# ACCESSIBILITY AND JSJ DECOMPOSITIONS OF GROUPS

by

Diane M. Vavrichek

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2008

Doctoral Committee:

Professor G. Peter Scott, Chair
Professor Mario Bonk
Professor Charles C. Brown
Professor Peter L. Duren
Professor Lizhen Ji

## ACKNOWLEDGEMENTS

I am extremely grateful to my advisor, Peter Scott, for his guidance over the past several years. I have enjoyed learning from him and working with him tremendously.

I thank Mario Bonk for his assistance during my work on the quasi-isometry invariance of the Scott-Swarup JSJ decomposition of groups, including his help in proving Proposition 4.5.7. I also thank Lars Louder for pointing out an error in an earlier version of Chapter III.

Lastly, I thank my family for their continuing support.

# TABLE OF CONTENTS

## LIST OF FIGURES

**Figure**

# CHAPTER I

## Introduction

In geometric group theory, one studies what can be said about the structure of a finitely generated group based on the structure of spaces that admit an action by the group, as well as what can be said about the structure of a space with a group action, based on the structure of the acting group. In this dissertation, we use the actions of groups on certain simplicial complexes to give us information about the structures of the groups.

More specifically, in Chapter III, we use actions of groups on trees in order to obtain a group accessibility result, i.e. that the process of decomposing a group in a certain manner cannot go on forever, but rather must eventually terminate, leaving a collection of subgroups that are indecomposable.

The process of decomposing groups that we work with is that of first decomposing a group maximally as a graph of groups over finite subgroups, then decomposing the resulting vertex groups maximally over two-ended subgroups, and then repeating these two steps on the new resulting vertex groups, and so on. This process results in what is called a hierarchy of the original group. In Theorem 3.2.9, we prove that the process must always terminate for hyperbolic groups with no 2-torsion, i.e. that all hierarchies associated to such decompositions are finite. The vertex groups remaining when the hierarchy is complete are indecomposable over finite and two-ended subgroups. This result has applications to 3-manifold topology (see Theorem 3.3.2) and to K-theory (see [16]).

In deriving this accessibility result, we make heavy use of the work of Delzant and Potyagailo [8]. They obtain a very general result about the existence of finite hierarchies for finitely presented groups with no 2-torsion (see Theorem 3.2.1). Their methods do not necessarily work when a group has 2-torsion, and it is for this reason that we must assume that the groups in Theorem 3.2.9 have no 2-torsion.

In light of the work done by Delzant and Potyagailo, we must prove two main things. The first is that their methods, which produce a hierarchy in which groups are decomposed over some fixed "elementary" family of subgroups, work when we alternate between decomposing maximally over finite subgroups and two-ended subgroups. This is seen in the proof of Theorem 3.2.9.

The second is that, although [8] only proves the existence of finite hierarchies, in fact *any* hierarchy as described above must be finite. Corollary 3.2.4 and Lemma 3.2.6 are used to show this, and the proofs of both of these facts make use of Stallings' folds on $G$-trees.

We remark that the contents of Chapter III will appear in Algebraic and Geometric Topology (see [31]).

In Chapter IV, we use the coarse geometry of a group's Cayley graph to determine some of the structure of a certain canonical decomposition of the group. The decomposition we are concerned with is the JSJ decomposition $\Gamma_1(G)$ of any one-ended, finitely presented group $G$, produced by Scott and Swarup in [25].

Our main result in this chapter is Theorem 4.5.4, which shows that commensurizer type vertex groups of these JSJ decompositions of groups are invariant under quasi-isometries. We prove this fact by analyzing "quasi-lines" inside Cayley graphs. We are aided in this analysis by the work of Papasoglu [21], who proved the invariance under quasi-isometries of other vertex groups of JSJ decompositions of groups. We note that our work, together with results of Papasoglu, shows the invariance under quasi-isometries of all $V_0$-vertex groups of the Scott-Swarup JSJ decompositions of groups.

In proving the invariance of commensurizer type vertex groups under quasi-isometries, we make use of a number of facts which we believe to be of independent interest. In Proposition 4.3.18, we prove that any 3-separating quasi-line in the Cayley graph of a finitely presented group that satisfies a condition about inessential components of its complement must be a finite Hausdorff distance from an infinite cyclic subgroup of the group. We also make use of a geometric characterization of a commensurizer – see Lemma 4.5.1. Another fact that we use in Chapter IV is Proposition 4.5.7, which is a coarse geometric characterization of when a subgroup of a finitely generated group is finitely generated.

# CHAPTER II

## Preliminaries

We shall begin with a discussion of the notation, definitions and background that we will need.

Throughout this work, we shall assume that all finitely generated (finitely presented, respectively) groups come equipped with fixed finite generating sets (finite presentations respectively). We shall assume that the given generating sets are symmetric, i.e. that if $s$ is in the generating set, then so is $s^{-1}$.

If $X$ is a metric space, $A$ a subset of $X$, and $R > 0$, we shall take the *$R$-neighborhood of $A$ in $X$* to be

$$N_R(A) = \{x \in X : \exists\ a \in A \text{ such that } d(x,a) \leq R\}.$$

If $p \in X$, we shall take the *$R$-ball about $p$ in $X$* to be

$$B_R(p) = \{x \in X : d(x,p) < R\}.$$

Thus neighborhoods of closed sets will be closed but balls will be open.

Recall that, if $Y$ is a subset of a metric space $X$, then $\overline{Y}$ denotes the closure of $Y$ in $X$.

We will use $d_{Haus}$ to denote the Hausdorff distance function. Thus, if $X$ is a metric space, and $A$ and $B$ are subsets of $X$, then

$$d_{Haus}(A, B) = \inf\{R \geq 0 : A \subset N_R(B) \text{ and } B \subset N_R(A)\}.$$

**Definition 2.0.1.** If $G$ is a finitely generated group, with finite generating set $S$, then the *Cayley graph of* $G$ (with respect to $S$), denoted $\mathscr{C}_G$, is a graph with vertex set equal to $G$, and, for each $(g, s) \in G \times S$, a single edge joining $g$ to $gs$.

Let $e(g, s)$ denote the edge corresponding to $(g, s)$. Note that, even if $s$ is of order two, so that the edges $e(g, s)$ and $e(gs, s)$ have the same endpoints, we do not identify $e(g, s)$ with $e(gs, s)$.

Observe that $G$ acts on $\mathscr{C}_G$ simplicially and freely on the left. We shall consider $\mathscr{C}_G$ to be a metric space by taking each edge to have length one. We remark that the Cayley graphs of $G$, with respect to different finite generating sets, are quasi-isometric. (See Definition 2.0.3 below. For a proof of this fact, see, for instance, Examples I.8.17 (1), (2) and (3) in [6].) Note also that subsets and subgroups of $G$ can be considered as subsets of $\mathscr{C}_G$.

Let $G$ be a finitely presented group, with finite presentation $\langle S : R \rangle$, and let $W$ denote a 2-dimensional CW complex with one vertex, and an edge corresponding to each pair $\{s, s^{-1}\}$ in $S$, with one orientation of the edge corresponding to $s$ and the other to $s^{-1}$. Let the 2-cells of $W$ correspond to the relations in $R$, with the boundary of the 2-cell corresponding to $r \in R$ glued to the 1-cells of $W$ along $r$, when thought of as a word in the alphabet $S$. Take each edge to be of length one, and take each 2-cell to be metrically a regular $n$-gon, where $n$ is the word length of the corresponding relation $r$.

**Definition 2.0.2.** Let $G$, $S$, $R$ and $W$ be as in the previous paragraph. The *Cayley complex of* $G$ (with respect to $\langle S : R \rangle$), denoted $\mathscr{CC} = \mathscr{CC}_G$, is the universal cover of $W$.

Note that the 1-skeleton of $\mathscr{C}\mathscr{C}$ is $\mathscr{C}_G$, and that $\mathscr{C}\mathscr{C}$ is simply connected.

The number of ends of a locally finite cell complex $X$, denoted $e(X)$, is the supremum, over all finite subcomplexes $K$ of $X$, of the number of infinite components of $X - K$. The number of ends of a finitely generated group $G$, denoted $e(G)$, is the number of ends of $\mathscr{C}_G$. This does not depend on the choice of finite generating sets for $G$. We recall that $e(G) = 2$ if and only if $G$ is virtually $\mathbb{Z}$, i.e. $G$ has a finite index subgroup that is infinite cyclic (see Theorem 5.12 of [27]).

A finitely generated group is said to be *hyperbolic* if there is some $\delta > 0$ and some finite generating set $S$ for $G$ such that, for any geodesic triangle in the Cayley graph of $G$ with respect to $S$, each side of the triangle is contained in the union of the $\delta$-neighborhoods of the other two sides. While the value of $\delta$ depends on our choice of a generating set, we note that hyperbolicity does not. For a proof of this fact, as well as an introduction to hyperbolic groups, we refer the reader to [6].

**Definition 2.0.3.** If $X$ and $Y$ are metric spaces and $f\colon X \to Y$ is a map of sets, then we say that $f$ is a $(\Lambda, K)$ *quasi-isometric embedding*, or merely a *quasi-isometric embedding*, if, for all points $x_1, x_2 \in X$,

$$\frac{1}{\Lambda} \cdot d_X(x_1, x_2) - K \leq d_Y(f(x_1), f(x_2)) \leq \Lambda \cdot d_X(x_1, x_2) + K.$$

We say that $f$ is a $(\Lambda, K)$ *quasi-isometry*, or a *quasi-isometry*, if $f$ is a $(\Lambda, K)$ quasi-isometric embedding, and

$$N_K(f(X)) = Y.$$

If $G$ and $G'$ are finitely generated groups, then we say that $G$ and $G'$ are *quasi-isometric* if there is a quasi-isometry between $\mathscr{C}_G$ and $\mathscr{C}_{G'}$.

We shall now discuss the basics of Bass-Serre theory, the theory of groups acting on trees, in order to define graph of groups decompositions of groups, hierarchies, and related concepts.

Let a group $G$ act simplicially on the left on a simplicial tree $\tau$, and let the action be without inversions, i.e. such that no element of $G$ preserves an edge of $\tau$, but swaps its vertices. Then we shall say that $\tau$ is a *$G$-tree*.

If $\tau$ is a $G$-tree, then the quotient $G\backslash\tau$ has a natural cell structure. Associate to each vertex $v_0$ of $\Gamma = G\backslash\tau$ the stabilizer $V$ of one of its preimages under the projection map $\tau \to G\backslash\tau$, and associate to each edge $e_0$ the stabilizer $E$ of one of its preimages as well. We shall call such $V$ and $E$ *vertex* and *edge groups*, respectively, and note that such groups associated to the vertices and edges of $G\backslash\tau$ are uniquely determined up to conjugacy.

To each pair $(v_0, e_0)$ of a vertex $v_0$ in $\Gamma$ and an oriented edge $e_0$ with terminal vertex $v_0$, associate an injective homomorphism from $E$ to $V$ induced by the inclusion of the stabilizer of a lift of $e_0$ into the stabilizer of a lift of $v_0$. Call $\Gamma$, together with this data, a *graph of groups* structure for $G$, and denote the graph and data also by $\Gamma$. We shall also say that $\Gamma$ is a *graph of groups decomposition*, or merely a *decomposition*, of $G$.

If $\tau$ is a $G$-tree, $G$ does not fix a point in $\tau$, and $\Gamma = G\backslash\tau$ is finite, then we call $\Gamma$ a *proper decomposition* of $G$. We shall say that $G$ acts *minimally* on a $G$-tree $\tau$ if $\tau$ contains no proper, $G$-invariant subtrees. Note that if $G$ is finitely generated and acts minimally on $\tau$, then $\Gamma$ is a finite graph. If, in addition, $\tau$ is not a vertex, then $\Gamma$ must be a proper decomposition of $G$.

Equivariant maps between $G$-trees will be very important in Chapter III, so we define the following:

**Definition 2.0.4.** Let a *$G$-map* be a simplicial, surjective, $G$-equivariant map between two $G$-trees that does not collapse any edge to a vertex.

We will now assume that $\tau$ and $\tau'$ are $G$-trees that are not vertices, and let $\Gamma = G\backslash\tau$ and $\Gamma' = G\backslash\tau'$.

**Definition 2.0.5.** If there is a simplicial, surjective, $G$-equivariant map $\tau' \to \tau$ (which may collapse edges to vertices), then we call the decomposition $\Gamma'$ a *refinement* of $\Gamma$.

**Definition 2.0.6.** Let $\tau' \to \tau$ be as in Definition 2.0.5 and not be a simplicial homeomorphism. Moreover, assume that for each edge $e$ of $\tau'$ that is collapsed to a vertex of $\tau$, either its vertices are in the same $G$-orbit, or both vertex stabilizers properly contain the stabilizer of $e$. Then we call $\Gamma'$ a *proper refinement* of $\Gamma$.

If all edge groups of a decomposition $\Gamma$ of $G$ are in some family $\mathscr{C}$ of subgroups of $G$, then we say that $\Gamma$ is a decomposition of $G$ *over* $\mathscr{C}$. Since the edge groups of $\Gamma$ are determined only up to conjugacy, $\mathscr{C}$ should be closed under conjugacy. Note that if $\Gamma$ is a decomposition of $G$ over $\mathscr{C}$, and $\Gamma'$ is a refinement of $\Gamma$ such that the associated map $\tau' \to \tau$ does not collapse any edges to vertices, then $\Gamma'$ is a decomposition of $G$ over the elements of $\mathscr{C}$ and their subgroups.

A decomposition of $G$ with one edge is a *splitting* of $G$, and a proper decomposition of $G$ with one edge is a *proper splitting* of $G$. If there exist no proper splittings of $G$ over a family $\mathscr{C}$ as above, then we say that $G$ is *unsplittable over* $\mathscr{C}$.

Note that if $G$ admits a decomposition $\Gamma'$ over $\mathscr{C}$, arising from an action on a $G$-tree $\tau'$, then for any edge $e$ of $\Gamma'$ with edge group $E$, there is a splitting $\Gamma$ of $G$ associated to $e$, where $\Gamma$ has one edge with edge group $E$, and $\Gamma'$ is a refinement of $\Gamma$. To see this, let $e$ be an edge in $\tau'$ with stabilizer $E$. Then, if $G{\cdot}\mathrm{int}(e)$ denotes the $G$-orbit of the interior of $e$ in $\tau'$, let $\tau$ be the $G$-tree obtained by collapsing the components of $\tau' - G{\cdot}\mathrm{int}(e)$ to vertices, with the action of $G$ induced from the action of $G$ on $\tau'$. Then we may take $\Gamma$ to be $G\backslash\tau$. We observe that if $G$ acts minimally on $\tau'$, then $\Gamma$ must be a proper splitting.

Next, we define the notion of a compatible decomposition, which leads us to the idea of a hierarchy for a group.

**Definition 2.0.7.** If $G$ has a decomposition $\Gamma$, and the vertex group of a vertex $v$ of $\Gamma$ admits a splitting, then we say that the splitting is *compatible* with the decomposition if there exists a refinement of $\Gamma$ in which $v$ is replaced with an edge that is associated to the splitting, as is described above. Equivalently, the splitting is compatible with the decomposition if a conjugate of each edge group of the edges incident to $v$ is contained in a vertex group of the splitting.

Consider a group $G$, and a family $\mathscr{C}$ of subgroups of $G$ which is closed under conjugacy.

**Definition 2.0.8.** A *hierarchy* for $G$ over $\mathscr{C}$ is a sequence $\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \ldots$ of finite sets of conjugacy classes of subgroups of $G$, defined inductively as follows. The set $\mathscr{G}_0$ contains only (the conjugacy class of) $G$. If $i > 0$, then for any conjugacy class in $\mathscr{G}_{i-1}$, either $\mathscr{G}_i$ contains that conjugacy class, or $\mathscr{G}_i$ contains the conjugacy classes of the vertex groups of some proper decomposition of a representative of that class over $\mathscr{C}$. We require that at least one representative from $\mathscr{G}_{i-1}$ be decomposed.

Thus, if a hierarchy of $G$ over $\mathscr{C}$ is finite, then its last set $\mathscr{G}_N$ contains only conjugacy classes of subgroups that are unsplittable over $\mathscr{C}$.

We note that the existence of a finite hierarchy over $\mathscr{C}$ does not, in general, imply the existence of any kind of maximal decomposition over $\mathscr{C}$, since the splittings of vertex groups need not be compatible with the decompositions producing those vertex groups.

**Definition 2.0.9.** If $\Gamma = G \backslash \tau$ is a finite decomposition of $G$ over $\mathscr{C}$, and there exists no proper refinement of $\Gamma$ over $\mathscr{C}$, then we say that $\Gamma$ is a *maximal decomposition* of $G$ over $\mathscr{C}$.

One might think of defining a maximal decomposition of $G$ over $\mathscr{C}$ to be a maximal collection of compatible splittings over $\mathscr{C}$. But often, such collections are infinite. For example, consider any group $G$ that has an infinite descending chain of subgroups $G \supset C_0 \supset C_1 \supset C_2 \supset \ldots$ Then we have

$$G = G *_{C_0} C_0 = G *_{C_0} C_0 *_{C_1} C_1 = G *_{C_0} C_0 *_{C_1} C_1 *_{C_2} C_2 = \ldots$$

Less trivially, consider the Baumslag-Solitar group $H = \mathrm{BS}(1,2) = \langle x, t : t^{-1}xt = x^2 \rangle$. The normal closure of $\langle x \rangle$ in $H$ is isomorphic to $\mathbb{Z}[\frac{1}{2}]$ under addition, by an isomorphism which takes $x$ to 1, and $t^i x t^{-i}$ to $\frac{1}{2^i}$ for each $i$. Let $A_i$ denote the infinite cyclic subgroup generated by $t^i x t^{-i}$, and let $K = H *_{A_0} H$. Note that $A_0 \subset A_1 \subset A_2 \subset \ldots$, and that $K$ is finitely presented. We can refine the given decomposition of $K$ as many times as we please, for we have that

$$K = H *_{A_0} (A_1 *_{A_1} H) = H *_{A_0} (A_1 *_{A_1} (A_2 *_{A_2} H)) = \ldots$$

with the splitting associated to each edge of any of these decompositions being proper. Thus both of the examples above have sequences of refinements that do not terminate. This concludes our review of Bass-Serre theory.

We recall that two subgroups $H$ and $H'$ of a group $G$ are said to be *commensurable* if their intersection is of finite index in both. The *commensurizer*, $\mathrm{Comm}_G(H)$, of $H$ in $G$ is the subgroup of elements $g$ of $G$ such that $H$ and $gHg^{-1}$ are commensurable. (We note that $\mathrm{Comm}_G(H)$ is called the "commensurator" of $H$ in $G$ by some authors.)

Finally, we recall notions related to almost invariant subsets of groups (see also [25]). Let $G$ be a finitely generated group, and let $H$ be a subgroup of $G$. A subset $X$ of $G$ is said to be *$H$-finite* if $X \subset HF$ for some finite subset $F$ of $G$. A subset $X \subset G$ is said to be *$H$-almost invariant*, or an *almost invariant set over $H$*, if $HX = X$, and the symmetric difference $X + Xg$ is $H$-finite, for all $g \in G$. $X$ is said to be *nontrivial* if neither $X$ nor its complement $X^*$ is $H$-finite.

If a group $G$ admits a splitting over $H$, then the group contains a nontrivial $H$-almost invariant set associated to that splitting. To see this, we shall follow Chapter 2 of [25]. Suppose that $G$ splits as $A *_H B$ or $A *_H$, let $\tau$ be the associated $G$-tree, and fix a base point $w \in \tau$. Define a map $\varphi$ from $G$ to the vertex set of $\tau$ by $\varphi(g) = g \cdot w$. Let $s$ be the edge of $\tau$ that is stabilized by $H$, and choose an orientation for $s$. Then $s$ determines a partition of the vertex set of $\tau$ into two sets; let $Y_s$ denote the vertices of the component of $\tau - \mathrm{int}(s)$ that meets the terminal vertex of $s$, and let $Y_s^*$ denote the vertices of the component that meets the initial vertex of $s$. Let $Z_s = \varphi^{-1}(Y_s)$ and let $Z_s^* = \varphi^{-1}(Y_s^*)$. It is shown in [25] that $Z_s$ and $Z_s^*$ are $H$-almost invariant, and that a different choice of base point $w$ results in $H$-almost invariant subsets whose symmetric differences with $Z_s$ and $Z_s^*$ are $H$-finite. Thus, up to $H$-finite symmetric difference and complementation, $Z_s$ is uniquely associated to the given splitting of $G$.

