MODELS OF TWISTED K-THEORY

by

José Manuel Gómez Guerra

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Doctoral Committee:

Professor Igor Kriz, Chair
Professor Mario Bonk
Associate Professor James T. Liu
Assistant Professor Valentina Joukhovitski
Assistant Professor Juan Souto

To my Mother and my Father “El burro”

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ABSTRACT

This thesis concerns geometrical models in complete generality of twistings in complex K-theory, in particular of higher twistings. In the first three chapters we treat the non-equivariant situation. In the rest of the chapters, we treat the equivariant situation with some restrictions, namely, we treat higher twistings on the Borel cohomology theory associated to equivariant K-theory. This amounts to considering the completion of equivariant K-theory with respect to the augmentation ideal. We only treat the particular case of a point.

Our model is based on the construction of a suitable classifying space for K-theory that has the structure of a semigroup with respect to two different operations that correspond to the tensor product and Whitney sum of vector bundles. We build such a space based on a category whose objects are Fredholm operators.

During the construction, there arises a problem of how to replace actions of topological semigroups with units up to homotopy on spaces, by topological group actions for groups of the same weak homotopy type. We give a positive answer to this problem under some conditions that hold in our particular application.

In the last chapter we show that when working with a compact Lie group the higher twistings of the completion of equivariant K-theory over a point vanish. This means that the only nontrivial twistings are those already known; those which correspond to twisted representations. Working in the Borel cohomology theory associated with equivariant K-theory allows us to consider the theory over general topological groups. For noncompact groups we give examples where the Borel cohomology K-theory higher twistings do not vanish. We also give examples where the corresponding analogue of the Atiyah-Hirzebruch spectral sequence has nontrivial higher differentials in arbitrarily large dimension.

CHAPTER 1

Introduction

1.1 The non-equivariant case

In general for a non-equivariant multiplicative cohomology theory E^*, where the multiplication is rigid enough (as we explain below), we can consider local coefficients or twistings. This procedure allows us to construct finer invariants out of the theory E^*. These invariants are well known and are used in many different contexts for the case of singular cohomology where they arise, for example in the Serre spectral sequence and the Poincaré isomorphism Theorem for compact manifolds without any orientability assumption.

In this work we explore the case where E^* is complex K-theory. In particular we present a model for the most general twistings of complex K-theory. We also discuss the equivariant situation but only in the case of the completion of equivariant K-theory with respect to the augmentation ideal and over a point.

In general a cohomology theory E^* is represented by a prespectrum $\{E_n\}_{n\geq 0}$; a sequence of spaces together with structural maps $E_n \to \Omega E_{n+1}$. We can assume that $\{E_n\}_{n\geq 0}$ is a spectrum after applying the spectrification functor L if necessary (see [27, Theorem 2.2]). Thus we assume that we have a sequence of spaces $\{E_n\}_{n\geq 0}$ together with homeomorphisms $E_n \to \Omega E_{n+1}$.

A multiplicative cohomology theory, such as singular cohomology with ring coefficients or K-theory, is represented by a ring spectrum. When E is an E_∞-ring spectrum then the zero space $Z = E_0$ is an E_∞-ring space. (See [32] and [34] for definitions and [34, Corollary 6.6].) If we write $Z = \coprod_{\alpha\in\pi_0(Z)} Z_\alpha$ then $Z_\otimes = GL_1 E = \coprod_{\alpha\in\pi_0(Z)^\times} Z_\alpha$, the space of units, is an infinite loop space by [34, Corollary 6.8].

The space $BZ_\otimes$ classifies the twistings for E^*. This means that for a space X and any map $f : X \to BZ_\otimes$, we have a twisting E^*_f of the theory E^* over X. (See [32, Chapter IV].) The groups $E^*_f(X)$ and $E^*_{f'}(X)$ are isomorphic through a possibly non-canonical isomorphism whenever f and f' are homotopic. In this sense we say that twistings of E^* over X are classified by the group $[X, BZ_\otimes]$. Twistings are supposed to be "bundles of E-module spectra". This is not so easy to make rigorous as it may seem. One approach to defining twistings in the most general setting is the theory of parametrized spectra of May and Sigurdsson (see [35].)

Complex K-theory is an important example of a cohomology theory. This theory is represented by the Ω-spectrum $\{K_n\}_{n\geq 0}$, where $K_n = BU \times \mathbb{Z}$ if n is even and $K_n = U$ if n is odd. In this case we have that $K_\otimes \simeq \mathbb{Z}/2 \times BU_\otimes$, where $BU_\otimes$ is the space BU with the structure of an H-space corresponding to the tensor product of vector bundles. Thus for a space X, twistings of K-theory over X are classified by homotopy classes of maps $X \to B(\mathbb{Z}/2 \times BU_\otimes) \simeq K(\mathbb{Z}/2, 1) \times BBU_\otimes$.

For the particular case of $BU_\otimes$, Segal proved in [42] that it is an infinite loop space and in [29], Madsen, Snaith and Tornehaveit proved that there is a factorization $BU_\otimes = K(\mathbb{Z}, 2) \times BSU_\otimes$ of the respective spectra. It follows that twistings in K-theory are classified by homotopy classes of maps

$$X \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BBSU_\otimes.$$

This means that for a compact space X we have twistings corresponding to elements in $H^1(X,\mathbb{Z}/2)$, $H^3(X,\mathbb{Z})$ and $[X, BBSU_{\otimes}] = bsu^1_{\otimes}(X)$.

