it suffices to prove the result for a graded right ideal $\mathcal{J}$ of $\mathcal{S}$, and we may assume that $\mathcal{J} \neq 0$. Let n_0 be such that $\mathcal{J}_{n_0} \neq 0$. Let Y be the subscheme of X defined by $\mathcal{J}_{n_0}$. By critical transversality, there is some $n_1 > n_0$ such that for $n \geq n_1$, we have $\sigma^{-n}(Z) \cap Y = \emptyset$.

For $n > n_1$, let $\mathcal{I}_n$ be the maximal ideal sheaf on X so that $\mathcal{I}_n \supseteq \mathcal{J}_n$ and so that $\mathcal{I}_n/\mathcal{J}_n$ is supported on

$$\sigma^{-(n_1+1)}Z \cup \cdots \cup \sigma^{-n}(Z).$$

Note that this implies that $(\mathcal{I}_n)_p = \mathcal{O}_{X,p}$ for all $p \in \sigma^{-j}Z$ with $j \geq n_1 + 1$. As

$$\mathcal{J}_n(\mathcal{S}_1)^{\sigma^n} = \mathcal{J}_n \mathcal{A}^{\sigma^n} \mathcal{C}^{\sigma^{n+1}} \subseteq \mathcal{J}_{n+1}$$

for any n, if $n > n_1$ then $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$. Further, if $n, j > n_1$, then

$$\mathcal{I}_n \supseteq \mathcal{J}_{n_0} \mathcal{A}^{\sigma^{n_0}} \mathcal{D}^{\sigma^{n_0+1}} \cdots \mathcal{D}^{\sigma^{n_1}}.$$

Therefore, $\mathcal{I}_n$ and $\mathcal{D}^{\sigma^j}$ are comaximal; thus $\mathcal{I}_n$ and $\mathcal{C}^{\sigma^n}$ are also comaximal. Therefore,

$$\mathcal{J}_n = \mathcal{I}_n \mathcal{D}^{\sigma^{n_1+1}} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n}$$

for $n \geq n_1 + 1$.

Let $\mathcal{I}$ be the maximal element in the chain of the $\mathcal{I}_n$. Let $m > n_1$ be such that $\mathcal{I}_n = \mathcal{I}$ for all $n \geq m$. Let $\mathcal{J}' = \mathcal{I}\mathcal{D}^{\sigma^{n_1+1}} \cdots \mathcal{D}^{\sigma^{m-1}}$. Then for $n > m$,

$$\mathcal{J}_n = \mathcal{I}\mathcal{D}^{\sigma^{n_1+1}} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n} = \mathcal{J}'\mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n}.$$

We have seen that $\mathcal{I}$ and $\mathcal{D}^{\sigma^n}$ are comaximal for all $n > n_1$, and in particular, for $n \geq m$. As $\mathcal{D}^{\sigma^j}$ and $\mathcal{D}^{\sigma^n}$ are comaximal if $j \neq n$, it follows that $\mathcal{J}'$ and $\mathcal{D}^{\sigma^n}$ are comaximal for $n \geq m$. $\square$

Corollary 4.2.10. *Suppose that the tuple $(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ is ADC data. Then the ADC ring $S(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ and the ADC bimodule algebra $\mathcal{S}(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ are left and right noetherian.*

Proof. Let

$$\mathcal{S} = \mathcal{S}(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}),$$

so that $\mathcal{S}_n = \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n}\mathcal{L}_n$ for $n \geq 1$. Let

$$S = S(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}).$$

Let Z be the subscheme of X defined by $\mathcal{D}$; by assumption, the ideal sheaves $\mathcal{A}$ and $\mathcal{C}$ define subschemes of Z. Since by Lemma 4.2.5 the sequence $\{(\mathcal{S}_n)_{\sigma^n}\}$ is left and right ample, by Theorem 2.3.12, to show that S is noetherian it suffices to show that the bimodule algebra $\mathcal{S}$ is left and right noetherian. By Lemma 2.3.14, this property does not depend on the invertible sheaf $\mathcal{L}$, so without loss of generality we may assume that $\mathcal{L} = \mathcal{O}_X$.

By symmetry, it suffices to prove that $\mathcal{S}$ is right noetherian. Let $\mathcal{J}$ be a graded right ideal of $\mathcal{S}$. By Proposition 4.2.9, there are an ideal sheaf $\mathcal{J}'$ on X and an integer $m \geq 0$ such that for $n > m$,

$$\mathcal{J}_n = \mathcal{J}'\mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n}.$$

We claim that $\mathcal{J}$ is generated by $\mathcal{J}_{\leq m+2}$.

This is a straightforward computation. Let $k \geq 2$. Note that $\mathcal{A}\mathcal{C}\mathcal{D}^{\sigma} + \mathcal{D}(\mathcal{A}\mathcal{C})^{\sigma} =$

$\mathcal{D}\mathcal{D}^\sigma$. Thus

$$\mathcal{J}_{m+1}\mathcal{S}_k^{\sigma^{m+1}} + \mathcal{J}_{m+2}\mathcal{S}_{k-1}^{\sigma^{m+2}} =$$
$$\mathcal{J}'\mathcal{D}^{\sigma^m}(\mathcal{A}\mathcal{C})^{\sigma^{m+1}}\mathcal{D}^{\sigma^{m+2}}\cdots\mathcal{D}^{\sigma^{m+k}}\mathcal{C}^{\sigma^{m+k+1}} +$$
$$\mathcal{J}'\mathcal{D}^{\sigma^m}\mathcal{D}^{\sigma^{m+1}}(\mathcal{A}\mathcal{C})^{\sigma^{m+2}}\mathcal{D}^{\sigma^{m+3}}\cdots\mathcal{D}^{\sigma^{m+k}}\mathcal{C}^{\sigma^{m+k+1}}$$
$$= \mathcal{J}'\mathcal{D}^{\sigma^m}\mathcal{D}^{\sigma^{m+1}}\mathcal{D}^{\sigma^{m+2}}\cdots\mathcal{D}^{\sigma^{m+k}}\mathcal{C}^{\sigma^{m+k+1}} = \mathcal{J}_{m+k+1}.$$

Thus $\mathcal{J}_{\geq m+1} = \mathcal{J}_{m+1}\mathcal{S} + \mathcal{J}_{m+2}\mathcal{S}$. The claim follows, and $\mathcal{J}$ is coherent. $\square$

Before proving that the rings $T(\mathbb{D})$ are noetherian, we give a result similar to Proposition 4.2.9 on the structure of left and right ideals of idealizer bimodule algebras.

Lemma 4.2.11. *Let X be a variety, let $\sigma \in \operatorname{Aut}(X)$, and let $\mathcal{L}$ be an invertible sheaf on X. Let*

$$\mathcal{S} = \bigoplus_{n\geq 0}(\mathcal{S}_n)_{\sigma^n}$$

be a noetherian sub-bimodule algebra of $\mathcal{B}(X, \mathcal{L}, \sigma)$, and let $\mathcal{I} = \bigoplus(\mathcal{I}_n)_{\sigma^n}$ be a graded right ideal of $\mathcal{S}$. Let $\mathcal{R} = \mathbb{I}_{\mathcal{S}}(\mathcal{I})$, and assume that $\mathcal{R}$ is also noetherian and that $\mathcal{R}_n = \mathcal{I}_n$ for $n \gg 0$. Let $\mathcal{J} = \bigoplus(\mathcal{J}_n)_{\sigma^n}$ be a graded right ideal of $\mathcal{R}$ and let $\mathcal{K} = \bigoplus(\mathcal{K}_n)_{\sigma^n}$ be a graded left ideal of $\mathcal{R}$. Then there are a right ideal $\mathcal{J}' \subseteq \mathcal{I}$ of $\mathcal{S}$ and a left ideal $\mathcal{K}'$ of $\mathcal{S}$ such that

$$\mathcal{J}_n = (\mathcal{J}')_n$$

and

$$\mathcal{K}_n = (\mathcal{I} \cap \mathcal{K}')_n = (\mathcal{I}\mathcal{K}')_n$$

for $n \gg 0$.

Proof. Since $\mathcal{R}$ is noetherian, there is an integer k such that both $\mathcal{J}$ and $\mathcal{K}$ are generated in degree $\leq k$. Let $\mathcal{J}' = \mathcal{J}\mathcal{I}$. Then $\mathcal{J}'$ is a right ideal of $\mathcal{S}$. Since $\mathcal{R}_n = \mathcal{I}_n$ for $n \gg 0$, we have

$$\mathcal{J}'_n = (\mathcal{J}\mathcal{I})_n = (\mathcal{J}_{\leq k}\mathcal{R})_n = \mathcal{J}_n$$

for $n \gg k$.

Let $\mathcal{K}' = \mathcal{S}\mathcal{K}$. A similar argument shows that for $n \gg k$ that $(\mathcal{I}\mathcal{K}')_n = \mathcal{K}_n$. By Proposition 3.3.3, since $\mathcal{R}$ is left noetherian, for $n \gg 0$ we have that $(\mathcal{I} \cap \mathcal{K}')_n = (\mathcal{I}\mathcal{K}')_n$. □

We are now ready to show that the rings $T(\mathbb{D})$, for transverse surface data $\mathbb{D}$, are noetherian. In fact, this is true even if the automorphism σ is not numerically trivial, and we prove it in that generality.

Definition 4.2.12. Let $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ be surface data. We say that $\mathbb{D}$ is *quasi-transverse* if

- $\mathcal{L}$ is σ-ample;
- all points in the cosupport of $\mathcal{D}$ have critically dense σ-orbits;
- $\{\sigma^n\Omega\}_{n\in\mathbb{Z}}$ is critically transverse; and
- both $\{\sigma^n\Lambda\}_{n\geq 0}$ and $\{\sigma^n\Lambda'\}_{n\leq 0}$ are critically transverse.

Proposition 4.2.13. *Suppose that the surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is quasi-transverse. Let $\mathcal{T} = \mathcal{T}(\mathbb{D})$ and let $T = T(\mathbb{D})$. Then both $\mathcal{T}$ and T are noetherian.*

Proof. By Lemma 4.2.7 the sequence of bimodules $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample. Thus by Theorem 2.3.12, it suffices to prove that $\mathcal{T}$ is right and left noetherian. By Lemma 2.3.14, without loss of generality we may assume that $\mathcal{L} = \mathcal{O}_X$.

If $\Lambda = \Lambda' = \Omega = \emptyset$ (that is, if $\mathcal{T}$ is an ADC bimodule algebra), then this is Corollary 4.2.10. Suppose that $\Lambda' = \emptyset$ but that Λ or Ω is nonempty. Let $\mathcal{S} = \mathcal{S}(X, \mathcal{O}_X, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$, and for $n \geq 0$ let

$$\mathcal{I}_n = \mathcal{S}_n \cap \mathcal{I}_\Omega \mathcal{I}_\Lambda.$$

Let

$$\mathcal{I} = \bigoplus_{n \geq 0} (\mathcal{I}_n)_{\sigma^n}.$$

Then $\mathcal{I}$ is a graded right ideal of $\mathcal{S}$. Let $\mathcal{J} \supseteq \mathcal{I}$ be another graded right ideal of $\mathcal{S}$. By Proposition 4.2.9, there are ideal sheaves $\mathcal{J}'$ and $\mathcal{I}'$ on X and an integer $m \geq 0$ such that for $n > m$,

$$\mathcal{J}_n = \mathcal{J}' \mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n}$$

and

$$\mathcal{I}_n = \mathcal{I}' \mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n},$$

and $\mathcal{I}'$ and $\mathcal{D}^{\sigma^n}$ are comaximal for $n \geq m$. Note that

$$\mathcal{I}' \subseteq \mathcal{A} \mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{m-1}} \cap \mathcal{I}_\Omega \mathcal{I}_\Lambda.$$

Let

$$\mathcal{F}_n = \mathcal{S}_n \cap \bigcap_{k \geq 0} (\mathcal{J}_{k+n} : \mathcal{I}_k^{\sigma^n}).$$

Then

$$\mathcal{F}_n \subseteq \mathcal{S}_n \cap \bigcap_{k > m} (\mathcal{J}' \mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{k+n-1}} \mathcal{C}^{\sigma^{k+n}} : (\mathcal{I}')^{\sigma^n} \mathcal{D}^{\sigma^{m+n}} \cdots \mathcal{D}^{\sigma^{k+n-1}} \mathcal{C}^{\sigma^{k+n}}).$$

For $n \gg 0$ and for any $k > m$, no primary component of $(\mathcal{I}')^{\sigma^n} \mathcal{D}^{\sigma^{m+n}} \cdots \mathcal{D}^{\sigma^{k+n-1}} \mathcal{C}^{\sigma^{k+n}}$ is contained in any associated prime of $\mathcal{J}'$, by assumption on the transversality of the defining data for $\mathcal{T}$. By Lemma 3.2.4(1), we see that for $k > m$,

$$(\mathcal{J}'\mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{k+n-1}}\mathcal{C}^{\sigma^{k+n}} : (\mathcal{I}')^{\sigma^n}\mathcal{D}^{\sigma^{m+n}} \cdots \mathcal{D}^{\sigma^{k+n-1}}\mathcal{C}^{\sigma^{k+n}})$$
$$\subseteq (\mathcal{J}' : (\mathcal{I}')^{\sigma^n}\mathcal{D}^{\sigma^{m+n}} \cdots \mathcal{D}^{\sigma^{k+n-1}}\mathcal{C}^{\sigma^{k+n}}) \subseteq \mathcal{J}'.$$

This implies that $\mathcal{F}_n = \mathcal{J}_n$ for $n \gg 0$.

In particular, letting $\mathcal{J} = \mathcal{I}$ we obtain that

$$\mathbb{I}_{\mathcal{S}}(\mathcal{I})_n = \mathcal{T}_n$$

for $n \gg 0$. By Lemma 3.2.9, $\mathbb{I}_{\mathcal{S}}(\mathcal{I})$ is right noetherian; thus $\mathcal{T}$ is right noetherian.

Now suppose that $\mathcal{K}$ is a graded left ideal of $\mathcal{S}$; by Proposition 4.2.9, there are an ideal sheaf $\mathcal{K}'$ on X and an integer m' so that for $n > m'$ we have that

$$\mathcal{K}_n = (\mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-m'}}(\mathcal{K}')^{\sigma^n}),$$

and $\mathcal{K}'$ and $\mathcal{D}^{\sigma^j}$ are comaximal for $j \leq -m$. Then for $n > N = m + m'$, we have that

$$(4.2.14) \qquad (\mathcal{I} \cap \mathcal{K})_n = \mathcal{I}'\mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-1}}\mathcal{C}^{\sigma^n} \cap \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-m'}}(\mathcal{K}')^{\sigma^n}.$$

Critical transversality of the defining data for $\mathcal{T}$ implies that

$$\mathcal{I}' \cap (\mathcal{K}')^{\sigma^n} = \mathcal{I}'(\mathcal{K}')^{\sigma^n}$$

for $n \gg 0$. Thus (4.2.14) is equal to

$$\mathcal{I}'\mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-m'}}(\mathcal{K}')^{\sigma^n}$$

for $n \gg 0$.

On the other hand, for $n \geq 2N + 1$ we have

$$(\mathcal{I}\mathcal{K})_n \supseteq \mathcal{I}_N(\mathcal{K}_{n-N})^{\sigma^N} + \mathcal{I}_{N+1}(\mathcal{K}_{n-N-1})^{\sigma^{N+1}} = \mathcal{I}'\mathcal{D}^{\sigma^m} \cdots \mathcal{D}^{\sigma^{n-m'}}(\mathcal{K}')^{\sigma^n}.$$

Thus we have

$$(\mathcal{IK})_n \supseteq (\mathcal{I} \cap \mathcal{K})_n$$

for $n \gg 0$. As the other containment is automatic, by Proposition 3.3.3 $\mathcal{T}$ is left noetherian.

We now consider the general case, except as before we let $\mathcal{L} = \mathcal{O}_X$. Given transverse surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$, let

$$\mathbb{E} = (X, \mathcal{O}_X, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \emptyset)$$

and let

$$\mathcal{R} = \mathcal{T}(\mathbb{E}).$$

We have seen above that $\mathcal{R}$ is left and right noetherian.

Define a left ideal $\mathcal{I}$ of $\mathcal{R}$ by

$$\mathcal{I} = \bigoplus_{n \geq 0} (\mathcal{R}_n \cap \mathcal{I}_{\Lambda'}^{\sigma^n})_{\sigma^n}.$$

Let $\mathcal{J} \supseteq \mathcal{I}$ be a graded left ideal of $\mathcal{R}$. By Proposition 4.2.9 and Lemma 4.2.11 there are ideal sheaves $\mathcal{J}'$ and $\mathcal{I}'$ and an integer j so that

$$\mathcal{J}_n = \mathcal{I}_\Omega \mathcal{I}_\Lambda \cap \mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-j}} (\mathcal{J}')^{\sigma^n} = \mathcal{I}_\Omega \mathcal{I}_\Lambda \cap \mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-j}} \cap (\mathcal{J}')^{\sigma^n}$$

and

$$\mathcal{I}_n = \mathcal{I}_\Omega \mathcal{I}_\Lambda \cap \mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-j}} (\mathcal{I}')^{\sigma^n} = \mathcal{I}_\Omega \mathcal{I}_\Lambda \cap \mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-j}} \cap (\mathcal{I}')^{\sigma^n}$$

for $n > j$. Further, we may assume that $\mathcal{D}^{\sigma^n}$ and $\mathcal{I}'$ are comaximal for $n \leq -j$.

By construction, the cosupport of $\mathcal{I}'$ and therefore of $\mathcal{J}'$ is 0-dimensional. For $m \gg 0$, the ideal sheaves $(\mathcal{J}')^{\sigma^m}$ and $\mathcal{I}'$ are comaximal, and computing locally we see that

$$(\mathcal{J}_{n+m} : \mathcal{I}_n)^{\sigma^{-n}} \subseteq (\mathcal{J}')^{\sigma^m}$$

for $n, m \gg 0$. Thus for $m \gg 0$ we have that

$$\mathcal{J}_m \subseteq \mathcal{R}_m \cap \bigcap_{n \geq 0} (\mathcal{J}_{n+m} : \mathcal{I}_n)^{\sigma^{-n}} \subseteq (\mathcal{J}')^{\sigma^m} \cap \mathcal{R}_m = \mathcal{J}_m,$$

and

$$\mathcal{R}_m \cap \bigcap_{n \geq 0} (\mathcal{J}_{n+m} : \mathcal{I}_m)^{\sigma^{-n}} = \mathcal{J}_m.$$

In particular, $\mathcal{T}$ and $\mathbb{I}_{\mathcal{R}}(\mathcal{I})$ are equal in large degree. The symmetric version of Lemma 3.2.9 for left idealizers implies that $\mathcal{T}$ is left noetherian.

Likewise, if $\mathcal{K}$ is a right ideal of $\mathcal{R}$, then there are an ideal sheaf $\mathcal{K}' \subseteq \mathcal{I}_\Omega \mathcal{I}_\Lambda$ and an integer k so that for $n > k$,

$$\mathcal{K}_n = \mathcal{K}' \mathcal{D}^{\sigma^k} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n}.$$

Choose $m > k, j$ so that if $n \geq m$, then $(\mathcal{I}')^{\sigma^n}$ and $\mathcal{K}'$ are comaximal, and $(\mathcal{I}')^{\sigma^n}$ and $\mathcal{I}_\Omega \mathcal{I}_\Lambda$ are also comaximal. Let N be such that the right ideal $\mathcal{K}_{\geq m}$ of $\mathcal{R}$ is generated in degrees $\leq m + N$. Let $n \geq 2m + N$.

We will show that

$$(\mathcal{K}\mathcal{I})_n \supseteq (\mathcal{K}_n \cap \mathcal{I}_n). \tag{4.2.15}$$

Certainly,

$$(\mathcal{K}\mathcal{I})_n \supseteq \mathcal{K}_m \mathcal{I}_{n-m}^{\sigma^m} = (\mathcal{K}' \mathcal{D}^{\sigma^k} \cdots \mathcal{D}^{\sigma^{m-1}} \mathcal{C}^{\sigma^m})(\mathcal{I}_\Omega \mathcal{I}_\Lambda \cap \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-m-j}})^{\sigma^m} (\mathcal{I}')^{\sigma^n}.$$

Let Y be the subscheme defined by $\mathcal{I}'$. Let $y \in \sigma^{-n}(Y)$. As $\mathcal{K}'$ and $(\mathcal{I}')^{\sigma^n}$ are comaximal, $(\mathcal{K}_n)_y = (\mathcal{R}_n)_y$. Further, $(\mathcal{I}_n)_y = ((\mathcal{I}')^{\sigma^n})_y$. Thus,

$$\begin{aligned}\big((\mathcal{K}' \mathcal{D}^{\sigma^k} \cdots \mathcal{D}^{\sigma^{m-1}} \mathcal{C}^{\sigma^m})(\mathcal{I}_\Omega \mathcal{I}_\Lambda \cap \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-m-j}})^{\sigma^m} (\mathcal{I}')^{\sigma^n}\big)_y \\ = (\mathcal{I}')_y^{\sigma^n} = (\mathcal{R}_n \cap \mathcal{I}_n)_y = (\mathcal{K}_n \cap \mathcal{I}_n)_y.\end{aligned}$$

Therefore (4.2.15) holds locally at y.

On the other hand, if $x \notin \sigma^{-n}(Y)$, then

$$(\mathcal{I}^{\sigma^{m+\ell}}_{n-m-\ell})_x = (\mathcal{R}^{\sigma^{m+\ell}}_{n-m-\ell})_x$$

for $\ell = 0 \dots N$. Therefore,

$$\big((\mathcal{K}\mathcal{I})_n\big)_x \supseteq \Big(\sum_{\ell=0}^{N} \mathcal{K}_{m+\ell}\mathcal{I}^{\sigma^{m+\ell}}_{n-m-\ell}\Big)_x = \Big(\sum_{\ell=0}^{N} \mathcal{K}_{m+\ell}\mathcal{R}^{\sigma^{m+\ell}}_{n-m-\ell}\Big)_x.$$

This is equal to $(\mathcal{K}_n)_x$ by assumption on N. Thus

$$\big((\mathcal{K}\mathcal{I})_n\big)_x \supseteq (\mathcal{K}_n)_x = (\mathcal{K}_n \cap \mathcal{R}_n)_x = (\mathcal{K}_n \cap \mathcal{I}_n)_x.$$

Thus (4.2.15) holds locally at x.

Since (4.2.15) holds locally at all points in X, it holds globally. Since the other inclusion is automatic, we have that

$$(\mathcal{K}\mathcal{I})_n = \mathcal{K}_n \cap \mathcal{I}_n$$

for all $n \geq 2m + N$. By the symmetric result to Proposition 3.3.3, $\mathcal{T}$ is right noetherian. □

Let $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ be transverse surface data. To end this section, we give some results about the two-sided ideals of $T(\mathbb{D})$, which we will need later in the chapter.

Lemma 4.2.16. *Let X be a projective surface, let σ be an automorphism of X, let $\mathcal{L}$ be an invertible sheaf on X, and let Ω be a curve on X such that the set $\{\sigma^n\Omega\}_{n\in\mathbb{Z}}$ is critically transverse. Let $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, \Omega)$ be the right idealizer bimodule algebra at Ω inside $\mathcal{B}(X, \mathcal{L}, \sigma)$, and let $\mathcal{K}$ be a graded ideal of $\mathcal{R}$. Then there is some σ-invariant ideal sheaf $\mathcal{K}'$ such that*

$$\mathcal{K}_n = \mathcal{I}_\Omega\mathcal{K}'\mathcal{L}_n = (\mathcal{I}_\Omega \cap \mathcal{K}')\mathcal{L}_n$$

for $n \gg 0$.

Proof. Let $\mathcal{I} = \mathcal{I}_\Omega$. By Lemma 2.3.14, is sufficient to prove the lemma in the case that $\mathcal{L} = \mathcal{O}_X$. Let $\mathcal{B} = \mathcal{B}(X, \mathcal{O}_X, \sigma)$. By Lemma 4.2.11, and by the equivalence between qgr-$\mathcal{B}$, $\mathcal{O}_X$-mod, and $\mathcal{B}$-qgr, there are ideal sheaves $\mathcal{K}'$ and $\mathcal{J}' \subseteq \mathcal{I}$ on X such that for $n \gg 0$,

$$\mathcal{K}_n = \mathcal{I} \cap (\mathcal{K}')^{\sigma^n} = \mathcal{I}(\mathcal{K}')^{\sigma^n} = \mathcal{J}'.$$

Since $\mathcal{I}$ is invertible by Lemma 3.5.7, we have that $\mathcal{I}^{-1}\mathcal{J}' = (\mathcal{K}')^{\sigma^n}$ for all $n \gg 0$. In particular, $(\mathcal{K}')^{\sigma^n}$ is constant for all $n \gg 0$. As a subscheme that is invariant under relatively prime powers of σ is σ-invariant, $\mathcal{K}'$ is σ-invariant. $\square$

If $p \in X$, we denote the σ-orbit of p by $O(p)$.

Proposition 4.2.17. *Suppose that the surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is quasi-transverse. Let $\mathcal{T} = \mathcal{T}(\mathbb{D})$. Let $\mathcal{K}$ be a graded ideal of $\mathcal{T}$. Then there are a σ-invariant ideal sheaf $\mathcal{J}$ on X and an integer $n_0 \geq 0$ such that if $n \geq n_0$, then $\mathcal{K}_n = \mathcal{T}_n \cap \mathcal{J}\mathcal{L}_n = \mathcal{J}\mathcal{T}_n$.*

Proof. We may assume that $\mathcal{K} \neq 0$. Without loss of generality, we may suppose that $\mathcal{L} = \mathcal{O}_X$. Let $\mathcal{B} = \mathcal{B}(X, \mathcal{O}_X, \sigma)$, and let $\mathcal{S} = \mathcal{O}_X \oplus \mathcal{I}_\Omega \mathcal{B}_+$. Let Z be the cosupport of $\mathcal{D}$ and let $\mathbb{W}$ be the union of the orbits of all points in $Z \cup \Lambda \cup \Lambda'$. For all $n \geq 0$, let $\hat{\mathcal{K}}_n \supseteq \mathcal{K}_n$ be the maximal ideal sheaf on X such that $\hat{\mathcal{K}}_n/\mathcal{K}_n$ is supported on a subset of $\mathbb{W}$. Note that, as $\mathcal{K}_n \subseteq \mathcal{I}_\Omega$ for all n, we have $\hat{\mathcal{K}}_n \subseteq \mathcal{I}_\Omega$ for all n.

As

$$\mathcal{T}_n \mathcal{K}_m^{\sigma^n} + \mathcal{K}_m \mathcal{T}_n^{\sigma^m} \subseteq \mathcal{K}_{m+n}$$

for all $m, n \geq 0$, one easily verifies that

$$\mathcal{I}_\Omega \hat{\mathcal{K}}_m^{\sigma^n} + \hat{\mathcal{K}}_m \mathcal{I}_\Omega^{\sigma^m} \subseteq \hat{\mathcal{K}}_{n+m}$$

for all $m, n \geq 0$. That is, the bimodule

$$\bigoplus_{n\geq 0}(\hat{\mathcal{K}}_n)_{\sigma^n}$$

is an ideal of the bimodule algebra $\mathcal{S}$. As $\{\sigma^n\Omega\}$ is critically transverse, $\mathcal{S}$ and $\mathcal{R}(X, \mathcal{O}_X, \sigma, \Omega)$ are equal in large degree; thus by Lemma 4.2.16, there are a σ-invariant ideal sheaf $\mathcal{J}$ and an integer n_0 such that if $n \geq n_0$, then $\hat{\mathcal{K}}_n = \mathcal{J} \cap \mathcal{I}_\Omega = \mathcal{J}\mathcal{I}_\Omega$.

We will show for $n \gg 0$ that

$$(\mathcal{K}_n)_q = (\mathcal{J} \cap \mathcal{T}_n)_q = (\mathcal{J}\mathcal{T}_n)_q \tag{4.2.18}$$

for all $q \in X$.

We first note that if $q \notin \mathbb{W}$, then $(\mathcal{T}_n)_q = \mathcal{I}_{\Omega,q}$. Thus if $n \geq n_0$ and $q \notin \mathbb{W}$, then

$$(\mathcal{K}_n)_q = (\hat{\mathcal{K}}_n)_q = (\mathcal{J} \cap \mathcal{I}_\Omega)_q = (\mathcal{J}\mathcal{I}_\Omega)_q = (\mathcal{J} \cap \mathcal{T}_n)_q = (\mathcal{J}\mathcal{T}_n)_q,$$

and (4.2.18) holds for q.

To show that (4.2.18) holds for $q \in \mathbb{W}$, it suffices to show that for any $p \in \mathbb{W}$ and for all $n \gg 0$ that (4.2.18) holds for all $q \in O(p)$. Now, by transversality of the surface data $\mathbb{D}$, the cosupport of $\mathcal{J}$ is disjoint from $\mathbb{W}$. Thus for any $q \in O(p)$, $\mathcal{J}_q = \mathcal{O}_{X,q}$. It therefore suffices to prove for $n \gg 0$ that

$$(\mathcal{K}_n)_q = (\mathcal{T}_n)_q \tag{4.2.19}$$

for all $q \in O(p)$.

Note that for any $p \in \mathbb{W}$, the cosupport of $\hat{\mathcal{K}}_{n_0}$ has finite intersection with $O(p)$ by assumption on the transversality of $\mathbb{D}$.

Sublemma 4.2.20. *Let X be a projective surface, let $\sigma \in \mathrm{Aut}(X)$, and let Ω be a curve on X. Let*

$$\mathcal{S} = \mathcal{O}_X \oplus \mathcal{I}_\Omega \cdot \mathcal{B}(X, \mathcal{O}_X, \sigma)_{\geq 1}$$

and let $\mathcal{T}$ be a finitely generated graded $(\mathcal{O}_X, \sigma)$-sub-bimodule algebra of $\mathcal{S}$ so that $\operatorname{Supp}(\mathcal{S}_n/\mathcal{T}_n)$ *is 0-dimensional and supported on infinite σ-orbits for all $n \geq 1$. Let $\mathcal{K}$ be a two-sided ideal of $\mathcal{T}$ and let $p \in X$ be a point of infinite order. Assume that for $n \geq n_0$, the cosupport of $\mathcal{K}_n$ meets $O(p)$ at only finitely many points. (In particular, this implies that $\Omega \cap O(p)$ is finite.)*

Let $\mathcal{O} = \mathcal{O}_{X,p}$. For all $n \geq 1$ and for all $i \in \mathbb{Z}$, let $\mathfrak{k}_i^n$ be the stalk of $\mathcal{K}_n$ at $\sigma^{-i}(p)$, considered as an ideal in $\mathcal{O}$ via σ^i. Similarly, let $\mathfrak{m}_i^n \subseteq \mathcal{O}$ be the stalk of $\mathcal{T}_n$ at $\sigma^{-i}(p)$. Our assumptions imply that the cosupport of $\mathcal{T}_1$ has finite intersection with $O(p)$, and so by reindexing the orbit of p, we may assume that $\mathfrak{m}_i^1 = \mathcal{O}$ if $i < 0$. Let

$$s = \max(\{i \mid \mathfrak{m}_i^1 \neq \mathcal{O}\} \cup \{0\}).$$

Then there are an ideal $\mathfrak{k}$ of $\mathcal{O}$ and integers $a' \leq a$, $b' \leq b$, and N so that if $n \geq N$ then:

(1) *if $i < a'$ or $i > n - b'$ then $\mathfrak{k}_i^n = \mathcal{O}$;*

(2) *if $a' \leq i \leq a$ then $\mathfrak{k}_i^n = \mathfrak{k}_i^N$;*

(3) *if $a \leq i \leq n - b$ then $\mathfrak{k}_i^n = \mathfrak{k}$;*

(4) *if $n - b \leq i \leq n - b'$ then $\mathfrak{k}_i^n = \mathfrak{k}_{i-n+N}^N$.*

Furthermore, we have

$$\mathfrak{k}_s^N \subseteq \mathfrak{k}_{s+1}^N \subseteq \cdots \subseteq \mathfrak{k} \tag{4.2.21}$$

and

$$\mathfrak{k}_{N-1}^N \subseteq \mathfrak{k}_{N-2}^N \subseteq \cdots \subseteq \mathfrak{k}. \tag{4.2.22}$$

We refer to the ideal $\mathfrak{k}$ of $\mathcal{O}$ constructed in Sublemma 4.2.20 as the *central stalk* of $\mathcal{K}$.

Proof of Sublemma 4.2.20. Since $\mathcal{K}$ is a two-sided ideal of $\mathcal{T}$, we certainly have for all $n, j \geq 0$ that

$$\mathcal{T}_j \mathcal{K}_n^{\sigma^j} + \mathcal{K}_n \mathcal{T}_j^{\sigma^n} \subseteq \mathcal{K}_{n+j}.$$

In terms of the stalks $\mathfrak{k}$ and $\mathfrak{m}$, this translates to the statement that

$$\mathfrak{m}_i^j \mathfrak{k}_{i-j}^n + \mathfrak{k}_i^n \mathfrak{m}_{i-n}^j \subseteq \mathfrak{k}_i^{n+j}.$$

Therefore,

$$\text{if } i \leq j-1 \text{ then } \mathfrak{k}_i^{j+1} \supseteq \mathfrak{k}_i^j \mathfrak{m}_{i-j}^1 = \mathfrak{k}_i^j, \tag{4.2.23}$$

and

$$\text{if } i \geq s \text{ then } \mathfrak{k}_{i+1}^{j+1} \supseteq \mathfrak{m}_{i+1}^1 \mathfrak{k}_i^j = \mathfrak{k}_i^j. \tag{4.2.24}$$

By assumption, $\{i \mid \mathfrak{k}_i^{n_0} \neq \mathcal{O}\}$ is finite. Let

$$a' = \min(\{i \mid \mathfrak{k}_i^{n_0} \neq \mathcal{O}\} \cup \{0\})$$

and let

$$b' = \min(\{j \mid \mathfrak{k}_{n_0-j}^{n_0} \neq \mathcal{O}\} \cup \{-s\}).$$

Then $\mathfrak{k}_i^{n_0} = \mathcal{O}$ for $i < a'$ or $i > n_0 - b'$, and the relations (4.2.23) and (4.2.24) imply that

$$\mathfrak{k}_i^n = \mathcal{O}$$

for $n \geq n_0$ and $i < a'$ or $i > n - b'$. Thus (1) holds for $n \geq n_0$.

For fixed i, (4.2.23) implies that

$$\mathfrak{k}_i^n = \mathfrak{k}_i^{n+1}$$

for $n \gg i$, and (4.2.24) implies that

$$\mathfrak{k}_{n-i}^n = \mathfrak{k}_{n+1-i}^{n+1}$$

for $n \gg i + s$. Furthermore, for $n \geq \max\{s, 1\}$ we have

$$\mathfrak{k}_n^{2n} \subseteq \mathfrak{k}_{n+1}^{2n+1} \subseteq \mathfrak{k}_{n+1}^{2n+2}.$$

Let

$$\mathfrak{k} = \bigcup_{n \geq s} \mathfrak{k}_n^{2n}.$$

Choose $m \geq \max\{n_0, s\}$ so that $\mathfrak{k}_m^{2m} = \mathfrak{k}$, and choose $N \geq 2m$ so that

$$\mathfrak{k}_i^n = \mathfrak{k}_i^N = \bigcup_{j > i} \mathfrak{k}_i^j$$

for $a' \leq i \leq m$ and $n \geq N$, and

$$\mathfrak{k}_i^n = \mathfrak{k}_{i-n+N}^N = \bigcup_{j \geq s+n-i} \mathfrak{k}_{j+i-n}^j$$

for $n - m \leq i \leq n - b'$ and $n \geq N$. By construction, (2) and (4) hold for $a = b = m$.

We now prove (3). We claim that

$$\mathfrak{k}_i^n = \mathfrak{k}$$

for $n \geq N$ and $m \leq i \leq n - m$. To see this, note that the claim is certainly true if $n = 2i$, by definition of $\mathfrak{k}$. We prove the claim for $n \neq 2i$; by symmetry, it suffices to consider the case $n > 2i$. If $n > 2i$, then

$$\mathfrak{k}_i^n \subseteq \mathfrak{k}_{i+(n-2i)}^{n+(n-2i)} = \mathfrak{k}_{n-i}^{2n-2i}$$

by (4.2.24), as $i \geq s$. This is equal to $\mathfrak{k}$, as $n - i \geq m$. On the other hand, we have

$$\mathfrak{k} = \mathfrak{k}_m^{2m} \subseteq \mathfrak{k}_{m+(i-m)}^{2m+(i-m)} = \mathfrak{k}_i^{m+i}$$

by (4.2.24), as $i \geq m \geq s$. Further,

$$\mathfrak{k}_i^{m+i} \subseteq \mathfrak{k}_i^n$$

by (4.2.23), as $n \geq m + i > i$. Thus we have

$$\mathfrak{k}_i^n = \mathfrak{k},$$

as claimed.