It is not true, however, that each $H$-almost invariant set is associated to a splitting of $G$. Related to this is the notion of the "crossing" of almost invariant sets: if $X$ is an $H$-almost invariant set and $Y$ is a $K$-almost invariant set in a group $G$, then it is said that $X$ *crosses* $Y$ if none of $X \cap Y$, $X \cap Y^*$, $X^* \cap Y$, or $X^* \cap Y^*$ is $K$-finite. If both $X$ and $Y$ are nontrivial, then it is proven in [23] that $X$ crosses $Y$ if and only if $Y$ crosses $X$. We refer the reader to [23] for more on crossings.

We can think of a subset $X$ of $G$ as a vertex set in $\mathscr{C}_G$, hence as a 0-cochain in $\mathscr{C}_G$ with $\mathbb{Z}_2$ coefficients. Thus the coboundary $\delta X$ is a collection of edges in $\mathscr{C}_G$. If $X$ and $Y$ are as above, then it is said that $X$ crosses $Y$ *strongly* if both $\delta X \cap Y$ and $\delta X \cap Y^*$ are infinite in $K\backslash\mathscr{C}_G$. If $X$ crosses $Y$ but not strongly, then we say that $X$ crosses $Y$ *weakly*.

# CHAPTER III

## Strong accessibility for hyperbolic groups

The theory of group accessibility is made up of "accessibility results" and "strong accessibility results". Accessibility results show that a group can be decomposed as a graph of groups in a maximal way over a specific family of subgroups. Strong accessibility results show that a group has a finite hierarchy over a family of subgroups.

When decomposing over finite subgroups, these two notions are equivalent. In 1940, Gruško proved in [14] that any finitely generated group admits a maximal free product decomposition, i.e., a maximal decomposition over the trivial group.

We shall say that a group is *accessible* over a family of subgroups if the group admits a maximal decomposition over that family. Wall coined this term in the early 1970's in the context of decomposing over finite groups, and conjectured that every finitely generated group is accessible over finite subgroups (see [32]). In 1985, Dunwoody proved in [9] that finitely presented groups are accessible over finite subgroups. (In fact, both Gruško and Dunwoody showed that any decomposition of a finitely generated or finitely presented group respectively over the appropriate family of subgroups has a maximal refinement over that family.) In [12], published in 1993, Dunwoody provided an example of a finitely generated group that is not accessible over finite subgroups.

As for the question of accessibility over more general families of subgroups, Bestvina and Feighn showed in [2] that, over any family of subgroups that are "small", any graph of groups decomposition of a finitely presented group can be refined to a maximal one. (Any group that does not contain a copy of the free group on two generators, for example, is small.)

In [8], Delzant and Potyagailo proved a very general strong accessibility result. They showed that any finitely presented group without 2-torsion admits a finite hierarchy over any family of "elementary" subgroups (see Definition 3.1.1). In this chapter, we use their work to prove the following strong accessibility result:

**Theorem 3.2.9** *Let $G$ be a hyperbolic group with no 2-torsion. Decompose $G$ maximally as a graph of groups over finite subgroups, and then take the resulting vertex groups, and decompose those maximally as graphs of groups over two-ended subgroups. Now repeat this process on the new vertex groups and so on. Then this process must eventually terminate, with subgroups of $G$ which are unsplittable over finite and two-ended subgroups.*

Swarup conjectured this result, without the assumption of $G$ having no 2-torsion. In Bestvina's *Questions in Geometric Group Theory* [1], this is referred to as Swarup's Strong Accessibility Conjecture.

This theorem is not a special case of the strong accessibility result of Delzant and Potyagailo for two reasons. Firstly, a hierarchy from [8] is over one fixed family of subgroups. For this result, however, we alternate between decomposing over finite subgroups and two-ended subgroups. Secondly, given a group and a family of elementary subgroups, [8] shows the existence of one finite hierarchy over the family.

For Swarup's conjecture, it must instead be shown that any hierarchy as described is finite. By analyzing equivariant maps between $G$-trees, we are able to overcome these difficulties.

As a corollary to Theorem 3.2.9, we get the following result about finite hierarchies in 3-manifolds:

**Theorem 3.3.2** *Let $M$ be an irreducible, orientable, compact 3-manifold with hyperbolic fundamental group. The process of decomposing $M$ along any maximal, disjoint collection of compressing disks, then decomposing the resulting manifolds along maximal, disjoint collections of essential annuli, then the resulting manifolds along compressing disks, then again along essential annuli and so on, must eventually terminate with a collection of manifolds, each of which has incompressible boundary and admits no essential annuli, or is a 3-ball.*

## 3.1 Elementary families of subgroups and Stallings' folds

Before proving these theorems, we will need to introduce a few more definitions and facts. In [8], Delzant and Potyagailo prove the existence of a finite hierarchy for any finitely presented group with no 2-torsion over any family of "elementary" subgroups, which are defined as follows.

**Definition 3.1.1.** If $G$ is a finitely presented group, and $\mathscr{C}$ a family of subgroups of $G$, then $\mathscr{C}$ is said to be *elementary* if the following conditions are satisfied:

1. If $C \in \mathscr{C}$, then all subgroups and conjugates of $C$ are in $\mathscr{C}$.

2. Each infinite element of $\mathscr{C}$ is contained in a unique maximal subgroup in $\mathscr{C}$.

3. Ascending unions of finite subgroups in $\mathscr{C}$ are contained in $\mathscr{C}$.

4. If any $C \in \mathscr{C}$ acts on a tree, then $C$ fixes a point in the tree, fixes a point in the boundary at infinity of the tree, or preserves but permutes two points in the boundary at infinity.

5. If $C \in \mathscr{C}$ is an infinite, maximal element of $\mathscr{C}$ and $gCg^{-1} = C$, then $g \in C$.

We will be interested in applying the results of [8] to a pair $(G, \mathscr{C})$, when $\mathscr{C}$ is the set of all finite and two-ended subgroups of $G$. The following proposition is the reason we assume hyperbolicity in Theorem 3.2.9.

**Proposition 3.1.2.** *If $G$ is a subgroup of a hyperbolic group, and $\mathscr{C}$ is the set of all finite and two-ended subgroups of $G$, then $\mathscr{C}$ is elementary.*

In order to show this, we must recall a few facts about two-ended and hyperbolic groups.

**Theorem 3.1.3.** *A finitely generated group $G$ is two-ended if and only if it is virtually $\mathbb{Z}$, i.e., it contains an infinite cyclic subgroup of finite index.*

See Theorem 5.12 of [27] for a proof of this fact. The next lemma follows from Lemmas 1.16 and 1.17 of [20].

**Lemma 3.1.4.** *Any two-ended subgroup $H$ of a hyperbolic group $G$ is contained in a unique maximal two-ended subgroup, which is equal to the commensurizer $Comm_G(H)$ of $H$ in $G$.*

This implies the following:

**Corollary 3.1.5.** *If $G$ is a subgroup of a hyperbolic group, and $H \subset G$ is two-ended, then $H$ is contained in a unique maximal two-ended subgroup of $G$, which is equal to $Comm_G(H)$.*

*Proof.* Let $G'$ be a hyperbolic group containing $G$, and let $H \subset G$ be two-ended. If $H' \subset G$ is two-ended and $H \subset H'$, then $H'$ must commensurize $H$, i.e. $H' \subset \mathrm{Comm}_G(H)$. Note also that $H \subset \mathrm{Comm}_G(H) \subset \mathrm{Comm}_{G'}(H)$ and $\mathrm{Comm}_{G'}(H)$ is two-ended by Lemma 3.1.4, so $\mathrm{Comm}_G(H)$ is two-ended. Thus $\mathrm{Comm}_G(H)$ is the unique maximal two-ended subgroup containing $H$. ∎

The next fact is Theorem III.Γ.3.2 in [6]:

**Theorem 3.1.6.** *If $G$ is a hyperbolic group, then $G$ contains only finitely many conjugacy classes of finite subgroups.*

We can now prove Proposition 3.1.2:

*Proof of Proposition 3.1.2.* Let $G$ be a subgroup of a hyperbolic group, with $\mathscr{C}$ the collection of all finite and two-ended subgroups of $G$. Then property 1 of Definition 3.1.1 is immediate, and property 2 is shown in Corollary 3.1.5. Since $G$ is contained in a hyperbolic group, property 3 follows from Theorem 3.1.6.

As for property 4, assume that a group $C$ acts on a tree. If $C$ is finite, then $C$ must fix a point of the tree. If $C$ is virtually $\mathbb{Z}$, then $C$ must have an axis, so $C$ preserves two points in the boundary of the tree. Thus $\mathscr{C}$ satisfies 4.

For property 5, let $C$ be any maximal, infinite two-ended subgroup of $G$. It follows from Corollary 3.1.5 that $C = \mathrm{Comm}_G(C)$. If $N_G(H)$ denotes the normalizer of any subgroup $H$ of $G$, then we always have that

$$H \subset N_G(H) \subset \mathrm{Comm}_G(H).$$

Thus $N_G(C) = C$, so property 5 follows. ∎

We shall conclude this section with a discussion of folds between $G$-trees, which were introduced by Stallings (see [30]). Here, as well as in later arguments, we shall denote vertices and edges with lower case letters, and their stabilizers with the capitalizations of those letters.

**Definition 3.1.7.** A *fold* on a $G$-tree $\tau$ is a $G$-map that identifies two adjacent edges $e$ and $f$ of $\tau$, identifies $g \cdot e$ with $g \cdot f$, for all $g \in G$, and makes no other identifications. If $e$ and $f$ meet at a vertex $v$, and are also incident to vertices $x$ and $y$ respectively, then the identification of $e$ with $f$ is such that $x$ is identified with $y$.

The next result shows that any $G$-map can often be decomposed into a series of folds. It follows from the proposition in Section 2 of [2].

**Proposition 3.1.8.** *If $\phi$ is a surjective $G$-map from a $G$-tree $\tau'$ to a $G$-tree $\tau$, $G\backslash\tau'$ is finite, and all the edge stabilizers of $\tau$ are finitely generated, then $\phi = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$, for some collection of folds $\{\phi_i\}$.*

As described by Bestvina and Feighn in [2], folds can be broken up into several different types, depending on whether $e$ and $f$ are in the same $G$-orbit, and whether $x$, $y$ and $v$ are in the $G$-orbits of one another. For simplicity, we shall work only with folds that are such that neither $x$ nor $y$ are in the $G$-orbit of $v$. Note that, by subdividing edges of our $G$-trees, we can always assume that any $G$-map $\phi$ as above is a composition of such folds.

If $\phi\colon\ \tau' \to \tau$ is such a fold, then $\phi$ must be one of three types, which, following [2], we will call types IA, IIA, and IIIA. These types correspond to the following three cases: when no $g \in G$ takes $x$ to $y$, when some $g \in G$ takes $x$ to $y$ and $e$ to $f$, and when some $g \in G$ takes $x$ to $y$, but does not take $e$ to $f$.

Let $\pi$ denote the projection map $\tau' \to \Gamma' = G\backslash\tau'$, and let $\Phi\colon\ \Gamma' \to \Gamma$ be the map induced from $\phi$. Our figures below indicate how, in each case, $\Phi$ will alter $\pi(e \cup f)$. Since $\Phi$ cannot alter the underlying graph, or edge or vertex groups, of $\Gamma' - \pi(e \cup f)$, these must describe $\Phi$ completely.

When no $g \in G$ takes $x$ to $y$, we will say that the fold is of type IA. In this case, $\pi(e \cup f)$ will change as indicated in Figure 3.1.

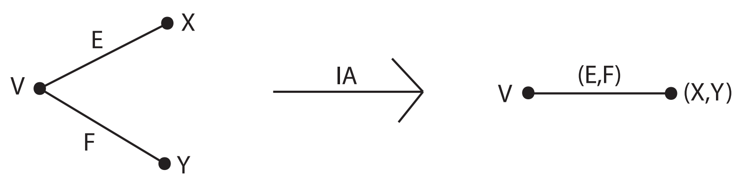


Figure 3.1: A fold of type IA, with vertices and edges labeled with their associated groups

A fold of type IIA occurs when some $g \in G$ takes $x$ to $y$ and takes $e$ to $f$, in which case we have that $g \in V$, the stabilizer of $v$. Here, the image under $\pi$ of the segment $e \cup f$ is a single edge, and folding changes only the labeling of $\Gamma'$. See Figure 3.2.

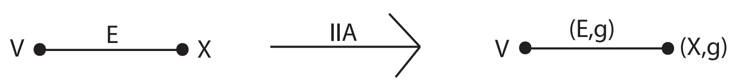


Figure 3.2: A fold of type IIA

Lastly, we have a fold of type IIIA when some $g \in G$ takes $x$ to $y$ and does not take $e$ to $f$. Note that then $g$ translates along an axis containing $e$ and $f$. In $\Gamma'$, we get what is shown in Figure 3.3.

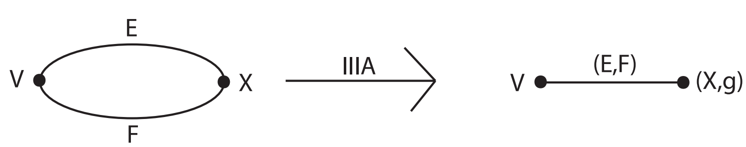


Figure 3.3: A fold of type IIIA

### 3.2 Strong accessibility

In this section, we shall prove Swarup's conjecture for hyperbolic groups with no 2-torsion. We shall first define the notion of complexity used by Delzant and Potyagailo in [8], and then carefully state their result.

Let $G$ be a finitely presented group, let $\mathscr{C}$ be a family of elementary subgroups of $G$, and note that $G$ is the fundamental group of a finite, two-dimensional simplicial orbihedron $\Pi$ for which vertex stabilizers are in $\mathscr{C}$. (For example, $G$ is the fundamental group of a finite, two-dimensional simplicial complex. In this case, vertex stabilizers are equal to the trivial group.) For any such $\Pi$, we define $T(\Pi)$ to be the number of faces of $\Pi$, and $b_1(\Pi)$ to be the first Betti number of the underlying space. Then we define the complexity of $\Pi$ to be

$$c(\Pi) = (T(\Pi), b_1(\Pi)).$$

The complexity of $G$ with respect to $\mathscr{C}$ is defined to be

$$c(G, \mathscr{C}) = c(G) = \min c(\Pi),$$

where the minimum is taken over all $\Pi$ with vertex groups elements of $\mathscr{C}$ and $G = \pi_1^{orb}(\Pi)$. Lexicographical ordering is used.

Proposition 3.4 of [8] shows that if $c(G) = (0, 0)$, then $G$ must be the fundamental group of a tree of groups (possibly just a vertex), with finite edge groups, and vertex groups in $\mathscr{C}$. We are taking $G$ to be finitely presented, so we note that any such tree will be finite.

A group is said to have a *dihedral* action on a tree if the group acts on the tree, has an axis, and some elements of the group interchange the endpoints of the axis. In [8], the following theorem is proven:

**Theorem 3.2.1.** [8] *Let $G$ be a finitely presented group, with $\mathscr{C}$ a family of elementary subgroups of $G$ and $c(G,\mathscr{C}) > (0,0)$. Suppose $G$ has a proper decomposition over $\mathscr{C}$, with $\tau$ the associated Bass-Serre tree, and suppose further that no $C \in \mathscr{C}$ has a dihedral action on $\tau$.*

*Then there is a proper decomposition of $G$ over $\mathscr{C}$ with associated tree $\tau'$ such that there is a $G$-map $\tau' \to \tau$, and, if $\{G_v\}$ denotes the vertex groups of $G\backslash\tau'$, then $\sum T(G_v) \leq T(G)$, and $\max_v c(G_v, \mathscr{C} \cap G_v) < c(G,\mathscr{C})$.*

With $\mathscr{C}$ defined to be the finite and two-ended subgroups of a group $G$, as long as $G$ has no 2-torsion, it follows that the action of any $C \in \mathscr{C}$ on any $G$-tree $\tau$ is not dihedral.

*Remark* 3.2.2. We will want to apply Proposition 3.1.8 to the map $\tau' \to \tau$. In order to do this, we need surjectivity. For the moment, we shall merely note that, if $G$ acts on $\tau$ minimally, then $\tau' \to \tau$ must be surjective. If the action is not minimal, then $\tau'$ maps onto a $G$-tree contained in $\tau$, and all of the edge and vertex groups of $\tau$ outside of this subtree are contained in $\mathscr{C}$.

We shall now present several lemmas, which will be used to bridge the gap between Theorem 3.2.1 and Swarup's Strong Accessiblity Conjecture for groups with no 2-torsion.

**Lemma 3.2.3.** *Let $G$ be a finitely generated group, and $\mathscr{C}$ a family of subgroups of $G$ which is closed under conjugation and subgroups. Suppose that $\phi\colon \tau' \to \tau$ is a surjective $G$-map between $G$-trees with all edge stabilizers in $\mathscr{C}$. Moreover, suppose $\phi$ is such that, for each edge $e$ of $\tau'$, stab$(e)$ is contained in stab$(\phi(e))$ with finite index. Let $\Gamma' = G\backslash\tau'$, and $\Gamma = G\backslash\tau$, and suppose that $\Gamma'$ is finite, and the edge groups of $\Gamma$ are all finitely generated. Then if $\Gamma'$ admits a proper refinement over $\mathscr{C}$, so does $\Gamma$,* *and the additional edge groups in the refinements are the same.*

From this, we immediately have the following:

**Corollary 3.2.4.** *If $G$, $\mathscr{C}$, $\Gamma$ and $\Gamma'$ are as in Lemma 3.2.3, and $\Gamma$ is a maximal proper decomposition of $G$, then $\Gamma'$ must be maximal as well.*

*Proof of Lemma 3.2.3.* We will start by showing that if $\phi$ is a fold, then a proper splitting of a vertex group of $\tau'$, which is compatible with $\Gamma'$, induces a proper splitting of the image of the vertex group which is compatible with $\Gamma$, over the same edge group. From this, it will follow that a proper refinement of $\Gamma'$ induces a proper refinement of $\Gamma$.

So assume that $\phi\colon \tau' \to \tau$ is a fold. We use our notation from above, so that $\phi$ identifies $e$ to $f$ and $x$ to $y$, where $e$ and $f$ meet at the vertex $v \in \tau'$, and similarly, identifies $g \cdot e$ to $g \cdot f$ for each $g$ in $G$. Let vertex $w \in \tau'$ be such that $W$, the stabilizer of $w$, admits a proper splitting over some $C \in \mathscr{C}$, which is compatible with $\Gamma'$. Thus there exists a tree $\overline{\tau'}$ and a $G$-equivariant map $\zeta'\colon \overline{\tau'} \to \tau'$ which merely collapses each edge in the orbit of $c$ to a vertex in the orbit of $w$. We would like to find a tree $\overline{\tau}$ such that there is a similar collapsing map $\zeta\colon \overline{\tau} \to \tau$, a fold $\overline{\phi}$ taking $\overline{\tau'}$ to $\overline{\tau}$, and such that the following diagram commutes:

$$\begin{array}{ccc} \overline{\tau'} & \xrightarrow{\overline{\phi}} & \overline{\tau} \\ \zeta' \downarrow & & \downarrow \zeta \\ \tau' & \xrightarrow{\phi} & \tau \end{array}$$

For our first case, assume that $w$ is not in the $G$-orbit of $v$, nor of $x$ nor $y$. Then we may define $\overline{\phi}$ to identify $\zeta'^{-1}(e)$ to $\zeta'^{-1}(f)$, and $\zeta'^{-1}(g \cdot e)$ to $\zeta'^{-1}(g \cdot f)$, for each $g$ in $G$. The edge $c$, as well as the edges in the $G$-orbit of $c$, are untouched by such a fold, so the above diagram must commute. Also because no edge gets identified to $c$

or any of its translates, and because the refinement $\overline{\Gamma'}$ of $\Gamma'$ is proper, it follows that $\overline{\phi}$ induces a refinement of $\Gamma$ which is proper.

Next, assume that $w$ is in the $G$-orbit of $x$, and not of $v$. ($w$ may be in the orbit of $y$.) Then we may again define $\overline{\phi}$ directly, taking that it identifies $\zeta'^{-1}(e)$ to $\zeta'^{-1}(f)$, and similarly for the $G$-orbits of $e$ and $f$. Define the map $\overline{\phi}_*$ to take the stabilizer of any vertex or edge $z$ in $\tau'$ to the stabilizer of $\overline{\phi}(z)$, and let $a$ and $b$ be the vertices of $c$. Recall that $A =$stab$(a)$, and so on. Then in this case, $\overline{\phi}_*(C) = C$, while $A \subseteq \overline{\phi}_*(A)$ and $B \subseteq \overline{\phi}_*(B)$. It follows again that $\overline{\Gamma}$ is a proper refinement of $\Gamma$ because $\overline{\Gamma'}$ is a proper refinement of $\Gamma'$. To see this, we note that if $C \hookrightarrow A$ and $C \hookrightarrow B$ are not isomorphisms, then neither are the new injections in $\overline{\tau}$. If instead $g \in V$ takes $a$ to $b$, then $g$ will take $\overline{\phi}(a)$ to $\overline{\phi}(b)$. Thus we have that $\overline{\Gamma}$ is a proper refinement of $\Gamma$.