During the past few years there has been an increasing interest in twisted K-theory because of its relations to mathematical physics and in particular with string theory. Around the 70's, Donovan and Karoubi [18], and in the 80's Rosenberg, [41] considered twisted K-theory for those twistings arising from elements of finite order in $H^3(X,\mathbb{Z})$. Around 2003 Atiyah and Hopkins studied what they called a variant of K-theory [8] which corresponds to those twistings arising from the factor $H^1(X,\mathbb{Z}/2)$. Also around 2004 Atiyah and Segal [9] introduced a model for twisted K-theory corresponding to the factors $H^1(X,\mathbb{Z}/2) \times H^3(X,\mathbb{Z})$ without any finite order assumption. Their model was based on bundles of Fredholm operators on a complex Hilbert space that is separable and of infinite dimension.

Around the same time, and continuing for the past few years, an important number of articles related to twisted K-theory have been published. For example, [10], [35], [17], [15], [16] and [44] give particular contextualizations and other models. Also [13], [46], [4] and [3] study twisted orbifold K-theory. We do not know of a reference in the literature where the most general case for twisted K-theory is discussed.

The aim of the present work is to give a model for the most general twistings of K-theory, first for the non-equivariant situation and later for the equivariant case, but only for the particular case of twistings of the completion of equivariant K-theory with respect to the augmentation ideal and over a point.

Our construction begins by considering the space of Fredholm operators $\mathcal{F}(\mathcal{H})$ for a complex separable Hilbert space $\mathcal{H}$ of infinite dimension. If we give $\mathcal{F}(\mathcal{H})$ the norm topology then we get a classifying space for complex K-theory (see [5] and [22].) On $\mathcal{F}(\mathcal{H})$ we have two H-space structures, corresponding either to the Whitney sum or the tensor product of vector bundles. These structures are better described by using $\mathcal{F}(\mathcal{H}\oplus\mathcal{H}) \simeq \mathcal{F}(\mathcal{H})$ and $\mathcal{F}((\mathcal{H}\otimes\mathcal{H})\oplus(\mathcal{H}\otimes\mathcal{H})) \simeq \mathcal{F}(\mathcal{H})$ respectively. This motivates the consideration of Fredholm operators for different Hilbert spaces.

Our starting point is the consideration of a topological category $\mathcal{C}$ whose objects are pairs $(\mathcal{H}, F)$ where $\mathcal{H}$ is a Hilbert space and F is a Fredholm operator. On this category we define two functors $\oplus$ and $\otimes$ that correspond to the Whitney sum and the tensor product of vector bundles. We show that these operations are commutative, associative and distributive up to natural isomorphisms. Thus we obtain a category satisfying all the axioms of a symmetric bimonoidal category (see [26]) except for coherent units. Generalizing a method of Maclane, (see [28, Chapter XI]), we replace this category by an equivalent category where both $\oplus$ and $\otimes$ are strictly associative and distributive on the left.

In addition, we show that the classifying space of $\mathcal{C}$ has the same homotopy type as the space of Fredholm operators $\mathcal{F}(\mathcal{H})$. This is an application of Kuiper's Theorem on the contractibility of the space of unitary operators on a Hilbert space $\mathcal{H}$ (see [25].) As a result of this we obtain a space representing complex K-theory on which we have two semigroup structures corresponding to the Whitney sum and the tensor product of vector bundles.

Using the category $\mathcal{C}$ we can find a suitable model for $K_{\otimes}$. For this we consider the full subcategory $\mathcal{D}$ consisting of pairs $(\mathcal{H}, F)$ where F has Index 1. We show that operation $\otimes$ leaves the subcategory $\mathcal{D}$ invariant. Associated to $\mathcal{D}$ is an equivalent category $\mathcal{D}^s$ where the operation $\otimes$ is strictly associative. By a considering $\mathbb{Z}/2\times B\mathcal{D}^s$ we get a suitable semigroup model of $K_{\otimes}$ acting on $B\mathcal{C}^s$.

Using this model of $K_{\otimes}$ we are able to define the most general twistings of K-theory over a space X. The motivating idea here is that given a space X and a map

$X \to BK_{\otimes}$ we want to associate to f a fibration $E_f \to X$ whose fibers are of the homotopy type of $BU \times \mathbb{Z}$.

On the other hand, the semigroup action of $B\mathcal{D}^s$ on $B\mathcal{C}^s$ leads us to the more general question of how to replace actions of topological semigroups on spaces by topological group actions for groups of the same weak homotopy type. We show in Theorem 5.1 that given a connected topological semigroup $\mathcal{M}$ satisfying some conditions we can find a topological group G, a semigroup $\mathcal{M}'$ and homomorphisms $i_1 : \mathcal{M}' \to \mathcal{M}$ and $i_2 : \mathcal{M}' \to G$ that are weak homotopy equivalences. Also we show that an action of $\mathcal{M}$ on a space X induces actions of $\mathcal{M}'$ and G on X compatible with the homomorphisms i_1 and i_2. In this sense we can replace the semigroup $\mathcal{M}$ by the group G.

