ARC VALUATIONS ON SMOOTH VARIETIES

by

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To my parents

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CHAPTER I

Introduction

Let X be an algebraic variety over a field $\mathbf{k}$. An arc on X is a morphism $\gamma :$ $\operatorname{Spec} \mathbf{k}[[t]] \to X$ of $\mathbf{k}$-schemes. The arc space of X, denoted by X_∞, is the set of all arcs on X, and it has a structure of a scheme. In this thesis, I study valuations $\operatorname{ord}_\gamma$ on a local ring $\mathcal{O}_{X,p}$ of X given by the order of vanishing along an arc $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ on X. Such valuations are the $\mathbb{Z}_{\geq 0} \cup \{\infty\}$-valued valuations with transcendence degree zero. I associate to such a valuation $\operatorname{ord}_\gamma$ several different natural subsets of the arc space X_∞, and show they are equal. Furthermore, I show this subset is irreducible, and the valuation given by the order of vanishing along a general arc of this subset is equal to the original valuation $\operatorname{ord}_\gamma$.

The motivation for this project was the discovery by Ein, Lazarsfeld, and Mustaţă [7, Thm. C] that divisorial valuations (equivalently, valuations with transcendence degree $\dim X - 1$) correspond to a special class of subsets of the arc space called cylinders. One can interpret our results as being complementary to those of Ein et. al. as follows. Both say that valuations are encoded in a natural way as closed subsets of the arc space. We address the case when the transcendence degree is zero, whereas Ein et. al. study the case of valuations with transcendence degree $\dim X - 1$.

I begin with some background on arc spaces and their usefulness in studying singularities. Recall from a first course in calculus that the tangents to X at a fixed point give a linear approximation to the shape of X near that point. By replacing linear approximations by quadratic, cubic or higher degree polynomial approximations, one can get a more accurate understanding of the local shape of X. An approximation by a power series is an arc on X, and can be considered as a limit of successive approximations by polynomials of increasing degree. The set of all arcs on X forms a rich geometric object X_∞ (in particular, a scheme) called the arc space of X. Information about the singularities of X (or a pair (X, D) where D is a divisor on X) can be recovered from the geometric structure of X_∞. In this thesis I investigate a small part of the wealth of information and structure contained in X_∞.

We give a basic and important example. Let $X = \mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ be affine n-space. An arc on X is a morphism $\gamma : \operatorname{Spec} \mathbb{C}[[t]] \to \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ of $\mathbb{C}$-schemes. Equivalently, an arc on X is given by a $\mathbb{C}$-algebra morphism $\gamma^* : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[[t]]$, and hence is determined by a collection of power series describing the image of each coordinate function:

$$\begin{aligned} \gamma^*(x_1) &= c_{1,0} + c_{1,1}t + c_{1,2}t^2 + \dots \\ &\dots \\ \gamma^*(x_n) &= c_{n,0} + c_{n,1}t + c_{n,2}t^2 + \dots \end{aligned} \tag{1.1}$$

for some numbers $c_{i,j} \in \mathbb{C}$. The arc space $(\mathbb{C}^n)_\infty$ is then an infinite dimensional affine space with coordinates $x_{i,j}$ for $1 \leq i \leq n$ and $0 \leq j$, i.e. $(\mathbb{C}^n)_\infty = \operatorname{Spec} \mathbf{k}[\{x_{i,j}\}_{1 \leq i \leq n,\ 0 \leq j}]$.

In algebraic geometry, the study of singularities is often approached via valuations. There is a body of research relating *divisorial* valuations on the function field $\mathbb{C}(X)$ of X to subsets of X_∞. Some of this work has been motivated by the Nash problem of understanding the relationship between irreducible components of

X_∞ that lie over the singularities of a variety X, and divisors appearing in every resolution of singularities of X (see [15, Problem 4.13] for the precise statement). Interest in the relationship between divisorial valuations and arc spaces also comes from higher dimensional birational geometry. For example, invariants coming from birational geometry (e.g. minimal log discrepancies) can be expressed in terms of the codimension of various subsets of the arc space (see [9, Thm 7.9] for a precise statement).

Ein, Lazarsfeld, and Mustaţă show in [7, Thm. C] that divisorial valuations over a nonsingular variety X arise from a special class of subsets of X_∞ called *cylinders.* More specifically, for a divisorial valuation val_E given by the order of vanishing along a divisor E over X, there is an irreducible cylinder $C_{\mathrm{div}}(E) \subseteq X_\infty$ such that for a general arc $\gamma \in C_{\mathrm{div}}(E)$, we have that the order of vanishing of any rational function $f \in \mathbb{C}(X)$ along γ equals its order of vanishing along E. In symbols, $\mathrm{ord}_t\, \gamma^*(f) = \mathrm{val}_E(f)$ for all $f \in \mathbb{C}(X)$. Conversely, it is shown in [7, Thm. C] that every valuation given by the order of vanishing along a general arc of a cylinder is a divisorial valuation.

The goal of this thesis is to investigate whether other types of valuations, besides divisorial ones, have a similar interpretation via the arc space. We find there is a nice answer for valuations given by the order of vanishing along an arc on a nonsingular variety X. If X is a surface, all valuations with value group $\mathbb{Z}^r$ (lexicographically ordered) for some r are equivalent to a valuation of this type.

To explain, we need to introduce some notation. Let X be a variety over a field $\mathbf{k}$ and let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc on X. The arc γ gives a $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(o)} \to \mathbf{k}[[t]]$, where o denotes the closed point of $\operatorname{Spec} \mathbf{k}[[t]]$. We will see that γ^* extends uniquely to a $\mathbf{k}$-algebra homomorphism $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to$

$\mathbf{k}[[t]]$ (Proposition IV.2). We define a valuation $\operatorname{ord}_\gamma : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $\operatorname{ord}_\gamma(f) = \operatorname{ord}_t \gamma^*(f)$ for $f \in \widehat{\mathcal{O}}_{X,\gamma(o)}$. If $\gamma^*(f) = 0$, we will adopt the convention that $\operatorname{ord}_\gamma(f) = \infty$.

Given an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ on X we set $\operatorname{ord}_\gamma(\mathfrak{a}) = \min_{f \in \mathfrak{a}_{\gamma(o)}} \operatorname{ord}_\gamma(f)$. For a nonnegative integer q, we define the q-th order *contact locus* of $\mathfrak{a}$ by

$$\operatorname{Cont}^{\geq q}(\mathfrak{a}) = \{\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X \mid \operatorname{ord}_\gamma(\mathfrak{a}) \geq q\}. \tag{1.2}$$

For $f, g \in \mathcal{O}_{X,\gamma(o)}$, notice that

$$\operatorname{ord}_\gamma(fg) = \operatorname{ord}_\gamma(f) + \operatorname{ord}_\gamma(g)$$

$$\operatorname{ord}_\gamma(f+g) \geq \min\{\operatorname{ord}_\gamma(f), \operatorname{ord}_\gamma(g)\}$$

These conditions are included in the definition of a discrete valuation (Definition III.1). However, the map $\operatorname{ord}_\gamma : \mathcal{O}_{X,\gamma(o)} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ generally cannot be extended to a valuation on the function field $\mathbf{k}(X)$ of X, but it comes close. The snag is the possible presence of $f \in \mathcal{O}_{X,\gamma(o)}$ with $\operatorname{ord}_\gamma(f) = \infty$. There are two possible approaches to circumvent this difficulty, and we will use both. One is to quotient out by the prime ideal $\mathfrak{p} = \{f \in \mathcal{O}_{X,\gamma(o)} \mid \operatorname{ord}_\gamma(f) = \infty\}$. Then $\operatorname{ord}_\gamma$ induces a discrete valuation on $\operatorname{Frac}(\mathcal{O}_{X,\gamma(o)}/\mathfrak{p}) \setminus \{0\}$. To describe this construction in more geometric terms, set $Y = \overline{\gamma(\eta)} \subseteq X$, where η is the generic point of $\operatorname{Spec} \mathbf{k}[[t]]$. Then $\operatorname{ord}_\gamma$ induces a discrete valuation $\mathbf{k}(Y) \to \mathbb{Z}$ on the function field of Y.

The second approach is to enlarge our notion of valuation by permitting the value ∞ for nonzero elements and allowing the domain of definition to be a ring. To be precise, our definition of valuation is the following:

Definition I.1. Let R be a $\mathbf{k}$-algebra that is a domain. A *valuation* on R is a map $v : R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that

1. $v(c) = 0$ for $c \in \mathbf{k}^*$

2. $v(0) = \infty$

3. $v(xy) = v(x) + v(y)$ for $x, y \in R$

4. $v(x + y) \geq \min\{v(x), v(y)\}$ for $x, y \in R$

5. v is not identically 0 on R^*.

If $R = \mathcal{O}_{X,p}$ is a local ring at a point p of a variety X, we will say that v is a valuation on X centered at the point p.

Working in the context of valuations taking value ∞ on nonzero elements is not without precedent (e.g. [11]). In Chapter IV we will say more about the relation between several different definitions of valuations found in the literature. Specifically, we will compare these definitions with regard to arc spaces. We will see that Definition I.1 seems to be the most useful one in the context of valuations arising from subsets of arc spaces. In fact, we will see (Proposition IV.12) that every such valuation is induced by a (not necessarily $\mathbf{k}$-valued) arc. Futhermore, we will see in Proposition IV.13 that such valuations are precisely the discrete valuations on the subvariety of X given by the ideal of elements with value infinity.

1.1 Valuations and subsets of the arc space

In this section, I begin by explaining the relationship between valuations on a variety $X/\mathbf{k}$ and subsets of the arc space X_∞ of X. I then construct several natural subsets of the arc space that one might associate to a valuation. One of the main results of this thesis is that for a large class of valuations, these different constructions agree, i.e. they define the same subset of the arc space.

The following definition appears in [7, p.3], and provided, at least for us, the initial link between valuations and arc spaces:

Definition I.2. Let $C \subseteq X_\infty$ be an irreducible subset. Assume C is a cylinder (Definition II.2). Define a valuation $\mathrm{val}_C : \mathbf{k}(X) \to \mathbb{Z}$ on the function field $\mathbf{k}(X)$ of X as follows. For $f \in \mathbf{k}(X)$, set

$$\mathrm{val}_C(f) = \mathrm{ord}_\gamma(f)$$

for general $\gamma \in C$. Equivalently, if $\alpha \in C$ is the generic point of C, then $\mathrm{val}_C(f) = \mathrm{ord}_\alpha(f)$. (Caveat: α need not be a $\mathbf{k}$-valued point of X_∞. See Remark II.5.)

It turns out that the condition that C is a cylinder implies that $\mathrm{val}_C(f)$ is always finite. If we drop the assumption that C is a cylinder, then the map ord_α (where α is the generic point of C) is a $\mathbb{Z}_{\geq 0} \cup \{\infty\}$-valued valuation on $\mathcal{O}_{X,\alpha(o)}$. In this thesis, we will allow such valuations.

We now describe a way to go from valuations centered on X to subsets of the arc space. Following Ishii [14, Definition 2.8], we associate to a valuation v a subset $C(v) \subseteq X_\infty$ in the following way.

Definition I.3. Let $p \in X$ be a (not necessarily closed) point. Let $v : \widehat{\mathcal{O}}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be a valuation. Define the *maximal arc set* $C(v)$ by

$$C(v) = \overline{\{\gamma \in X_\infty \mid \mathrm{ord}_\gamma = v,\ \gamma(o) = p\}} \subseteq X_\infty,$$

where the bar denotes closure in X_∞. We will see in Proposition IV.12 that $C(v)$ is non-empty. Let $p \in X$ be a (not necessarily closed) point. Let $v : \widehat{\mathcal{O}}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be a valuation. Define the *maximal arc set* $C(v)$ by

$$C(v) = \overline{\{\gamma \in X_\infty \mid \mathrm{ord}_\gamma = v,\ \gamma(o) = p\}} \subseteq X_\infty,$$

where the bar denotes closure in X_∞. We will see in Proposition IV.12 that $C(v)$ is non-empty.

If we start with an irreducible subset C, we get a valuation val_C by Definition I.2. We can then form the subset $C(\mathrm{val}_C)$ as in Definition I.3. We have $C \subseteq C(\mathrm{val}_C)$ because $C(\mathrm{val}_C)$ contains the generic point of C. In general, we do not have equality.

We can associate another subset of X_∞ to a valuation v on a nonsingular variety X as follows. Let $\{E_q\}_{q\geq 1}$ be the sequence of divisors formed by blowing up successive centers of v (see Definition III.3). Following [7, Example 2.5], to each divisor E_q we associate a cylinder $C_q = C_{\mathrm{div}}(E_q) \subseteq X_\infty$. Using notation we will explain in Chapter V, we will define $C_q = \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$. In words, C_q is simply the set of arcs on X whose lift to X_{q-1} (a model of X formed by blowing up $q-1$ successive centers of v) has the same center on X_{q-1} as v. This collection of cylinders forms a decreasing nested sequence. We take their interesection, $\bigcap_{q\geq 1} C_q$, to get another subset of X_∞ that is reasonable to associate with v.

On the other hand, another way the valuation v can be studied is through its valuation ideals $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p} \mid v(f) \geq q\}$, where q ranges over the positive integers. The set $\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ is yet another reasonable set to associate with v.

Given an arc $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$, we have an induced map $\alpha^* : \widehat{\mathcal{O}}_{X,\alpha(o)} \to \mathbf{k}[[t]]$. We associate to ord_α the set

$$\mathcal{I} = \{\gamma \in X_\infty \mid \gamma(o) = \alpha(o), \ker(\alpha^*) \subseteq \ker(\gamma^*) \subseteq \widehat{\mathcal{O}}_{X,\alpha(o)}\} \tag{1.3}$$

In words, $\mathcal{I}$ is the set of arcs γ with $\mathrm{ord}_\gamma(f) = \infty$ for all $f \in \widehat{\mathcal{O}}_{X,\alpha(o)}$ with $\mathrm{ord}_\alpha(f) = \infty$.

Finally, let $R = \{\alpha \circ h \in X_\infty \mid h : \operatorname{Spec} \mathbf{k}[[t]] \to \operatorname{Spec} \mathbf{k}[[t]]\}$. In words, R is the set of $\mathbf{k}$-arcs that are reparametrizations of α.

The main result of this thesis is that for $\mathbf{k}$-arc valuations $v = \mathrm{ord}_\alpha$, all five of these closed subsets ($C(v)$, $\bigcap_{q\geq 1} C_q$, $\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$, $\mathcal{I}$, R) are equal. Furthermore, this subset is irreducible, and the valuation given by the order of vanishing along a general arc of this subset is equal to v.

For convenience, we will assume the arc α we begin with is normalized, that is, the set $\{v(f) \mid f \in \widehat{\mathcal{O}}_{X,p},\ 0 < v(f) < \infty\}$ (where $v = \mathrm{ord}_\alpha$) is non-empty and the greatest common factor of its elements is 1. Every arc valuation taking some value strictly between 0 and ∞ is a scalar multiple of a normalized valuation.

Also, we restrict ourselves to considering the $\mathbf{k}$-arcs in the sets described above. We denote by $(X_\infty)_0$ the subset of points of X_∞ with residue field equal to $\mathbf{k}$. If $D \subseteq X_\infty$, then we set $D_0 = D \cap (X_\infty)_0$.

Theorem I.4. *Let $\alpha : \mathrm{Spec}\,\mathbf{k}[[t]] \to X$ be a normalized arc on a nonsingular variety X ($\dim X \geq 2$) over an algebraically closed field $\mathbf{k}$ of characteristic zero. Set $v = \mathrm{ord}_\alpha$. Then the following closed subsets of X_∞ are equal:*

$$(C(v))_0 = (\bigcap_{q\geq 1} C_q)_0 = (\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q))_0 = \mathcal{I}_0 = R.$$

Furthermore, the valuation given by the order of vanishing along a general arc of this subset is equal to v.

When X is a surface, we recover the construction for divisorial valuations given in [7, Example 2.5]:

Remark I.5. If X is a surface and if v is a divisorial valuation, then $\bigcap_{q>0} C_q$ equals the cylinder C_r associated to v in [7, Example 2.5], where r is such that p_r is a divisor.

1.2 Outline of thesis

In Chapter II, we define jet schemes and arc spaces. We also recall standard constructions and theorems related to arc spaces. In Sections 2.2 and 2.3 we study how contact loci transform after blowing up. We also recall results of Ishii on divisorial valuations and arcs. In a later chapter we will see that some of the constructions Ishii makes for divisorial valuations extend to arbitrary arc valuations.

In Chapter III we present the background material from valuation theory that we will need. We present the classically known description of all the valuations on a smooth surface. On surfaces, there are four general classes of valuations: divisorial valuations, curve valuations, irrational valuations, and infinitely singular valuations. Of these, the first two are arc valuations. On the other hand, irrational valuations have value groups (isomorphic to) $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{R}$ where $\tau \in \mathbb{R} \setminus \mathbb{Q}$, while infinitely singular valuations have value groups (isomorphic to) subgroups of $\mathbb{R}$ that are not finitely generated. There are many different approaches to studying all four types of valuations. For example, this classification can be studied by sequences of centers of the valuation, or by sequences of key polynomials, or by Hamburger-Noether expansions. The article of Spivakovsky [21] is gives a detailed exposition of the classification, building on work of Zariski [23] and Abhyankar [1]. In Chapter III, we describe the classification of surface valuations via sequences of key polynomials (SKP). Our source for this material is a book by Favre and Jonsson [11, Chapter 2], which we follow closely. However, the original source cited in [11, Chapter 2] is MacLane's paper [16].

Chapter IV explores arc valuations. We begin by defining arc valuations and establishing some of their basic properties. We point out that divisorial valuations are arc valuations. We also define the notion of the transcendence degree of a valuation, and study the transcendence degree of an arc valuation. We also show that a normalized $\mathbf{k}$-arc valuation on a nonsingular variety X over $\mathbf{k}$ can be desingularized. More precisely, a normalized $\mathbf{k}$-valued arc γ can be lifted after finitely many blowups (of its centers) to an arc γ_r that is nonsingular.

Chapter V, in which we prove the main results of our thesis, studies $\mathbf{k}$-arc valuations on a nonsingular variety X over an algebraically closed field $\mathbf{k}$ of characteristic zero. In Section 5.3, we prove our main result in the special case that our valuation is nonsingular. We do this by reducing to the case $X = \mathbb{A}^n$, and then explicitly calculating the ideals of the various sets we associate to a valuation. In Section 5.4, we prove our main result, Theorem V.17, using the special case considered in Section 4.3.

In Chapter VI, we turn our attention to K-arc valuations, where $\mathbf{k} \subseteq K$ is an extension of fields. By a K-arc valuation we mean the order of vanishing along a K-arc $\operatorname{Spec} K[[t]] \to X$. By changing the base field to K, we are able to use our analysis from Chapter V. We establish inclusions between various subsets of the arc space associated with a K-arc valuation.

Chapter VII considers valuations that are not arc valuations. We restrict our attention to surfaces, and use the classification of surface valuations presented in Chapter III. A natural question is, what do the sets $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ and $\bigcap_q \mu_{q,\infty}(\mathrm{Cont}^{\geq 1}(E_q))$, which were the focus of Chapter V, look like for valuations that are not arc valuations? We begin by computing the sets

$$\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q) \ \text{ and } \bigcap_{q\geq 1} \mu_{q,\infty}(\mathrm{Cont}^{\geq 1}(E_q))$$

for irrational valuations on $X = \mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x, y]$. We have seen that these sets are equal for nonsingular arc valuations (Proposition V.2). However, for irrational valuations, these sets are not equal. In fact, we will see that for an irrational valuation on $\mathbb{A}^2$, the set $\bigcap_q \mu_{q,\infty}(\mathrm{Cont}^{\geq 1}(E_q))$ contains only the trivial arc. On the other hand, we will see that $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ is an irreducible cylinder. However, one cannot recover the original irrational valuation from $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$. More precisely, there are infinitely many irrational valuations whose corresponding sets $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ are equal.

When working with subsets of arc spaces, it is often useful to measure, in some way, the size of any subset. In Chapter VIII, we calculate the motivic measure of the maximal arc sets that we associate (in Chapter V) to an arc valuation. The motivic measure of a subset of the arc space is an element in the completion of a localization of the Grothendieck group of varieties. We find that the motivic measure cannot distinguish between the sets we associate to divisorial and irrational valuations.

Finally, in Chapter IX, we present open questions and futher directions of research. One direction of further research is the extension of the results of this thesis to singular varieties. Another direction is the study of generalized arcs, which we will define. The goal is to use these generalized arcs to extend the work of this thesis to more general (e.g. non-discrete) valuations.

CHAPTER II

Background on Arc spaces

In this chapter, we establish the facts about arc spaces that we will use. We begin by defining arc spaces. We then define an important class of subsets of the arc space called contact loci. When then show a technical result (Lemma II.9) we will later need about how these contact loci transform with respect to blowups.

2.1 Construction of the Arc Space

2.1.1 Jet spaces

We will construct arc spaces as a limit of jet spaces. We begin by describing jet spaces. Let X be a scheme of finite type over a base field $\mathbf{k}$. All morphisms between schemes over $\operatorname{Spec} \mathbf{k}$ will assumed to be morphisms of $\mathbf{k}$-schemes. For any nonnegative integer n and $\mathbf{k}$-algebra A, an A-valued n-jet on X is a morphism of $\mathbf{k}$-schemes $\operatorname{Spec} A[t]/(t^{n+1}) \to X$. The set of all n-jets on X can be parametrized by a $\mathbf{k}$-scheme X_n, called the *jet space* of X. More precisely, X_n is a scheme that represents the contravariant functor

$$\operatorname{Hom}(- \times \operatorname{Spec} \mathbf{k}[t]/(t^{n+1}), X) : (\mathbf{k} - schemes) \to (Sets)$$

sending a $\mathbf{k}$-scheme Y to the set $\mathrm{Hom}_{\mathrm{Spec}\,\mathbf{k}}(Y \times \mathrm{Spec}\,\mathbf{k}[t]/(t^{n+1}), X)$. In particular, X_n is uniquely determined up to isomorphism, and satisfies a functorial bijection

$$\mathrm{Hom}(\mathrm{Spec}\,A, X_n) = \mathrm{Hom}(\mathrm{Spec}\,A[t]/(t^{n+1}), X) \tag{2.1}$$

for every $\mathbf{k}$-algebra A. Note that the $\mathbf{k}$-valued points of X_n are given by the set of n-jets, $\mathrm{Hom}(\mathrm{Spec}\,\mathbf{k}[t]/(t^{n+1}), X)$. For nonnegative integers $m < n$, we have a canonical projection map $\pi_{n,m} : X_n \to X_m$ induced by the truncation map $A[t]/(t^{n+1}) \to A[t]/(t^{m+1})$ sending $a_0 + a_1 t + \ldots a_n t^n$ to $a_0 + a_1 t + \ldots a_m t^m$. Note that we may identify X_0 with X.