It remains to show that (4.2.21) and (4.2.22) hold. Let $s \leq j \leq m-1$. We have

$$\mathfrak{k}_j^N \subseteq \mathfrak{k}_{j+1}^{N+1}$$

by (4.2.24), as $j \geq s$. As $j+1 \leq m$,

$$\mathfrak{k}_{j+1}^{N+1} = \mathfrak{k}_{j+1}^N$$

by our choice of N. Thus $\mathfrak{k}_j^N \subseteq \mathfrak{k}_{j+1}^N$. Note that $\mathfrak{k}_m^N = \mathfrak{k}$. Thus (4.2.21) holds. The proof that (4.2.22) holds is symmetric. $\square$

We return to the proof of Proposition 4.2.17. Our assumption that $\mathbb{D}$ is transverse implies that the hypotheses of Sublemma 4.2.20 hold for $\mathcal{T}$, p, and $\mathcal{K}$. They hold also for $\mathcal{K} = \mathcal{T}_+$, with $n_0 = 1$.

Let $\mathcal{O} = \mathcal{O}_{X,p}$. For all $n \geq 1$ and $i \in \mathbb{Z}$ define ideals $\mathfrak{m}_i^n$ and $\mathfrak{k}_i^n$ of $\mathcal{O}$ as in the statement of Sublemma 4.2.20. By applying Sublemma 4.2.20 to the ideals $\mathcal{T}_+$ and $\mathcal{K}$, we obtain integers a, b, and N and ideals $\mathfrak{k}$ and $\mathfrak{d}$ of $\mathcal{O}$ so that if $n \geq N$ then

- if $i \leq a$ then $\mathfrak{k}_i^n = \mathfrak{k}_i^N$ and $\mathfrak{m}_i^n = \mathfrak{m}_i^N$;
- if $a \leq i \leq n-b$ then $\mathfrak{k}_i^n = \mathfrak{k}$ and $\mathfrak{m}_i^n = \mathfrak{d}$;
- if $i \geq n-b$ then $\mathfrak{k}_i^n = \mathfrak{k}_{i-n+N}^N$ and $\mathfrak{m}_i^n = \mathfrak{m}_{i-n+N}^N$.

For fixed i, by taking $j \gg 0$ we have $\mathfrak{k}_{i-j}^N = \mathcal{O}$. Thus if $i \leq a$, by taking $j \gg N$ we obtain that

$$\mathfrak{m}_i^N \supseteq \mathfrak{k}_i^N = \mathfrak{k}_i^{N+j} \supseteq \mathfrak{m}_i^j \mathfrak{k}_{i-j}^N = \mathfrak{m}_i^j = \mathfrak{m}_i^N,$$

so $\mathfrak{k}_i^N = \mathfrak{m}_i^N$. In particular,

$$\mathfrak{d} = \mathfrak{m}_a^N = \mathfrak{k}_a^N = \mathfrak{k}.$$

The argument that if $i \geq N - b$ then

$$\mathfrak{k}_i^N = \mathfrak{m}_i^N$$

is symmetric. Thus $\mathfrak{k}_i^N = \mathfrak{m}_i^N$ for all i, and so $\mathfrak{k}_i^n = \mathfrak{m}_i^n$ for all i and for all $n \geq N$. This precisely says that (4.2.19) holds, as we sought to prove. $\square$

Corollary 4.2.25. *Suppose that the surface data*

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

is quasi-transverse. Let $\mathcal{T} = \mathcal{T}(\mathbb{D})$. Recall our convention (4.2.1) *that*

$$T(\mathbb{D}) = \bigoplus_{n \geq 0} H^0(\mathcal{T}_n)z^n.$$

Let K be a graded ideal of T. Then there are a σ-invariant ideal sheaf $\mathcal{J}$ on X and an integer $n_0 \geq 0$ such that if $n \geq n_0$, then

$$K_n = H^0(\mathcal{J}\mathcal{T}_n)z^n = H^0(\mathcal{J}\mathcal{L}_n \cap \mathcal{T}_n)z^n.$$

Proof. By Lemma 4.2.7, the sequence of bimodules $\{(\mathcal{T}_n)_{\sigma^n}\}$ is left and right ample. Theorem 2.3.12 then implies that there is some graded ideal $\mathcal{K}$ of $\mathcal{T}$ so that

$$K_n = H^0(\mathcal{K}_n)z^n$$

or

$$\overline{K}_n = H^0(\mathcal{K}_n)$$

for $n \gg 0$. Note that $\mathcal{T}_n$ and $\mathcal{K}_n$ are globally generated for $n \gg 0$.

From the inclusion

$$T_n K_m \subseteq K_{n+m},$$

we obtain that

$$\mathcal{T}_n(\mathcal{K}_m)^{\sigma^n} \subseteq \mathcal{K}_{n+m}$$

for all $n, m \gg 0$. By Proposition 4.2.13, the right ideal $\mathcal{K}$ of $\mathcal{T}$ is generated by a coherent $\mathcal{O}_X$-submodule $\mathcal{F}$. Therefore

$$(\mathcal{T}\mathcal{F}\mathcal{T})_n = (\mathcal{T}\mathcal{K})_n = \mathcal{K}_n$$

for $n \gg 0$. That is, without loss of generality we may assume that $\mathcal{K}$ is a two-sided ideal of $\mathcal{T}$. Proposition 4.2.17 implies that there is a σ-invariant ideal sheaf $\mathcal{J}$ on X so that $\mathcal{K}_n = \mathcal{J}\mathcal{T}_n = \mathcal{J}\mathcal{L}_n \cap \mathcal{T}_n$ for $n \gg 0$. Thus

$$\overline{K}_n = H^0(\mathcal{K}_n) = H^0(\mathcal{J}\mathcal{T}_n) = H^0(\mathcal{J}\mathcal{L}_n \cap \mathcal{T}_n)$$

for $n \gg 0$. □

4.3 Approximating birationally commutative surfaces in codimension 1

Let R be a birationally commutative projective surface with function field K, as in Definition 4.1.1. We now turn to constructing surface data

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

that will correspond to R. The central problem is to find the correct surface X.

Fortunately, we have a place to start. The graded quotient ring of R is isomorphic to $K[z, z^{-1}; \sigma]$, where σ is a $\Bbbk$-automorphism of K; since R has GK-dimension 3, K has transcendence degree 2. We say that σ is *geometric* if there is a projective surface X with $K = \Bbbk(X)$ such that σ is induced by an automorphism of X. We call such a pair (X, σ) a *model* for R. We note that not all automorphisms of fields of transcendence degree 2 are geometric; for example, by [DF01, Remark 7.3], the automorphism $(x, y) \mapsto (x, xy)$ of $\mathbb{C}(x, y)$ is not geometric.

Suppose that X and X' are birationally equivalent surfaces; let σ, respectively σ', be an automorphism of X, respectively X'. We say that σ and σ' are *conjugate* if they induce (up to conjugacy) the same automorphism of $\Bbbk(X) \cong \Bbbk(X')$; that is, if there is a birational map $\pi : X' \to X$ so that $\pi\sigma' = \sigma\pi$ as birational maps from X' to X.

Rogalski and Stafford note that it is an easy consequence of the existence of resolutions of singularities for surfaces (see [Lip69]) that any geometric automorphism of a field of transcendence degree 2 is conjugate to an automorphism of a nonsingular surface.

Lemma 4.3.1. ([RS06, Lemma 6.2]) *If K is a field of transcendence degree 2 over $\Bbbk$ and $\sigma \in \mathrm{Aut}_{\Bbbk}(K)$ is a geometric automorphism of K, then there is a nonsingular surface X with $\Bbbk(X) = K$ such that σ is induced from an automorphism of X. In particular, if a birationally commutative projective surface has a model, it has a nonsingular model.* □

A result of Rogalski ensures that in our situation, R has a model (X, σ); results of Artin and Van den Bergh then allow us to get precise information on the numerical action of the automorphism σ of X. Recall that two Cartier divisors D and D' on a projective scheme X are *numerically equivalent* (written $D \equiv D'$) if $D.C = D'.C$ for any irreducible curve C on X. An automorphism σ of X is *numerically trivial* if $\sigma D \equiv D$ for any Cartier divisor D on X. We will say that an σ is *quasi-trivial* if there is some integer $r > 0$ so that σ^r is numerically trivial.

If X is a projective scheme, we denote the group of Cartier divisors on X modulo numerical equivalence by $\mathrm{NS}(X)$.

Theorem 4.3.2. (Rogalski, Artin-Van den Bergh) *Let $K/\Bbbk$ be a finitely generated* *field extension where K has transcendence degree 2, and let $\sigma \in \mathrm{Aut}_{\Bbbk}(K)$. Then every locally finite $\mathbb{N}$-graded domain R such that $Q_{\mathrm{gr}}(R) = K[z, z^{-1}; \sigma]$ has the same GK-dimension $d \in \{3, 4, 5, \infty\}$. Moreover, $d \in \{3, 5\}$ if and only if σ is geometric. Further, $d = 3$ if and only if for any model (X, σ) for R, the automorphism σ is quasi-trivial.*

Proof. The first and second statements are [Rog07, Theorem 1.1]. Now suppose that σ is geometric, and let (X, σ) be a model for R. Let $P \in O(\mathrm{NS}(X))$ be the matrix giving the numeric action of σ on $\mathrm{NS}(X)$. By [Rog07, Theorem 7.1] and [Rog07, Lemma 2.12], all eigenvalues of P have modulus 1; now by [AV90, Lemma 5.3], the eigenvalues of P are all roots of unity. Let $\mathcal{L}$ be an ample invertible sheaf on X. Then [AV90, Theorem 1.7] implies that $\mathcal{L}$ is σ-ample, and that the GK-dimension of $B(X, \mathcal{L}, \sigma)$, which is equal to d, is 3 if and only if σ is quasi-trivial. The result follows. $\square$

As we have assumed that R has GK-dimension 3, Theorem 4.3.2 implies that there is a model (X, σ) for R. By Lemma 4.3.1, we may also, if we choose, assume that X is nonsingular.

We begin be establishing notation for the geometric data determined by R. If X is a projective variety and $V \subseteq K = \Bbbk(X)$ is a finite-dimensional $\Bbbk$-vector space, we will denote the coherent subsheaf of the constant sheaf K on X generated by the elements of V by

$$V \cdot \mathcal{O}_X.$$

We note that any Veronese subring $R^{(k)}$ of R has the same function field as R and is also a birationally commutative projective surface; that is, $R^{(k)}$ is noetherian and of GK-dimension 3. Thus, by replacing R by an appropriate Veronese subring, we may assume that $R_1 \neq 0$.

Assumption-Notation 4.3.3. *We assume that R is a birationally commutative projective surface with $R_1 \neq 0$. Let K be the function field of R and let (X, σ) be a model for R. Fix $z \neq 0 \in R_1$. For all $n \geq 0$, we define $\overline{R}_n = (R_n) \cdot z^{-n} \subseteq K$, so that*

$$R = \bigoplus_{n \geq 0} \overline{R}_n z^n \subseteq K[z, z^{-1}; \sigma].$$

Let $\mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$.

Example 4.3.4. Before beginning to work with our noncommutative ring R, suppose for a moment that $R = \Bbbk[x, y, z]$. We know, of course, that $R \cong B(\mathbb{P}^2, \mathcal{O}(1), 1) = B(\mathbb{P}^2, \mathcal{O}(1))$ and that $\mathbb{P}^2 = \operatorname{Proj} R$. However, we cannot construct the variety $\operatorname{Proj} R$ directly using noncommutative techniques. Instead, we will construct the defining data $(\mathbb{P}^2, \mathcal{O}(1))$ from the graded pieces of R.

The function field of R is $K = \Bbbk(x/z, y/z)$. Consider the model $X = \mathbb{P}^1 \times \mathbb{P}^1$ for K, where we think of X as Proj of the bigraded ring $\Bbbk[s, t][u, v]$. We will let $s/t = x/z$ and $u/v = y/z$ in K.

Let $\overline{R}_1 = R_1 z^{-1} \subset K$. Then

$$\overline{R}_1 = \{\frac{x}{z}, \frac{y}{z}, 1\} = \{\frac{sv}{tv}, \frac{tu}{tv}, \frac{tv}{tv}\}.$$

Let $D \cong \mathcal{O}(1, 1)$ be the divisor on X defined by the equation $tv = 0$. On X, the rational functions in $\overline{R}_1$ correspond to sections of $\mathcal{O}_X(D)$, and they generate

$$\overline{R}_1 \cdot \mathcal{O}_X = \mathcal{I}_{[1:0] \times [1:0]} \mathcal{O}_X(D).$$

We will modify X by blowing up the base locus of $\overline{R}_1$, considered as a vector space of sections of D.

Let $\pi : \widetilde{X} \to X$ be the blowup of X at $[1:0] \times [1:0]$. Let $E = \pi^{-1}([1:0] \times [1:0])$ be the exceptional locus of π, and let F_1 and F_2 be the strict transforms of $[1:0] \times \mathbb{P}^1$

and $\mathbb{P}^1 \times [1:0]$ respectively. Then F_1, F_2 and E are the three (-1) curves on $\widetilde{X}$, and on $\widetilde{X}$, $\overline{R}_1$ generates the invertible sheaf

$$\mathcal{L} = \overline{R}_1 \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(F_1 + F_2 + E) \cong \mathcal{I}_E \mathcal{O}_{\widetilde{X}}(\pi^* D).$$

One may check that R and the section ring $B(\widetilde{X}, \mathcal{L})$ are isomorphic. However, $\mathcal{L}$ is not ample. By the Nakai-Moishezon criterion [Har77, Theorem V.1.10], the failure of ampleness of $\mathcal{L}$ is equivalent to the existence of an effective curve C so that $(F_1 + F_2 + E).C = 0$. One checks that $(F_1 + F_2 + E).F_1 = (F_1 + F_2 + E).F_2 = 0$. That is, the curves F_1 and F_2 are contracted by the morphism defined by the base point free linear system $\overline{R}_1 \subseteq H^0(\mathcal{O}_{\widetilde{X}}(F_1 + F_2 + E))$ on $\widetilde{X}$. The image of $\widetilde{X}$ under this morphism is, of course, $\mathbb{P}^2$, the "correct" model for R.

We now return to the setting of a noncommutative projective surface R. We assume Assumption-Notation 4.3.3. It is immediate that the bimodule

$$\mathcal{R}(X) = \bigoplus_{n \geq 0} (\mathcal{R}_n(X))_{\sigma^n}$$

is in fact a graded $(\mathcal{O}_X, \sigma)$-bimodule algebra, and of course $R \subseteq H^0(\mathcal{R}(X))$. While ultimately we wish to understand R, our fundamental technique will be to approach R by analyzing the bimodule algebra $\mathcal{R}(X)$ on a suitable model (X, σ) for R; to construct X, we will mimic the steps carried out in Example 4.3.4.

For all $n \geq 0$, let $\mathcal{R}_n = \mathcal{R}_n(X)$. We note immediately that we have

$$\mathcal{R}_n \mathcal{R}_m^{\sigma^n} \subseteq \mathcal{R}_{n+m} \tag{4.3.5}$$

for all $n, m \geq 0$. Since R is an affine $\Bbbk$-algebra, there is some $r \geq 1$ such that for all $n > r$ we have:

$$\mathcal{R}_n = \sum_{i=1}^{r} \mathcal{R}_i \mathcal{R}_{n-i}^{\sigma^i}. \tag{4.3.6}$$

We introduce some notation and terminology on divisors associated to finite-dimensional spaces of rational functions; see [Laz04, Chapter 1] for a more detailed discussion.

Definition 4.3.7. If X is a normal projective variety and f is a rational function on X, we will denote its associated Weil divisor by $\operatorname{div}_X(f)$. We note that if $\sigma \in \operatorname{Aut} X$ and $f \in \Bbbk(X)$, then $\operatorname{div}_X(f^\sigma) = \sigma^{-1}\operatorname{div}_X(f)$. For any finite dimensional $\Bbbk$-vector space $V \subseteq K$, and for any normal projective model X for K, let $D^X(V)$ be the minimal Weil divisor on X such that $\operatorname{div}_X(f) + D \geq 0$ for all $f \in V$. That is, $\mathcal{O}_X(D^X(V))$ is canonically isomorphic to the double dual $(V \cdot \mathcal{O}_X)^{**}$.

Now suppose that X is an arbitrary projective variety and let $K = \Bbbk(X)$. Let D be a Cartier divisor on X. Recall [Har77, p. 144] that to D is associated an invertible subsheaf $\mathcal{O}_X(D)$ of the constant sheaf K on X. We will denote $H^0(\mathcal{O}_X(D))$ by $|D|$; this is the *complete linear system* associated to D.

Let $V \subset K$ be a finite-dimensional $\Bbbk$-vector space. Note that V may be contained in many complete linear systems. If $V \subseteq |D|$ for some Cartier divisor D, we define the image of the natural map

$$V \otimes \mathcal{O}_X(-D) \to \mathcal{O}_X$$

to be the *base ideal of V with respect to D*. The closed subscheme of X that it defines is called the *base locus of V with respect to D*. We write it $\operatorname{Bs}^D(V)$. If $(V \cdot \mathcal{O}_X)^{**}$ is an invertible sheaf, then it corresponds to an effective Cartier divisor D with $V \subseteq |D|$. This is the minimal such D, and in this situation we refer to the base ideal (respectively base locus) of V with respect to D simply as the *base ideal of V* (respectively, the *base locus of V*). We write the base locus of V as $\operatorname{Bs}(V)$.

If the base locus of the complete linear system $|D|$ is empty, we say that D and

$|D|$ are *base point free.* A divisor D is base point free if and only if the sheaf $\mathcal{O}_X(D)$ is globally generated.

If X is nonsingular or X is normal and $D^X(V)$ is Cartier, then the base ideal and base locus of V are always defined. Note that if either of these holds, then the base locus of V must have codimension at least 2.

Lemma 4.3.8. *Let X be a normal surface and let $K = \Bbbk(X)$. Let $\sigma \in \operatorname{Aut} X$, and let $V, W \subseteq K$ be finite-dimensional $\Bbbk$-vector spaces.*

(1) $D^X(VW) = D^X(V) + D^X(W)$.

(2) *For every n, $D^X(V^{\sigma^n}) = \sigma^{-n}(D^X(V))$.*

Proof. (1) For any $f \in V$ and $g \in W$, we have

$$\operatorname{div}_X(fg) + D^X(V) + D^X(W) = \operatorname{div}_X(f) + \operatorname{div}_X(g) + D^X(V) + D^X(W) \geq 0,$$

and so

$$D^X(V) + D^X(W) \geq D^X(VW). \tag{4.3.9}$$

Now fix $f \in V$. Since for any $g \in W$, we have $D^X(VW) + \operatorname{div}_X(f) + \operatorname{div}_X(g) \geq 0$, we see that $D^X(VW) + \operatorname{div}_X(f) \geq D^X(W)$. As this holds for any $f \in V$, we obtain that

$$D^X(VW) - D^X(W) \geq D^X(V). \tag{4.3.10}$$

Combining (4.3.9) and (4.3.10), we have proved (1).

(2) is a consequence of the equality $\operatorname{div}_X(f^\sigma) = \sigma^{-1}\operatorname{div}_X(f)$. $\square$

We introduce some more notation for data associated to R_n, in the situation that we are working on a normal model for R.

Assumption-Notation 4.3.11. *Assume that R is a birationally commutative projective surface with $R_1 \neq 0$. Let K be the function field of R and let (X, σ) be a normal model for R. Fix $z \neq 0 \in R_1$. Let $\overline{R}_n = R_n \cdot z^{-n}$ and let $\mathcal{R}_n = \mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$ for all $n \geq 0$.*

For all $n \geq 0$, let $D_n = D^X(\mathcal{R}_n)$. If $n < 0$, let $D_n = 0$. If D_n is Cartier for all $n \geq 1$ (for example, if X is nonsingular), then for $n \geq 1$ we further let $\mathcal{I}_n$ be the base ideal of $\overline{R}_n$ and let W_n be the base locus of $\overline{R}_n$.

The following purely combinatorial lemma is a restatement of results of Artin and Stafford on the combinatorics of divisors on smooth curves.

Lemma 4.3.12. *(Artin-Stafford) Let $A = \mathbb{Z}/(k)$ for some $k \in \mathbb{Z}$ (possibly $k = 0$). Let M be the free abelian group on the generating set $\{P_i \mid i \in A\}$; define a partial order $\geq$ on M by saying that $E \geq 0$ if $E = \sum n_i P_i$ where $n_i \geq 0$ for all i. Define an automorphism σ of M by $\sigma(P_i) = P_{i+1}$.*

Suppose there is a sequence of elements $\{E_i \mid i \in \mathbb{Z}\}$ in M satisfying:

(i) *$E_i \geq 0$ for all $i \geq 0$, and $E_i = 0$ if $i < 0$.*

(ii) *There exists an integer r such that*

$$E_n = \sup_{i=1}^{r}(E_i + \sigma^{-i}E_{n-i})$$

for all $n \geq 1$.

Then:

(1) *If $k = 0$, so $A = \mathbb{Z}$, there is an element $\Psi \geq 0 \in M$ and an integer $t \geq 0$ such that*

$$E_{m+n} = E_m + \sigma^{-m}(E_n) + \sigma^{-m}(\Psi)$$

for all $m, n \geq t$.

(2) *If $k = 1$, then there is an integer ℓ so that $E_{n\ell} = nE_\ell$ for all $n \geq 1$.*

Proof. (1) is [AS95, Corollary 2.12]. (2) is [AS95, Lemma 2.7]. $\square$

Lemma 4.3.13. *Assume Assumption-Notation 4.3.11. Then there are Weil divisors $0 \leq \Omega \leq D$ on X and an integer $k \geq 1$ such that for all $n \geq 1$ we have*

$$D_{kn} = D + \sigma^{-k}D + \cdots + \sigma^{-k(n-1)}D - \Omega, \tag{4.3.14}$$

and so that no irreducible component of Ω is fixed by any power of σ. Furthermore,

$$D_{(n+m)k} = D_{nk} + \sigma^{-nk}(D_{mk}) + \sigma^{-nk}(\Omega) \tag{4.3.15}$$

for all $n, m \geq 1$.

Proof. We note that it suffices to prove the lemma for a Veronese subring of R; it then holds for R by changing k and D.

We claim that for all $n, m \geq 0$ we have

$$D_{n+m} \geq D_n + \sigma^{-n}D_m \tag{4.3.16}$$

and that there is $r \geq 1$ such that for all $n \geq 1$, we have

$$D_n = \sup_{i=1}^{r}\Big(D_i + \sigma^{-i}(D_{n-i})\Big). \tag{4.3.17}$$

To see this, fix $m, n \geq 0$. Let $D' = D^X(\overline{R}_n(\overline{R}_m)^{\sigma^n})$. By Lemma 4.3.8, $D' = D^X(\overline{R}_n) + \sigma^{-n}D^X(\overline{R}_m)$. Because $\overline{R}_n(\overline{R}_m^{\sigma^n}) \subseteq \overline{R}_{n+m}$, we have that $D_{n+m} \geq D'$. This gives (4.3.16). Because $1 \in \overline{R}_1$, we have $D_{n+1} \geq D_n$ for all n. Let $r \geq 1$ be such that for all $n \geq r$, we have $R_n = \sum_{i=1}^r R_i R_{n-i}$. Then (4.3.17) follows.

Let $\mathrm{WDiv}(X)$ denote the group of Weil divisors on X. Equation 4.3.17 implies that there are only finitely many σ-orbits of prime divisors in $\mathrm{WDiv}(X)$ on which some D_n is nonzero. In particular, there are only finitely many such σ-orbits that are finite. Thus for some ℓ, each σ^ℓ-orbit of $\mathrm{WDiv}(X)$ on which some $D_{n\ell}$ is nonzero is either infinite or consists of one point. Without loss of generality, we may replace R by $R^{(\ell)}$ and assume that all curves of finite order that appear in some D_n are σ-invariant. Note that $R^{(\ell)}$ is still a birationally commutative surface, and, in particular, is finitely generated.

Let $E \in \mathrm{WDiv}(X)$ be a σ-invariant irreducible curve such that some $D_n \geq E$. There are only finitely many such E. Let $E_n = D_n|_E$. Equations 4.3.16 and 4.3.17 imply that $\{E_n\}$ satisfies the hypotheses of Lemma 4.3.12, with $k = 1$. Thus, by Lemma 4.3.12(2), there is an integer $m \geq 1$ such that for all $n \geq 1$, we have

$$D_{nm}|_E = n(D_m|_E).$$

If $E \in \mathrm{WDiv}(X)$ is of finite order under σ but not σ-invariant, then

$$D_{nm}|_E = 0 = n(D_m|_E)$$

for all m. Thus, by replacing R by $R^{(m)}$, we may assume that

$$D_n|_E = n(D_1|_E)$$

for all irreducible curves E that are of finite order under σ.

Let $\{P^1, \ldots, P^s\}$ be irreducible generators of the finitely many distinct infinite σ-orbits in $\mathrm{WDiv}(X)$ on which some D_n is nonzero. Fix $1 \leq i \leq s$, and let M be the subgroup of $\mathrm{WDiv}(X)$ generated by $\{\sigma^n(P^i)\}_{n\in\mathbb{Z}}$. Let $E_n = D_n|_M$. As before, $\{E_n\}$ satisfies the hypotheses of Lemma 4.3.12. Thus there exist t and Ψ as in the statement of Lemma 4.3.12(1). By varying i, we obtain integers $t^1, \ldots, t^s$ and divisors $\Psi^1, \ldots, \Psi^s$, with Ψ^i supported on $\{\sigma^n P^i\}_{n\in\mathbb{Z}}$. Let $k = \max\{t^i\}$ and let $\Omega = \Psi^1 + \cdots + \Psi^s$. By construction, Ω contains no components of finite order under σ, and

$$D_{m+n} = D_m + \sigma^{-m}(D_n) + \sigma^{-m}(\Omega) \tag{4.3.18}$$

for all $n, m \geq k$. Define $D = D_k + \Omega$. Note that (4.3.15) holds for all $n, m \geq 1$.

We claim that (4.3.14) holds for all $n \geq 1$. The claim is true for $n = 1$; assume it holds for $n - 1$. By (4.3.18),

$$D_{kn} = D_{k(n-1)} + \sigma^{-k(n-1)}(D_k) + \sigma^{-k(n-1)}(\Omega).$$

This is equal to

$$\begin{aligned}\big(D + \sigma^{-k}(D) + \cdots + \sigma^{-k(n-2)}(D) - \Omega\big) + \sigma^{-k(n-1)}(D - \Omega) + \sigma^{-k(n-1)}(\Omega) \\ = D + \sigma^{-k}(D) + \cdots + \sigma^{-k(n-1)}(D) - \Omega\end{aligned}$$

by induction. $\square$

Definition 4.3.19. Assume Assumption-Notation 4.3.3; in particular, fix $0 \neq z \in R_1$. Let (X, σ) be a normal model for R. Let $D_n = D^X(\overline{R}_n)$. If there are effective Weil divisors D and Ω on X and an integer k so that (4.3.14) and (4.3.15) hold for all $n, m \gg 0$, we follow the terminology of [AS95] and say that Ω is a *gap divisor for R on X associated to z* (or more briefly a *gap divisor for R on X*), and that D is a *coordinate divisor for R on X (associated to z).*

Note that Ω is a gap divisor for R associated to z if and only if it is a gap divisor associated to z^n for some $R^{(n)}$. We note that this gap divisor is unique (at least up to choice of z).

Lemma 4.3.20. *Assume Assumption-Notation 4.3.11. For a fixed $z \neq 0 \in \overline{R}_1$, there is exactly one Weil divisor Ω that is a gap divisor for R associated to z.*

Proof. By Lemma 4.3.13, there is a gap divisor Ω for R on X associated to z. Suppose that there are Weil divisors Ω and Ω' so that for some $k, k' \geq 1$ we have

$$D_{k(n+m)} - D_{kn} - \sigma^{-kn}(D_{km}) = \sigma^{-kn}(\Omega)$$

and

$$D_{k'(n+m)} - D_{k'n} - \sigma^{-k'n}(D_{k'm}) = \sigma^{-k'n}(\Omega')$$

for all $n, m \geq 1$. Then

$$\sigma^{-kk'}(\Omega) = D_{2kk'} - D_{kk'} - \sigma^{-kk'}(D_{kk'}) = \sigma^{-k'k}(\Omega').$$

Thus $\Omega = \Omega'$. □

Initially, it will be more convenient to work on a nonsingular model for R. By Lemma 4.3.13 and Theorem 4.3.2, we may replace R by a Veronese subring to assume without loss of generality that we are in the following situation:

Assumption-Notation 4.3.21. *Assume that R is a birationally commutative projective surface with $R_1 \neq 0$. Let K be the function field of R and assume that there is a nonsingular model (X, σ) for R so that σ is numerically trivial. As usual, we will identify Weil and Cartier divisors. Fix $z \neq 0 \in R_1$. Let $\overline{R}_n = R_n \cdot z^{-n}$ and let $\mathcal{R}_n = \mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$ for all $n \geq 0$. For all $n \geq 0$, let $D_n = D^X(\mathcal{R}_n)$. If $n < 0$, let $D_n = 0$. Let $\mathcal{I}_n$ be the base ideal of $\overline{R}_n$ and let W_n be the base locus of $\overline{R}_n$.*

Further assume that there are a gap divisor Ω and a coordinate divisor D associated to z so that (4.3.14) and (4.3.15) hold with $k = 1$ for all $n \geq 1$, and that $\Omega \cap \sigma^k \Omega$ is finite for all $k \neq 0$.

Let $\mathcal{L} = \mathcal{O}_X(D)$. Recall that $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$. For $n \geq 0$, let

$$\Delta_n = D + \cdots + \sigma^{-(n-1)}D,$$

so $\mathcal{L}_n = \mathcal{O}_X(\Delta_n)$.

Our assumptions imply that

$$\mathcal{R}_n = \mathcal{I}_n \mathcal{I}_\Omega \mathcal{L}_n = \mathcal{I}_n(\Delta_n - \Omega)$$

for all $n \geq 1$.

We note that if Assumption-Notation 4.3.21 holds for R, then it holds for any Veronese $R^{(n)}$ of R, by replacing σ by σ^n. (The effect of this change is to also replace D by Δ_n.) Also note that in the setting of Assumption-Notation 4.3.21, we may regard R as a subring of $B(X, \mathcal{L}, \sigma) = \bigoplus H^0(\mathcal{L}_n)z^n$, even if $\mathcal{L}$ is not ample or σ-ample. That is, elements of $\overline{R}_n$ correspond to global sections of $\mathcal{L}_n$. We will make this identification throughout the rest of the chapter.

4.4 Points of finite order

The model X that we chose was picked quite arbitrarily, and in general we cannot expect that X is the space to which R is actually associated. Thus in the rest of this chapter, we will work to gradually modify X and to construct the other data that will define the ring R. In this section, we will show that we can modify X to remove any points of finite order in the the base loci of the rational functions $\overline{R}_n$.

Lemma 4.4.1. *Assume Assumption-Notation 4.3.21. Then there is a finite set V so that W_n is supported on $V \cup \cdots \sigma^{-(n-1)}(V)$ for all $n \geq 1$. In fact, we may take*

$$V = W_1 \cup W_2 \cup (\sigma^{-1}\Omega \cap \sigma^{-2}\Omega). \tag{4.4.2}$$

Proof. Recall that $\mathcal{I}_n$ is the base ideal of the vector space $\overline{R}_n$ of rational functions. For all $m, n \geq 0$, the equation

$$\mathcal{I}_\Omega \mathcal{I}_n \mathcal{I}_\Omega^{\sigma^n} \mathcal{I}_m^{\sigma^n} \mathcal{L}_{n+m} = \mathcal{R}_n \mathcal{R}_m^{\sigma^n} \subseteq \mathcal{R}_{n+m} = \mathcal{I}_\Omega \mathcal{I}_{n+m} \mathcal{L}_{n+m}$$

gives a set-theoretic containment

$$W_{n+m} \subseteq W_n \cup \sigma^{-n}(W_m) \cup \sigma^{-n}(\Omega). \tag{4.4.3}$$

Define V as in (4.4.2). As $\sigma^{-1}\Omega \cap \sigma^{-2}\Omega$ is finite by assumption, V is finite.

Assume that for all $j \leq n$, we have $W_j \subseteq V \cup \cdots \cup \sigma^{-(j-1)}(V)$. By construction, this is true for $n = 1, 2$. For $n \geq 2$, (4.4.3) gives that

$$W_{n+1} \subseteq (W_1 \cup \sigma^{-1} W_n \cup \sigma^{-1}\Omega) \cap (W_2 \cup \sigma^{-2} W_{n-1} \cup \sigma^{-2}\Omega).$$

By induction, we therefore have

$$W_{n+1} \subseteq V \cup \cdots \cup \sigma^{-n} V \cup (\sigma^{-1}\Omega \cap \sigma^{-2}\Omega) = V \cup \cdots \cup \sigma^{-n} V.$$

□

We give an elementary lemma on how base ideals transform under birational morphisms of projective varieties.

Lemma 4.4.4. *Let $\pi : X' \to X$ be a birational morphism of projective varieties, and let D be an effective (Cartier) divisor on X. Let $V \subseteq |D|$. Then the base ideal of V on X' with respect to $\pi^* D$ is the expansion to X' of the base ideal of V on X with respect to D. If X and X' are normal, $D^X(V)$ is Cartier, and the indeterminacy locus of π^{-1} consists of smooth points of X, then*

$$\pi^* D^X(V) - D^{X'}(V)$$

is effective and supported on the exceptional locus of π.

Proof. Note that the elements of V are also elements of the linear system $|\pi^* D|$. Let $\mathcal{I}$ be the base ideal of V with respect to D; this is the image of the natural map

$$V \otimes \mathcal{O}_X(-D) \to \mathcal{O}_X.$$

Let $\mathcal{J}$ be the base ideal of V on X' with respect to $\pi^* D$; that is, the image of the natural map

$$V \otimes \mathcal{O}_{X'}(-\pi^* D) \to \mathcal{O}_{X'}.$$

Now, if we pull back the surjection

$$V \otimes \mathcal{O}_X(-D) \twoheadrightarrow \mathcal{I}$$

to X', we obtain, by right exactness of pullbacks, a surjection

$$V \otimes \mathcal{O}_{X'}(-\pi^* D) \twoheadrightarrow \pi^* \mathcal{I}.$$

Composing this with the natural map from $\pi^* \mathcal{I} \to \pi^* \mathcal{O}_X = \mathcal{O}_{X'}$, we obtain the map

$$V \otimes \mathcal{O}_{X'}(-\pi^* D) \to \mathcal{O}_{X'}$$

defining $\mathcal{J}$. The image of $\pi^* \mathcal{I}$ in $\mathcal{O}_{X'}$ is precisely $\mathcal{I}\mathcal{O}_{X'}$; that is, $\mathcal{J}$ is the expansion of $\mathcal{I}$ to $\mathcal{O}_{X'}$.

Suppose now that X and X' are normal, $D^X(V)$ is Cartier, and the indeterminacy locus of π^{-1} consists of smooth points of X. Let $F = D^X(V)$, and let $\mathcal{I}$ be the base ideal of V on X. Then by the above,

$$V \cdot \mathcal{O}_{X'} = \mathcal{I}\mathcal{O}_{X'}(\pi^* F),$$

and so $D^{X'}(V) = \pi^* F - C$ for some effective Weil divisor C contained in the subscheme of X' defined by $\mathcal{I}\mathcal{O}_{X'}$. Thus C is supported on the exceptional locus of π. $\square$

Suppose now that X is a surface and $\sigma \in \operatorname{Aut}(X)$. Let $Z = \{p, \sigma(p), \ldots, \sigma^{k-1}(p)\}$ be a finite σ-orbit in X. We record an easy result on automorphisms of blowups.

Lemma 4.4.5. *Let X be a smooth surface, let $\sigma \in \operatorname{Aut}(X)$ and let $Z \subseteq X$ be a finite (reduced) σ-orbit. Let $\pi : X' \to X$ be the blowup of X at Z. Then σ lifts to an automorphism σ' of X'.*

Proof. Let $\mathcal{I} = \mathcal{I}_Z$ be the ideal sheaf defining Z. As Z is σ-invariant, we have $\mathcal{I}^\sigma = \mathcal{I}$ and so σ induces an automorphism of $\mathcal{I}$. It therefore induces an automorphism of the blowup of X at Z; see [Har77, p. 163]. By construction, we have $\pi\sigma' = \sigma\pi$. $\square$

We now begin the process of modifying X to remove points of finite order from the base loci W_n. We will do this through a series of blowups at finite orbits, and we begin by studying the effect of blowing up on the gap divisor Ω.