1.2 The equivariant case

From the point of view of the present work we can consider similar questions equivariantly; that is, we can consider multiplicative coholomology theories with an action of a group G. The theories that we will consider will be those represented by a G-equivariant E_∞-ring spectrum E_G. For compact Lie groups G, there is an established theory of G-equivariant cohomology and E_∞-ring spectra, (see, for example, [27] and [33].) The main feature of this theory is that we can consider groups $E_G^V X$ where X is a space with G-action and V is a real G-representation; that is, the "dimensions" of equivariant cohomology are elements of the real representation ring of G. In this context what we call a twisted E_G-cohomology is much less clear (it is not only a matter of calculation, but also definition). The question is at present unanswered in general even on the philosophical level. The general feeling is that we do not wish to restrict ourselves to 1 dimensional bundles of E_G-modules, but instead want to consider GL_1E_G as a "G-equivariant group"; that is, a group with a G-action, and we want to define twistings as G-equivariant bundles with this structure group. This is more general, (for example see [9] for the case of "lower twistings" for a single point), and these groups capture the groups of virtual projective representations twisted by a given cocycle.

Also, general lower twistings are classified by the 3rd Borel cohomology group with coefficients in $\mathbb{Z}$. (See [9].) The problem with considering higher twistings in this way, however, seems to be that it depends on the selection of a model of GL_1E_G as a G-equivariant group, and the "right" choice has not been identified yet.

Because of this, we consider here a simpler scenario, namely Borel cohomology theories. For a G-equivariant cohomology theory E_G, the corresponding Borel cohomology of a G-space X is $E_G^*(EG \times X)$, where EG is a free contractible G-CW complex. In the case of equivariant K-theory, the Borel cohomology turns out, by the Atiyah-Segal completion Theorem (see [6]), to be just the corresponding equivariant K-theory completed at the augmentation ideal, so not much information is lost. Further, the advantage of G Borel cohomology theories is that G now does not have to be a compact Lie group, but can be a general topological group.

If E is a G-equivariant E_∞-ring spectrum, with G-compact Lie, then for a G-space X, the twistings of the Borel theory corresponding to E over X are then classified by $E^1_{\otimes,\mathrm{Borel},G}(X)$. In the case of a split spectrum we have a G-equivariant map $i_*E_{\otimes,e} \to E_{\otimes,G}$, where i_* is the functor adjoint to i^* induced by the inclusion map $i : U_{G-\mathrm{trivial}} \to U$ from a G-trivial universe $U_{G-\mathrm{trivial}}$ to a universe U. Here by a universe U we mean an ambient real inner product space U of countably infinite dimension such that G acts on U through isometries and U is the direct sum of its finite dimensional G-invariant sub inner product spaces. In addition, $U_{G-\mathrm{trivial}}$ is a universe where G acts trivially. (See [27] for details and definitions.) Therefore, the group $E^1_{\otimes,\mathrm{Borel},G}(X)$ is isomorphic to $E^1_{\otimes,e}(EG\times_G X)$. Without further rigidification, however, this group only produces twistings of the Borel cohomology theory associated with E_G. (See Greenless and May [21].) On the other hand, if we only consider twistings of the Borel theory, we are not restricted to the case of G compact Lie. This idea works for any topological group of the homotopy type of a CW complex. A case where the rigidification is known is for the " lower twistings" of equivariant K-theory. (See [9], [35]). Therefore one can define twisted equivariant K-theory where the twisting group is

$$H^1_{\mathrm{Borel},G}(X,\mathbb{Z}/2)\times H^3_{\mathrm{Borel},G}(X,\mathbb{Z}). \tag{1.1}$$

In this work we do not resolve the issue of rigidification of higher twistings of K-theory, and thus only work on twistings of the Borel cohomology associated to K-theory. One aspect of this is that we are not restricted to the case when G is compact Lie. On the other hand, for G-compact Lie, the Atiyah-Segal completion Theorem should extend to twisted theories, so even if the untwisted theory existed, the theory we produce should be given by its completion at the augmentation ideal of $R(G)$.

Most of these constructions generalize to the equivariant case for a compact Lie group G. In this way we can obtain an equivariant analogue $B\mathcal{C}^s_G$ of $B\mathcal{C}^s$. We show by generalizing the classical Bott periodicity Theorem that we can find a G-equivariant spectrum E_G together with an action of a topological group replacement of $K_\otimes$. We use the spectrum E_G to give a description of the general twistings for the Borel cohomology associated to equivariant K-theory. For reasons of simplicity, we work out only the case of a point here. The twistings over a point of the Borel theory associated to K-theory over a point are classified by

$$Gl_1(K)^1(BG) = [BG, BK_{\otimes}] \approx H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z}) \times bsu^1_{\otimes}(BG).$$

An interesting fact is that $bsu^1_{\otimes}(BG) = 0$ for a compact Lie group, which implies that there are no higher twistings of the Borel cohomology associated to K-theory over a point. (I thank Professor Robert Bruener for explaining to me that $k^5(BG) = 0$ for a compact Lie group, which is used to prove this result.) This is interesting as the groups in (1.1) have a geometric interpretation in the case where $X = *$ and G is a finite group acting trivially on $\mathbb{Z}$. In this case cohomology classes in $H^3(BG, \mathbb{Z})$ are in one to one correspondence with the equivalence classes of central extensions

$$1 \to \mathbb{T} \to \tilde{G} \to G \to 1.$$

(Here $\mathbb{T}$ is the multiplicative group of elements of norm 1 in $\mathbb{C}$). Thus given such a central extension representing a cohomology class $\tau \in H^3(BG, \mathbb{Z})$, $K_G^{0,\tau}(*)$ is defined as the K-group of the semigroup of τ-twisted G-representations; that is, the set of isomorphism classes of representations of $\tilde{G}$ with an action of $\mathbb{T}$ by scalar multiplication.

On the other hand, since we are not restricted to the case when G is a compact Lie group, it is easy to produce examples of noncompact groups where non-trivial higher twistings over a point do exist and the corresponding Borel K_G-groups are different depending on the twistings. In fact, the example we provide also shows that the higher differentials on the Atiyah-Hirzebruch spectral sequence are not trivial in general.