It remains to consider the case in which $w$ is in the $G$-orbit of $v$. Without loss of generality, we assume that $w = v$. By abuse of notation, we will denote $\zeta'^{-1}(e)$ by $e$, and $\zeta'^{-1}(f)$ by $f$. Suppose that $e$ and $f$ are adjacent in $\overline{\tau'}$, so both contain either $a$ or $b$. Here again, we may simply define $\overline{\phi}$ to identify $e$ to $f$, and extend equivariantly. Then $\overline{\phi_*}$ takes $A$, $B$ and $C$ to themselves, and if there is some $g \in V$ which takes $a$ to $b$, then $g$ must also take $\overline{\phi}(a)$ to $\overline{\phi}(b)$. Hence, this induced refinement $\overline{\Gamma}$ must be proper.

So for our last case, assume that $w = v$, and that $e$ and $f$ are not adjacent in $\overline{\tau'}$. Without loss of generality, take that $e$ contains $a$ and $f$ contains $b$, i.e. $E \subseteq A$ and $F \subseteq B$. Here, we will use our hypothesis that $E$ and $F$ are of finite index in $\phi_*(E) = \phi_*(F)$ to show that either $E \subseteq C$ or $F \subseteq C$. If $E \subseteq C$, then we may alter $\overline{\tau'}$ by 'sliding' $e$ so that it is incident to $b$ instead of $a$, and do the same with the $G$-orbit of $e$. If $F \subseteq C$, then we can slide $f$ instead. By doing this, we are able to create a proper refinement of $\Gamma'$ of the type discussed in the previous paragraph, and may refer now to that argument.

To show that this is possible, assume that neither $E$ nor $F$ is contained in $C$, and choose elements $g_E \in E - C$ and $g_F \in F - C$. Then the subset of $\overline{\tau'}$ which is fixed pointwise by $g_E$ is a subtree of $\overline{\tau'}$ which is disjoint from the subtree of points fixed by $g_F$. Thus $g_E g_F$ acts by translation on an axis in $\overline{\tau'}$. Both $E$ and $F$ are contained in $\phi_*(E)$, hence so is $g_E g_F$, but because $g_E g_F$ has an axis, it is of infinite order, and no power $(g_E g_F)^n$ is contained in $E$ or $F$, except when $n = 0$. This means that $E$ and $F$ must be of infinite index in $\phi_*(E)$, which is a contradiction. Thus either $E \subseteq C$ or $F \subseteq C$ as desired.

We have seen now that if $\phi\colon\ \Gamma' \to \Gamma$ is a fold, and if $\Gamma'$ admits a proper refinement by a splitting over a subgroup $C$, then $\Gamma$ must also admit a proper refinement by a splitting which is also over $C$. For general $\phi$, Proposition 3.1.8 implies that $\phi$ is a composition of folds. If $\Gamma'$ admits a proper refinement, then by what we have shown, the refinement pushes through each fold, giving a proper refinement of $\Gamma$, as desired.

■

Next, we note the following fact, which we shall make use of with $n = 2$:

**Lemma 3.2.5.** *Let $G$ be a finitely generated group, with a $G$-tree $\sigma$ and associated decomposition $\Sigma$, identified with $G\backslash\sigma$. Let $V_1, \ldots, V_n$ be stabilizers of vertices $v_1, \ldots, v_n$ of $\sigma$, and let $\sigma_0$ be the smallest subtree of $\sigma$ containing $\{v_1, \ldots, v_n\}$. Then the orbit of $\sigma_0$ under $\langle V_1, \ldots, V_n\rangle$ is connected, thus a subtree of $\sigma$.*

*Proof.* Fix any $w \in \langle V_1, \ldots, V_n\rangle$. It will suffice to show that $w \cdot \sigma_0$ is connected to $\sigma_0$ in $\langle V_1, \ldots, V_n\rangle \cdot \sigma_0$.

We can write $w = w_1 w_2 \cdots w_{m-1} w_m$, where each $w_i$ is contained in some $V_{j_i}$. Then $w_m \cdot \sigma_0$ intersects $\sigma_0$ at the vertex stabilized by $V_{j_m}$, the subtree $w_{m-1} \cdot (w_m \cdot \sigma_0 \cup \sigma_0) =$

$(w_{m-1}w_m \cdot \sigma_0) \cup (w_{m-1} \cdot \sigma_0)$ intersects $\sigma_0$ at the vertex stabilized by $V_{j_{m-1}}$, the subtree $w_{m-2} \cdot ((w_{m-1}w_m \cdot \sigma_0) \cup (w_{m-1} \cdot \sigma_0) \cup \sigma_0) = (w_{m-2}w_{m-1}w_m \cdot \sigma_0) \cup (w_{m-2}w_{m-1} \cdot \sigma_0) \cup (w_{m-2} \cdot \sigma_0)$ intersects $\sigma_0$ at the vertex stabilized by $V_{j_{m-2}}$, and so on. Continuing in this manner, it follows that the translates $w \cdot \sigma_0 = w_1 w_2 \cdots w_m \cdot \sigma_0$, $w_1 w_2 \cdots w_{m-1} \cdot \sigma_0$, $\ldots$, $w_1 w_2 \cdot \sigma_0$, $w_1 \cdot \sigma_0$, $\sigma_0$ make a subtree, hence $w \cdot \sigma_0$ is connected to $\sigma_0$ in $\langle V_1, \ldots, V_n \rangle \cdot \sigma_0$.

■

From this, it follows that if $\sigma$ is a $G$-tree, and $v_1, \ldots, v_n$ are vertices of $\sigma$ with respective stabilizers $V_1, \ldots, V_n \subset G$, then the $\langle V_1, \ldots, V_n \rangle$-orbit of the smallest subtree containing $\{v_1, \ldots, v_n\}$ is a $\langle V_1, \ldots, V_n \rangle$-tree.

We can now prove the following:

**Lemma 3.2.6.** *Let $\Gamma = G \backslash \tau$ be a maximal proper decomposition of a finitely presented group $G$ over a family $\mathscr{C}$ which is closed under conjugation and subgroups. Let $\Gamma' = G \backslash \tau'$ be the decomposition from Theorem 3.2.1, with $\phi\colon\ \tau' \to \tau$ the associated $G$-map. Assume that, for each edge $e$ of $\tau'$, stab$(e)$ is contained in stab$(\phi(e))$ with finite index. Then, for each vertex group $V$ of $\Gamma$, either $V$ is a vertex group of $\Gamma'$, or $V \in \mathscr{C}$.*

*Proof.* By Remark 3.2.2, we can assume that $\phi\colon\ \tau' \to \tau$ from Theorem 3.2.1 is a surjection.

We may subdivide the edges of $\tau$ and $\tau'$ so that, for each edge of $\tau$ and $\tau'$, the vertices of that edge are in different $G$-orbits, yet still $\phi\colon\ \tau' \to \tau$ is a $G$-map. Again, from Proposition 3.1.8, $\phi$ is a composition of folds. Our subdivision of the edges of $\tau$ and $\tau'$ ensures that $\phi$ is, in fact, a composition of folds of types IA, IIA, and IIIA.

Assume first that $\phi$ is a fold of type IA, IIA, or IIIA. Using that $\Gamma$ is maximal, we will show that, for any vertex group $Z$ of $\Gamma$, either $Z$ is isomorphic by the given injection to one of its edge groups, or $Z$ is a vertex group of $\Gamma'$, hence has smaller complexity than $G$. Thus for a composition of such folds, a vertex group of the target decomposition is either a vertex group of the source decomposition, or is in $\mathscr{C}$.

We employ our previous notation, so that $\phi$ is a fold which takes edge $e$ of $\tau'$ to edge $f$, and vertex $x$ to vertex $y$, with $e$ and $f$ sharing the additional vertex $v$. It is immediate that, for all vertices $z'$ of $\tau'$, $\text{stab}(z') = \text{stab}(\phi(z'))$ if $z'$ is not in the $G$-orbit of $x$ or $y$. Hence it suffices to show the above statement for $Z = \text{stab}(\phi(x))$.

Consider the case in which $\phi$ is a fold of type IA. Recall that $\phi(x) = \phi(y)$ has stabilizer $Z = (X, Y)$, and consider the action of $Z$ on $\tau'$. Lemma 3.2.5 implies that this gives the decomposition of $Z$ that is pictured in Figure 3.4. Thus $(X, Y) =$

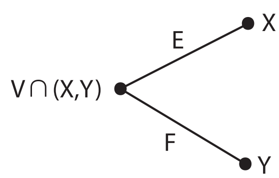


Figure 3.4: Decomposition of $Z$ in the case when $\phi$ is a fold of type IA.

$X *_E (V \cap (X, Y)) *_F Y$. If this decomposition gives a proper splitting of $Z$ which is compatible with $\Gamma$, i.e. the edge stabilizer of any edge adjacent to $\phi(x)$ is contained in a vertex group of the splitting, then this splitting would induce a proper refinement of $\Gamma$. This would be a contradiction, however, because $\Gamma$ is assumed to be maximal.

We claim first that the decomposition is compatible with $\Gamma$, hence either splitting from the decomposition is compatible with $\Gamma$. This follows because $E$ and $F$ are contained in $V$, so the stabilizer $(E, F)$ of $\phi(e)$ is contained in $V \cap (X, Y)$, and any other edge incident to $\phi(x)$ is untouched by the fold, hence has stabilizer either contained in $X$ or contained in $Y$.

Therefore this decomposition of $Z$ must not give a proper splitting. $[X *_E (V \cap$

$(X,Y))] *_F Y$ not being proper implies that either $Z = Y$ or $Y = F$. If $Z = Y$, then $Z$ is a vertex group of $\tau'$. Otherwise, $Y = F$. But also $X *_E [(V \cap (X,Y)) *_F Y]$ is not a proper splitting, so either $Z = X$ or $X = E$. If $Z = X$, then, as before, $Z$ is a vertex group of $\tau'$. Otherwise, $Y = F$ and $X = E$, so $Z = (X,Y) = (E,F)$, and hence $Z$ is an edge group of $\tau$. Thus if $\phi$ is a fold of type $IA$, then either $Z$ is isomorphic to a vertex group of $\Gamma'$, or an edge group of $\Gamma$.

Consider next the case in which $\phi$ is a fold of type IIA. There is some $g \in G$ taking $e$ to $f$, and fixing $v$, and $\phi(x)$ is stabilized by $(X,g)$. The action of this subgroup on $\tau'$ gives the splitting of $Z = (X,g)$ that is in Figure 3.5, and hence


Figure 3.5: Decomposition of $Z$ in the case when $\phi$ is a fold of type IIA.

$(X,g) = (V \cap (X,g)) *_E X$. It is clear that $(E,g) \subset (V \cap (X,g))$, so if we show that any other edge group of $\Gamma$ contained in $(X,g)$ is contained in one of the new vertex groups, then the compatibility of this splitting of $(X,g)$ with $\Gamma$ will follow. But as above, since the fold only affects the edge group labeled $(E,g)$, then any other edge group incident to the vertex labeled $(X,g)$ must have been contained in $X$.

We note that since $g \in V \cap (E,g)$, but $g \notin E$, this splitting induces a proper refinement of $\Gamma$ unless $X = E$, in which case $(X,g) = (E,g)$. Thus if $\phi$ is of type IIA, $Z = (X,g)$ must be an edge group of $\Gamma$.

If $\phi$ is a fold of type IIIA, then there is some $g \in G$ taking $x$ to $y$, but not taking $e$ to $f$. Recall that $Z = \mathrm{stab}(\phi(x))$ is $(X,g)$, and consider the action of $(X,g)$ on $\tau'$. The quotient by this action contains the decomposition of $(X,g)$ given in Figure 3.6, thus $(X,g) = ((V \cap (X,g)) *_F X) *_E$, where this HNN extension is by $g$. A

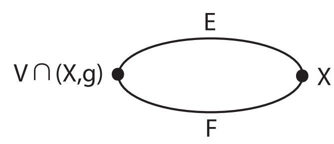


Figure 3.6: Decomposition of $Z$ in the case when $\phi$ is a fold of type IIIA.

refinement by an HNN extension must always be proper, so it remains to show that this splitting induces a refinement of $\Gamma$, i.e. is compatible with the other splittings of $\Gamma$. To do this, we must show that the stabilizer of any edge incident to $\phi(x)$ is contained in $((V \cap (X, g)) *_F X)$. The argument for this is similar to the above: except for $\phi(e)$, any edge $d$ incident to $\phi(x)$ is again untouched by the fold, hence has stabilizer equal to the stabilizer of $\phi^{-1}(d)$, which is contained in $X$, as $\phi^{-1}(d)$ is incident to $x$. $X \subset ((V \cap (X, g)) *_F X)$, so our splitting is compatible with the splitting over $D$.

Now recall that $\phi(e)$ is stabilized by $(E, F)$. But both $E$ and $F$ stabilize $v$, hence are in $V$. Also, $E$ and $F$, when conjugated by $g$, stabilize $x$, hence $(E, F)$ is in $(X, g)$. Thus $\mathrm{stab}(\phi(e)) = (E, F) \subset (V \cap (X, g)) \subset ((V \cap (X, g)) *_F X)$, so the given splitting of $(X, g)$ is compatible with the other splittings of $\Gamma$. But this means that there is a proper refinement of $\Gamma$, a contradiction. Hence $\phi$ cannot be a fold of type IIIA.

We now address the situation in which $\phi = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$, where each $\phi_i$ is a fold of type IA, IIA, or IIIA. Let $\Gamma_i$ denote the decomposition $G \backslash \phi_i \circ \phi_{i-1} \circ \ldots \circ \phi_1(\tau')$. Lemma 3.2.3, and the fact that $\Gamma$ is maximal, imply that the decompositions $\Gamma'$, $\Gamma_1$, $\Gamma_2$, …, $\Gamma_{n-1}$ are all maximal, and since $\Gamma$ and $\Gamma'$ are proper decompositions, each $\Gamma_i$ is also proper. Thus, for each $i$, the vertex groups of $\Gamma_i$ are edge groups of $\Gamma_i$, or are vertex groups of $\Gamma_{i-1}$. It follows that any vertex group of $\Gamma$ is isomorphic to either a vertex group of $\Gamma'$, or an edge group of some $\Gamma_i$, thus is in $\mathscr{C}$. (Note that our early subdivision of edges of $\Gamma'$ only adds edge groups to the collection of vertex groups of

$\Gamma'$, hence does not affect this result.) ■

Before proving Theorem 3.2.9, we shall need two additional facts. The first is a result from Scott and Swarup [25] about the existence of maximal decompositions over two-ended subgroups:

**Theorem 3.2.7.** *Let $G$ be a one-ended, finitely presented group, and let $\Gamma$ be a proper decomposition of $G$ over two-ended subgroups. Then $\Gamma$ admits a refinement $\Sigma$ which is a maximal proper decomposition of $G$ over two-ended subgroups.*

*Proof.* Let $\tau$ be the $G$-tree corresponding to $\Gamma$. For any vertex $v$ of valence two of $\Gamma$ which is not the vertex of a circuit and has incident edges $e$ and $f$ such that $E = V = F$ by the given injections, collapse either $e$ or $f$. Continue this process until no such vertices remain, and denote the resulting decomposition by $\overline{\Gamma}$. We have now removed enough redundancy from $\Gamma$ to be able to apply Theorem 7.11 of [25], with corrected statement in [24], giving us that $\overline{\Gamma}$ has a maximal refinement $\overline{\Sigma}$.

We claim now that $\overline{\Sigma}$ induces a maximal refinement $\Sigma$ of $\Gamma$, i.e. that we may put the collapsed edges back into $\overline{\Sigma}$ corresponding to their location in $\Gamma$. This can be done by merely subdividing each edge of $\overline{\Sigma}$ which corresponds to an edge $e$ (respectively $f$) of $\Gamma$ when, as in our notation above, the edge $f$ (respectively $e$) was collapsed to a point. ■

The second result we will need is the following:

**Lemma 3.2.8.** *If $G$ is a finitely presented group, and $\Gamma$ is a decomposition of $G$ over finitely generated subgroups, then the vertex group(s) of $\Gamma$ are also finitely presented.*

For a proof of this, we refer the reader to Lemma 1.1 in [3].

We can now prove Swarup's conjecture for hyperbolic groups with no 2-torsion:

**Theorem 3.2.9.** *Let $G$ be a hyperbolic group with no 2-torsion. Decompose $G$ maximally over finite subgroups, and then take the resulting vertex groups, and decompose those maximally over two-ended subgroups. Now repeat this process on the new vertex groups and so on. Then this process must eventually terminate, with subgroups of $G$ which are unsplittable over finite and two-ended subgroups.*

*Remark* 3.2.10. We note that the proof below also goes through if $G$ is a finitely presented subgroup of a hyperbolic group.

*Proof.* First, we will note that the above process must terminate for any finitely generated group $H$ such that $c(H) = (0, 0)$, with respect to the family of finite and two-ended subgroups of $H$. Recall that in this case, $H$ is the fundamental group of a tree of groups with finite edge groups, and finite or two-ended vertex groups. If the tree consists of just one vertex, then $H$ is finite or two-ended. When $H$ is finite, then it is unsplittable over all subgroups and hence the process terminates. If $H$ is two-ended, then $H$ admits one nontrivial decomposition, which is over a finite subgroup and has finite vertex groups, thus the above process must also terminate.

More generally, let $H$ be the fundamental group of a tree of groups as described above. The only vertex groups of the tree which admit any splittings are the two-ended groups. As noted above, each splits over a finite subgroup, and the resulting vertex groups are finite, hence unsplittable. Any collection of splittings of $H$ over finite subgroups are compatible, hence we may combine any splittings of vertex groups of the tree with the splittings of $H$ determined by the edges of the tree to get a decomposition of $H$ over finite subgroups with vertex groups which are completely unsplittable. It follows that the process terminates for any $H$ such that $c(H) = (0, 0)$.

Now we let $G$ be any hyperbolic group. Then $G$ must be finitely presented, thus, by [9], it has a maximal decomposition over finite subgroups. Choose such a decomposition (which must be finite), and let $\tau$ be the associated tree. Let $\mathscr{C}$ be the family of all finite and two-ended subgroups of $G$, and let $\tau'$ be the $G$-tree resulting from an application of Theorem 3.2.1. Note that since the map $\tau' \to \tau$ collapses no edges to vertices, stabilizers of edges of $\tau'$ are subgroups of stabilizers of edges of $\tau$, hence the decomposition of $G$ associated to $\tau'$ is over finite subgroups of $G$.

Thus Lemma 3.2.6, applied taking the family of elementary subgroups to be the collection of finite subgroups of $G$, implies that any vertex stabilizer $V_1$ of $\tau$ either is finite or is a vertex stabilizer of $\tau'$, hence is of smaller complexity (with respect to the family of finite and two-ended subgroups of $V_1$) than $G$. Certainly the process described above must terminate for finite groups, so we may assume that $V_1$ is not finite.

By Lemma 3.2.8, $V_1$ must be finitely presented. Let $\mathscr{C}_1$ be the collection of finite and two-ended subgroups of $V_1$, i.e. $\mathscr{C}_1 = \mathscr{C} \cap V_1$. By Proposition 3.1.2, $\mathscr{C}_1$ is elementary in $V_1$. Note that $V_1$ must have one end, so by Theorem 3.2.7, $V_1$ has a maximal decomposition over two-ended subgroups. By Remark 3.2.2, we can assume that this decomposition is finite. Let $\tau_1$ be the corresponding $V_1$-tree, and $\tau_1'$ the tree from Theorem 3.2.1.

Since $V_1$ has one end, the edge groups of $\tau_1'$ are also two-ended, and thus any edge group of $\tau_1'$ is of finite index in the image edge group from the map $\tau_1' \to \tau_1$. Therefore, Lemma 3.2.6 gives us that if $V_2$ is a vertex group of $\tau_1$, then $V_2$ is in $\mathscr{C}_1$ or has smaller complexity than $V_1$, with respect to the family $\mathscr{C}_2 = \mathscr{C}_1 \cap V_2$ of the finite and two-ended subgroups of $V_2$. We note that $V_2$ is finitely presented, and $\mathscr{C}_2$ is elementary in $V_2$.