We outline the construction of X_n, and refer the reader to [9, Section 2] for more details. The first step is to assume X_n exists, and notice that if U is an open subset of X, then $\pi_{n,0}^{-1}(U)$ satisfies

$$\mathrm{Hom}(\mathrm{Spec}\,A, \pi_{n,0}^{-1}(U)) = \mathrm{Hom}(\mathrm{Spec}\,A[t]/(t^{n+1}), U).$$

Hence $U_n = \pi_{n,0}^{-1}(U)$. This implies X_n can be constructed by gluing together n-jets schemes of each set in an open cover of X. Hence we have reduced the problem to proving X_n exists when X is affine. Suppose X is affine, say $X = \mathrm{Spec}\,\mathbf{k}[x_1, \ldots, x_m]/I$. Define the polynomial ring $R = \mathbf{k}[x_{ij|1\le i\le m, 0\le j\le n}]$, where the x_{ij} are indeterminates. For $f = f(x_1, \ldots, x_m) \in I$, let Φ_f be the set of $n+1$ elements of R that are the coefficients of $1, t, \ldots, t^n$ in $f(\sum_{j=0}^{j=n} x_{1j}t^j, \ldots, \sum_{j=0}^{j=n} x_{mj}t^j) \in R[t]$. Set $\Phi = \cup_{f\in I}\Phi_f$. Let $J \subseteq R$ be the ideal of R generated by the elements of Φ. I claim $X_n = \mathrm{Spec}\,R/J$.

In other words, for any $\mathbf{k}$-algebra A, I claim there is a functorial bijection

$$\mathrm{Hom}(\mathrm{Spec}\,A, \mathrm{Spec}\,R/J) = \mathrm{Hom}(\mathrm{Spec}\,A[t]/(t^{n+1}), X). \tag{2.2}$$

For $\theta \in \mathrm{Hom}(R/J, A)$, define $\theta' \in \mathrm{Hom}(\mathbf{k}[x_1, \ldots, x_m], A[t]/(t^{n+1}))$, by $\theta'(x_i) =$

$\sum_{j=0}^{j=n}\theta(x_{ij})t^j$. I claim that θ' induces a $\mathbf{k}$-algebra homomorphism

$$\theta' \in \mathrm{Hom}(\mathbf{k}[x_1,\ldots,x_m]/I, A[t]/(t^{n+1})).$$

Suppose $f \in I$. We have

$$\begin{aligned}
\theta'(f) &= f(\theta'(x_1),\ldots,\theta'(x_m)) \\
&= f(\sum_{j=0}^{j=n}\theta(x_{1j})t^j,\ldots,\sum_{j=0}^{j=n}\theta(x_{mj})t^j) \\
&= \theta(f(\sum_{j=0}^{j=n}x_{1j}t^j,\ldots,\sum_{j=0}^{j=n}x_{mj}t^j)) \\
&= 0
\end{aligned}$$

The last line follows from the fact that the coefficients of $1, t, \ldots, t^n$ in

$$f(\sum_{j=0}^{j=n}x_{1j}t^j,\ldots,\sum_{j=0}^{j=n}x_{mj}t^j)$$

are the elements of Φ_f, and therefore θ vanishes on them since θ vanishes on J by assumption. We leave to the reader to check that the map $\theta \to \theta'$ gives an isomorphism $\mathrm{Hom}(R/J, A) \simeq \mathrm{Hom}(\mathbf{k}[x_1,\ldots,x_m]/I, A[t]/(t^{n+1}))$.

Hence we have isomorphisms

$$\begin{aligned}
\mathrm{Hom}(\mathrm{Spec}\,A, \mathrm{Spec}\,R/J) &= \mathrm{Hom}(R/J, A) \\
&\simeq \mathrm{Hom}(\mathbf{k}[x_1,\ldots,x_m]/I, A[t]/(t^{n+1})) \\
&= \mathrm{Hom}(\mathrm{Spec}\,A[t]/(t^{n+1}), X).
\end{aligned}$$

We leave it to the reader to check the functoriality of this isomorphism. Granting this, we conclude X_n exists. Notice that our proof shows that if X is affine, then X_n is affine.

2.1.2 Arc spaces

We now define arc spaces. First note that the map $\pi_{n,n-1} : X_n \to X_{n-1}$ is affine. Indeed, we saw in the proof of the existence of X_n that if U is an affine open subscheme of X, then $U_{n-1} = \pi_{n-1,0}^{-1}(U)$ is an affine open subscheme of X_{n-1}, and so $\pi_{n,n-1}^{-1}(U_{n-1}) = \pi_{n-1,0}^{-1}(U) = U_n$ is an affine open subscheme of X_n. Since the map $\pi_{n,n-1} : X_n \to X_{n-1}$ is affine, the inverse limit of the inverse system $\{\pi_{n,n-1} : X_n \to X_{n-1}\}$ of jet spaces exists in the category of $\mathbf{k}$-schemes, and is called the arc space X_∞ of X:

$$X_\infty := \varprojlim X_n.$$

Let $\mathbf{k} \subseteq K$ be a field extension. The arc space X_∞ is a scheme over $\mathbf{k}$ whose K-valued points are morphisms $\operatorname{Spec} K[[t]] \to X$ of $\mathbf{k}$-schemes, since we have

$$\operatorname{Hom}(\operatorname{Spec} K, X_\infty) = \operatorname{Hom}(\operatorname{Spec} K[[t]], X). \tag{2.3}$$

In particular, when X is affine, giving a K-valued point of X_∞ is the same thing as giving a homomorphism of $\mathbf{k}$-algebras $\Gamma(X, \mathcal{O}_X) \to K[[t]]$.

Definition II.1. Let $\mathbf{k} \subseteq K$ be a field extension. A K-*arc* is a morphism of $\mathbf{k}$-schemes $\operatorname{Spec} K[[t]] \to X$.

If $\mu : X' \to X$ is a morphism of $\mathbf{k}$-schemes, then we have an induced morphism $\mu_\infty : X'_\infty \to X_\infty$ sending γ to $\mu \circ \gamma$. Let $\pi_n : X_\infty \to X_n$ be the canonical morphism arising from the definition of inverse limit.

Definition II.2. A *cylinder* is a subset of X_∞ of the form $(\pi_n)^{-1}(A)$ where A is a constructible subset of X_n. (Recall that a constructible subset of a variety is one that can be written as a finite disjoint union of locally closed subsets [12, Exercise II.3.18].)

The following notation will be used often.

Notation II.3. Let K be a field. We denote the closed point of $\operatorname{Spec} K[[t]]$ by o.

An arc $\gamma : \operatorname{Spec} K[[t]] \to X$ gives homomorphism of $\mathbf{k}$-algebras $\gamma^* : \mathcal{O}_{X,\gamma(o)} \to K[[t]]$. Define $\operatorname{ord}_\gamma : \mathcal{O}_{X,\gamma(o)} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $\operatorname{ord}_\gamma(f) = \operatorname{ord}_t \gamma^*(f)$ for $f \in \mathcal{O}_{X,\gamma(o)}$. If $\gamma^*(f) = 0$, we adopt the convention that $\operatorname{ord}_\gamma(f) = \infty$.

Proposition II.4. *Let X be a variety over a field $\mathbf{k}$. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be a $\mathbf{k}$-arc. Then $\gamma(o) \in X$ is a closed point of X with residue field $\mathbf{k}$.*

Proof. Set $p = \gamma(o)$, and let $\kappa(p)$ denote the residue field of $p \in X$. We have a local $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathcal{O}_{X,p} \to \mathbf{k}[[t]]$. Taking the quotient by the maximal ideals, we get a $\mathbf{k}$-algebra homomorphism $\kappa(p) \hookrightarrow \mathbf{k}$ that is an isomorphism on $\mathbf{k} \subseteq \kappa(p)$. Hence $\kappa(p) = \mathbf{k}$. Since $\operatorname{tr.deg}_{\mathbf{k}} \kappa(p) = 0$, it follows that p is a closed point. $\square$

2.1.3 Points of the arc space

We next make a couple of remarks about the notion of a *point of the arc space.*

Remark II.5. Let X be a scheme of finite type over a field $\mathbf{k}$. Let $\alpha \in X_\infty$ be a (not necessarily closed) point of the scheme X_∞. That is, in some open affine patch of X_∞, α corresponds to a prime ideal. Let $\kappa(\alpha)$ denote the residue field at the point α of the scheme X_∞. There is a canonical morphism $\Theta_\alpha : \operatorname{Spec} \kappa(\alpha) \to X_\infty$ induced by the canonical $\mathbf{k}$-algebra homomorphism $\mathcal{O}_{X_\infty,\alpha} \to \kappa(\alpha)$. By Equation 2.3, the morphism Θ_α corresponds to a $\kappa(\alpha)$-arc $\theta_\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$. We will abuse notation and refer to this arc $\theta_\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$ by $\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$. That is, given a point $\alpha \in X_\infty$, we have a canonical $\kappa(\alpha)$-arc $\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$.

Remark II.6. We now examine the reverse of the construction given in Remark II.5. Let $\mathbf{k} \subseteq K$ be some extension of fields. Given a K-arc $\theta : \operatorname{Spec} K[[t]] \to X$, by

Equation 2.3, we get a morphism $\Theta : \operatorname{Spec} K \to X_\infty$. The image $\Theta(\mathtt{pt})$ of the only point $\mathtt{pt}$ of $\operatorname{Spec} K$ is a point of X_∞, call it α. By Remark II.5, we associate to α a $\kappa(\alpha)$-arc $\Theta_\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$. Note that $\Theta : \operatorname{Spec} K \to X_\infty$ factors through $\Theta_\alpha : \operatorname{Spec} \kappa(\alpha) \to X_\infty$, since on the level of rings, the $\mathbf{k}$-algebra map $\Theta^* : \mathcal{O}_{X_\infty,\alpha} \to K$ induces a map $\kappa(\alpha) \to K$. Hence $\theta : \operatorname{Spec} K[[t]] \to X$ factors through $\theta_\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$. To summarize, K-arcs on X correspond to K-valued points of X_∞. To each K-valued point of X_∞, we can assign a point of X_∞. If we let K range over all field extensions on $\mathbf{k}$, this assignment is surjective onto the set of points of X_∞, but it is not injective. To a point α of X_∞, we assign (as described in Remark II.5) a canonical $\kappa(\alpha)$-valued point of X_∞. The point of X_∞ that we assign to this $\kappa(\alpha)$-valued point is α.

2.2 Contact loci

Let $\gamma : \operatorname{Spec} K[[t]] \to X$ be an arc on X, and let $x = \gamma(o)$. Given an ideal sheaf $\mathfrak{a}$ on X we define $\operatorname{ord}_\gamma(\mathfrak{a}) = \min_{f \in \mathfrak{a}_x} \operatorname{ord}_\gamma(f)$. For a nonnegative integer p, define the p-th order *contact locus* of $\mathfrak{a}$ by

$$\operatorname{Cont}^{\geq p}(\mathfrak{a}) = \{\gamma : \operatorname{Spec} K[[t]] \to X \mid \operatorname{ord}_\gamma(\mathfrak{a}) \geq p\}. \tag{2.4}$$

If Z is a closed subscheme of X defined by the ideal sheaf $\mathcal{I}$, we write $\operatorname{Cont}^{\geq p}(Z)$ to mean $\operatorname{Cont}^{\geq p}(\mathcal{I})$. If a closed subscheme structure on a closed subset of X has not been specified, we implicitly give it the reduced subscheme structure.

Given an arc $\gamma : \operatorname{Spec} K[[t]] \to X$, the local $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(o)} \to K[[t]]$ extends uniquely to a $\mathbf{k}$-algebra homomorphism $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to K[[t]]$, where $\widehat{\mathcal{O}}_{X,\gamma(o)}$ is the completion of $\mathcal{O}_{X,\gamma(o)}$ at its maximal ideal. For $f \in \widehat{\mathcal{O}}_{X,\gamma(o)}$ we define $\operatorname{ord}_\gamma(f) = \operatorname{ord}_t \gamma^*(f)$. For an ideal $\mathfrak{a}$ of $\widehat{\mathcal{O}}_{X,\gamma(o)}$, we define $\operatorname{ord}_\gamma(\mathfrak{a}) =$

$\min_{f \in \mathfrak{a}} \operatorname{ord}_\gamma(f)$. For $x \in X$ and an ideal $\mathfrak{a}$ of $\widehat{\mathcal{O}}_{X,x}$ we define

$$\operatorname{Cont}^{\geq p}(\mathfrak{a}) = \{\gamma : \operatorname{Spec} K[[t]] \to X \mid \gamma(o) = x,\ \operatorname{ord}_\gamma(\mathfrak{a}) \geq p\}. \tag{2.5}$$

Lemma II.7. *Let p be a closed point of an n-dimensional nonsingular variety X over a field $\mathbf{k}$, and fix generators $x_1, \ldots, x_n$ of the maximal ideal of $\mathcal{O}_{X,p}$. Let $\mathbf{k} \subseteq K$ be an extension of fields. To give an arc $\gamma : \operatorname{Spec} K[[t]] \to X$ such that $\gamma \in \operatorname{Cont}^{\geq 1}(p)$ it is equivalent to give a homomorphism of $\mathbf{k}$-algebras $\widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[[x_1, \ldots, x_n]] \to K[[t]]$ sending each x_i into $(t)K[[t]]$.*

Proof. Let $\gamma : \operatorname{Spec} K[[t]] \to X$ satisfy $\gamma \in \operatorname{Cont}^{\geq 1}(p)$. I claim $\gamma(o) = p$. Let $\mathfrak{p} \subset \mathcal{O}_X$ be the ideal sheaf of the closed point p. Note that γ gives a local $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(o)} \to K[[t]]$, where o denotes the closed point of $\operatorname{Spec} K[[t]]$. By Equation 2.4, the assumption $\gamma \in \operatorname{Cont}^{\geq 1}(p)$ implies $\gamma^*(\mathfrak{p}_{\gamma(o)}) \subseteq (t)$. Hence $\mathfrak{p}_{\gamma(o)}$ is contained in the maximal ideal of $\mathcal{O}_{X,\gamma(o)}$, and therefore $(\mathcal{O}_X/\mathfrak{p})_{\gamma(o)} \neq 0$. That is, $\gamma(o)$ is contained in the support of $\mathcal{O}_X/\mathfrak{p}$. Since $\mathcal{O}_X/\mathfrak{p}$ is supported only at the point p, we have $\gamma(o) = p$.

Fix generators $x_1, \ldots x_n$ for the maximal ideal of $\mathcal{O}_{X,p}$. Since γ^* is a local homomorphism, we see that γ^* sends each x_i into the maximal ideal of $K[[t]]$. The map $\gamma^* : \mathcal{O}_{X,p} \to K[[t]]$ extends to a homomorphism of $\mathbf{k}$-algebras $\gamma^* : \widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[[x_1, \ldots, x_n]] \to K[[t]]$.

Conversely, suppose we have a homomorphism of $\mathbf{k}$-algebras $\widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[[x_1, \ldots, x_n]] \to K[[t]]$ defined by sending $x_i \to f_i \in (t)K[[t]]$. By restricting this homomorphism to $\mathcal{O}_{X,p}$, we get a local homomorphism $\mathcal{O}_{X,p} \to K[[t]]$, which yields an arc γ on X by the composition $\operatorname{Spec} K[[t]] \to \operatorname{Spec} \mathcal{O}_{X,p} \to X$ (where the last morphism is the canonical one). We have $\operatorname{ord}_\gamma(\mathfrak{p}) = \min_{1 \leq i \leq n} \{\operatorname{ord}_t f_i\} \geq 1$, that is, $\gamma \in \operatorname{Cont}^{\geq 1}(p)$.

$\square$

2.3 Contact loci and blowups

In this section we show that contact loci (defined in Equation 2.4) behave nicely under blowups.

Definition II.8. We say an arc $\gamma : \operatorname{Spec} K[[t]] \to X$ is a *trivial arc* if the maximal ideal of $\widehat{\mathcal{O}}_{X,\gamma(o)}$ equals the kernel of the map $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to K[[t]]$.

Lemma II.9. *Let X be a nonsingular variety of dimension n ($n \geq 2$). Let $\mu : X' \to X$ be the blowup of a closed point $p \in X$. Let E be the exceptional divisor. Let $x_1, \ldots, x_n$ be local algebraic coordinates centered at p.*

1. *Let $\gamma : \operatorname{Spec} K[[t]] \to X$ be an arc such that $\gamma \in \mathrm{Cont}^{\geq 1}(p)$, and suppose γ is not the trivial arc. Then there exists a unique arc $\gamma' : \operatorname{Spec} K[[t]] \to X'$ lifting γ, i.e. $\gamma = \mu \circ \gamma'$. Furthermore, $\gamma' \in \mathrm{Cont}^{\geq 1}(E)$.*

2. *If γ is as in part 1 and additionally $K = \mathbf{k}$, then the residue field at $\gamma'(o) \in X'$ equals $\mathbf{k}$. Furthermore, if $\mathrm{ord}_\gamma(x_1) \leq \mathrm{ord}_\gamma(x_i)$ for all $2 \leq i \leq n$, then there exist $c_i \in \mathbf{k}$ (for $2 \leq i \leq n$) such that x_1 and $\frac{x_i}{x_1} - c_i$ for $2 \leq i \leq n$ are local algebraic coordinates at $\gamma'(o)$.*

3. *$\mu_\infty(\mathrm{Cont}^{\geq 1}(E)) = \mathrm{Cont}^{\geq 1}(p)$.*

Proof. (1) Let $f_i(t) \in K[[t]]$ be defined by $\gamma^*(x_i) = f_i(t)$ for $1 \leq i \leq n$. By Lemma II.7 we have $f_i(t) \in (t)K[[t]]$. Assume without loss of generality that $\mathrm{ord}_t f_1 \leq \mathrm{ord}_t f_i$ for all $2 \leq i \leq n$. Consider the patch U of X' with coordinates $x_1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}$. The arc γ' on U given by $x_1 \to f_1$ and $\frac{x_i}{x_1} \to \frac{f_i}{f_1}$ is a lift of γ. Since E is given in the patch U by $x_1 = 0$, we have $\gamma' \in \mathrm{Cont}^{\geq 1}(E)$. For the uniqueness, note that the center of a lift of γ must lie in the patch with coordinates $x_1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}$, since $\mathrm{ord}_t f_1 \leq \mathrm{ord}_t f_i$ for $2 \leq i \leq n$. The lift must send $x_1 \to f_1$, and this forces $\frac{x_i}{x_1} \to \frac{f_i}{f_1}$.

(2) If $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$, let $c_i \in \mathbf{k}$ be the constant coefficient of $\frac{f_i}{f_1}$ for $2 \le i \le n$. Then since $\operatorname{ord}_t f_1 \ge 1$ and $\operatorname{ord}_t(\frac{f_i}{f_1} - c_i) \ge 1$ for $2 \le i \le n$, we have that $\gamma'(o)$ is the closed point with coordinates $x_1 = 0$ and $\frac{x_i}{x_1} = c_i$ for $2 \le i \le n$.

(3) Suppose that $\gamma : \operatorname{Spec} K[[t]] \to X$ is such that $\gamma \in \operatorname{Cont}^{\ge 1}(p)$. By part (1), there exists $\gamma' : \operatorname{Spec} K[[t]] \to X'$ such that $\gamma = \mu \circ \gamma'$. We have $\gamma^*(\mathfrak{p}) = \gamma'^*(\mu^*_{\gamma'(o)}(\mathfrak{p})) = \gamma'^*(\mathcal{O}_{X'}(-E)_{\gamma'(o)})$. Since $\gamma^*(\mathfrak{p}) \subseteq (t)K[[t]]$, we have $\gamma' \in \operatorname{Cont}^{\ge 1}(\mathcal{O}_{X'}(-E))$. So $\gamma = \mu_\infty(\gamma') \in \mu_\infty(\operatorname{Cont}^{\ge 1}(E))$.

Conversely, let $\gamma' : \operatorname{Spec} K[[t]] \to X'$, and suppose $\gamma' \in \operatorname{Cont}^{\ge 1}(E)$. Set $\gamma = \mu_\infty(\gamma')$. Then $\gamma^*(\mathfrak{p}) = \gamma'^*(\mu^*_{\gamma'(o)}(\mathfrak{p})) = \gamma'^*(\mathcal{O}_{X'}(-E)_{\gamma'(o)})$, and the condition that $\gamma' \in \operatorname{Cont}^{\ge 1}(E)$ means $\gamma'^*(\mathcal{O}_{X'}(-E)_{\gamma'(o)}) \subseteq (t)K[[t]]$. Hence $\operatorname{ord}_\gamma(\mathfrak{p}) \ge 1$, i.e. $\gamma \in \operatorname{Cont}^{\ge 1}(p)$. $\square$

2.4 Fat arcs

We describe the notion of fat arcs, introduced by Ishii [13, Definition 2.4], and some related facts.

Definition II.10. ([13, Definition 2.4]). Let η denote the generic point of $\operatorname{Spec} K[[t]]$. An arc $\gamma : \operatorname{Spec} K[[t]] \to X$ is called *fat* if $\gamma(\eta)$ is the generic point of X.

Let $\gamma : \operatorname{Spec} K[[t]] \to X$ be an arc. Then γ is a fat arc if and only if the ring homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(o)} \to K[[t]]$ is injective [13, Prop. 2.5i]. When γ^* is injective, it extends to a homomorphism $\gamma^* : \mathbf{k}(X) \to K((t))$ on the function field $\mathbf{k}(X)$ of X. Furthermore, $\operatorname{ord}_\gamma : \mathbf{k}(X)^* \to \mathbb{Z}$ is a valuation.

Example II.11. ([13, Example 2.12]). Let $X = \mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x, y]$. The arc $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ given by the $\mathbf{k}$-algebra homomorphism $\mathbf{k}[x, y] \to \mathbf{k}[[t]]$ sending $x \to t$ and $y \to e^t - 1 = \sum_{i \ge 1} \frac{t^i}{i!}$ is a fat arc. The valuation $\operatorname{ord}_\gamma$ on $\mathbf{k}(X) = \mathbf{k}(x, y)$ has transcendence degree 0 (see Definition IV.15), and is not a divisorial valuation, since divisorial valuations have transcendence degree $\dim X - 1$ [13, Proposition 2.10].

If γ is a fat arc and $\phi : Y \to X$ is a proper birational morphism, then γ can be lifted to a fat arc on Y, and such a lift is unique and a fat arc [13, Prop. 2.5ii]. Indeed, since ϕ is a birational map, the generic point η_Y of Y is the unique point of Y mapped by ϕ to the generic point η_X of X. The generic point η of $\operatorname{Spec} K[[t]]$ is mapped by γ to η_X, and so by the valuative criteria for properness there is a unique lift γ' of γ to Y such that $\gamma'(\eta) = \eta_Y$.