Lemma 4.4.6. *Assume Assumption-Notation 4.3.21; in particular, fix $0 \neq z \in R_1$, which we will use to calculate gap divisors, and let Ω be the gap divisor of R on X.*

(1) *Let $\pi : \widetilde{X} \to X$ be the blowup of X at a finite σ-orbit. Then the gap divisor of R on $\widetilde{X}$ is the strict transform of Ω.*

(2) *There are a nonsingular projective surface X' and a birational morphism $\pi : X' \to X$ so that there is an automorphism σ' of X' with $\pi\sigma' = \sigma\pi$, and so that the gap divisor of R on X' contains no points of finite order under σ'. That is, by changing our smooth model X, without loss of generality we may assume that Ω contains no points of finite order.*

Proof. (1) By assumption,

$$D_n + \sigma^{-n} D_m + \sigma^{-n}\Omega = D_{n+m} \tag{4.4.7}$$

for all $n, m \geq 1$. For all $n \geq 1$, let $F_n = D^{\widetilde{X}}(\overline{R}_n)$; let $\mathcal{J}_n$ be the base ideal of $\overline{R}_n$ on $\widetilde{X}$, so

$$\mathcal{J}_n = \mathcal{R}_n(\widetilde{X})(-F_n) \subseteq \mathcal{O}_{\widetilde{X}}.$$

By [Har77, Proposition V.3.1], $\widetilde{X}$ is nonsingular. Let $\widetilde{\sigma}$ be the automorphism of $\widetilde{X}$ that is conjugate to σ, given by Lemma 4.4.5. By Lemma 4.3.13, let $\widetilde{\Omega}$ be the gap divisor of R on $\widetilde{X}$. All components of $\widetilde{\Omega}$ are of infinite order under $\widetilde{\sigma}$, and there is some $k \geq 1$ so that

$$F_{nk} + \widetilde{\sigma}^{-nk} F_{mk} + \widetilde{\sigma}^{-nk} \widetilde{\Omega} = F_{(n+m)k} \tag{4.4.8}$$

for all $n, m \geq 1$.

For all $n \geq 0$, let $E_n = \pi^* D_n - F_n$. By Lemma 4.4.4, E_n is effective and supported on the exceptional locus of π. Pulling back (4.4.7) to $\widetilde{X}$, we obtain that

$$\pi^* D_n + \widetilde{\sigma}^{-n}(\pi^* D_m) + \widetilde{\sigma}^{-n}(\pi^* \Omega) = \pi^*(D_{n+m})$$

for all $n, m \geq 1$. Comparing this to (4.4.8), we see that

$$E_{nk} + \widetilde{\sigma}^{-nk}(E_{mk}) + \widetilde{\sigma}^{-nk}(\pi^* \Omega - \widetilde{\Omega}) = E_{(n+m)k}$$

for all $n, m \geq 1$. Thus $\pi^* \Omega - \widetilde{\Omega}$ is supported on the exceptional locus of π. All its components are thus of finite order under $\widetilde{\sigma}$; as $\widetilde{\Omega}$ contains no components of finite order under $\widetilde{\sigma}$, we see that $\widetilde{\Omega}$ is the strict transform of Ω.

(2) Suppose that Ω contains a point p of finite order, and let $\pi : \widetilde{X} \to X$ be the blowup of X at the orbit of p. Let $\widetilde{\sigma}$ be the automorphism of $\widetilde{X}$ conjugate to σ. Let $\widetilde{\Omega}$ be the gap divisor of R on $\widetilde{X}$. By (1), $\widetilde{\Omega}$ is the strict transform of Ω.

Note that $\widetilde{\sigma}$ is quasi-trivial. Thus we may choose k so that $\widetilde{\sigma}^k$ is numerically trivial. By assumption, $\widetilde{\Omega} \cap \widetilde{\sigma}^k \widetilde{\Omega}$ is finite. Then we have:

$$\Omega^2 > (\widetilde{\Omega})^2 = \widetilde{\Omega}.\widetilde{\sigma}^k(\widetilde{\Omega}) \geq 0. \tag{4.4.9}$$

If $\widetilde{\Omega}$ contains any points of finite order, we may repeat this process and reduce $(\widetilde{\Omega})^2$ further. Since (4.4.9) shows that the gap divisor always has non-negative self-intersection, this process must terminate after finitely many steps. That is, after finitely many steps we must obtain a gap divisor containing no points of finite order.

$\square$

We are ready to prove that there is some model of R on which the base loci of all $\overline{R}_n$ consist of points of infinite order. Before doing so, we recall some terminology from commutative algebra. Let (S, M) be a regular local ring of dimension 2, and let I be an M-primary ideal of S. Recall [Eis95, Section 12.1] that the *Hilbert-Samuel function* of S with respect to I is defined as

$$H_I(n) = \operatorname{len} I^n / I^{n+1}.$$

Recall further [Eis95, Exercise 12.6] that the *multiplicity* of I, written $e(I)$, is defined as

$$e(I) = (2 = 2!) \times (\text{the leading coefficient of } H_I).$$

This is a positive integer that may be defined more geometrically as follows: let $a, b \in I$ be a regular sequence. Then $e(I)$ is the intersection multiplicity of two general members of the ideal $aS + bS$.

Now let X be a nonsingular surface and let Z be a 0-dimensional subscheme of X. We define the *multiplicity* $e(Z)$ of Z to be the sum of the multiplicities of the defining ideal of Z at all points in $\operatorname{Supp}(Z)$. By definition, $e(Z) \geq 0$, and $e(Z) = 0$ if and only if $Z = \emptyset$. Let $p \in Z$ and let $\pi : X' \to X$ be the blowup of X at p; let $Z' \subseteq X'$ be the strict transform of Z. The identification of $e(Z)$ with an intersection multiplicity shows that $e(Z')$ is strictly less than $e(Z)$.

Proposition 4.4.10. *Let R be a birationally commutative surface with $R_1 \neq 0$. There is a smooth model (X, σ) for some Veronese $R^{(r)}$ of R such that σ is numerically trivial, the gap divisor of $R^{(r)}$ on X contains no points of finite order, and so that for all $n \geq 1$ the base locus of $\overline{R}_{nr}$ on X is supported on points of infinite order.*

Proof. Choose a smooth model (X, σ) for R. By Lemma 4.3.13, by replacing R by a Veronese subring, we may assume that we are in the situation of Assumption-

Notation 4.3.21. By Lemma 4.4.6, by changing X and possibly replacing R by a further Veronese (to ensure that Assumption-Notation 4.3.21 still holds), we may further assume that Ω contains no points of finite order.

Let M be such that R and $\mathcal{R} = \mathcal{R}(X)$ are generated in degrees $\leq M$. If there is some $1 \leq i \leq M$ such that W_i contains a point p of finite order under σ, replace X by the blowup of X at the orbit of p. As $e(W_i)$ is reduced each time, continuing finitely many times, we may assume that there is a surface $\widetilde{X}$ with a morphism $\pi : \widetilde{X} \to X$ and an automorphism $\widetilde{\sigma}$ of $\widetilde{X}$, conjugate to σ, so that for $i = 1 \ldots M$ the base locus of $\overline{R}_i$ on $\widetilde{X}$ contains no points of finite order.

For all $n \geq 1$, let $F_n = D^{\widetilde{X}}(\overline{R}_n)$, and let $\mathcal{J}_n = \mathcal{R}_n(\widetilde{X})(-F_n)$ be the base ideal of $\overline{R}_n$ on $\widetilde{X}$. We caution that (4.3.14) and (4.3.15) may not hold for the F_i with $k = 1$, although they do, of course, hold for some k. On the other hand, $D_n = \Delta_n - \Omega$ for all $n \geq 1$. By Lemma 4.4.4, for all $n \geq 1$ the divisor $E_n = \pi^* D_n - F_n$ is effective and supported on the exceptional locus of π. In particular, all components of any E_n are of finite order under σ. Note that as the indeterminacy locus of π^{-1} consists of points of finite order, it is disjoint from Ω. Thus, $\pi^*\Omega \cap E_n = \emptyset$ for all n. By Lemma 4.4.6(1), the gap divisor of R on $\widetilde{X}$ is equal to $\pi^*\Omega$ and contains no points of finite order under $\widetilde{\sigma}$.

The bimodule algebra $\mathcal{R}(\widetilde{X})$ on $\widetilde{X}$ is still generated in degrees $\leq M$. That is,

$$\mathcal{R}_n(\widetilde{X}) = \sum_{i=1}^{M} \mathcal{R}_i(\widetilde{X})\mathcal{R}_{n-i}(\widetilde{X})^{\widetilde{\sigma}^i} = \mathcal{R}_n(\widetilde{X}) \tag{4.4.11}$$

for all $n > M$. As

$$\mathcal{R}_n(\widetilde{X}) = \mathcal{J}_n(F_n) = \mathcal{J}_n\mathcal{O}_{\widetilde{X}}(-\pi^*\Omega - E_n + \pi^*\Delta_n)$$

for all $n \geq 1$, we may rewrite (4.4.11) as

$$\mathcal{J}_n \mathcal{O}_{\widetilde{X}}(-\pi^*\Omega - E_n + \pi^*\Delta_n) = \sum_{i=1}^{M} \mathcal{J}_i \mathcal{O}_{\widetilde{X}}(-\pi^*\Omega - E_i + \pi^*\Delta_i) \cdot \mathcal{J}_{n-i}^{\widetilde{\sigma}^i} \mathcal{O}_{\widetilde{X}}(\widetilde{\sigma}^{-i}(-\pi^*\Omega - E_{n-i} + \pi^*\Delta_{n-i}))$$

for all $n > M$. Since $\pi^*\Delta_n = \pi^*\Delta_i + \widetilde{\sigma}^{-i}\pi^*\Delta_{n-i}$, this may be rewritten as

$$\mathcal{J}_n \mathcal{I}_{E_n} = \mathcal{J}_n \mathcal{O}_{\widetilde{X}}(-E_n) = \sum_{i=1}^{M} \mathcal{J}_i \mathcal{J}_{n-i}^{\widetilde{\sigma}^i} \mathcal{O}_{\widetilde{X}}(-E_i - \widetilde{\sigma}^{-i}(E_{n-i} + \pi^*\Omega)). \tag{4.4.12}$$

for all $n > M$.

For all n, let $\mathcal{K}_n$ be the minimal ideal sheaf on $\widetilde{X}$ that contains $\mathcal{J}_n$ and is cosupported at points on the exceptional locus of π; this exists because $\mathcal{J}_n$ is coartinian. Now, $\widetilde{\sigma}^{-i}(\pi^*\Omega)$ is disjoint from the exceptional locus of π. This means that by restricting (4.4.12) to the exceptional locus of π, we obtain that

$$\mathcal{K}_n \mathcal{I}_{E_n} = \sum_{i=1}^{M} \mathcal{K}_i \mathcal{K}_{n-i}^{\widetilde{\sigma}^i} \mathcal{O}_{\widetilde{X}}(-E_i - \widetilde{\sigma}^{-i} E_{n-i}) = \sum_{i=1}^{M} \mathcal{K}_i \mathcal{K}_{n-i}^{\widetilde{\sigma}^i} \mathcal{I}_{E_i} \mathcal{I}_{E_{n-i}}^{\widetilde{\sigma}^i}$$

for all $n > M$.

For all n, let $\hat{\mathcal{K}}_n$ be the minimal ideal sheaf on $\widetilde{X}$ containing $\mathcal{K}_n$ and cosupported at points of finite order. Now, there is some k so that the ideal sheaf $\sum_{i=1}^{M} \mathcal{I}_{E_i} \mathcal{I}_{E_{n-i}}^{\widetilde{\sigma}^i}$ is $\widetilde{\sigma}^k$-invariant, as all E_i are of finite order under $\widetilde{\sigma}$. This implies that

$$\hat{\mathcal{K}}_n \mathcal{I}_{E_n} = \sum_{i=1}^{M} \hat{\mathcal{K}}_i \hat{\mathcal{K}}_{n-i}^{\widetilde{\sigma}^i} \mathcal{I}_{E_i} \mathcal{I}_{E_{n-i}}^{\widetilde{\sigma}^i}$$

for all $n > M$. But for $1 \leq i \leq M$, the base locus of $\overline{R}_i$ on $\widetilde{X}$ contains no points of finite order, and so $\hat{\mathcal{K}}_i = \mathcal{O}_{\widetilde{X}}$. Thus

$$\hat{\mathcal{K}}_n \mathcal{I}_{E_n} = \sum_{i=1}^{M} \hat{\mathcal{K}}_{n-i}^{\widetilde{\sigma}^i} \mathcal{I}_{E_i} \mathcal{I}_{E_{n-i}}^{\widetilde{\sigma}^i} \tag{4.4.13}$$

for $n > M$. Let ℓ be such that $\widetilde{\sigma}^\ell$ fixes all irreducible exceptional curves and so all components of $E_1, \ldots, E_M$. Then (4.4.13) and an easy induction imply that for all n, $\hat{\mathcal{K}}_n$ is $\widetilde{\sigma}^\ell$-invariant.

Let $\mathcal{S}$ be the graded $(\mathcal{O}_{\widetilde{X}}, \widetilde{\sigma}^\ell)$-bimodule algebra defined by

$$\mathcal{S} = \bigoplus_{n \geq 0} (\mathcal{S}_n)_{\widetilde{\sigma}^{n\ell}},$$

where

$$\mathcal{S}_n = \hat{\mathcal{K}}_{n\ell} \mathcal{I}_{E_{n\ell}}.$$

As all $\mathcal{S}_n$ are $\widetilde{\sigma}^\ell$-invariant, $\mathcal{S}$ is a *commutative* bimodule algebra; that is, $\mathcal{S}$ is a sheaf of (commutative) graded algebras on $\widetilde{X}$. Now, as R is noetherian, $\mathcal{R}(\widetilde{X})^{(\ell)}$ is finitely generated. Thus there is $N \geq 1$ so that

$$\mathcal{R}_{n\ell}(\widetilde{X}) = \sum_{i=1}^{N} \mathcal{R}_{i\ell}(\widetilde{X}) \mathcal{R}_{(n-i)\ell}(\widetilde{X})^{\widetilde{\sigma}^{i\ell}}$$

for all $n > N$. Restricting to the exceptional locus of π and to finite orbits, we obtain that

$$\mathcal{S}_n = \sum_{i=1}^{N} \mathcal{S}_i \mathcal{S}_{n-i}^{\widetilde{\sigma}^{i\ell}} = \sum_{i=1}^{N} \mathcal{S}_i \mathcal{S}_{n-i}$$

for all $n > N$, and so $\mathcal{S}$ is finitely generated.

Let $U_1, \ldots, U_n$ be a finite affine cover of X. As is well-known (see [Bou98, Section III.1.3, Proposition 3]), for each $1 \leq j \leq n$ there is some e_j so that the graded ring $\mathcal{S}(U_j)$ is generated by $\mathcal{S}_{e_j}(U_j)$ and $\mathcal{O}_{\widetilde{X}}(U_j)$. Let $e = e_1 \cdots e_n$. Then all $\mathcal{S}^{(e)}(U_i)$ are generated in degree 1, so $\mathcal{S}^{(e)}$ is generated in degree 1.

That is,

$$\hat{\mathcal{K}}_{n\ell e} \mathcal{I}_{E_{n\ell e}} = \mathcal{S}_{ne} = (\mathcal{S}_e)^n = (\hat{\mathcal{K}}_{\ell e} \mathcal{I}_{E_{\ell e}})^n$$

for all $n \geq 1$. As $\hat{\mathcal{K}}_{\ell e}$ is $\widetilde{\sigma}^\ell$-invariant, we may resolve it by a sequence of point blowups at finite $\widetilde{\sigma}$-orbits. We obtain a nonsingular surface X' with a birational morphism $\pi : X' \to X$ so that $\widetilde{\sigma}$ is conjugate to an automorphism σ' of X' and so that the expansion of $\hat{\mathcal{K}}_{\ell e}$ to X' is invertible. Thus the expansion of $\hat{\mathcal{K}}_{n\ell e}$ to X' is invertible for all $n \geq 1$.

Recall that $\mathcal{J}_n$ is the base ideal of $\overline{R}_n$ on $\widetilde{X}$. For all n, there is an ideal sheaf $\mathcal{C}_n$ so that

$$\mathcal{J}_n = \hat{\mathcal{K}}_n \mathcal{C}_n.$$

Necessarily, $\mathcal{C}_n$ is cosupported at points of infinite order. Let Z_n be the subscheme of $\widetilde{X}$ defined by $\mathcal{C}_n$. Lemma 4.4.4 implies that the base locus of $\overline{R}_{n\ell e}$ on X' is $\pi^{-1}(Z_{n\ell e})$, as the expansion of $\hat{\mathcal{K}}_{n\ell e}$ to X' is invertible. This contains no points of finite order for any $n \geq 1$. By Lemma 4.4.6, the gap divisor of R on X' contains no points of finite order under σ'. □

We will be considering the rational maps to projective space defined by the rational functions in $\overline{R}_n$ and $|\Delta_n|$. We record here the elementary result that these are birational onto their image for $n \gg 0$.

Lemma 4.4.14. *Let R be a birationally commutative projective surface with function field K. Assume that $R_1 \neq 0$ and fix $0 \neq z \in R_1$. For some n, the rational functions in $\overline{R}_n$ generate K as a field and so induce a birational map of X onto its image.*

Proof. Let $f_1, \dots, f_k$ be rational functions that generate K. For each i, there are homogeneous elements a_i, b_i of some R_{n_i} so that $f_i = a_i b_i^{-1}$. By putting all the f_i over a common denominator, we may assume that there are some $c_1, \dots, c_k, b \in R_n$ with $f_i = c_i b^{-1}$ for all i. Thus $\overline{R}_n$ generates the field K. □

By Proposition 4.4.10 and Lemma 4.4.14, we may pass to a further Veronese subring to strengthen our assumptions on R.

Assumption-Notation 4.4.15. *Assume that R is a birationally commutative projective surface with function field K so that $R_1 \neq 0$. Fix $0 \neq z \in R_1$, and define $\overline{R}_n = R_n z^{-n}$. Assume that $\overline{R}_1$ generates K as a field.*

Assume also that there is a nonsingular model (X, σ) for R so that σ is numerically trivial. Define $\mathcal{R}_n(X)$, D_n, $\mathcal{I}_n$, and W_n as in Assumption-Notation 4.3.11. Further assume that there are a gap divisor Ω and a coordinate divisor D associated to z so that (4.3.14) *and* (4.3.15) *hold with $k = 1$ for all $n, m \geq 1$, and that $\Omega \cap \sigma^k \Omega$ is finite for all $k \neq 0$. We further assume that Ω and all W_n are disjoint from finite σ-orbits.*

We continue to define $\Delta_n = D + \cdots + \sigma^{-(n-1)} D$ and $\mathcal{L} = \mathcal{O}_X(D)$.

We remark that if Assumption-Notation 4.4.15 holds for R, it holds for any Veronese $R^{(k)}$ of R, by replacing σ by σ^k and D by Δ_k.

4.5 An ample model for R

Let (X, σ) be a normal model for R. If a coordinate divisor of R on X is σ-ample, we refer to X or to the pair (X, σ) as an *ample model* for R. The goal of this section is to show that an ample model for R exists.

We begin by giving the σ-twisted versions of some results about big and nef divisors. Recall that a divisor D on a projective surface X is *big* if

$$h^0(\mathcal{O}_X(nD)) = \dim H^0(\mathcal{O}_X(nD))$$

grows as $O(n^2)$, and D is *nef* if $D.C \geq 0$ for any curve C on X. We refer the reader to [Laz04] for the basic properties of big and nef divisors.

Recall also that we denote linear equivalence of divisors by $\sim$ and numerical equivalence by $\equiv$. If D is a divisor on X and $m \geq 1$, let $\Delta_m = D + \sigma^{-1} D + \cdots + \sigma^{-(m-1)} D$.

Definition 4.5.1. Let σ be a quasi-trivial automorphism of the projective surface X. We say that a divisor D is *σ-big* if $h^0(\mathcal{O}_X(\Delta_n))$ grows at least as $O(n^2)$.

We note that for any normal model (X, σ) for R, if D is a coordinate divisor for R on X, then D is σ-big by assumption on the GK-dimension of R.

Lemma 4.5.2. (Kodaira's Lemma; cf. [Laz04, Proposition 2.2.6]) *Let σ be a quasi-trivial automorphism of a smooth projective surface X, and let D be a σ-big divisor on X. Let F be an effective divisor on X. Then $H^0(\mathcal{O}_X(\Delta_m - F)) \neq 0$ for all sufficiently large m.*

Proof. We consider the exact sequence

$$0 \to H^0(\mathcal{O}_X(\Delta_m - F)) \to H^0(\mathcal{O}_X(\Delta_m)) \xrightarrow{\phi_m} H^0(\mathcal{O}_F(\Delta_m)).$$

By Theorem 2.5.3, there are constants n and c such that if E is a divisor on X with $E.F \geq n$, then $h^0(\mathcal{O}_F(E)) = F.E + c$. Since σ is quasi-trivial, $\Delta_m.F$ grows no faster than $O(m)$, and thus $h^0(\mathcal{O}_F(\Delta_m))$ grows no faster than $O(m)$. Since D is σ-big, for $m \gg 0$ we have that $h^0(\mathcal{O}_X(\Delta_m)) > h^0(\mathcal{O}_F(\Delta_m))$ and therefore the map $\phi_m : H^0(\mathcal{O}_X(\Delta_m)) \to H^0(\mathcal{O}_F(\Delta_m))$ must have a kernel. This gives a section of $\mathcal{O}_X(\Delta_m - F)$. □

Corollary 4.5.3. (cf. [Laz04, Corollary 2.2.7]) *Let σ be a quasi-trivial automorphism of the smooth projective surface X, and let D be a σ-big divisor on X. Let A be an ample divisor on X. Then there is some $m > 0$ and some effective divisor N on X such that $\Delta_m \sim A + N$.*

Proof. Choose r such that $(r + 1)A$ and rA are both effective. Using Lemma 4.5.2, choose m such that $H^0(\mathcal{O}_X(\Delta_m - (r + 1)A)) \neq 0$. Thus there is some effective N' with

$$\Delta_m - (r + 1)A \sim N'.$$

That is, $\Delta_m \sim A + (rA + N')$. Since rA and N' are both effective, the theorem is proved for $N = rA + N'$. □

Lemma 4.5.4. (Wilson's Theorem; cf. [Laz04, Theorem 2.3.9]) *Let σ be a quasi-trivial automorphism of the smooth projective surface X, and let D be a σ-big and nef divisor on X. Then there are an effective divisor N and an integer $m_0 > 0$ such that for every $m \geq m_0$, both $\Delta_m - N$ and $\Delta_m - \sigma^{-(m-m_0)}(N)$ are base point free.*

Proof. By Theorem 2.5.1, there is a very ample divisor B such that $H^i(\mathcal{O}_X(B+P)) = 0$ for all nef P and for all $i \geq 1$. Note that the same property holds for all $\sigma^n B$. Corollary 4.5.3 implies that there is some $m_0 > 0$ so that $\Delta_{m_0} \sim 3B + N$ for N effective. Then for $m \geq m_0$,

$$\Delta_m - N \sim 3B + \sigma^{-m_0}\Delta_{m-m_0}.$$

As D is nef, all $\sigma^i D$ are also nef. Nef divisors form a cone, so $\sigma^{-m_0}\Delta_{m-m_0}$ is nef. Thus

$$\Delta_m - N \sim B + 2B + \text{nef}.$$

Our assumption on B implies that

$$H^i(\mathcal{O}_X(\Delta_m - N - iB)) = 0$$

for $i = 1, 2$. That is, $\Delta_m - N$ is 0-regular with respect to B (in the sense of Section 2.5), and so by Theorem 2.5.2, $\mathcal{O}_X(\Delta_m - N)$ is globally generated.

Similarly,

$$\Delta_m - \sigma^{-(m-m_0)}N \sim \Delta_{m-m_0} + 3\sigma^{-(m-m_0)}B,$$

and so $H^i(\mathcal{O}_X(\Delta_m - \sigma^{-(m-m_0)}N - i\sigma^{-(m-m_0)}B)) = 0$ for $i = 1, 2$. That is, $\Delta_m - \sigma^{-(m-m_0)}N$ is 0-regular with respect to the very ample divisor $\sigma^{-(m-m_0)}B$. Applying Theorem 2.5.2 again, we have that $\mathcal{O}_X(\Delta_m - \sigma^{-(m-m_0)}N)$ is globally generated. $\square$

Lemma 4.5.5. *Assume Assumption-Notation 4.4.15, so D is the coordinate divisor of R on X and σ is numerically trivial. Then Δ_n is big and nef for all $n \geq 1$.*

Proof. It is enough to show that D is big and nef. Suppose that D is not nef. Then there is some effective curve C such that $C.D < 0$. For $n \gg 0$ we have $(\Delta_n - \Omega).C = nD.C - \Omega.C < 0$. This implies that if $\Gamma \sim \Delta_n - \Omega$ is effective, then $C \leq \Gamma$; that is, C is contained in the base locus of $|\Delta_n - \Omega|$. But $\mathrm{Bs}(|\Delta_n - \Omega|) \subseteq \mathrm{Bs}(\overline{R}_n) = W_n$, and this is 0-dimensional. Thus D is nef.

By assumption on R, we know that D is σ-big. By Corollary 4.5.3 we have that some $\Delta_n \sim A + F$ for some ample A and some effective F. Thus Δ_n is big by [Laz04, Corollary 2.2.7]. Since σ is numerically trivial and bigness is numeric [Laz04, Corollary 2.2.8], we see that nD and therefore D are big. $\square$

Theorem 4.5.6. *Assume Assumption-Notation 4.4.15. Then there is some k so that Δ_{nk} is base point free for $n \gg 0$.*

We note that if R is commutative (so $\Omega = 0$ and $\sigma = \mathrm{Id}_X$), then this follows from Zariski's result [Zar62, Theorem 6.2] that if $\mathcal{L}$ is a line bundle on a projective variety with a 0-dimensional base locus, then some tensor power of $\mathcal{L}$ is globally generated.

Proof. For all n, let $Z_n = \mathrm{Bs}(|\Delta_n|)$. We want to show that for some k, $Z_{nk} = \emptyset$ for $n \gg 0$.

We first show that Z_n is 0-dimensional for $n \gg 0$. Let C_n be the 1-dimensional component of Z_n. The coordinate divisor D is σ-big by assumption, and nef by Lemma 4.5.5. By Lemma 4.5.4, we know that there is some effective N such that for all $m \gg 0$, both $\Delta_m - N$ and $\Delta_m - \sigma^{-(m-m_0)}N$ are base point free. Thus $C_m \subseteq N \cap \sigma^{m-m_0}N$ for all $m \gg 0$, and so for all $m \gg 0$, C_m is a union of components of N that are of finite order under σ. Now,

$$C_m \subseteq \mathrm{Bs}(|\Delta_m|) \subseteq \Omega \cup \mathrm{Bs}(|\Delta_m - \Omega|).$$

As $\mathrm{Bs}(|\Delta_m - \Omega|) \subseteq \mathrm{Bs}(\overline{R}_n) = W_n$ is 0-dimensional, we also have that $C_m \leq \Omega$. Since Ω has no components of finite order, $C_m = 0$ for $m \gg 0$.

By passing to a Veronese subring, and replacing D by some Δ_k and σ by σ^k, we may assume that Z_n is 0-dimensional for all $n \geq 1$. Let $\phi = \phi_{|D|}$ be the rational map from X to some $\mathbb{P}^N$ defined by the complete linear system $|D|$. Let Y be the closure of $\phi(X)$; we will abuse notation and refer to ϕ as a rational map from X to Y. Note that ϕ is birational by assumption, as $\overline{R}_1 \subseteq H^0(\mathcal{O}_X(D))$ generates K.

By blowing up the finite base locus of $|D|$, we obtain a surface X' and a diagram of birational maps

$$\begin{tikzcd} & X' \arrow[dl, "\pi"'] \arrow[dr, "\phi'"] & \\ X \arrow[rr, dashed, "\phi"'] & & Y \end{tikzcd}$$

such that π and ϕ' are morphisms. Let C be a reduced and irreducible hyperplane section of Y that avoids the finitely many points with positive-dimensional preimage in X or X' and does not contain any component of the singular locus of Y. Such C exist by Bertini's theorem and [Har77, Remark III.7.9.1]. Then $\pi(\phi')^{-1}(C) = D'$ is a reduced and irreducible curve that is linearly equivalent to D. Without loss of generality, we may replace D by D' and assume that D is reduced and irreducible.

We will show that $\mathcal{O}_X(\Delta_m)$ is globally generated for all $m \gg 0$. The proof is based on repeated applications of the following long exact cohomology sequence. Let B be an effective divisor on X and let A and A' be divisors such that $A' \sim A - B$. Then the exact sequence

$$0 \to \mathcal{O}_X(A') \to \mathcal{O}_X(A) \to \mathcal{O}_B(A) \to 0$$

induces a long exact cohomology sequence

$$(4.5.7)\quad 0 \to H^0(\mathcal{O}_X(A')) \to H^0(\mathcal{O}_X(A)) \to H^0(\mathcal{O}_B(A)) \\ \to H^1(\mathcal{O}_X(A')) \to H^1(\mathcal{O}_X(A)) \to H^1(\mathcal{O}_B(A)).$$

In particular, for all $m \geq 0$ there are homomorphisms

$$H^1(\mathcal{O}_X(\sigma^{-1}(\Delta_m))) \to H^1(\mathcal{O}_X(\Delta_{m+1})) \to H^1(\mathcal{O}_D(\Delta_{m+1})).$$

Now D is irreducible and $D.\Delta_m = mD^2$, as σ is numerically trivial. Since D is big and nef, $D^2 > 0$ by [Laz04, Theorem 2.2.16]. Applying Theorem 2.5.3, there is an integer m_0 such that if $m \geq m_0$, then $H^1(\mathcal{O}_D(\Delta_m)) = 0$. Thus if $m \geq m_0$ we have that $h^1(\mathcal{O}_X(\Delta_m)) = h^1(\mathcal{O}_X(\sigma^{-1}\Delta_m)) \geq h^1(\mathcal{O}_X(\Delta_{m+1}))$. Therefore, there are some $m_1 \geq m_0$ and some non-negative integer a such that if $m \geq m_1$, we have that

$$h^1(\mathcal{O}_X(\Delta_m)) = a.$$

Applying Theorem 2.5.3 again, we see that by possibly increasing m_1 further, we may also assume that if H is any divisor on X with $D.H \geq m_1 D^2$, then for any j, $\mathcal{O}_{\sigma^j D}(H)$ is globally generated and $H^1(\mathcal{O}_{\sigma^j D}(H)) = 0$.

Suppose that $m \geq 2m_1$. We claim that $\mathcal{O}_X(\Delta_m)$ is globally generated; that is, $\mathrm{Bs}(|\Delta_m|) = 0$. Since $\mathrm{Bs}(|\Delta_m|) \subseteq D \cup \sigma^{-1}(D) \cup \cdots \cup \sigma^{-(m-1)}(D)$, it is enough to show that $\mathcal{O}_X(\Delta_m)$ is globally generated at every point in $\sigma^{-i}(D)$ for $i = 0 \ldots m-1$.

We claim that for any such i, we have that

$$(4.5.8)\qquad h^1(\mathcal{O}_X(\Delta_m - \sigma^{-i}(D))) = a.$$

We will do the case when $m = 2m'$ for $m' \geq m_1$ and $i \leq m' - 1$; similar arguments work for other choices for m and i. For $j = 0 \ldots i$, let

$$C_j = \Delta_m - \sigma^{-i}(D) - \cdots - \sigma^{-j}(D).$$

Define $C_{i+1} = \Delta_m$. Thus for $j = 0 \ldots i$, we have $C_j = C_{j+1} - \sigma^{-j}(D)$. For all $j = 0 \ldots i$, we have $C_{j+1} \geq \sigma^{-m'}\Delta_{m'}$. Thus $\sigma^{-j}D \cdot C_{j+1} \geq m_1 D^2$ and so by the choice of m_1 we have that $H^1(\mathcal{O}_{\sigma^{-j}D}(C_{j+1})) = 0$. Thus the long exact cohomology sequence (4.5.7) gives an exact sequence

$$H^1(\mathcal{O}_X(C_j)) \to H^1(\mathcal{O}_X(C_{j+1})) \to H^1(\mathcal{O}_{\sigma^{-j}D}(C_{j+1})) = 0.$$

We obtain that

$$h^1(\mathcal{O}_X(C_0)) \geq h^1(\mathcal{O}_X(C_1)) \geq \cdots \geq h^1(\mathcal{O}_X(C_i)) \geq h^1(\Delta_m) = a.$$

Since $C_0 = \Delta_m - \Delta_{i+1} = \sigma^{-(i+1)}(\Delta_{m-i-1})$ and $m - i - 1 \geq m' \geq m_1$, we have that $h^1(\mathcal{O}_X(C_0)) = a$; so $h^1(\mathcal{O}_X(C_i)) = a$. The claim (4.5.8) is proved.

Now let $0 \leq i \leq m-1$ be arbitrary. As a special case of (4.5.7), we obtain the long exact sequence

$$0 \to H^0(\mathcal{O}_X(\Delta_m - \sigma^{-i}(D))) \to H^0(\mathcal{O}_X(\Delta_m)) \xrightarrow{\phi} H^0(\mathcal{O}_{\sigma^{-i}(D)}(\Delta_m)) \to$$
$$H^1(\mathcal{O}_X(\Delta_m - \sigma^{-i}(D))) \to H^1(\mathcal{O}_X(\Delta_m)) \to H^1(\mathcal{O}_{\sigma^{-i}(D)}(\Delta_m)).$$

By assumption on m, we have $H^1(\mathcal{O}_{\sigma^{-i}D}(\Delta_m)) = 0$, and we have seen that

$$h^1(\mathcal{O}_X(\Delta_m - \sigma^{-i}(D))) = h^1(\mathcal{O}_X(\Delta_m)) = a.$$

Thus the map

$$\phi : H^0(\mathcal{O}_X(\Delta_m)) \to H^0(\mathcal{O}_{\sigma^{-i}(D)}(\Delta_m))$$

is surjective. Since we have taken m sufficiently large so that $\mathcal{O}_{\sigma^{-i}(D)}(\Delta_m)$ is globally generated, $\mathrm{Bs}(|\Delta_m|)$ must be disjoint from $\sigma^{-i}(D)$. Since this holds for all i, we see that $|\Delta_m|$ is base point free. $\square$

We are almost ready to construct the ample model for R. We first prove two lemmas about birational maps.

Lemma 4.5.9. *Let D be a Cartier divisor on a normal projective variety X and let $V \subseteq |D|$ be a subspace of dimension $d \geq 2$. Let $\phi = \phi_V$ be the rational map to $\mathbb{P}^{d-1}$ defined by V, and let Γ be an irreducible curve on X that is disjoint from the base locus of V with respect to D. Then ϕ contracts Γ if and only if $D \cdot \Gamma = 0$. Further, if ϕ contracts Γ, then for any $v \in V$, either v never vanishes on Γ or $v|_\Gamma \equiv 0$.*

Proof. Suppose that ϕ contracts Γ to a point. By making a linear change of coordinates, without loss of generality we may assume that $\phi(\Gamma) = [1 : 0 : \cdots : 0]$. This is the same as choosing a basis $\{v_1, \ldots, v_d\}$ of V such that $v_1|_\Gamma$ is never 0 and that $v_i|_\Gamma \equiv 0$ for all $i \geq 2$. In particular, the divisor of zeroes of v_1 is disjoint from Γ and so $D.\Gamma = 0$.

Conversely, suppose that $D.\Gamma = 0$. Then choose $x, y \in \Gamma$ and $v \in V$. If $v(x) \neq 0$ but $v(y) = 0$ then we have that $\Gamma.D > 0$; thus v vanishes at some point of Γ if and only if $v|_\Gamma \equiv 0$. Now, since Γ does not meet $\mathrm{Bs}^D(V)$, there is some $v \in V$ such that $v(x) \neq 0$. We may choose a basis $\{v, v_2 \ldots, v_d\}$ for V such that $v_i(x) = 0$ for all $i \geq 2$. By the above, in these coordinates $\phi(\Gamma) = [1 : 0 : \cdots : 0]$. $\square$

We obtain as a corollary that any curve Γ such that $\Gamma.\Delta_n = 0$ must be disjoint from the gap divisor Ω and from the base loci W_m.