If $V_2$ is in $\mathscr{C}_1$, then $V_2$ could admit one nontrivial maximal decomposition, which would be over a finite subgroup and would have finite vertex groups. Otherwise, we can repeat the arguments above, decomposing $V_2$ maximally over finite subgroups, decomposing the resulting vertex groups maximally over 2-ended subgroups, etc. Complexity of the resulting groups continues to decrease, so we must eventually reach a collection of subgroups of $G$ which are unsplittable over any finite or two-ended subgroups, as desired. ■

### 3.3 Application to 3-manifolds

We will now use this result to get the hierarchy theorem for 3-manifolds stated earlier. First, we recall that a surface $N$ in a 3-manifold $M$ is said to be essential if $N$ is properly embedded in $M$, 2-sided, $\pi_1$-injective into $M$, and is not properly homotopic into the boundary of $M$.

**Lemma 3.3.1.** *Let $M$ be a compact, connected 3-manifold, and let $\mathscr{A} = \{A_i\}_{i \in I}$ be a nonempty, finite collection of disjoint, non-parallel, essential surfaces in $M$, such that $\{\pi_1(A_i)\}$ is contained in a family $\mathscr{C}$ of subgroups of $G = \pi_1(M)$ which is closed under subgroups and conjugation. Suppose further that $\mathscr{A}$ is maximal with respect to collections of disjoint, non-parallel essential surfaces of $M$ with fundamental groups in $\mathscr{C}$. Let $\Gamma$ be the decomposition of $G$ which is dual to $\mathscr{A}$. Then $\Gamma$ is a maximal proper decomposition of $G$ over $\mathscr{C}$.*

*Proof.* Assume for the contrapositive that $\Gamma$ is not maximal. Then there exists some vertex group $V$ of $\Gamma$ which admits a proper splitting over some $C \in \mathscr{C}$ which is compatible with $\Gamma$. Let $L$ be the graph of groups for such a splitting of $V$, and let $p$ denote the midpoint of the edge of $L$. Let $N$ denote the union of the component of $M - \mathscr{A}$ which corresponds to $V$ with the surfaces $A_i$ which correspond to the edge groups incident to $V$. We can define a map from $N$ to $L$ which is an isomorphism on $\pi_1$, with each $A_i$ in $N$ mapped to a vertex of $L$, and such that the map is transverse to $p$. Note that each component of the inverse image of $p$ is a properly embedded, 2-sided surface in $N$. Furthermore, Stallings showed in [29] that we can homotope this map on $N$ rel $\partial N$ to a new map $f$ such that the surfaces comprising $f^{-1}(p)$ are $\pi_1$-injective in $M$ (see also [15]).

We may further assume that these components are not parallel to the boundary of $N$, because of the following. Let $S$ denote a component of $f^{-1}(p)$ which is boundary parallel in $N$, and let $R$ be the region made up of $S$ and the component of $N - S$ through which $S$ can be homotoped to $\partial N$, so $R$ is homeomorphic to $S \times I$. Then we may homotope $f$ to take $R$ to $p$, and then to take a small neighborhood of $R$ past $p$, so that $p$ is not contained in $f(R)$. We may then homotope $f$ to map the elements of $\mathscr{A} \cap R$ to the other vertex of $L$, so that still $p$ is not in $f(R)$, and still $f$ is an isomorphism on $\pi_1$. Note that, because $L$ is the graph of groups of a proper splitting, and $f$ is surjective on $\pi_1$, this process will never make $f^{-1}(p)$ empty.

We have arranged that the components of $f^{-1}(p)$ are essential in $M$. Because $f$ is $\pi_1$-injective, the fundamental group of each component of $f^{-1}(p)$ is conjugate to a subgroup of $C$ and so is in $\mathscr{C}$. Since $f$ maps the $A_i$'s to vertices of $L$, the surfaces $f^{-1}(p)$ are disjoint from $\mathscr{A}$. Also, as components of $f^{-1}(p)$ are not boundary parallel in $N$, they are not parallel to elements of $\mathscr{A}$. Hence $\mathscr{A}$ is not maximal. ■

We note that each component of $f^{-1}(p)$ induces a refinement of $\Gamma$. Suppose, in addition to the hypotheses on $M$ in the above lemma, that $M$ is irreducible. Then we can homotope $f$ to remove any sphere components of $f^{-1}(p)$, so that any simply connected component of $f^{-1}(p)$ must be a compressing disk for $M$. Thus, a maximal collection of compressing disks in an irreducible, connected 3-manifold $M$ induces a maximal proper decomposition of $G$ over the trivial group.

It also follows that, if $\mathscr{A}$ is a maximal collection of annuli in $M$, and $M$ is as in the above lemma, has incompressible boundary and is irreducible, then the graph of groups $\Gamma$ corresponding to $\mathscr{A}$ must be maximal over the family generated by all infinite cyclic subgroups of $\pi_1(M)$.

Recall that, if $M$ is orientable and irreducible and $\pi_1(M) = G$ is infinite, then $G$ has no torsion (see Hempel [15]). Hence any essential surface in $M$ with finite fundamental group must be simply connected, and any essential surface with two-ended fundamental group must be an annulus. We also note that it follows from the Geometrization Theorem proven by Perelman (see [7], [19]) that $M$ as above has a hyperbolic fundamental group if and only if $M$ is hyperbolic and has no torus boundary components.

These observations, together with Theorem 3.2.9, imply the following theorem.

**Theorem 3.3.2.** *Let $M$ be an irreducible, orientable, compact 3-manifold with hyperbolic fundamental group. The process of decomposing $M$ along any maximal, disjoint collection of compressing disks, then decomposing the resulting manifolds along maximal, disjoint collections of essential annuli, then the resulting manifolds along compressing disks, then again along essential annuli and so on, must eventually terminate with a collection of manifolds, each of which has incompressible boundary and admits no essential annuli, or is a 3-ball.*

# CHAPTER IV

## On the quasi-isometry invariance of the Scott-Swarup JSJ decompositions of groups

Roughly speaking, a JSJ decomposition of a group is a graph of groups decomposition that encapsulates the structure of the different splittings the group admits over two-ended subgroups, in the same way that a JSJ decomposition of a 3-manifold indicates the structure of the essential maps of annuli and tori into the manifold. Many different versions of these decompositions have been defined (see [17], [28], [22], [4], [5], [11], [13]). In particular, Dunwoody and Sageev defined a version in [10], and, more recently, Scott and Swarup defined a version in [25]. In the most general cases, the decompositions have been shown to exist for one-ended, finitely presented groups.

In [21], Papasoglu proved that Dunwoody and Sageev's JSJ decompositions of one-ended, finitely presented groups are invariant under quasi-isometries. By this, we mean that if $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ is a quasi-isometry, then the image under $f$ of any vertex group of a JSJ decomposition of $G$ is a finite Hausdorff distance from a vertex group of a JSJ decomposition of $G'$. In the following, we show the quasi-isometry invariance of vertex groups of the JSJ decompositions of Scott and Swarup for which the methods of [21] are not sufficient.

We also show that properties of these vertex groups, such as whether or not they are finitely or infinitely generated, are preserved under quasi-isometry. We do this by analyzing "quasi-lines" in the Cayley graphs of groups that separate the graphs into at least three "essential" components, making use of several results from [21]. It turns out that this simple separation property gives the rigidity necessary to draw algebraic conclusions from the geometry of the groups.

### 4.1 The Scott-Swarup JSJ decomposition

In [25], Scott and Swarup construct a canonical JSJ decomposition $\Gamma_1(G)$ of any one-ended, finitely presented group $G$, in which the vertex groups "enclose" all splittings of $G$ over two-ended subgroups, and moreover, enclose all nontrivial almost invariant subsets of $G$ over two-ended subgroups. We will now describe $\Gamma_1(G)$.

$\Gamma_1(G)$ is a regular neighborhood, as defined in [25], of all the nontrivial almost invariant subsets of $G$ over two-ended subgroups. Thus $\Gamma_1(G)$ is a bipartite graph of groups with fundamental group $G$, and with vertices said to be either $V_0$-*vertices* or $V_1$-*vertices*. If $v$ is a vertex of $\Gamma_1(G)$, then its vertex group, $G(v)$, is defined up to conjugacy, and is said to be either a $V_0$- or $V_1$-*vertex group*, depending on whether $v$ is a $V_0$- or $V_1$-vertex.

Furthermore, each nontrivial almost invariant subset of $G$ over a two-ended subgroup is "enclosed" by some $V_0$-vertex. In the case that such an almost invariant set is associated to a splitting of $G$, this means that the enclosing $V_0$-vertex group admits a splitting that is compatible with $\Gamma_1(G)$. Moreover, when $\Gamma_1(G)$ is refined by this splitting, the added edge is associated to the given splitting of $G$. Each $V_0$-vertex of $\Gamma_1(G)$ encloses at least one such splitting of $G$ over a two-ended subgroup.

Each $V_0$-vertex $v$ is one of three types:

1. $v$ is isolated

2. $v$ is of Fuchsian type, or

3. $v$ is of commensurizer type.

If $v$ is isolated, then $v$ is of valence two. Moreover, if we let $e_1$ and $e_2$ denote the edges incident to $v$, then the inclusions of $G(e_1)$ and $G(e_2)$ into $G(v)$ are isomorphisms, and all three subgroups are two-ended.

If $v$ is of Fuchsian type, then $G(v)$ is finite-by-Fuchsian, where the Fuchsian group is a discrete group of isometries of the hyperbolic plane or of the Euclidean plane, but is not finite nor two-ended. Associated to each peripheral subgroup of $G(v)$ there is exactly one corresponding edge $e$ incident to $v$, and $G(e)$ is conjugate to that subgroup.

Lastly, if $v$ is of commensurizer type, then $v$ is not isolated nor of Fuchsian type, and there is a two-ended subgroup $H$ of $G$ such that $G(v) = \mathrm{Comm}_G(H)$. Only in this case is it possible that the subgroups carried by the edges incident to $v$ are not two-ended, and in fact they may not even be finitely generated. From this, one is able to see that the $V_1$-vertex groups of $\Gamma_1(G)$ may not be finitely generated either.

We will now discuss properties of almost invariant sets which depend on the types of the vertices by which the sets are enclosed. In [18], given a group $G$ with a subgroup $H$, Kropholler and Roller defined the number of coends of $H$ in $G$ to be

$$\tilde{e}(G,H) = \dim_{\mathbb{F}_2}(\mathcal{P}G/\mathcal{F}_H G)^G,$$

where $\mathcal{P}G$ is the power set of all subsets of $G$, and $\mathcal{F}_H G$ is the set of all $H$-finite subsets of $G$. $\mathcal{P}G/\mathcal{F}_H G$ forms a vector space over $\mathbb{Z}/2\mathbb{Z}$ under the operation of symmetric difference. Thus a subset $X$ of $G$ represents an element of $(\mathcal{P}G/\mathcal{F}_H G)^G$ if and only if the symmetric difference $X + Xg$ is $H$-finite for all $g \in G$. (Note that this is the same as the definition of an $H$-almost invariant set, without the assumption that $HX = X$.) This is equivalent to the coboundary $\delta X$ of $X$ being an $H$-finite set of edges. We call any such $X$ an *$H$-KR almost invariant set.*

All almost invariant subsets of $G$ discussed in the remainder of this subsection are over two-ended subgroups.

If $v$ is isolated, then the only almost invariant sets enclosed by $v$ are those from the splitting of $G$ associated to the edges incident to $v$, hence $v$ does not enclose any crossing almost invariant sets. Conversely, if a $V_0$-vertex $v$ does not enclose any crossing almost invariant sets over two-ended subgroups, then $v$ is isolated. Moreover, if $X$ is an $H$-almost invariant set of $G$ which is enclosed by $v$, then by part 1 of Theorem 1.8 from [26], $\tilde{e}(G, H)$ must be 2 or 3.

If $v$ is of Fuchsian type, then $v$ is not isolated, and any almost invariant sets enclosed by $v$ that cross do so strongly. (See Propositions 7.2, 7.4 and 7.5 of [25].) Also, Theorem 7.8 of [25], tells us that, if $X$ is an $H$-almost invariant set that is enclosed by a vertex $v$ of Fuchsian type, then we have that either $\tilde{e}(G, H) = 2$, or $X$ is associated to the splitting given by an edge incident to $v$.

If $v$ is of commensurizer type, then any two almost invariant sets enclosed by $v$ that cross do so weakly. Moreover, if $X$ and $Y$ are almost invariant sets over subgroups $H$ and $K$ that are enclosed by $v$, then $H$ and $K$ are commensurable (see Propositions 7.3 and 7.5 of [25]), $G(v) = \mathrm{Comm}_G(H) = \mathrm{Comm}_G(K)$, and $\tilde{e}(G, H) = \tilde{e}(G, K)$ is at least 4. Conversely, if $H$ is a two-ended subgroup of $G$ such that $\tilde{e}(G, H) \geq 4$, then there is a commensurizer vertex group of $\Gamma_1(G)$ that is equal to $\mathrm{Comm}_G(H)$. (See part 1 of Theorem 1.8 from [26].)

## 4.2 Papasoglu's result for $\Gamma_{DS}$

We now consider JSJ decompositions of quasi-isometric one-ended finitely presented groups. We shall first discuss the existing results for the JSJ decomposition given by Dunwoody and Sageev in [10].

The JSJ decomposition of a group $G$ as given in [10] is a graph of groups decomposition of $G$, say $\Gamma_{DS}(G)$, which is bipartite. Call the two types of vertex groups white and black, and then all the black vertex groups are either of Fuchsian type or of isolated type (see the previous section). Thus all of the edge groups of $\Gamma_{DS}(G)$ are two-ended. $\Gamma_{DS}(G)$ describes all the splittings of $G$ over two-ended subgroups, in that if $G$ splits over a two-ended subgroup $C$, either as $A *_C B$ or $A*_C$, then $C$ is conjugate into a vertex group of $\Gamma_{DS}$, has a finite index subgroup which is contained in a black vertex group, and each white vertex group is conjugate into $A$ or $B$.

In [21], Papasoglu proves the quasi-isometry invariance of this JSJ decomposition. Specifically, the author proves the following.

**Theorem 4.2.1.** [21] *Let $G$ and $G'$ be one-ended, finitely presented groups. Suppose that $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ is a quasi-isometry. Then there is a constant $C > 0$ such that if $A$ is a subgroup of $G$ conjugate to a vertex group, a vertex group of Fuchsian type, or an edge group of the graph of groups $\Gamma_{DS}(G)$, then $f(A)$ has Hausdorff distance less than or equal to $C$ from a subgroup of $G'$ conjugate to, respectively, a vertex group, a vertex group of Fuchsian type, or an edge group of $\Gamma_{DS}(G')$.*

Given any one-ended, finitely presented group $G$, $\Gamma_{DS}(G)$ differs from $\Gamma_1(G)$ as follows. The Fuchsian type vertex groups of $\Gamma_{DS}(G)$ and the Fuchsian type vertex groups of $\Gamma_1(G)$ are the same (up to conjugacy), and have the same edge groups. Also, the isolated vertex groups of $\Gamma_1(G)$ are vertex groups of $\Gamma_{DS}(G)$, and have the same edge groups. Thus $V_1$-vertices adjacent only to Fuchsian and isolated vertices of $\Gamma_1(G)$ are the same as the corresponding white vertex groups of $\Gamma_{DS}(G)$. So $\Gamma_1(G)$ differs from $\Gamma_{DS}(G)$ only at the vertices of commensurizer type, and the adjacent edges and $V_1$-vertices.

The edges and $V_1$-vertices adjacent to vertices of commensurizer type need not have any special structure. We do, however, know a good deal about the structure of the vertex groups of commensurizer type, so we shall consider those vertex groups in the following.

### 4.3 Quasi-lines

In order to study the geometry of vertex groups of $\Gamma_1(G)$ of commensurizer type, we will make use of the "quasi-lines" that were fundamental in [21]. In this section, we define these objects, and discuss some of their properties.

Let $X$ be a metric space, and let $l\colon \mathbb{R} \to X$ be injective, continuous, and parametrized by arc length (that is, for each $x, y \in \mathbb{R}$, $\text{length}(l[x,y]) = d_{\mathbb{R}}(x,y)$). Suppose further that $l$ is a uniformly proper map, i.e. that for every $M > 0$, there exists an $N > 0$ such that if $A \subset X$ with $\text{diam}(A) < M$, then $\text{diam}(l^{-1}(A)) < N$. Then we shall say that $l$ is a *line*.

To a line $l$, we associate the distortion function $D_l(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$, where

$$D_l(t) = \sup\{\text{diam}(l^{-1}(A)) : \text{diam}(A) \leq t\}.$$

Let $L$ be a closed, path connected subspace of $X$ containing $l$, with $N > 0$ such that any point in $L$ can be joined to $l$ by a path in $L$ of length less than or equal to $N$. If $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a proper, increasing function and, for all $t > 0$, $D_l(t) \leq \phi(t)$, then we will say that $L$ is a $(\phi, N)$ *quasi-line*, or simply, a *quasi-line*. We shall refer to $\phi$ and $N$ as *constants* for $L$.

The following lemma shows that the restriction that a line be embedded is not an important one.

**Lemma 4.3.1.** *Let $l'$ be a uniformly proper simplicial map from $\mathbb{R}$ into a graph $X$ (taking $\mathbb{R}$ to be a simplicial complex with vertex set $\mathbb{Z}$). Then there is a line $l\colon \mathbb{R} \to X$ with Im$(l) \subset$ Im$(l')$, and such that $d_{Haus}($Im$(l)$,Im$(l')) < \infty$.*

*Proof.* As $l'$ is uniformly proper, there is some maximal $n = n(l') \in \mathbb{R}$ such that the preimage of some point in $\mathrm{Im}(l')$ has diameter $n$. Because $l'$ is simplicial, note that $n \in \mathbb{Z}$. We shall induct on $n$. Note that we are done if $n = 0$, for then $l'$ is already an embedding.

Let $S$ denote a maximal disjoint set of closed intervals of size $n$ in $\mathbb{R}$ such that the endpoints of each interval are sent to the same vertex of $X$ by $l'$. Let $\iota\colon \mathbb{R} \to \mathbb{R}$ denote the quotient map attained by identifying each component of $S \subset \mathbb{R}$ to a point, and define $\iota^{-1}$ to take each such point to an endpoint of its full preimage. Since the endpoints of each component of $S$ are identified by $l'$, there is a well-defined, continuous map $l_1 : \mathbb{R} \to X$ defined by $l_1(t) = l' \circ \iota^{-1}(t)$.

Clearly $l_1$ is simplicial, thus is parameterized by arc length, and we note that $l_1$ is uniformly proper, for if $A$ is any subset of $X$, then $\mathrm{diam}(l_1^{-1}(A)) \leq \mathrm{diam}(l'^{-1}(A))$. Furthermore, we have that $d_{Haus}(\mathrm{Im}(l_1), \mathrm{Im}(l')) \leq \frac{1}{2}n(l')$.

It remains to show that $n(l_1) < n(l')$. To see this, let us suppose that there are $t_0, t_1 \in \mathbb{R}$ such that $|t_0 - t_1| \geq n(l')$, and $l_1(t_0) = l_1(t_1)$. Suppose that $t_0$ is the image of a collapsed segment under $\iota$. Then there exist two points $s, s' \in \mathbb{R}$ that are the endpoints of this segment, with $\iota(s) = \iota(s') = t_0$, $l'(s) = l'(s')$, and $|s - s'| = n(l')$. If $\iota^{-1}(t_1)$ is a point, then let $s_1$ denote that point. If $\iota^{-1}(t_1)$ is a segment, then let $s_1$ denote an endpoint of that segment. Then $l'(s_1) = l'(s) = l'(s')$, and either $|s_1 - s| > |s - s'|$ or $|s_1 - s'| > |s - s'|$. But $|s - s'| = n(l')$, so this contradicts the definition of the function $n$.

Thus we may suppose that $\iota^{-1}(t_0)$ and $\iota^{-1}(t_1)$ are single points. If $\iota$ collapses no segments in the interval $[\iota^{-1}(t_0), \iota^{-1}(t_1)]$ then we reach another contradiction, for $[\iota^{-1}(t_0), \iota^{-1}(t_1)]$ must be an interval of size $n(l')$, whose endpoints are mapped to the same vertex of $X$ by $l'$, and that is disjoint from $S$. This contradicts the maximality of $S$.

Finally, suppose that $\iota$ collapses a segment $[s, s']$ in $[\iota^{-1}(t_0), \iota^{-1}(t_1)]$. Then $|s-s'| = n(l')$, so $|\iota^{-1}(t_0) - \iota^{-1}(t_1)| > n(l')$. The endpoints $\iota^{-1}(t_0)$ and $\iota^{-1}(t_1)$ must share the same image under $l'$, and again this contradicts the definition of the function $n$.

Thus $n(l_1) < n(l')$, and the lemma follows. ■

In this paper, we will be concerned with two-ended subgroups, their (left) cosets, and images of these under quasi-isometries. The following lemmas indicate why quasi-lines will be relevant to our discussion.