2.4.1 Divisorial valuations and fat arcs

Definition II.12. Let X be a variety. We say D *is a prime divisor over* X if there is a proper birational morphism $\phi : Y \to X$ such that $D \subset Y$ is a prime divisor on Y.

Definition II.13. A valuation v on the function field $\mathbf{k}(X)$ of a variety X over a field $\mathbf{k}$ is called a divisorial valuation if there is a normal variety Y, a prime divisor D on Y, a proper birational morphism $\phi : Y \to X$, and a positive integer q such that $v = q \cdot \mathrm{val}_D$ on $\mathbf{k}(Y) = \mathbf{k}(X)$, where val_D is the valuation given by the order of vanishing along D.

Proposition II.14. *([13, Proposition 2.11]). Let D be a prime divisor over a variety X, and let K be the residue field of the local ring at the generic point of D. Then there is a fat arc $\gamma : \operatorname{Spec} K[[t]] \to X$ such that $\mathrm{ord}_\gamma = \mathrm{val}_D$ on $\mathbf{k}(X)$. Also, we have* $\operatorname{tr.deg}_{\mathbf{k}} K = \dim X - 1$.

Proof. (Due to Ishii [13, Proposition 2.11]). Let $\phi : Y \to X$ and $D \subset Y$ be as in Definition II.13. Since Y is normal, $\mathcal{O}_{Y,D}$ is a rank one discrete valuation ring, and hence its completion $\widehat{\mathcal{O}_{Y,D}}$ is isomorphic to $K[[t]]$ where $K = \kappa(D)$ is the residue field at the generic point of D ([18, p.206 Corollary 2]). Hence $\operatorname{tr.deg}_{\mathbf{k}} K = \dim X - 1$. Also, the injective maps

$$\mathcal{O}_X \to \mathcal{O}_Y \hookrightarrow \mathcal{O}_{Y,D} \hookrightarrow \widehat{\mathcal{O}_{Y,D}} \simeq K[[t]]$$

give rise to a fat arc $\gamma : \operatorname{Spec} K[[t]] \to X$ such that $\operatorname{ord}_\gamma = \operatorname{val}_D$. □

Ishii introduced the following definition:

Definition II.15. ([14, Definition 2.8]). Let $v : \mathbf{k}(X) \to \mathbb{Z}$ be a divisorial valuation on X. Define the *maximal divisorial set* associated to v by

$$C(v) = \overline{\{\alpha \in X_\infty \mid \operatorname{ord}_\alpha = v\}} \subseteq X_\infty,$$

where the bar denotes closure in X_∞.

We will later consider the set $C(v)$ when v has a transcendence degree 0. For divisorial valuations v, Ishii proves the following results about $C(v)$.

Theorem II.16. *([14, Prop. 3.4, Prop. 4.1]). Let $v = q \cdot \operatorname{val}_D$ be a divisorial valuation on X, where $\phi : Y \to X$ is a proper birational morphism, Y is nonsingular, and $D \subset Y$ is a divisor on Y. Then:*

1. $C(v) = \overline{\phi_\infty(\operatorname{Cont}^q(D))}$
2. *$C(v)$ is an irreducible subset of X_∞*
3. *If $X = \operatorname{Spec} A$, then* $C(v) = \overline{\bigcap_{f \in A - \{0\}} \operatorname{Cont}^{v(f)}(f)}$
4. *If $X = \operatorname{Spec} A$ then $C(v)$ is an irreducible component of* $\bigcap_{f \in A - \{0\}} \operatorname{Cont}^{\geq v(f)}(f)$.

CHAPTER III

Background on Valuations

3.1 Definition of valuations

In this chapter, we establish the terminology and state the background results we use about valuations. We begin with the definition of a discrete valuation.

Definition III.1. Let K be a field and set $K^* = K \setminus \{0\}$. A *discrete valuation* on K is a map $v : K^* \to \mathbb{Z}$ such that

1. $v(xy) = v(x) + v(y)$ for all $x, y \in K^*$
2. $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K^*$

In this thesis, it will be useful to consider more general valuations. For example, if $\gamma : \operatorname{Spec} K[[t]] \to X$ is an arc, then the map $\operatorname{ord}_\gamma : \mathcal{O}_{X,\gamma(o)} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ satisfies conditions 1 and 2. However, because of the possible presence of nonzero $f \in \mathcal{O}_{X,\gamma(o)}$ with $\operatorname{ord}_\gamma(f) = \infty$, the map $\operatorname{ord}_\gamma$ cannot be extended to the function field $\mathbf{k}(X)$ of X. Since we are primarily interested in functions of the form $\operatorname{ord}_\gamma$, we need to use a more general notion of valuation. We next give a very general definition of a valuation. However, in the construction that follows, the reader should keep in mind the case $\Gamma = \mathbb{Z}_{\geq 0}$, which is the primary situation we will be interested in.

Let $(\Gamma, +, <)$ be a totally ordered abelian monoid. Give $\Gamma \cup \{\infty\}$ the structure of an ordered monoid as follows. Extend the order $<$ on Γ to an order $<$ on $\Gamma \cup \{\infty\}$ by setting $x < \infty$ for $x \in \Gamma$. Extend the binary operation $+$ on Γ to a binary operation $+$ on $\Gamma \cup \{\infty\}$ by setting $x + \infty = \infty$ for every $x \in \Gamma \cup \{\infty\}$.

We will always work over a base field $\mathbf{k}$.

Definition III.2. Let R be a $\mathbf{k}$-algebra and Γ a totally ordered abelian monoid. A *valuation* on R is a map $v : R \to \Gamma \cup \{\infty\}$ such that

1. $v(c) = 0$ for $c \in \mathbf{k}^*$ and $v(0) = \infty$, i.e. v extends the trivial valuation on $\mathbf{k}$
2. $v(xy) = v(x) + v(y)$ for $x, y \in R$
3. $v(x + y) \geq \min\{v(x), v(y)\}$ for $x, y \in R$
4. v is not identically 0 on R^*.

We now describe a geometric construction, called the sequence of centers of a valuation, that is useful in studying valuations, especially those on smooth surfaces. We give the definition only for valuations given by the order of vanishing along an arc $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$, as this is the case we will be interested in. For a general valuation, the definition is similar to the one given in [12, Exer. II.4.12].

Definition III.3 (Sequences of centers of an arc valuation)**.** Let X be a nonsingular variety over a field $\mathbf{k}$. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc on X. Assume γ is not the trivial arc (Definition II.8). Set $p_0 = \gamma(o)$ (where o is the closed point of $\operatorname{Spec} \mathbf{k}[[t]]$) and $v = \operatorname{ord}_\gamma$. By Proposition II.4, the point p_0 is a closed point (with residue field $\mathbf{k}$) of X. The point p_0 is called the *center* of v on X. Blowup p_0 to get a model X_1 with exceptional divisor E_1. By Lemma II.9 the arc γ has a unique lift to an arc $\gamma_1 : \operatorname{Spec} \mathbf{k}[[t]] \to X_1$. Let p_1 be the closed point $\gamma_1(o)$. Inductively define a sequence of closed points p_i and exceptional divisors E_i on models X_i and lifts $\gamma_i : \operatorname{Spec} \mathbf{k}[[t]] \to X_i$ of γ as follows. Blowup $p_{i-1} \in X_{i-1}$, to get a model X_i. Let

E_i be the exceptional divisor of this blowup. Let $\gamma_i : \operatorname{Spec} \mathbf{k}[[t]] \to X_i$ be the lift of $\gamma_{i-1} : \operatorname{Spec} \mathbf{k}[[t]] \to X_{i-1}$. Let p_i be the closed point $\gamma_i(o)$. Let $\mu_i : X_i \to X$ be the composition of the first i blowups. We call $\{p_i\}_{i \geq 0}$ the *sequence of centers* of v. This sequence is classically called the sequence of infinitely near points of v.

3.2 Classification of valuations on a smooth surface

There is a complete classification of valuations on a smooth surface. There are many different approaches to this classification, such as sequences of centers, sequences of key polynomials, and Hamburger-Noether expansions.

We describe the classification of surface valuations via sequences of key polynomials (SKP). Our source for this material is [11, Chapter 2], which we follow closely. However, the original source that Favre and Jonsson cite is MacLane's paper [16]. The simple idea behind SKPs is nicely explained in [8, Example 3.15]. Briefly, the idea is that we want to find a minimal subset of polynomials such that v is determined by its value on these polynomials.

Definition III.4. [11, Definition 2.1] A sequence of polynomials $(U_j)_{j=0}^{k}, 1 \leq k \leq \infty$, in $\mathbf{k}[x, y]$ is called a *sequence of key polynomials* (SKP) if it satisfies:

(P0) $U_0 = x$ and $U_1 = y$

(P1) for each U_j there is a number $\tilde{\beta}_j \in [0, \infty]$ (not all ∞) such that

$$\tilde{\beta}_{j+1} > n_j \tilde{\beta}_j = \sum_{l=0}^{l=j-1} m_{j,l} \tilde{\beta}_l \quad \text{for} \quad 1 \leq j < k \tag{3.1}$$

where $n_j \in \mathbb{N}^* = \{n \in \mathbb{Z} \mid n > 0\}$ and $m_{j,l} \in \mathbb{N}$ satisfy, for $j < l$ and $1 \leq l < j$,

$$n_j = \min\{l \in \mathbb{N}^* \mid l\tilde{\beta}_j \in \mathbb{Z}\tilde{\beta}_0 + \cdots \mathbb{Z}\tilde{\beta}_{j-1}\} \quad \text{and} \quad 0 \leq m_{j,l} < n_l \tag{3.2}$$

(P2) for $1 \le j < k$ there exists $\theta_j \in \mathbf{k}^*$ such that

$$U_{j+1} = U_j{}^{n_j} - \theta_j \cdot U_0{}^{m_{j,0}} \cdots U_{j-1}{}^{m_{j,j-1}} \tag{3.3}$$

Given a finite SKP $(U_j)_{j=0}^k$, we associate a valuation ν_k to it via the following theorem.

Theorem III.5. *[11, Theorem 2.8] Let $\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\}$ be a SKP with $k < \infty$. Then there exists a unique valuation $\nu_k : \mathbf{k}[[x, y]] \to [0, \infty]$ centered on the maximal ideal $\mathfrak{m} = (x, y)$ satisfying*

(Q1) $\nu_k(U_j) = \tilde{\beta}_j$ for $0 \le j \le k$

(Q2) $\nu_k \le \nu$ for any valuation $\nu : \mathbf{k}[[x, y]] \to [0, \infty]$ centered on $\mathfrak{m}$ and satisfying Q1. Further, if $l < k$, then $\nu_l \le \nu_k$.

Given an infinite SKP $(U_j)_{j=0}^\infty$, we associate a valuation ν_∞ to it by the following theorem.

Theorem III.6. *[11, Theorem 2.22] Let $\{(U_j)_0^\infty, (\tilde{\beta}_j)_0^\infty\}$ be an infinite SKP and let ν_k be the valuation associated to $\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\}$ for $k \ge 1$ by Theorem III.5.*

(i) If $n_j \ge 2$ for infinitely many j, then for any $\phi \in \mathbf{k}[[x, y]]$ there exists $k_0 = k_0(\phi)$ such that $\nu_k(\phi) = \nu_{k_0}(\phi)$ for all $k \ge k_0$. In particular, ν_k converges to a valuation ν_∞.

(ii) If $n_j = 1$ for $j >> 1$, then U_k converges in $\mathbf{k}[[x, y]]$ to an irreducible formal power series U_∞ and ν_k converges to a valuation ν_∞. For $\phi \in \mathbf{k}[[x, y]]$ prime to U_∞ we have $\nu_k(\phi) = \nu_{k_0}(\phi) < \infty$ for $k \ge k_0 = k_0(\phi)$, and if U_∞ divides ϕ, then $\nu_k(\phi) \to \infty$.

Given an SKP $\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\}$, where $1 \leq k \leq \infty$, we denote the associated valuation ν_k defined in the previous theorems by $\mathrm{val}(\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\})$.

Theorem III.7. *[11, Theorem 2.29] For any valuation $\nu : \mathbf{k}[[x, y]] \to [0, \infty]$ centered on $\mathfrak{m}$, there exists a unique SKP $\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\}$, where $1 \leq k \leq \infty$, such that $\nu = \mathrm{val}(\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\})$. We have $\nu(U_j) = \tilde{\beta}_j$ for all j.*

We now describe the classification of valuations of $\mathbf{k}[[x, y]]$ based on SKPs given in [11, Definition 2.23].

Definition III.8. [11, Definition 2.23] Let $\nu = \mathrm{val}(\{(U_j)_0^k, (\tilde{\beta}_j)_0^k\})$ (where $1 \leq k \leq \infty$ is fixed) be a valuation (with values in $[0, \infty]$) on $\mathbf{k}[[x, y]]$ given by an SKP. Assume that ν is normalized in the sense that $\nu(\mathfrak{m}) = 1$, where $\nu(\mathfrak{m}) := \min_{z \in \mathfrak{m}} \nu(z)$. We then say that ν is

(i) *monomial* (in coordinates (x, y)) if $k = 1$, $\tilde{\beta}_0 < \infty$, and $\tilde{\beta}_1 < \infty$

(ii) *quasimonomial* if $k < \infty$, $\tilde{\beta}_0 < \infty$, and $\tilde{\beta}_k < \infty$

(iii) *divisorial* if ν is quasimonomial and $\tilde{\beta}_k \in \mathbb{Q}$

(iv) *irrational* if ν is quasimonomial but not divisorial

(v) *infinitely singular* if $k = \infty$ and $d_j \to \infty$ where $d_j = \deg_y(U_j)$

(vi) *curve valuation* if $k = \infty$ and $d_j \nrightarrow \infty$, or $k < \infty$ and $\max\{\tilde{\beta}_0, \tilde{\beta}_k\} = \infty$.

Next we state some properties of the various types of valuations defined above. My source for this material is [11, Chapters 1, 2], to which we refer the reader for proofs. Before stating these properties, we need to introduce some useful invariants associated to a valuation $v : \mathbf{k}[x, y]^* \to G$, where G is an ordered abelian group and $v(x), v(y) > 0$. (More precisely, such a valuation is called a centered Krull valuation, and how it relates to valuations as defined in Definition III.2 is explained in Remark IV.17.)

The rational rank of v, denoted by $\operatorname{rat.rk}(v)$ is the $\mathbb{Q}$-vector space dimension of $G \otimes_{\mathbb{Z}} \mathbb{Q}$. The rank of v, denoted by $\operatorname{rk}(v)$ is the Krull dimension of the valuation ring $R_v := \{r \in \operatorname{Frac} R^* \mid v(r) \geq 0\} \cup \{0\}$. Let m_v be the maximal ideal of R_v. The transcendence degree of v, denoted by $\operatorname{tr.deg}(v)$ is equal to $\operatorname{tr.deg}_{\mathbf{k}} R_v/m_v$.

3.2.1 Quasimonomial valuations

A quasimonomial valuation v has the property that there is some finite number r and local coordinates x', y' at the center p_r of v on X_r such that v is a monomial valuation in x', y'. Quasimonomial valuations can be divided into two types: divisorial valuations and irrational valuations.

Divisorial valuations

Divisorial valuations are also given by the order of vanishing along a divisor on some normal variety over X. For a divisorial valuation v we have $\operatorname{rk}(v) = 1$, $\operatorname{tr.deg}(v) = 1$, and $\operatorname{rat.rk}(v) = 1$.

Irrational valuations

For an irrational valuation v, we have $\operatorname{rk}(v) = 1$, $\operatorname{tr.deg}(v) = 0$, and $\operatorname{rat.rk}(v) = 2$. For example, the monomial valuation on $\mathbf{k}[x, y]$ with $v(x) = 1$ and $v(y) = \pi$ is an irrational valuation.

3.2.2 Infinitely singular valuations

Let v be an infinitely singular valuation. We have $\operatorname{rk}(v) = 1$, $\operatorname{tr.deg}(v) = 0$, and $\operatorname{rat.rk}(v) = 1$. These three conditions characterize infinitely singular valuations. Another characterization is given in terms of (generalized) Puiseux series. Namely, there exist local coordinates x, y at the center of v on X and a generalized power series $\phi = \sum_{i=1}^{i=\infty} a_i t^{\tilde{\beta}_i}$ where the $a_i \in \mathbf{k}^*$ and $(\tilde{\beta}_i)_1^\infty$ is a sequence of strictly increasing positive rational numbers with unbounded denominators when expressed as the ratio of two relatively prime positive integers. Then for $\psi(x, y) \in \mathbf{k}[[x, y]]$, we have $v(\psi) = \operatorname{ord}_t(\psi(t, \phi))$. For several other equivalent characterizations, see [11, Appendix A].

3.2.3 Curve valuations

We give an equivalent but more geometric definition of a curve valuation than the one given by SKPs. Let $\phi \in \mathfrak{m} \subset \mathbf{k}[[x, y]]$ be an irreducible element. We call such an element a curve. Let $m(\phi)$ be the highest power of $\mathfrak{m}$ that contains ϕ. Define a valuation $v = v_\phi : \mathbf{k}[[x, y]] \to [0, \infty]$ by

$$v(\psi) = \frac{1}{m(\phi)} \dim_{\mathbf{k}}(\mathbf{k}[[x, y]]/(\phi, \psi)).$$

In other words, v is the normalized intersection number of ψ with a fixed curve ϕ. (The normalization is done so that $v(\mathfrak{m}) = 1$, but this is not essential.) Note that $v(\psi) = \infty$ if and only if ϕ divides ψ. We can associate a Krull valuation to v as follows. Write $\psi = \phi^k \tilde{\psi}$ where $k \in \mathbb{N}$ and ϕ is prime to $\tilde{\psi}$. Define the associated Krull valuation $v' : \mathbf{k}[[x, y]] \to \mathbb{Z} \times \mathbb{Q}$ (lexicographically ordered) by $v'(\psi) = (k, v(\tilde{\psi}))$. We have $\operatorname{rk}(v') = 2$, $\operatorname{tr.deg}(v') = 0$, and $\operatorname{rat.rk}(v') = 2$.

Example III.9. Let v be the curve valuation defined by $\phi = y$. Then for $\psi(x, y) \in \mathbf{k}[[x, y]]$, $v(\psi) = \operatorname{ord}_x(\psi(x, 0))$. The associated Krull valuation v' satisfies $v'(x) = (0, 1)$ and $v'(y) = (1, 0)$. Note that v' sends the monomial $x^a y^b$ to (b, a) and hence sends distinct monomials to distinct values.

3.2.4 Exceptional curve valuations

Let $\mu : X' \to X$ be a proper birational morphism between nonsingular surfaces, and suppose there is a closed point $p \in X$ such that μ is an isomorphism over $X \setminus \{p\}$.

Let E be an irreducible component of the exceptional divisor $\mu^{-1}(p)$ and q a point on E. Let v_E denote the Krull valuation on $\mathcal{O}_{X',q}$ associated to the curve valuation defined by E. Then the Krull valuation $\mu_* v_E = v_E \circ \mu$ is called an exceptional curve valuation. Exceptional curve valuations are the only valuations on $\mathbf{k}[[x, y]]$ that are not equivalent to a valuation with value monoid contained in $[0, \infty]$ [11, Lemma 1.5]. For an exceptional curve valuation v, we have $\operatorname{rk}(v) = 2$, $\operatorname{tr.deg}(v) = 0$, and $\operatorname{rat.rk}(v) = 2$.

CHAPTER IV

Arc valuations

In this chapter, we begin the study of arc valuations, which are the central object of this thesis. We begin with some background that will motivate the definition. In algebraic geometry, a fundamental type of valuation is a rank one discrete valuation on the function field $\mathbf{k}(X)$ of a variety X. For example, the valuation given by the order of vanishing along a prime divisor of normal variety is of this form. Consequently, one can define the Weil divisor associated to a function, and from this definition the notions of linear equivalence of Weil divisors and the ideal class group of a variety follow. In addition, the valuation ring associated to a rank one discrete valuation can be interpreted geometrically as the local ring of a point on some nonsingular curve [12, Cor. I.6.6].

Now consider the slightly general notion of a valuation $v : \mathcal{O}_X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with value semigroup $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ on a variety X. Then v induces a rank one discrete valuation on the subscheme of X given by the ideal sheaf $\mathcal{I} = \{f \in \mathcal{O}_X \mid v(f) = \infty\}$. This motivates the study of valuations $v : \mathcal{O}_X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We will see in Proposition IV.12 that such a valuation v is also given by ord_γ for some arc γ. This motivates the definition of arc valuations, which we now present.

4.1 Arc valuations: definitions and basic properties

Definition IV.1 (Arc valuations)**.** Let X be a variety over a field $\mathbf{k}$, and let $p \in X$ be a (not necessarily closed) point. An *arc valuation* v on X centered at p is a map $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that there exists an arc $\gamma : \operatorname{Spec} K[[t]] \to X$ (where $\mathbf{k} \subseteq K$ is an extension of fields) sending the closed point o of $\operatorname{Spec} K[[t]]$ to p and $v(f) = \operatorname{ord}_\gamma(f)$ for $f \in \mathcal{O}_{X,p}$. In this case, we say v is a K-arc valuation. Since $\operatorname{ord}_\gamma$ extends uniquely to $\widehat{\mathcal{O}}_{X,p}$ (the completion of $\mathcal{O}_{X,p}$ at its maximal ideal), we can extend v to $\widehat{\mathcal{O}}_{X,p}$ as well. We show below in Proposition IV.2 that this extension does not depend on the choice of arcs γ satisfying $v = \operatorname{ord}_\gamma$ on $\mathcal{O}_{X,p}$. Therefore we will also regard arc valuations as maps $v : \widehat{\mathcal{O}}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ without additional comment.