The present thesis is organized as follows. In Chapter II we give a general construction of the category $\mathcal{C}$ and compute its homotopy type. In Chapter III we give the general definition of non-equivariant twisted K-theory and show that this way we get abelian groups. In Chapter IV we extend the categorical construction to the equivariant case. In Chapter V we take a little detour to prove general theorems about replacing semigroups by topological groups. In Chapter VI we show that our construction can be used to get a spectrum representing complex K-theory. Finally in Chapter VII we give the definition of the twistings of the Borel cohomology associated to K-theory and give some computations.

CHAPTER 2

Construction

In this chapter we construct a topological groupoid $\mathcal{C}$ and show that its classifying space has the same homotopy type as $BU \times \mathbb{Z}$.

For a complex infinite dimensional separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{F}(\mathcal{H})$ the space of Fredholm operators on $\mathcal{H}$; that is, the space of bounded operators $F : \mathcal{H} \to \mathcal{H}$ with closed image and such that both $\mathrm{Ker}F$ and $\mathrm{Coker}F$ are finite dimensional. By giving $\mathcal{F}(\mathcal{H})$ the norm topology we obtain a classifying space for complex K-theory, which means that for every compact space X there is a natural isomorphism between $K^0(X)$ and $[X, \mathcal{F}(\mathcal{H})]$. (See [5, Appendix] and [22].) We review some of the important facts about these operators in the appendix.

$\mathcal{F}(\mathcal{H})$ has two structures as an H-space corresponding to both the Whitney sum and the tensor product of vector bundles. The goal of this chapter is to construct a space of the same homotopy type as $\mathcal{F}(\mathcal{H})$ together with explicit formulas for both of these H-space structures.

2.1 The Category $\mathcal{C}$

Let us fix from now on a set $\mathfrak{U}$ of complex infinite dimensional Hilbert spaces that is closed under direct sums and tensor products. Such a set obviously exists and from now on when we speak of a Hilbert space $\mathcal{H}$ we will assume that $\mathcal{H} \in \mathfrak{U}$. Two different choices of "universes" will give rise to different categories, but this is not of importance as we get equivalent constructions for any such categories.

Definition 2.1. Consider $\mathcal{C}$ the category whose objects are tuples of the form $\mathfrak{a} = (\mathcal{H}, F)$, where $\mathcal{H} \in \mathfrak{U}$ and $F : \mathcal{H} \to \mathcal{H}$ is a Fredholm operator.

For objects $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ in $\mathcal{C}$, define a morphism $(\alpha, \beta) : \mathfrak{a} \to \mathfrak{b}$ to be a pair of unitary operators $\alpha, \beta : \mathcal{H} \to \mathcal{K}$ such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{H} \arrow[r, "F"] \arrow[d, "\alpha"'] & \mathcal{H} \arrow[d, "\beta"] \\ \mathcal{K} \arrow[r, "G"] & \mathcal{K}. \end{tikzcd}$$

The composition of two morphisms $(\alpha, \beta) : \mathfrak{a} \to \mathfrak{b}$ and $(\delta, \eta) : \mathfrak{b} \to \mathfrak{c}$ is defined by

$$(\alpha, \beta) \circ (\delta, \eta) = (\delta\alpha, \eta\beta),$$

that is, the composition of morphisms has as components the composites of the respective components. This defines the category $\mathcal{C}$.

Note that $\mathcal{C}$ is a small category and that every morphism is an isomorphism.

Definition 2.2.

- A topological category $\mathcal{X}$ is a small category together with topologies on the object set $\mathcal{X}_0$ and the morphism set $\mathcal{X}_1$ in such a way that the source map s, the target map t, the composition $\circ$ and the identity or unit map u are continuous.
- A topological groupoid is a topological category in which every morphism is invertible and the inverse map i is continuous.

Now we want to give topologies to the object set $\mathcal{C}_0$ and the arrow set $\mathcal{C}_1$ so as to get a topological groupoid.

For a Hilbert space $\mathcal{H} \in \mathfrak{U}$ let

$$\mathcal{C}_0(\mathcal{H}) = \{\, \mathfrak{a} \in \mathcal{C}_0 /\ \mathfrak{a} = (\mathcal{H}, F) \text{ for a Fredholm operator } F\}\,.$$

We give each $\mathcal{C}_0(\mathcal{H})$ the norm topology. Notice that

$$\mathcal{C}_0 = \coprod_{\mathcal{H} \in \mathfrak{U}} \mathcal{C}_0(\mathcal{H})$$

and thus we topologize the set $\mathcal{C}_0$ as a disjoint union of the topological spaces $\mathcal{C}_0(\mathcal{H})$.

We consider now the arrow set $\mathcal{C}_1$. Notice that an element f of $\mathcal{C}_1$ consists of the following data:

- the source of f, which is an object $s(f) = (\mathcal{H}_f, F_f)$ of $\mathcal{C}$,
- the target, which is an object $t(f) = (\mathcal{K}_f, G_f)$,
- two unitary operators $\alpha_f : \mathcal{H}_f \to \mathcal{K}_f$ and $\beta_f : \mathcal{H}_f \to \mathcal{K}_f$.