**Lemma 4.3.2.** *Let $G$ be a finitely generated group, and $H \subset G$ a two-ended subgroup. If $R > 0$ is large enough so that $N_R(H) \subset \mathscr{C}_G$ is connected, then it is a quasi-line.*

*Proof.* Let $H$ be a two-ended subgroup of $G$, let $\langle h \rangle \cong \mathbb{Z}$ be a finite index subgroup of $H$, and let $R > 0$ be such that $N_R(H) \subset \mathscr{C}_G$ is connected. Let $p$ be a simplicial path in $N_R(H)$ from the identity to $h$. Let $l_0 \colon \mathbb{R} \to N_R(H) \subset \mathscr{C}_G$ be the natural map onto $\cup_{n\in\mathbb{Z}} h^n \cdot p$ that is parameterized by arc length.

Note that there is some $N > 0$ depending on $\langle h \rangle$ and $R$ such that each point of $N_R(H)$ can be connected to $\mathrm{Im}(l_0)$ by a path in $N_R(H)$ of length less than or equal to $N$. If we can show that $l_0$ is uniformly proper, then it will follow from Lemma 4.3.1 that $N_R(H)$ is a quasi-line.

We claim first that there is a bound on the diameter of the preimage under $l_0$ of any vertex in $\mathscr{C}_G$. Let $v_1, \ldots, v_k$ be the vertices of $p$. Since $G$ acts freely on $\mathscr{C}_G$, for any distinct indices $i, j$, there is at most one nonzero value of $n$ such that $v_i = h^n \cdot v_j$. Thus $l_0^{-1}(v_i)$ is a set of at most $k$ points, say of diameter $d_i$. Since $\langle h \rangle$ acts on $\mathrm{Im}(l_0)$ by simplicial isometries, we also have that $\mathrm{diam}(l_0^{-1}(h^n \cdot v_i)) = d_i$ for any $n$. It follows that the diameter of the preimage of any vertex of $\mathscr{C}_G$ is bounded by $\max_i\{d_i\}$.

Now fix some $M > 0$, and suppose that there are subsets $Y_i$ of $\mathscr{C}_G$ such that $\mathrm{diam}(l_0^{-1}(Y_i)) \to \infty$, with $\mathrm{diam}(Y_i) < M$ for all $i$. It follows that there are points $a_i, b_i \in \mathbb{R}$ such that $d(a_i, b_i) \to \infty$, while $d(l_0(a_i), l_0(b_i)) < M$ for all $i$. We can further assume that $l_0(a_i)$ and $l_0(b_i)$ are vertices of $\mathscr{C}_G$, for each $i$.

By translating the pairs $(a_i, b_i)$ by the action of elements of $\langle h \rangle$, we may assume that the image of each $a_i$ is contained in $p$. Since $p$ has only finitely many vertices, by passing to a subsequence, we can further assume that all $a_i$ get mapped to the same vertex $v$. Let $A$ denote $l_0^{-1}(v)$, and recall from the above that $\mathrm{diam}(A)$ is bounded.

Then $d_{Haus}(A, b_i) \to \infty$, and the $l_0(b_i)$ are all contained in a ball about $v$ with radius $M$. Any such ball has only finitely many vertices, so infinitely many $b_i$ get mapped to some vertex $v'$. But $l_0^{-1}(v')$ also has finite diameter, and this contradicts that $d_{Haus}(A, b_i) \to \infty$. ■

**Lemma 4.3.3.** *Let $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ be a $(\Lambda, K)$ quasi-isometry and let $L \subset \mathscr{C}_G$ be a $(\phi, N)$ quasi-line. Then there is some $R_0 = R_0(\Lambda, K, \phi, N) > 0$ such that, for all $R \geq R_0$, $N_R(f(L))$ is a quasi-line.*

*Proof.* Let $l$ be the line associated to $L$. Note that $f \circ l|_{\mathbb{Z}}$ is a uniformly proper map of $\mathbb{Z}$ into $f(L) \subset \mathscr{C}_{G'}$. Let $z_i$ be a vertex in $\mathscr{C}_{G'}$ that is closest to $f \circ l(i)$, for each $i \in \mathbb{Z}$, and let $d$ be an upper bound for $d(z_i, z_{i+1})$, for all $i \in \mathbb{Z}$. Note that $d$ can be taken to depend only on the constants associated to $L$ and $f$.

Then there is some $R_0 \geq d$ (which can also be taken to depend only on $\Lambda, K, \phi$, and $N$) such that $L' = N_{R_0}(f(L))$ is connected. For each $i \in \mathbb{Z}$, $N_{R_0}(f(L))$ contains a shortest (simplicial) path $p_i$ from $z_i$ to $z_{i+1}$. Let $l''$ denote the natural map of $\mathbb{R}$ onto $\cup_i p_i$ that is parameterized by arc length. Since the restriction of $l''$ to the preimage of $\{z_i\}$ is uniformly proper, and each path $p_i$ is a geodesic, it follows that $l''$ is uniformly proper.

Note that there is some $N > 0$ such that each point of $N_{R_0}(f(L))$ can be connected to $\mathrm{Im}(l'')$ by a path in $N_{R_0}(f(L))$ of length less than or equal to $N$. Thus Lemma 4.3.1 implies that $N_{R_0}(f(L))$ is a quasi-line. Certainly if $R > R_0$, then $N_R(f(L))$ is also a quasi-line, so Lemma 4.3.3 follows. ∎

*Remark* 4.3.4. In this paper, we shall work with a quasi-isometry $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$, and a two-ended subgroup $H$ of $G$. We will discuss a neighborhood of $N_R(H)$ that is a quasi-line in $\mathscr{C}_G$, as well as translates of this quasi-line under the action of $G$. We will also discuss quasi-lines in $\mathscr{C}_{G'}$ that are neighborhoods of the images under $f$ of these translates, given by Lemma 4.3.3.

As a group acts on its Cayley graph by isometries, all of the translates of $N_R(H)$ will be quasi-lines, and shall share the same constants. Since these are all isometric, the argument given in Lemma 4.3.3 shows that the quasi-lines in $\mathscr{C}_{G'}$ that we shall consider will also have the same constants as one another.

We note next that quasi-lines are two-ended:

**Lemma 4.3.5.** *Let $L$ be a quasi-line contained in a locally finite simplicial graph $X$. Then $e(L) = 2$.*

*Proof.* Let $\phi$ and $N$ be constants for $L$, and let $l \subset L$ be the line associated to $L$. Then every point in $L$ can be connected by a path of length less than or equal to $N$ to $l$, and $e(l) = 2$, so $e(L) \leq 2$. As $l$ is injective and of unit speed, and $X$ is a locally finite graph, we must have that $e(L) \geq 1$.

To see that $e(L) = 2$, first note that if $a, b \in \mathbb{R}$ are such that $|a - b| > \phi(2N)$, then $d(l(a), l(b)) > 2N$. Thus if we fix $a' < b' \in \mathbb{R}$ such that $b' - a' > \phi(2N)$, then, for any $q \in l((-\infty, a'])$ and $q' \in l([b', \infty))$, $d(q, q') > 2N$. Let $K$ be the set of all points $p \in L$ such that there is a path of length less than or equal to $N$ contained in $L$ that connects $p$ to $l((a', b'))$. Since $X$ is locally finite, $K$ is compact. Thus $L - K$ contains two infinite components - one intersecting $l((-\infty, a'])$ and one intersecting $l([b', \infty))$. ■

**Definition 4.3.6.** If $L$ is a quasi-line in a metric space $X$, then a connected component $C$ of $X - L$ is said to be *essential* if $C \cup L$ has one end. Otherwise, $C$ is said to be *inessential*.

If $C$ is not contained in $N_R(L)$, for any $R \geq 0$, then we shall say that $C$ is *nearly essential*.

**Definition 4.3.7.** If there is some $m_0 > 0$ such that, for each $p \in L$, each essential component of the complement of $L$ intersects a vertex of $B_{m_0}(p)$, then we say that $L$ satisfies $ess(m_0)$.

**Definition 4.3.8.** If $m_1 > 0$ is such that each inessential component of the complement of $L$ is contained in the $m_1$-neighborhood of $L$, then we say that $L$ satisfies $iness(m_1)$.

$L$ is said to be *$n$-separating* if the complement of $L$ has at least $n$ essential components.

Scott and Swarup show that, if $\mathrm{Comm}_G(H)$ is a vertex group of $\Gamma_1(G)$ of commensurizer type, then $\tilde{e}(G, H) \geq 4$ (see [26], Theorem 1.8). We will see in Lemma 4.4.1 that this is equivalent to there being some $R > 0$ such that $N_R(H)$ is a 4-separating quasi-line.

Next, we shall show that any quasi-line in a one-ended finitely presented group has only finitely many essential components in its complement, and moreover satisfies $ess(m_0)$ for some $m_0$. In doing this, we will make use of the following:

**Definition 4.3.9.** Let $L$ and $C$ denote subsets of a metric space $X$, let $n > 0$, and let $x, y \in C \cap L$. Then we shall say that $x$ and $y$ are connected by an $(L, n)$*-chain in* $C \cap L$ if there are points $x = z_0, z_1, \ldots, z_k = y$ in $C \cap L$ such that, for each $i$, there is a path in $L$ connecting $z_i$ to $z_{i+1}$ of length less than or equal to $n$.

In the proofs of the next lemmas, we shall work in the Cayley complex, $\mathscr{C}\mathscr{C}$, for $G$. Recall that, if $G = \langle S : R \rangle$ is a finite presentation for $G$, then the associated Cayley complex is the universal cover of the 2-complex made up of one vertex, an edge corresponding to each set $\{s, s^{-1}\}$ of elements of $S$, and a disk glued on corresponding to each relation in $R$. The 1-skeleton of $\mathscr{C}\mathscr{C}$ is the Cayley graph $\mathscr{C}_G$ for $G$ with respect to $S$, and $\mathscr{C}\mathscr{C}$ is simply connected.

**Lemma 4.3.10.** *Let $G$ be a finitely presented group with finite presentation $\langle S : R \rangle$, let $L$ be a $(\phi, N)$ quasi-line in $\mathscr{C}_G$, let $0 < \epsilon \ll 1$ and let $N_\epsilon(L)$ denote the $\epsilon$-neighborhood of $L$ in $\mathscr{C}\mathscr{C}$. Then let $L'$ be an open set in $\mathscr{C}\mathscr{C}$ such that $L \subset L' \subset N_\epsilon(L)$ and such that $\overline{L'}$ deformation retracts onto $L$.*

*Then there is some $n_0 = n_0(S, R, \phi, N, \epsilon)$ such that, if $C'$ denotes any component of $\mathscr{C}\mathscr{C} - L'$, and $x, y \in C' \cap \overline{L'}$, then $x$ and $y$ are connected by an $(\overline{L'}, n_0)$-chain $\{z_i\}$ in $C' \cap \overline{L'}$. Moreover, the path in $\overline{L'}$ connecting any two $z_i$ and $z_{i+1}$ can be taken to be in $L$, outside of an initial segment containing $z_i$ and a final segment containing $z_{i+1}$, each of length less than or equal to $\epsilon$.*

To prove this, we first need the following.

**Lemma 4.3.11.** *Let $G, \mathscr{C}\mathscr{C}, L, L'$, and $\epsilon$ be as in Lemma 4.3.10. Let $\mathscr{C}\mathscr{C}''$ denote the union of $\mathscr{C}\mathscr{C}$ together with a disk added at each simple closed edge path of $\overline{L'}$ of length less than or equal to $\phi(2(N+\epsilon)+1)+2(N+\epsilon)+1$, and let $\overline{L''}$ denote the union of $\overline{L'}$ with these disks. Then $\overline{L''}$ is simply connected.*

*Proof.* Let $l$ be the line associated to $L$, so that $l \subset L \subset L'$, and note that $\overline{L'}$ is a $(\phi, N+\epsilon)$ quasi-line. Recall that the edges of $\mathscr{C}\mathscr{C}$ are of length one, and metrically, the 2-cells of $\mathscr{C}\mathscr{C}$ are regular polygons. Let the disks added to create $\mathscr{C}\mathscr{C}''$ and $\overline{L''}$ metrically be regular polygons.

Assume that there is some simple closed curve $\gamma$ in $\overline{L''}$ that is not null-homotopic. Note that $\gamma$ can be homotoped in $\overline{L''}$ to a simplicial path that is contained in $L' \cap \mathscr{C}\mathscr{C}^{(1)} = L' \cap (\mathscr{C}\mathscr{C}'')^{(1)}$. Let $\gamma$ now denote the image curve under this homotopy.

There exists a further homotopy of $\gamma$ to a curve that is contained in $l$ except for finitely many segments in $\overline{L'} \cap \mathscr{C}\mathscr{C}^{(1)}$ of length bounded by $2(N+\epsilon)+1$ as follows: suppose that $\gamma$ meets $l$, and that $\sigma$ is a component of $\gamma - (l \cap \gamma)$ of length greater than $2(N+\epsilon)+1$. Recall that each point of $\sigma$ can be joined to $l$ by a path in $\overline{L'}$ of length no more than $(N+\epsilon)$. Let $s_0, s_1, \ldots, s_k$ be points in $\sigma$ such that $s_0$ and $s_k$ are the endpoints of $\sigma$, and, for each $i$, the portion $p_i$ of $\sigma$ between $s_i$ and $s_{i+1}$ is of length no more than 1. Let $q_i$ denote a shortest edge path from $s_i$ to $l$ (so $q_i$ is constant for $i = 0, k$), and homotope $\sigma$ so that each $p_i$ is replaced with $-q_i \cup p_i \cup q_{i+1}$, where $-q_i$ denotes the path $q_i$, traversed in reverse. Repeat this process on any remaining segments of $\gamma - (l \cap \gamma)$ of length greater than $2(N+\epsilon)+1$, and it follows that the resulting path is homotopic in $\overline{L''}$ to $\gamma$, and is contained in $l$ except for segments of length bounded by $2(N+\epsilon)+1$.

If $\gamma$ does not meet $l$, then we can instead apply the above process, with $\sigma = \gamma$, to points $s_0, \ldots, s_k$ of $\sigma$ such that, for each $i$, if $j$ denotes $i$ mod $(k+1)$ and $j'$ denotes

$(i+1) \bmod (k+1)$, then the portion of $\sigma$ between $s_j$ and $s_{j'}$ is of length no more than 1.

Now consider the set of all closed edge paths that are homotopic to $\gamma$ within $\overline{L''}$, that meet $l$, and that are contained in $l$ except for finitely many segments in $\overline{L'} \cap \mathscr{C}\mathscr{C}^{(1)}$ of length bounded by $2(N+\epsilon)+1$, and choose one that minimizes the number of such segments. If it is not simple, then we may consider instead a simple closed subcurve of it that is not null-homotopic. Denote this again by $\gamma$.

Let $\Sigma = \{\sigma_i\}$ be the set of segments of $\gamma - (l \cap \gamma)$. We shall induct on $|\Sigma|$. We note that, since $\gamma$ is simple and $l$ is an embedding, $|\Sigma| > 0$.

If $|\Sigma| = 1$, then let $\Sigma = \{\sigma\}$, and let $p$ and $q$ denote the endpoints of $\sigma$. Recall that $d(l^{-1}(p), l^{-1}(q)) \leq \phi(d(p,q)) \leq \phi(2(N+\epsilon)+1)$. Thus $\gamma$ represents a word in the generators of $G$ of length less than or equal to $\phi(2(N+\epsilon)+1)+2(N+\epsilon)+1$. But this means that $\overline{L''}$ must contain a 2-cell attached along $\gamma$, so $\gamma$ is null-homotopic.

Fix $i > 1$, and assume that any simple closed edge path in $\overline{L''}$ with $|\Sigma| < i$ is null-homotopic in $\overline{L''}$. Let $\gamma$ now be a simple closed edge path in $\overline{L''}$, homotoped as mentioned above, with $|\Sigma| = i$. Fix $\sigma \in \Sigma$ and let $p$ and $q$ again be the endpoints of $\sigma$. Then we again have that $d(l^{-1}(p), l^{-1}(q)) \leq \phi(d(p,q)) \leq \phi(2(N+\epsilon)+1)$. Thus $\{\sigma \cup l([l^{-1}(p), l^{-1}(q)])\} \subset (\sigma \cup l)$ is a closed curve in $\overline{L'}$ of length less than or equal to $\phi(2(N+\epsilon)+1)+2(N+\epsilon)+1$. But then $\overline{L''}$ contains a disk $D$ with boundary attached to this curve.

Thus $\sigma$ can be homotoped, rel $\partial\sigma$, across $D$ to lie in $l$. Doing this decreases $|\Sigma|$. The resulting closed curve may no longer be simple, but can be homotoped to remove spikes, so that the resulting curve is equal to a union of closed, simple subcurves and segments connecting them. Each subcurve has less than $|\Sigma|$ segments that are not contained in $l$, thus are null-homotopic in $\overline{L''}$. It follows that $\gamma$ was originally null-homotopic in $\overline{L''}$. Thus $\overline{L''}$ is simply connected. ∎

*Proof of Lemma 4.3.10.* Again let $l$ be the line associated to $L$, and let $\mathscr{C}\mathscr{C}''$ and $\overline{L''}$ be as in Lemma 4.3.11.

We shall first show that, if this lemma holds for $\overline{L''}$ in $\mathscr{C}\mathscr{C}''$, then it holds for $L'$ in $\mathscr{C}\mathscr{C}$. Thus assume that we can find some $n_0'' > 0$ so that, for any component $C''$ of the complement of $\overline{L''}$ in $\mathscr{C}\mathscr{C}''$, an $(\overline{L''}, n_0'')$-chain in $\overline{C''} \cap \overline{L''}$ connects any two points in $\overline{C''} \cap \overline{L''}$. Suppose also, as in the statement of the lemma, that paths in $\overline{L''}$ connecting points in the $(\overline{L''}, n_0)$-chains can be taken to be in $L$, outside of $\epsilon$-neighborhoods of their endpoints.

Recall that $\overline{L''}$ differs from $\overline{L'}$ only by disks with boundary in $\overline{L'}$, and these disks are not contained in $\mathscr{C}\mathscr{C}$. It follows that the intersections of $\overline{L'}$ with the components of $\mathscr{C}\mathscr{C} - L'$ are the same as the intersections of $\overline{L''}$ with the closures of the components of $\mathscr{C}\mathscr{C}'' - \overline{L''}$. Let $C'$ be a component of $\mathscr{C}\mathscr{C} - L'$. Then there is automatically an $(\overline{L'}, n_0)$-chain as desired between any two points of $C' \cap \overline{L'}$, for $n_0 \geq n_0''$. Thus, the lemma will follow for $n_0 = n_0''$.

Now, we shall prove the lemma for $\overline{L''}$, which is simply connected. If $\{C_\alpha\}$ denotes all of the components of the complement of $\overline{L''}$, then we can apply Van Kampen's theorem to $\{\overline{C_\alpha} \cup \overline{L''}\}$ to see that each $\overline{C_\alpha} \cup \overline{L''}$ is simply connected. Since $\overline{L''}$ and each $\overline{C_\alpha}$ are connected, we have that $\overline{C_\alpha} \cap \overline{L''}$ is connected. Thus for any fixed $x, y \in \overline{C_\alpha} \cap \overline{L''}$, there exists a path $p$ from $x$ to $y$ contained in $\overline{C_\alpha} \cap \overline{L''}$.

Without loss of generality, we can assume that the frontier of $\overline{L'}$ (which equals the frontier of $\overline{L''}$) meets any edge of $(\mathscr{C}\mathscr{C}'')^{(1)}$ in only finitely many points. The group $G$ has one end and $\mathscr{C}\mathscr{C}$ is simply connected, thus each edge of $\mathscr{C}\mathscr{C}$ is contained in a 2-cell. The same is true for $\mathscr{C}\mathscr{C}''$, thus we can take $|\{p \cap (\mathscr{C}\mathscr{C}'')^{(1)}\}|$ to be finite, with $p$ still contained in $\overline{C_\alpha} \cap \overline{L''}$. Let $z_0, z_1, \ldots, z_k$ denote the elements of $p \cap (\mathscr{C}\mathscr{C}'')^{(1)}$, numbered in the order in which they are traversed by $p$, traveling from $x$ to $y$, with $z_0 = x, z_k = y$.

Since $G$ is finitely presented, there is some $K_0 = K_0(S, R)$ such that $K_0$ is the maximal perimeter of a 2-cell in $\mathscr{C}\mathscr{C}$. Thus $K_0'' = \max\{K_0, \phi(2(N+\epsilon)+1)+2(N+\epsilon)+1\}$ is the maximum perimeter of 2-cells in $\mathscr{C}\mathscr{C}''$. Any component of $p - (\mathscr{C}\mathscr{C}'')^{(1)}$ must be contained in such a cell, thus its interior can be replaced by a segment in $(\mathscr{C}\mathscr{C}'')^{(1)}$ with length less than $K_0''$. As $p \subset \overline{C_\alpha} \cap \overline{L''} = \overline{C_\alpha} \cap \overline{L'}$, and $\overline{L'}$ deformation retracts to $L$, we can further assume that this segment is contained in $\overline{L'}$. Hence each such path is contained in $\overline{L''}$, so $\{z_i\}$ is an $(\overline{L''}, K_0'')$-chain from $x$ to $y$ in $\overline{C_\alpha} \cap \overline{L''}$.