Proposition IV.2. *Let* $\gamma_1 : \operatorname{Spec} K_1[[t]] \to X$ *and* $\gamma_2 : \operatorname{Spec} K_2[[t]] \to X$ *be arcs both sending the closed points to the same point* $p \in X$*, such that* $\operatorname{ord}_{\gamma_1} = \operatorname{ord}_{\gamma_2}$ *on* $\mathcal{O}_{X,p}$*, where* $p = \gamma_1(o)$ *and* $\mathbf{k} \subseteq K_1, K_2$*. Then* $\operatorname{ord}_{\gamma_1} = \operatorname{ord}_{\gamma_2}$ *on* $\widehat{\mathcal{O}}_{X,p}$*.*

Proof. Let $a_1, \ldots, a_r$ be generators of the maximal ideal of $\mathcal{O}_{X,p}$. Let $f \in \widehat{\mathcal{O}}_{X,p}$. Let $m = \min\limits_{i=1,2} \operatorname{ord}_{\gamma_i}(f)$. If $m = \infty$, then $\operatorname{ord}_{\gamma_1}(f) = \operatorname{ord}_{\gamma_2}(f) = \infty$. So we may assume m is finite, and $\operatorname{ord}_{\gamma_1}(f) \leq \operatorname{ord}_{\gamma_2}(f)$. Since

$$\widehat{\mathcal{O}}_{X,p} \simeq \mathcal{O}_{X,p}[[X_1, \ldots, X_r]]/(X_1 - a_1, \ldots, X_r - a_r) \tag{4.1}$$

[17, Theorem 8.12], there is a power series $P(X_1, \ldots, X_r) \in \mathcal{O}_{X,p}[[X_1, \ldots, X_r]]$ whose image $\overline{P} \in \mathcal{O}_{X,p}[[X_1, \ldots, X_r]]/(X_1 - a_1, \ldots, X_r - a_r)$ corresponds to f under the isomorphism 4.1. Let $P_m \in \mathcal{O}_{X,p}[X_1, \ldots, X_r]$ be a polynomial such that $P - P_m \in (X_1, \ldots, X_r)^{m+1}\mathcal{O}_{X,p}[[X_1, \ldots, X_r]]$, i.e. P_m is the part of P of degree less than or equal to m. For $i = 1, 2$, the map $\gamma_i^* : \widehat{\mathcal{O}}_{X,p} \to K_i[[t]]$ corresponds under the isomorphism 4.1 to the homomorphism $\gamma_i^* : \mathcal{O}_{X,p}[[X_1, \ldots, X_r]]/(X_1 - a_1, \ldots, X_r -$

$a_r) \to K_i[[t]]$ which sends $X_j \to \gamma_i^*(a_j)$ for $j = 1, \ldots, r$ and extends $\gamma_i^* : \mathcal{O}_{X,p} \to K_i[[t]]$. In particular, $\gamma_i^*(\overline{P - P_m}) \in (t)^{m+1}$. We have

$$\gamma_i^*(f) = \gamma_i^*(\overline{P}) = \gamma_i^*(\overline{P - P_m}) + \gamma_i^*(\overline{P_m}). \tag{4.2}$$

We have $\operatorname{ord}_{\gamma_1}(f) = \operatorname{ord}_t \gamma_1^*(f) = m$, and hence $\operatorname{ord}_t \gamma_1^*(\overline{P_m}) = m$. Since P_m is a polynomial, we have $\operatorname{ord}_t P_m(\gamma_1^*(a_1), \ldots, \gamma_1^*(a_r)) = m$. Also since P_m is a polynomial, we have $P_m(a_1, \ldots, a_r) \in \mathcal{O}_{X,p}$. Hence by assumption, $\gamma_1^*(P_m(a_1, \ldots, a_r)) = \gamma_2^*(P_m(a_1, \ldots, a_r))$. Since P_m is a polynomial, we have

$$\gamma_i^*(P_m(a_1, \ldots, a_r)) = P_m(\gamma_i^*(a_1), \ldots, \gamma_i^*(a_r))$$

for $i = 1, 2$. Hence $\operatorname{ord}_t P_m(\gamma_2^*(a_1), \ldots, \gamma_2^*(a_r)) = m$, i.e. $\operatorname{ord}_t \gamma_2^*(P_m) = m$. So by Equation 4.2, we have $\operatorname{ord}_{\gamma_2}(f) = m$. Hence $\operatorname{ord}_{\gamma_1}(f) = \operatorname{ord}_{\gamma_2}(f)$. $\square$

Example IV.3. Proposition II.14 shows that every divisorial valuation is an arc valuation.

Definition IV.4 (Normalized arc valuations)**.** We call an arc valuation v centered at a point $p \in X$ *normalized* if the set $\{v(f) \mid f \in \widehat{\mathcal{O}}_{X,p},\ 0 < v(f) < \infty\}$ is non-empty and the greatest common factor of its elements is 1. Every arc valuation taking some value strictly between 0 and ∞ is a scalar multiple of a normalized valuation. We say an arc $\gamma : \operatorname{Spec} K[[t]] \to X$ is *normalized* if $\operatorname{ord}_\gamma : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is a normalized arc valuation.

Notation IV.5. Let X be a nonsingular variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc centered at $p \in X$ and let $\gamma^* : \widehat{\mathcal{O}}_{X,p} \to \mathbf{k}[[t]]$ be the corresponding $\mathbf{k}$-algebra morphism. Assume γ is not a trivial arc (Definition II.8). Define a $\mathbf{k}$-algebra A_γ by $A_\gamma = \widehat{\mathcal{O}}_{X,p} / \ker(\gamma^*)$. Let $\tilde{A}_\gamma$ be the normalization of A_γ. Then γ^* induces an injective $\mathbf{k}$-algebra map $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[[t]]$

which extends to an injective $\mathbf{k}$-algebra homomorphism $\overline{\gamma}^* : \tilde{A}_\gamma \hookrightarrow \mathbf{k}[[t]]$. We denote by $\mathrm{ord}_{\overline{\gamma}}$ the composition $\mathrm{ord}_t \circ \overline{\gamma}^* : \tilde{A}_\gamma \to \mathbb{Z}_{\geq 0}$. Note that for $f \in \widehat{\mathcal{O}}_{X,p} \setminus \ker(\gamma^*)$, $\mathrm{ord}_\gamma(f) = \mathrm{ord}_{\overline{\gamma}}(\overline{f})$. We will show in Lemma IV.7 that there exists $\phi \in \mathbf{k}[[t]]$ such that the image of $\overline{\gamma}^* : \tilde{A}_\gamma \hookrightarrow \mathbf{k}[[t]]$ equals $\mathbf{k}[[\phi]] \subseteq \mathbf{k}[[t]]$.

Lemma IV.6. *Let X be a nonsingular variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc centered at $p \in X$. Assume γ is not the trivial arc. Use notation IV.5. Then the ring homomorphism $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[[t]]$ makes $\mathbf{k}[[t]]$ module finite over A_γ. In particular, A_γ has Krull dimension one.*

Proof. Choose local coordinates $x_1, \ldots, x_n$ at p such that $\gamma^*(x_1) \neq 0$. We have $\gamma^*(x_1) = t^r u$ for some positive integer r and unit $u \in \mathbf{k}[[t]]$. Since $\mathbf{k}$ is algebraically closed and has characteristic zero, there exists a unit $v \in \mathbf{k}[[t]]$ such that $v^r = u$. Indeed, we may use the binomial series and take $v = u^{1/r}$. To be precise, write $u = u_0(1 + u_1(t))$, with $u_1(t) \in (t)\mathbf{k}[[t]]$ and $u_0 \neq 0$. Then $u^{1/r} = u_0^{1/r}(1 + u_1(t))^{1/r} = u_0^{1/r}(1 + \sum_{i \geq 1} \binom{1/r}{i} u_1^i)$, where $u_0^{1/r}$ denotes any root of $X^r - u_0 = 0$.

Let $\tau : \mathbf{k}[[t]] \to \mathbf{k}[[t]]$ be the $\mathbf{k}$-algebra automorphism of $\mathbf{k}[[t]]$ defined by $\tau(t) = tv^{-1}$. Then $\tau(\gamma^*(x_1)) = \tau(t^r u) = t^r v^{-r} u = t^r$. Therefore, we may assume without loss of generality that $\gamma^*(x_1) = t^r$.

I claim $1, t, \ldots, t^{r-1}$ generate $\mathbf{k}[[t]]$ as a module over A_γ. Let $f(t) = \sum_{i \geq 0} f_i t^i \in \mathbf{k}[[t]]$ with $f_i \in \mathbf{k}$ for all $i \geq 0$. For $0 \leq j \leq r$, define a power series $p_j(X) \in \mathbf{k}[[X]]$ by $p_j(X) = \sum_{i \geq 0} f_{j+ir} X^i$.

Then

$$\begin{aligned}\sum_{j=0}^{j=r-1} \gamma^*(p_j(x_1))t^j &= \sum_{j=0}^{j=r-1} p_j(\gamma^*(x_1))t^j \\ &= \sum_{j=0}^{j=r-1} p_j(t^r)t^j \\ &= \sum_{j=0}^{j=r-1} \sum_{i\geq 0} f_{j+ir}t^{j+ir} = \sum_{i\geq 0} f_i t^i = f(t).\end{aligned}$$

Hence $1, t, \ldots, t^{r-1}$ generate $\mathbf{k}[[t]]$ considered as a module over A_γ via the ring homomorphism $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[[t]]$. Since $\mathbf{k}[[t]]$ has dimension one and module finite ring homomorphisms preserve dimension, we conclude A_γ has dimension one. $\square$

Lemma IV.7. *We continue using the setup and hypotheses of Lemma IV.6. There exists $\phi \in \mathbf{k}[[t]]$ such that the image of $\overline{\gamma}^* : \tilde{A}_\gamma \hookrightarrow \mathbf{k}[[t]]$ equals $\mathbf{k}[[\phi]] \subseteq \mathbf{k}[[t]]$.*

Proof. Since an integral extension of rings preserves dimension ([10, Proposition 9.2]), we have that $\tilde{A}_\gamma$ has dimension one. Since $\mathbf{k}[[t]]$ is normal (in fact it is a DVR), the local $\mathbf{k}$-algebra map $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[[t]]$ extends to a $\mathbf{k}$-algebra map $\overline{\gamma}^* : \tilde{A}_\gamma \hookrightarrow \mathbf{k}[[t]]$.

I claim the ring $\tilde{A}_\gamma$ is a complete local domain. The local ring A_γ is complete since it is the image of a complete local ring. The normalization of an excellent ring A (in our case, the complete local domain A_γ) is module finite over A [18, p.259]. A module finite domain over a complete local domain is local and complete (apply [10, Corollary 7.6] and use the domain hypothesis to conclude there is only one maximal ideal). Hence $\tilde{A}_\gamma$ is a complete local domain.

Since $\tilde{A}_\gamma$ is a complete normal 1-dimensional local domain containing the field $\mathbf{k}$, it is isomorphic to a power series over $\mathbf{k}$ in one variable [18, Cor. 2, p.206]. That is, there exists $\phi \in \mathbf{k}[[t]]$ such that the image of $\overline{\gamma}^* : \tilde{A}_\gamma \hookrightarrow \mathbf{k}[[t]]$ equals $\mathbf{k}[[\phi]] \subseteq \mathbf{k}[[t]]$. $\square$

The following result was pointed out to me by Mel Hochster.

Proposition IV.8. *Assume the setup of Notation IV.5. Let d be the greatest common divisor of the elements of the non-empty set* $\{\mathrm{ord}_\gamma(f) \mid f \in \widehat{\mathcal{O}}_{X,p},\ 0 < \mathrm{ord}_\gamma(f) < \infty\}$. *Then* $d = \mathrm{ord}_t(\phi)$. *In particular,* ord_γ *is a normalized arc valuation if and only if* $\mathrm{ord}_t(\phi) = 1$.

Proof. For $f, g \in A_\gamma$ such that $\frac{f}{g} \in \tilde{A}_\gamma \subseteq \mathrm{Frac}(A_\gamma)$, we have $\mathrm{ord}_{\overline{\gamma}}(\frac{f}{g}) = \mathrm{ord}_{\overline{\gamma}}(f) - \mathrm{ord}_{\overline{\gamma}}(g)$, and hence d divides $\mathrm{ord}_{\overline{\gamma}}(\frac{f}{g})$. In particular d divides $\mathrm{ord}_t(\phi)$. We have $\overline{\gamma}^*(A_\gamma) \subseteq \overline{\gamma}^*(\tilde{A}_\gamma) = \mathbf{k}[[\phi]] \subseteq \mathbf{k}[[t]]$ and hence $\mathrm{ord}_t(\phi)$ divides $\mathrm{ord}_\gamma(f)$ for all $f \in A_\gamma$. So $\mathrm{ord}_t(\phi)$ divides d. Hence $d = \mathrm{ord}_t(\phi)$. $\square$

Definition IV.9 (Nonsingular arc valuations)**.** Let v be an arc valuation centered at p, and let $\mathfrak{m}_p$ denote the maximal ideal of $\mathcal{O}_{X,p}$. We call v *nonsingular* if

$$\min_{f \in \mathfrak{m}_p} v(f) = 1. \tag{4.3}$$

If $\gamma \in X_\infty$, then we say γ is nonsingular if ord_γ is a nonsingular valuation.

Let C be an irreducible subset of X_∞, and let α be the generic point of C. By Remark II.5, we get an arc $\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$. Following Ein, Lazarsfeld, and Mustaţă [7, p.3], we define a map $\mathrm{val}_C : \mathcal{O}_{X,\alpha(o)} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by setting for $f \in \mathcal{O}_{X,\alpha(o)}$

$$\mathrm{val}_C(f) = \min\{\mathrm{ord}_\gamma(f) \mid \gamma \in C \text{ such that } f \in \mathcal{O}_{X,\gamma(o)}\} \tag{4.4}$$

Proposition IV.10. *Let $C \subseteq X_\infty$ be an irreducible subset and let α be its generic point. Let* $\alpha : \operatorname{Spec} \kappa(\alpha)[[t]] \to X$ *be the arc corresponding to α, as explained in Remark II.5. Then* $\mathrm{val}_C = \mathrm{ord}_\alpha$ *on* $\mathcal{O}_{X,\alpha(o)}$. *In particular,* val_C *is an arc valuation.*

Proof. Fix $f \in \mathcal{O}_{X,\alpha(o)}$, and let $U \subseteq X$ be the maximal open set on which f is regular. We have $\mathrm{ord}_\alpha(f) \geq \mathrm{val}_C(f)$ by Equation (4.4). Let $\alpha' \in C$ be such that

$\mathrm{val}_C(f) = \mathrm{ord}_{\alpha'}(f)$. Let $\pi : X_\infty \to X$ be the canonical morphism sending $\gamma \to \gamma(o)$. If $\mathrm{ord}_\alpha(f) > \mathrm{val}_C(f)$, then $C \cap \mathrm{Cont}^{\geq \mathrm{ord}_\alpha(f)}(f)$ is a closed subset of the irreducible set $C \cap \pi^{-1}(U)$, containing α but not $\alpha' \in C$, contradicting $\overline{\{\alpha\}} = C$. Hence $\mathrm{ord}_\alpha(f) = \mathrm{val}_C(f)$ for all $f \in \mathcal{O}_{X,\alpha(o)}$.

□

Next, we show arc valuations are the same as $\mathbb{Z}_{\geq 0} \cup \{\infty\}$-valued valuations, which are defined as follows:

Definition IV.11. Let R be a $\mathbf{k}$-algebra. A $\mathbb{Z}_{\geq 0} \cup \{\infty\}$-*valued valuation* on R is a map $v : R \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that

1. $v(c) = 0$ for $c \in \mathbf{k}^*$ and $v(0) = \infty$, i.e. v extends the trivial valuation on $\mathbf{k}$
2. $v(xy) = v(x) + v(y)$ for $x, y \in R$
3. $v(x + y) \geq \min\{v(x), v(y)\}$ for $x, y \in R$
4. v is not identically 0 on R^*.

Note that arc valuations given by nontrivial arcs are $\mathbb{Z}_{\geq 0} \cup \{\infty\}$-valuations. We will see in Proposition IV.12 that the converse is true.

Let $p \in X$ be a (not necessarily closed) point of X, and let $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be a $\mathbb{Z}_{\geq 0} \cup \{\infty\}$-valued valuation. Set $\mathfrak{p} = \{f \in \mathcal{O}_{X,p} \mid v(f) = \infty\}$. We have an induced valuation $\tilde{v} : \mathcal{O}_{X,p}/\mathfrak{p} \setminus \{0\} \to \mathbb{Z}$ that extends to a discrete valuation $\tilde{v} : \mathrm{Frac}(\mathcal{O}_{X,p}/\mathfrak{p})^* \to \mathbb{Z}$. Let $R_{\tilde{v}} = \{f \in \mathrm{Frac}(\mathcal{O}_{X,p}/\mathfrak{p})^* \mid \tilde{v}(f) \geq 0\} \cup \{0\}$ be the valuation ring of $\tilde{v}$. $R_{\tilde{v}}$ is a discrete valuation ring. Let $\mathfrak{m}_{\tilde{v}}$ be the maximal ideal of $R_{\tilde{v}}$, and let $\kappa(v) = R_{\tilde{v}}/\mathfrak{m}_{\tilde{v}}$.

Proposition IV.12. *Let $p \in X$ be a (not necessarily closed) point of X. If $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is a valuation as in Definition IV.11, then v is an arc valuation* *on X. In fact, there exists an arc $\gamma : \operatorname{Spec} \kappa(v)[[t]] \to X$ such that $\gamma(o) = p$ and* $\mathrm{ord}_\gamma = v$ *on $\mathcal{O}_{X,p}$.*

Proof. The completion $\widehat{R}_{\tilde{v}}$ of $R_{\tilde{v}}$ with respect $\mathfrak{m}_{\tilde{v}}$ is again a discrete valuation ring ([17, Exercise 11.3]). The complete regular local $\mathbf{k}$-algebra $\widehat{R}_{\tilde{v}}$ is isomorphic to the power series ring $\kappa(v)[[t]]$ ([18, p.206 Corollary 2]). The composition of the canonical homomorphisms $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{p} \to R_{\tilde{v}} \to \widehat{R}_{\tilde{v}} = \kappa(v)[[t]]$ gives an arc $\gamma : \operatorname{Spec} \kappa(v)[[t]] \to X$. Tracing through the constructions, we see that $\mathrm{ord}_\gamma = v$ on $\mathcal{O}_{X,p}$. $\square$

Proposition IV.13. *Let $p \in X$ be a (not necessarily closed) point of X. If $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is a valuation as in Definition IV.11, then there is a subvariety Y of X such that v restricts to a discrete valuation $v : \mathbf{k}(Y) \to \mathbb{Z}$ on the function field of Y.*

Proof. By Proposition IV.12, there exists an arc $\gamma : \operatorname{Spec} \kappa(v)[[t]] \to X$ such that $\gamma(o) = p$ and $\mathrm{ord}_\gamma = v$ on $\mathcal{O}_{X,p}$. Let $U \subseteq X$ be an open set containing $\gamma(o)$. Set $Y = \overline{\gamma(\eta)}$, where η is the generic point of $\operatorname{Spec} \kappa(v)[[t]]$. We have $o \in \overline{\eta}$, hence $\gamma(o) \in \gamma(\overline{\eta}) \subseteq \overline{\gamma(\eta)} = Y$. Hence $U \cap Y$ is nonempty and therefore as an open subset of Y contains the generic point $\gamma(\eta)$ of Y. The $\mathbf{k}$-algebra map $\gamma^* : \mathcal{O}_X(U) \to \kappa(v)[[t]]$ induces a map $\gamma^* : \mathcal{O}_Y(U \cap Y) \hookrightarrow \kappa(v)[[t]]$ after taking the quotient of $\mathcal{O}_X(U)$ by the kernel of γ^*. Localizing at $\gamma(\eta)$ gives a map $\gamma^* : \mathbf{k}(Y) \to \kappa(v)((t))$. Composing this map with $\mathrm{ord}_t : \kappa(v)((t)) \to \mathbb{Z}$ gives the required valuation $v : \mathbf{k}(Y) \to \mathbb{Z}$. $\square$

Remark IV.14. If $C \subseteq X_\infty$ is an irreducible cylinder, then $\mathrm{val}_C : K(X)^* \to \mathbb{Z}$ is a valuation. Ein, Lazarsfeld, and Mustaţă [7, Thm. 2.7] show that if X is a nonsingular variety $C \subseteq X_\infty$ is an irreducible cylinder then val_C is a divisorial valuation, i.e. there is a divisor D on a normal variety Y and a proper birational map $\mu : Y \to X$

such that on $K(Y) = K(X)$, val_C equals val_D, the valuation given by the order of vanishing along D. Ishii [13, Example 3.9] has given another proof of this result. On the other hand, Ein et. al. ([7, Example 2.5]) show that $C_1 := \overline{\mu_\infty(\mathrm{Cont}^{\geq 1}(D))}$ is an irreducible cylinder of X_∞ with $\mathrm{val}_{C_1} = \mathrm{val}_D$.

Definition IV.15 (transcendence degree)**.** Given an arc valuation $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the *transcendence degree* of v over $\mathbf{k}$, denoted $\mathrm{tr.deg}\, v$, is the transcendence degree of $\kappa(v)$ over $\mathbf{k}$. By Proposition IV.12, there exists an arc $\gamma : \mathrm{Spec}\, \kappa(v)[[t]] \to X$ such that $\gamma(o) = p$ and $\mathrm{ord}_\gamma = v$ on $\mathcal{O}_{X,p}$. In particular, if $\mathrm{tr.deg}\, v = 0$, then there is an arc $\gamma : \mathrm{Spec}\, \mathbf{k}[[t]] \to X$ such that $v = \mathrm{ord}_\gamma$ on $\mathcal{O}_{X,p}$.

Lemma IV.16. *Let* $\gamma : \mathrm{Spec}\, K[[t]] \to X$ *be an arc on* X*. Then* $\mathrm{tr.deg}\, \mathrm{ord}_\gamma \leq \mathrm{tr.deg}\, K/\mathbf{k}$*. In particular, if* $K = \mathbf{k}$*, then* ord_γ *has transcendence degree* 0*.*

Proof. We have a local $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(o)} \to K[[t]]$. Taking quotients by the maximal ideals gives a map of fields $\kappa(\mathrm{ord}_\gamma) \hookrightarrow K$. Hence $\mathrm{tr.deg}\, \kappa(\mathrm{ord}_\gamma) \leq \mathrm{tr.deg}\, K/\mathbf{k}$. □

Remark IV.17. Following [11], a *Krull valuation* V is a map $V : \mathbf{k}(X)^* \to \Gamma$, where $\mathbf{k}(X)$ is the function field of X and Γ is a totally ordered abelian group, satisfying equations (1), (3), (4), (5) of Definition IV.11. For a discussion of the differences between Krull valuations and valuations (as defined in Definition IV.11) in the case of surfaces, see [11, Section 1.6]. For example, Favre and Jonsson associate to any Krull valuation $V : \mathbb{C}[[x,y]] \to \Gamma$ other than an exceptional curve valuation, a unique (up to scalar multiple) valuation $v : \mathbb{C}[[x,y]] \to \mathbb{R} \cup \{\infty\}$ [11, Prop. 1.6].

To any Krull valuation $V : \mathbf{k}(X)^* \to \mathbb{Z}^r$ (where $\mathbb{Z}^r$ is lexicographically ordered with $(0,\ldots,0,1)$ as the smallest positive element) with center p (that is, the valuation ring $R_V := \{f \in \mathbf{k}(X)^* \mid V(f) \geq 0\} \cup \{0\}$ dominates $\mathcal{O}_{X,p}$), we associate an arc valuation $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as follows. Set $v(0) = \infty$. For $f \in \mathcal{O}_{X,p}$, suppose $V(f) = (a_1, \ldots, a_r)$. If $a_1 = a_2 = \ldots = a_{r-1} = 0$, set $v(f) = a_r$. Otherwise, set $v(f) = \infty$.