The data for f is uniquely determined by the source of f and the unitary operators α_f and β_f. Thus we have a bijection

$$\mathcal{C}_1 \overset{\approx}{\to} \coprod_{\mathcal{H},\mathcal{K}} \coprod_{\alpha,\beta} \mathcal{F}(\mathcal{H}) \times \mathcal{U}(\mathcal{H},\mathcal{K}) \times \mathcal{U}(\mathcal{H},\mathcal{K}).$$

Here $\mathcal{U}(\mathcal{H},\mathcal{K})$ is the set of unitary operators $\mathcal{H} \to \mathcal{K}$. Thus if we give the sets $\mathcal{F}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H},\mathcal{K})$ the norm topology then we can give $\mathcal{C}_1$ a topology via this bijection. It is easy to check that the structural maps are continuous and thus $\mathcal{C}$ is a topological category. Moreover, as the inverse function for arrows is continuous, $\mathcal{C}$ is a topological groupoid.

We will now construct continuous functors $\oplus, \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that will correspond to the Whitney sum and the tensor product of vector bundles.

Definition 2.3. Suppose $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ are in $\mathcal{C}_0$. Then we define

$$\mathfrak{a} \oplus \mathfrak{b} = (\mathcal{H} \oplus \mathcal{K}, F \oplus G).$$

Suppose that $(\alpha, \beta) : \mathfrak{a} \to \mathfrak{b}$ and $(\delta, \eta) : \mathfrak{c} \to \mathfrak{d}$ are two morphisms in $\mathcal{C}$ and define

$$(\alpha, \beta) \oplus (\delta, \eta) = (\alpha \oplus \delta, \beta \oplus \eta).$$

The functor $\oplus$ is well defined, as if F and G are Fredholm operators then so is $F \oplus G$. Similarly, if α and β are unitary operators then so is $\alpha \oplus \beta$. The fact that $\oplus$ is continuous follows easily from our definition of the topologies on $\mathcal{C}_0$ and $\mathcal{C}_1$.

Definition 2.4. Suppose that $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ are in $\mathcal{C}_0$. Define

$$\mathfrak{a} \otimes \mathfrak{b} := (\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P_{F,G}),$$

where

$$P_{F,G} : \mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}$$

is given by the matrix

$$\begin{bmatrix} -I \otimes G & F^* \otimes I \\ F \otimes I & I \otimes G^* \end{bmatrix}.$$

If $(\alpha, \beta) : \mathfrak{a} \to \mathfrak{b}$ and $(\delta, \eta) : \mathfrak{c} \to \mathfrak{d}$ are two morphisms in $\mathcal{C}$, define

$$(\alpha, \beta) \otimes (\delta, \eta) = (\alpha \otimes \delta \oplus \beta \otimes \eta, \alpha \otimes \eta \oplus \beta \otimes \delta).$$

To see that $\otimes$ is well defined we need to show that $P_{F,G}$ is a Fredholm operator whenever F, G are also Fredholm. In [22] Janich showed that this is the case by a direct computation. In the appendix we present a different approach to this. By a direct computation it follows at once that if α, β, δ and η are unitary operators then so are

$$\alpha \otimes \delta \oplus \beta \otimes \eta, \ \alpha \otimes \eta \oplus \beta \otimes \delta.$$

On the other hand it is clear from the definition that the functor $\otimes$ is continuous.

Remark 2.5. Atiyah and Janich (see [5, Appendix] and [22]) proved that for every compact space X there is a natural isomorphism $I : K^0(X) \xrightarrow{\approx} [X, \mathcal{F}(\mathcal{H})]$. By considering an isomorphism between $\mathcal{H} \oplus \mathcal{H}$ and $\mathcal{H}$, we can see the operation $\oplus$ as giving the structure of an H-space to $\mathcal{F}(\mathcal{H})$. Similarly, by considering an isomorphism between $\mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H}$ and $\mathcal{H}$, we can see the operation $\otimes$ as giving an additional structure of an H-space to $\mathcal{F}(\mathcal{H})$. In [22], Janich showed that these operations correspond to the Whitney sum and the tensor product of vector bundles and that the isomorphism I gives an isomorphism of rings.

Such explicit descriptions of the sum and product operations in K^0 have been in long use for computational purposes in K-theory. For example, in her Ph.D. thesis [39], Marta B. Pecuch used these formulas to obtain formulas for the Dyer-Lashof cohomology operations for $(BU \times \mathbb{Z}, \otimes)$.

In this work we will use these formulas to obtain a description of the most general twistings in twisted K-theory.

We discuss now the properties of the operations $\oplus$ and $\otimes$. We claim that these functors are associative, commutative and satisfy the distributive property up coherent natural isomorphisms; that is, we claim the existence of natural transformations

$$\begin{aligned}
\gamma^{\oplus}_{\mathfrak{a},\mathfrak{b}} &: \mathfrak{a} \oplus \mathfrak{b} \xrightarrow{\approx} \mathfrak{b} \oplus \mathfrak{a} \\
\gamma^{\otimes}_{\mathfrak{a},\mathfrak{b}} &: \mathfrak{a} \otimes \mathfrak{b} \xrightarrow{\approx} \mathfrak{b} \otimes \mathfrak{a} \\
\eta_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} &: (\mathfrak{a} \oplus \mathfrak{b}) \oplus \mathfrak{c} \xrightarrow{\approx} \mathfrak{a} \oplus (\mathfrak{b} \oplus \mathfrak{c}) \\
\iota_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} &: (\mathfrak{a} \otimes \mathfrak{b}) \otimes \mathfrak{c} \xrightarrow{\approx} \mathfrak{a} \otimes (\mathfrak{b} \otimes \mathfrak{c}) \\
\delta^{l}_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} &: (\mathfrak{a} \oplus \mathfrak{b}) \otimes \mathfrak{c} \xrightarrow{\approx} (\mathfrak{a} \otimes \mathfrak{c}) \oplus (\mathfrak{b} \otimes \mathfrak{c}) \\
\delta^{r}_{\mathfrak{a},\mathfrak{b},\mathfrak{c}} &: \mathfrak{a} \otimes (\mathfrak{b} \oplus \mathfrak{c}) \xrightarrow{\approx} (\mathfrak{a} \otimes \mathfrak{b}) \oplus (\mathfrak{a} \otimes \mathfrak{c})
\end{aligned}$$

for $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathcal{C}_0$ that satisfy coherences similar to those of a bisymmetric monoidal category, except for those involving units (see [26].)