It remains to show that there are paths between consecutive $z_i$'s that are contained in $L$ except near their endpoints and are of bounded length. Recall that $l$ is the line associated to $L$. Let $p_i$ be the path in $\overline{L''} \cap (\mathscr{C}\mathscr{C}'')^{(1)}$ of length less than or equal to $K_0''$ connecting $z_i$ to $z_{i+1}$, and let $w_j$ be a point in $l$ that can be connected to $z_j$ by a path $q_j$ of length less than or equal to $N + \epsilon$. We can take $q_j$ to be contained in $L$, except for initial and terminal segments of length less than or equal to $\epsilon$. Then $d(w_i, w_{i+1}) \leq K_0'' + 2(N+\epsilon)$, so there is a path $p_i'$ in $l$ from $w_i$ to $w_{i+1}$, of length less than or equal to $\phi(K_0'' + 2(N+\epsilon))$. Then $q_i \cup p_i' \cup q_{i+1}$ gives us a path between $z_i$ and $z_{i+1}$ of length less than or equal to $K_0'' + 2(N+\epsilon) + \phi(K_0'' + 2(N+\epsilon))$, that is contained in $L$ except for initial and terminal segments no longer than $\epsilon$.

Thus, if $C_\alpha$ denotes any complementary component of $\overline{L''}$, then, for any $n_0'' > [K_0'' + 2(N+\epsilon) + \phi(K_0'' + 2(N+\epsilon))]$, any two points of $\overline{C_\alpha} \cap \overline{L''}$ can be connected by an $(\overline{L''}, n_0'')$-chain (which is also a $(\overline{L'}, n_0'')$-chain) of paths contained in $L$, outside of initial and final subpaths of length no more than $\epsilon$ in $\overline{L'}$. Hence the lemma follows, for any $n_0 > [K_0'' + 2(N+\epsilon) + \phi(K_0'' + 2(N+\epsilon))]$. ■

We can now prove that quasi-lines in finitely presented groups satisfy $ess(m_0)$.

**Lemma 4.3.12.** *Let $G = \langle S : R \rangle$ be a one-ended finitely presented group, with $L$ a $(\phi, N)$ quasi-line in $\mathscr{C}_G$. Then $\mathscr{C}_G - L$ contains only finitely many essential components. Moreover, there is some $m_0 = m_0(S, R, \phi, N)$ such that $L$ satisfies $ess(m_0)$.*

*Proof.* We shall prove that $L$ satisfies $ess(m_0)$, for some $m_0 > 0$. Since $\mathscr{C}_G$ is locally finite, it will follow that the complement of $L$ contains finitely many essential components.

Let $C$ be a component of the complement of $L$ in $\mathscr{C}_G$. We shall use Lemma 4.3.10 to show that there is some $n > 0$ (not depending on our choice of $C$) such that any $x, y \in \overline{C} \cap L$ are connected by an $(L, n)$-chain in $\overline{C} \cap L \subset \mathscr{C}_G \subset \mathscr{C}\mathscr{C}$. (Recall that in Lemma 4.3.10, we proved a similar statement in $\mathscr{C}\mathscr{C}$.) Let $L'$ be as defined in Lemma 4.3.10, so, for some $0 < \epsilon \ll 1$, $L \subset L' \subset N_\epsilon(L)$, $L'$ is open in $\mathscr{C}\mathscr{C}$, and $\overline{L'}$ deformation retracts onto $L$.

Fix any such $x$ and $y$, and, as $\overline{C}$ is connected, there is a simple oriented edge path $p$ in $\overline{C}$ connecting them. Recall that $\overline{L'}$ deformation retracts onto $L$, so each edge in $p$ must intersect some component $C'$ of $\mathscr{C}\mathscr{C} - L'$, with $(C' \cap \mathscr{C}\mathscr{C}^{(1)}) \subset C$. Thus $p$ is a union of segments $p_1, p_2, \ldots, p_k$ such that, for each $i$, the terminal vertex of $p_i$ is equal to the initial vertex of $p_{i+1}$, and each $p_i$ intersects $L'$ in components of length no more than $\epsilon$ containing its initial and terminal vertices, with the rest of $p_i$ contained in some component $C'$ of the complement of $L'$.

By Lemma 4.3.10, the endpoints of each $p_i$ can be connected by an $(\overline{L'}, n_0)$-chain $\{z'_j\}$ in $C' \cap \overline{L'}$. Recall that, moreover, a path of length no more than $n_0$ between any two consecutive points in the chain is in $L$, outside of initial and final segments of length no more than $\epsilon$, and that $\overline{L'}$ deformation retracts onto $L$. Thus, each $z'_j$ is a distance of no more than $\epsilon$ from a point $z_j \in \overline{C} \cap L$ such that $\{z_j\}$ forms an

$(L, n_0 + 2\epsilon)$-chain in $\overline{C} \cap L$, connecting the endpoints of $p_i$. Concatenating these chains, we see that, for $n = n_0 + 2\epsilon$, $x$ and $y$ can be connected by an $(L, n)$-chain in $\overline{C} \cap L$ as desired.

From now on, we shall work only in $\mathscr{C}_G$, not $\mathscr{C}\mathscr{C}$.

We shall now find an $m_0 > 0$ such that there is an $(L, n)$-chain in the frontier of each essential component $C$ of the complement of $L$ that must intersect the $m_0$-ball about any given point of $L$.

Fix any $a \in L$ and $R \gg 0$. As $C$ is essential, $e(C \cup L) = 1$, and, from Lemma 4.3.5, recall that $L$ must have two ends. It follows then that $C$ must intersect both unbounded components of $L - B_R(a)$; let $x$ be in the intersection of $\overline{C} \cap L$ with one, and $y$ in the intersection of $\overline{C} \cap L$ with the other. By the work above, there exists an $(L, n)$-chain, $\{z_i\}$, from $x$ to $y$ in $\overline{C} \cap L$.

Recall that $L$ is a $(\phi, N)$ quasi-line, and let $l$ be the line associated to $L$. Then, for each $i$, there is a path in $L$ of length less than or equal to $N$ connecting $z_i$ to some $w_i \in l$. For each $i$, $d(z_i, z_{i+1}) \leq n$, thus $d(w_i, w_{i+1}) \leq n + 2N$, and thus there is a path in $l$ between any two adjacent $w_i$'s, of length less than or equal to $\phi(n + 2N)$.

Let $a_0 \in l$ be of distance less than or equal to $N$ from $a \in L$. As $R \gg 0$, $x$ and $y$ are such that there is some $i$ with $l^{-1}(w_i) \leq l^{-1}(a_0) \leq l^{-1}(w_{i\pm 1})$, and hence, for some $j$, $d(a_0, w_j) \leq \frac{1}{2}\phi(n + 2N)$. Thus

$$d(a, z_j) \leq d(a, a_0) + d(a_0, w_j) + d(w_j, z_j) \leq \frac{1}{2}\phi(n + 2N) + 2N.$$

Since $z_j \in C$, and $z_j$ is of distance less than 1 from a vertex of $C$, it follows that, for any $m_0 \geq [\frac{1}{2}\phi(n + 2N) + 2N + 1]$, $C$ intersects $B_{m_0}(a)$ in a vertex. Thus $L$ satisfies $ess(m_0)$. ■

We note that the argument above also proves the following:

**Corollary 4.3.13.** *Let $G = \langle S : R \rangle$ be a one-ended finitely presented group, with $L$ a $(\phi, N)$ quasi-line in $\mathscr{C}_G$ and $C$ a component of $\mathscr{C}_G - L$, which need not be essential. Let $m_0 = m_0(S, R, \phi, N)$ be as in Lemma 4.3.12.*

*If $K \subset L$ is such that the $2N$-neighborhood of $K$ separates $L$ into two infinite components and $C$ meets both of those components, then $B_{m_0}(x)$ meets $C$ in a vertex, for each $x \in K$.*

*Remark* 4.3.14. We note that, by Lemma 4.3.12, since all quasi-lines with which we are concerned in any one Cayley graph will have the same constants, they will all satisfy $ess(m_0)$ for some fixed $m_0$.

Next, we shall see in Proposition 4.3.18 that any 3-separating quasi-line satisfying $ess(m_0)$ and $iness(m_1)$ is a finite Hausdorff distance from an infinite cyclic subgroup of $G$. As we are going to see in Lemma 4.3.21, Lemma 4.3.22 and Remark 4.3.23, all of the quasi-lines that we are concerned with satisfy $iness(m_1)$, for some $m_1$, so Proposition 4.3.18 will apply to our setting.

Note that such a statement need not be true for quasi-lines that are not 3-separating. For example, consider the nearest-point projection of a line $l_0$ in $\mathbb{R}^2$ with irrational slope into the Cayley graph of $\mathbb{Z}^2$, where the vertices are taken to be the integer lattice points in $\mathbb{R}^2$. Let $L$ denote a large enough neighborhood in $\mathscr{C}_{\mathbb{Z}^2}$ of the projection of $l_0$ so that $L$ is connected. Then $L$ is a 2-separating quasi-line in $\mathscr{C}_{\mathbb{Z}^2}$ that satisfies $ess(m_0)$ and $iness(m_1)$ for some $m_0$ and $m_1$, and is an infinite Hausdorff distance from any subgroup of $\mathbb{Z}^2$.

In order to prove Proposition 4.3.18, we will need to know that 3-separating quasi-lines do not cross one another in an essential way. Following [21], we say that $a, b \in \mathscr{C}_G$ are *$K_0$-separated by a quasi-line $L$* if $d(a, L) > K_0$, $d(b, L) > K_0$, and $a$ and $b$ are in distinct essential components of $\mathscr{C}_G - L$. The following is Proposition

2.1 from [21]:

**Proposition 4.3.15.** [21] *Let $L$ be a 3-separating $(\phi, N)$ quasi-line contained in the Cayley graph $\mathscr{C}_G$ of a finitely presented group $G = \langle S : R\rangle$, and suppose that $L$ satisfies iness$(m_1)$. Let $L' \subset \mathscr{C}_G$ be a 2-separating $(\phi', N')$ quasi-line satisfying ess$(m_0')$ and iness$(m_1')$.*

*Then there exist $r = r(S, R, \phi, N, m_1, \phi', N', m_0', m_1')$ and $K_0 = K_0(S, R, \phi, N, m_1, \phi', N', m_0', m_1')$ such that for any two points $a, b \in L$, if $d_{\mathscr{C}_G}(a, b) > r$, then $a$ and $b$ are not $K_0$-separated by $L'$.*

In other words, if there is some $a \in L$ that is in an essential component $C$ of the complement of $L'$, $d_{\mathscr{C}_G}(a, L') > K_0$, and $b \in L$ is in a different essential component of the complement of $L'$, with $d_{\mathscr{C}_G}(a, b) > r$, then $d_{\mathscr{C}_G}(b, L') \leq K_0$.

Let $K' = \max\{K_0, m_1', r - K_0\}$, and note that, since $L'$ satisfies $iness(m_1')$, it follows that $L$ is contained in the $K'$-neighborhood of $L' \cup C$. Let $K = K' + 2m_0'$, and since $L'$ satisfies $ess(m_0')$, we also have that $L$ is contained in the $K$-neighborhood of $C$.

Thus we have the following corollary to Proposition 4.3.15:

**Corollary 4.3.16.** *Let $L$ and $L'$ be* 3*-separating quasi-lines in the Cayley graph of a finitely presented group $G = \langle S : R\rangle$ such that $L$ is a $(\phi, N)$ quasi-line satisfying iness$(m_1)$, and $L'$ is a $(\phi', N')$ quasi-line that satisfies iness$(m_1')$. Then there is some $K = K(S, R, \phi, N, m_1, \phi', N', m_1')$ such that $L$ is contained in the $K$-neighborhood of an essential component of the complement of $L'$.*

We shall also need the following lemma, both to prove Proposition 4.3.18 and also to prove another later result.

**Lemma 4.3.17.** *Let $G = \langle S : R\rangle$ be a one-ended, finitely presented group, and let*

*$\{L^i\}$ be a collection of 3-separating $(\phi, N)$ quasi-lines in $\mathscr{C}_G$ satisfying $iness(m_1)$. Suppose that $\cap_i L^i$ contains a vertex.*

*Then there is some constant $x = x(S, R, \phi, N, m_1)$ such that if, for all $i, j$, $L^i \nsubseteq N_x(L^j)$, then $\{L^i\}$ is finite.*

*Proof.* Let $m_0 = m_0(S, R, \phi, N)$ be as in Lemma 4.3.12, so that each $L^i$ satisfies $ess(m_0)$. Let $K = K(S, R, \phi, N, m_1, \phi, N, m_1)$ be as in Corollary 4.3.16, so that, for each $i, j$, $L^i$ is contained in the $K$-neighborhood of an essential component of the complement of $L^j$. Furthermore, let $m_0' = m_0(S, R, \phi, N + K)$, so that, for any $i$, $N_K(L^i)$ (which is a $(\phi, N + K)$ quasi-line) satisfies $ess(m_0')$. Then let $x > \max\{K, m_0'\}$, and assume that $L^i \nsubseteq N_x(L^j)$ for all $i$ and $j$.

Let $\mathscr{L}_0$ denote $\{L^i\}$, and suppose that $\mathscr{L}_0$ is infinite. Then choose any element $L_0$ from $\mathscr{L}_0$. As $L_0$ satisfies $ess(m_0)$, the complement of $L_0$ has only finitely many essential components, so there is some essential component $B_0$ whose $K$-neighborhood contains infinitely many elements of $\mathscr{L}_0$. Let $\mathscr{L}_1 = \{L \in [\mathscr{L}_0 - L_0] : L \subset N_K(B_0)\}$. Choose $L_1$ from $\mathscr{L}_1$, and let $B_1'$ be the essential component of the complement of $L_1$ whose $K$-neighborhood contains $L_0$. Note that $x > K$ implies that $B_1'$ is unique.

As $\mathscr{L}_1$ is infinite, there is some essential component of the complement of $L_1$ whose $K$-neighborhood contains infinitely many elements of $\mathscr{L}_1$. Let $B_1$ denote this component, and let $\mathscr{L}_2$ denote $\{L \in [\mathscr{L}_1 - \{L_0, L_1\}] : L \subset N_K(B_1)\}$. Choose $L_2$ from $\mathscr{L}_2$, and continue on in this manner. This produces an infinite sequence of quasi-lines $\{L_i\}$ and subsets of $\mathscr{C}$, $\{B_i\}$ and $\{B_i'\}$, such that, for each $i$, $B_i$ is an essential component of the complement of $L_i$ such that $L_j \subset N_K(B_i)$ for all $j > i$, and $B_i'$ is an essential component of the complement of $L_i$ such that $L_j \subset N_K(B_i')$ for all $j < i$ (with perhaps $B_i = B_i'$). Each $L_i$ is 3-separating, so we may set $D_i$ to be an essential component of the complement of $L_i$ that is not equal to $B_i$ nor $B_i'$, for each $i$.

We shall see next that the $D_i$'s are basically disjoint. Let $i \neq j$, and note that, since $L_i$ is not contained in the $x$-neighborhood of $L_j$, there must be some point $p \in L_i$ such that $B_x(p)$ does not intersect $L_j$. Thus $B_x(p)$ is contained in $B_j$ or $B'_j$.

Note that, for each $i$, $D_i - N_K(L_i)$ is a collection of essential and inessential components of the complement of $N_K(L_i)$. Since $D_i$ is an essential component of the complement of $L_i$, it is not contained in $N_{K+m'_1}(L_i)$, so $D_i - N_K(L_i)$ must contain an essential component $E_i$ of the complement of $N_K(L_i)$. As $x > m'_0$, $B_x(p)$ must meet each essential component of the complement of $N_K(L_i)$, so, in particular, $B_x(p)$ meets $E_i$, hence $B_x(p) \cup E_i$ is connected.

The quasi-line $L_j$ is disjoint from $D_i - N_K(L_i)$, hence does not meet $E_i$, or the union $B_x(p) \cup E_i$. It follows that this union is contained in $B_j$ or $B'_j$, so is disjoint from $D_j$, and hence from $E_j \subset D_j$. Thus, the $E_i$'s are disjoint.

Now we recall that $\cap_i L_i$ contains a vertex $y \in \mathscr{C}_G$, and hence $B_{m'_0}(y)$ intersects each $E_i$. Since these regions are disjoint, $B_{m'_0}(y)$ must contain a collection of vertices in bijection with $\{L_i\}$. But $G$ is finitely generated, hence $B_{m'_0}(y)$ has only finitely many vertices, and we have reached a contradiction. ∎

**Proposition 4.3.18.** *Let $L$ be a 3-separating $(\phi, N)$ quasi-line in the Cayley graph, $\mathscr{C}_G$, of a finitely presented group $G$, and suppose that $L$ satisfies iness$(m_1)$ for some $m_1$. Then there is some subgroup $H \cong \mathbb{Z}$ of $G$ such that $d_{Haus}(L, H) < \infty$.*

*Proof.* Let $L$ be as in the statement of the proposition, and recall from Lemma 4.3.12 that there is some $m_0$ such that $L$ satisfies $ess(m_0)$. Then case 1 of section 6 of [21] shows that either $L$ is a finite Hausdorff distance from an infinite cyclic subgroup of $G$, or there is a different $(\phi, N)$ quasi-line $L_1$ such that $\mathscr{C}_G - L_1$ has more essential components than $\mathscr{C}_G - L$.

Furthermore, $L_1$ is a limit of translates of $L$, in the following sense. Fix some $y \in L$, and choose a sequence $\{y_i\} \subset L$ such that $d(y, y_i) \to \infty$. Let $g_i$ be such that $g_i y_i = y$, and, by passing to a subsequence, we may assume that, for all $i > j$,

$$g_j L \cap B_j(y) = g_i L \cap B_j(y),$$

where $B_j(y)$ denotes the ball of radius $j$ about $y$. Then we can define $L_1$ to equal the set of points $p$ for which there is some $i_0 = i_0(p)$ such that, for all $i > i_0$, $p \in g_i L$.

As is argued in [21], in this manner, we may get a sequence of $(\phi, N)$ quasi-lines $L = L_0, L_1, L_2, \ldots$, all satisfying $ess(m_0)$ and $iness(m_1)$, such that no quasi-line is a finite Hausdorff distance from an infinite cyclic subgroup of $G$, and the number of essential components of their complements is strictly increasing. However, since the constants of these quasi-lines are not changing, and $G$ is finitely presented and has one end, by Lemma 4.3.12, there is an upper bound to how many essential components can be in the complement of each of these quasi-lines. Thus, this sequence must terminate, and, for some $k$, $L_k$ is a finite Hausdorff distance from a copy of $\mathbb{Z}$.

We will show now that this is not possible unless $k = 0$, i.e. that $L$ must be a finite Hausdorff distance from a copy of $\mathbb{Z}$. Assume instead that $k > 0$, so that $L_k$ is a limit of translates of $L_{k-1}$, and $L_{k-1}$ is not a bounded distance from any copy of $\mathbb{Z}$ in $\mathscr{C}_G$. Without loss of generality, we may assume that $k = 1$.

Recall the sequence $\{g_i\}$ from above. If there is some $i$ such that $d_{Haus}(g_i L, g_j L)$ is less than or equal to any fixed constant for infinitely many $g_j$, then it is shown in [21] that there is some $g$ contained in the subgroup generated by these $g_j$ such that $\langle g \rangle \cong \mathbb{Z}$, and $g_i L$ is a finite Hausdorff distance from $\langle g \rangle$. Thus $d_{Haus}(L, g_i^{-1}\langle g \rangle) < \infty$. Since $d_{Haus}(g_i^{-1}\langle g \rangle, g_i^{-1}\langle g \rangle g_i)$ is bounded by the word length of $g_i$, it follows that $L$ is a finite Hausdorff distance from $g_i^{-1}\langle g \rangle g_i \cong \mathbb{Z}$.

So, by passing to a subsequence, we may assume that, for each $i$ and $j$, $g_i L$ is not in the $x$-neighborhood of $g_jL$, and $g_jL$ is not in the $x$-neighborhood of $g_iL$, for any fixed $x$. It follows that there exists an infinite subsequence of $\{g_iL\}$ that satisfies the hypotheses of Lemma 4.3.17, which is a contradiction. ■

Recall from Definition 4.3.6 that a component $C$ of the complement of a quasi-line $L$ is said to be essential if $e(C \cup L) = 1$, and is said to be nearly essential if $C$ is contained in no finite neighborhood of $L$.