When $\dim X = 2$, the above association $V \to v$ gives a bijection between Krull valuations $V : \mathbf{k}(X)^* \to \mathbb{Z}^2$ centered at p and arc valuations centered at p [11, Prop. 1.6].

The following example shows this association $V \to v$ is not injective in general.

Example IV.18. Let $X = \operatorname{Spec} \mathbf{k}[x, y, z]$ and let $V_1 : \mathbf{k}(X)^* \to \mathbb{Z}^2$ and $V_2 : \mathbf{k}(X)^* \to \mathbb{Z}^3$ be Krull valuations defined by $V_1(\sum c_{ijk}x^iy^jz^k = \min\{(j + 2k, i) \mid c_{ijk} \neq 0\}$ and $V_2(\sum c_{ijk}x^iy^jz^k = \min\{(0, j + k, i) \mid c_{ijk} \neq 0\}$. Then V_1, V_2 both have transcendence degree 0 over $\mathbf{k}$, and have the same sequence of centers. The arc valuations associated (in the manner described above) to V_1 and V_2 both equal the arc valuation $\operatorname{ord}_\gamma$ where $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ is the arc given by $x \to t$, $y \to 0$, and $z \to 0$.

4.2 The arcs corresponding to an arc valuation

In this section, given an arc valuation v we study the set of irreducible subsets $C \subseteq X_\infty$ such that $\operatorname{val}_C = v$. By Proposition IV.10, it is equivalent to consider the set of arcs $\alpha \in X_\infty$ such that $\operatorname{ord}_\alpha = v$.

We begin by examining the situation for the divisorial valuation v on $\mathbb{A}^2$ given by the order of vanishing at the origin. We see that there are many irreducible sets C such that $\operatorname{val}_C = v$ on $\mathcal{O}_{X,p} = \mathbf{k}[x, y]_{(x,y)}$, and not all of these sets are cylinders. There is however a maximal irreducible set $C(v)$ with $\operatorname{val}_{C(v)} = v$ – that is, $C(v)$ contains all irreducible sets C such that $\operatorname{val}_C = v$.

Example IV.19. Let $v : \mathbf{k}(x, y)^* \to \mathbb{Z}$ be the valuation given by the order of vanishing at the origin p in $\mathbf{k}^2 = \operatorname{Spec} \mathbf{k}[x, y]$. Let $x_0, x_1, \ldots, y_0, y_1, \ldots$ be indeterminate variables over $\mathbf{k}$. Identify $(\mathbf{k}^2)_\infty$ with $\operatorname{Spec}\mathbf{k}[x_0, x_1, \ldots, y_0, y_1, \ldots]$ as follows. Let $\mathbf{k} \subseteq K$ be an extension of fields. Given an arc $\gamma : \operatorname{Spec} K[[t]] \to \operatorname{Spec}\mathbf{k}[x, y]$, let the corresponding $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathbf{k}[x, y] \to K[[t]]$ be given by $\gamma^*(x) = \sum_{i\geq 0} a_i t^i$ and $\gamma^*(y) = \sum_{i\geq 0} b_i t^i$, where $a_i, b_i \in K$, for all $i \geq 0$. Then γ corresponds to the K-valued point of $\operatorname{Spec}\mathbf{k}[x_0, x_1, \ldots, y_0, y_1, \ldots]$ given by the $\mathbf{k}$-algebra homomorphism $\mathbf{k}[x_0, x_1, \ldots, y_0, y_1, \ldots] \to K$ sending $x_i \to a_i$ and $y_i \to b_i$ for all $i \geq 0$.

For $q \geq 0$, the ideal of $\operatorname{Cont}^{\geq q}(p)$ in $\mathbf{k}[x_0, x_1, \ldots, y_0, y_1, \ldots]$ is the prime ideal $(x_0, \ldots, x_{q-1}, y_0, \ldots, y_{q-1})$, and hence $\operatorname{Cont}^{\geq q}(p)$ is an irreducible cylinder. The generic point of $\operatorname{Cont}^{\geq q}(p)$ is the arc $\gamma : \operatorname{Spec}\mathbf{k}(x_q, x_{q+1}, \ldots, y_q, y_{q+1}, \ldots)[[t]] \to \operatorname{Spec}\mathbf{k}[x, y]$ given by $\gamma^*(x) = x_q t^q + x_{q+1} t^{q+1} + \cdots$ and $\gamma^*(y) = y_q t^q + y_{q+1} t^{q+1} + \cdots$. The valuation $\operatorname{val}_{\operatorname{Cont}^{\geq q}(p)}$ is given by $\operatorname{ord}_\gamma$ (Proposition IV.10). Also, $\operatorname{ord}_\gamma = qv$. Let α be the arc given by $x \to x_q t^q$ and $y \to y_q t^q$. Then $\operatorname{ord}_\alpha = qv$. Note that $\overline{\{\alpha\}}$ is a set of infinite codimension, and its ideal is $(x_0, x_1, \ldots, x_{q-1}, x_{q+1}, \ldots, y_0, y_1, \ldots y_{q-1}, y_{q+1} \ldots)$ (notice that x_q, y_q are left out). Note that $\overline{\alpha}$ is not a cylinder, but qv is a divisorial valuation. Also, $\overline{\alpha}$ does not contain $\operatorname{Cont}^{\geq r}(p)$ for any r.

There are many arcs β such that $\operatorname{ord}_\beta = v$. For example, let $\beta : \operatorname{Spec}\mathbf{k}(x_1, x_2, \ldots, y_1)[[t]] \to \operatorname{Spec}\mathbf{k}[x, y]$ be the arc given by $\beta^*(x) = x_1 t + x_2 t^2 + \ldots$ and $\beta^*(y) = y_1 t + f_2(X) t^2 + f_3(X) t^3 + \ldots$ where $f_i(X)$ is any polynomial in the x_i. Then $v = \operatorname{ord}_\beta$. The maximal irreducible set $C \subseteq X_\infty$ with $\operatorname{val}_C = v$ is given by $\operatorname{Cont}^{\geq 1}(p)$. Indeed, if γ is an arc such that $\operatorname{ord}_\gamma = v$, then $\gamma \in \operatorname{Cont}^1(p)$. Hence $\overline{\{\gamma\}} \subseteq \operatorname{Cont}^{\geq 1}(p)$. By Proposition IV.10, $\operatorname{Cont}^{\geq 1}(p)$ contains every irreducible cylinder $D \subseteq X_\infty$ with $\operatorname{val}_D = v$. On the other hand, the calculation in the previous paragraph (with $q = 1$) shows $\operatorname{val}_{\operatorname{Cont}^{\geq 1}(p)} = v$.

Definition IV.20. Let X be a scheme of finite type over a field $\mathbf{k}$, and let C be an irreducible subset of X_∞. We define the *dimension of* C to equal $\operatorname{tr.deg}_{\mathbf{k}} K \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, where K is the residue field at the generic point of C.

Example IV.21. Note that a $\mathbf{k}$-valued point of X_∞ has dimension 0.

Proposition IV.22. *Let X be a variety over a field $\mathbf{k}$ and $p \in X$ be a (not necessarily closed) point. Let $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be a valuation. Let $C \subseteq X_\infty$ be an irreducible set with generic point $\gamma : \operatorname{Spec} K[[t]] \to X$ such that $\gamma(o) = p$ and $\operatorname{val}_C = v$ on $\mathcal{O}_{X,p}$. Then $\dim C \geq \operatorname{tr.deg} v$.*

Proof. We have $\dim C = \operatorname{tr.deg}_{\mathbf{k}} K$ by definition and $\operatorname{tr.deg}_{\mathbf{k}} K \geq \operatorname{tr.deg} \operatorname{ord}_\gamma$ by Lemma IV.16. We have $\operatorname{ord}_\gamma = v$ by Proposition IV.10. Hence $\dim C \geq \operatorname{tr.deg} v$.

□

4.3 Desingularization of normalized k-arc valuations

In this section, we prove that a normalized $\mathbf{k}$-arc valuation on a nonsingular variety X over a field $\mathbf{k}$ can be desingularized. Specifically, the goal of this section is to prove Proposition IV.27, which says that a normalized $\mathbf{k}$-valued arc can be lifted after finitely many blowups to an arc that is nonsingular. Our proof is based on Hamburger-Noether expansions.

Let X be a nonsingular variety of dimension n ($n \geq 2$) over a field $\mathbf{k}$ and let $p_0 \in X$ be a closed point. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc such that $\gamma(o) = p_0$ and $v := \operatorname{ord}_\gamma$ is a normalized arc valuation (Definition IV.4). Let $p_i \in X_i$ ($i \geq 0$) be the sequence of centers of v, as described in Definition III.3. If γ_r denotes the unique lift of γ to X_r (by Lemma II.9), then note that v extends to the valuation $\widehat{\mathcal{O}}_{X_r,p_r} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ associated to γ_r. Hence for $f \in \widehat{\mathcal{O}}_{X_r,p_r}$, we will write $v(f)$ to mean $\operatorname{ord}_{\gamma_r}(f)$.

4.3.1 Hamburger-Noether expansions

We will use a list of equations known as Hamburger-Noether expansions (HNEs) to keep track of local coordinates of the sequences of centers of v. We explain HNEs in this section. Our source for this material is [5, Section 1], where the presentation is given for arbitrary valuations on a nonsingular surface.

HNEs are constructed by repeated application of Lemma II.9 part 2, which we recall:

Lemma IV.23. *Let X be a nonsingular variety of dimension n ($n \geq 2$) over a field $\mathbf{k}$ and let $p_0 \in X$ be a closed point. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc such that $\gamma(o) = p_0$ and $v := \operatorname{ord}_\gamma$ is a normalized arc valuation (Definition IV.4). Let $x_1, x_2, \ldots, x_n$ be local algebraic coordinates at p_0 such that $1 \leq v(x_1) \leq v(x_i)$ for $2 \leq i \leq n$. Then for $2 \leq i \leq n$, there exists $a_{i,1} \in \mathbf{k}$ such that if we let $y_i = \frac{x_i}{x_1} - a_{i,1} \in \mathbf{k}(X)$, then $x_1, y_2, \ldots, y_n$ generate the maximal ideal of $\mathcal{O}_{X_1,p_1} \subseteq \mathbf{k}(X) = \mathbf{k}(X_1)$.*

We now describe how to write down the HNEs, following [5, Section 1]. Let $x_i, a_{i,1}, y_i$ be as in Lemma IV.23. We have $x_i = a_{i,1}x_1 + x_1 y_i$. If $v(x_1) \leq v(y_i)$ for every $2 \leq i \leq n$, then with the local algebraic coordinates $x_1, y_2, \ldots, y_n$ at p_1 we are in a similar situation as before, and we repeat the process of applying Lemma IV.23 to get local algebraic coordinates at p_2. Suppose that after h steps we have local algebraic coordinates $x_1, y'_2, \ldots y'_n$ at p_h such that $v(x_1) > v(y'_j)$ for some $2 \leq j \leq n$. We may choose j such that $v(y'_j) \leq v(y'_i)$ for $2 \leq i \leq n$. There are $a_{i,k} \in \mathbf{k}$ such that

$$x_i = a_{i,1}x_1 + a_{i,2}x_1^2 + \ldots + a_{i,h}x_1^h + x_1^h y'_i \tag{4.5}$$

for $2 \leq i \leq n$, $1 \leq k \leq h$. The assumption that p_h is a closed point implies $v(y'_i) > 0$ for $2 \leq i \leq n$. Let $z_1 = y'_j$, and we repeat the procedure of applying Lemma IV.23 with the local coordinates $z_1, x_1, y'_2, \ldots, y'_{j-1}, y'_{j+1}, \ldots, y'_n$ (note that we brought z_1

to the front of the list because it is the coordinate with smallest value). We will refer to such a change in the first coordinate (in this case, from x_1 to z_1) of our list as an iteration.

If we do not arrive at a situation where $v(x_1) > v(y'_j)$ for some $2 \leq j \leq n$, then there exist $a_{i,k} \in \mathbf{k}$ (for $2 \leq i \leq n$, and all $k \geq 1$) such that

$$v\left(\frac{x_i - \sum_{k=1}^{N} a_{i,k} x_1^k}{x_1^N}\right) \geq v(x_1),$$

and hence (since $v(x_1) \geq 1$)

$$v\left(x_i - \sum_{k=1}^{N} a_{i,k} x_1^k\right) > N \tag{4.6}$$

for all $N > 0$.

Let $z_0 = x_1$, and for $l > 0$ let z_l be the first listed local coordinate at the l-th iteration. We have $v(z_l) < v(z_{l-1})$ since an iteration occurs when the smallest value of the local coordinates at the center decreases in value after a blowup. So $\{v(z_l)\}_{l \geq 0}$ is a strictly decreasing sequence of positive integers, and hence must be finite, say $v(z_0), v(z_1), \ldots, v(z_L)$.

For notational convenience, redefine $x_1, \ldots, x_n$ to be the local algebraic coordinates after the final iteration, with $x_1 = z_L$. So $x_1, \ldots, x_n$ are local algebraic coordinates centered at p_r on X_r for some r, and Equation 4.6 becomes

$$v(x_i - \sum_{k=1}^{N} c_{i,k} x_1^k) > N \tag{4.7}$$

for $2 \leq i \leq n$, $c_{i,k} \in \mathbf{k}$, and all $N > 0$.

Definition IV.24. Let $P_1(t) = t$, and for $2 \leq i \leq n$ define $P_i(t) \in \mathbf{k}[[t]]$ by $P_i(t) = \sum_{k=1}^{\infty} c_{i,k} t^k$.

Remark IV.25. Equation 4.7 implies $v(x_i - P_i(x_1)) = \infty$ for $2 \leq i \leq n$.

Lemma IV.26. *For every $\psi = \psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X_r,p_r} \simeq \mathbf{k}[[x_1, \ldots, x_n]]$, we have $v(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t))$.*

Proof. Since $\mathbf{k}[[x_1, \ldots, x_n]]/(x_2 - P_2(x_1), \ldots, x_n - P_n(x_1)) \simeq \mathbf{k}[[x_1]]$, we may write $\psi(x_1, \ldots, x_n) = q(x_1) + \sum_{i=2}^{n}(x_i - P_i(x_1))h_i$ for $h_i \in \mathbf{k}[[x_1, \ldots, x_n]]$ and $q(x_1) \in \mathbf{k}[[x_1]]$. Note that $q(x_1) = \psi(x_1, P_2(x_1), \ldots, P_n(x_1))$. We have $v(\psi) \geq \min\{v(q), v((x_2 - P_2(x_1))h_2), \ldots, v((x_n - P_n(x_1))h_n)\}$. Since $v((x_i - P_i(x_1))h_i) = \infty$, we have $v(\psi) = v(q)$, since in general, if $v(a) \neq v(b)$, then $v(a + b) = \min\{v(a), v(b)\}$.

Let $n = \operatorname{ord}_{x_1} q(x_1)$. We claim $v(q) = nv(x_1)$. If $n = \infty$, then $q = 0$ and both sides of $v(q) = nv(x_1)$ are ∞. If $n < \infty$, then $q = x_1^n u$ for a unit u in $\mathbf{k}[[x_1]]$. We have $v(u) = 0$, since $0 = v(1) = v(uu^{-1}) = v(u) + v(u^{-1})$ and $v(u), v(u^{-1}) \geq 0$. Hence $v(q) = nv(x_1)$.

So we have $v(\psi) = v(q) = (\operatorname{ord}_{x_1} q(x_1))v(x_1) = \operatorname{ord}_{x_1} \psi(x_1, P_2(x_1) \ldots, P_n(x_1)) \cdot v(x_1)$. Since ψ was arbitrary, we have that the image of $v : \mathbf{k}[[x_1, \ldots, x_n]] \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ equals $\mathbb{Z}_{\geq 0} \cdot v(x_1) \cup \{\infty\}$. Since v was normalized so that the image of v had 1 as the greatest common factor of its elements, we have $v(x_1) = 1$ and $v(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t))$. $\square$

Summarizing the discussion so far, we have:

Proposition IV.27. *Let v be a normalized $\mathbf{k}$-arc valuation on a nonsingular variety X over a field $\mathbf{k}$. Then there exists a nonnegative integer r and local algebraic coordinates $x_1, \ldots, x_n$ at the center p_r of v on X_r and*

$$P_i(t) \in (t)\mathbf{k}[[t]]$$

for $2 \leq i \leq n$ such that for every $\psi = \psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X_r,p_r} \simeq \mathbf{k}[[x_1, \ldots, x_n]]$, we *have*

$$v(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t)).$$

Roughly speaking, this result says that a normalized $\mathbf{k}$-arc valuation can be desingularized. More precisely, a normalized $\mathbf{k}$-valued arc γ can be lifted after finitely many blowups (of its centers) to an arc γ_r that is nonsingular (see Definition IV.9 for the definition of nonsingular arc). Using the notation of Proposition IV.27, the arc $\gamma_r : \operatorname{Spec} \mathbf{k}[[t]] \to X_r$ is given by the $\mathbf{k}$-algebra map $\widehat{\mathcal{O}}_{X_r,p_r} \to \mathbf{k}[[t]]$ with $\operatorname{ord}_{\gamma_r}(x_1) = 1$ and $x_i \to P_i(\gamma_r^*(x_1))$ for $2 \leq i \leq n$. Since $\operatorname{ord}_{\gamma_r}(x_1) = 1$, we have γ_r is a nonsingular arc.

If the arc γ is nonsingular, we can take $r = 0$ in Proposition IV.27, and we have the following result.

Proposition IV.28. *Let* $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ *be a nonsingular* $\mathbf{k}$*-arc on a nonsingular variety* X *over a field* $\mathbf{k}$*. Let* $x_1, \ldots, x_n$ *be local algebraic coordinates at* $p = \gamma(o)$ *on* X *with* $\operatorname{ord}_\gamma(x_1) = 1$ *(Definition IV.9). Then there exists*

$$P_i(t) \in (t)\mathbf{k}[[t]]$$

for $2 \leq i \leq n$ *such that* $\gamma^*(x_i) = P_i(\gamma^*(x_1))$ *for* $2 \leq i \leq n$*. Furthermore, for every* $\psi = \psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[[x_1, \ldots, x_n]]$*, we have*

$$\operatorname{ord}_\gamma(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t)).$$

Proof. Since $\operatorname{ord}_\gamma(x_1) = 1$, there can be no iterations in the Hamburger-Noether algorithm for $v = \operatorname{ord}_\gamma$. Hence Equation 4.7 holds, and in particular, Remark IV.25 applies. That is, if the $P_i(t)$ for $2 \leq i \leq n$ are as in Definition IV.24, we have $\operatorname{ord}_\gamma(x_i - P_i(x_1)) = \infty$ for $2 \leq i \leq n$. So $\gamma^*(x_i - P_i(x_1)) = 0$, and therefore $\gamma^*(x_i) = \gamma^*(P_i(x_1)) = P_i(\gamma^*(x_1))$ for $2 \leq i \leq n$. According to Lemma IV.26, for every $\psi = \psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[[x_1, \ldots, x_n]]$, we have

$$\operatorname{ord}_\gamma(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t)).$$

$\square$

We will see in the next chapter that for a nonsingular $\mathbf{k}$-valued arc γ, one can explicitly compute the ideals of $\bigcap_{q \geq 1} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$ and $\bigcap_{q \geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$, where $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,\gamma(o)} \mid \operatorname{ord}_\gamma(f) \geq q\}$. We will see that these ideals are the same, and thus these two sets are equal.

CHAPTER V

Main results: k-arc valuations on a nonsingular k-variety

5.1 Introduction

In this chapter, we present the main results of the thesis. Let X be a nonsingular variety of dimension n ($n \geq 2$) over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an normalized arc. Set $v = \mathrm{ord}_\alpha$ and $p = \alpha(o)$, where o denotes the closed point of $\operatorname{Spec} \mathbf{k}[[t]]$. We associate to v several different subsets of the arc space X_∞. In notation we will explain later in the chapter, these subsets are $C(v)$, $\bigcap_{q\geq 1} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$, $\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$, $\{\gamma \in X_\infty \mid \gamma(o) = \alpha(o), \ker(\alpha^*) \subseteq \ker(\gamma^*) \subseteq \widehat{\mathcal{O}}_{X,\alpha(o)}\}$, and $R = \{\alpha \circ h \in X_\infty \mid h : \operatorname{Spec} \mathbf{k}[[t]] \to \operatorname{Spec} \mathbf{k}[[t]]\}$. Our main result is that these five subsets are all equal. We first analyze the case when v is a nonsingular arc valuation (Definition IV.9). We then consider the general case where we drop the hypothesis of nonsingularity.

5.2 Setup

Throughout this chapter, we fix the following notation. Let X be a nonsingular variety of dimension n ($n \geq 2$) over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be a normalized arc valuation on X (see Definition IV.4). Set $v = \mathrm{ord}_\alpha$.

In Definition III.3, we defined the sequence of centers of a $\mathbf{k}$-arc valuation. To set notation for the rest of this chapter, we recall this definition.

Definition V.1 (Sequences of centers of a $\mathbf{k}$-arc valuation)**.** Let X be a nonsingular variety over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc on X. Assume α is not the trivial arc (Definition II.8). Set $p_0 = \alpha(o)$ (where o is the closed point of $\operatorname{Spec} \mathbf{k}[[t]]$) and $v = \operatorname{ord}_\alpha$. By Proposition II.4, the point p_0 is a closed point (with residue field $\mathbf{k}$) of X. The point p_0 is called the *center* of v on $X_0 := X$. Blowup p_0 to get a model X_1 with exceptional divisor E_1. By Lemma II.9 the arc α has a unique lift to an arc $\alpha_1 : \operatorname{Spec} \mathbf{k}[[t]] \to X_1$. Let p_1 be the closed point $\alpha_1(o)$. Inductively define a sequence of closed points p_i and exceptional divisors E_i on models X_i and lifts $\alpha_i : \operatorname{Spec} \mathbf{k}[[t]] \to X_i$ of α as follows. Blowup $p_{i-1} \in X_{i-1}$, to get a model X_i. Let E_i be the exceptional divisor of this blowup. Let $\alpha_i : \operatorname{Spec} \mathbf{k}[[t]] \to X_i$ be the lift of $\alpha_{i-1} : \operatorname{Spec} \mathbf{k}[[t]] \to X_{i-1}$. Let p_i be the closed point $\alpha_i(o)$. Let $\mu_i : X_i \to X$ be the composition of the first i blowups. We call $\{p_i\}_{i \geq 0}$ the *sequence of centers* of v.

5.3 Simplified situation

We first consider the special case when the arc $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ is nonsingular (Definition IV.9).