We will show in detail how to construct the natural map $\gamma^{\oplus}$. The other maps are handled in a similar way. In addition, in the appendix we include a method for dealing with those transformations involving the operation $\otimes$.

To begin, note that if $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G) \in \mathcal{C}_0$, then by definition

$$\mathfrak{a} \oplus \mathfrak{b} = (\mathcal{H} \oplus \mathcal{K}, F \oplus G) \text{ and } \mathfrak{b} \oplus \mathfrak{a} = (\mathcal{K} \oplus \mathcal{H}, G \oplus F).$$

Consider $\tau : \mathcal{H} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}$ the natural isomorphism of Hilbert spaces that switches the factors. τ is a unitary operator and we define

$$\gamma^{\oplus}_{\mathfrak{a},\mathfrak{b}} = (\tau, \tau) : \mathfrak{a} \oplus \mathfrak{b} \to \mathfrak{b} \oplus \mathfrak{a}.$$

We need to show that this is a well defined morphism in $\mathcal{C}$. This amounts to showing that the following diagram is commutative,

$$\begin{tikzcd}
\mathcal{H} \oplus \mathcal{K} \arrow[r, "F \oplus G"] \arrow[d, "\tau"'] & \mathcal{H} \oplus \mathcal{K} \arrow[d, "\tau"] \\
\mathcal{K} \oplus \mathcal{H} \arrow[r, "G \oplus F"'] & \mathcal{K} \otimes \mathcal{H}
\end{tikzcd}$$

which is straight forward from the definition. The fact that $\gamma^{\oplus}$ is a natural transformation follows from the fact that τ is a natural transformation in the category of Hilbert spaces and linear operators.

In general, the way to construct the transformations $\gamma^{\otimes}_{\mathfrak{a},\mathfrak{b}}$, $\eta_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$, $\iota_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$, $\delta^{l}_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ and $\delta^{r}_{\mathfrak{a},\mathfrak{b},\mathfrak{c}}$ is to choose the natural isomorphism between the underlying Hilbert spaces (which exists because the tensor product and direct sum of Hilbert spaces are commutative and associative up to natural isomorphisms which are unitary operators). By a direct but lengthly computation one shows that these natural transformations define morphisms in the category $\mathcal{C}$. In the appendix we show a simple way to do this. To see that in fact these natural transformations satisfy the required coherences, we only need to note that the composition of two morphisms in $\mathcal{C}$ is the composition of the respective components. Thus the problem of checking the coherences is reduced to the case of Hilbert spaces and unitary operations. It is well known that the category whose objects are Hilbert spaces and whose morphisms are unitary operators forms a symmetric bimonoidal category. Thus if we restrict to the case of Hilbert spaces that are separable and of infinite dimension we obtain our required coherences.

2.2 The homotopy type of $B\mathcal{C}$

In this section we show that the space $B\mathcal{C}$ has the same homotopy type as $BU \times \mathbb{Z}$. To do so, we consider any Hilbert space $\mathcal{H} \in \mathfrak{U}$ and show that the map $i : \mathcal{F}(\mathcal{H}) \to B\mathcal{C}$ given by $F \mapsto [(\mathcal{H}, F)]$ is a homotopy equivalence. This is done in the following theorem.

Theorem 2.6. *If we see $\mathcal{C}$ as a topological category then the map $i : \mathcal{F}(\mathcal{H}) \to B\mathcal{C}$ is a homotopy equivalence. Moreover, $B\mathcal{C}$ is compactly generated Hausdorff.*

Proof. Let us fix a Hilbert space $\mathcal{H}_o \in \mathfrak{U}$ and denote by $\mathcal{C}(\mathcal{H}_o)$ the full subcategory of $\mathcal{C}$ whose objects are tuples of the form $(\mathcal{H}_o, F) \in \mathcal{C}_0$. For each $\mathcal{H} \in \mathfrak{U}$ choose a unitary operator $\alpha_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}_o$ in such a way that $\alpha_{\mathcal{H}_o}$ is the identity. As a first step we will show that $B\mathcal{C} \simeq B\mathcal{C}(\mathcal{H}_o)$. In fact we will show that $B\mathcal{C}(\mathcal{H}_o)$ is a deformation retract of $B\mathcal{C}$. To show this it will be enough to show that $\mathcal{C}$ and $\mathcal{C}(\mathcal{H}_o)$ are equivalent categories under continuous functors $\mathcal{P} : \mathcal{C} \to \mathcal{C}(\mathcal{H}_o)$ and $j : \mathcal{C}(\mathcal{H}_o) \to \mathcal{C}$, where j is the inclusion functor.