**Lemma 4.3.19.** *Let $G$ be a one-ended, finitely generated group, let $H$ be a two-ended subgroup with $R$ such that $N_R(H)$ is a quasi-line, and assume that $C$ is a component of $\mathscr{C}_G - N_R(H)$. Then $C$ is essential if and only if $C$ is nearly essential.*

*Proof.* As $N_R(H)$ has two ends, if $C$ is essential, then $C$ is nearly essential. Thus our efforts here will be to prove the converse.

Note that if $C$ is a nearly essential component of the complement of $N_R(H)$, then, for any $g \in H$, $g \cdot C$ is also a nearly essential component of the complement of $N_R(H)$.

Suppose that $C$ is not essential. Then $C \cup N_R(H)$ has more than one end, so there is a compact $K \subset (C \cup N_R(H))$ such that $(C \cup N_R(H)) - K$ has more than one infinite component.

Let $m$ denote the number of infinite components of $(C \cup N_R(H)) - K$, and suppose that $m > 2$. Since $e(G) = 1$, each of these components must meet $N_R(H)$, and as $e(N_R(H)) = 2$, the intersection of $N_R(H)$ with at least $(m-2)$ of these components must be finite. Let $M$ be the union of $K$ with these finite regions of $N_R(H)$, and note that at least $(m-2)$ components of the complement of $M$ in $C \cup N_R(H)$ do not intersect $N_R(H)$. Thus $\mathscr{C}_G - M$ has at least $m-1$ infinite components, i.e. $e(G) > 1$, a contradiction.

It follows that $e(C \cup N_R(H)) = 2$. We shall show next that we can find a finite index subgroup of $H$ that fixes $C$. Let $\langle h \rangle$ be a finite index subgroup of $H$, and suppose that the $\langle h \rangle$-orbit of $C$ contains infinitely many nearly essential components of the complement of $N_R(H)$.

Suppose, in addition, that $C$ does not meet $N_R(H)$ along its entire length, i.e. that there is some compact region $K' \subset N_R(H)$ and an infinite component $L_+$ of $N_R(H) - K'$ such that $C$ does not meet $L_+$. As $e(\mathscr{C}_G) = 1$, the intersection of $C$ with $N_R(H)$ must be infinite, so $N_R(H) - K'$ must have another infinite component, call it $L_-$, and $C$ must meet $L_-$. Moreover, for any point $q \in N_R(H)$ and any $r > 0$, $C$ must meet $L_-$ outside of $B_r(q)$.

Let $\phi$ and $N$ be constants for $N_R(H)$ and let $m_0 = m_0(S, R, \phi, N)$ be as in Lemma 4.3.12. Then, by Corollary 4.3.13, there is some $s > 0$ such that, for any point $p \in L_-$ that is of distance more than $s$ from $K'$, $C$ must meet $B_{m_0}(p)$ in a vertex. Fix such a point $p$.

As we have assumed that $\langle h \rangle \cdot C$ consists of infinitely many components, choose $\{n_i\}$ such that $\{h^{n_i} \cdot C\}$ are distinct. We can moreover choose the $\{n_i\}$ such that $L_- \subset h^{n_i} \cdot L_-$, for all $i$.

But then each $h^{n_i} \cdot C$ must meet $B_{m_0}(p)$ in a vertex. $\mathscr{C}_G$ is finitely generated, hence there are only finitely many vertices in $B_{m_0}(p)$, but the translates $h^{n_i} \cdot C$ are disjoint, thus we have reached a contradiction.

It follows that either the $\langle h \rangle$-orbit of $C$ is finite, or that, for any compact subset $K'$ of $N_R(H)$, $C$ meets both infinite components of $N_R(H) - K'$. If the latter condition holds, then Corollary 4.3.13 shows that, although $C$ need not a priori be essential, if $p$ is any point in $N_R(H)$, then $B_{m_0}(p)$ must meet $C$ in a vertex.

Similarly $B_{m_0}(p)$ must meet any translate of $C$ by an element of $H$ in a vertex. But $B_{m_0}(p)$ has only finitely many vertices, so we have reached a contradiction. Thus the $H$-orbit of $C$ must be a finite collection of nearly essential components, and in particular the $\langle h\rangle$-orbit of $C$ must also be a finite collection.

We have now seen that it will always be the case that the $\langle h\rangle$-orbit of $C$ is a finite collection of components. Thus, by passing to a finite index subgroup of $\langle h\rangle$ if necessary, we can assume that $\langle h\rangle$ fixes $C$.

Recall that we showed above that $C \cup N_R(H)$ has two ends. The subgroup $\langle h\rangle$ acts on this union by isometries, so the quotient of $C \cup N_R(H)$ by this action must be compact. It follows that $C$ is contained in the $r$-neighborhood of $N_R(H)$ for some $r > 0$, hence is not nearly essential.

Thus $C$ is essential if and only $C$ is nearly essential. ■

**Lemma 4.3.20.** *Let $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ be a quasi-isometry between the Cayley graphs of one-ended, finitely presented groups $G$ and $G'$, and let $L \subset \mathscr{C}_G$ be a quasi-line such that, for any $R \geq 0$, $N_R(L)$ satisfies iness$(m_1)$ for some $m_1 = m_1(R)$. Suppose further that, if $C$ is any component of $\mathscr{C}_G - N_R(L)$, then $C$ is essential if and only if $C$ is nearly essential.*

*If $R' \geq 0$ is such that $L' = N_{R'}(f(L))$ is a quasi-line, and $C'$ is a component of $\mathscr{C}_{G'} - L'$, then $C'$ is essential if and only if $C'$ is nearly essential.*

*Proof.* Fix $R'$ so that $L' = N_{R'}(f(L))$ is a quasi-line. Let $f^{-1}$ be a quasi-inverse to $f$, and note that, for any $R > 0$, each component of $\mathscr{C}_{G'} - L'$ gets mapped by $f^{-1}$ either into $N_R(L)$ or into the union of $N_R(L)$ with components of its complement. We claim that we may choose $R$ large enough that, if $C'$ is a component of $\mathscr{C}_{G'} - L'$ such that $f^{-1}(C')$ meets a component $C$ of $\mathscr{C}_G - N_R(L)$, then the preimage of no other component of $\mathscr{C}_{G'} - L'$ will meet $C$.

To see this, let $\Lambda, \kappa, \delta$ be such that $f^{-1}$ is a $(\Lambda, \kappa)$ quasi-isometry, with $f^{-1}(L') \subset N_\delta(L)$. Let $\{C'_\alpha\}$ be the components of $\mathscr{C}_{G'} - L'$, and let $R_1 > \Lambda\kappa$. Note that, if $\alpha \neq \beta$, and $C'_\alpha - N_{R_1}(L')$ and $C'_\beta - N_{R_1}(L')$ are nonempty, then any points $p_\alpha \in C'_\alpha - N_{R_1}(L')$, $p_\beta \in C'_\beta - N_{R_1}(L')$ are at least a distance of $2\Lambda\kappa$ apart.

Let $R > (\delta + \Lambda R_1 + \kappa)$, and note that $f^{-1}(N_{R_1}(L')) \subset N_R(L)$. Recall that $f^{-1}$ is coarsely surjective, with $N_\kappa(f^{-1}(\mathscr{C}_{G'})) = \mathscr{C}_G$. Suppose that there is some component of $\mathscr{C}_G - N_R(L)$ that is met by more than one image $f^{-1}(C'_\alpha)$. Then there are two such, call them $f^{-1}(C'_\alpha)$ and $f^{-1}(C'_\beta)$, such that, for some $p_\alpha \in C'_\alpha - N_{R_1}(L'), p_\beta \in C'_\beta - N_{R_1}(L')$, $d(f^{-1}(p_\alpha), f^{-1}(p_\beta)) < \kappa$. But this means that $\frac{1}{\Lambda}d(p_\alpha, p_\beta) - \kappa < \kappa$, i.e. that $d(p_\alpha, p_\beta) < 2\Lambda\kappa$, which is a contradiction.

Thus, with $R$ chosen as above, we have that the images under $f^{-1}$ of different components of the complement of $L'$ shall not meet the same component of the complement of $N_R(L)$.

Suppose now that $C'$ is a component of $\mathscr{C}_{G'} - L'$ that is nearly essential. Let $C_0$ be the union of the components of $\mathscr{C}_G - N_R(L)$ that are met by $f^{-1}(C')$. Since $N_R(L)$ satisfies $iness(m_1(R))$, $C_0$ must contain an essential component of the complement of $N_R(L)$. Let $C_0^e$ denote the essential components in the complement of $N_R(L)$ that are met by $f^{-1}(C')$, and now we have that $C_0^e$ is nonempty.

Observe that $C' \cup L'$ is quasi-isometric to $C_0 \cup f^{-1}(L')$, which is quasi-isometric to $C_0 \cup L$. Certainly this is quasi-isometric to $C_0 \cup N_R(L)$, which in turn must be quasi-isometric to $C_0^e \cup N_R(L)$, since $N_R(L)$ satisfies $iness(m_1(R))$.

But $N_R(L)$ satisfies $ess(m_0)$ for some $m_0$, so the union of $N_R(L)$ with any nonempty collection of essential complementary components must be one-ended. Thus $1 = e(C_0^e \cup N_R(L)) = e(C' \cup L')$, so $C'$ is essential. ∎

Using Lemma 4.3.19, we note next that quasi-lines associated with two-ended groups satisfy $iness(m_1)$ for some $m_1$.

**Lemma 4.3.21.** *Let $G$ be a one-ended finitely generated group, and let $H$ be a two-ended subgroup of $G$. For any $R > 0$ such that $N_R(H)$ is an $(\phi, N)$ quasi-line, $N_R(H)$ satisfies $iness(m_1)$, for some $m_1$ depending only on $\phi, N$, and $R$.*

*Proof.* Let $C$ be an inessential component of $\mathscr{C}_G - N_R(H)$. Then, by Lemma 4.3.19, $C$ projects onto a bounded component of $H\backslash\mathscr{C}_G - H\backslash N_R(H)$. As $H\backslash N_R(H)$ is compact and $H\backslash\mathscr{C}_G$ is locally finite, there are only finitely many components of $H\backslash\mathscr{C}_G - H\backslash N_R(H)$, thus there is some $R' > 0$ such that each bounded component is contained in the $R'$-neighborhood of the image of $H$ in $H\backslash\mathscr{C}_G$. It follows that $C$ is contained in the $R'$-neighborhood of $H$, and hence $N_R(H)$ satisfies $iness(m_1)$, for $m_1 = R' - R$. ∎

We show below that the conclusion of Lemma 4.3.21 is invariant under quasi-isometries:

**Lemma 4.3.22.** *Let $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ be a $(\Lambda, K)$ quasi-isometry between the Cayley graphs of one-ended, finitely presented groups $G$ and $G'$, and let $L \subset \mathscr{C}_G$ be a quasi-line such that, for each $R \geq 0$, there is some $m_1 = m_1(R)$ such that $N_R(L)$ satisfies $iness(m_1)$. Suppose also that, for each $N_R(L)$, a component $C$ of the complement of $N_R(L)$ is essential if and only if $C$ is nearly essential.*

*If $R' \geq 0$ is such that $L' = N_{R'}(f(L))$ is a quasi-line in $\mathscr{C}_{G'}$, then $L'$ must satisfy $iness(m_1')$, for some $m_1'$ depending on $\Lambda, K$, the values $m_1(R)$, and $R'$.*

*Proof.* Let $f^{-1}$ denote a quasi-inverse to $f$, and note that, as we saw in the proof of Lemma 4.3.20, we may choose $R$ large enough that, if $C'$ is a component of $\mathscr{C}_{G'} - L'$ such that $f^{-1}(C')$ meets a component $C$ of $\mathscr{C}_G - N_R(L)$, then the image under $f^{-1}$ of no other component of $\mathscr{C}_{G'} - L'$ will meet $C$.

As $f^{-1}$ is coarsely surjective, we have that a finite neighborhood of the image under $f^{-1}$ of any component of the complement of $L'$ is equal to a subset of $N_R(L)$, together with a collection of components of the complement of $N_R(L)$. As $N_R(L)$ satisfies $iness(m_1(R))$, it follows that there is some $m_1' > 0$ such that any component $C'$ of the complement of $L'$ is either contained in the $m_1'$-neighborhood of $L'$, or is contained in no finite neighborhood of $L'$. Thus, by Lemma 4.3.20, $C'$ must be contained in $N_{m_1'}(L')$ or else is essential, i.e. $L'$ satisfies $iness(m_1')$. ■

*Remark* 4.3.23. We note in this remark that the conclusions of the last four lemmas hold for all the quasi-lines with which we are concerned. If $N_R(H)$ is a quasi-line, then, by Lemma 4.3.19, its complementary components are essential if and only if they are nearly essential. Thus any of its translates $g \cdot N_R(H) = N_R(gH)$ shall also satisfy this statement. It is then an immediate consequence of Lemma 4.3.20 that quasi-lines that are finite neighborhoods of images of such translates, under a quasi-isometry, will also satisfy the statement.

Similarly, any such $N_R(H)$ satisfies $iness(m_1)$ for some $m_1$, and hence any translate $N_R(gH)$ also satisfies $iness(m_1)$. As for the images of such translates under a quasi-isometry, the constant $m_1'$ in Lemma 4.3.22 depends only on the constants of $L$, $L'$, the values $m_1(R)$ such that $N_R(L)$ satisfies $iness(m_1(R))$, and the quasi-isometry $f$. Recalling Remark 4.3.4, it follows that all of the quasi-lines that are derived from the images of the quasi-lines $N_R(gH)$ satisfy $iness(m_1')$ for some $m_1'$.

## 4.4 Quasi-isometry invariance of the existence of vertices of commensurizer type

We shall prove in this section that, if $G$ and $G'$ are one-ended, finitely presented, quasi-isometric groups, then $\Gamma_1(G)$ has a vertex of commensurizer type if and only if $\Gamma_1(G')$ does. We will need the following lemmas.

**Lemma 4.4.1.** *Let $G$ be a finitely generated group with two-ended subgroup $H$, and let $n < \infty$. Then $\tilde{e}(G, H) \geq n$ if and only if there is some $R > 0$ so that $N_R(H)$ is a quasi-line in $\mathscr{C}_G$ that is $n$-separating.*

*Proof.* By Lemma 4.3.2, there is some $R_0 > 0$ such that, for any $R \geq R_0$, $N_R(H)$ is a quasi-line.

Recall that a subset $X$ of $G$ represents an element in the $\mathbb{Z}/2\mathbb{Z}$-vector space $(\mathcal{P}G/\mathcal{F}_H G)^G$ if and only if $X$ is an $H$-KR almost invariant set, i.e. $\delta X$ is an $H$-finite set of edges in $\mathscr{C}_G$, i.e. $\delta X$ is contained in a finite neighborhood of $H$ in $\mathscr{C}_G$.

Essential components of the complement of any quasi-line of the form $N_R(H)$ naturally correspond to elements of $(\mathcal{P}G/\mathcal{F}_H G)^G$: let $\hat{Y}$ be an essential component of the complement of $N_R(H)$, and let $Y$ denote the vertex set of $\hat{Y}$. Then clearly for any $\epsilon > 0$, $\partial \hat{Y} \subset N_{R+\epsilon}(H)$, hence $\delta Y \subset N_{R+1}(H)$, thus $Y$ is an $H$-KR almost invariant set. Lemma 4.3.19 tells us that $Y$ must be nontrivial in $(\mathcal{P}G/\mathcal{F}_H G)^G$.

Let $N_R(H)$ be $n$-separating, and let $Y_1, \ldots, Y_n$ be essential components of the complement of $N_R(H)$. They are disjoint, hence represent independent elements of $(\mathcal{P}G/\mathcal{F}_H G)^G$, and thus $\tilde{e}(G, H) \geq n$.

If $\tilde{e}(G, H) \geq n$, then we can find representatives $X_1, \ldots, X_n$ of elements of a basis for $(\mathcal{P}G/\mathcal{F}_H G)^G$. Thus there is some $R > 0$ such that, in $\mathscr{C}_G$, $\delta X_i \subset N_R(H)$, for all $i$. Then note that each $X_i$ is equivalent in $(\mathcal{P}G/\mathcal{F}_H G)^G$ to a union of components of $\mathscr{C}_G - N_R(H)$. Recall from Lemma 4.3.21 that, for some $m_1 > 0$, $N_R(H)$ satisfies $iness(m_1)$, hence each $X_i$ is equivalent to a union of essential components of $\mathscr{C}_G - N_R(H)$. Since the $X_i$'s are independent, $n$ of these essential components must be disjoint, so the complement of $N_R(H)$ has at least $n$ distinct essential components, i.e. $N_R(H)$ is $n$-separating. ■

**Lemma 4.4.2.** *Let $f \colon \mathscr{C}_G \to \mathscr{C}_{G'}$ be a quasi-isometry between the Cayley graphs of* *one-ended, finitely presented groups $G$ and $G'$, and let $L$ be a quasi-line in $\mathscr{C}_G$ such that, for each $R \geq 0$, there is some $m_1 = m_1(R)$ such that $N_R(L)$ satisfies iness$(m_1)$. Suppose also that, for each $N_R(L)$, a component $C$ of the complement of $N_R(L)$ is essential if and only if $C$ is nearly essential.*

*Then there is some $R_0' > 0$ such that, for all $R' \geq R_0'$, $N_{R'}(f(L))$ is a quasi-line in $\mathscr{C}_{G'}$ that has at least as many essential complementary components as $L$ does. In other words, if $L$ is $n$-separating, then $N_{R'}(f(L))$ is also $n$-separating.*

*Proof.* Lemma 4.3.3 shows that we can find some $R_0'' > 0$ so that, for all $R' \geq R_0''$, $N_{R'}(f(L))$ is a quasi-line. By Lemma 4.3.22, we also have that $N_{R'}(f(L))$ satisfies *iness*$(m_1')$ for some $m_1'$ (depending on $R'$). For any such $R'$, the images of components of $\mathscr{C}_G - L$ will be contained in the union of $N_{R'}(f(L))$ and components of its complement. As in the proof of Lemma 4.3.20, there is some $R_0' \geq R_0''$ such that, for any $R' \geq R_0'$, the images of distinct components of $\mathscr{C}_G - L$ do not meet the same components of $\mathscr{C}_{G'} - N_{R'}(f(L))$. For such an $R'$, let $L' = N_{R'}(f(L))$.

Recall that, if $C$ denotes an essential component of $\mathscr{C}_G - L$, then $C$ is nearly essential, so is not contained in the $R_1$-neighborhood of $L$, for any $R_1$. By Lemma 4.3.20, the same statement is true for $L'$. Thus, as $f$ is coarsely surjective, the image of any essential component in the complement of $L$ meets an essential component in the complement of $L'$. As no two components of the complement of $L$ meet the same components of the complement of $L'$, it follows that the complement of $L'$ contains at least as many essential components as the complement of $L$. ■

We now can prove the following:

**Proposition 4.4.3.** *Let $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ be a quasi-isometry between the Cayley graphs of one-ended, finitely presented groups $G$ and $G'$, and assume that $\Gamma_1(G)$ has a* *commensurizer vertex group* $Comm_G(H)$*, for some two-ended subgroup* $H$ *of* $G$*. Then* $\Gamma_1(G')$ *has a commensurizer vertex group* $Comm_{G'}(H')$ *for some two-ended subgroup* $H'$ *of* $G'$ *such that* $d_{Haus}(f(H), H') < \infty$*.*

*Proof.* We recall from [25] and [26] that for any one-ended, finitely presented group $G_0$, $\Gamma_1(G_0)$ has a commensurizer vertex group if and only if $G_0$ contains a two-ended subgroup $H_0$ such that $\tilde{e}(G_0, H_0) \geq 4$. Moreover, this vertex group is equal to $\mathrm{Comm}_{G_0}(H_0)$.

So assume that $H$ is as in the hypothesis of this proposition. By Lemmas 4.3.2, 4.3.12 and 4.3.21, there is some $R_0 > 0$ such that, for any $R \geq R_0$, $N_R(H)$ is a $(\phi, N)$ quasi-line that satisfies $ess(m_0)$ and $iness(m_1)$, where $\phi, N, m_0$, and $m_1$ all depend on $R$. As $\tilde{e}(G, H) \geq 4$, it follows from Lemma 4.4.1 that we can further choose $R$ so that $N_R(H)$ is 4-separating. Let $L = N_R(H)$ for some such $R$.

Then, by Lemmas 4.3.3, 4.3.12, 4.3.19, 4.3.22, and 4.4.2, there is some $R'$ such that $N_{R'}(f(L))$ is a quasi-line satisfying $ess(m_0')$ and $iness(m_1')$, and $N_{R'}(f(L))$ is 4-separating. Let $L'$ denote $N_{R'}(f(L))$ for some such $R'$.