Proposition V.2. *Let X be a nonsingular variety of dimension n ($n \geq 2$) over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ a nonsingular arc (Definition IV.9). Set $v = \operatorname{ord}_\alpha$ and $p_0 = \alpha(o)$. Let $C = \bigcap_{q \geq 1} \mu_{q\infty}(\operatorname{Cont}^{\geq 1}(E_q))$. Then*

1. *C is an irreducible subset of X_∞.*
2. *Let $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p_0} \mid v(f) \geq q\}$. Then $C = \bigcap_{q \geq 1} \operatorname{Cont}^{\geq q}(\mathfrak{a}_q)$.*
3. *$\operatorname{val}_C = v$ on $\widehat{\mathcal{O}}_{X,p_0}$.*

Notation V.3. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,p_0}$. Since α is nonsingular, there exists $x_1 \in \mathfrak{m}$ such that $\operatorname{ord}_\alpha(x_1) = 1$. Since $\operatorname{ord}_\alpha(x_1) = 1$, we have $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$.

Choose $x_2, \dots, x_n$ in $\mathfrak{m}$ so that $x_1, \dots, x_n$ are local algebraic coordinates at p_0 (i.e. generators of $\mathfrak{m}$). For $2 \le i \le n$, let $P_i(t) \in (t)\mathbf{k}[[t]]$ be as in Proposition IV.28. Write $P_i(t) = \sum_{j \ge 1} c_{i,j} t^j \in (t)\mathbf{k}[[t]]$ for $2 \le i \le n$ and $c_{i,j} \in \mathbf{k}$. By Proposition IV.28, for every $\psi(x_1, \dots, x_n) \in \widehat{\mathcal{O}}_{X,p_0} \simeq \mathbf{k}[[x_1, \dots, x_n]]$, we have

$$v(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \dots, P_n(t)). \tag{5.1}$$

For $2 \le i \le n$, we also have

$$\begin{aligned} \alpha^*(x_i) &= P_i(\alpha^*(x_1)) \\ &= \sum_{j \ge 1} c_{i,j} (\alpha^*(x_1))^j \end{aligned} \tag{5.2}$$

We break up the proof of Proposition V.2 into several steps. For the remainder of this section, v, $x_1, \dots, x_n$, $P_2(t), \dots, P_n(t)$ and $c_{i,j}$ are as in Proposition V.2 and Notation V.3.

Lemma V.4. *With the notation in Definition V.1, Proposition V.2, and Notation V.3, the functions x_1 and $\frac{x_i - c_{i,1}x_1 - c_{i,2}x_1^2 \cdots - c_{i,q-1}x_1^{q-1}}{x_1^{q-1}} \in \mathbf{k}(X)$ for $2 \le i \le n$ form local algebraic coordinates on X_{q-1} centered at p_{q-1}.*

Proof. These n functions are elements of positive value under $\operatorname{ord}_{\alpha_q}$ (by Equation 5.2), and hence lie in the maximal ideal of the n-dimensional regular local ring $\mathcal{O}_{X_{q-1},p_{q-1}}$. The ideal $\mathfrak{n} \subseteq \mathcal{O}_{X_{q-1},p_{q-1}}$ they generate satisfies $\mathcal{O}_{X_{q-1},p_{q-1}}/\mathfrak{n} \simeq \mathbf{k}$, and hence $\mathfrak{n}$ is a maximal ideal. □

5.3.1 Reduction to $X = \mathbb{A}^n$

We denote the affine line $\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[T]$ simply by $\mathbb{A}^1$. We show that we may reduce many computations about the arc space of the nonsingular n-dimensional variety X to the case $X = \mathbb{A}^n$.

Proposition V.5. *Let X be a nonsingular variety and $p \in X$. Let $\pi : X_\infty \to X$ be the canonical morphism sending an arc γ to its center $\gamma(o)$. Then $\pi^{-1}(p) \simeq (\mathbb{A}^n_{\kappa(p)})_\infty$, where $\kappa(p)$ is the residue field at $p \in X$. In particular, if $\kappa(p) = \mathbf{k}$ then $\pi^{-1}(p) \simeq (\mathbb{A}^n)_\infty$.*

Proof. Since X is nonsingular, there exists an open affine neighborhood U of p and an étale morphism $\phi : U \to \operatorname{Spec} \mathbf{k}[X_1, \dots, X_n] = \mathbb{A}^n$ ([19, Prop. 3.24b]). We will use the following fact ([9, p.7]): if $f : X \to Y$ is an étale morphism, then $X_\infty = X \times_Y Y_\infty$. Applied to the open inclusion $U \to X$, we have $U_\infty = U \times_X X_\infty$. Applied to the étale map $U \to \mathbb{A}^n$ we have $U_\infty = U \times_{\mathbb{A}^n} \mathbb{A}^n_\infty$. Hence we have

$$\pi^{-1}(U) = U \times_X X_\infty = U_\infty = U \times_{\mathbb{A}^n} \mathbb{A}^n_\infty.$$

Hence

$$\pi^{-1}(p) = \operatorname{Spec} \kappa(p) \times_U \pi^{-1}(U) = \operatorname{Spec} \kappa(p) \times_{\mathbb{A}^n} (\mathbb{A}^n)_\infty = (\mathbb{A}^n_{\kappa(p)})_\infty.$$

□

We resume considering Proposition V.2, where now it is sufficient to assume $X = \mathbb{A}^n = \operatorname{Spec} \mathbf{k}[x_1, \dots, x_n]$, and the $\mathbf{k}$-valued point p_0 corresponds to the maximal ideal $(x_1, \dots, x_n)$. We write $(\mathbb{A}^n)_\infty = (\operatorname{Spec} \mathbf{k}[x_1, \dots, x_n])_\infty = \operatorname{Spec} \mathbf{k}[\{x_{i,j}\}_{1 \le i \le n,\ j \ge 0}]$, where the last equality comes from parametrizing arcs on $\operatorname{Spec} \mathbf{k}[x_1, \dots, x_n]$ by $x_i \to \sum_{j \ge 0} x_{i,j} t^j$ for $1 \le i \le n$. Note that $\pi : X_\infty \to X$ (defined in Proposition V.5) maps C to p_0. Hence

$$C \subseteq \pi^{-1}(p_0) = (\mathbb{A}^n)_\infty = \operatorname{Spec} S,$$

where

$$S = \mathbf{k}[\{x_{i,j}\}_{1 \le i \le n,\ j \ge 1}] \tag{5.3}$$

Definition V.6. For $2 \leq i \leq n$ and $q \geq 1$, let $f_{i,q}(X_1, \ldots, X_q)$ be the polynomial that is the coefficient of t^q in

$$\sum_{j=1}^{q} c_{i,j}(X_1 t + X_2 t^2 + \cdots)^j.$$

(Recall that the $c_{i,j}$ were defined in Notation V.3).

Definition V.7. For each positive integer q, let I_q be the ideal of S generated by

1. $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $2 \leq i \leq n$ and $1 \leq j \leq q-1$.

Note that I_q is a prime ideal of S, since $S/I_q = \mathbf{k}[\{x_{1,j}\}_{j\geq 1}, \{x_{i,j}\}_{2\leq i\leq n, q\leq j}]$.

Notation V.8. If J is an ideal of S, we denote by $V(J)$ the closed subscheme of $\operatorname{Spec} S$ defined by the ideal J.

Definition V.9. Let I be the ideal of S defined by $I = \bigcup_{q\geq 1} I_q$. Since I is the ideal of S generated by $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $2 \leq i \leq n$ and $1 \leq j$, we have $S/I = \mathbf{k}[\{x_{1,j}\}_{1\leq j}]$. In particular, I is a prime ideal of S.

Lemma V.10. *For each positive integer q, the ideal of $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$ in S is I_q. (Note: I_q is defined in Definition V.7.)*

Proof. Note that $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$ is irreducible (e.g. [7, p.9]). Since I_q is a prime ideal, we need to show

$$\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))} = V(I_q).$$

First we show $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))} \subseteq V(I_q)$ by showing that the generic point of $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$ lies in $V(I_q)$. Suppose $\beta' : \operatorname{Spec} K[[t]] \to X_q$ is the generic point of $\mathrm{Cont}^{\geq 1}(E_q)$. To be precise, β' is the canonical arc (described in Remark II.5) associated to the generic point of $\mathrm{Cont}^{\geq 1}(E_q)$. Also, K is the residue field at the generic point of $\mathrm{Cont}^{\geq 1}(E_q)$. By Lemma II.9 part 3, the pushdown of β' to X_{q-1}

is an arc $\beta : \operatorname{Spec} K[[t]] \to X_{q-1}$ that is the generic point of $\mathrm{Cont}^{\geq 1}(p_{q-1})$. By the description of local coordinates at p_{q-1} given in Lemma V.4, the arc β corresponds (by Lemma II.9) to a map $x_1 \to x_{1,1}t + x_{1,2}t^2 + \cdots$ and $\frac{x_i - c_{i,1}x_1 - c_{i,2}x_1^2 \cdots - c_{i,q-1}x_1^{q-1}}{x_1^{q-1}} \to a_{i,1}t + a_{i,2}t^2 + \cdots$ for $2 \leq i \leq n$ and some $a_{i,j} \in K$. The pushdown of β to X is the arc given by $x_1 \to x_{1,1}t + x_{1,2}t^2 + \cdots$ and $x_i \to \sum_{j=1}^{j=q-1} c_{i,j}(x_{1,1}t + x_{1,2}t^2 + \cdots)^j + r(t)$ where $r(t) \in (t^q) \subseteq K[[t]]$. In particular, the pushdown of β' to X corresponds to a prime ideal in S containing the ideal I_q of S generated by $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $1 \leq j \leq q-1$ and $2 \leq i \leq n$. That is, the generic point of $\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$ lies in $V(I_q)$. Hence $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))} \subseteq V(I_q)$.

Conversely, we show that $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))} \supseteq V(I_q)$. The generators of I_q listed in Definition V.7 show that the coordinate ring of $V(I_q)$ is $S/I_q = \mathbf{k}[\{x_{1,j}\}_{j\geq 1}, \{x_{i,j}\}_{2\leq i \leq n, q \leq j}]$. Let $\beta : \operatorname{Spec} K[[t]] \to X$ be the arc corresponding (see Remark II.5) to the generic point of $V(I_q)$, where $K = \mathbf{k}(\{x_{1,j}\}_{j\geq 1}, \{x_{i,j}\}_{2\leq i\leq n, q\leq j})$. We have $\beta^*(x_1) = x_{1,1}t + x_{1,2}t^2 + \ldots$. Since I_q contains $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $1 \leq j \leq q-1$ and $2 \leq i \leq n$, we have that $\beta^*(x_i) = \displaystyle\sum_{j\geq 1}^{q-1} f_{i,j}(x_{1,1}, \ldots, x_{1,j})t^j + t^q r_i(t)$ for some $r_i(t) \in K[[t]]$ and for each $2 \leq i \leq n$. Hence $\beta^*(x_i) = \sum_{j\geq 1}^{q-1} c_{i,j}(\beta^*(x_1))^j + t^q s_i(t)$ for some $s_i(t) \in K[[t]]$, by Definition V.6.

Therefore

$$\mathrm{ord}_\beta(x_i - c_{i,1}x_1 - c_{i,2}x_1^2 \cdots - c_{i,q-1}x_1^{q-1}) \geq q = \mathrm{ord}_\beta(x_1^{q-1}) + 1,$$

where the last equality follows from the fact $\mathrm{ord}_\beta(x_1) = 1$ as $x_{1,1} \neq 0 \in K$. In particular, the unique lift of β to an arc on X_{q-1} has center p_{q-1}, by Lemma V.4. Hence $\beta \in \mu_{q-1\infty}(\mathrm{Cont}^{\geq 1}(p_{q-1})) = \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$. Hence $V(I_q) = \overline{\{\beta\}} \subseteq \overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$. $\square$

Lemma V.11. *The ideal of C in S is I. (Note: C is defined in Proposition V.2, S*

is defined in Equation 5.3, and I is defined in Definition V.9.)

Proof. Since I is a prime ideal, we need to show $C = V(I)$. We have

$$\bigcap_{q\geq 1} V(I_q) = V(\bigcup_{q\geq 1} I_q) = V(I)$$

and

$$C = \bigcap_{q\geq 1} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q)) \subseteq \bigcap_{q\geq 1} V(I_q)$$

by Lemma V.10. It remains to show $\bigcap_{q\geq 1} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q)) \supseteq \bigcap_{q\geq 1} V(I_q)$.

Let $\beta : \operatorname{Spec} K[[t]] \to X$ be an arc corresponding to a point in $\bigcap_{q\geq 1} V(I_q)$. We may assume β is not the trivial arc, since the trivial arc lies in $\bigcap_{q\geq 1} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$. Say $\beta^*(x_1) = \sum_{j\geq 1} a_{1,j}t^j$, where $a_{1,j} \in K$. Since I_q contains $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $1 \leq j \leq q-1$ and $2 \leq i \leq n$, we have that $\beta^*(x_i) = \sum_{j\geq 1}^{\infty} f_{i,j}(a_{1,1}, \ldots, a_{1,j})t^j$ for each $2 \leq i \leq n$. Hence $\beta^*(x_i) = \sum_{j\geq 1}^{\infty} c_{i,j}(\beta^*(x_1))^j$, by Definition V.6. Hence

$$\mathrm{ord}_\beta(x_i - c_{i,1}x_1 - c_{i,2}x_1^2 \cdots - c_{i,q-1}x_1^{q-1}) = \mathrm{ord}_\beta(\sum_{j\geq q} c_{i,j}x_1^j) = \mathrm{ord}_\beta\, x_1^q \geq \mathrm{ord}_\beta(x_1^{q-1})+1.$$

In particular, the unique lift of β to an arc on X_{q-1} has center p_{q-1}, by Lemma V.4. Hence $\beta \in \mu_{q-1\infty}(\mathrm{Cont}^{\geq 1}(p_{q-1})) = \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$. Hence $\bigcap_{q\geq 1} V(I_q) \subseteq \bigcap_{q\geq 1} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$. □

Lemma V.12. *For a positive integer q, let $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p_0} \mid v(f) \geq q\}$. Set $z_i = x_i - \sum_{j=1}^{q-1} c_{i,j}x_1^j$ for $2 \leq i \leq n$. Then $\mathfrak{a}_q$ is generated (as an ideal in $\widehat{\mathcal{O}}_{X,p_0}$) by $x_1^q, z_2, \ldots, z_n$.*

Proof. By Equation 5.1, we have $v(x_1^q), v(z_i) \geq q$ for $2 \leq i \leq n$. Suppose $f \in \mathfrak{a}_q$. Since $\mathbf{k}[[x_1, \ldots, x_n]]/(z_2, \ldots, z_n) \simeq \mathbf{k}[[x_1]]$, we can write $f = \sum_{i\geq 2}^{i=n} h_i z_i + g(x_1)$, where $h_i \in \mathbf{k}[[x_1, \ldots, x_n]]$ and $g(x_1) \in \mathbf{k}[[x_1]]$. Then since $v(f) \geq q$, and $v(z_i) \geq q$, we must have $v(g) \geq q$. By Equation 5.1, we conclude x_1^q divides $g(x_1)$ in $\mathbf{k}[[x_1]]$. Hence f is in the ideal generated by $x_1^q, z_2, \ldots, z_n$. □

Lemma V.13. *For every positive integer q, the ideal of* $\mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ *in S is I_q.*

Proof. First we show $\mathrm{Cont}^{\geq q}(\mathfrak{a}_q) \subseteq V(I_q)$. Suppose $\beta : \operatorname{Spec} K[[t]] \to X$ is an arc corresponding (via Remark II.5) to a generic point of $\mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$. Write $\beta^*(x_i) = \overline{x}_{i,1}t + \overline{x}_{i,2}t^2 + \cdots$ for $1 \leq i \leq n$, where $\overline{x}_{i,j} \in K$ denotes the image in K of $x_{i,j} \in S$. Since $\mathfrak{a}_q$ is generated by $x_1^q, z_2, \ldots, z_n$ (Lemma V.12) (recall that $z_i = x_i - \sum_{j=1}^{q-1} c_{i,j}x_1^j$ for $2 \leq i \leq n$), we have

$$\overline{x}_{i,1}t + \overline{x}_{i,2}t^2 + \cdots - \sum_{j=1}^{j=q-1} c_{i,j}(\overline{x}_{1,1}t + \overline{x}_{1,2}t^2 + \cdots)^j \in (t^q). \tag{5.4}$$

The coefficient of t^j in Equation 5.4 is $\overline{x}_{i,j} - f_{i,j}(\overline{x}_{1,1}, \ldots, \overline{x}_{1,j})$. Hence β corresponds to a prime ideal of S containing the ideal I_q of S generated by $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $2 \leq i \leq n$ and $1 \leq j \leq q-1$. Thus $\mathrm{Cont}^{\geq q}(\mathfrak{a}_q) \subseteq V(I_q)$.

Conversely, suppose $\beta : \operatorname{Spec} K[[t]] \to X$ corresponds (via Remark II.5) to the generic point of $V(I)$. The coordinate ring of $V(I_q)$ is $S/I_q = \mathbf{k}[\{x_{1,j}\}_{j\geq 1}, \{x_{i,j}\}_{2\leq i\leq n, q\leq j}]$ (Definition V.7). Hence K, the residue field at the generic point of $V(I_q)$, equals $K = \mathbf{k}(\{x_{1,j}\}_{j\geq 1}, \{x_{i,j}\}_{2\leq i\leq n, q\leq j})$. We have $\beta^*(x_1) = x_{1,1}t + x_{1,2}t^2 + \cdots \in K[[t]]$. Since I_q contains $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j})$ for $1 \leq j \leq q-1$ and $2 \leq i \leq n$, we have that $\beta^*(x_i) = \displaystyle\sum_{j\geq 1}^{q-1} f_{i,j}(x_{1,1}, \ldots, x_{1,j})t^j + t^q r_i(t)$ for some $r_i(t) \in K[[t]]$ and for each $2 \leq i \leq n$. Since $\sum_{j\geq 1} c_{i,j}(x_{1,1}t + x_{1,2}t^2 + \cdots)^j = \sum_{j\geq 1} f_{i,j}(x_{1,1}, \ldots, x_{1,j})t^j$ for $2 \leq i \leq n$ (Notation V.3), we have that β^* maps $x_i - c_{i,1}x_1 - c_{i,2}x_1^2 \cdots - c_{i,q-1}x_1^{q-1}$ into the ideal $(t^q) \subseteq K[[t]]$. Hence by Lemma V.12, we have $\beta \in \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$. So $V(I_q) = \overline{\{\beta\}} \subseteq \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$.

□

Lemma V.14. *The ideal of* $\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ *in S is I. (Note: S is defined in Equation 5.3, and I is defined in Definition V.9, and $\mathfrak{a}_q$ is defined in Proposition V.2 (2).)*

Proof. Since I is a prime ideal, it is enough to show $\bigcap_{q\geq 1}\mathrm{Cont}^{\geq q}(\mathfrak{a}_q) = V(I)$. By Lemma V.13, we have

$$\bigcap_{q\geq 1}\mathrm{Cont}^{\geq q}(\mathfrak{a}_q) = \bigcap_{q\geq 1} V(I_q) = V(\bigcup_{q\geq 1} I_q) = V(I).$$

□

We now finish the proof of Proposition V.2.

Proof of Proposition V.2. Since $S/I \simeq \mathbf{k}[\{x_{1,j}\}_{j\geq 1}]$ is a domain, the ideal I is a prime ideal. By Lemma V.11, the ideal of C is I. Hence C is irreducible. We have $C = \bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ because by Lemmas V.11 and V.14, their ideals are the same.

It remains to show $\mathrm{val}_C = v$. Let $\gamma : \mathrm{Spec}\,\mathbf{k}[[t]] \to X$ be the arc centered at p_0 with $\gamma^*(x_1) = t$ and $\gamma^*(x_i) = P_i(t)$ for $2 \leq i \leq n$. Then $\gamma \in C$ since the ideal in S corresponding to γ, namely the ideal generated by $x_{1,0}$, $x_{1,1} - 1$, $x_{1,m}$, $x_{i,0}$, and $x_{i,j} - c_{i,j}$ for $m \geq 2$, $2 \leq i \leq n$, and $j \geq 1$ contains I. Hence for any $f \in \mathcal{O}_{X,p_0}$, we have $\mathrm{val}_C(f) \leq \mathrm{ord}_\gamma(f) = v(f)$.

For the reverse inequality, first suppose $f \in \mathcal{O}_{X,p_0}$ is such that $s := v(f) < \infty$. Let $\gamma \in C$ be such that $\mathrm{val}_C(f) = \mathrm{ord}_\gamma(f)$. Since $f \in \mathfrak{a}_s$ and $\gamma \in \mathrm{Cont}^{\geq s}(\mathfrak{a}_s)$, we have $\mathrm{ord}_\gamma(f) \geq s$, i.e. $\mathrm{val}_C(f) \geq v(f)$.

Next suppose $v(f) = \infty$. Set $\phi_i = x_i - P_i(x_1)$ for $2 \leq i \leq n$. Since

$$\mathbf{k}[[x_1, \ldots, x_n]]/(\phi_2, \ldots, \phi_n) \simeq \mathbf{k}[[x_1]],$$

we can write $f = \sum_{i=2}^{n} \phi_i h_i + g(x_1)$ for $h_i \in \mathbf{k}[[x_1, \ldots, x_n]]$ and $g \in \mathbf{k}[[x_1]]$. Since $v(f) = \infty$, we have $g = 0$ by Equation 5.1. Let $\gamma \in C$, and write $\gamma^*(x_1) = \sum_{j\geq 1} a_j t^j$. Since $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j}) \in I$ for $2 \leq i \leq n$ and $j \geq 1$, we have $\gamma^*(x_i) = \sum_{j\geq 1} f_{i,j}(a_1, \ldots, a_j)t^j = \sum_{j\geq 1} c_{i,j}(a_1 t + a_2 t^2 + \ldots)^j = p_i(\gamma^*(x_1)) = \gamma^*(p_i(x_1))$. Hence $\gamma^*(\phi_i) = 0$, and so $\gamma^*(f) = \gamma^*(\sum_{i=2}^{n} \phi_i h_i) = 0$. So $\mathrm{ord}_\gamma(f) = \infty$. Since $\gamma \in C$ was arbitrary, we have $\mathrm{val}_C(f) = \infty$, as desired. □

5.4 General case

Lemma V.15. *Let X be a nonsingular variety of dimension n ($n \geq 2$) over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be a normalized arc (Definition IV.4). Set $p_0 = \alpha(o)$. Let $\alpha^* : \widehat{\mathcal{O}}_{X,p_0} \to \mathbf{k}[[t]]$ be the $\mathbf{k}$-algebra homomorphism induced by α. Suppose $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ satisfies $\gamma(o) = p_0$ and $\ker(\alpha^*) \subseteq \ker(\gamma^*)$, where $\gamma^* : \widehat{\mathcal{O}}_{X,p_0} \to \mathbf{k}[[t]]$ is the $\mathbf{k}$-algebra homomorphism induced by γ. Assume γ is not the trivial arc (Definition II.8). Then*

1. *There exists a morphism $h : \operatorname{Spec} \mathbf{k}[[t]] \to \operatorname{Spec} \mathbf{k}[[t]]$ such that $\gamma = \alpha \circ h$, i.e. γ is a reparametrization of α.*

2. *$h^* : \mathbf{k}[[t]] \to \mathbf{k}[[t]]$ is a local homomorphism.*

3. *Set $N = \operatorname{ord}_t(h)$. Then $\operatorname{ord}_\gamma = N \operatorname{ord}_\alpha$ on $\widehat{\mathcal{O}}_{X,p_0}$. (We use the convention that $\infty = N \cdot \infty$.)*

Proof. (Due to Mel Hochster.) We use Notation IV.5. Suppose γ is not the trivial arc. By Lemma IV.6, A_γ has dimension one, and so $\ker(\gamma^*)$ is a prime ideal of height $n-1$. The same is true for $\ker(\alpha^*)$, and so our assumption $\ker(\alpha^*) \subseteq \ker(\gamma^*)$ implies $\ker(\alpha^*) = \ker(\gamma^*)$. Hence $A_\alpha = A_\gamma$. By Lemma IV.7, the map α^* (resp. γ^*) induces an isomorphism $\overline{\alpha^*} : \tilde{A}_\alpha \to \mathbf{k}[[\phi_\alpha]]$ (resp. $\overline{\gamma^*} : \tilde{A}_\gamma \to \mathbf{k}[[\phi_\gamma]]$) for some $\phi_\alpha \in \mathbf{k}[[t]]$ (resp. $\phi_\gamma \in \mathbf{k}[[t]]$). Since α is normalized, we have $\operatorname{ord}_t(\phi_\alpha) = 1$ by Proposition IV.8.