Corollary 4.5.10. *Assume Assumption-Notation 4.4.15. Suppose that $|\Delta_n|$ is base point free. Let ϕ_n be the morphism to projective space defined by $|\Delta_n|$. If ϕ_n contracts an irreducible curve Γ, then there is some $f \in \overline{R}_n$ so that*

$$(\mathrm{div}_X(f) + \Delta_n) \cap \Gamma = (\mathrm{div}_X(f) + \Omega + (\Delta_n - \Omega)) \cap \Gamma = \emptyset.$$

In particular, $W_n \cup \Omega$ is disjoint from Γ.

Proof. As X is nonsingular, we may identify Cartier and Weil divisors. Lemma 4.5.9 implies that the set of irreducible curves contracted by ϕ_n is precisely the set of irreducible curves Γ with $\Gamma.\Delta_n = 0$. As σ is numerically trivial, $\Gamma.\Delta_n = 0$ if and only if $\sigma\Gamma.\Delta_n = 0$. Thus the set of curves contracted by the morphism ϕ_n is σ-invariant.

By assumption ϕ_n is birational onto its image. Thus there are finitely many such curves and so all are of finite order under σ. In particular, if Γ is such a curve, then $\Gamma \not\leq \Omega$.

Now, set-theoretically we have

$$\operatorname{Supp}(\Omega \cup W_n) = \bigcap_{f \in \overline{R}_n} \operatorname{div}_X(f) + \Delta_n. \tag{4.5.11}$$

Fix an irreducible curve Γ with $\Delta_n.\Gamma = 0$. As $\Gamma \not\subseteq \operatorname{Supp}(\Omega \cup W_n)$, we have some $f \in \overline{R}_n$ so that $\Gamma \not\leq \operatorname{div}_X(f) + \Delta_n$. Thus $\Gamma \cap \operatorname{div}_X(f) + \Delta_n = \emptyset$ by Lemma 4.5.9. By (4.5.11), $\Omega \cap \Gamma = \emptyset$ and $W_n \cap \Gamma = \emptyset$. □

Lemma 4.5.12. (Compare [AS95, Lemma 3.2].) *Let X be a normal variety, and let G_1, G_2, and G_3 be effective (Cartier) divisors on X; let $E = G_3 - G_1 - G_2$. For $i = 1 \dots 3$, let $U_i \subseteq |G_i|$ be a vector space of dimension at least 2, and suppose that $U_1U_2 \subseteq U_3$. Let $\phi_i : X \to \mathbb{P}^{N_i}$ be the rational map defined by the sections U_i of G_i and let Y_i be the closure of $\operatorname{Im}\phi_i$ in $\mathbb{P}^{N_i}$. Further assume that $\phi_3 : X \to Y_3$ is birational. Then there is an induced rational map $\pi : Y_3 \to Y_1$ so that $\pi\phi_3 = \phi_1$ and so that if $x \notin \operatorname{Bs}^{G_i}(U_i)$ for $i = 1 \dots 3$ and $x \notin \operatorname{Supp} E$, then π is defined at $\phi_3(x)$.*

Proof. We repeat the proof of [AS95, Lemma 3.2], to note that it works in our situation as well. As rational maps, $\pi = \phi_1(\phi_3)^{-1}$. Let $x \in X \smallsetminus (\operatorname{Supp} E \cup \operatorname{Bs}^{G_1}(U_1) \cup \operatorname{Bs}^{G_2}(U_2) \cup \operatorname{Bs}^{G_3}(U_3))$; then all the maps ϕ_i are defined at x. We may thus choose elements $u_0 \in U_1$ and $v \in U_2$ so that, locally at x, $D_1 = -\operatorname{div}_X(u_0)$ and $D_2 = -\operatorname{div}_X(v)$. Our assumptions imply that, locally at x, $D_3 = -\operatorname{div}_X(u_0v)$. Let $\{u_0, \dots, u_r\}$ be a basis for U_1. Locally at x, ϕ_1 is defined by $[u_0 : \cdots : u_r]$; we may also define it by $[u_0v : \cdots : u_rv]$. Then if $\{u_0v, \dots, u_rv, w_{r+1}, \dots, w_s\}$ is a basis for U_3, then the rational map π is given by projection onto the first $r+1$ coordinates. This is defined locally at $\phi_3(x)$ by construction. $\square$

Theorem 4.5.13. *Assume Assumption-Notation 4.4.15. Then there are a normal surface X', a birational morphism $\theta : X \to X'$, and an ample invertible sheaf $\mathcal{L}'$ on X' such that for some $k \geq 1$, σ^k is conjugate to a numerically trivial automorphism σ' of X' and $\theta^*(\mathcal{L}') \cong \mathcal{L}_k$. In particular, $\mathcal{L}'$ is σ'-ample.*

Let Ω' be the gap divisor of $R^{(k)}$ on X'. Then Ω' is Cartier and contains no points or components of finite order. Furthermore, for all $n \geq 1$, the base locus of $\overline{R}_{nk}$ on X' contains no points of finite order.

Proof. For all n, let α_n be the rational map from X to some projective space given by $|\Delta_n|$; let X_n be the closure of the image of X under α_n. By Theorem 4.5.6, we may replace R by a Veronese subring to assume that $|\Delta_n|$ is base point free for all $n \geq 1$, so α_n is a birational morphism for all $n \geq 1$. Assumption-Notation 4.4.15 continues to hold.

For all n, we have $\Delta_n + \sigma^{-n}D = \Delta_{n+1}$ and $|\Delta_n| \cdot |\sigma^{-n}D| \subseteq |\Delta_{n+1}|$. Using Lemma 4.5.12 with $E = 0$, for each $n \geq 1$ we obtain a birational morphism $\pi_n : X_{n+1} \to X_n$ so that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_{n+1}"] \arrow[dr, "\alpha_n"'] & X_{n+1} \arrow[d, "\pi_n"] \\ & X_n \end{tikzcd}$$

commutes. Likewise, the equation $D + \sigma^{-1}\Delta_n = \Delta_{n+1}$ gives a birational morphism $\rho_n : X_{n+1} \to X_n$ so that

$$\begin{tikzcd} X \arrow[r, "\alpha_{n+1}"] \arrow[dr, "\alpha_n \circ \sigma"'] & X_{n+1} \arrow[d, "\rho_n"] \\ & X_n \end{tikzcd}$$

commutes.

Let Γ be an irreducible curve on X. Then, as σ is numerically trivial,

$$\Delta_{n+1}.\Gamma = (n+1)D.\Gamma = \frac{n+1}{n}\Delta_n.\Gamma,$$

so $\Delta_{n+1}.\Gamma = 0$ if and only if $\Delta_n.\Gamma = 0$. By Lemma 4.5.9, α_{n+1} and $\alpha_n = \pi_n \circ \alpha_{n+1}$ contract the same curves; thus $\pi_n : X_{n+1} \to X_n$ does not contract any curves and is a finite morphism. Likewise, ρ_n is a finite morphism. By finiteness of the integral closure, there is some k such that if $n \geq k$, then both π_n and ρ_n are isomorphisms.

Let $\overline{X} = X_k$, and let $\alpha = \alpha_k : X \to \overline{X}$. Define $\overline{\sigma} = (\rho_k \pi_k^{-1})^k$. Then $\overline{\sigma}$ is an automorphism of $\overline{X}$, and we have that

$$\overline{\sigma} \circ \alpha = \alpha \circ \sigma^k.$$

Clearly $\overline{\sigma}$ is numerically trivial.

Let $\pi : X' \to \overline{X}$ be the normalization of $\overline{X}$. Since X is normal by assumption, the morphism α factors through π — that is, there is a birational morphism $\theta : X \to X'$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\theta"] \arrow[dr, "\alpha"'] & & X' \arrow[dl, "\pi"] \\ & \overline{X} & \end{tikzcd}$$

commutes. Note that if θ is finite at $x \in X$, then θ is a local isomorphism at x. By the universal property of normalizations, $\overline{\sigma}$ lifts uniquely to an automorphism σ' of X', which is also numerically trivial.

By construction, $\overline{X}$ carries a very ample line bundle $\overline{\mathcal{L}}$ such that

$$\mathcal{L}_k \cong \mathcal{O}_X(\Delta_k) \cong \alpha^*\overline{\mathcal{L}} \cong \theta^*\pi^*\overline{\mathcal{L}}.$$

Let $\mathcal{L}' = \pi^*\overline{\mathcal{L}}$. Then $\mathcal{L}'$ is the pullback of an ample line bundle by a finite map and so is ample by [Har77, Exercise III.5.7(d)]. Further, $\mathcal{L}'$ is σ'-ample by [AV90, Theorem 1.7]. By the projection formula [Har77, Exercise II.5.1.(d)], $\theta_*(\mathcal{L}_k) \cong \mathcal{L}'$.

Let C be the union of the finitely many curves in X that are contracted by θ. Note that θ is an isomorphism from the open subset $X \smallsetminus C$ of X onto an open subset of X'. Note also that, by Corollary 4.5.10, $\Omega \subseteq X \smallsetminus C$. Let $\Omega' = \theta(\Omega)$ be the scheme-theoretic image of Ω. Thus Ω' is a Cartier divisor on X'.

Let D' be the Cartier divisor on X' corresponding to the invertible sheaf $\mathcal{L}'$. The singular locus of X' consists of finitely many points, as X' is normal. Fix $n \geq 1$. By restricting the Weil divisor $D^{X'}(\overline{R}_{nk})$ to the open set where X' is smooth, we obtain that

$$D^{X'}(\overline{R}_{nk}) = D' + (\sigma')^{-1}(D') + \cdots + (\sigma')^{-(n-1)}(D') - \Omega'.$$

By Lemma 4.3.20, Ω' is the gap divisor of $R^{(k)}$ on X' associated to z. That Ω' contains no points or components of finite order follows directly from the corresponding properties for Ω.

Fix $n \geq 1$. We have seen that $D^{X'}(\overline{R}_{nk})$ is Cartier. Let Z_n be the base locus of $\overline{R}_{nk}$ on X'. Let $x \in X'$ be a point of finite order under σ', and let Γ be an irreducible component of $\theta^{-1}(x)$. If Γ is a curve, then by Corollary 4.5.10, there is some $f \in \overline{R}_{nk}$ so that $\Gamma \cap (\mathrm{div}_X(f) + (\Delta_{nk} - \Omega)) = \emptyset$. If $\Gamma = \{p\}$ is a point, then it is of finite order, and so by assumption $p \notin W_n$. Again, there is an $f \in \overline{R}_{nk}$ so that $p \notin \mathrm{div}_X(f) + (\Delta_{nk} - \Omega)$. In either case, f gives a section of $\mathcal{L}'_n(-\Omega')$ that does not vanish at x, so $x \notin Z_n$. Thus Z_n contains no points of finite order. □

We comment that in the commutative setting, $\overline{X}$ would be normal automatically; see [Laz04, Theorem 2.1.27, Example 2.1.15]. We do not know if this is true for our construction.

4.6 Stabilizing 0-dimensional data

We are ready to start working with the infinite order 0-dimensional data defining X. In this section, we construct an ADC ring $S(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$ so that (some Veronese of) R is a subring of S, and give surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ such that the bimodule algebras $\mathcal{R}(X)$ and $\mathcal{T}(\mathbb{D})$ are equal up to finite dimension.

By Theorem 4.5.13, by replacing R by a further Veronese subring, we may without loss of generality make the following assumptions:

Assumption-Notation 4.6.1. *We assume that R is a birationally commutative projective surface with function field K and fix $0 \neq z \in R_1$. Let $\overline{R}_n = R_n z^{-n}$, and assume that $\overline{R}_1$ generates K as a field. Let (X, σ) be a normal model for R with σ numerically trivial, and let $\mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$. Assume also that there are an ample and σ-ample invertible sheaf $\mathcal{L}$ on X, an effective locally principal Weil divisor Ω on X containing no points or components of finite order under σ, and 0-dimensional subschemes W_n of X, disjoint from finite σ-orbits, such that for all $n \geq 1$, $\mathcal{R}_n(X) = \mathcal{I}_{W_n} \mathcal{I}_\Omega \mathcal{L}_n$.*

To begin, we show that our assumptions imply that Ω meets orbits only finitely often.

Proposition 4.6.2. *Assume Assumption-Notation 4.6.1. Let $p \in X$ be a point of infinite order under σ; let $O(p)$ denote the σ-orbit of p. Then Ω intersects $O(p)$ only finitely often.*

Proof. Suppose that $O(p) \cap \Omega$ is infinite. We will show that R is not noetherian.

First suppose that for infinitely many $d \leq 0$, we have $\sigma^d(p) \in \Omega$. By Lemma 4.4.1 there is a finite set V such that, for all $n \geq 1$, we have $W_n \subseteq V \cup \cdots \cup \sigma^{-(n-1)} V$. We define a point q as follows: if $O(p) \cap V = \emptyset$, let $q = p$. If $O(p)$ meets V, let

$c = \min\{d \mid \sigma^d(p) \in V\}$ and let $q = \sigma^{c-1}(p)$. In either case, for all $n \geq 1$ and $1 \leq m \leq n$, we have $\sigma^{-n}(q) \notin W_m$.

Define a left ideal J of R by letting $J = \bigoplus \overline{J}_n z^n$, where

$$\overline{J}_n = H^0(\mathcal{L}_n \cdot \mathcal{I}_q^{\sigma^n}) \cap \overline{R}_n.$$

If $\sigma^{-n}(q) \in \Omega$, then $\mathcal{R}_n \subseteq \mathcal{L}_n \mathcal{I}_q^{\sigma^n}$ and so $J_n = R_n$. On the other hand, since $\sigma^{-n}(q) \notin W_n = \mathrm{Bs}(\overline{R}_n)$ by construction, if $\sigma^{-n}(q) \notin \Omega$ then there is some section of $\mathcal{L}_n$ in $\overline{R}_n$ that does not vanish at $\sigma^{-n}(q)$. Thus $J_n \subsetneqq R_n$. That is, $J_n = R_n$ if and only if $\sigma^{-n}(q) \in \Omega$.

For all $i < n$ we have $R_{n-i} J_i \subseteq H^0(\mathcal{I}_\Omega \cdot \mathcal{I}_q^{\sigma^n} \mathcal{L}_n) z^n$. Fix $m \geq 1$ and $n > m$ such that $\sigma^{-n}(q) \in \Omega$. Then

$$(R \cdot J_{\leq m})_n \subseteq H^0(\mathcal{I}_\Omega \cdot \mathcal{I}_q^{\sigma^n} \mathcal{L}_n) z^n.$$

As $\sigma^{-n}(q) \notin W_n = \mathrm{Bs}(\overline{R}_n)$, we have that

$$H^0(\mathcal{I}_\Omega \cdot \mathcal{I}_q^{\sigma^n} \mathcal{L}_n) \neq \overline{R}_n = \overline{J}_n.$$

Thus J is not finitely generated.

Now suppose that for infinitely many $d \geq 0$, we have $\sigma^d(p) \in \Omega$. Let

$$\mathcal{L}' = \mathcal{I}_\Omega \otimes \mathcal{L} \otimes (\mathcal{I}_\Omega^\sigma)^{-1}.$$

Then $\mathcal{R}_n = \mathcal{I}_{W_n} (\mathcal{I}_\Omega)^{\sigma^n} (\mathcal{L}')_n$. That is, R is also contained in a *left* idealizer at Ω inside $B' = B(X, \mathcal{L}', \sigma)$. Define a point $q' \in O(p)$ as follows: if $O(p) \cap V = \emptyset$, let $q' = p$. Otherwise, let $c = \max\{d \mid \sigma^d(p) \in V\}$, and let $q' = \sigma^{c+1}(p)$. Then $q' \notin W_m$ for any m. Let

$$J' = \bigoplus_{n \geq 0} \big(H^0(\mathcal{I}_{q'}(\mathcal{L}')_n) \cap \overline{R}_n\big) z^n.$$

Then J' is a right ideal of R, and a symmetric argument to the one above shows that J' is not finitely generated. $\square$

We now analyze the 0-dimensional schemes $(\Omega \cup W_n) \cap O(p)$. To simplify our computations, we will pass to a Veronese subring so that our data may be presented in a standard form. That we may do so is the content of the following elementary lemma.

For any $k \geq 1$, and for any $p \in X$, we will let $O_k(p)$ denote the σ^k-orbit of p.

Lemma 4.6.3. *Assume Assumption-Notation 4.6.1. Then there is some positive integer k such that, for any $p \in X$, either $O_k(p)$ is disjoint from all W_n or there is a point $q \in O_k(p)$ so that $O_k(p) \cap \Omega \subseteq \{q\}$ and*

$$\{q\} \subseteq \big(O_k(p) \cap (\Omega \cup W_k)\big) \subseteq \{q, \sigma^{-k}(q)\}.$$

We first prove:

Sublemma 4.6.4. *Suppose that q is a point of infinite order and that*

$$(\Omega \cup W_1) \cap O(q) \subseteq \{q, \dots, \sigma^{-s}(q)\}.$$

Then

$$(\Omega \cup W_n) \cap O(q) \subseteq \{q, \sigma^{-1}(q), \dots, \sigma^{-(n+s-1)}(q)\}$$

for all $n \geq 1$.

Proof. It clearly suffices to prove that $W_n \cap O(q) \subseteq \{q, \dots, \sigma^{-(n+s-1)}(q)\}$. But by (4.4.3),

$$W_n \cap O(q) \subseteq \big((\Omega \cup W_1) \cup \dots \cup \sigma^{-(n-1)}(\Omega \cup W_1)\big) \cap O(q) \subseteq \{q, \dots, \sigma^{-(n-1)-s}(q)\}.$$

□

Proof of Lemma 4.6.3. By Lemma 4.4.1 we know that

$$\bigcup_{n \geq 1} W_n$$

is contained in finitely many infinite σ-orbits. By Proposition 4.6.2 each of those orbits meets Ω only finitely often. Thus there is some $s \geq 1$ such that for any $p \in \bigcup_{n\geq 1} W_n$, we have that

$$(\Omega \cup W_1) \cap O(p) \subseteq \{\sigma^{-i}(p), \sigma^{-(i+1)}(p), \ldots \sigma^{-(i+s)}(p)\}$$

for some $i \in \mathbb{Z}$.

Let p be a point of $\bigcup_{n\geq 1} W_n$. Let

$$m = \max\{n \in \mathbb{Z} \mid \sigma^n(p) \in \Omega \cup W_1\},$$

and let $q = \sigma^m(p)$. Then the hypotheses of Sublemma 4.6.4 hold, and therefore

$$(\Omega \cup W_n) \cap O(q) \subseteq \{q, \ldots, \sigma^{-(n+s-1)}(q)\}$$

for all $n \geq 1$. Thus, for any $n \geq s$ and any $0 \leq i \leq n-1$, we have that

$$\big((\Omega \cup W_n) \cap O_n(\sigma^{-i}(q))\big) \subseteq \{\sigma^{-i}(q), \sigma^{-(i+n)}(q)\}.$$

In particular, for $i = 0 \ldots 2s-1$, as

$$O_{2s}(\sigma^{-i}(q)) \subset O_s(\sigma^{-i}(q)),$$

we have

$$\Omega \cap O_{2s}(\sigma^{-i}(q)) \subseteq \{\sigma^{-i}(q), \sigma^{-(i+2s)}(q)\} \cap \{\sigma^{-i}(q), \sigma^{-(i+s)}(q)\} = \{\sigma^{-i}(q)\}.$$

The lemma holds for $k = 2s$. $\square$

Lemma 4.6.3 allows us to replace R by a Veronese subring so that without loss of generality we may make the following assumptions:

Assumption-Notation 4.6.5. *We assume that R is a birationally commutative projective surface with function field K and fix $0 \neq z \in R_1$. Let $\overline{R}_n = R_n z^{-n}$,* *and assume that $\overline{R}_1$ generates K as a field. Let (X,σ) be a normal model for R with σ numerically trivial, and let $\mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$. Assume also that there are an ample and σ-ample invertible sheaf $\mathcal{L}$ on X, an effective locally principal Weil divisor Ω on X containing no points or components of finite order under σ, and 0-dimensional subschemes W_n of X, disjoint from finite σ-orbits, such that for all $n \geq 1$, $\mathcal{R}_n(X) = \mathcal{I}_{W_n}\mathcal{I}_\Omega\mathcal{L}_n$.*

In addition, we assume that for any orbit $O(p)$ that meets $\bigcup_{n\geq 1} W_n$, there is some $q \in O(p)$ such that

$$\{q\} \subseteq O(p) \cap (W_1 \cup \Omega) \subseteq \{q, \sigma^{-1}(q)\},$$

and $O(p) \cap \Omega \subseteq \{q\}$.

Lemma 4.6.6. *Assume Assumption-Notation 4.6.5. Let $p \in \bigcup_{n\geq 1} W_n$; note that Sublemma 4.6.4 implies that $\Omega \cup W_1$ must therefore meet $O(p)$. Let $\mathcal{O} = \mathcal{O}_{X,p}$ be the local ring of X at p, with maximal ideal $\mathfrak{p}$. For all $j \geq 1$ and all $i \in \mathbb{Z}$, define $\mathfrak{m}_i^j$ to be the stalk of the ideal sheaf $\mathcal{R}_j\mathcal{L}_j^{-1} = \mathcal{I}_\Omega\mathcal{I}_{W_n}$ at $\sigma^{-i}(p)$, considered as an ideal in $\mathcal{O}$ via the isomorphism $\sigma^i : \mathcal{O}_{X,\sigma^{-i}p} \to \mathcal{O}$. Our assumptions imply that by reindexing the orbit of p if necessary, we may assume that $\mathfrak{m}_i^1 = \mathcal{O}$ for all $i < 0$ and $i > 1$, that $\mathfrak{m}_0^1 \neq \mathcal{O}$, and that $\Omega \cap O(p) \subseteq \{p\}$.*

Then there are integers $t, N \geq 1$, ideals $\mathfrak{a}_1, \dots \mathfrak{a}_{t-1}, \mathfrak{d}, \mathfrak{c}_{t-1}, \dots \mathfrak{c}_0$ of $\mathcal{O}$ that are either $\mathfrak{p}$-primary or equal to $\mathcal{O}$, and an ideal $\mathfrak{a}_0$ of $\mathcal{O}$ so that for all $n \geq N$, we have:

$$\mathfrak{m}_i^n = \mathfrak{a}_i \text{ for } 0 \leq i < t$$

$$\mathfrak{m}_i^n = \mathfrak{d} \text{ for } t \leq i \leq n-t$$

$$\mathfrak{m}_i^n = \mathfrak{c}_{n-i} \text{ for } n-t < i \leq n$$

$$\mathfrak{m}_i^n = \mathcal{O} \text{ for } i < 0 \text{ and } i > n.$$

Further, we have $\mathfrak{a}_0\mathfrak{c}_0 \subseteq \mathfrak{d}$ *and*

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_{t-1} \subseteq \mathfrak{d} \supseteq \mathfrak{c}_{t-1} \supseteq \cdots \supseteq \mathfrak{c}_1.$$

Proof. Define $\mathcal{T}$ to be the graded $(\mathcal{O}_X, \sigma)$-bimodule algebra

$$\mathcal{T} = \bigoplus_{n \geq 0} \mathcal{R}_n \mathcal{L}_n^{-1}.$$

Then $\mathcal{T}$ is a sub-bimodule-algebra of $\mathcal{B}(X, \mathcal{O}_X, \sigma)$, and $\mathfrak{m}_i^j$ is by definition the stalk of $\mathcal{T}_j$ at $\sigma^{-i}(p)$. Let $\mathcal{K} = \mathcal{T}_+$. For all n, the cosupport of $\mathcal{K}_n$ is (set-theoretically) equal to $\Omega \cup W_n$; by assumption, this meets $O(p)$ in at most finitely many points.

Thus $\mathcal{T}$, $\mathcal{K}$, and p satisfy the hypotheses of Sublemma 4.2.20, with $s = 0$ or 1. Let N, a', b', a, and b be the integers given by Sublemma 4.2.20. Note that by Sublemma 4.6.4, if $i < 0$ or $i > n$ then $\mathfrak{m}_i^n = \mathcal{O}$. Thus we may take $a' = b' = 0$. Let $t = \max\{a, b\}$. For $i = 0, \ldots t-1$ let $\mathfrak{a}_i = \mathfrak{m}_i^N$ and let $\mathfrak{c}_i = \mathfrak{m}_{N-i}^N$. Let $\mathfrak{d} = \mathfrak{m}_t^N$ be the central stalk of $\mathcal{K}$.

Since $O(p) \cap \Omega \subseteq \{p\}$ by assumption, if $i \neq 0$ the stalk of $\mathcal{T}_n$ at $\sigma^{-i}(p)$ is either $\mathcal{O}$ or is $\mathfrak{p}$-primary. Because $\mathcal{T}_n \mathcal{T}_n^{\sigma^n} \subseteq \mathcal{T}_{2n}$, we have that

$$\mathfrak{m}_n^n \mathfrak{m}_0^m \subseteq \mathfrak{m}_n^{2n}.$$

By taking $n \gg 0$ this relation gives that $\mathfrak{a}_0\mathfrak{c}_0 \subseteq \mathfrak{d}$. The rest of the conclusions of the lemma follow directly from Sublemma 4.2.20. $\square$

We may now give the defining data for the bimodule algebra $\mathcal{R} = \mathcal{R}(X)$.

Definition 4.6.7. We will say that the surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ is *normal* if X is normal, σ is numerically trivial, $\mathcal{L}$ is ample and σ-ample, Ω contains no points or components of finite order under σ, and Λ and Λ' are disjoint from finite σ-orbits. In particular, if $\mathbb{D}$ is normal, then Ω is locally principal.

Theorem 4.6.8. *Let R be a birationally commutative projective surface. Then there are normal surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ and integers $N, k \geq 1$ so that*

$$(\mathcal{R}(X)^{(k)})_{\geq N} = \mathcal{T}(\mathbb{D})_{\geq N}.$$

Proof. By Proposition 4.4.10, after replacing R by a Veronese subring we may assume that we are in the situation of Assumption-Notation 4.4.15. By Theorem 4.5.13, by replacing R by a further Veronese subring and possibly changing X, we may assume that R and X satisfy Assumption-Notation 4.6.1.

By Lemma 4.6.3, we may replace R by a further Veronese subring to assume that for all p such that $O(p)$ meets W_1, there is a $q \in O(p)$ so that

$$\{q\} \subseteq O(p) \cap (\Omega \cup W_1) \subseteq \{q, \sigma^{-1}(q)\}$$

and $O(p) \cap \Omega \subseteq \{q\}$. By Sublemma 4.6.4,

$$O(p) \cap (\Omega \cup W_n) \subseteq \{q, \dots, \sigma^{-n}(q)\}$$

for all $n \geq 1$. By Lemma 4.4.1, there are only finitely many orbits to consider; that is, there are points $q^1, \dots, q^r$ with orbits $O^j = O(q^j)$, so that

$$\bigcup_{n \geq 1} W_n \subseteq \bigcup_{j=1}^{r} O^j.$$

For each O^j, let $\mathcal{A}^j, \mathcal{D}^j$, and $\mathcal{C}^j$ be the ideal sheaves that are cosupported at q^j and locally at q^j are equal to, respectively, the ideals $\mathfrak{a}_0$, $\mathfrak{d}$, and $\mathfrak{c}_0$ of $\mathcal{O}_{X,q^j}$ produced by Lemma 4.6.6. Define

$$\mathcal{A}' = \prod_j \mathcal{A}^j,$$

$$\mathcal{D} = \prod_j \mathcal{D}^j,$$

and

$$\mathcal{C}' = \prod_j \mathcal{C}^j.$$

Finally, choose ideal sheaves $\mathcal{A} \supseteq \mathcal{A}'$ and $\mathcal{C} \supseteq \mathcal{C}'$ such that the pair $(\mathcal{A}, \mathcal{C})$ is maximal with respect to the containment $\mathcal{AC} \subseteq \mathcal{D}$. Let $\mathcal{S}$ be the ADC bimodule algebra $\mathcal{S}(X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C})$.

For $j = 1 \dots r$ let t^j and N^j be the integers produced by Lemma 4.6.6 applied to O^j; let the ideals in $\mathcal{O}_{X,q^j}$ produced by Lemma 4.6.6 be $\mathfrak{d}^j$, $\mathfrak{a}_i^j$, and $\mathfrak{c}_i^j$ for $1 \leq i \leq t^j - 1$. Let $t = \max\{t^j\}$ and let $N = \max\{N^j, 2t\}$. For integers i with $t^j \leq i \leq t-1$, define $\mathfrak{a}_i^j = \mathfrak{c}_i^j = \mathfrak{d}^j$. By Lemma 4.6.6 we have

$$\mathfrak{a}_1^j \subseteq \cdots \subseteq \mathfrak{a}_{t-1}^j \subseteq \mathfrak{d} \supseteq \mathfrak{c}_{t-1}^j \supseteq \cdots \supseteq \mathfrak{c}_1^j$$

for all j.

We define an ideal sheaf $\mathcal{J} \subseteq \mathcal{I}_\Omega$ so that $\mathcal{I}_\Omega / \mathcal{J}$ is supported on

$$\{\sigma^{-i}(q^j) \mid 0 \leq i \leq t-1, 1 \leq j \leq r\}$$

by setting the stalk of $\mathcal{J}$ at $\sigma^{-i}(q^j)$ to be isomorphic to $\mathfrak{a}_i^j$. Similarly, we define an ideal sheaf $\mathcal{J}'$, cosupported on

$$\{\sigma^{i}(q^j) \mid 0 \leq i \leq t-1, 1 \leq j \leq r\},$$

by setting the stalk of $\mathcal{J}'$ at $\sigma^{i}(q^j)$ to be isomorphic to $\mathfrak{c}_i^j$. (Note that the definitions of $\mathcal{J}$ and $\mathcal{J}'$ differ by a sign!)

Let Λ be the subscheme defined by $\mathcal{I}_\Omega^{-1}\mathcal{J}$ and let Λ' be the subscheme defined by $\mathcal{J}'$. Let Z be the subscheme defined by $\mathcal{D}$. Then Λ, Λ', and Z are 0-dimensional and supported at points of infinite order. In fact, we have

$$\operatorname{Supp} \sigma^{-n}(\Lambda') \cap O^j \subseteq \{\sigma^{-(n-(t-1))}(q^j), \dots, \sigma^{-n}(q^j)\} \text{ for all } n \in \mathbb{Z} \tag{4.6.9}$$

and

$$\operatorname{Supp}(\Lambda \cup \Omega) \cap O^j \subseteq \{q^j, \dots, \sigma^{-(t-1)}(q^j)\}. \tag{4.6.10}$$

Further,

$$\mathcal{I}_{\Lambda'} \subseteq \mathcal{C}' \subseteq \mathcal{C}$$

and

$$\mathcal{I}_{\Omega}\mathcal{I}_{\Lambda} \subseteq \mathcal{A}' \subseteq \mathcal{A}.$$

Fix $n \geq N$, so that $t-1 < n-(t-1)$ and (4.6.9) and (4.6.10) imply that $\sigma^{-n}(\Lambda') \cap (\Omega \cup \Lambda) = \emptyset$. Now, if $0 \leq i \leq t-1$, then

$$(\mathcal{R}_n\mathcal{L}_n^{-1})_{\sigma^{-i}(q^j)} \cong \mathfrak{a}_i^j \cong (\mathcal{I}_{\Omega}\mathcal{I}_{\Lambda})_{\sigma^{-i}(q^j)} = \big((\mathcal{I}_{\Omega}\mathcal{I}_{\Lambda}\mathcal{I}_{\Lambda'}^{\sigma^n}) \cap \mathcal{S}_n\big)_{\sigma^{-i}(q^j)}.$$

If $n-(t-1) \leq i \leq n$, then

$$(\mathcal{R}_n\mathcal{L}_n^{-1})_{\sigma^{-i}(q^j)} \cong \mathfrak{c}_{n-i}^j \cong (\mathcal{I}_{\Lambda'}^{\sigma^n})_{\sigma^{-i}(q^j)} = \big((\mathcal{I}_{\Omega}\mathcal{I}_{\Lambda}\mathcal{I}_{\Lambda'}^{\sigma^n}) \cap \mathcal{S}_n\big)_{\sigma^{-i}(q^j)}.$$

And if $t \leq i \leq n-t$, then $\sigma^{-i}(q^j) \notin \Omega \cup \Lambda \cup \sigma^{-n}\Lambda'$ and so

$$(\mathcal{R}_n\mathcal{L}_n^{-1})_{\sigma^{-i}(q^j)} \cong \mathfrak{d}^j \cong \mathcal{D}_{\sigma^{-i}(q^j)} = \big((\mathcal{I}_{\Omega}\mathcal{I}_{\Lambda}\mathcal{I}_{\Lambda'}^{\sigma^n}) \cap \mathcal{S}_n\big)_{\sigma^{-i}(q^j)}.$$

Thus for $n \geq N$,

$$\mathcal{R}_n = \mathcal{S}_n \cap \mathcal{I}_{\Omega}\mathcal{I}_{\Lambda}\mathcal{I}_{\Lambda'}^{\sigma^n}\mathcal{L}_n.$$

That is, if $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$, then

$$\mathcal{R}_{\geq N} = \mathcal{T}(\mathbb{D})_{\geq N}.$$

$\square$

We record for future reference an elementary observation on how surface data transforms upon taking Veronese subrings.

Lemma 4.6.11. *Suppose that*

$$\mathbb{D}' = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

is surface data. Let $n \geq 1$, and let

$$\mathbb{D} = (X, \mathcal{L}_n, \sigma^n, \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}}, \mathcal{D}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}}, \mathcal{C}, \Omega, \Lambda, \Lambda').$$

Then $\mathbb{D}$ is surface data, and

$$\mathcal{T}(\mathbb{D}')^{(n)} = \mathcal{T}(\mathbb{D}).$$

Furthermore, if the surface data $\mathbb{D}'$ is normal, respectively transverse, then the surface data $\mathbb{D}$ is normal, respectively transverse.

Proof. This is an elementary computation, which we leave to the reader. □

Corollary 4.6.12. *Let R be a birationally commutative projective surface. Then there are normal surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ and an integer $\ell \geq 1$ so that*

$$(\mathcal{R}(X)^{(\ell)}) = \mathcal{T}(\mathbb{D}).$$

Proof. By Theorem 4.6.8, there are normal surface data $\mathbb{D}'$ and integers $k, N \geq 1$ so that

$$\mathcal{R}(X)^{(k)}_{\geq N} = \mathcal{T}(\mathbb{D}')_{\geq N}.$$

Let $\ell = kN$, and by Lemma 4.6.11 let $\mathbb{D}$ be the normal surface data corresponding to $\mathcal{T}(\mathbb{D}')^{(N)}$. Then

$$\mathcal{R}(X)^{(\ell)} = \mathcal{T}(\mathbb{D}')^{(N)} = \mathcal{T}(\mathbb{D}).$$

□

4.7 Transversality of the defining data

In Section 4.6, we constructed normal surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ such that (up to a Veronese) we have that $\mathcal{R}(X) = \mathcal{T}(\mathbb{D})$. In this section, we show that the data $\mathbb{D}$ is in fact transverse, and that $T(\mathbb{D})$ is a finite module over (a

Veronese of) R. This allows us to consider $T(\mathbb{D})$ as some sort of normalization of R, and further justifies the term "normal surface data."

Assumption-Notation 4.7.1. *We assume that R is a birationally commutative projective surface with $R_1 \neq 0$ and fix $0 \neq z \in R_1$. As usual, we define $\overline{R}_n = R_n z^{-n}$. In addition, we assume that $\overline{R}_1$ generates K as a field, and that there is surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$, normal in the sense of Definition 4.6.7, so that if $\mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$, then*

$$\mathcal{R}(X) = \mathcal{T}(\mathbb{D}).$$

We will continue to let W_n be the base locus of $\overline{R}_n$ for $n \geq 1$, so that W_n is defined by

$$\mathcal{I}_\Omega^{-1}(\mathcal{I}_\Omega \mathcal{I}_\Lambda \mathcal{I}_{\Lambda'}^{\sigma^n} \cap \mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n})$$

for all $n \geq 1$.

Assumption-Notation 4.7.1 implies in particular that if Z is the subscheme defined by $\mathcal{D}$, then

$$W_n \subseteq \Lambda \cup \sigma^{-n}\Lambda' \cup Z \cup \cdots \cup \sigma^{-n} Z$$

for all $n \geq 1$.

We first prove the unsurprising result that in this situation Ω has good transversality properties.