Let us define the functor $\mathcal{P}$. For $\mathfrak{a} = (\mathcal{H}, F)$ in $\mathcal{C}_0$ define

$$\mathcal{P}(\mathfrak{a}) = (\mathcal{H}_o, \alpha_H F \alpha_{\mathcal{H}}^{-1}).$$

$\mathcal{P}(\mathfrak{a})$ is defined so that the pair $(\alpha_{\mathcal{H}}, \alpha_{\mathcal{H}})$ is a morphism $\mathfrak{a} \to \mathcal{P}(\mathfrak{a})$.

If $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ are in $\mathcal{C}_0$ and $(\beta, \gamma) : \mathfrak{a} \to \mathfrak{b}$ is a morphism define

$$\mathcal{P}(\beta, \gamma) : \mathcal{P}(\mathfrak{a}) \to \mathcal{P}(\mathfrak{b})$$

to be the morphism given by

$$\mathcal{P}(\beta, \gamma) = (\alpha_{\mathcal{K}} \beta \alpha_{\mathcal{H}}^{-1}, \alpha_{\mathcal{K}} \gamma \alpha_{\mathcal{H}}^{-1}).$$

Defined this way we obtain a functor $\mathcal{P}$ that is easily seen to be continuous.

On the other hand, let us denote by $j : \mathcal{C}(\mathcal{H}_o) \to \mathcal{C}$ the inclusion functor. Since we chose $\alpha_{\mathcal{H}_o}$ to be the identity it follows that $\mathcal{P} \circ j = Id$. Thus it only remains to construct a natural transformation $\eta : Id \to j \circ \mathcal{P}$. To define η, take $\mathfrak{a} = (\mathcal{H}, F)$ to be an object of $\mathcal{C}$. Define $\eta_{\mathfrak{a}} : \mathfrak{a} \to \mathcal{P}(\mathfrak{a}) = j \circ \mathcal{P}(\mathfrak{a})$ by $\eta_{\mathfrak{a}} = (\alpha_{\mathcal{H}}, \alpha_{\mathcal{H}})$. Let us see that this in fact defines a natural transformation.

Indeed, take $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ objects in $\mathcal{C}$ and $(\beta, \gamma) : \mathfrak{a} \to \mathfrak{b}$ a morphism. We need to check that the following diagram commutes

$$\begin{tikzcd}
\mathfrak{a} \arrow[r, "\eta_{\mathfrak{a}}"] \arrow[d, "(\beta,\gamma)"'] & \mathcal{P}(\mathfrak{a}) \arrow[d, "\mathcal{P}(\beta,\gamma)"] \\
\mathfrak{b} \arrow[r, "\eta_{\mathfrak{b}}"'] & \mathcal{P}(\mathfrak{b}).
\end{tikzcd}$$

By definition we have that

$$\eta_{\mathfrak{b}} \circ (\beta, \gamma) = (\alpha_{\mathcal{K}}, \alpha_{\mathcal{H}}) \circ (\beta, \gamma) = (\alpha_{\mathcal{K}} \beta, \alpha_{\mathcal{H}} \gamma),$$

$$\mathcal{P}(\beta, \gamma) \circ \eta_{\mathfrak{a}} = (\alpha_{\mathcal{K}} \beta \alpha_{\mathcal{H}}^{-1}, \alpha_{\mathcal{K}} \gamma \alpha_{H}^{-1}) \circ (\alpha_{\mathcal{H}}, \alpha_{\mathcal{H}}) = (\alpha_{\mathcal{K}} \beta, \alpha_{\mathcal{K}} \gamma).$$

We conclude that the transformation $\eta : Id \to j \circ \mathcal{P}$ is natural. This proves that $B\mathcal{C} \simeq B\mathcal{C}(H_o)$. In fact it shows that $B\mathcal{C}(\mathcal{H}_o)$ is a deformation retract of $B\mathcal{C}$.

As our next step we will show that $B\mathcal{C}(\mathcal{H}_o)$ has the same homotopy type as $\mathcal{F}(\mathcal{H}_o)$ by showing that the inclusion map $i : \mathcal{F}(\mathcal{H}_o) \to B\mathcal{C}(\mathcal{H}_o)$ is a homotopy equivalence. This will prove the proposition.

From the space $\mathcal{F}(\mathcal{H}_o)$ we can construct a topological category by just considering as objects the Fredholm operators $F : \mathcal{H}_o \to \mathcal{H}_o$ and as the only morphisms the identity morphisms. Let us denote by $\mathfrak{F}(\mathcal{H}_o)$ to this associated category, which is a subcategory of $\mathcal{C}(\mathcal{H}_o)$. Since there are no non-identity morphisms in $\mathfrak{F}(\mathcal{H}_o)$ we have that $B\mathfrak{F}(\mathcal{H}_o) = \mathcal{F}(\mathcal{H}_o)$.

Let $i : \mathfrak{F}(\mathcal{C}(\mathcal{H}_o)) \to \mathcal{C}(\mathcal{H}_o)$ be the inclusion functor. Consider the nerves $X_* = N_*(\mathfrak{F}(\mathcal{H}_o))$ and $X'_* = N_*(\mathcal{C}(\mathcal{H}_o))$ of the categories $\mathfrak{F}(\mathcal{H}_o)$ and $\mathcal{C}(\mathcal{H}_o)$ respectively. These are simplicial spaces, as we regard $\mathfrak{F}(\mathcal{H}_o)$ and $\mathcal{C}(\mathcal{H}_o)$ as topological categories.

The functor i induces maps $i_n : X_n \to X'_n$. We show now that these maps are homotopy equivalences.