I claim that the inclusion $\mathbf{k}[[\phi_\alpha]] \subseteq \mathbf{k}[[t]]$ is actually an equality. It suffices to find $a_j \in \mathbf{k}$ such that $t = \sum_{j \geq 1} a_j (\phi_\alpha)^j$. Suppose $\phi_\alpha = \sum_{j\geq 1} b_j t^j$, where $b_j \in \mathbf{k}$ and $b_1 \neq 0$. We proceed to define a_j by induction on j. Set $a_1 = {b_1}^{-1}$. Suppose $a_1, \ldots, a_{d-1}$ have been specified. The coefficient of t^d in $\sum_{j\geq 1} a_j (\phi_\alpha)^j$ is $a_d b_1^d + Q_d(a_1, \ldots, a_{d-1}, b_1, \ldots, b_d)$ for some polynomial Q_d. We require this coefficient to be

0. We can solve the equation

$$a_d b_1^d + Q_d(a_1, \ldots, a_{d-1}, b_1, \ldots, b_d) = 0$$

for a_d since $b_1 \neq 0$. This completes the induction, and we have $t = \sum_{j \geq 1} a_j (\phi_\alpha)^j$.

Let $h : \operatorname{Spec} \mathbf{k}[[t]] \to \operatorname{Spec} \mathbf{k}[[t]]$ be induced by the $\mathbf{k}$-algebra homomorphism $h^* : \mathbf{k}[[t]] \to \mathbf{k}[[t]]$ defined by the composition

$$\mathbf{k}[[t]] = \mathbf{k}[[\phi_\alpha]] \xrightarrow{(\overline{\alpha^*})^{-1}} \tilde{A}_\alpha = \tilde{A}_\gamma \xrightarrow{\overline{\gamma^*}} \mathbf{k}[[\phi_\gamma]] \subseteq \mathbf{k}[[t]].$$

The last inclusion is an inclusion of local $\mathbf{k}$-algebras and all other maps are isomorphisms. Hence h^* is a local homomorphism. For $f \in \widehat{\mathcal{O}}_{X,p_0}$, we have $\gamma^*(f) = \overline{\gamma^*}(f) = h^* \circ \overline{\alpha^*}(f) = h^* \circ \alpha^*(f)$, and hence $\gamma = \alpha \circ h$. If $\operatorname{ord}_t(h) = N$ and $a = \operatorname{ord}_\alpha(f)$, then the order of t in $\gamma^*(f) = h^* \circ \alpha^*(f)$ is Na, i.e. $\operatorname{ord}_\gamma(f) = N \operatorname{ord}_\alpha(f)$. $\square$

Notation V.16. We denote by $(X_\infty)_0$ the subset of points of X_∞ with residue field equal to $\mathbf{k}$. If $D \subseteq X_\infty$, then we set $D_0 = D \cap (X_\infty)_0$.

Here is the main theorem of this paper.

Theorem V.17. *Let X be a nonsingular variety of dimension n ($n \geq 2$) over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be a normalized arc (Definition IV.4). Set $p_0 = \alpha(o)$ and $v = \operatorname{ord}_\alpha$. Let E_i and p_i be the sequence of divisors and centers, respectively, of v (described in Definition III.3). Let $\mu_q : X_q \to X$ be the composition of the first q blowups of centers of v. Let*

$$C = \bigcap_{q > 0} \mu_{q\infty}(\operatorname{Cont}^{\geq 1}(E_q)) \subseteq X_\infty. \tag{5.5}$$

Let $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p_0} \mid v(f) \geq q\}$. Let

$$C'' = \bigcap_{q \geq 1} \operatorname{Cont}^{\geq q}(\mathfrak{a}_q) \subseteq X_\infty.$$

Set $C(v) = \overline{\{\gamma \in X_\infty \mid \operatorname{ord}_\gamma = v,\ \gamma(o) = p\}} \subseteq X_\infty$.

For an arc $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$*, let* $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbf{k}[[t]]$ *be the induced* $\mathbf{k}$*-algebra homomorphism. Set* $\mathcal{I} = \{\gamma \in X_\infty \mid \gamma(o) = \alpha(o), \ker(\alpha^*) \subseteq \ker(\gamma^*) \subseteq \widehat{\mathcal{O}}_{X,\alpha(o)}\}$.

Let $R = \{\alpha \circ h \in X_\infty \mid h : \operatorname{Spec} \mathbf{k}[[t]] \to \operatorname{Spec} \mathbf{k}[[t]]\}$*, where* h *is a morphism of* $\mathbf{k}$*-schemes.*

Then

1. C *is an irreducible subset of* X_∞ *and* $\operatorname{val}_C = v$.

2. *Assume* $\mathbf{k}$ *is algebraically closed and has characteristic zero. The following closed subsets of* $(X_\infty)_0$ *are equal (we use Notation V.16):*

$$C(v)_0 = C_0 = C''_0 = (\mathcal{I})_0 = R.$$

Proof of Theorem V.17. (Part 1) Let r be a nonnegative integer such that the lift of α to X_r is a nonsingular arc. For $q > r$, let $\mu_{q,r} : X_q \to X_r$ be the composition of the blowups along the centers of v, starting at $X_{r+1} \to X_r$ and ending at the blowup $X_q \to X_{q-1}$. Let

$$C' = \bigcap_{q>r} \mu_{q,r\infty}(\operatorname{Cont}^{\geq 1}(E_q)) \subseteq (X_r)_\infty.$$

Note that

$$C = \mu_{r\infty}(C') \subseteq X_\infty.$$

By Proposition V.2, C' is irreducible. Hence C is irreducible. Since the generic point of C' maps to the generic point of C, we have that $\operatorname{val}_{C'} = \operatorname{val}_C$, i.e. $\operatorname{val}_{C'}(\mu_r^*(f)) = \operatorname{val}_C(f)$ for $f \in \mathcal{O}_{X,p_0}$. Since $v = \operatorname{val}_{C'}$ by Proposition V.2, we conclude $v = \operatorname{val}_C$.

(Part 2) We show $C(v)_0 \subseteq C''_0 \subseteq C_0 \subseteq C(v)_0$. Separately we will establish $C''_0 = \mathcal{I}_0$.

First we check $C(v) \subseteq C''$. If $\gamma \in X_\infty$ is such that $\gamma(o) = p$ and $\mathrm{ord}_\gamma = v$, then $\gamma \in \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ for every $q \geq 1$, and so $\gamma \subseteq C''$. Since C'' is closed, we have $C(v) \subseteq C''$.

Now we show $C''_0 \subseteq C_0$. Let $\gamma \in C''_0$, and assume without loss of generality that γ is not the trivial arc. We claim that $\ker(\alpha^*) \subseteq \ker(\gamma^*)$. Let $f \in \ker(\alpha^*)$. Then $v(f) = \infty$, and so $f \in \mathfrak{a}_q$ for every $q \in \mathbb{Z}_{\geq 0}$. Hence $\mathrm{ord}_\gamma(f) \geq q$ for all $q \in \mathbb{Z}_{\geq 0}$. Therefore $\mathrm{ord}_\gamma(f) = \infty$, so $f \in \ker(\gamma^*)$. By Lemma V.15 there exists $h : \mathrm{Spec}\,\mathbf{k}[[t]] \to \mathbf{k}[[t]]$ such that $\gamma = \alpha \circ h$. It follows that γ has the same sequence of centers as α. Indeed, if $\gamma_q : \mathrm{Spec}\,\mathbf{k}[[t]] \to X_q$ is the unique lift of γ to an arc on X_q, then $\gamma_q \circ h$ is the unique lift of α to an arc on X_q. Since h^* is a local homomorphism, we have that h maps the closed point of $\mathrm{Spec}\,\mathbf{k}[[t]]$ to the closed point of $\mathrm{Spec}\,\mathbf{k}[[t]]$. Hence the center of γ_q is the same as the center of $\gamma_g \circ h$. We conclude $\gamma \in C$. Note that this argument also shows $C''_0 \subseteq R$, and Lemma V.15 shows that $C''_0 \subseteq R$.

To see that $C \subseteq C(v)$, let β be the generic point of C. Note that $\mathrm{ord}_\beta = v$ and $\pi(\beta) = p_0$, and so $\beta \in C(v)$. Hence $C \subseteq C(v)$.

Now we show $C''_0 = (\mathcal{I})_0$. Let J be the kernel of the map $\alpha^* : \widehat{\mathcal{O}}_{X,p_0} \to \mathrm{Spec}\,\mathbf{k}[[t]]$. If $f \in J$, then $\mathrm{ord}_\alpha = \infty$ and hence $f \in \mathfrak{a}_q$ for every $q \geq 1$. Let $\gamma \in C''_0$. Since $\mathfrak{a}_1$ is the maximal ideal of $\widehat{\mathcal{O}}_{X,p_0}$, we have $\gamma(o) = p_0$, i.e. $\gamma \in \pi^{-1}(p_0)$. Also, since $\mathrm{ord}_\gamma(f) \geq q$ for every $q \geq 1$, we have $\mathrm{ord}_\gamma(f) = \infty$. Hence $\gamma \in (\mathcal{I})_0$.

For the reverse inclusion $C''_0 \supseteq (\mathcal{I})_0$, let $\gamma \in (\mathcal{I})_0$. Then $J \subseteq \ker(\gamma^*)$, and hence by Lemma V.15 we have that either γ is the trivial arc or $\mathrm{ord}_\gamma = N\,\mathrm{ord}_\alpha$ for some positive integer N. In both cases we have $\gamma \in C''_0$.

□

Remark V.18. If X is a surface and if v is a divisorial valuation, then the set

$$C = \bigcap_{q>0} \mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))$$

equals the cylinder associated to v in [7, Example 2.5], namely $\mu_{r\infty}(\mathrm{Cont}^{\geq 1}(E_r))$, where r is such that p_r is a divisor.

Proof. If r is such that $p_r \in X_r$ (Definition III.3) is a divisor, then $C = \mu_{r\infty}(\mathrm{Cont}^{\geq 1}(E_r))$ since $\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q)) \supseteq \mu_{q+1\infty}(\mathrm{Cont}^{\geq 1}(E_{q+1}))$, and for $q > r$ we have equality since the maps $\mu_{q,r}$ are isomorphisms. Hence $C = \mu_{r\infty}(\mathrm{Cont}^{\geq 1}(E_r))$, which is the set in [7, Example 2.5]. $\square$

CHAPTER VI

K-arc valuations on a nonsingular k-variety

In this chapter, we consider arc valuations v of the form $v = \mathrm{ord}_\gamma$, where $\gamma : \operatorname{Spec} K[[t]] \to X$ is an arc and $\mathbf{k} \subseteq K$ is an extension of fields. Such arcs arise naturally (via Remark II.5) as generic points of irreducible subsets of the arc space X_∞. To analyze these valuations, we perform a base change $\operatorname{Spec} K \to \operatorname{Spec} \mathbf{k}$. The arc γ gives rise to an arc $\gamma_K : \operatorname{Spec} K[[t]] \to X_K = X \times \operatorname{Spec} K$. We then apply our results (Theorem V.17) for K-arc valuations on a K-variety to this situation. In particular, we give a description of the K-valued points of the *maximal arc set* (defined below).

Following Ishii [14, Definition 2.8], we associate to a valuation v a subset $C(v) \subseteq X_\infty$ in the following way.

Definition VI.1. Let $p \in X$ be a (not necessarily closed) point. Let $v : \widehat{\mathcal{O}}_{X,p} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be a valuation. Define the *maximal arc set* $C(v)$ by

$$C(v) = \overline{\{\gamma \in X_\infty \mid \mathrm{ord}_\gamma = v,\ \gamma(o) = p\}} \subseteq X_\infty,$$

where the bar denotes closure in X_∞.

Lemma VI.2. *Let $C \subseteq X_\infty$ be an irreducible subset. We have $C \subseteq C(\mathrm{val}_C)$. (See Chapter IV Equation 4.4 for the definition of* val_C*).*

Proof. Let α be the generic point of C. By Proposition IV.10, $\mathrm{ord}_\alpha = \mathrm{val}_C$ and hence $\alpha \in C(\mathrm{val}_C)$. Hence $C = \overline{\{\alpha\}} \subseteq C(\mathrm{val}_C)$. $\square$

Let X be a smooth variety over a field $\mathbf{k}$. Let $\gamma : \operatorname{Spec} K[[t]] \to X$ be a normalized arc on X, where $\mathbf{k} \subseteq K$ is an extension of fields. Let $X_K = X \times_{\operatorname{Spec}\mathbf{k}} \operatorname{Spec} K$ and $f : X_K \to X$ the canonical map. Let $\gamma_K : \operatorname{Spec} K[[t]] \to X_K$ be given by $\gamma_K = \gamma \times \iota$ where $\iota : \operatorname{Spec} K[[t]] \to \operatorname{Spec} K$ is the natural map.

With the notation introduced above, we have ord_{γ_K} is a normalized K-arc valuation on the K-variety X_K, and X_K is a nonsingular variety.

Definition VI.3. Let $p_{K,i}$ for $i \geq 0$ be the sequence of infinitely near points of ord_{γ_K}, with $p_{K,i}$ lying on the i-th blowup $X_{K,i}$ of X_K. Let $E_{K,i} \subset X_{K,i}$ be the exceptional divisor of the ith blowup $\mu_{K,i,i-1} : X_{K,i} \to X_{K,i-1}$. Let $\mu_{K,i} : X_{K,i} \to X_K$ be the composition of the first i blowups.

By Theorem V.17 part 1, the set

$$D := \bigcap_{q>1} \mu_{K,q,\infty}(\mathrm{Cont}^{\geq 1}(E_{K,q})) \tag{6.1}$$

is an irreducible subset of $X_{K\infty}$ with $\mathrm{val}_D = \mathrm{ord}_{\gamma_K}$ on $\mathcal{O}_{X_K,p_{K,0}}$. Hence $C' := \overline{f_\infty(D)}$ is an irreducible subset of X_∞, where $f : X_K \to X$ is the canonical map. Let $\alpha \in X_{K\infty}$ be the generic point of D. We have $\mathrm{ord}_\alpha = \mathrm{val}_D = \mathrm{ord}_{\gamma_K}$. Applying f_∞ we get $\mathrm{ord}_{f_\infty(\alpha)} = \mathrm{val}_{C'} = \mathrm{ord}_\gamma$, where we have used $f_\infty(\gamma_K) = \gamma$. Hence $f_\infty(\alpha) \in C(\mathrm{val}_C)$, hence $C' \subseteq C(\mathrm{val}_C)$. Also, by $f_\infty(\gamma_K) = \gamma$ and the fact that $\gamma_K \in D$, we have $C \subseteq C'$. To summarize, we have proven:

Proposition VI.4. *Let X be a smooth variety over a field $\mathbf{k}$. Let $\gamma : \operatorname{Spec} K[[t]] \to X$ be a normalized arc on X, where $\mathbf{k} \subseteq K$ is an extension of fields. Let $C = \overline{\{\gamma\}} \subseteq X_\infty$.*

Using the notation of Definition VI.3, let

$$C' = \overline{f_\infty(\bigcap_{q>0} \mu_{K,q\infty}(\mathrm{Cont}^{\geq 1}(E_{K,q})))}.$$

Then C' is an irreducible subset of X_∞ with $\mathrm{val}_{C'} = \mathrm{ord}_\gamma$ *and*

$$C \subseteq C' \subseteq C(\mathrm{val}_C).$$

CHAPTER VII

Other valuations

In this chapter, we turn our attention to valuations that are *not* arc valuations. We restrict our attention to surfaces, where there is a complete classification of valuations. This classification is presented in Chapter III Definition III.8. On surfaces, there are four general classes of valuations: divisorial valuations, curve valuations, irrational valuations, and infinitely singular valuations. Of these, the first two are arc valuations. On the other hand, irrational valuations have value groups (isomorphic to) $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{R}$ where $\tau \in \mathbb{R} \setminus \mathbb{Q}$, while infinitely singular valuations have value groups (isomorphic to) subgroups of $\mathbb{R}$ that are not finitely generated. A natural question is, what do the sets $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ and $\bigcap_q \overline{\mu_{q,\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$, which were the focus of Chapter V, look like for these valuations?

In this chapter, we begin by computing the sets

$$\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q) \text{ and } \bigcap_q \overline{\mu_{q,\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$$

for irrational valuations on $X = \mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x, y]$. We have seen that these sets are equal for nonsingular arc valuations (Proposition V.2). However, for irrational valuations, these sets are not equal. In fact, in Proposition VII.2, we will see that for an irrational valuation on $\mathbb{A}^2$, the set $\bigcap_q \overline{\mu_{q,\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$ contains only the trivial arc. On the other hand, we will see that $C = \bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ is an irreducible cylinder. However, one cannot recover the original irrational valuation from C. More precisely, there are infinitely many irrational valuations whose corresponding sets $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ are equal.

These results suggest that arc spaces are not well-suited to the study of valuations that are not arc valuations. However, irrational valuations can be expressed as the order of vanishing along *generalized arcs*. For example, the irrational valuation v on $\mathbf{k}[x, y]$ given by $v(x) = 1$ and $v(y) = \pi$ is given by the order of vanishing along a generalized arc $\gamma : \operatorname{Spec} \mathbf{k}[[t, t^\pi]] \to \operatorname{Spec} \mathbf{k}[x, y]$ given by $x \to t$, $y \to t^\pi$. This suggests generalizing the notion of arc spaces to spaces of generalized arcs. We sketch this idea later in Chapter IX.

7.1 Irrational valuations

The valuation $v : \mathbf{k}(x, y)^* \to \mathbb{R}$ on $X = \mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x, y]$ given by $v(x) = 1$ and $v(y) = \tau$ where $\tau > 1$ is an irrational number is an example of an irrational valuation. Note that v takes on distinct values on distinct monomials, and hence is a monomial valuation. Furthermore, the center of v on X_q will be a $\mathbf{k}$-valued point with local coordinates of the form $x^a y^b$, where x, y are local coordinates of the center v on X and $a, b \in \mathbb{Z}$. To give the exact expression, we need to discuss the continued fraction expansions of τ. This material is rather straightforward. The author made these calculations independently, but makes no claims of originality.

7.1.1 Continued fractions

Let $\tau > 1$ be an irrational number.

Consider the continued fraction expansion of τ,

$$\tau = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}, \tag{7.1}$$

where $a_0 = \lfloor \tau \rfloor$ and all the a_i are (uniquely determined) positive integers. Let b_i be the i-th convergent – that is, the truncation of Equation 7.1 to the partial fraction involving only $a_0, a_1, \ldots, a_i$. For example $b_0 = a_0$, $b_1 = a_0 + \frac{1}{a_1}$, and $b_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}$. We recall some elementary facts about these continued fractions. We have that $b_{2i} < b_{2i+2} < \tau < b_{2i+3} < b_{2i+1}$ for all $i \geq 0$ ([20, Theorem 7.6]). We also have $\lim_{i\to\infty} b_{2i} = \lim_{i\to\infty} b_{2i+1} = \tau$ ([20, p.335]). Let c_i, d_i be relatively prime positive integers such that $b_i = c_i/d_i$, for $i \geq 0$. Set $c_{-2} = 0$, $c_{-1} = 1$, $d_{-2} = 1$ and $d_{-1} = 0$. Then we have the recursion relations $c_i = c_{i-2} + a_i c_{i-1}$ and $d_i = d_{i-2} + a_i d_{i-1}$ for $i \geq 1$ ([20, p.335]). We also have $c_i d_{i+1} - c_{i+1} d_i = (-1)^{i+1}$.