Lemma 4.7.2. *The set $\{\sigma^n \Omega\}_{n \in \mathbb{Z}}$ is critically transverse.*

Proof. By Lemma 3.3.12, it is sufficient to check that for any reduced and irreducible subscheme Y of X, we have

$$\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n \Omega}, \mathcal{O}_Y) = 0 \text{ for all } |n| \gg 0. \tag{4.7.3}$$

Let $Y \subseteq X$ be reduced and irreducible. If $Y = \{p\}$ is a point of infinite order, then by Proposition 4.6.2, $p \notin \sigma^n \Omega$ for $|n| \gg 0$, and so (4.7.3) holds for Y. If $Y = \{p\}$ is a point of finite order, then $\Omega \cap O(p) = \emptyset$ by assumption, so (4.7.3) also holds for Y. In particular, Ω is disjoint from the singular locus of X.

Thus if Y is a curve and (4.7.3) fails for Y, then Y must be contained in infinitely many $\sigma^n \Omega$. This is impossible, as Ω has no components of finite order under σ. □

We next prove two lemmas that will, in many cases, allow us to work with the full algebra $T(\mathbb{D})$ instead of the subalgebra R. The first is an easy generalization of a lemma of Rogalski and Stafford.

Lemma 4.7.4. (Compare [RS06, Lemma 9.3]) *Let X be a projective scheme with automorphism σ. Let $\{(\mathcal{R}_n)_{\sigma^n}\}$ be a left and right ample of sequence of bimodules on X such that for each n, the set where $\mathcal{R}_n$ is not locally free has dimension ≤ 0. Let $\mathcal{F}$ be a globally generated coherent sheaf on X and let $V \subseteq H^0(\mathcal{F})$ be a vector space that generates $\mathcal{F}$. Let $i \in \mathbb{Z}$. Then for $n \gg 0$, the natural homomorphism*

$$\alpha : V \otimes_k H^0(\mathcal{R}_n^{\sigma^i}) \to H^0(\mathcal{F} \otimes \mathcal{R}_n^{\sigma^i})$$

is surjective.

Proof. By assumption, there is an exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0.$$

Tensoring with $\mathcal{R}_n^{\sigma^i}$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{T}or_1^X(\mathcal{F}, \mathcal{R}_n^{\sigma^i}) \longrightarrow \mathcal{H} \otimes \mathcal{R}_n^{\sigma^i} \xrightarrow{\theta_n} V \otimes \mathcal{R}_n^{\sigma^i} \longrightarrow \mathcal{F} \otimes \mathcal{R}_n^{\sigma^i} \longrightarrow 0.$$

Let $\mathcal{K}_n = \operatorname{Im} \theta_n$. Our assumptions on $\mathcal{R}$ imply that $\mathcal{T}or_1^X(\mathcal{F}, \mathcal{R}_n^{\sigma^i})$ is supported on a set of dimension 0, and so $H^i(\mathcal{T}or_1^X(\mathcal{F}, \mathcal{R}_n^{\sigma^i})) = 0$ for all n and for all $i \geq 1$.

Thus $H^1(\mathcal{K}_n) \cong H^1(\mathcal{H} \otimes \mathcal{R}_n^{\sigma^i})$. By ampleness of $\{(\mathcal{R}_n)_{\sigma^n}\}$, this vanishes for $n \gg 0$. Then the exact sequence

$$0 \longrightarrow H^0(\mathcal{K}_n) \to V \otimes_k H^0(\mathcal{R}_n^{\sigma^i}) \xrightarrow{\alpha} H^0(\mathcal{F} \otimes \mathcal{R}_n^{\sigma^i}) \longrightarrow H^1(\mathcal{K}_n)$$

gives that α is surjective for $n \gg 0$. □

Lemma 4.7.5. *Let X be a projective scheme, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Suppose that R is a finitely generated graded subalgebra of $B(X, \mathcal{L}, \sigma)$. For all $n \geq 1$ let $\mathcal{R}_n \subseteq \mathcal{L}_n$ be the sheaf generated by the sections in $\overline{R}_n$. Let $T = \bigoplus_{n \geq 0} H^0(\mathcal{R}_n) z^n$.*

Suppose that for all n, the set where $\mathcal{R}_n$ is not locally free has dimension ≤ 0 and that the sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left and right ample. Then T is finitely generated as a left and right R-module.

Proof. By symmetry, it suffices to prove that ${}_R T$ is finitely generated.

Let k be such that

$$R_n = \sum_{i=1}^{k} R_i R_{n-i}$$

for all $n > k$. Then

$$\mathcal{R}_n = \sum_{i=1}^{k} \mathcal{R}_i \mathcal{R}_{n-i}^{\sigma^i}$$

for all $n > k$; taking global sections we have

$$T_n = \sum_{i=1}^{k} H^0(\mathcal{R}_i \mathcal{R}_{n-i}^{\sigma^i}) z^n \tag{4.7.6}$$

for all $n > k$.

For each $1 \leq i \leq k$, the sections in $\overline{R}_i$ generate $\mathcal{R}_i$. Applying Lemma 4.7.4, we obtain that there is some n_0, which we may take to be greater than k, so that the multiplication map

$$R_i \otimes T_{n-i} \to H^0(\mathcal{R}_i \otimes \mathcal{R}_{n-i}^{\sigma^i}) z^n$$

is surjective for $n \geq n_0$ and $1 \leq i \leq k$.

Now consider the exact sequence

$$0 \to \mathcal{J}_{i,n} \to \mathcal{R}_i \otimes \mathcal{R}_{n-i}^{\sigma^i} \to \mathcal{R}_i \mathcal{R}_{n-i}^{\sigma^i} \to 0.$$

The kernel $\mathcal{J}_{i,n}$ is supported at finitely many points, and so $H^1(\mathcal{J}_{i,n}) = 0$. Thus the induced map from $H^0(\mathcal{R}_i \otimes \mathcal{R}_{n-i}^{\sigma^i}) \to H^0(\mathcal{R}_i \mathcal{R}_{n-i}^{\sigma^i})$ is surjective. Therefore, for all $n \geq n_0$, the natural map

$$R_i \otimes T_{n-i} \to H^0(\mathcal{R}_i \mathcal{R}_{n-i}^{\sigma^i}) z^n$$

is surjective. Applying (4.7.6), we see that for $n \geq n_0$,

$$T_n = \sum_{i=1}^{k} R_i T_{n-i}.$$

By induction, T is generated as a left R-module by $T_{\leq n_0}$. $\square$

The next step in proving transversality of the data $\mathbb{D}$ is to show that $\mathcal{D}$ is cosupported on points with dense orbits.

Proposition 4.7.7. *Assume Assumption-Notation 4.7.1. Let Z be the subscheme of X defined by $\mathcal{D}$. Then all points in the support of Z have dense σ-orbits.*

Proof. Suppose that there is a point in Z without a dense orbit. We claim that R is not noetherian.

Let C be the Zariski closure of the orbits of all points without dense orbits in $\mathrm{Supp}(\Lambda \cup Z \cup \Lambda')$. Then C is a reduced but not necessarily irreducible curve on X. Let Γ be an irreducible component of C. For all $n \geq 1$, let Π_n be the closed subscheme of X defined by $\mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^{\sigma^n}$. We note that the scheme-theoretic intersections $\Pi_n \cap \Gamma$ and $Z \cap \Gamma$ are supported on points of infinite order, which are therefore nonsingular points of Γ (and of X). Note also that because $\mathcal{A}$ and $\mathcal{C}$ are maximal with respect to the inclusion $\mathcal{A}\mathcal{C} \subseteq \mathcal{D}$, we have that

$$\deg_\Gamma(\Pi_n \cap \Gamma) = n \deg_\Gamma(Z \cap \Gamma) \tag{4.7.8}$$

for all $n \geq 1$.

Fix $0 \neq f \in \overline{R}_1$, and let $F = \mathrm{div}_X(f) + \Delta_1$. As

$$f \in H^0(\mathcal{R}_1) \subseteq H^0(\mathcal{A}\mathcal{C}^\sigma\mathcal{L}),$$

we have that $F \cap \Gamma \supseteq \Pi_1 \cap \Gamma$. Thus

$$\deg_\Gamma(\mathcal{L}|_\Gamma) = \Delta_1.\Gamma = F.\Gamma \geq \deg_\Gamma(\Pi_1 \cap \Gamma) = \deg_\Gamma(Z \cap \Gamma).$$

We first suppose that this inequality is strict for some Γ, and so

$$\deg_\Gamma(\mathcal{L}|_\Gamma) > \deg_\Gamma(Z \cap \Gamma). \tag{4.7.9}$$

For some k, $\sigma^k\Gamma = \Gamma$. It is enough to prove that $R^{(k)}$ is not noetherian, so we may pass without loss of generality to a Veronese subalgebra and assume that Γ is σ-invariant.

Now, the sheaves $\mathcal{R}_n|_\Gamma$ are invertible on Γ; their sections give an idealizer at points of infinite order on the curve Γ, which will be noetherian by [AS95]. However, for a sufficiently high multiple $d\Gamma$ of Γ, the sheaves $\mathcal{R}_n|_{d\Gamma}$ will not be invertible; they will correspond, roughly speaking, to attempting to naïvely blow up a point on $d\Gamma$. Here we will not have critical transversality, and so we do not expect the corresponding factor ring of R to be noetherian. We will show that it is not.

For any $p \in Z \cap \Gamma$, consider the closed subscheme $Z_p = Z|_{\{p\}}$ of Z supported at p. For any such p, there is an integer d_p such that the Zariski closure of $\{\sigma^n(Z_p)\}_{n\in\mathbb{Z}}$ is $d_p\Gamma$. Let

$$d = \min\{d_p \mid d_p > 0\},$$

and let $x \in Z$ be a point with $d_x = d$.

Let Σ be the $2d$-uple curve defined by the Weil divisor $2d\Gamma$. The action of σ on X restricts to an automorphism of Σ, which we also denote by σ. There is a natural map

$$\phi : B(X, \mathcal{L}, \sigma) \to B(\Sigma, \mathcal{L}|_{\Sigma}, \sigma).$$

Let $S = \phi(R)$. That is,

$$S = \bigoplus_{n \geq 0} \left(\frac{\overline{R}_n}{H^0(\mathcal{I}_{\Sigma}\mathcal{L}_n) \cap \overline{R}_n} \right) z^n = \bigoplus_{n \geq 0} \overline{S}_n z^n$$

We claim that S is not noetherian. This implies that R is not noetherian, giving a contradiction.

Let $\mathcal{M}_n = \mathcal{L}_n|_{\Sigma}$. For all n, let $\mathcal{S}_n$ be the image of $\mathcal{R}_n \otimes \mathcal{O}_{\Sigma}$ under the natural map

$$\mathcal{R}_n \otimes \mathcal{O}_{\Sigma} \to \mathcal{M}_n.$$

The sections in $\overline{S}_n$ generate the subsheaf $\mathcal{S}_n$ of $\mathcal{M}_n$. Let

$$k = \deg_{\Gamma}(Z \cap \Gamma) = \deg_{\Gamma}(\Pi_1 \cap \Gamma),$$

and let

$$\ell = \deg_{\Gamma}(\mathcal{L}|_{\Gamma}).$$

By (4.7.9), $\ell > k$.

One can easily see that the data Ω, Λ, and Λ' give a constant $c \geq 0$ so that

$$\deg_{\Gamma}(\Omega \cap \Gamma + W_n \cap \Gamma) = nk + c$$

for all $n \gg 0$. Thus

$$\deg_{\Gamma}(\mathcal{R}_n|_{\Gamma}) = n(\ell - k) - c$$

for all $n \gg 0$.

We will work with the nonreduced scheme Σ carefully. Fix n; let Z_n be the subscheme of X defined by $\mathcal{I}_\Omega \mathcal{I}_{W_n}$. Let P be the scheme-theoretic intersection $\Gamma \cap Z_n$. Let $\{p_1, \ldots, p_r\} = \operatorname{Supp} P$. Recall that Γ is nonsingular at all p_i. Therefore, considered as a subscheme of Γ,

$$P = m_1 p_1 + \cdots + m_r p_r$$

for some integers $m_i \geq 1$.

If $f \in \mathcal{O}_{X,p_i}$, let $\overline{f}$ be its image in $\mathcal{O}_{\Gamma,p_i}$. Then for $i = 1, \ldots, r$, there are elements $f_i \in (\mathcal{R}_n \mathcal{L}_n^{-1})_{p_i} \subseteq \mathcal{O}_{X,p_i}$ so that the valuation of $\overline{f_i}$ in the discrete valuation ring $\mathcal{O}_{\Gamma,p_i}$ is m_i. In particular, the image of f_i in $\mathcal{O}_{\Sigma,p_i}$ is not in the nilradical and so is a non-zerodivisor. By taking the locally free rank 1 ideal sheaf on Σ generated by the images of the f_i in $\mathcal{O}_{\Sigma,p_i}$, we obtain an invertible ideal sheaf $\mathcal{N}_n$ on Σ. The sheaf $\mathcal{N}_n$ defines a locally principal subscheme Q of Σ so that the scheme-theoretic intersection $Q \cap \Gamma$ is equal to P. Let $\mathcal{N}'_n = \mathcal{N} \otimes \mathcal{M}_n$; then $\mathcal{N}'_n$ is an invertible subsheaf of $\mathcal{S}_n$ with

$$\deg_\Gamma(\mathcal{N}'_n|_\Gamma) = \deg_\Gamma(\mathcal{R}_n|_\Gamma).$$

Thus

$$\deg_\Gamma(\mathcal{N}'_n|_\Gamma) = n(l-k) - c$$

for $n \gg 0$.

As

$$\lim_{n \to \infty} n(l-k) - c = \infty,$$

by Lemma 4.2.2 the sequence of bimodules $\{(\mathcal{N}'_n)_{\sigma^n}\}_{n \geq 0}$ is a left and right ample sequence on Σ. Since for any coherent sheaf $\mathcal{H}$ on X, the kernel and cokernel of

$$\mathcal{H} \otimes \mathcal{N}'_n \to \mathcal{H} \otimes \mathcal{S}_n$$

are supported on sets of dimension 0, by Lemma 4.2.4, $\{(\mathcal{S}_n)_{\sigma^n}\}$ is also a left and right ample sequence on Σ.

Let

$$T = \bigoplus_{n \geq 0} H^0(\Sigma; \mathcal{S}_n) z^n.$$

By Lemma 4.7.5, T is finitely generated as a left and right S-module. Thus it suffices to prove that T is not noetherian.

Let $\mathcal{J}$ be the ideal sheaf of $d\Gamma$ on Σ. Note that $\mathcal{J}$ is σ-invariant. Let J be the ideal

$$J = \bigoplus_{n \geq 0} H^0(\Sigma; \mathcal{J}\mathcal{M}_n \cap \mathcal{S}_n) z^n$$

of T. Let $\mathcal{E}$ be the subsheaf $\mathcal{D}\mathcal{O}_\Sigma$ of $\mathcal{O}_\Sigma$.

There are integers $a, n_0 \geq 0$ so that

$$(\mathcal{S}_n)_{\sigma^{-a}(x)} = (\mathcal{E}^{\sigma^a})_{\sigma^{-a}(x)}$$

for all $n \geq n_0$. As $d\Gamma$ is the Zariski closure of $\{\sigma^n(Z_x)\}$, we have containments

$$(\mathcal{J}\mathcal{M}_n)_{\sigma^{-a}(x)} \subseteq (\mathcal{S}_n)_{\sigma^{-a}(x)} \subseteq (\mathcal{M}_n)_{\sigma^{-a}(x)}.$$

Thus for any $m \geq n_0$ and $n \geq 1$, we have

$$T_m J_n \subseteq H^0(\Sigma; \mathcal{E}^{\sigma^a} \mathcal{M}_m \mathcal{J}^{\sigma^m} \mathcal{M}_n^{\sigma^m}) z^{m+n} = H^0(\Sigma; \mathcal{E}^{\sigma^a} \mathcal{J}\mathcal{M}_{n+m}) z^{m+n}.$$

Let

$$K = J_{\geq n_0}.$$

The kernel and cokernel of

$$\mathcal{J} \otimes \mathcal{S}_n \to \mathcal{J}\mathcal{M}_n \cap \mathcal{S}_n$$

are supported on sets of dimension 0, and $\{(\mathcal{S}_n)_{\sigma^n}\}$ is a left and right ample sequence on Σ. Thus by Lemma 4.2.4, there is n_1 so that the sheaf $\mathcal{J}\mathcal{M}_n \cap \mathcal{S}_n$ is globally generated for $n \geq n_1$. We may assume that $n_1 \geq n_0$. Then for any $n > m \geq n_1$, we have

$$((K_{\leq m}) \cdot T)_n \subseteq H^0(\Sigma; \mathcal{E}^{\sigma^a} \mathcal{J} \mathcal{M}_{n+m}) z^{m+n} \subsetneqq K_{n+m}.$$

Thus K is not finitely generated as a right ideal of T.

It remains to consider the case that for all irreducible components Γ of C,

$$\deg_\Gamma(Z \cap \Gamma) = \deg_\Gamma(\mathcal{L}|_\Gamma).$$

By (4.7.8), this implies that for all $n \geq 1$,

$$\deg_\Gamma(\Pi_n \cap \Gamma) = \deg_\Gamma(\mathcal{L}_n|_\Gamma). \tag{4.7.10}$$

Let Γ be an irreducible component of C; by passing to a Veronese subalgebra of R as above we may assume that Γ is σ-invariant.

Fix a point $p \in Z \cap \Gamma$. For all i, let

$$p_i = \sigma^{-i}(p).$$

By reindexing the orbit of p if necessary, we may assume that $p_i \notin Z$ for $i < 0$. By assumption on the defining data for $\mathcal{R}$, if $i \geq 2$, then $p_i \notin \Omega \cup W_1$. Let $\mathcal{O} = \mathcal{O}_{X,p}$. As usual, we will identify all $\mathcal{O}_{X,p_i}$ with $\mathcal{O}$. Note that $\mathcal{O}$ is a regular local ring of dimension 2, since X is normal by assumption and the orbit of p is infinite. Let $\mathfrak{d}$ be the central stalk of $\mathcal{R}_+$ at $O(p)$; that is, there are integers b, which we assume to be at least 1, and N, which we assume to be at least $2b$, so that for $n \geq N$ and $b \leq i \leq n - b$ we have that

$$(\mathcal{R}_n)_{p_i} = \mathfrak{d}.$$

The point p has infinite order on Γ, and so Γ is also nonsingular at p. Let y be the local equation of Γ in $\mathcal{O}$. Thus there is some $x \in \mathcal{O}$ so that x and y generate the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$.

Our assumptions imply that $\overline{R}_N \not\subseteq H^0(\mathcal{I}_\Gamma \mathcal{L}_N)$. Let $f \in \overline{R}_N \smallsetminus H^0(\mathcal{I}_\Gamma \mathcal{L}_N)$ and let $F = \operatorname{div}_X(f) + \Delta_N$. The germ of F at p_b is in $\mathfrak{d} \smallsetminus y\mathcal{O}$ and is thus equal to $xr + ys$ for some $r \in \mathcal{O} \smallsetminus y\mathcal{O}$. As

$$\deg_\Gamma(F \cap \Gamma) = \deg_\Gamma(\Delta_N \cap \Gamma) = \deg_\Gamma(\Pi_N \cap \Gamma)$$

by (4.7.10), and

$$f \in H^0(\mathcal{A}\mathcal{D}^\sigma \cdots \mathcal{D}^{\sigma^{N-1}} \mathcal{C}^{\sigma^N} \mathcal{L}_N),$$

we see that F does not vanish at p_{b+Nj} unless $j = 0$.

Let $d = \min\{i \mid y^i \in \mathfrak{d}\}$. Then there is some $h \in \overline{R}_N$ so that the germ of $H = \operatorname{div}_X(h) + \Delta_n$ at p_b is equal to y^d. Thus we have $H = d\Gamma + G$, where $G(p_b) \neq 0$.

Let $m \geq 2$. For $i = 1 \ldots m-1$, define $\gamma_i \in \overline{R}_{Nm}$ by

$$\gamma_i = f f^{\sigma^N} \cdots f^{\sigma^{N(i-1)}} h^{\sigma^{Ni}} f^{\sigma^{N(i+1)}} \cdots f^{\sigma^{N(m-1)}}.$$

Then

$$\begin{aligned} &\operatorname{div}_X(\gamma_i) + \Delta_{Nm} = \\ &F + \sigma^{-N}(F) + \cdots + \sigma^{-N(i-1)}(F) + \sigma^{-Ni}(H) + \sigma^{-N(i+1)}(F) + \cdots + \sigma^{-N(m-1)}(F) \\ &\quad = d\Gamma + F + \cdots + \sigma^{-N(i-1)}(F) + \sigma^{-Ni}(G) + \sigma^{-N(i+1)}(F) + \cdots + \sigma^{-N(m-1)}(F). \end{aligned}$$

Fix $1 \leq i \leq m-1$. The local equation of $\operatorname{div}_X(\gamma_i) + \Delta_{Nm}$ at p_{b+Ni} is equal to y^d. On the other hand, if $j \neq i$, the local equation of $\operatorname{div}_X(\gamma_j) + \Delta_{Nm}$ at p_{b+Ni} is equal to $(xr + ys)y^d\beta$ for some $0 \neq \beta \in \mathcal{O}$. In particular, we see that modulo y^{d+1}, the set

$$\{\gamma_j \mid j \neq i\}$$

does not generate γ_i.

For all $i \geq 1$, let

$$J(i) = \bigoplus_{n \geq 1} (\overline{R}_n \cap H^0(\mathcal{I}_{i\Gamma}\mathcal{L}_n)) z^n \subseteq R.$$

Let

$$A = R/J(d+1).$$

The computations above imply that the images of the elements $\gamma_i z^{Nm}$ in A_{Nm} are linearly independent, so $\dim A_{Nm} \geq m$. Therefore, the GK-dimension of A is at least 2.

We show that this contradicts our assumption that R is noetherian. Let

$$B = R/J(1) \subseteq \bigoplus_{n \geq 0} H^0(\Gamma; \mathcal{R}_n|_\Gamma) z^n.$$

As the sections in $\overline{R}_n$ do not vanish identically on Γ, we have $\dim B_n \geq 1$ for all n. On the other hand, recall from (4.7.10) that $\deg_\Gamma(\Pi_n \cap \Gamma) = \deg_\Gamma(\mathcal{L}_n|_\Gamma)$ for all $n \geq 1$. As $\Omega \cap \Gamma + W_n \cap \Gamma \supseteq \Pi_n \cap \Gamma$, we have

$$\deg_\Gamma(\mathcal{L}_n|_\Gamma) \geq \deg_\Gamma(\Omega \cap \Gamma + W_n \cap \Gamma) \geq \deg_\Gamma(\Pi_n \cap \Gamma) = \deg_\Gamma(\mathcal{L}_n|_\Gamma)$$

for all $n \geq 1$. That is,

$$\deg_\Gamma(\mathcal{R}_n|_\Gamma) = \deg_\Gamma(\mathcal{L}_n|_\Gamma) - \deg_\Gamma(\Omega \cap \Gamma + W_n \cap \Gamma) = 0$$

for all $n \geq 1$, and $\dim H^0(\Gamma; \mathcal{R}_n|_\Gamma) = 1$ for all n. That is, $\dim B_n = 1$ for all n, and so $\operatorname{GKdim} B = 1$.

Since R is noetherian, each $J(i)/J(i+1)$ is a finitely generated R-module. The R-action on $J(i)/J(i+1)$ factors through B. Thus each $J(i)/J(i+1)$ is also a finitely generated B-module and thus has GK-dimension 1. As an R-module, A has a finite filtration by modules of the form $J(i)/J(i+1)$; therefore A has GK-dimension 1. This gives a contradiction. $\square$

Corollary 4.7.11. *Assume Assumption-Notation 4.7.1. Then the sequence of bimodules*

$$\{(\mathcal{R}_n(X))_{\sigma^n}\} = \{(\mathcal{T}_n(\mathbb{D}))_{\sigma^n}\}$$

is left and right ample, and $T(\mathbb{D})$ is a finitely generated left and right R-module.

Proof. Let Z be the closed subscheme defined by $\mathcal{D}$. We have seen in Lemma 4.7.2 that $\{\sigma^n\Omega\}$ is critically transverse and in Proposition 4.7.7 that all points in Z have dense σ-orbits. Since all points in Z have dense orbits and σ is numerically trivial, by Lemma 4.2.7(2) the sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left and right ample. By Lemma 4.7.5, $T(\mathbb{D})$ is a finitely generated left and right R-module. $\square$

Theorem 4.7.12. *Assume Assumption-Notation 4.7.1. Then the surface data*

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

is transverse.

Proof. Let $T = T(\mathbb{D})$. Recall our convention (4.2.1) that

$$T = \bigoplus_{n \geq 0} H^0(\mathcal{T}_n) z^n.$$

Let Z be the closed subscheme of X defined by $\mathcal{D}$. We have seen that $\{\sigma^n\Omega\}$ is critically transverse and that all points in Z have dense σ-orbits. It remains to show that the sets $\{\sigma^n(\Lambda)\}_{n\geq 0}$, $\{\sigma^n(\Lambda')\}_{n\leq 0}$, and $\{\sigma^n(Z)\}_{n\in\mathbb{Z}}$ are critically dense.

By Corollary 4.7.11, the sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left and right ample, and T is a finitely generated left and right R-module. Therefore, T is noetherian. We claim that this implies that the sets $\{\sigma^n(\Lambda)\}_{n\geq 0}$, $\{\sigma^n(\Lambda')\}_{n\leq 0}$, and $\{\sigma^n(Z)\}_{n\in\mathbb{Z}}$ are critically transverse.

By symmetry, it suffices to prove that $\{\sigma^n\Lambda\}_{n\geq 0}$ and $\{\sigma^n(Z)\}_{n\geq 0}$ are critically transverse. Corollary 3.3.15 implies that it suffices to prove that if there is some $p \in Z \cup \Lambda$ so that $\{\sigma^n(p)\}_{n\geq 0}$ is not critically dense, then T is not noetherian. Suppose some such p exists. Let $d = \max\{j \mid \sigma^j(p) \in \Lambda \cup Z\}$ and let $q = \sigma^d(p)$. There is a curve Γ on X (which we may take to be irreducible but possibly nonreduced) so that the germ of $\mathcal{I}_\Gamma^{\sigma^n} = \mathcal{I}_{\sigma^{-n}\Gamma}$ at q is contained in the germ of $\mathcal{R}_n$ at q for infinitely many $n \geq 0$; let $A \subseteq \mathbb{N}$ be the (infinite) set of $n \geq 0$ where this occurs.

For all $n \geq 1$, let $\mathcal{J}_n = \mathcal{I}_\Gamma^{\sigma^n}\mathcal{L}_n \cap \mathcal{R}_n$. The left ampleness of $\{(\mathcal{R}_n)_{\sigma^n}\}$ implies that for $n \gg 0$, $\mathcal{J}_n$ is globally generated. Let

$$J = \bigoplus_{n \geq 0} H^0(\mathcal{J}_n)z^n,$$

so J is a left ideal of T.

For any $k \in \mathbb{N}$ and for any $n > k$, we have that

$$(R \cdot J_{\leq k})_n \subseteq H^0(\mathcal{I}_q\mathcal{I}_\Gamma^{\sigma^n}\mathcal{L}_n)z^n.$$

On the other hand,

$$(\mathcal{I}_\Gamma^{\sigma^n}\mathcal{L}_n \cap \mathcal{R}_n)_q = (\mathcal{I}_\Gamma^{\sigma^n}\mathcal{L}_n)_q$$

for any $n \in A$. As $\mathcal{J}_n$ is globally generated for $n \gg 0$, we see that $(R \cdot J_{\leq k})_n \neq J_n$ for any $n \gg k \in A$. Thus J is not a finitely generated left ideal of T, and T is not left noetherian. $\square$

Corollary 4.7.13. *Let R be a birationally commutative projective surface. Then there are transverse surface data*

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda'),$$

where X is normal, and an integer $\ell > 1$ so that

$$R^{(\ell)} \subseteq T(\mathbb{D})$$

and $T(\mathbb{D})$ is a finitely generated left and right module over $R^{(\ell)}$.

Proof. By Corollary 4.6.12, there are a positive integer ℓ and normal surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ so that

$$\mathcal{R}(X)^{(\ell)} = \mathcal{T}(\mathbb{D}).$$

By possibly increasing ℓ, we may also assume that $\overline{R}_1$ generates $K = \Bbbk(X)$. Thus Assumption-Notation 4.7.1 holds for $R^{(\ell)}$. By Corollary 4.7.11, $T(\mathbb{D})$ is a finitely generated left and right $R^{(\ell)}$-module. Theorem 4.7.12 shows that data $\mathbb{D}$ are in fact transverse. $\square$

The most difficult part of the proof of Theorem 4.7.12 is Proposition 4.7.7, and specifically working in the situation where R and $T(\mathbb{D})$ may not be equal in large degree. To end this section, we show directly that the full section ring of a naïve blowup bimodule algebra is noetherian exactly when the orbits of the defining data are all critically dense. This gives the converse to [RS07, Theorem 3.1].

Proposition 4.7.14. *Let X be a projective variety of dimension ≥ 2, let σ be an automorphism of X, and let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Z be a 0-dimensional scheme supported at points with infinite orbits and let $\mathcal{I}$ be the ideal sheaf of Z. As in* (1.4.2)*, let*

$$S = S(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} H^0(\mathcal{I}\mathcal{I}^{\sigma} \cdots \mathcal{I}^{\sigma^n} \mathcal{L}_n)$$

be the naïve blowup algebra of X at Z. Then S is noetherian if and only if all points in the support of Z have critically dense orbits.

Proof. If all points have dense orbits, then this is [RS07, Proposition 3.16]. Suppose that there is some $z \in Z$ whose orbit is not dense. We show S is not noetherian.

Let Z^d be the maximal subscheme of Z supported on points with dense orbits, and let Z^c be the maximal subscheme of Z supported on points with non-dense orbits. By assumption, $Z^c \neq \emptyset$.

For all $n \geq 1$, let V_n be the closed subscheme defined by $\mathcal{I}_{Z^c} \cdots \mathcal{I}_{Z^c}^{\sigma^{n-1}}$. Let C be the Zariski closure of the V_n; note that C is a proper subscheme of X. Since C is

σ-invariant it is disjoint from Z^d; thus by computing locally we have

$$\mathcal{I}_C\mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{n-1}}\mathcal{L}_n \cong \mathcal{I}_C \otimes \mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{n-1}}\mathcal{L}_n.$$

Now by [RS07, Theorem 3.1], the sequence $\{(\mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{n-1}}\mathcal{L}_n)_{\sigma^n}\}$ is left and right ample, so for $n \gg 0$, $\mathcal{I}_C\mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{n-1}}\mathcal{L}_n$ is globally generated. It is contained in $\mathcal{I}\cdots\mathcal{I}^{\sigma^{n-1}}\mathcal{L}_n$ by construction.

Let

$$I = \bigoplus_{n\geq 1} H^0(\mathcal{I}_C\mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{n-1}}\mathcal{L}_n).$$

Then I is an ideal of S. Choose k such that if $n \geq k$, then $\mathcal{I}_C\mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{n-1}}\mathcal{L}_n$ is globally generated. Let $p \in Z^c$. Then for any $m > n \geq k$, we have that

$$(S \cdot I_{\leq n})_m \subseteq H^0(\mathcal{I}_p\mathcal{I}_C\mathcal{I}_{Z^d}\cdots\mathcal{I}_{Z^d}^{\sigma^{m-1}}\mathcal{L}_m) \subsetneqq I_m.$$

Thus I is not finitely generated as a left ideal and S is not left noetherian. $\square$

4.8 The correct model for R

Let us review our progress towards proving Theorem 4.1.4. In Theorem 4.6.8, we constructed surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ for an appropriate bimodule algebra $\mathcal{R}$ associated to R; in Theorem 4.7.12 we showed that this data is actually transverse, and that $T(\mathbb{D})$ is a finite left and right module over some $R^{(k)}$. In some sense, we may think of $T(\mathbb{D})$ as an "integral extension" of R; note that the variety X given in Theorem 4.6.8 is normal. Of course, there is no guarantee that R is really associated to a normal variety. In this section we show how to modify X to find the true surface associated to R.

We will assume that we are in the situation of Assumption-Notation 4.7.1. Let $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$. By Theorem 4.7.12 the data $\mathbb{D}$ is transverse.

Notation 4.8.1. We establish notation that we will use throughout the section. For any $n \geq 1$, $\overline{R}_n$ defines a rational map

$$X \overset{\beta_n}{\dashrightarrow} \mathbb{P}^N$$

that is birational onto its image. Let Y_n be the closure of the image of X; we write the induced birational map from X to Y_n as β_n, as well. If we let $\alpha'_n : X'_n \to X$ be the blowup of X at the base locus W_n of $\overline{R}_n$, then by [Har77, Example II.7.17.3] there is a birational morphism $\gamma'_n : X'_n \to Y_n$ such that the diagram

$$\begin{tikzcd} X'_n \arrow[d, "\alpha'_n"'] \arrow[dr, "\gamma'_n"] & \\ X \arrow[r, dashed, "\beta_n"'] & Y_n \end{tikzcd}$$

commutes. Let $\lambda_n : X_n \to X'_n$ be the normalization of X'_n, and let $\alpha_n = \alpha'_n \lambda_n$ and $\gamma_n = \gamma'_n \lambda_n$. Thus we have

$$\begin{tikzcd} X_n \arrow[d, "\alpha_n"'] \arrow[dr, "\gamma_n"] & \\ X \arrow[r, dashed, "\beta_n"'] & Y_n \end{tikzcd}$$

for all $n \geq 1$. Note, that as X is normal, α_n^{-1} is defined at all points in $X \smallsetminus W_n$.

Let Z be the closed subscheme defined by $\mathcal{D}$ and let

$$\mathbb{W} = \bigcup_{p \in Z \cup \Lambda \cup \Lambda'} O(p).$$

That is, $\mathbb{W}$ is the union of the finitely many (dense) orbits that meet some W_n. Let $U = X \smallsetminus \mathbb{W}$. Note that all β_n are defined at all points in U. For any $n \geq 1$, let E_n be the exceptional locus of α_n. Then α_n induces an isomorphism from $X_n \smallsetminus E_n \to X \smallsetminus W_n$. Let $U_n = \alpha_n^{-1}(U)$. We caution that U_n is *not* $X_n \smallsetminus E_n$.

For all $n \geq 1$, let A_n be the set of points $p \in U$ such that β_n is not a local isomorphism at p; that is, the set of p such that the induced map from $\mathcal{O}_{Y_n, \beta_n(p)} \to$

$\mathcal{O}_{X,p}$ is not an isomorphism. We write A_n as the disjoint union $A_n = C_n \sqcup Q_n \sqcup P_n$, where C_n is the intersection of a curve in X with U, Q_n is 0-dimensional and supported on points of infinite order under σ, and P_n is 0-dimensional and supported on points of finite order. By assumption on the cardinality of $\Bbbk$, any curve in X must meet U in uncountably many points, and so the sets C, P, and Q are well-defined.

If $N > n \geq 1$, let $\pi_n^N : Y_N \to Y_n$ be the birational map induced from the multiplication $\overline{R}_n(\overline{R}_{N-n})^{\sigma^n} \subseteq \overline{R}_N$ and Lemma 4.5.12, with $E = \sigma^{-n}(\Omega)$. That is, the diagram of birational maps

$$\begin{tikzcd} X \arrow[r, dashed, "\beta_N"] \arrow[dr, dashed, "\beta_n"'] & Y_N \arrow[d, dashed, "\pi_n^N"] \\ & Y_n \end{tikzcd}$$

commutes, and for any $x \in U \smallsetminus \sigma^{-n}(\Omega)$, π_n^N is defined at $\beta_N(x)$. Likewise, the multiplication $\overline{R}_{N-n}(\overline{R}_n)^{\sigma^{N-n}} \subseteq \overline{R}_N$ gives a commuting diagram of birational maps

$$\begin{tikzcd} X \arrow[r, dashed, "\beta_N"] \arrow[dr, dashed, "\beta_n \circ \sigma^{N-n}"'] & Y_N \arrow[d, dashed, "\rho_n^N"] \\ & Y_n. \end{tikzcd}$$

The map ρ_n^N is defined at $\beta_N(x)$ if $x \in U \smallsetminus \sigma^{-(N-n)}(\Omega)$.

We record for future reference an elementary lemma on birational maps.