Proposition 4.3.18 implies that there is some $H' \cong \mathbb{Z}$ that is a finite Hausdorff distance from $L'$. Let $L'' = N_{R''}(H')$, with $R'' > 0$ such that $L''$ contains $L'$. Then $L''$ is 4-separating. By Lemma 4.4.1, $\tilde{e}(G', H') \geq 4$. Hence $\Gamma_1(G')$ has a commensurizer vertex group equal to $\mathrm{Comm}_{G'}(H')$, with $d_{Haus}(f(H), H') < \infty$ as required. ∎

## 4.5 Quasi-isometry invariance of the commensurizer vertex stabilizers

We have seen in the last section that the existence of commensurizer vertex groups in $\Gamma_1$ is a quasi-isometry invariant. Moreover, if $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ is a quasi-isometry between the Cayley graphs of one-ended, finitely presented groups $G$ and $G'$ with commensurizer vertex groups in their JSJ decompositions, and one such subgroup of $G$ is $\mathrm{Comm}_G(H)$, then one such subgroup of $G'$ is $\mathrm{Comm}_{G'}(H')$, with $H'$ a finite Hausdorff distance from the image of $H$ in the Cayley graph $\mathscr{C}_{G'}$.

In this section, we shall see that in fact $\mathrm{Comm}_{G'}(H')$ is a finite Hausdorff distance from the image under $f$ of $\mathrm{Comm}_G(H)$. From this, we will see that it follows that $\mathrm{Comm}_{G'}(H')$ is "small" if and only if $\mathrm{Comm}_G(H)$ is, and $\mathrm{Comm}_{G'}(H')$ is finitely generated if and only if $\mathrm{Comm}_G(H)$ is.

We shall first observe the geometric structure of commensurizers:

**Lemma 4.5.1.** *If $G$ is a finitely generated group with subgroup $H$, then*

$$\mathrm{Comm}_G(H) = \{g \in G : d_{Haus}(H, gH) < \infty\}.$$

*Proof.* Let $l(g)$ be the minimal word length of representatives for $g \in G$, with respect to the given finite generating set for $G$. Then note that, for all points $x \in \mathscr{C}_G$ and all $g \in G$, $d(x, xg) = d(e, g) = l(g)$. Thus $d_{Haus}(gH, gHg^{-1}) \leq l(g^{-1})$, so it suffices to show that $g \in Comm_G(H)$ if and only if $d_{Haus}(H, gHg^{-1}) < \infty$.

Let $H^g$ denote $gHg^{-1}$. If $d_{Haus}(H, H^g) = M < \infty$, then, for any $x \in H$, there is some $y \in H^g$ such that $d(x, y) \leq M$, i.e. $d(y^{-1}x, e) = l(y^{-1}x) \leq M$. Let $L(M) = \{k \in G : l(k) \leq M\}$. It follows that

$$H \subset \cup_{k \in L(M)} H^g k. \tag{4.1}$$

and similarly that

$$H^g \subset \cup_{k \in L(M)} Hk. \tag{4.2}$$

Observe that in fact (4.1) and (4.2) are equivalent to having $d_{Haus}(H, H^g) \leq M$.

$G$ is finitely generated, so $L(M)$ is finite, and it follows that there are finitely many elements $h_1, \ldots, h_n$ in $H$ such that

$$H \subset \cup_{i=1}^{n} H^g h_i.$$

Thus $H = \cup_{i=1}^{n}(H \cap H^g)h_i$, i.e. $(H \cap H^g)$ is of finite index in $H$. Similarly $(H \cap H^g)$ is of finite index in $H^g$, so $H$ and $H^g$ are commensurable, hence $g \in \mathrm{Comm}_G(H)$.

Conversely, if $g \in \mathrm{Comm}_G(H)$, then there are elements $h_1, \ldots, h_n$ in $H$ such that $H = \cup_{i=1}^{n}(H \cap H^g)h_i$, and elements $h_1', \ldots, h_n'$ in $H^g$ such that $H^g = \cup_{i=1}^{n'}(H \cap H^g)h_i'$. In particular, (4.1) and (4.2) hold if we take $M$ to be the maximal word length of the $h_i$'s and $(h_i')$'s. Thus $d_{Haus}(H, H^g) \leq M$, so we have shown the lemma. ■

*Remark* 4.5.2. As we saw in the proof of Proposition 4.4.3, if $\mathrm{Comm}_G(H)$ is a vertex group of $\Gamma_1(G)$ of commensurizer type, then there is some $R$ such that $N_R(H)$ is a 3-separating (actually, 4-separating) $(\phi, N)$ quasi-line satisfying $ess(m_0)$ and $iness(m_1)$, for some $\phi, N, m_0$, and $m_1$. Thus, by Lemma 4.5.1 and since $G$ acts on its Cayley graph by isometries on the left,

$$N_R(\mathrm{Comm}_G(H)) = \cup_{g \in \mathrm{Comm}_G(H)} N_R(gH) = \cup_{g \in \mathrm{Comm}_G(H)} g \cdot N_R(H)$$

is a union of isometric copies of $N_R(H)$ that are pairwise of finite Hausdorff distance from one another. Hence we may think of $\mathrm{Comm}_G(H)$ as a collection of "parallel" 3-separating $(\phi, N)$ quasi-lines that satisfy $ess(m_0)$ and $iness(m_1)$.

We can now prove the following, which is the main result of this section:

**Proposition 4.5.3.** *Let $G$ be a finitely presented, one-ended group with $\Gamma_1(G)$ containing a commensurizer vertex with vertex group equal to $C = Comm_G(H) = \coprod_i g_i H$, where $H \subset G$ has two ends. If $L$ is a 3-separating $(\phi, N)$ quasi-line in $\mathscr{C}_G$ satisfying $ess(m_0)$ and $iness(m_1)$, and such that $d_{Haus}(L, H) < \infty$, then there is some constant $x = x(\phi, N, m_0, m_1, H)$ such that, for some $i$, $d_{Haus}(L, g_i H) < x$.*

Assuming this proposition for the moment, we shall see how it implies the invariance of the commensurizer type vertex groups under quasi-isometries.

Suppose that $f\colon \mathscr{C}_{G_1} \to \mathscr{C}_{G_2}$ is a quasi-isometry between the Cayley graphs of one-ended, finitely presented groups $G_1$ and $G_2$, and that $C_1 = \mathrm{Comm}_{G_1}(H_1)$ is a commensurizer type vertex group of $\Gamma_1(G_1)$. Then, by Remark 4.5.2, we have that some neighborhood $N_R(C_1)$ of $C_1$ is a union of pairwise finite Hausdorff distance, 3-separating $(\phi, N)$ quasi-lines $\{L_i\}$, all of which satisfy $ess(m_0)$ and $iness(m_1)$, for some $m_0, m_1$.

By Proposition 4.4.3, there is a two-ended subgroup $H_2$ of $G_2$ such that $C_2 = \mathrm{Comm}_{G_2}(H_2)$ is a vertex group of $\Gamma_1(G_2)$ of commensurizer type, and $d_{Haus}(f(H_1), H_2)$ is finite. By Lemma 4.3.3, there exists $R'$ such that the $R'$-neighborhood of each $f(L_i)$ is a $(\phi', N')$ quasi-line, for some $\phi'$ and $N'$ depending on $R'$. By Lemma 4.4.2, we can choose $R'$ so that each $N_{R'}(f(L_i))$ is also 3-separating. By Remarks 4.3.14 and 4.3.23 we can further suppose that each $N_{R'}(f(L_i))$ satisfies $ess(m_0')$ and $iness(m_1')$, for some fixed constants $m_0'$ and $m_1'$. Thus we may apply Proposition 4.5.3 to get some $x$ such that each $N_{R'}(f(L_i))$ is contained in $N_x(C_2)$. It follows that $N_{R'}(f(N_R(C_1))) \subset N_x(C_2)$, i.e. that $f(C_1)$ is contained in a finite neighborhood of $C_2$.

As was the case for $C_1$, recall that a neighborhood of $C_2$ is a union of quasi-lines as above. Thus, by running the same argument on a quasi-inverse to $f$, it follows that $d_{Haus}(f(C_1), C_2) < \infty$. Hence we have the following.

**Theorem 4.5.4.** *If $f\colon \mathscr{C}_{G_1} \to \mathscr{C}_{G_2}$ is a $(\Lambda, K)$-quasi isometry between finitely presented, one-ended groups, and $\Gamma_1(G_1)$ has a vertex group $C_1 = Comm_{G_1}(H_1)$ of commensurizer type, then there is some constant $y = y(G_1, H_1, \Lambda, K)$ such that $G_2$ has a vertex group $C_2$ of commensurizer type with $d_{Haus}(f(C_1), C_2) < y$.*

*Proof of Proposition 4.5.3.* Let $\mathscr{L}$ be the set of 3-separating $(\phi, N)$ quasi-lines in $\mathscr{C}_G$ that satisfy $ess(m_0)$ and $iness(m_1)$, and are a finite Hausdorff distance from $H$. If

$\mathscr{L}$ is finite, then we are done, so assume that $\mathscr{L}$ is infinite, and that no such $x$ exists. Then we can find a sequence $\{L_i\}$ of elements of $\mathscr{L}$ such that

$$\min_{g \in C} d_{Haus}(L_i, gH) \to \infty,$$

as $i \to \infty$.

Let $c_i = g \in C$ realize the minimum above for $L_i$, and fix any $x > 0$. Then we can pass to a subsequence so that, for all $j > i$,

$$d_{Haus}(L_j, c_j H) > d_{Haus}(L_i, c_i H) + x. \tag{4.3}$$

Then, for all $g, g' \in G$ and $i \neq j$, we have

$$d_{Haus}(gL_i, g'L_j) > x, \tag{4.4}$$

by the following argument. Firstly, note that it suffices to show that $d_{Haus}(L_i, gL_j) > x$, for any $g \in G$ and $i < j$. If $g \notin C$, then $d_{Haus}(H, gH) = \infty$. But $d_{Haus}(L_i, H)$ and $d_{Haus}(gL_j, gH)$ are finite, so $d_{Haus}(L_i, gL_j) = \infty$.

Assume then that $g \in C$, and $d_{Haus}(L_i, gL_j) \leq x$. Then

$$d_{Haus}(gL_j, c_i H) \leq d_{Haus}(gL_j, L_i) + d_{Haus}(L_i, c_i H) \leq x + d_{Haus}(L_i, c_i H).$$

Thus

$$d_{Haus}(L_j, g^{-1}c_i H) = d_{Haus}(gL_j, c_i H) \leq x + d_{Haus}(L_i, c_i H).$$

But note that $d_{Haus}(L_j, c_j H) \leq d_{Haus}(L_j, g^{-1}c_i H)$ by the definition of $c_j$, so we have that

$$d_{Haus}(L_j, c_j H) \leq x + d_{Haus}(L_i, c_i H),$$

contradicting (4.3). Thus (4.4) holds for all $g \in G$.

By translating the $L_i$'s, we can obtain a new set of quasi-lines that each contain $e \in G$, and for which (4.4) holds for all $g \in G$, though the quasi-lines may no longer be a finite Hausdorff distance from $H$. This new sequence of quasi-lines satisfies the hypotheses of Lemma 4.3.17. This leads to a contradiction, since we had assumed $\mathscr{L}$ to be infinite. ∎

We recall from [25] that a vertex group $C = \mathrm{Comm}_G(H)$ of $\Gamma_1(G)$ of commensurizer type is said to be *small* if $H$ is of finite index in $C$, and otherwise, $C$ is said to be *large*. Thus $C$ is small if and only if $e(C) = 2$. Hence the following is an immediate corollary to Theorem 4.5.4.

**Corollary 4.5.5.** *If $f \colon \mathscr{C}_{G_1} \to \mathscr{C}_{G_2}$ is a quasi-isometry between the Cayley graphs of finitely presented, one-ended groups $G_1$ and $G_2$, then $\Gamma_1(G_1)$ has a vertex group of small commensurizer type if and only if $\Gamma_1(G_2)$ does, and $\Gamma_1(G_1)$ has a vertex group of large commensurizer type if and only if $\Gamma_1(G_2)$ does.*

We shall next prove that the finite or infinite generation of commensurizer vertex groups is also preserved under quasi-isometries. In order to do this, we shall use the following lemma to prove Proposition 4.5.7, which gives a coarse geometric characterization of finitely generated subgroups of finitely generated groups.

**Lemma 4.5.6.** *Let $C$ be a subgroup of a finitely generated group $G$. Then $C$ is finitely generated if and only if there exists some $A_0 > 0$ such that, for any $g, h \in C$, there is some sequence $s_0, s_1, \dots s_n \subset C$ so that $g = s_0$, $h = s_n$, and for all $i$, $d(s_i, s_{i+1}) < A_0$.*

*Proof.* Call a sequence $\{s_i\}$ as in the statement of the lemma an $A_0$*-chain from $g$ to $h$*. If $C$ is finitely generated, then fix a generating set $S_C$ for $C$, and note that the generators of $C$ have word length in $\mathscr{C}_G$ less than some constant $A_0$. For any $g, h \in C$, we can represent $g^{-1}h$ by a word $s_1 s_2 \cdots s_m$ with each $s_i$ in $S_C$, and then the sequence $e$, $s_1$, $s_1 s_2$, $\dots$, $s_1 s_2 \cdots s_m = g^{-1}h$ is a $A_0$-chain from $e$ to $g^{-1}h$, and hence $g, gs_1, gs_1s_2, \ldots, gs_1s_2\cdots s_m = h$ is a $A_0$-chain in $C$ from $g$ to $h$.

Assume now that $C$ contains a $A_0$-chain between any two of its elements, for some $A_0$, and let $S_C = C \cap B_{A_0}(e)$. Since $G$ is finitely generated, $S_C$ is finite, and we claim that $S_C$ generates $C$. Fix any $h \in C$, and let $e = s_0, s_1, \ldots s_{n-1}, s_n = h$ be a $A_0$-chain in $C$ from $e$ to $h$. Then $h = s_0(s_0^{-1}s_1)(s_1^{-1}s_2)\cdots(s_{n-2}^{-1}s_{n-1})(s_{n-1}^{-1}s_n)$, with $s_0 = e$ and $(s_i^{-1}s_{i+1})$ in $S_C$ for each $i$. Thus $S_C$ generates $C$, so we are done. ■

**Proposition 4.5.7.** *Let $f\colon \mathscr{C}_G \to \mathscr{C}_{G'}$ be a quasi-isometry, and let $C$ and $C'$ be subgroups of $G$ and $G'$ respectively, with $d_{Haus}(f(C), C') < \infty$. Then $C$ is finitely generated if and only if $C'$ is.*

*Proof.* Let $f$ be a $(\Lambda, \kappa)$-quasi-isometry. Let $n = d_{Haus}(f(C), C')$, and assume that $C$ is finitely generated. Since$f$ has a quasi-inverse, it suffices to prove that $C'$ must also be finitely generated.

Fix $g', h' \in C'$, and let $s_0, \ldots s_m$ be a sequence of vertices in $C$ such that $d(f(s_0), g') < n$, $d(f(s_m), h') < n$, and $d(s_i, s_{i+1}) < A_0$ for all $i$. Let $s_i' = f(s_i)$, and let $s_i'' \in C'$ be such that $d(s_i', s_i'') < n$. As $d(s_i', s_{i+1}') < \Lambda A_0 + \kappa$, we must have that $d(s_i'', s_{i+1}'') < \Lambda A_0 + \kappa + 2n$. Then the consecutive terms of the sequence $g', s_0'', s_1'', \ldots, s_n'', h'$ are less than $(\Lambda A_0 + \kappa + 2n)$ apart, thus by the lemma above, $C'$ is finitely generated. ■

Theorem 4.5.4 and Proposition 4.5.7 combine to prove the following corollary, showing the invariance under quasi-isometry of finitely generated, respectively infinitely generated, vertex groups of commensurizer type in $\Gamma_1$. Note that if $C$ is a vertex group of small commensurizer type, then $C$ must be finitely generated, so this result distinguishes between different types of vertex groups of large commensurizer type.

**Corollary 4.5.8.** *If $f\colon \mathscr{C}_{G_1} \to \mathscr{C}_{G_2}$ is a quasi-isometry between the Cayley graphs of* *finitely presented, one-ended groups* $G_1$ *and* $G_2$*, then* $\Gamma_1(G_1)$ *has a finitely generated vertex group of commensurizer type if and only if* $\Gamma_1(G_2)$ *does, and* $\Gamma_1(G_1)$ *has an infinitely generated vertex group of commensurizer type if and only if* $\Gamma_1(G_2)$ *does.*

# BIBLIOGRAPHY

# BIBLIOGRAPHY


[1] Mladen Bestvina. Questions in geometric group theory. <http://www.math.utah.edu/~bestvina/eprints/ questions-updated.pdf>.

[2] Mladen Bestvina and Mark Feighn. Bounding the complexity of simplicial group actions on trees. *Invent. Math.*, 103(3):449–469, 1991.

[3] B. H. Bowditch. Connectedness properties of limit sets. *Trans. Amer. Math. Soc.*, 351(9):3673–3686, 1999.

[4] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.*, 180(2):145–186, 1998.

[5] Brian H. Bowditch. Splittings of finitely generated groups over two-ended subgroups. *Trans. Amer. Math. Soc.*, 354(3):1049–1078 (electronic), 2002.

[6] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[7] Huai-Dong Cao and Xi-Ping Zhu. Hamilton-Perelman's proof of the Poincaré Conjecture and the Geometrization Conjecture. arXiv:math/0612069.

[8] Thomas Delzant and Leonid Potyagailo. Accessibilité hiérarchique des groupes de présentation finie. *Topology*, 40(3):617–629, 2001.

[9] M. J. Dunwoody. The accessibility of finitely presented groups. *Invent. Math.*, 81(3):449–457, 1985.

[10] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.*, 135(1):25–44, 1999.

[11] M. J. Dunwoody and E. L. Swenson. The algebraic torus theorem. *Invent. Math.*, 140(3):605–637, 2000.

[12] Martin J. Dunwoody. An inaccessible group. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 75–78. Cambridge Univ. Press, Cambridge, 1993.

[13] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.*, 16(1):70–125, 2006.

[14] I. A. Gruško. On the bases of a free product of groups. *Mat. Sbornik*, 8:169–182, 1940.

[15] John Hempel. 3-*Manifolds*. Princeton University Press, Princeton, N. J., 1976. Ann. of Math. Studies, No. 86.

[16] Daniel Juan-Pineda, Jean Lafont, Ivonne Ortiz, and Diane M. Vavrichek. Kleinian groups: lattice retracts, accessibility, and the Farrell-Jones isomorphism conjecture. In preparation.

[17] P. H. Kropholler. An analogue of the torus decomposition theorem for certain Poincaré duality groups. *Proc. London Math. Soc. (3)*, 60(3):503–529, 1990.

[18] P. H. Kropholler and M. A. Roller. Relative ends and duality groups. *J. Pure Appl. Algebra*, 61(2):197–210, 1989.

[19] John Morgan and Gang Tian. *Ricci flow and the Poincaré conjecture*, volume 3 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2007.

[20] A. Yu. Olʹshanskiĭ. On residualing homomorphisms and $G$-subgroups of hyperbolic groups. *Internat. J. Algebra Comput.*, 3(4):365–409, 1993.

[21] Panos Papasoglu. Quasi-isometry invariance of group splittings. *Ann. of Math. (2)*, 161(2):759–830, 2005.

[22] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.

[23] Peter Scott. The symmetry of intersection numbers in group theory. *Geom. Topol.*, 2:11–29 (electronic), 1998.

[24] Peter Scott and Gadde A. Swarup. Errata for "Regular neighbourhoods and canonical decompositions for groups". <http://www.math.lsa.umich.edu/∼pscott/regularn/errataregularn.pdf>.

[25] Peter Scott and Gadde A. Swarup. Regular neighbourhoods and canonical decompositions for groups. *Astérisque*, (289):vi+233, 2003.

[26] Peter Scott and Gadde A. Swarup. Annulus-Torus decompositions for Poincaré duality pairs. math.GR/0703890, 2007.

[27] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.

[28] Z. Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.*, 7(3):561–593, 1997.

[29] John Stallings. On fibering certain 3-manifolds. In *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, pages 95–100. Prentice-Hall, Englewood Cliffs, N.J., 1962.

[30] John R. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.

[31] Diane M. Vavrichek. Strong accessibility for hyperbolic groups. arXiv:math/0701544. To appear in *Algebraic and Geometric Topology.*

[32] C. T. C. Wall. Pairs of relative cohomological dimension one. *J. Pure Appl. Algebra*, 1(2):141–154, 1971.