For $i \geq -1$, let

$$z_i = x^{(-1)^{i+1} c_i} y^{(-1)^i d_i} \in \mathbf{k}(X).$$

We have $z_{2i} = x^{-c_{2i}} y^{d_{2i}}$, and so $v(z_{2i}) = -c_{2i} + \tau d_{2i} > 0$ where the inequality follows from $\frac{c_{2i}}{d_{2i}} = b_{2i} < \tau$. Also, we have $z_{2i+1} = x^{c_{2i+1}} y^{-d_{2i+1}}$. Hence $v(z_{2i+1}) = c_{2i+1} - \tau d_{2i+1} > 0$ where the inequality follows from $\frac{c_{2i+1}}{d_{2i+1}} = b_{2i+1} > \tau$. Thus $v(z_i) > 0$ for all $i \geq -1$.Also note that the equations $c_i = c_{i-2} + a_i c_{i-1}$ and $d_i = d_{i-2} + a_i d_{i-1}$ for $i \geq 1$ imply $z_i = z_{i-2} z_{i-1}^{-a_i}$. Since $v(z_i)$ are positive, we have $v(z_{i-2} z_{i-1}^{-a_i}) > 0$. Also, the equation $c_i d_{i+1} - c_{i+1} d_i = (-1)^{i+1}$ gives

(7.2) $$x = {z_i}^{d_{i+1}} {z_{i+1}}^{d_i}$$

(7.3) $$y = {z_i}^{c_{i+1}} {z_{i+1}}^{c_i}$$

Proposition VII.1. *Let $q_{-1} = 0$ and let $q_i = \sum_{j=0}^{j=i} a_i$. Then (z_{i-1}, z_i) form local coordinates at the center of v on X_{q_i}, for $i \geq -1$.*

Proof. We prove the result by induction on i. When $i = -1$, the statement is that $(z_{-2}, z_{-1}) = (y, x)$ form local coordinates at the center of v on $X_0 = X$. Since

$v(x), v(y) > 0$, the result is true for $i = -1$. Now fix $i > -1$ and assume the result is true for a $i-1$, i.e. we have a model $X_{q_{i-1}}$ on which we have local coordinates (z_{i-2}, z_{i-1}) centered at the center of v on $X_{q_{i-1}}$. Recall that $z_i = z_{i-2}z_{i-1}^{-a_i}$ and $v(z_i) > 0$. Blowup the center of v on X_{q-1}. The center of v will be given by $(z_{i-1}, z_{i-2}/z_{i-1})$ as v is positive on both these generators. Performing $a_i - 1$ more blowups, we find that the center of v on X_{q_i} has z_{i-1}, z_i as local algebraic coordinates. This completes the induction. $\square$

7.1.2 Irrational valuations and arc spaces

Proposition VII.2. *Let* $X = \operatorname{Spec} \mathbf{k}[x, y]$ *and let* $v : \mathbf{k}(x, y)^* \to \mathbb{R}$ *be the valuation defined by* $v(x) = 1$ *and* $v(y) = \tau$ *where* $\tau > 1$ *is irrational. Then* $\bigcap_q \operatorname{Cont}^{\geq q}(\mathfrak{a}_q) = \operatorname{Cont}^{\geq 1}(x) \cap \operatorname{Cont}^{\geq \lceil \tau \rceil}(y)$. *In particular, this intersection is an irreducible cylinder of codimension* $\lceil \tau \rceil + 1$. *On the other hand, the only arc in* $\bigcap_q \overline{\mu_{q\infty}(\operatorname{Cont}^{\geq 1}(\mathcal{I}_{E_q}))}$ *is the trivial arc (Definition II.8).*

Proof. Let $\gamma \in \bigcap_q \operatorname{Cont}^{\geq q}(\mathfrak{a}_q)$. Since $x \in \mathfrak{a}_1$ and $\gamma \in \operatorname{Cont}^{\geq 1}(\mathfrak{a}_1)$ it follows that $\operatorname{ord}_\gamma(x) \geq 1$. I claim $\operatorname{ord}_\gamma(y) > \lfloor \tau \rfloor$. For a contradiction, suppose $\operatorname{ord}_\gamma(y) \leq \lfloor \tau \rfloor$. Since $\tau - \lfloor \tau \rfloor > 0$, there exists $s \in \mathbb{N}$ such that $s(\tau - \lfloor \tau \rfloor) > 1$. Hence there exists $q \in \mathbb{N}$ such that $s\lfloor \tau \rfloor < q < s\tau$. Since $v(y^s) = s\tau > q$, we have $y^s \in \mathfrak{a}_q$. Since $\gamma \in \operatorname{Cont}^{\geq q}(\mathfrak{a}_q)$, we have $q \leq \operatorname{ord}_\gamma(y^s) = s \operatorname{ord}_\gamma(y) \leq s\lfloor \tau \rfloor$. This contradicts $s\lfloor \tau \rfloor < q$. So $\operatorname{ord}_\gamma(x) \geq 1$ and $\operatorname{ord}_\gamma(y) > \lfloor \tau \rfloor$ are required conditions for an arc γ to lie in $\bigcap_q \operatorname{Cont}^{\geq q}(\mathfrak{a}_q)$.

I claim they are also sufficient. Let $\gamma \in X_\infty$ be such that $\operatorname{ord}_\gamma(x) \geq 1$ and $\operatorname{ord}_\gamma(y) \geq \lfloor \tau \rfloor + 1$. Note that $\mathfrak{a}_\mathfrak{q}$ is the ideal generated by the monomials $x^a y^b$ with $a + b\tau \geq q$. (This last observation uses the general fact that for any valuation v, if r_1, r_2 are elements of the valuation ring such that $v(r_1) \neq v(r_2)$ then $v(r_1 + r_2) =$

$\min\{v(r_1), v(r_2)\}$.) We have $\mathrm{ord}_\gamma(x^a y^b) \geq a + b(\lfloor\tau\rfloor + 1)$. Hence $\gamma \in \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ for all q. It follows that the ideal of $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ is given by $(x_0, y_0, y_1, \ldots, y_{\lfloor\tau\rfloor})$. Hence $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q) = \mathrm{Cont}^{\geq 1}(x) \cap \mathrm{Cont}^{\geq\lceil\tau\rceil}(y)$. This intersection is also the preimage in X_∞ of the subset of $X_{\lfloor\tau\rfloor} = \operatorname{Spec}\mathbf{k}[x_0, x_1, \ldots x_{\lfloor\tau\rfloor}, y_0, y_1, \ldots, y_{\lfloor\tau\rfloor}]$ given by $(x_0, y_0, y_1, \ldots, y_{\lfloor\tau\rfloor})$. In particular, we see that $\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ is an irreducible cylinder of codimension $\lceil\tau\rceil + 1$.

Now we show that the trivial arc is the only arc in $\bigcap_q \overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(\mathcal{I}_{E_q}))}$.

If $\gamma \in \mathrm{Cont}^{\geq 1}(p_{q_i})$, then γ is given by a map $z_{i-1} \to b_1 t + b_2 t^2 + \cdots$ and $z_i \to b'_1 t + b'_2 t^2 + \ldots$. By equations 7.2 and 7.3, we have that $\mu_{q_i} \circ \gamma$ is an arc on X contained in $\mathrm{Cont}^{\geq d_{i-1}+d_i}(x) \cap \mathrm{Cont}^{\geq c_{i-1}+c_i}(y)$. Hence $\mu_{q_i\infty}(\mathrm{Cont}^{\geq 1}(p_{q_i})) \subseteq \mathrm{Cont}^{\geq d_{i-1}+d_i}(x) \cap \mathrm{Cont}^{\geq c_{i-1}+c_i}(y)$. Since the right hand side of this inclusion is a closed subset of X_∞, we have $\overline{\mu_{q_i\infty}(\mathrm{Cont}^{\geq 1}(p_{q_i}))} \subseteq \mathrm{Cont}^{\geq d_{i-1}+d_i}(x) \cap \mathrm{Cont}^{\geq c_{i-1}+c_i}(y)$. Hence $\bigcap_i \overline{\mu_{q_i\infty}(\mathrm{Cont}^{\geq 1}(p_{q_i}))} \subseteq \bigcap_i \mathrm{Cont}^{\geq d_{i-1}+d_i}(x) \cap \mathrm{Cont}^{\geq c_{i-1}+c_i}(y)$. Since $c_i, d_i \to \infty$ as $i \to \infty$, we have that the right hand side equals $\mathrm{Cont}^{\infty}(x) \cap \mathrm{Cont}^{\infty}(y)$, which contains only the trivial arc.

$\square$

CHAPTER VIII

Motivic measure

When working with subsets of arc spaces, it is often useful to measure, in some way, the size of any subset. For example, if $A \subseteq X_m$ is a closed subset of codimension d (where X_m is the m-th jet scheme of X) then we define the codimension of the cylinder $C = \pi_m^{-1}(A) \subseteq X_\infty$ to equal d. Invariants coming from birational geometry (e.g. minimal log discrepancies) can be expressed in terms of the codimension of various subsets of the arc space (see [9, Thm 7.9] for a precise statement). The set C in Theorem V.2 is not a cylinder, but it is the intersection of cylinders $C_q = \overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$ with $\mathrm{codim}_{q\to\infty} C_q = \infty$. (By Lemma V.10, the coordinate ring of C_q is isomorphic to the polynomial ring over $\mathbf{k}$ in the indeterminates $x_{1,j}$ for $1 \leq j$ and $x_{i,j}$ for $2 \leq i \leq n$ and $q \leq j$. Hence the codimension of C_q in X_∞ is $n + (q-1)(n-1)$.) One may say that the codimension of C is infinite.

In an effort to find a more meaningful quantity to attach to C, we consider the motivic measure of C. The motivic measure of a subset of the arc space is an element in the completion of a localization of the Grothendieck group of varieties. In this chapter, we compute the motivic measure of the set C from Theorem V.17 for valuations on $\mathbb{A}^2$.

8.1 Generalities on motivic measure

Following [22], we recall the basic definitions of motivic integration while fixing the notation. Let $K_0(\mathrm{Var}_{\mathbf{k}})$ denote the Grothendieck group of algebraic varieties over a field $\mathbf{k}$. This group is the abelian group generated by symbols $[V]$, where V is an algebraic variety over $\mathbf{k}$, with the relations $[V] = [W]$ if V and W are isomorphic, and $[V] = [Z] + [V \backslash Z]$ if Z is a Zariski-closed subvariety of V. Place a ring structure on $K_0(\mathrm{Var}_{\mathbf{k}})$ by $[V] \cdot [W] = [V \times W]$. Set $1 := [\mathit{point}]$, $\mathbb{L} := [\mathbb{A}^1]$, and

$$\mathcal{M}_{\mathbf{k}} := K_0(\mathrm{Var}_{\mathbf{k}})_{\mathbb{L}},$$

the ring obtained from $K_0(\mathrm{Var}_{\mathbf{k}})$ by inverting $\mathbb{L}$. For $m \in \mathbb{Z}$ let F^m be the subgroup of $\mathcal{M}_{\mathbf{k}}$ generated by the elements $\frac{[V]}{\mathbb{L}^i}$ with $\dim V \leq i - m$. Define

$$\hat{\mathcal{M}}_{\mathbf{k}} := \varprojlim \mathcal{M}_{\mathbf{k}}/F^m.$$

Let X be an algebraic variety (over a field $\mathbf{k}$) of pure dimension d. Let A be a cylinder in X_∞. Let $\psi_n : X_\infty \to X_n$ be the canonical projection morphism. Define the *motivic measure* of A by $\mu(A) := \lim\limits_{n\to\infty} \dfrac{[\psi_n(A)]}{\mathbb{L}^{nd}}$. It is a theorem of Denef and Loeser [6, Theorem 5.1] that this limit exists in $\hat{\mathcal{M}}_{\mathbf{k}}$. We extend μ to the Boolean algebra generated by the cylinders in X_∞ by requiring μ to be a σ-additive measure.

8.2 Motivic measures of subsets associated to valuations on $\mathbb{A}^2$

Let $X = \mathbb{A}^2 = \operatorname{Spec} \mathbf{k}[x, y]$. We compute the motivic measure of various subsets of $(\mathbb{A}^2)_\infty$. We write $(\mathbb{A}^2)_\infty = (\operatorname{Spec} \mathbf{k}[x, y])_\infty = \operatorname{Spec} \mathbf{k}[x_0, x_1, \ldots, y_0, y_1, \ldots]$, where the last equality comes from parametrizing arcs on $\operatorname{Spec} \mathbf{k}[x, y]$ by $x \to \sum_{j\geq 0} x_j t^j$ and $y \to \sum_{j\geq 0} y_j t^j$.

For a valuation $v : \mathbf{k}[x, y] \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and integer q, we define the valuation ideal $\mathfrak{a}_q = \{f \in \mathbf{k}[x, y] \mid v(f) \geq q\}$.

Proposition VIII.1. *Let v be the monomial valuation on $X = \operatorname{Spec} \mathbf{k}[x, y]$ given by $v(x) = 1, v(y) = Q$. Then $\mu(\mathrm{Cont}^{\geq Q}(\mathfrak{a}_Q)) = \mathbb{L}^{-Q+1}$.*

Proof. We have $\mathfrak{a}_Q = (x^Q, y)$. Therefore the generic point of $\mathrm{Cont}^{\geq Q}(\mathfrak{a}_Q)$ corresponds to an arc sending $x \to x_1 t + x_2 t^2 + \ldots$ and $y \to y_Q t^Q + y_{Q+1} t^{Q+1} + \ldots$. Hence the ideal of $\mathrm{Cont}^{\geq Q}(\mathfrak{a}_Q)$ is given by $(x_0, y_0, \ldots, y_{Q-1})$. Let $\psi_n : X_\infty \to X_n$ denote the canonical projection morphism to X_n, the n-th jet space X. For $n > Q$, the coordinate ring of $\psi_n(\mathrm{Cont}^{\geq Q}(\mathfrak{a}_Q)) \subseteq X_n$ is

$$\mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_n]/(x_0, y_0, \ldots, y_{Q-1}) = \mathbf{k}[x_1, \ldots, x_n, y_Q, \ldots, y_n].$$

So $\psi_n(\mathrm{Cont}^{\geq Q}(\mathfrak{a}_Q)) \simeq \mathbb{A}^{n+n-Q+1} = \mathbb{A}^{2n-Q+1}$. Hence

$$\mu(\mathrm{Cont}^{\geq Q}(\mathfrak{a}_Q)) = \lim_{n\to\infty} \mathbb{L}^{2n-Q+1}/\mathbb{L}^{2n} = \mathbb{L}^{-Q+1}.$$

□

Proposition VIII.2. *Lexicographically order $\mathbb{Z} \oplus \mathbb{Z}$ with $(0, 1) < (1, 0)$. Let $v : \mathbf{k}[x, y] \to \mathbb{Z}\oplus\mathbb{Z}$ be the monomial valuation on $X = \operatorname{Spec} \mathbf{k}[x, y]$ given by $v(x) = (0, 1)$ and $v(y) = (1, 0)$. Let $C = \bigcap_{q\geq 1} \overline{\mu_{q\infty}(\mathrm{Cont}^{\geq 1}(E_q))}$, where μ_q and E_q are as in Definition V.1. Then $\mu(C) = 0$.*

Proof. The sequence of centers of v is the same as that of the $\mathbf{k}$-arc valuation ord_α on $\operatorname{Spec} \mathbf{k}[x, y]$ defined by $\alpha^*(x) = t$ and $\alpha^*(y) = 0$. Hence by Proposition V.2, we have $C = \mathrm{Cont}^{\geq 1}(x) \cap \mathrm{Cont}^{\infty}(y)$. The ideal of C is $(x_0, y_0, y_1, \ldots) \subset \mathbf{k}[x_0, x_1, \ldots, y_0, y_1, \ldots]$. Hence the coordinate ring of $\psi_n(C)$ is $\mathbf{k}[x_1, \ldots, x_n]$. Hence $\mu(C) = \lim_{n\to\infty} \mathbb{L}^n/\mathbb{L}^{2n} = 0$. □

Proposition VIII.3. *Let $X = \operatorname{Spec} \mathbf{k}[x, y]$ and let $\alpha : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be an arc centered at the origin and with $\alpha^*(x) = t$. Set $v = \mathrm{ord}_\alpha$. Then $\mu(\mathrm{Cont}^{\geq q}(\mathfrak{a}_\mathfrak{q})) = \mathbb{L}^{-q+1}$. Furthermore, $\mu(\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)) = 0$.*

Proof. Let $\alpha^*(y) = \sum_{j\geq 1} c_j t^j$. Set $S = \mathbf{k}[x_1, x_2, \ldots, y_1, y_2, \ldots]$. For $q \geq 1$, let $f_q(X_1, \ldots, X_q)$ be the polynomial that is the coefficient of t^q in

$$\sum_{j=1}^{q} c_j (X_1 t + X_2 t^2 + \cdots)^j.$$

Let I_q be the ideal of S generated by $y_j - f_j(x_1, \ldots, x_j)$ for $1 \leq j \leq q-1$. By Lemma V.13, the ideal of $\mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ in S is I_q.

Hence for $n > q$, the coordinate ring of $\psi_n(\mathrm{Cont}^{\geq q}(\mathfrak{a}_q))$ is isomorphic to

$$\mathbf{k}[x_1, \ldots, x_n, y_q, \ldots, y_n].$$

Hence $\psi_n(\mathrm{Cont}^{\geq q}(\mathfrak{a}_q)) \simeq \mathbb{A}^{2n-q+1}$. Hence

$$\mu(\mathrm{Cont}^{\geq q}(\mathfrak{a}_q)) = \lim_{n\to\infty} \frac{[\psi_n(\mathrm{Cont}^{\geq q}(\mathfrak{a}_q))]}{\mathbb{L}^{2n}} = \mathbb{L}^{-q+1}.$$

So $\mu(\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)) = \lim_{q\to\infty} \mathbb{L}^{-q+1} = 0$.

$\square$

Proposition VIII.4. *Let $X = \operatorname{Spec} \mathbf{k}[x, y]$ and let v be the monomial valuation defined by $v(x) = 1$ and $v(y) = \tau$ where $\tau > 1$ is irrational. Let $A = \bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$. Then $\mu(A) = \mathbb{L}^{-\lfloor \tau \rfloor}$.*

Proof. By Proposition VII.2, the coordinate ring of $\psi_n(\bigcap_q \mathrm{Cont}^{\geq q}(\mathfrak{a}_q))$ is

$$\mathbf{k}[x_1, \ldots, x_n, y_{\lceil \tau \rceil}, \ldots, y_n].$$

Hence $\mu(A) = \lim_{n\to\infty} \mathbb{L}^{n+n-\lfloor \tau \rfloor}/\mathbb{L}^{2n} = \mathbb{L}^{-\lfloor \tau \rfloor}$.

$\square$

CHAPTER IX

Further directions

In this chapter, we outline some directions of future research.

9.1 Spaces of generalized arcs

Proposition VII.2 suggests that irrational valuations v (i.e. surface valuations with transcendence degree zero, rank one, and rational rank two) do not have a natural interpretation within the arc space. Specifically, $C(v)$ contains only one arc (namely, the constant arc at the center of v) while $\bigcap_{q\geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$ is an unexpectedly large set whose general arc does not recover v. The arc space is too coarse an object to use to detect these specialized valuations, and that a refinement of the notions of arcs and arc spaces may be more suitable. We now describe one possible refinement.

Let G be a totally ordered abelian group and let $G^+ = \{g \in G \mid g \geq 0\}$. The ring of *generalized power series*, denoted by $\mathbf{k}[[t^G]]$, is the set of formal sums $\sum_{i\in G^+} a_i t^i$ where $a_i \in \mathbf{k}$ and the support $\{i \mid a_i \neq 0\}$ is a well ordered set. Addition and multiplication are defined as usual for power series. Let $\mathbf{k}(t^G) = \mathrm{Frac}(\mathbf{k}[[t^G]])$ be the fraction field of $\mathbf{k}[[t^G]]$.

For example, when $G = \mathbb{Z}$, we have $\mathbf{k}[[t^G]] = \mathbf{k}[[t]]$, the formal power series ring. When G is a finitely generated subgroup of $\mathbb{Q}$, generalized power series are known as Puiseux series. They appear classically in the study of singularities of plane curves (e.g. [2, Chapter 2]).

There is a valuation $\overline{v} : \mathbf{k}(t^G)^* \to G$ given by $\overline{v}(\sum_{i\in G^+} a_i t^i) = \min\limits_{a_i\neq 0} i$ ([3, p.52]).

By analogy with the definition of arcs, I define a *generalized arc* on a variety X to be a morphism $\operatorname{Spec} \mathbf{k}[[t^G]] \to X$. We recover the usual notion of an arc when we take $G = \mathbb{Z}$ in this definition of generalized arc. One can consider the notion of a *generalized arc space* as a space parametrizing the generalized arcs on X. It is not clear if this space exists as a scheme.

Generalized arcs have been considered before. The following result of Kaplansky equates transcendence degree 0 valuations with the order of vanishing along generalized arcs.

Theorem IX.1. *([3, p.52]) Let X be a variety over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let $K = \mathbf{k}(X)$ be the function field of X. Let $v : K \to G$ be a valuation of $K/\mathbf{k}$ with $\operatorname{tr.deg} v = 0$. Then we have an embedding $K \subset \mathbf{k}(t^G)$ such that $V_{\overline{v}} \cap K = V_v$, where V_v (resp., $V_{\overline{v}}$) denotes the valuation ring of v (resp., $\overline{v}$).*

To the author's knowledge, a detailed theory of generalized arcs has not been done. The following questions are interesting to study:

- What structure can be put on the generalized arc space? For example, is it a scheme?

- Can one do geometry on generalized arc spaces? For example, can one define analogs of cylinders, codimension, and contact loci? What can these notions tell us about the geometry of X?

- Can the theory of motivic integration be extended to generalized arc spaces?

- What sorts of valuations appear in generalized arc spaces?

I now outline some methods that could be used to investigate these questions. First, one should analyze the proofs of the constructions and theorems related to arc spaces and motivic integration and see if they extend to generalized arcs. The proof of the existence of generalized arc spaces given above suggests that some of the proofs remain essentially the same. However, I expect other results to rely on some property of $\mathbb{Z}$, such as being well-ordered or topologically discrete, and thus some results may extend only if G has similar properties.

One can also investigate valuations that arise from irreducible subsets of generalized arc spaces. It would be interesting to see if Theorem V.17 extends to the setting of generalized arc spaces. One tool that could be used is the sequence of key polynomials (SKPs) (Definition III.4) associated to a valuation. SKPs provide an algebraic description of a valuation. The usefulness of SKPs stems from the fact that the algorithm to find the SKPs provides a systematic way to find the generators of the ideals $\mathfrak{a}_q$, where $q \in G^+$. These generators provide a tractable description of the ideals $\mathfrak{a}_q$. In particular, we can use these generators to compute $\bigcap_{q \geq 1} \mathrm{Cont}^{\geq q}(\mathfrak{a}_q)$.

It would be interesting to see if the classical studies of curve singularities, where Puiseux series appear, or the works of Abhyankar and Zariski, where non-divisorial valuations are analyzed geometrically, can be rephrased in terms of generalized arc spaces.

9.2 Arc valuations on singular varieties

Recently, T. de Fernex, L. Ein, and S. Ishii [4] have studied divisorial valuations via arc spaces of singular varieties. They extend many of the results of [7] from the non-singular case to the singular case. One possible research direction is to extend work on non-divisorial valuations to singular varieties.

There are two approaches one might try. The first is to see if one can extend the methods of [4] to non-divisorial valuations. The key idea is to blow up the smooth part of the variety along the smooth part of the center. The resulting blow-up is smooth, and its arc space is easier to manage than the arc space of the singular variety. With this idea, it would be interesting to see if Theorem V.17 to the setting when X is singular.

The second approach is to use the description of non-divisorial valuations via SKPs, described in Chapter III. The authors of [4] note that some of their results have previously been obtained by an alternate method based on SKPs, and they suggest investigating the connection between their approach and the approach via SKPs.

In both approaches, one might begin by looking at the case when the singular variety can be described as the quotient of $\mathbb{A}^2$ by a finite subgroup G of SL_2 or GL_2.

Understanding the arc space of a singular variety X, in particular identifying the irreducible components of the fiber over the singularities of X, has been of interest due in part to a problem raised by Nash. Nash's problem [14, problem 4.13] studies the relationship between these irreducible components and divisorial valuations. Formulating a generalization of the Nash problem to non-divisorial valuations is an interesting goal for future work.

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