Lemma 4.8.2. *Let $\beta : X \to Y$ be a birational map of projective varieties that is defined and is a local isomorphism at $x \in X$; let $y = \beta(x)$. Then β^{-1} is defined at y; in particular, if $x' \in X$ with $\beta(x') = y$, then $x' = x$.*

Proof. This is almost tautological. The fact that β induces an isomorphism between the local rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ means that there are open neighborhoods $x \in V \subseteq X$ and $y \in V' \subseteq Y$ so that β restricts to an isomorphism between V and V'. This means that β^{-1} gives a well-defined map

$$V' \to V \subseteq X.$$

This precisely says that the birational map $\beta^{-1} : Y \to X$ is defined at y. $\square$

Proposition 4.8.3. *There is some m_1 such that A_{m_1} is σ-invariant and $A_n = A_{m_1}$ for all $n \geq m_1$; further, $\overline{C}_{m_1} \subseteq U$ and $Q_{m_1} = \emptyset$.*

Proof. Let $y \in U$ and let $N > n \geq 1$. If π_n^N is defined at $\beta_N(y)$ and β_n is a local isomorphism at y, then from the inclusions

$$\mathcal{O}_{Y_n,\beta_n(y)} \subseteq \mathcal{O}_{Y_N,\beta_N(y)} \subseteq \mathcal{O}_{X,y}$$

clearly β_N is a local isomorphism at y. As π_n^N is defined on $\beta_n(U \smallsetminus \sigma^{-n}\Omega)$, we see that

$$A_N \subseteq A_n \cup \sigma^{-n}\Omega.$$

Making the same argument with the map ρ_n^N, we obtain that

$$A_N \subseteq \sigma^{-(N-n)}(A_n) \cup \sigma^{-(N-n)}\Omega.$$

Thus

$$A_{n+m} \subseteq \sigma^{-m}(\Omega \cup A_n) \cap (A_n \cup \sigma^{-n}\Omega) \tag{4.8.4}$$

for any $n, m \geq 1$. In particular,

$$C_{n+m} \subseteq \sigma^{-m}(\Omega \cup C_n) \cap (C_n \cup \sigma^{-n}\Omega)$$

for all $n, m \geq 1$.

Now, $\{\sigma^m\Omega\}_{m\in\mathbb{Z}}$ is critically transverse. Thus $\sigma^{-n}\Omega \cap \sigma^{-m}\Omega$ is finite for $m \gg 0$. Further, $C_n \cap \sigma^{-m}\Omega$ and $\sigma^{-m}C_n \cap \sigma^{-n}\Omega$ are finite for $m \gg 0$. Thus for $m \gg 0$, we have

$$C_{n+m} \subseteq C_n \cap \sigma^{-m}C_n.$$

Thus there is some n_1 such that if $n \geq n_1$, then $C_n = C_{n_1}$; further, $\sigma(C_{n_1}) = C_{n_1}$. Let $C = C_{n_1}$.

Now C is a curve in U; let $\overline{C}$ be its closure in X. Then $\overline{C}$ is also σ-stable, and since all orbits in $\mathbb{W}$ are Zariski-dense in X, we have that $\mathbb{W} \cap \overline{C} = \emptyset$. Thus $C = \overline{C} \subseteq U$.

For $n, m \geq n_1$ we have

$$Q_{n+m} \subseteq A_{n+m} \subseteq \sigma^{-m}(\Omega \cup C \cup Q_n \cup P_n) \cap (Q_n \cup C \cup P_n \cup \sigma^{-n}\Omega).$$

As

$$Q_n \cap C = Q_n \cap \sigma^{-m}(C) = \emptyset$$

for $n \geq n_1$, and $Q_n \cap \sigma^k(P_m) = \emptyset$ for all n, m, k by definition, we see that

$$Q_{n+m} \subseteq \sigma^{-m}(\Omega \cup Q_n) \cap (Q_n \cup \sigma^{-n}\Omega)$$

for $n \geq n_1$ and $m \geq 1$.

Choose k such that

$$\Omega \cap \sigma^{-1}\Omega \cap \cdots \cap \sigma^{-k}\Omega = \emptyset. \tag{4.8.5}$$

Such k exists because, by transversality of the data $\mathbb{D}$, Ω contains no forward σ-orbits. Choose $n_2 \geq n_1$ such that if $n \geq n_2$, then we have for $i = 0 \ldots k$ that

$$Q_{n_1+i} \cap \sigma^{-(n-(n_1+i))}(Q_{n_1+i} \cup \Omega) = \emptyset.$$

We may do this because each finite set Q_{n_1+i} is supported on infinite orbits and $\{\sigma^n\Omega\}$ is critically transverse.

So we have that for $i = 0 \ldots k$ and for $r \geq n_2$ that

$$\begin{aligned} Q_r &\subseteq \sigma^{-(r-(n_1+i))}(Q_{n_1+i} \cup \Omega) \cap (Q_{n_1+i} \cup \sigma^{-(n_1+i)}\Omega) \\ &\subseteq \sigma^{-(n_1+i)}\Omega \end{aligned}$$

and so $Q_r = \emptyset$ for $r \geq n_2$ by (4.8.5).

Finally, Ω does not contain any points of finite order, and by construction P_n is disjoint from C and from Q_n. Thus (4.8.4) implies that

$$P_{n+m} \subseteq P_n \cap \sigma^{-m} P_n$$

for all $n \geq n_1$ and $m \geq 1$. Thus there is some $n_3 \geq n_2$ such that P_{n_3} is σ-invariant and $P_n = P_{n_3}$ if if $n \geq n_3$. The result is proven for $m_1 = n_3$. $\square$

Notation 4.8.6. Let m_1 be the integer given by Proposition 4.8.3. Let $A = A_{m_1}$, $C = C_{m_1}$, and $P = P_{m_1}$. Recall that $Q_n = \emptyset$ for all $n \geq m_1$.

Corollary 4.8.7. *For $n \geq m_1$, the only curves in X_n that are contracted by γ_n are contained in the exceptional locus E_n of α_n. In particular, the map γ_n is finite at all points of U_n, and β_n is finite at all points of U.*

Proof. Suppose that $n \geq m_1$ and that γ_n contracts some irreducible curve Γ that is not contained in E_n. By assumption on the cardinality of $\Bbbk$, Γ meets U_n. By construction, we have that $\Gamma \cap U_n \subseteq \alpha_n^{-1} C_n = \alpha_n^{-1} C$. Now, as α_n is an isomorphism away from E_n, the curve $\alpha_n^{-1}(C)$ is closed in X_n. Thus

$$\Gamma = \overline{\Gamma \cap U_n} \subseteq \overline{\alpha_n^{-1}(C)} = \alpha_n^{-1} C.$$

This means that $\alpha_n(\Gamma) \subseteq C \subset U$, so $\alpha_n(\Gamma)$ is disjoint from $\mathrm{Bs}(\overline{R}_n)$. As β_n contracts the curve $\alpha_n(\Gamma)$, Lemma 4.5.9 implies that $\Delta_n . \alpha_n(\Gamma) = 0$. This contradicts the ampleness of Δ_n by the Nakai-Moishezon criterion ([Har77, Theorem V.I.10]; see [Laz04, Theorem 1.2.23] for a reference that includes singular surfaces). As α_n is a local isomorphism at all points in U_n, the statement on β_n follows immediately. $\square$

We recall some terminology from commutative algebra. Let R be a commutative noetherian $\Bbbk$-algebra, and let T be its normalization. Recall that the *S_2-ification* of

R is the unique minimal $\Bbbk$-algebra $S \subseteq T$ such that $R \subseteq S$ and S satisfies Serre's condition S_2. More explicitly,

$$S = \bigcap_{\substack{P \in \operatorname{Spec} R \\ \text{ht P}=1}} R_P.$$

See [Kol85, Definition 2.2.2(ii)] and subsequent discussion.

We give a lemma on the domain of definition of birational maps of S_2-ifications.

Lemma 4.8.8. *Let T be a normal commutative domain that is a finitely generated $\Bbbk$-algebra, and let $R, R' \subseteq T$ be finitely generated subalgebras so that T is the normalization of both R and R'. Let S, respectively S', be the S_2-ification of R, respectively R'. Suppose that the induced birational map*

$$\pi : \operatorname{Spec} R \dashrightarrow \operatorname{Spec} R'$$

is defined away from a locus of codimension 2. Then the induced birational map

$$\zeta : \operatorname{Spec} S \dashrightarrow \operatorname{Spec} S'$$

is defined everywhere; that is, $S' \subseteq S$.

Proof. Because π is defined in codimension 2, for every height 1 prime P of R, π is defined at the generic point of $V(P) \subseteq \operatorname{Spec} R$. That is, for every height 1 P, we have $R' \subseteq R_P$. Thus

$$R' \subseteq \bigcap_{\substack{P \in \operatorname{Spec} R \\ \text{ht P}=1}} R_P = S.$$

By definition, $S' \subseteq S$. $\square$

Proposition 4.8.9. *If $n \gg 0$, then the rational maps $\pi_n^{n+1}, \rho_n^{n+1} : Y_{n+1} \to Y_n$ are local isomorphisms everywhere on the image of U; in particular, they are defined everywhere on the image of U.*

Proof. We continue to let m_1, A, C, and P be as in Notation 4.8.6. In particular, if $n \geq m_1$ then $\gamma_n : X_n \to Y_n$ is a local isomorphism at all points in $U_n \smallsetminus \alpha_n^{-1}(C \cup P)$.

Let $n \geq m_1$. Recall that if $x \in U \smallsetminus \sigma^{-n}(\Omega)$, then π_n^{n+1} is defined at $\beta_{n+1}(x)$. As $\sigma^{-n}(\Omega)$ contains no points of finite order, π_n^{n+1} is defined at all points in $\beta_{n+1}(P)$. We saw in Corollary 4.8.7 that β_m is finite at all $p \in U$, and in particular, at all $p \in P$. By finiteness of the integral closure, there is some $m_2 \geq m_1$ so that if $n \geq m_2$, then π_n^{n+1} is a local isomorphism at all points in $\beta_{n+1}(P)$.

Let $n \geq m_2$. Now, let $x \in U \smallsetminus A$. Then β_{n+1} and β_n are local isomorphisms at x, and thus by Lemma 4.8.2, β_{n+1}^{-1} is defined at $\beta_{n+1}(x)$. Thus $\pi_n^{n+1} = \beta_n \beta_{n+1}^{-1}$ is defined and is a local isomorphism at $\beta_{n+1}(x)$.

The only points where π_n^{n+1} may not be defined thus lie in $\beta_{n+1}(C \cap \sigma^{-n}(\Omega))$. If $x \in U \smallsetminus (\sigma^{-n}(\Omega) \cap C)$, then π_n^{n+1} is defined and is a local isomorphism at $\beta_{n+1}(x)$.

The intersection $\sigma^{-n}(\Omega) \cap C$ is finite by transversality of Ω, since C is σ-invariant. Thus π_n^{n+1} is defined at the generic point of each component of $\beta_{n+1}(C)$. As γ_n is finite at each point of U_n, by finiteness of the integral closure there is $m_3 \geq m_2$ such that if $n \geq m_3$, then π_n^{n+1} is an isomorphism at the generic point of each component of $\beta_{n+1}(C)$.

For each $n \geq m_3$, let $\delta_n : Z_n \to Y_n$ be the projective variety obtained by taking the S_2-ification of Y_n at all points in $\beta_n(C)$. Now, X_n is normal, so the birational morphism $\gamma_n : X_n \to Y_n$ factors through Z_n, and there are birational morphisms $\epsilon_n : X_n \to Z_n$ and a birational map $\eta_n : X \to Z_n$ so that the diagram

$$\begin{tikzcd}
& X_n \arrow[dl, "\alpha_n"'] \arrow[d, "\epsilon_n"] \arrow[dr, "\gamma_n"] & \\
X \arrow[r, dashed, "\eta_n"] \arrow[rr, dashed, bend right, "\beta_n"'] & Z_n \arrow[r, "\delta_n"] & Y_n
\end{tikzcd}$$

commutes. In particular, η_n is defined at all points of U.

Let $\zeta_n^{n+1} : Z_{n+1} \to Z_n$ be the induced birational map such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\eta_{n+1}"] \arrow[dr, dashed, "\eta_n"'] & Z_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, dashed, "\zeta_n^{n+1}"] & Y_{n+1} \arrow[d, dashed, "\pi_n^{n+1}"] \\ & Z_n \arrow[r, "\delta_n"'] & Y_n \end{tikzcd} \tag{4.8.10}$$

commutes. We claim that for $n \geq m_3$, that the rational map ζ_n^{n+1} is defined at all points of $\eta_{n+1}(U)$.

Let $p \in U \smallsetminus (C \cup P)$. Then β_{n+1} and therefore η_{n+1} is a local isomorphism at p, so, using Lemma 4.8.2, $\zeta_n^{n+1} = \eta_n \eta_{n+1}^{-1}$ is defined at $\eta_{n+1}(p)$.

If $p \in P$ and $\beta_{n+1}(p) \notin \beta_{n+1}(C)$, then by our choice of n, the map π_n^{n+1} is a local isomorphism at $\beta_{n+1}(p)$. Thus $\beta_n(p) \notin \beta_n(C)$. By construction δ_n is a local isomorphism at $\eta_n(p)$ and so $\zeta_n^{n+1} = \delta_n^{-1}\pi_n^{n+1}\delta_{n+1}$ is defined at $\eta_{n+1}(p)$.

Now let $p \in C$. We have seen that π_n^{n+1} is defined on $\beta_{n+1}(U)$, except at a 0-dimensional locus contained in $\beta_{n+1}(C)$. It follows from Lemma 4.8.8 that ζ_n^{n+1} is defined at $\eta_{n+1}(p)$. This completes the proof of the claim.

Using finiteness of the integral closure again, we may choose $m_4 \geq m_3$ so that ζ_n^{n+1} is a local isomorphism at all points of $\eta_{n+1}(U)$ for $n \geq m_4$. For $n \geq m_4$, let

$$F_n = \{p \in C \mid \delta_n \text{ is not a local isomorphism at } \eta_n(p)\}.$$

As δ_n is finite, the set F_n is finite. Arguing as in the proof of Proposition 4.8.3, and using the maps ζ_n^{n+m}, we have that

$$F_{n+m} \subseteq \big(F_n \cup \sigma^{-n}(\Omega \cap C)\big) \cap \sigma^{-m}\big(F_n \cup (\Omega \cap C)\big)$$

if $n \geq m_4$. Since $\Omega \cap C$ consists of finitely many points of infinite order, for $n, m \gg 0$ we have that

$$\sigma^{-n}(\Omega \cap C) \cap \sigma^{-m}(F_n \cup (\Omega \cap C)) = F_n \cap \sigma^{-m}(\Omega \cap C) = \emptyset.$$

Thus for $m, n \gg 0$, we have that

$$F_{n+m} \subseteq F_n \cap \sigma^{-m} F_m,$$

and so for $n \gg 0$, we have that $F_n = F_{n+1}$ is σ-invariant. In particular, $F_n \cap \sigma^{-n}\Omega = \emptyset$. This means that δ_{n+1} is a local isomorphism at all points of $\eta_{n+1}(\sigma^{-n}\Omega \cap U)$. By (4.8.10), $\pi_n^{n+1} = \delta_n \zeta_n^{n+1} \delta_{n+1}^{-1}$ is defined everywhere in $\beta_{n+1}(\sigma^{-n}\Omega \cap U)$. Thus π_n^{n+1} is defined everywhere in $\beta_{n+1}(U)$ for $n \geq m_4$.

The argument that for $n \gg 0$, ρ_n^{n+1} is defined everywhere on $\beta_{n+1}(U)$ is completely symmetric. By finiteness of the integral closure, we see that for $n \gg 0$ both π_n^{n+1} and ρ_n^{n+1} are local isomorphisms at every point of $\beta_{n+1}(U)$. $\square$

We establish some more notation, which we will use in the next few results.

Notation 4.8.11. Assume Assumption-Notation 4.7.1. Let m be such that for $n \geq m-1$, the rational maps π_n^{n+1} and ρ_n^{n+1} are defined and are local isomorphisms at every point in $\gamma_{n+1}(U)$. We call Y_m a *stable scheme* for R. Let $C \subset U$ be the σ-invariant curve where γ_m is not a local isomorphism, and let $P \subset U$ be the σ-invariant 0-dimensional subscheme where γ_m is not a local isomorphism. Let F be the σ-invariant subset of C that maps onto points where Y_m does not satisfy S_2.

For all $n \geq 2$, we define a birational automorphism τ_n of Y_n by setting

$$\tau_n = (\pi_{n-1}^n)^{-1} \rho_{n-1}^n.$$

By construction, $\tau_n \gamma_n = \gamma_n \sigma$ as birational maps. Proposition 4.8.9 implies that for $n \geq m$, τ_n is an automorphism of $\gamma_n(U_n)$.

For all $n, m \geq 1$, we define birational maps

$$p_n^{n+m}, r_n^{n+m} : X_{n+m} \to X_n,$$

where $p_n^{n+m} = (\alpha_n)^{-1}\alpha_{n+m}$ and $r_n^{n+m} = \alpha_n^{-1}\sigma^m\alpha_{n+m}$. By construction, we have

$$\pi_n^{n+m}\gamma_{n+m} = \gamma_n p_n^{n+m}$$

and

$$\rho_n^{n+m}\gamma_{n+m} = \gamma_n r_n^{n+m}$$

as birational maps from X_{n+m} to Y_n.

Let Z be the subscheme of X defined by $\mathcal{D}$. Recall that $\mathbb{W} = \bigcup_{p \in Z \cup \Lambda \cup \Lambda'} O(p)$ and that $U = X \smallsetminus \mathbb{W}$. The map β_m is defined and finite at every point of U. Heuristically, it may fold U along C or pinch U into a cusp at some point of P. At points of F, β_m does some additional cusping, since there Y_m fails S_2. We will construct the variety Y by cusping and folding U along C, P, and F, and gluing in the points of $\mathbb{W}$, which correspond to orbits on which β_m is not always defined, to obtain a finite morphism from X to Y.

There is one technicality still to dispose of: in order to glue as described, we need the sets

$$\gamma_m(U_m) = \beta_m(U)$$

and

$$\gamma_m(\alpha_m^{-1}(\mathbb{W}))$$

to be disjoint, at least for large m. Proving this is the content of the next few results.

We will need to look carefully at how our various birational maps affect $\mathbb{W}$, and we establish some more notation. For any $w \in \mathbb{W}$ and $n \in \mathbb{Z}$, we let $w_n = \sigma^{-n}(w)$.

Definition 4.8.12. Let $w \in \mathbb{W}$. Let $\mathcal{O} = \mathcal{O}_{X,w}$, and let $\mathfrak{m}_i^n$ be the germ of $\mathcal{R}_n\mathcal{L}_n^{-1}$ at w_i, regarded as an ideal in $\mathcal{O}$, as usual. We say that the orbit $O(w)$ is *nice* if there is an integer j so that $\Omega \cap O(w) \subseteq \{w_j\}$ and so that there are ideals $\mathfrak{a}$, $\mathfrak{d}$, and $\mathfrak{c}$ of $\mathcal{O}$ so that for all $n \geq 1$, we have

- $\mathfrak{m}^n_j = \mathfrak{a}$;
- if $1 \leq i \leq n-1$ then $\mathfrak{m}^n_{i+j} = \mathfrak{d}$;
- $\mathfrak{m}^n_{n+j} = \mathfrak{c}$; and
- if $i < 0$ or $i > n$ then $\mathfrak{m}^n_{i+j} = \mathcal{O}$.

If $O(w)$ is nice and $j = 0$, then we say that w itself is *nice*; by reindexing, if $O(w)$ is nice we may always assume that w is nice.

Niceness is a purely formal notion; if $O(w)$ is nice, it is easier to analyze the behavior of the loci $\gamma_n(\alpha_n^{-1}(O(w))$ for various n. We carry out this analysis in the next three lemmas.

We first establish some more notation.

Notation 4.8.13. Let $w \in \mathbb{W}$ be a nice point. Define curves

$$E_{\mathfrak{a}} = \alpha_2^{-1}(w_0),$$

$$E_{\mathfrak{d}} = \alpha_2^{-1}(w_1),$$

and

$$E_{\mathfrak{c}} = \alpha_2^{-1}(w_2).$$

That is, $E_{\mathfrak{d}}$ is the exceptional locus obtained by blowing up the ideal $\mathfrak{d}$ and normalizing, and similarly for $E_{\mathfrak{a}}$ and $E_{\mathfrak{c}}$.

For any $n \geq 1$ and $i \in \mathbb{Z}$, define $E^n_i \subseteq Y_n$ by

$$E^n_i = \gamma_n(\alpha_n^{-1}(w_i)).$$

Lemma 4.8.14. *Suppose that $w \in \mathbb{W}$ is nice. Then for all $n, m \geq 1$ the map p^{n+m}_n is defined at all points in $\alpha^{-1}_{n+m}(O(w) \smallsetminus \{w_n\})$, and is a local isomorphism at* *all points in $\alpha_{n+m}^{-1}(O(w) \smallsetminus \{w_n, w_{n+1}, \ldots, w_{n+m}\})$. Likewise, r_n^{n+m} is defined at all points in $\alpha_{n+m}^{-1}(O(w) \smallsetminus \{w_m\})$ and is a local isomorphism at all points in $\alpha_{n+m}^{-1}(O(w) \smallsetminus \{w_0, \ldots, w_m\})$.*

Proof. Fix $n, m \geq 1$ and let $w \in \mathbb{W}$ be a nice point. By definition, $\operatorname{Supp} W_n \subseteq \{w_0, \ldots, w_n\}$,and so the map α_n is a local isomorphism at all points in $\alpha_n^{-1}(O(w) \smallsetminus \{w_0, \ldots, w_n\})$. Furthermore, as $\mathfrak{m}_0^n = \mathfrak{a}$, we have that $\alpha_n^{-1}(w_0) \cong E_{\mathfrak{a}}$. Likewise, $\alpha_n^{-1}(w_n) \cong E_{\mathfrak{c}}$, and for $1 \leq i \leq n-1$, $\alpha_n^{-1}(w_i) \cong E_{\mathfrak{d}}$.

Therefore, if $i < 0$ or $i > n+m$, then $p_n^{n+m} = \alpha_n^{-1}\alpha_{n+m}$ is defined and is a local isomorphism at the point $\alpha_{n+m}^{-1}(w_i)$. For $0 \leq i \leq n-1$, the stalks $\mathfrak{m}_i^{n+m}$ and $\mathfrak{m}_i^n$ are isomorphic. Thus $\alpha_n^{-1}\alpha_{n+m}$ extends to a map that is defined and a local isomorphism at all points of $\alpha_{n+m}^{-1}(w_i)$. For $n+1 \leq i \leq n+m$, α_n^{-1} is defined at w_i, so p_n^{n+1} is defined on $\alpha_{n+m}^{-1}(w_i)$, although it is not necessarily a local isomorphism.

We repeat this analysis for the maps r_n^{n+m}. If $i < 0$ or $i > n+m$, then α_{n+m} is a local isomorphism at the point $\alpha_{n+m}^{-1}(w_i)$, and α_n^{-1} is defined (and is thus a local isomorphism) at $w_{i-m} = \sigma^m(w_i)$. Thus $r_n^{n+m} = \alpha_n^{-1}\sigma^m\alpha_{n+m}$ is a local isomorphism at $\alpha_{n+m}^{-1}(w_i)$. If $m+1 \leq i \leq n+m$, then $\alpha_{n+m}^{-1}(w_i) \cong \alpha_n^{-1}(w_{i-m})$ and r_n^{n+m} extends to a local isomorphism at all points in $\alpha_{n+m}^{-1}(w_i)$. Finally, if $0 \leq i \leq m-1$, then α_n^{-1} is defined at w_{i-m} and so r_n^{n+m} is defined on $\alpha_{n+m}^{-1}(w_i)$. $\square$

Lemma 4.8.15. *Suppose that $w \in \mathbb{W}$ is a nice point.*

(1) *For $m, n \geq 1$, if $i \neq n$ then π_n^{n+m} is defined at all points in $E_i^{n+m} = \gamma_{n+m}(\alpha_{n+m}^{-1}(w_i))$, and $\pi_n^{n+m}(E_i^{n+m}) = E_i^n$.*

(2) *For $m, n \geq 1$, if $i \neq m$ then ρ_n^{n+m} is defined at all points in $E_i^{n+m} = \gamma_{n+m}(\alpha_{n+m}^{-1}(w_i))$, and $\rho_n^{n+m}(E_i^{n+m}) = E_{i-m}^n$.*

Proof. (1) On X_{n+m}, the rational functions in $\overline{R}_{n+m}$ define the morphism

$$\gamma_{n+m} : X_{n+m} \longrightarrow Y_{n+m} .$$

The rational map induced by the rational functions in $\overline{R}_n$ is easily seen to be

$$\gamma_n p_n^{n+m} = \pi_n^{n+m} \gamma_{n+m} : X_{n+m} \dashrightarrow Y_n ,$$

and the rational map induced by the rational functions in $\overline{R}_m^{\sigma^n}$ is

$$\gamma_m r_m^{n+m} = \rho_m^{n+m} \gamma_{n+m} : X_{n+m} \dashrightarrow Y_m .$$

If $i \neq n$, then p_n^{n+m} and r_m^{n+m} are defined at all points in $\alpha_{n+m}^{-1}(w_i)$.

We wish to apply Lemma 4.5.12. To do so, we must calculate the divisors and base loci on X_{n+m} associated to the vector spaces $\overline{R}_{n+m}$, $\overline{R}_n$, and $\overline{R}_m^{\sigma^n}$.

For $0 \leq i \leq m+n$, let F_i be the effective exceptional Weil divisor $\alpha_{n+m}^{-1}(w_i)$. Now, $\mathcal{I}_{F_i}$ is the expansion of $\mathfrak{m}_i^{n+m}$ to X_{n+m}. By [Har77, Proposition 7.1], the expansion of $\mathfrak{m}_i^{n+m}$ to X'_{n+m} is Cartier; the ideal sheaf $\mathcal{I}_{F_i}$ is its pullback to X_{n+m} and is thus also Cartier.

By Lemma 4.4.4, we have

$$D^{X_{n+m}}(\overline{R}_{n+m}) = \alpha_{n+m}^* D_{n+m} - F_0 - \cdots - F_{n+m}.$$

Let $G_3 = D^{X_{n+m}}(\overline{R}_{n+m})$. Let

$$G_1 = \alpha_{n+m}^* D_{n+m} - F_0 - \cdots - F_{n-1},$$

and let

$$G_2 = \alpha_{n+m}^* D_{n+m} - F_{n+1} \cdots - F_{n+m}.$$

The niceness of w implies that $G_1 - D^{X_{n+m}}(\overline{R}_n)$ and $G_2 - D^{X_{n+m}}(\overline{R}_m^{\sigma^n})$ are both effective and supported on F_n. That is, the base locus of the rational functions in $\overline{R}_n$

with respect to the Cartier divisor G_1 is contained in F_n. Likewise, the base locus of the rational functions in $\overline{R}_m^{\sigma^n}$ with respect to the Cartier divisor G_2 is also contained in F_n.

We now apply Lemma 4.5.12 to the multiplication $\overline{R}_n\overline{R}_m^{\sigma^n} \subseteq \overline{R}_{n+m}$. We have that

$$G_3 - G_1 - G_2 = -F_n + \alpha_{n+m}^* \sigma^{-n}\Omega.$$

Recall that $\sigma^{-n}\Omega \cap O(w) \subseteq \{w_n\}$. Thus by Lemma 4.5.12, the rational map

$$\pi_n^{n+m} = (\pi_n^{n+m}\gamma_{n+m})\gamma_{n+m}^{-1} : Y_{n+m} \dashrightarrow Y_n$$

is defined at every point of $\gamma_{n+m}(F_i)$ for every $i \neq n$. That is, if $i \neq n$, then π_n^{n+m} is defined at all points of E_i^{n+m}. That the image of E_i^{n+m} is E_i^n is immediate.

The proof of (2) is symmetric: we use the multiplication $\overline{R}_m\overline{R}_n^{\sigma^m} \subseteq \overline{R}_{n+m}$. $\square$

Lemma 4.8.16. *Suppose that $w \in \mathbb{W}$ is a nice point. Then there are integers n_1 and $b \geq 1$ so that:*

(1) *If $n \geq n_1$ and $i \leq n - b$ then for all $m \geq 1$, π_n^{n+m} is a local isomorphism at all points of E_i^{n+m}.*

(2) *If $n \geq n_1$ then for all $m \geq 1$ and $i \geq m + b$, ρ_n^{n+m} is a local isomorphism at all points of E_i^{n+m}.*

Proof. Fix $i \in \mathbb{Z}$. For $n \geq i+1$, the map π_n^{n+1} is defined on E_i^{n+1} by Lemma 4.8.15, and

$$\pi_n^{n+1}(E_i^{n+1}) = E_i^n.$$

We claim that for $n \gg 0$, the map π_n^{n+1} is finite at all points of E_i^{n+1}. This is clear if $i < 0$, as then by Corollary 4.8.7 γ_{n+1} and $\gamma_n p_n^{n+1}$ are finite at $\alpha_{n+1}^{-1}(w_i)$.

Now suppose that $n > i \geq 0$. By Lemma 4.8.15(1), π_n^{n+1} is defined on E_i^{n+1}. If it is not finite on E_i^{n+1}, it must contract some component of it by Corollary 4.8.7. Let

$E = \alpha_{i+1}^{-1}(w_i)$; thus E is isomorphic to either $E_{\mathfrak{a}}$ or $E_{\mathfrak{d}}$. We have an infinite series of surjections

(4.8.17)
$$\begin{tikzcd} & & E \arrow[dl] \arrow[d] \arrow[dr] & \\ \cdots \arrow[r] & E_i^{i+3} \arrow[r, "\pi_{i+2}^{i+3}"'] & E_i^{i+2} \arrow[r, "\pi_{i+1}^{i+2}"'] & E_i^{i+1} \end{tikzcd}$$

where E surjects onto all terms. Since E has only finitely many components, for $n \gg 0$ the map π_n^{n+1} must be finite at all points of E_i^{n+1}, and the claim is proved.

Fix $a \in \mathbb{Z}$. We claim that there is some integer N_a so that if $n \geq N_a$ and $i \leq a$, then π_{n-1}^n is a local isomorphism at all points in E_i^n.

Note that if $i < 0$, then β_1 is defined and finite at w_i. Let

$$\mathbb{A} = \{i < 0 \,|\, \beta_1 \text{ is not a local isomorphism at } w_i\}.$$

The set $\mathbb{A}$ is finite. If $i < 0$ and $i \notin \mathbb{A}$, then $\mathcal{O}_{Y_1,\beta_1(w_i)}$ is integrally closed. Thus for all $n \geq 1$ the map π_1^n, which is finite at the point E_i^n, is automatically a local isomorphism at E_i^n. By finiteness of the integral closure, there is some N so that for $n \geq N$, π_{n-1}^n is a local isomorphism at all points in the finite point set

$$\bigcup_{i \in \mathbb{A}} E_i^n.$$

Then for any $i \leq -1$ and $n \geq N$, π_{n-1}^n is a local isomorphism at any point in E_i^n, and we may take $N_a = N$ for any $a \leq -1$.

If $a \geq 0$, choose N' so that π_{n-1}^n is finite at all points of E_i^n for $0 \leq i \leq a$ and $n \geq N'$. By finiteness of the integral closure, there is some $N'' \geq N'$ so that π_{n-1}^n is a local isomorphism at all points of E_i^n for $0 \leq i \leq a$ and $n \geq N''$. Let $N_a = \max\{N'', N_{-1}\}$.

Repeating this analysis for the ρ_{n-1}^n, for any a we can find M_a so that for all $n \geq M_a$ and $i \geq n - a$, the map ρ_{n-1}^n is a local isomorphism at all points in E_i^n.

Let $n \geq 2$. As $n \neq 2n-1$, by Lemma 4.8.15(1) the map π^{2n}_{2n-1} is defined at all points of E^{2n}_n. It maps E^{2n}_n onto E^{2n-1}_n. Likewise, by Lemma 4.8.15(2), as $n \neq 1$ the map ρ^{2n-1}_{2n-2} is defined at all points of E^{2n-1}_n. Thus, the map

$$q^n = \rho^{2n-1}_{2n-2}\pi^{2n}_{2n-1}$$

is defined at all points of E^{2n}_n. It is finite unless it contracts a component of E^{2n}_n. As $E_{\mathfrak{d}}$ surjects onto all E^{2n}_n, arguing as above we obtain that there is some b so that for all $n > b$, q^n is a local isomorphism at all points of E^{2n}_n. We may take $b \geq 1$.

Sublemma 4.8.18. *For any $m \geq b$, for all j, n with $b < j \leq m$ and $b+j \leq n \leq m+j$, the map ρ^n_{n-1} is defined and is a local isomorphism at all points of E^n_j. For all j, n with $b \leq j \leq m$ and $b+j < n \leq m+j$, the map π^n_{n-1} is defined and is a local isomorphism at all points of E^n_j.*

Proof of Sublemma 4.8.18. We prove the sublemma by inducting on m; note that it is vacuously true for $m = b$. Assume the sublemma holds for m. We show it holds for $m+1$. It suffices to prove the following:

(i) For all $b \leq j \leq m$, the map π^{j+m+1}_{j+m} is defined and a local isomorphism at all points of E^{j+m+1}_j.

(ii) For all $b < j \leq m+1$, the map ρ^{j+m+1}_{j+m} is defined and a local isomorphism at all points of E^{j+m+1}_j.

(iii) For all $b+m+1 < n \leq 2m+2$, the map π^n_{n-1} is defined and a local isomorphism at all points of E^n_{m+1}.

(iv) For all $b+m+1 \leq n \leq 2m+1$, the map ρ^n_{n-1} is defined and a local isomorphism at all points of E^n_{m+1}.

By symmetry it suffices to prove only (i) and (ii). We first verify that the maps are defined. In case (i), as $m \neq 0$ and so $j+m \neq j$, by Lemma 4.8.15(1) the map

π_{j+m}^{j+m+1} is defined on E_j^{j+m+1}. In case (ii), as $j \neq 1 = (j+m+1) - (j+m)$, by Lemma 4.8.15(2) the map ρ_{j+m}^{j+m+1} is defined on E_j^{j+m+1}.

Fix $b \leq j \leq m$, and consider the compositions

$$f = \pi_{j+m}^{j+m+1} \rho_{j+m+1}^{j+m+2} \cdots \rho_{2m+1}^{2m+2}$$

and

$$g = \pi_{2j}^{2j+1} \cdots \pi_{j+m-1}^{j+m}.$$

(If $j = m$, we define $g = \mathrm{Id}_{Y_{2m}}$.) By induction, g is defined at all points of E_j^{j+m}, and we have seen that f is defined at all points of E_{m+1}^{2m+2}. Further,

$$f(E_{m+1}^{2m+2}) = \pi_{j+m}^{j+m+1} \rho_{j+m+1}^{j+m+2}(E_{j+1}^{j+m+2}) = \pi_{j+m}^{j+m+1}(E_j^{j+m+1}) = E_j^{j+m}.$$

Thus gf is defined at all points in E_{m+1}^{2m+2}. Now, the rational map $q^{j+1} \cdots q^{m+1}$ is a local isomorphism at all points of E_{m+1}^{2m+2}. As gf and $q^{j+1} \cdots q^{m+1}$ agree where both are defined, we see that f is a local isomorphism at all points of E_{m+1}^{2m+2}. Thus all composition factors of f are local isomorphisms. In particular, π_{j+m}^{j+m+1} is a local isomorphism at all points of E_j^{j+m+1}, and ρ_{j+m+1}^{j+m+2} is a local isomorphism at all points of E_{j+1}^{j+m+2}. This proves that (i) and (ii) hold. $\square$

We return to the proof of Lemma 4.8.16. It follows from the sublemma (by letting m go to infinity) that if $b+1 \leq j \leq n-b-1$, both π_{n-1}^{n} and ρ_{n-1}^{n} are defined and are local isomorphisms at all points in E_j^n. This b is the integer we seek in the statement of the lemma; now let $n_1 = \max\{N_b, M_b, 2b\}$. $\square$

Lemma 4.8.19. *There is some integer k so that for all $w \in \mathbb{W}$, the σ^k-orbit $O_k(w)$ is nice.*

Proof. Clearly it suffices to prove that there is an integer k that works for all points in one σ-orbit $O(w)$. For one orbit, we may let k be the integer N from Lemma 4.6.6. $\square$

We are finally ready to prove:

Proposition 4.8.20. *Assume Assumption-Notation 4.7.1 and Notation 4.8.11. For all $n \gg 0$, the sets $\gamma_n(\alpha_n^{-1}(\mathbb{W}))$ and $\gamma_n(U_n)$ are disjoint.*

Proof. Fix $t \geq 1$. It is clearly sufficient to prove that for all $w \in \mathbb{W}$, for $n \gg 0$ the sets

$$\gamma_n(U_n)$$

and

$$\gamma_n(\alpha_n^{-1}(O_t(w)))$$

are disjoint. Applying Lemma 4.8.19 and Lemma 4.6.11, by letting t be sufficiently large we may thus reduce without loss of generality to considering nice points.

Let $w \in \mathbb{W}$ be nice, and adopt Notation 4.8.13. Let n_1 and b be the integers constructed in Lemma 4.8.16; let $N \geq \max\{n_1, 2b\}$ be such that for $n \geq N$, τ_n is an automorphism of $\gamma_n(U)$. This exists by Proposition 4.8.9.

Suppose there is some $e \in E_N$ and $u \in U_N$ such that $\gamma_N(e) = \gamma_N(u) = x$. As e and u are in different connected components of $\gamma_N^{-1}(x)$, clearly x is of finite order, say k, under τ. Let i be such that $\alpha_N(e) = w_i$.

First suppose that $i \leq N - b$. If $i \geq 0$, let $n = N + (i+1)k$; if $i < 0$, let $n = N + k$. As $i < N < n$, $(p_N^n)^{-1}$ is defined at e; let $e' = (p_N^n)^{-1}(e)$. Let $u' = (p_N^n)^{-1}(u)$. Then

$$\pi_N^n \gamma_n(u') = \gamma_N p_N^n(u') = x = \gamma_N p_N^n(e') = \pi_N^n \gamma_n(e').$$

Note that $i < N$, so π_N^n is defined at $\gamma_n(e')$. By the choice of N and n, π_N^n is one-to-one on $\gamma_n(U_n) \cup E_i^n$ and so

$$\gamma_n(e') = \gamma_n(u').$$

But now, as $(\tau_n)^k(x) = x$, we have

$$x = \pi_N^n \gamma_n(u') = \pi_N^n (\tau_n)^{n-N} \gamma_n(u') = \rho_N^n \gamma_n(u') = \rho_N^n \gamma_n(e') = \gamma_N r_N^n(e').$$

Our assumption on i ensures that $n - N \neq i$ and so $r_N^n(e')$ is well-defined. As

$$\alpha_N r_N^n(e') = w_{i-(n-N)},$$

we see that $r_N^n(e') \notin \{e', u\}$. We have produced a new point in $\gamma_N^{-1}(x)$. Continuing, we may produce infinitely many such points, which is impossible. Thus $i > N - b$.

Arguing symmetrically, we obtain that $i < b$. Since $N \geq 2b$, we see that no such e can exist. $\square$

Theorem 4.8.21. *Assume Assumption-Notation 4.7.1. Then there are a projective variety Y and a finite birational morphism $\theta : X \to Y$ such that for all $n \gg 0$ the rational map from Y to $\mathbb{P}^{N_n}$ induced by the rational functions $\overline{R}_n$ is a closed immersion at every point of $Y \smallsetminus \theta(\mathbb{W})$. Further, there are a numerically trivial automorphism ϕ of Y such that $\theta\sigma = \phi\theta$, an ample and ϕ-ample invertible sheaf $\mathcal{L}'$ on Y so that $\theta^*\mathcal{L}' = \mathcal{L}$, and a locally principal subscheme Ω' of Y so that $\Omega = \theta^*\Omega'$. Furthermore, for $n \gg 0$, the rational functions in $\overline{R}_n$ correspond to sections of the invertible sheaf $\mathcal{I}_{\Omega'} \otimes \mathcal{L}'(\mathcal{L}')^{\phi} \cdots (\mathcal{L}')^{\phi^{n-1}}$, and their base locus is equal to $\theta(W_n)$.*

Proof. We continue to use Notation 4.8.1 and Notation 4.8.11, so m is such that Y_m is stable, and $C \cup P$ is the subset of U on which β_m is not a local isomorphism. By Proposition 4.8.20, by increasing m if necessary we may assume also that

$$\gamma_m(\alpha_m^{-1}(\mathbb{W})) \cap \gamma_m(U) = \emptyset. \tag{4.8.22}$$

Let τ be the birational automorphism τ_m of Y_m.

Let $H \subseteq \mathbb{W}$ be the set

$$\{x \in \mathbb{W} \mid \text{ either } \beta_m \text{ is undefined at } x \text{ or } \beta_m \text{ is not a local isomorphism at } x\}.$$

Thus $H = \{h_1, \ldots, h_s\}$ is the finite set of "bad points" of β_m that do not lie on $C \cup P$. Let $G = \alpha_m^{-1}(H)$.

We claim that the sets $\beta_m(U \smallsetminus (C \cup P))$, $\beta_m(C \cup P)$, $\gamma_m \alpha_m^{-1}(H)$, and $\gamma_m \alpha_m^{-1}(\mathbb{W} \smallsetminus H)$ are pairwise disjoint. To see this, recall that β_m is a local isomorphism at all points of $X \smallsetminus (C \cup P \cup H)$. Thus if $x \in U \smallsetminus (C \cup P)$, then β_m^{-1} is defined at $\beta_m(x)$. As α_m^{-1} is defined at x, if $x' \in X_m$ with $\gamma_m(x') = \beta_m(x)$, then $x' = \alpha_m^{-1}(x)$. Thus $\beta_m(U \smallsetminus (C \cup P))$ is disjoint from the other three sets. That $\beta_m(C \cup P)$ is disjoint from the other sets follows; recall that $\beta_m(U) \cap \gamma_m \alpha_m^{-1}(\mathbb{W}) = \emptyset$.

If $x \in \alpha_m^{-1}(\mathbb{W} \smallsetminus H)$ and $x' \in \alpha_m^{-1}(\mathbb{W})$ with $\gamma_m(x) = \gamma_m(x')$, then note that β_m^{-1} is defined at $\gamma_m(x)$. Therefore,

$$\alpha_m(x') = \beta_m^{-1} \gamma_m(x) = \alpha_m(x)$$

and $x' \notin \alpha_m^{-1}(H)$. This completes the proof of the claim.

To construct Y, let

$$V_1 = X \smallsetminus (C \cup P)$$

and let

$$V_2 = Y_m \smallsetminus \gamma_m(G).$$

Let $V_{12} = V_1 \cap (\alpha_m \gamma_m^{-1}(V_2))$, and let $V_{21} = V_2 \cap \gamma_m \alpha_m^{-1}(V_1)$. By the claim just previous, $V_{12} = V_1 - H$ and $V_{21} = V_2 - \beta_m(C \cup P)$. Further, $\beta_m(V_{12}) = V_{21}$; note that β_m is defined and is a local isomorphism at all $x \in V_{12}$.

As β_m defines a bijection between V_{12} and V_{21} that is a local isomorphism at each point, it is an isomorphism between V_{12} and V_{21}. By [Har77, Example 2.3.5] there is a scheme Y given by glueing V_1 and V_2 along the isomorphism $\beta_m : V_{12} \to V_{21}$. For $i = 1, 2$ let ψ_i be the induced map from V_i to Y.

We now construct the automorphism ϕ of Y. Let

$$V_{22} = V_2 \smallsetminus \gamma_m(\alpha_m^{-1}(\sigma^{-1}(H))).$$

We define morphisms

$$\phi_1 = \psi_1\sigma : V_1 \to Y,$$

$$\phi_{21} = \psi_1\sigma\beta_m^{-1} : V_{21} \to Y,$$

and

$$\phi_{22} = \psi_2\tau : V_{22} \to Y.$$

We check that ϕ_1, ϕ_{21}, and ϕ_{22} are well-defined; that is, that they are in fact morphisms. First, V_1 is σ-invariant by construction, so $\sigma(V_1) \subseteq V_1$ and ϕ_1 is well-defined. Since $\beta_m^{-1}(V_{21}) = V_{12} \subseteq V_1$, ϕ_{21} is also well-defined. Now, if $y \in V_{22} \cap \gamma_m(U_m)$, then, using (4.8.22), we have that $\tau(y) \in \gamma_m(U_m) \subseteq V_2$ and so ϕ_{22} is defined at y. Finally, if $y \in V_{22} \cap \gamma_m\alpha_m^{-1}(\mathbb{W})$, then β_m^{-1} is defined at y. Let $x = \beta_m^{-1}(y) \in \mathbb{W} \smallsetminus H \smallsetminus \sigma^{-1}(H)$. As $\sigma(x) \notin H$, the map $\tau = \beta_m\sigma\beta_m^{-1}$ is defined at y. Further, β_m is a local isomorphism at $\sigma(x)$, and so $\beta_m\sigma(x) \notin \gamma_m(G)$ and $\tau(y) \in V_2$. Thus ψ_2 is defined at $\tau(y)$.

We next claim that $V_{21} \cup V_{22} = V_2$. To see this, let $y \in V_2 \smallsetminus V_{22} = \gamma_m\alpha_m^{-1}(\sigma^{-1}H) \cap V_2$. Then there is $x \in \sigma^{-1}(H)$ so that $y \in \gamma_m\alpha_m^{-1}(x)$; as $y \in V_2$, therefore $x \notin H$. As x is certainly not in $C \cup P$, we see that $x \in V_{12}$, and $\beta_m(x) = y \in V_{21}$.

The diagram

$$\begin{tikzcd}
& V_2 \arrow[dl, dashed, "\sigma\beta_m^{-1}"'] \arrow[dr, dashed, "\tau"] & \\
X \arrow[rr, dashed, "\beta_m"] \arrow[dr, dashed, "\psi_1"'] & & Y_m \arrow[dl, dashed, "\psi_2"] \\
& Y &
\end{tikzcd}$$

of rational maps commutes by construction. Note that the left side of this diagram gives ϕ_{21} and the right side gives ϕ_{22}, considered as rational maps from V_2 to Y. Thus ϕ_{21} and ϕ_{22} agree where both are defined; in particular, they agree on $V_{21} \cap V_{22}$. By [Har77, page 88], the morphisms ϕ_{21} and ϕ_{22} glue to give a birational morphism $\phi_2 : V_2 \to Y$. It is clear that $\phi_1 = \phi_2 \beta_m$ on V_{12}, and so ϕ_1 and ϕ_2 glue via $\beta_m : V_{12} \to V_{21}$ to give a morphism $\phi : Y \to Y$. As ϕ is a local isomorphism at every point of Y, it is an automorphism of Y by Lemma 4.8.2.

Now let $V_3 = X \smallsetminus H$. Note that β_m is defined on V_3, and $\beta_m(V_3) = V_2$, by (4.8.22). Define

$$\psi_3 = \psi_2 \beta_m : V_3 \to Y.$$

Now, $V_3 \cup V_1 = X$, and $V_3 \cap V_1 = V_{12}$. By construction, $\psi_3 = \psi_1$ on V_{12}. Thus we may glue ψ_1 and ψ_3 to obtain a morphism $\theta : X \to Y$. Clearly $\theta\sigma = \phi\theta$. Furthermore, as both ψ_3 and ψ_1 are finite maps, θ is finite.

Clearly Y is integral. We claim that Y is also separated. To see this, consider the diagonal $\Delta_Y = \{(y, y)\} \subseteq Y \times Y$. This is the image of the diagonal $\Delta_X \subseteq X \times X$ under the finite morphism $\theta \times \theta$. As X is separated, Δ_X is closed. By [Har77, Exercise 3.5] the finite morphism $\theta \times \theta$ is closed. Thus Δ_Y is also closed, and so Y is separated. Thus Y is a variety.

For all $n \geq 1$, the rational functions $\overline{R}_n$ induce a rational map $\mu_n : Y \to \mathbb{P}^{N_n}$. By construction, for $n \geq m$, the indeterminacy locus of μ_n is equal to $\theta(W_n)$. In particular, it is contained in $\theta(\mathbb{W})$ and so supported at smooth points of Y. Further, note that if $n \geq m$ and $x \in \theta(U)$, that locally at x the rational map μ_n factors through the local isomorphism

$$(\pi_m^n)^{-1} \psi_2^{-1} : Y \dashrightarrow Y_n \subseteq \mathbb{P}^{N_n}.$$

Thus μ_n is locally a closed immersion at any point of $Y \smallsetminus \theta(\mathbb{W})$.

By resolving the indeterminacy locus of μ_n, we obtain a variety Y'_n, a morphism $\xi_n : Y'_n \to Y$ and a morphism $\nu_n : Y'_n \to \mathbb{P}^{N_n}$ so that the diagram

$$\begin{tikzcd} Y'_n \arrow[d, "\xi_n"'] \arrow[dr, "\nu_n"] & \\ Y \arrow[r, dashed, "\mu_n"] & \mathbb{P}^{N_n} \end{tikzcd}$$

commutes. For all n, let $\mathcal{N}_n = \nu_n^* \mathcal{O}(1)$ and let

$$\mathcal{K}_n = \big((\xi_n)_* \mathcal{N}_n\big)^{**}.$$

Away from the indeterminacy locus of μ_n, $\mathcal{K}_n$ is isomorphic to $\mu_n^* O(1)$ and is invertible. As any rank 1 reflexive module over a regular local ring is invertible, $\mathcal{K}_n$ is invertible on the indeterminacy locus of μ_n as well, and therefore is an invertible sheaf on Y for all $n \geq m$. Thus $\overline{R}_n \subseteq H^0(\mathcal{K}_n)$, and the (set-theoretic) base locus of the sections $\overline{R}_n$ of $\mathcal{K}_n$ is precisely $\theta(W_n)$ for $n \geq m$.

For $n \geq m$, consider the Weil divisor corresponding to the invertible sheaf $\theta^* \mathcal{K}_n$ on X. Away from the finitely many points in $\operatorname{Supp} W_n$, this is equal to $\Delta_n - \Omega$. As X is smooth at all points of $\operatorname{Supp} W_n$, by extending this equality to all of X, we obtain that

$$\mathcal{I}_\Omega \mathcal{L}_n = \mathcal{O}_X(\Delta_n - \Omega) = \theta^* \mathcal{K}_n$$

for $n \geq m$.

Let $\mathcal{L}' = \big(\mathcal{K}_m (\mathcal{K}_{m+1})^{-1}\big)^{\phi^{-m}}$, and let $\mathcal{M} = \mathcal{K}_m^{-1} \otimes \mathcal{L}' \otimes (\mathcal{L}')^{\phi} \otimes \cdots \otimes (\mathcal{L}')^{\phi^{m-1}}$. Then

$$\theta^* \mathcal{L}' = \big((\mathcal{I}_\Omega \mathcal{L}_m)(\mathcal{I}_\Omega \mathcal{L}_{m+1})^{-1}\big)^{\sigma^{-m}} \cong \big(\mathcal{L}^{\sigma^m}\big)^{\sigma^{-m}} \cong \mathcal{L},$$

and

$$\theta^* \mathcal{M} = (\mathcal{I}_\Omega \mathcal{L}_m)^{-1} \mathcal{L}_m \cong (\mathcal{I}_\Omega)^{-1}.$$

As $\theta^*\mathcal{M}$ corresponds to an effective Cartier divisor, so does $\mathcal{M}$; that is, $\mathcal{M}^{-1}$ is an ideal sheaf defining a locally principal curve on Y. We will denote this curve by Ω'; by construction, $\theta^*\Omega' = \Omega$. Note that $\mathcal{K}_n \cong \mathcal{I}_{\Omega'}(\mathcal{L}')_n$.

Recall that $\mathcal{L}$ is ample. As θ is finite, $\mathcal{L}'$ is ample by [Gro61, Proposition 2.6.2]. Thus Y carries an ample line bundle and so is projective. The numeric action of ϕ is clearly still trivial, and so $\mathcal{L}'$ is also ϕ-ample by [AV90, Theorem 1.7]. $\square$

We remark that the fact that Y is a projective variety may also be deduced from [RS06, Proposition 7.4].

4.9 The proof of the main theorem

At this point, we are very close to finishing the proof of Theorem 4.1.4. Starting with a birationally commutative projective surface R, we have produced transverse surface data $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ so that (after replacing R by a Veronese subring) R is contained in $T(\mathbb{D})$, and the bimodule algebras $\mathcal{R}(X)$ and $\mathcal{T}(\mathbb{D})$ are equal. We then showed that there are another surface Y and a finite birational morphism $\theta : X \to Y$, so that Y has an automorphism ϕ conjugate to σ and carries a ϕ-ample line bundle $\mathcal{L}'$ that pulls back under θ to $\mathcal{L}$. We further showed that the rational functions in $\overline{R}_n$ define a closed immersion at any point of $Y \smallsetminus \theta(\mathbb{W}) = \theta(U)$ for $n \gg 0$.

We claim that we may construct transverse surface data

$$\mathbb{E} = (Y, \mathcal{L}', \phi, \mathcal{A}', \mathcal{D}', \mathcal{C}', \Omega', \Phi, \Phi')$$

on Y so that some Veronese of R is actually equal to the ring $T(\mathbb{E})$. We do this in the next few propositions. We then combine our results to prove Theorem 4.1.4.

Let $\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$ be transverse surface data, and suppose that $R \subseteq T(\mathbb{D})$ is a graded ring. We begin by establishing sufficient conditions for R

to actually be equal to $T(\mathbb{D})$ in large degree. Our methods involve reducing the question to one involving subrings of twisted homogeneous coordinate rings of σ-invariant curves in X. We wish to use the results of [AS95] on subrings of idealizer rings on curves; however, as those were proved only for reduced and irreducible curves, we repeat the proofs here in a more general context.

Theorem 4.9.1. *Suppose that the surface data*

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

is transverse. Let $T = T(\mathbb{D})$ and let $\mathcal{T} = \mathcal{T}(\mathbb{D})$. Let R be a subalgebra of T with $R_1 \neq 0$, and fix $0 \neq z \in R_1$. Let $\overline{R}_n = R_n z^{-1}$ and let $\mathcal{R}_n(X) = \overline{R}_n \cdot \mathcal{O}_X$.

Suppose that $\mathcal{R}_n(X) = \mathcal{T}_n$ for $n \gg 0$. Let Z be the cosupport of $\mathcal{D}$ and let

$$\mathbb{W} = \bigcup_{p \in \Lambda \cup \Lambda' \cup Z} O(p).$$

Further assume that for all $n \gg 0$, the rational map defined on X by the rational functions in $\overline{R}_n$ is birational onto its image and is a closed immersion at each point in $X \smallsetminus \mathbb{W}$. Then $R_n = T_n$ for $n \gg 0$.

We will prove Theorem 4.9.1 in several steps. We first establish some notation. If Σ is a σ-invariant proper subscheme of X, then σ restricts to an automorphism of Σ, which we also denote by σ. For any such Σ, let $B_\Sigma = B(\Sigma, \mathcal{L}|_\Sigma, \sigma)$. We may consider T and R to be subrings of $B(X, \mathcal{L}, \sigma)$; we will let T_Σ, respectively R_Σ, be the image of T, respectively R, under the natural map from $B(X, \mathcal{L}, \sigma)$ to B_Σ.

Proof of Theorem 4.9.1. By Lemma 4.2.7, the sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left and right ample; thus by Lemma 4.7.5, T is a finitely generated left and right R-module. Let $J_l = \mathrm{l.ann_R(T/R)}$ and let $J_r = \mathrm{r.ann_R(T/R)}$. Note that J_l is a graded right ideal of T and that J_r is a graded left ideal of T. Our assumptions imply that

R and T have the same graded quotient ring, and thus J_l and J_r are nonzero. Let $K = J_r J_l$. Then $K \neq 0$ is a nonzero graded ideal both of R and of T. Note also that by Proposition 4.2.13, both $\mathcal{T}$ and T are left and right noetherian.

By Corollary 4.2.25, there is a σ-invariant ideal sheaf $\mathcal{K}$ on X such that for $n \gg 0$, we have that $K_n = H^0(\mathcal{K}\mathcal{R}_n)z^n$. Let Σ be the σ-invariant closed subscheme defined by $\mathcal{K}$; then $\dim \Sigma \leq 1$. By transversality of the defining data for $\mathcal{R}$, the σ-invariant subscheme Σ is disjoint from $\mathbb{W}$, and $\Omega \cap \Sigma$ consists of points of infinite order. Let $\mathcal{J}$ be the ideal sheaf on Σ of the scheme-theoretic intersection $\Omega \cap \Sigma$. Since $\mathcal{T}or_1^X(\mathcal{O}_\Omega, \mathcal{O}_\Sigma) = 0$ by critical transversality of $\{\sigma^n \Omega\}$, the natural map from

$$\mathcal{I}_\Omega \mathcal{L}_n \otimes \mathcal{O}_\Sigma \to \mathcal{L}_n \otimes \mathcal{O}_\Sigma$$

is injective, and we see that $\mathcal{R}_n|_\Sigma = \mathcal{J}(\mathcal{L}_n|_\Sigma)$ for $n \geq 1$.

Note that R/K and R_Σ are equal in large degree, and T/K and T_Σ are equal in large degree. Note also that as for $n \gg 0$ the rational functions in $\overline{R}_n$ define a closed immersion at all points of $X \smallsetminus \mathbb{W}$, that their restrictions to $\Sigma \subseteq X \smallsetminus \mathbb{W}$ also define a closed immersion for $n \gg 0$.

We claim that R_Σ and T_Σ are equal in large degree. Before proving this claim, we give a lemma generalizing a result of Artin and Stafford.

Lemma 4.9.2. (cf. [AS95, Lemma 4.6]) *Suppose, in addition, that there are no proper σ-invariant subschemes Y of Σ so that $(T_Y)/(R_Y)$ is infinite-dimensional, and that there are σ-invariant ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_\ell \subseteq \mathcal{O}_\Sigma$ so that $\mathcal{I}_1\mathcal{I}_2 \cdots \mathcal{I}_\ell = 0$ on Σ. Then T_Σ/R_Σ is finite-dimensional.*

Proof. The proof is similar to the proof of [AS95, Lemma 4.6]; we give it in detail because some of the details are different in our slightly more general context.

Suppose, in contrast, that T_Σ/R_Σ is infinite-dimensional. We first note that if J is a nonzero graded ideal of T_Σ, then there is a graded ideal $J' \supseteq K$ of T so that $J = J'/K$ in large degree. By Corollary 4.2.25, in large degree J' consists of sections of $\mathcal{R}_n$ that vanish on some σ-invariant proper subscheme Y of Σ, and so (in large degree) $T_\Sigma/J = T_Y$. If J were also an ideal of R_Σ, then as by hypothesis R_Y and T_Y are equal in large degree, we would have that R_Σ and T_Σ are equal in large degree.

Thus R_Σ and T_Σ have no nonzero ideals in common. By induction, we may assume that $\ell = 2$. Let Z_1 and Z_2, respectively, be the subschemes of Y defined by $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Let $\mathcal{M} = \mathcal{L}|_\Sigma$ and let $B = B(\Sigma, \mathcal{M}, \sigma)$. For $i = 1, 2$, let

$$K_i = \Gamma^*(\mathcal{I}_i) = \bigoplus_{n \geq 0} H^0(\Sigma; \mathcal{I}_i \mathcal{M}_n) z^n \subseteq B,$$

and let $M_i = K_i \cap T_\Sigma$. Note that the M_i are two-sided ideals of T_Σ. As $\mathcal{I}_1$ is an $\mathcal{O}_{Z_2}$-module, the right and left actions of T_Σ on M_1 factor through $T_2 = T_\Sigma/M_2$.

Now, M_1 is a finitely generated left and right T_2-module, because T_2 is a factor of the noetherian ring T and is therefore noetherian. Let $R_2 = (R_\Sigma + M_2)/M_2 \subseteq T_2$. By hypothesis, R_2 and T_2 are equal in large degree. Thus R_2 is noetherian, and both M_1 and $N = R_\Sigma \cap M_1$ are finitely generated left and right R_2-modules. Let $N' = T_2 N T_2 \subseteq M_1$. Let V be a finite-dimensional subspace of N such that $T_2 V T_2 = N'$. Then, as R_2 and T_2 are equal in large degree, we have that $(T_2 V T_2)_n = (R_2 V R_2)_n$ for $n \gg 0$. Thus for $n \gg 0$, we have

$$N_n \subseteq N'_n = (T_2 V T_2)_n = (R_2 V R_2)_n \subseteq N_n.$$

Thus N'/N is finite-dimensional. There is thus some n_0 so that $N_{\geq n_0} = N'_{\geq n_0}$ is a left and right T_2-module. That is, $N_{\geq n_0}$ is an ideal of T_Σ. As $N_{\geq n_0} \subseteq R_\Sigma$ and R_Σ and T_Σ have no nonzero ideals in common, $N_{\geq n_0} = 0$.

Therefore, $(R_\Sigma)_{\geq n_0} \cap \Gamma^*(\mathcal{I}_1) = 0$. That is, $(R_\Sigma)_{\geq n_0} \hookrightarrow T_1$. This implies that the map defined by the sections $(\overline{R}_\Sigma)_n$ of $\mathcal{M}_n$ factors through Z_1 for $n \geq n_0$, and so is not an embedding. This gives a contradiction. $\square$

We return to the proof of Theorem 4.9.1. We show that R_Σ and T_Σ are equal in large degree. By noetherian induction on Σ, we may assume that for any proper σ-invariant closed subscheme $Y \subseteq \Sigma$, that R_Y has finite codimension in T_Y.

We first suppose that Σ is not irreducible. Let k be such that σ^k fixes all irreducible components of Σ. The hypotheses of Lemma 4.9.2 thus hold for $R_\Sigma^{(k)}$ and $T_\Sigma^{(k)}$. Applying Lemma 4.9.2, we see that $T_\Sigma^{(k)}/R_\Sigma^{(k)}$ is finite-dimensional.

We show that this implies that T_Σ/R_Σ is finite-dimensional. Let $\mathcal{F}_n = \mathcal{R}_n|_\Sigma$, and let

$$\mathcal{F} = \bigoplus_{n \geq 0} (\mathcal{F}_n)_{\sigma^n}.$$

The noetherian property of $\mathcal{T}$ descends to the $\mathcal{O}_\Sigma$-bimodule algebra $\mathcal{F}$, and so $\mathcal{F}$ and its Veronese $\mathcal{F}^{(k)}$ are noetherian. As the restriction of an ample sequence to a σ-invariant subscheme, the sequence of bimodules $\{(\mathcal{F}_{rk})_{\sigma^{rk}}\}$ is left and right ample.

Recall that $\mathcal{J} = \mathcal{I}_\Omega \mathcal{O}_\Sigma \subseteq \mathcal{O}_\Sigma$ and that $\mathcal{M} = \mathcal{L}|_\Sigma$. Fix $0 \leq i \leq k-1$, and let

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{J}\mathcal{M}_{i+nk}.$$

The sheaf $\mathcal{J}\mathcal{M}_{k-i}$ is an invertible sheaf on Σ, so by Lemma 2.3.14, the submodule lattices of the right $\mathcal{F}^{(k)}$-modules $\mathcal{P}$ and

$$(\mathcal{J}\mathcal{M}_{k-i})_{\sigma^{k-i}} \otimes \mathcal{P} \cong \bigoplus_{n \geq 0} (\mathcal{J}\mathcal{J}^{\sigma^{k-i}} \mathcal{M}_{nk})_{\sigma^{nk}} \subseteq \mathcal{F}^{(k)}$$

are isomorphic. In particular, $\mathcal{P}$ is a coherent right $\mathcal{F}^{(k)}$-module.

Fix n_0 so that if $n \geq i + n_0 k$, then $(\overline{R}_\Sigma)_n$ generates $\mathcal{P}_n$. Let $n_1 \geq n_0$ be such that

$$\mathcal{F}_{i+n_0k} \oplus \cdots \oplus \mathcal{F}_{i+n_1k}$$

generates $\mathcal{P}_{\geq i+n_0k}$ as a right $\mathcal{F}^{(k)}$-module. Then for $r \geq n_1$, we have

$$(\overline{T}_\Sigma)_{i+rk} = H^0(\sum_{j=n_0}^{n_1} \mathcal{F}_{i+jk}\mathcal{F}_{(r-j)k}^{\sigma^{i+jk}}) = \sum_{j=n_0}^{n_1} H^0(\mathcal{F}_{i+jk}\mathcal{F}_{(r-j)k}^{\sigma^{i+jk}}).$$

By Lemma 4.7.4, for fixed ℓ and for $r \gg 0$, we have

$$(R_\Sigma)_\ell (T_\Sigma)_{rk} = H^0(\mathcal{F}_\ell \mathcal{F}_{rk}^{\sigma^\ell}) z^{\ell+rk}.$$

Recall that $R_\Sigma^{(k)}$ and $T_\Sigma^{(k)}$ are equal in large degree. Thus, by taking $r \gg 0$, we obtain that

$$(R_\Sigma)_{i+rk} \subseteq (T_\Sigma)_{i+rk} = \sum_{j=n_0}^{n_1} (R_\Sigma)_{i+jk}(T_\Sigma)_{(r-j)k} = \sum_{j=n_0}^{n_1} (R_\Sigma)_{i+jk}(R_\Sigma)_{(r-j)k} \subseteq (R_\Sigma)_{i+rk}.$$

Since this holds for $0 \leq i \leq k-1$, R_Σ has finite codimension in T_Σ.

Now suppose that Y is irreducible but not reduced. Then the nilradical $\mathcal{N}$ of $\mathcal{O}_\Sigma$ is a σ-invariant nilpotent ideal sheaf on Σ; so the hypotheses of Lemma 4.9.2 hold for R_Σ and T_Σ, with $\mathcal{I}_1 = \mathcal{I}_2 = \cdots = \mathcal{I}_\ell = \mathcal{N}$. We see again that T_Σ/R_Σ is finite-dimensional.

Thus we have reduced to considering the case that Σ is reduced and irreducible. Now, if $\Omega \cap \Sigma = \emptyset$, then R_Σ and T_Σ are equal in large degree by [AS95, Theorem 4.1]; in particular, this holds if Σ is a point. If $\Omega \cap \Sigma$ is nonempty, and Σ is a reduced and irreducible curve, then T_Σ/R_Σ is finite-dimensional by [AS95, Proposition 5.4].

We have thus shown that there is an ideal K of T that is contained in R and so that $(R_K)_n = (R_\Sigma)_n = (T_\Sigma)_n = (T/K)_n$ for $n \gg 0$. Thus $R_n = T_n$ for $n \gg 0$. $\square$

We now prove Theorem 4.1.4.

Proof of Theorem 4.1.4. One direction is Proposition 4.2.13. For the other direction, suppose that R is a birationally commutative projective surface. By Corollary 4.7.13, there are a positive integer ℓ and transverse surface data

$$\mathbb{D} = (X, \mathcal{L}, \sigma, \mathcal{A}, \mathcal{D}, \mathcal{C}, \Omega, \Lambda, \Lambda')$$

so that X is normal, $R_\ell \neq 0$, $\overline{R}_\ell$ generates K, and

$$\mathcal{R}(X)^{(\ell)} = \mathcal{T}(\mathbb{D}).$$

Note that Assumption-Notation 4.7.1 holds for $R^{(\ell)}$.

Let Z be the cosupport of $\mathcal{D}$. Recall that for any $p \in X$, we denote the σ^ℓ-orbit of p by $O_\ell(p)$. Let

$$\mathbb{W} = \bigcup_{p \in \Lambda \cup \Lambda' \cup Z} O_\ell(p).$$

By Theorem 4.8.21 there are a projective variety Y, a numerically trivial automorphism ϕ of Y, an ample invertible sheaf $\mathcal{L}'$ on Y, a locally principal subscheme Ω' of Y, and a finite birational morphism $\theta : X \to Y$ so that for $n \gg 0$ the rational functions $\overline{R}_{n\ell}$ induce a closed immersion into projective space at every point of $Y \smallsetminus \theta(\mathbb{W})$ and so that $\theta\sigma^\ell = \phi\theta$, $\theta^*\mathcal{L}' = \mathcal{L}_\ell$, and $\theta^*\Omega' = \Omega$; further, set-theoretically the base locus of the sections $\overline{R}_{n\ell}$ of $\mathcal{I}_{\Omega'}(\mathcal{L}')_n$ on Y is equal to $\theta(W_{n\ell})$.

Now, θ is a local isomorphism at every point of $\mathbb{W}$. Let $\mathcal{A}'$ be the ideal sheaf on Y that is cosupported on $\theta(Z)$ and so that for every $w \in Z$, the stalks $\mathcal{A}_w$ and ${\mathcal{A}'}_{\theta(w)}$ are isomorphic; similarly define ideal sheaves $\mathcal{D}'$ and $\mathcal{C}'$. The ideal sheaves $\mathcal{A}'$, $\mathcal{D}'$, and $\mathcal{C}'$ on Y pull back to $\mathcal{A}, \mathcal{D}$, and $\mathcal{C}$ respectively. Furthermore, by working locally at each point of $\theta(Z)$, we see that $\mathcal{A}'\mathcal{C}' \subseteq \mathcal{D}'$, and the pair $(\mathcal{A}', \mathcal{C}')$ is maximal with respect to this property. As distinct points in the cosupport of $\mathcal{D}$ have distinct σ-orbits, distinct points in the cosupport of $\mathcal{D}'$ have distinct ϕ-orbits.

Let Φ be the scheme-theoretic image of Λ under θ, and let Φ' be the scheme-theoretic image of Λ'. Let

$$\mathbb{D}' = (Y, \mathcal{L}', \phi, \mathcal{A}', \mathcal{D}', \mathcal{C}', \Omega', \Phi, \Phi').$$

By construction, $\mathcal{R}(Y)^{(\ell)} = \mathcal{T}(\mathbb{D}')$. We claim that the data $\mathbb{D}'$ is transverse.

Let Z' be the subscheme of Y defined by $\mathcal{D}'$. We first show that $\{\phi^n(\Phi)\}_{n\geq 0}$, $\{\phi^n(Z')\}_{n\in\mathbb{Z}}$, and $\{\phi^n(\Phi')_{n\leq 0}$ are critically transverse. Applying Corollary 3.3.15 and using symmetry, it suffices to show that if $w \in \Lambda \cup Z$, then $\{\phi^n\theta(w)\}_{n\geq 0}$ is critically dense. Fix $w \in \Lambda \cup Z$, and suppose there is some nonzero curve $\Gamma \subset Y$ so that for infinitely many $n \geq 0$, $\phi^n\theta(w) \in \Gamma$. Therefore, for infinitely many $n \geq 0$, $\sigma^{n\ell}(w) \in \theta^{-1}(\Gamma)$. This contradicts the transversality of the data $\mathbb{D}$ on X.

We now show that $\{\phi^n\Omega'\}_{n\in\mathbb{Z}}$ is critically transverse. By Lemma 3.3.12, it suffices to prove that

$$\{n \mid \mathcal{T}or_1^Y(\mathcal{O}_{\Omega'}, \mathcal{O}_{\phi^n\Gamma}) \neq 0\} \tag{4.9.3}$$

is finite for all reduced and irreducible $\Gamma \subseteq Y$. As Ω' is locally principal and Γ is reduced and irreducible, (4.9.3) is equal to

$$\{n \mid \phi^n\Gamma \subseteq \Omega'\} \tag{4.9.4}$$

for any reduced and irreducible $\Gamma \subseteq Y$.

Suppose that (4.9.4) is infinite for some reduced and irreducible $\Gamma \subseteq Y$. Pulling back to X, we obtain that

$$\{n \mid \sigma^{n\ell}\Gamma' \subseteq \Omega\}$$

is infinite for some irreducible component Γ' of $\theta^{-1}\Gamma$. This does not happen, by transversality of $\mathbb{D}$. Thus (4.9.4) is finite for all Γ, and $\{\phi^n\Omega'\}_{n\in\mathbb{Z}}$ is critically transverse. Thus the surface data $(Y, \mathcal{L}', \phi, \mathcal{A}', \mathcal{D}', \mathcal{C}', \Omega', \Phi, \Phi')$ is transverse.

We have seen that for $n \gg 0$, the sections in $\overline{R}_{n\ell}$ define a closed immersion at all points of $Y \smallsetminus \theta(\mathbb{W})$. Theorem 4.9.1 now implies that there is some $k \geq 1$ so that

$$R_{n\ell} = T_n$$

for $n \geq k$. Thus if $\mathbb{D}''$ is the surface data given by Lemma 4.6.11 so that

$$\mathcal{T}(\mathbb{D}')^{(k)} = \mathcal{T}(\mathbb{D}''),$$

then

$$R^{(k\ell)} = T(\mathbb{D}'').$$

This is precisely what we sought to prove. $\square$

To end this chapter, we make a few remarks on a possible extension of Theorem 4.1.4 to rings of GK-dimension 5: that is, to graded noetherian domains R whose graded quotient ring is of the form

$$K[z, z^{-1}; \sigma]$$

for some field K of transcendence degree 2 and geometric, but non-quasi-trivial automorphism σ of K. (Recall from Theorem 4.3.2 that such a ring must have GK-dimension 5.)

There some significant technical issues involved in extending Theorem 4.1.4 to the GK-dimension 5 case. For example, Lemma 4.5.5, where we prove that the coordinate divisor is, in fact, nef, depends on the quasi-triviality of σ. Although this result is elementary in the GK-dimension 3 case, we have not been able to extend it to GK-dimension 5. The GK 3 assumption is also used in Theorem 4.5.13, to show that the set of curves contracted by Δ_n is σ-invariant.

We conjecture that a similar result to Theorem 4.1.4 holds in this case; that is, that all such R correspond (up to a Veronese, of course) to quasi-transverse surface data. One possible avenue of approach is to use the Enriques classification of projective surfaces, which we have not so far used significantly. This puts strong constraints on the situations where GK 5 automorphisms can occur. We plan to pursue this further in future work.

CHAPTER V

A general homological Kleiman-Bertini theorem

5.1 Introduction

All schemes that we consider in this chapter are of finite type over a fixed field, which we denote in this chapter by k; we make no assumptions on the characteristic, cardinality, or algebraic closure of k.

Recall that two subschemes Y and Z of X are *homologically transverse* if, for all $j \geq 1$, we have that $\mathcal{T}or_j^X(\mathcal{O}_Y, \mathcal{O}_Z) = 0$. In this chapter, we investigate geometric questions relating to homological transversality. These questions were motivated by the investigations of idealizers in Chapter III. In that chapter, we saw that if X is a projective variety, σ an automorphism of X, $\mathcal{L}$ a σ-ample invertible sheaf on X, and Z a closed subscheme of X, then one may form the *geometric idealizer*

$$R(X, \mathcal{L}, \sigma, Z) \subseteq B(X, \mathcal{L}, \sigma),$$

and that the properties of $R(X, \mathcal{L}, \sigma, Z)$ are controlled by the *critical transversality* of the set $\{\sigma^n Z\}$: for any closed subscheme Y of X, one wants $\sigma^n Z$ and Y to be homologically transverse for all but finitely many n. One is naturally, then, led to ask how often homological transversality can be considered "generic" behavior, and what conditions on Z ensure this.

Our intuition leads us to believe that two subvarieties in general position, in the appropriate sense, will be homologically transverse. This is often true, and can be made more precise in many situations by the following Bertini-type result of Miller and Speyer. We will say that two coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on X are homologically transverse if their higher $\mathcal{T}or$ sheaves all vanish.

Theorem 5.1.1. [MS06] *Let X be a variety with a transitive left action of a smooth algebraic group G. Let $\mathcal{F}$ and $\mathcal{E}$ be coherent sheaves on X, and for all k-points $g \in G$, let $g\mathcal{F}$ denote the pushforward of $\mathcal{F}$ along multiplication by g. Then there is a dense Zariski open subset U of G such that, for all k-rational points $g \in U$, the sheaves $g\mathcal{F}$ and $\mathcal{E}$ are homologically transverse.*

As Miller and Speyer remark, their result is a homological generalization of the Kleiman-Bertini theorem: in characteristic 0, if $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{E} = \mathcal{O}_Y$ are structure sheaves of smooth subvarieties of X and G acts transitively on X, then gZ and Y meet transversally for generic g, implying that gZ and Y are homologically transverse.

Homological transversality has a geometric meaning if $\mathcal{F} = \mathcal{O}_Z$ and $\mathcal{E} = \mathcal{O}_Y$ are structure sheaves of closed subschemes of X. If P is a component of $Y \cap Z$, then Serre's formula for the multiplicity of the intersection of Y and Z at P [Har77, p. 427] is:

$$i(Y, Z; P) = \sum_{j \geq 0} (-1)^j \operatorname{len}_P(\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})),$$

where the length $\operatorname{len}_P(\_\_)$ is taken over the local ring at P. Thus if Y and Z are homologically transverse, their intersection multiplicity at P is simply the length of their scheme-theoretic intersection over the local ring at P.

It is natural to ask what conditions on the action of G are necessary to conclude that homological transversality holds generically in the sense of Theorem 5.1.1. In particular, the restriction to transitive actions is unfortunately strong, as it excludes important situations such as the torus action on $\mathbb{P}^n$. On the other hand, suppose that $\mathcal{F}$ is the structure sheaf of the closure of a non-dense orbit. Then for all k-points $g \in G$, we have $\mathcal{T}or_1^X(g\mathcal{F}, \mathcal{F}) = \mathcal{T}or_1^X(\mathcal{F}, \mathcal{F}) \neq 0$, and so the conclusion of Theorem 5.1.1 fails (as long as $G(k)$ is dense in G). Thus for non-transitive group actions some additional hypothesis is necessary.

The main result of this chapter is that there is a simple condition for homological transversality to be generic. This is:

Theorem 5.1.2. *Let X be a variety with a left action of a smooth algebraic group G, and let $\mathcal{F}$ be a coherent sheaf on X. Let $\overline{k}$ be an algebraic closure of k. Consider the following conditions:*

(1) *For all closed points $x \in X \times \overline{k}$, the pullback of $\mathcal{F}$ to $X \times \overline{k}$ is homologically transverse to the closure of the $G(\overline{k})$-orbit of x;*

(2) *For all coherent sheaves $\mathcal{E}$ on X, there is a Zariski open and dense subset U of G such that for all k-rational points $g \in U$, the sheaf $g\mathcal{F}$ is homologically transverse to $\mathcal{E}$.*

Then (1) $\Rightarrow$ (2)*. If k is algebraically closed, then* (1) *and* (2) *are equivalent.*

If g is not k-rational, the sheaf $g\mathcal{F}$ can still be defined; in Section 5.2 we give this definition and a generalization of (2) that is equivalent to (1) in any setting (see Theorem 5.2.1).

If G acts transitively on X in the sense of [MS06], then the action is geometrically transitive, and so (1) is trivially satisfied. Thus Theorem 5.1.1 follows from Theorem 5.1.2. Since transversality of smooth subvarieties in characteristic 0 implies homological transversality, Theorem 5.1.2 also generalizes the following result of Robert Speiser:

Theorem 5.1.3. [Spe88, Theorem 1.3] *Suppose that k is algebraically closed of characteristic 0. Let X be a smooth variety, and let G be a (necessarily smooth) algebraic group acting on X. Let Z be a smooth closed subvariety of X. If Z is transverse to every G-orbit in X, then for any smooth closed subvariety $Y \subseteq X$, there is a dense open subset U of G such that if $g \in U$, then gZ and Y are transverse.*

We remark that for the set U we construct in Theorem 5.1.2, for any extension k' of k and any k'-rational $g \in U \times k'$, then $g\mathcal{F}$ will be homologically transverse to $\mathcal{E}$ on $X \times k'$. Further, in many situations U will automatically contain a k-rational point of G. This holds, in particular, if k is infinite, G is connected and affine, and either k is perfect or G is reductive, by [Bor91, Corollary 18.3].

We make some remarks on notation. If x is any point of a scheme X, we denote the skyscraper sheaf at x by k_x. For schemes X and Y, we will write $X \times Y$ for the product $X \times_k Y$. If k' is a field containing k, then we write $X \times k'$ for $X \times \operatorname{Spec} k'$. Finally, if X is a scheme with a (left) action of an algebraic group G, we will always denote the multiplication map by $\mu : G \times X \to X$.

5.2 Generalizations

We begin this section by defining homological transversality more generally. If W and Y are schemes over a scheme X, with (quasi)coherent sheaves $\mathcal{F}$ on W and $\mathcal{E}$ on Y respectively, then for all $j \geq 0$ there is a (quasi)coherent sheaf $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})$ on $W \times_X Y$. This sheaf is defined locally. Suppose that $X = \operatorname{Spec} R$, $W = \operatorname{Spec} S$ and $Y = \operatorname{Spec} T$ are affine. Let $(\_)\tilde{}$ denote the functor that takes an R-module (respectively S- or T-module) to the associated quasicoherent sheaf on X (respectively W or Y). If F is an S-module and E is a T-module, we define $\mathcal{T}or_j^X(\widetilde{F}, \widetilde{E})$ to be $(\operatorname{Tor}_j^R(F, E))\tilde{}$. That these glue properly to give sheaves on $W \times_X Y$ for general W, Y, and X is

[Gro63, 6.5.3]. As before, we will say that $\mathcal{F}$ and $\mathcal{E}$ are *homologically transverse* if the sheaf $\mathcal{T}or_j^X(\mathcal{F},\mathcal{E})$ is zero for all $j \geq 1$.

We caution the reader that the maps from W and Y to X are implicit in the definition of $\mathcal{T}or_j^X(\mathcal{F},\mathcal{E})$; at times we will write $\mathcal{T}or_j^{W\to X\leftarrow Y}(\mathcal{F},\mathcal{E})$ to make this more obvious. We also remark that if $Y = X$, then $\mathcal{T}or_j^X(\mathcal{F},\mathcal{E})$ is a sheaf on $W\times_X X = W$. As localization commutes with Tor, for any $w \in W$ lying over $x \in X$ we have in this case that $\mathcal{T}or_j^X(\mathcal{F},\mathcal{E})_w = \mathrm{Tor}_j^{\mathcal{O}_{X,x}}(\mathcal{F}_w,\mathcal{E}_x)$.

Now suppose that $f : W \to X$ is a morphism of schemes and G is an algebraic group acting on X. Let $\mathcal{F}$ be a (quasi)coherent sheaf on W and let g be any point of G. We will denote the pullback of $\mathcal{F}$ to $\{g\}\times W$ by $g\mathcal{F}$. There is a map

$$\{g\}\times W \longrightarrow G\times W \xrightarrow{1\times f} G\times X \xrightarrow{\mu} X.$$

If Y is a scheme over X and $\mathcal{E}$ is a (quasi)coherent sheaf on Y, we will write $\mathcal{T}or_j^X(g\mathcal{F},\mathcal{E})$ for the (quasi)coherent sheaf $\mathcal{T}or_j^{\{g\}\times W\to X\leftarrow Y}(g\mathcal{F},\mathcal{E})$ on $W\times_X Y\times k(g)$. Note that if $W = X$ and g is k-rational, then $g\mathcal{F}$ is simply the pushforward of $\mathcal{F}$ along multiplication by g.

In this context, we prove the following relative version of Theorem 5.1.2:

Theorem 5.2.1. *Let X be a scheme with a left action of a smooth algebraic group G, let $f : W \to X$ be a morphism of schemes, and let $\mathcal{F}$ be a coherent sheaf on W. We define maps:*

$$\begin{array}{ccc} G\times W & \xrightarrow{\rho} & X \\ {\scriptstyle p}\downarrow & & \\ W & & \end{array}$$

where ρ is the map $\rho(g,w) = gf(w)$ induced by the action of G and p is projection onto the second factor.

Then the following are equivalent:

(1) *For all closed points $x \in X \times \overline{k}$, the pullback of $\mathcal{F}$ to $W \times \overline{k}$ is homologically transverse to the closure of the $G(\overline{k})$-orbit of x;*

(2) *For all schemes $r : Y \to X$ and all coherent sheaves $\mathcal{E}$ on Y, there is a Zariski open and dense subset U of G such that for all closed points $g \in U$, the sheaf $g\mathcal{F}$ on $\{g\} \times W$ is homologically transverse to $\mathcal{E}$.*

(3) *The sheaf $p^*\mathcal{F}$ on $G \times W$ is ρ-flat over X.*

A related relative version of Theorem 5.1.3 is given in [Spe88].

Our general approach to Theorem 5.2.1 mirrors that of [Spe88], although the proof techniques are quite different. We first generalize Theorem 5.1.1 to apply to any flat map $f : W \to X$; this is a homological version of [Kle74, Lemma 1] and may be of independent interest.

Theorem 5.2.2. *Let X, Y, and W be schemes, let A be a generically reduced scheme, and suppose that there are morphisms:*

$$\begin{tikzcd} & Y \arrow[d, "r"] \\ W \arrow[r, "f"] \arrow[d, "q"'] & X \\ A. & \end{tikzcd}$$

Let $\mathcal{F}$ be a coherent sheaf on W that is f-flat over X, and let $\mathcal{E}$ be a coherent sheaf on Y. For all $a \in A$, let W_a denote the fiber of W over a, and let $\mathcal{F}_a = \mathcal{F} \otimes_W \mathcal{O}_{W_a}$ be the fiber of $\mathcal{F}$ over a.

Then there is a dense open $U \subseteq A$ such that if $a \in U$, then $\mathcal{F}_a$ is homologically transverse to $\mathcal{E}$.

We note that we have not assumed that X, Y, W, or A is smooth.

5.3 Proofs

In this section we prove Theorem 5.1.2, Theorem 5.2.1, and Theorem 5.2.2. We begin by establishing some preparatory lemmas.

Lemma 5.3.1. *Let*

$$X_1 \xrightarrow{\alpha} X_2 \xrightarrow{\gamma} X_3$$

be morphisms of schemes, and assume that γ is flat. Let $\mathcal{G}$ be a quasicoherent sheaf on X_1 that is flat over X_3. Let $\mathcal{H}$ be any quasicoherent sheaf on X_3. Then for all $j \geq 1$, we have $\mathcal{T}or_j^{X_2}(\mathcal{G}, \gamma^*\mathcal{H}) = 0$.

Proof. We may reduce to the local case. Thus let $x \in X_1$ and let $y = \alpha(x)$ and $z = \gamma(y)$. Let $S = \mathcal{O}_{X_2,y}$ and let $R = \mathcal{O}_{X_3,z}$. Then $(\gamma^*\mathcal{H})_y \cong S \otimes_R \mathcal{H}_z$. Since S is flat over R, we have

$$\mathrm{Tor}_j^R(\mathcal{G}_x, \mathcal{H}_z) \cong \mathrm{Tor}_j^S(\mathcal{G}_x, S \otimes_R \mathcal{H}_z) = \mathcal{T}or_j^{X_2}(\mathcal{G}, \gamma^*\mathcal{H})_x$$

by flat base change. The left-hand side is 0 for $j \geq 1$ since $\mathcal{G}$ is flat over X_3. Thus for $j \geq 1$ we have $\mathcal{T}or_j^{X_2}(\mathcal{G}, \gamma^*\mathcal{H}) = 0$. $\square$

To prove Theorem 5.2.2, we show that a suitable modification of the spectral sequences used in [MS06] will work in our situation. Our key computation is the following lemma; compare to [MS06, Proposition 2].

Lemma 5.3.2. *Given the notation of Theorem 5.2.2, there is an open dense $U \subseteq A$ such that for all $a \in U$ and for all $j \geq 0$ we have*

$$\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^*k_a) \cong \mathcal{T}or_j^X(\mathcal{F}_a, \mathcal{E})$$

as sheaves on $W \times_X Y$.

Note that $\mathcal{F} \otimes_X \mathcal{E}$ is a sheaf on $W \times_X Y$ and thus $\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a)$ is a sheaf on $W \times_X Y \times_W W = W \times_X Y$ as required.

Proof. Since A is generically reduced, we may apply generic flatness to the morphism $q : W \to A$. Thus there is an open dense subset U of A such that both W and $\mathcal{F}$ are flat over U. Let $a \in U$. Away from $q^{-1}(U)$, both sides of the equality we seek to establish are zero, and so the result is trivial. Since $\mathcal{F}|_{q^{-1}(U)}$ is still flat over X, without loss of generality we may replace W by $q^{-1}(U)$; that is, we may assume that both W and $\mathcal{F}$ are flat over A.

The question is local, so assume that $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} T$, and $W = \operatorname{Spec} S$ are affine. Let $E = \Gamma(Y, \mathcal{E})$ and let $F = \Gamma(W, \mathcal{F})$. Let $Q = \Gamma(W, q^* k_a)$; then $\Gamma(W, \mathcal{F}_a) = F \otimes_S Q$. We seek to show that

$$\operatorname{Tor}_j^S(F \otimes_R E, Q) \cong \operatorname{Tor}_j^R(F \otimes_S Q, E)$$

as $S \otimes_R T$-modules.

We will work on $W \times X$. For clarity, we lay out the various morphisms and corresponding ring maps in our situation. We have morphisms of schemes

$$\begin{tikzcd} W \times X \arrow[d, "p"'] & Y \arrow[d, "r"] \\ W \arrow[r, "f"'] \arrow[u, bend right, "\phi"'] & X \end{tikzcd}$$

where p is projection onto the first factor and the morphism ϕ splitting p is given by the graph of f. Letting $B = S \otimes_k R$, we have corresponding maps of rings

$$\begin{tikzcd} B \arrow[d, bend left, "\phi^{\#}"] & T \\ S \arrow[u, "p^{\#}"] & R, \arrow[l, "f^{\#}"] \arrow[u, "r^{\#}"'] \end{tikzcd}$$

where $p^{\#}(s) = s \otimes 1$ and $\phi^{\#}(s \otimes r) = s \cdot f^{\#}(r)$. We make the trivial observation that

$$B \otimes_R E = (S \otimes_k R) \otimes_R E \cong S \otimes_k E.$$

Let $K_\bullet \to F$ be a projective resolution of F, considered as a B-module via the map $\phi^{\#} : B \to S$. As E is an R-module via the map $r^{\#} : R \to T$, there is a B-action on $S \otimes_k E$; let $L_\bullet \to S \otimes_k E$ be a projective resolution over B.

Let $P_{\bullet,\bullet}$ be the double complex $K_\bullet \otimes_B L_\bullet$. We claim the total complex of $P_{\bullet,\bullet}$ resolves $F \otimes_B (S \otimes_k E)$. To see this, note that the rows of $P_{\bullet,\bullet}$, which are of the form $K_\bullet \otimes_B L_j$, are acyclic, except in degree 0, where the homology is $F \otimes_B L_j$. The degree 0 horizontal homology forms a vertical complex whose homology computes $\operatorname{Tor}_j^B(F, S \otimes_k E)$. But $S \otimes_k E \cong B \otimes_R E$, and B is a flat R-module. Therefore $\operatorname{Tor}_j^B(F, S \otimes_k E) \cong \operatorname{Tor}_j^B(F, B \otimes_R E) \cong \operatorname{Tor}_j^R(F, E)$ by the formula for flat base change for Tor. Since F is flat over R, this is zero for all $j \geq 1$. Thus, via the spectral sequence

$$H_j^v(H_i^h P_{\bullet,\bullet}) \Rightarrow H_{i+j} \operatorname{Tot} P_{\bullet,\bullet}$$

we see that the total complex of $P_{\bullet,\bullet}$ is acyclic, except in degree 0, where the homology is $F \otimes_B S \otimes_k E \cong F \otimes_R E$.

Consider the double complex $P_{\bullet,\bullet} \otimes_S Q$. Since $\operatorname{Tot} P_{\bullet,\bullet}$ is a B-projective and therefore S-projective resolution of $F \otimes_R E$, the homology of the total complex of this double complex computes $\operatorname{Tor}_j^S(F \otimes_R E, Q)$.

Now consider the row $K_\bullet \otimes_B L_j \otimes_S Q$. As L_j is B-projective and therefore B-flat, the i'th homology of this row is isomorphic to $\operatorname{Tor}_i^S(F, Q) \otimes_B L_j$. Since W and $\mathcal{F}$ are flat over A, by Lemma 5.3.1 we have $\operatorname{Tor}_i^S(F, Q) = 0$ for all $i \geq 1$. Thus this row is acyclic except in degree 0, where the homology is $F \otimes_B L_j \otimes_S Q$. The vertical differentials on the degree 0 homology give a complex whose j'th homology is isomorphic to $\mathrm{Tor}_j^B(F \otimes_S Q, S \otimes_k E)$. As before, this is simply $\mathrm{Tor}_j^R(F \otimes_S Q, E)$.

Thus (via a spectral sequence) we see that the homology of the total complex of $P_{\bullet,\bullet} \otimes_S Q$ computes $\mathrm{Tor}_j^R(F \otimes_S Q, E)$. But we have already seen that the homology of this total complex is isomorphic to $\mathrm{Tor}_j^S(F \otimes_R E, Q)$. Thus the two are isomorphic.

$\square$

Proof of Theorem 5.2.2. By generic flatness, we may reduce without loss of generality to the case where W is flat over A. Since $\mathcal{F}$ and $\mathcal{E}$ are coherent sheaves on W and Y respectively, $\mathcal{F} \otimes_X \mathcal{E}$ is a coherent sheaf on $W \times_X Y$. Applying generic flatness to the composition $W \times_X Y \to W \to A$, we obtain a dense open $V \subseteq A$ such that $\mathcal{F} \otimes_X \mathcal{E}$ is flat over V. Therefore, by Lemma 5.3.1, if $a \in V$ and $j \geq 1$, we have $\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a) = 0$.

We apply Lemma 5.3.2 to choose a dense open $U \subseteq A$ such that for all $j \geq 1$, if $a \in U$, then $\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a) \cong \mathcal{T}or_j^X(\mathcal{F}_a, \mathcal{E})$. Thus if a is in the dense open set $U \cap V$, then for all $j \geq 1$ we have

$$\mathcal{T}or_j^X(\mathcal{F}_a, \mathcal{E}) \cong \mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a) = 0,$$

as required. $\square$

We now turn to the proof of Theorem 5.2.1; for the remainder of this section, we will adopt the hypotheses and notation given there.

Lemma 5.3.3. *Let $R, R', S,$ and T be commutative rings, and let*

$$\begin{array}{ccc} R' & \longrightarrow & T \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

be a commutative diagram of ring homomorphisms, such that R'_R and T_S are flat.

Let N be an R-module. Then for all $j \geq 0$, we have that

$$\mathrm{Tor}_j^{R'}(N \otimes_R R', T) \cong \mathrm{Tor}_j^R(N, S) \otimes_S T.$$

Proof. Let $P_\bullet \to N$ be a projective resolution of N. Consider the complex

$$P_\bullet \otimes_R R' \otimes_{R'} T \cong P_\bullet \otimes_R T \cong P_\bullet \otimes_R S \otimes_S T. \tag{5.3.4}$$

Since R'_R is flat, $P_\bullet \otimes_R R'$ is a projective resolution of $N \otimes_R R'$. Thus the j'th homology of (5.3.4) computes $\mathrm{Tor}_j^{R'}(N \otimes_R R', T)$. Since T_S is flat, this homology is isomorphic to $H_j(P_\bullet \otimes_R S) \otimes_S T$. Thus $\mathrm{Tor}_j^{R'}(N \otimes_R R', T) \cong \mathrm{Tor}_j^R(N, S) \otimes_S T$. □

Lemma 5.3.5. *Let x be a closed point of X. Consider the multiplication map*

$$\mu_x : G \times \{x\} \to X.$$

Then for all $j \geq 0$ we have

$$\mathcal{T}or_j^X(\mathcal{F}, \mathcal{O}_{G\times\{x\}}) \cong \mathcal{T}or_j^{G\times X}(p^*\mathcal{F}, \mu^* k_x) \tag{5.3.6}$$

If k is algebraically closed, then we also have

$$\mathcal{T}or_j^{G\times X}(p^*\mathcal{F}, \mu^* k_x) \cong \mathcal{T}or_j^X(\mathcal{F}, \mathcal{O}_{\overline{Gx}}) \otimes_X \mathcal{O}_{G\times\{x\}}. \tag{5.3.7}$$

All isomorphisms are of sheaves on $G \times W$.

Proof. Note that μ_x maps $G \times \{x\}$ onto a locally closed subscheme of X, which we will denote Gx. Since all computations may be done locally, without loss of generality we may assume that Gx is in fact a closed subscheme of X.

Let $\nu : G \to G$ be the inverse map, and let $\psi = \nu \times \mu : G \times X \to G \times X$. Consider the commutative diagram:

$$\begin{tikzcd}
G \times W \arrow[r, "1\times f"] \arrow[d, "p"'] & G \times X \arrow[d, "p"'] & G \times \{x\} \arrow[l, "\psi"'] \arrow[dl, "\mu_x"'] \arrow[d, "\pi"] \\
W \arrow[r, "f"'] & X & Gx \arrow[l, "\supseteq"']
\end{tikzcd} \tag{5.3.8}$$

where π is the induced map and p is projection onto the second factor. Since $\psi^2 = \mathrm{Id}_{G\times X}$ and $\mu = p \circ \psi$, we have that $\mu^* k_x \cong \psi^* p^* k_x \cong \psi_* \mathcal{O}_{G\times\{x\}}$, considered as sheaves on $G \times X$. Then the isomorphism (5.3.6) is a direct consequence of the flatness of p and Lemma 5.3.3. If k is algebraically closed, then π is also flat, and so the isomorphism (5.3.7) also follows from Lemma 5.3.3. □

Proof of Theorem 5.2.1. (3) ⇒ (2). Assume (3). Let $\mathcal{E}$ be a coherent sheaf on Y. Consider the maps:

$$\begin{tikzcd} & Y \arrow[d, "r"] \\ G \times W \arrow[r, "\rho"] \arrow[d, "q"'] & X \\ G, & \end{tikzcd}$$

where q is projection on the first factor.

Since G is smooth, it is generically reduced. Thus we may apply Theorem 5.2.2 to the ρ-flat sheaf $p^*\mathcal{F}$ to obtain a dense open $U \subseteq G$ such that if $g \in U$ is a closed point, then ρ makes $(p^*\mathcal{F})_g$ homologically transverse to $\mathcal{E}$. But $\rho|_{\{g\}\times W}$ is the map used to define $\mathcal{T}or_j^X(g\mathcal{F}, \mathcal{E})$; that is, considered as sheaves over X, $(p^*\mathcal{F})_g \cong g\mathcal{F}$. Thus (2) holds.

(2) ⇒ (3). The morphism ρ factors as

$$G \times W \xrightarrow{1\times f} G \times X \xrightarrow{\mu} X.$$

Since the multiplication map μ is the composition of an automorphism of $G \times X$ and projection, it is flat. Therefore for any quasicoherent $\mathcal{N}$ on X and $\mathcal{M}$ on $G \times W$ and for any closed point $z \in G \times W$, we have

$$\mathcal{T}or_j^{G\times X}(\mathcal{M}, \mu^*\mathcal{N})_z \cong \mathcal{T}or_j^{\mathcal{O}_{X,\rho(z)}}(\mathcal{M}_z, \mathcal{N}_{\rho(z)}), \tag{5.3.9}$$

as in the proof of Lemma 5.3.1.

If $p^*\mathcal{F}$ fails to be flat over X, then flatness fails against the structure sheaf of some closed point $x \in X$, by the local criterion for flatness [Eis95, Theorem 6.8]. Thus to check that $p^*\mathcal{F}$ is flat over X, it is equivalent to test flatness against structure sheaves of closed points of X. By (5.3.9), we see that $p^*\mathcal{F}$ is ρ-flat over X if and only if

$$\mathcal{T}or_j^{G\times X}(p^*\mathcal{F}, \mu^* k_x) = 0 \quad \text{for all closed points } x \in X \text{ and for all } j \geq 1. \tag{5.3.10}$$

Applying Lemma 5.3.5, we see that the flatness of $p^*\mathcal{F}$ is equivalent to the vanishing

$$\mathcal{T}or_j^{X}(\mathcal{F}, \mathcal{O}_{G\times\{x\}}) = 0 \quad \text{for all closed points } x \in X \text{ and for all } j \geq 1. \tag{5.3.11}$$

Assume (2). We will show that (5.3.11) holds for all $x \in X$. Fix a closed point $x \in X$ and consider the morphism $\mu_x : G \times \{x\} \to X$. By assumption, there is a closed point $g \in G$ such that $g\mathcal{F}$ is homologically transverse to $\mathcal{O}_{G\times\{x\}}$. Let $k' = k(g)$ and let g' be the canonical k'-point of $G \times k'$ lying over g. Let $G' = G \times k'$ and let $X' = X \times k'$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $W' = W \times k'$. Consider the commutative diagram

$$\begin{tikzcd}
G \times \{x\} \times k' \arrow[r, "\mu_x \times 1"] \arrow[d] & X' \arrow[d] & \{g'\} \times_{k'} W' \arrow[l, "\rho"'] \arrow[d, "\cong"'] \\
G \times \{x\} \arrow[r, "\mu_x"] & X & \{g\} \times W. \arrow[l, "\rho"']
\end{tikzcd}$$

Since the vertical maps are faithfully flat and the left-hand square is a fiber square, by Lemma 5.3.3 we have that $g'\mathcal{F}'$ is homologically transverse to

$$G \times \{x\} \times k' \cong G' \times \{x\}.$$

By $G(k')$-equivariance, $\mathcal{F}'$ is homologically transverse to $(g')^{-1}G' \times \{x\} = G' \times \{x\}$. Since

$$G' \times \{x\} \longrightarrow X' \xleftarrow{f} W'$$

is base-extended from

$$G \times \{x\} \longrightarrow X \xleftarrow{f} W,$$

we obtain that $\mathcal{F}$ is homologically transverse to $G \times \{x\}$. Thus (5.3.11) holds.

(1) $\Rightarrow$ (3). The ρ-flatness of $\mathcal{F}$ is not affected by base extension, so without loss of generality we may assume that k is algebraically closed. Then (3) follows directly from Lemma 5.3.5 and the criterion (5.3.10) for flatness.

(3) $\Rightarrow$ (1). As before, we may assume that k is algebraically closed. Let x be a closed point of X. We have seen that (3) and (2) are equivalent; by (2) applied to $\mathcal{E} = \mathcal{O}_{\overline{Gx}}$ there is a closed point $g \in G$ such that $g\mathcal{F}$ and $\overline{Gx}$ are homologically transverse. By $G(k)$-equivariance, $\mathcal{F}$ and $g^{-1}\overline{Gx} = \overline{Gx}$ are homologically transverse. □

Proof of Theorem 5.1.2. If $\mathcal{F}$ is homologically transverse to orbit closures upon extending to $\overline{k}$, then, using Theorem 5.2.1(2), for any $\mathcal{E}$ there is a dense open $U \subseteq G$ such that, in particular, for any k-rational $g \in U$ we have that $g\mathcal{F}$ and $\mathcal{E}$ are homologically transverse.

The equivalence of (1) and (2) in the case that k is algebraically closed follows directly from Theorem 5.2.1. □

5.4 Applications to critical transversality

We have seen repeatedly in this thesis that the algebraic properties of birationally commutative rings defined by geometric data are largely controlled by the motion of the defining data under σ. In particular, recall (Definition 3.3.8) that if X is a projective variety, $\sigma \in \operatorname{Aut} X$, and Z is a closed subscheme of X, that the set

$$\{\sigma^n Z\}_{n \in \mathbb{Z}}.$$

is *critically transverse* if for any closed subscheme Y of X, we have that $\sigma^n Z$ and Y are homologically transverse for all but finitely many n. We saw in Chapter III that the properties of the idealizer $R(X, \mathcal{L}, \sigma, Z)$ are controlled by the critical transversality of the set $\{\sigma^n Z\}$. In this section, we apply Theorem 5.1.2 to obtain a simple criterion for critical transversality, at least in characteristic 0. It turns out that in many situations, critical transversality is, in a suitable sense, generic behavior.

We will use the following result of Cutkosky and Srinivas.

Theorem 5.4.1. ([CS93, Theorem 7]) *Let G be a connected abelian algebraic group defined over a field k of characteristic 0. Suppose that $g \in G$ is such that the cyclic subgroup $\langle g \rangle$ is dense in G. Then any infinite subset of $\langle g \rangle$ is dense in G.* □

Theorem 5.4.2. *Let k be an algebraically closed field of characteristic 0, let X be a variety of finite type over k, let Z be a closed subscheme of X, and let σ be an element of an algebraic group G that acts on X. Then $\{\sigma^n Z\}$ is critically transverse if and only if Z is homologically transverse to all reduced σ-invariant subschemes of X.*

Proof. If $\{\sigma^n Z\}$ is critically transverse, then Z is obviously homologically transverse to σ-invariant subschemes. We prove the converse. Assume that Z is homologically transverse to all σ-invariant subschemes of X. We consider the abelian subgroup

$$H = \overline{\langle \sigma^n \rangle} \subseteq G$$

Now, the closures of H-orbits in X are σ-invariant and reduced. Thus, by assumption, Z is homologically transverse to all H-orbit closures, and we may apply Theorem 5.1.2. Fix a closed subscheme Y of X. By Theorem 5.1.2, there is a dense open $U \subseteq H$ such that if $g \in U$, then gZ and Y are homologically transverse.

Let H^o be the connected component of the identity in H, so the components of H are $H^o, \sigma H^o, \ldots, \sigma^{c-1} H^o$ for some $c \geq 1$. As $\langle \sigma^c \rangle$ is dense in H^o, it is critically dense by Theorem 5.4.1.

Fix $0 \leq j \leq c-1$. The set

$$U_j = \sigma^{-j}(U \cap \sigma^j H^o)$$

is an open dense subset of H^o. By critical density, the set

$$\{m \mid \sigma^{mc} \notin U_j\}$$

is finite. Thus

$$\{n \mid \sigma^n \notin U\} = \bigcup_{j=0}^{c-1} \{n \mid n \equiv j \pmod{c} \text{ and } \sigma^{n-j} \notin U_j\}$$

is also finite. That is to say, for all but finitely many n, $\sigma^n \in U$ and $\sigma^n Z$ is homologically transverse to Y. As Y was arbitrary, $\{\sigma^n Z\}$ is critically transverse. □

We note that the case of Theorem 5.4.2 where Z is a point is proved in [KRS05, Theorem 11.2].

Suppose that k is uncountable and that X is a variety over k. We say that $x \in X$ is *very general* if there are proper subvarieties $\{Y_i \mid i \in \mathbb{Z}\}$ so that

$$x \notin \bigcup_i Y_i.$$

Corollary 5.4.3. *Assume that k is uncountable and algebraically closed and that* $\operatorname{char} k = 0$. *Let Z be a subscheme of $\mathbb{P}^d$, and let $\mathcal{X}$ be the $\mathbb{P}GL_{d+1}$-orbit of Z in the Hilbert scheme of $\mathbb{P}^d$. Let $\mathcal{Y} = \mathbb{P}GL_{d+1} \times \mathcal{X}$. Then if (σ, Z') is a very general element of $\mathcal{Y}$, then the set $\{\sigma^n Z'\}$ is critically transverse.*

Proof. By avoiding a countable union of proper subvarieties of $\mathbb{P}GL_{d+1}$, we may ensure that the eigenvalues of σ are distinct and algebraically independent over $\mathbb{Q}$. This implies that the Zariski closure of $\{\sigma^n\}$ in $\mathbb{P}GL_{d+1}$ is the torus $\mathbb{T}^d$, and that the only reduced subschemes fixed by σ are unions of coordinate linear subspaces.

There are finitely many of these; by repeated applications of Theorem 5.1.1 we see that there is a dense open $U \subseteq \mathbb{P}GL_{d+1}$ such that for all $\tau \in U$, the subscheme $Z' = \tau Z$ is homologically transverse to all unions of coordinate linear subspaces. By Theorem 5.4.2, the set $\{\sigma^n Z'\}$ is critically transverse. □

Corollary 5.4.4. *Let k be an algebraically closed field of characteristic 0, let X be a projective variety, and let σ be an element of an algebraic group G that acts on X. Let $\mathcal{L}$ be a σ-ample invertible sheaf on X. Let Z be a closed subscheme of X such that the components of Z^{red} have infinite order under σ. Then the idealizer ring $R(X, \mathcal{L}, \sigma, Z)$ is noetherian if and only if Z is homologically transverse to all reduced σ-invariant subschemes of X.*

Proof. First suppose that there is $x \in X$ so that $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is infinite. Then by Proposition 3.5.2, R is not right noetherian. Furthermore, $\{\sigma^n Z\}_{n\in\mathbb{Z}}$ is certainly not critically transverse, and so by Theorem 5.4.2 there is a reduced σ-invariant subscheme that is not homologically transverse to Z. Thus the result holds.

Thus we may assume that no such x exists; by Proposition 3.5.2, R is right noetherian. Note also that Assumption-Notation 3.3.1 is satisfied.

If there is a σ-invariant subvariety Y such that Z is not homologically transverse to Y, then by Proposition 3.5.6 $R(X, \mathcal{L}, \sigma, Z)$ is not left noetherian. If Z is homologically transverse to all reduced σ-invariant subschemes, then by Theorem 5.4.2, $\{\sigma^n Z\}_{n\in ZZ}$ is critically transverse. By Proposition 3.5.5, $R(X, \mathcal{L}, \sigma, Z)$ is left noetherian. □

Theorem 5.4.2 suggests the following conjecture:

Conjecture 5.4.5. *Let k be an algebraically closed field of characteristic 0, and let X be a projective variety defined over k. Let $\sigma \in \operatorname{Aut} X$ and let $Z \subseteq X$ be a closed subvariety. Then $\{\sigma^n Z\}$ is critically transverse if and only if Z is homologically* *transverse to all σ-invariant subschemes of X.*

If Z is 0-dimensional, then this conjecture reduces to Bell's recent result [Bel08, Corollary 1.3] that in characteristic 0, the orbit of a point under an automorphism is dense exactly when it is critically dense. If σ is an element of an algebraic group that acts on X, the conjecture is Theorem 5.4.2. In positive characteristic, the conjecture is known to be false; see [Rog04a, Example 12.9] for an example of an automorphism $\sigma \in \mathbb{P}GL_n$ in positive characteristic with a dense but not critically dense orbit.

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BIBLIOGRAPHY

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