For $n = 0$ we have that $i_0 : X_0 \to X'_0$ is the identity map. Thus we can assume that $n > 0$. By Kuiper's Theorem (see [25]), we have that $\mathcal{U}(\mathcal{H}_o)$ with norm topology is contractible, and thus we can find a null homotopy $h : \mathcal{U}(\mathcal{H}_o) \times [0,1] \to \mathcal{U}(\mathcal{H}_o)$ from the identity to the constant map

$$\begin{aligned} c_{Id_{\mathcal{H}_o}} : \mathcal{U}(\mathcal{H}_o) &\to \mathcal{U}(\mathcal{H}_o) \\ \alpha &\mapsto Id_{\mathcal{H}_o}. \end{aligned}$$

Moreover, we can assume that h is such that $h(Id, t) = Id$ for all $0 \leq t \leq 1$. To see this take $h' : \mathcal{U}(\mathcal{H}_o) \times [0,1] \to \mathcal{U}(\mathcal{H}_o)$ any homotopy from the identity to the constant map $c_{Id_{\mathcal{H}_o}}$. Let $f : [0,1] \to \mathcal{U}(\mathcal{H}_o)$ be defined by $f(t) = h(Id, t)$. Then f is a path in $\mathcal{U}(\mathcal{H}_o)$ from Id to itself. We can define

$$h(\alpha, t) = h'(\alpha, t) \circ f(t)^{-1}.$$

h is clearly continuous and satisfies

$$h(\alpha, 0) = h'(\alpha, 0) \circ f(0)^{-1} = \alpha$$

$$h(\alpha, 1) = h'(\alpha, 1) \circ f(1)^{-1} = Id$$

$$h(Id, t) = h'(I, t) \circ f(t)^{-1} = f(t) \circ f(t)^{-1} = Id.$$

Hence h satisfies the additional condition.

To define $j_n : X'_n \to X_n$ note that an element $x \in X'_n$ is a sequence of n composable morphisms in $\mathcal{C}(H_o)$. Thus $x = (f_1, ..., f_n)$, where $f_i : \mathfrak{a}_i \to \mathfrak{b}_i$ with $\mathfrak{b}_i = \mathfrak{a}_{i+1}$ for $1 \leq i \leq n-1$. Write

$$\mathfrak{a}_i = (\mathcal{H}_o, F_{f_i}) \text{ and } \mathfrak{b}_i = (\mathcal{H}_o, G_{f_i})$$

and $f_i = (\alpha_i, \beta_i)$, where α_i, β_i are unitary operators on $\mathcal{H}_o$. Define

$$j_n(x) = j_n(f_1, ..., f_n) = (Id_{\mathfrak{a}_1}, ..., Id_{\mathfrak{a}_1}).$$

The claim is that j_n is a homotopy inverse of i_n.

To see this notice that by definition $i_n \circ j_n = Id$. On the other hand,

$$j_n \circ i_n(f_1, ..., f_n) = (Id_{\mathfrak{a}_1}, ..., Id_{a_1}).$$

Define

$$\begin{aligned}
g_1(x_n, t) &= (h(\alpha_1, t), h(\beta_1, t)) : \\
&\quad (\mathcal{H}_o, F_{f_1}) \to (\mathcal{H}_o, h(\beta_1, t) \circ F_{f_1} \circ h(\alpha_1, t)^{-1}), \\
g_i(x_n, t) &= (h(\alpha_1 ... \alpha_{i-1}, t)^{-1} h(\alpha_1 ... \alpha_i, t), h(\beta_1 ... \beta_{i-1}, t)^{-1} h(\beta_1 ... \beta_i, t)) : \\
&\quad (\mathcal{H}_o, h(\beta_1 ... \beta_{i-1}, t) \circ F_{f_1} \circ h(\alpha_1 ... \alpha_{i-1}, t)^{-1}) \\
&\to (\mathcal{H}_o, h(\beta_1 ... \beta_i, t) \circ F_{f_1} \circ h(\alpha_1 ... \alpha_i, t)^{-1})
\end{aligned}$$

for $2 \leq i \leq n$. It follows that $s(g_{i+1}(x_n,t)) = t(g_i(x_n,t))$ for all $t \in [0,1]$, and thus $(g_1(x_n,t),...,g_n(x_n,t)) \in X'_n$ for all t. We also have $g_i(x_n,0) = (\alpha_i,\beta_i) = f_i$ and $g_i(x_n,1) = (Id,Id) = Id_{\mathfrak{a}_1}$. Hence if we define $k : X'_n \times [0,1] \to X'_n$ by

$$k(x,t) = (g_1(x,t),...,g_n(x,t)),$$

we have that k is a continuous map such that

$$k(x,0) = (g_1(x,0),...,g_n(x,0)) = x,$$
$$k(x,1) = (g_1(x,1),...,g_n(x,1)) = (Id_{\mathfrak{a}_1},...,Id_{\mathfrak{a}_1})$$

that is, k is a homotopy from Id to $j_n \circ i_n$.

What we have shown is that we have two simplicial spaces X_*, X'_* and a simplicial map $i_* : X_* \to X'_*$ such that for all $n \geq 0$ $i_n : X_n \to X'_n$ is a homotopy equivalence. In the lemma below we show that X, X' are proper simplicial spaces (see definition below) and thus by [31, Theorem A.4], it follows that the map induced on the classifying spaces $|i| : |X| = \mathcal{F}(\mathcal{H}_o) \to |X'| = B\mathcal{C}(\mathcal{H}_o)$ is a homotopy equivalence.

Moreover we can conclude something stronger. If we denote by $F_k|X'|$ the image of $\coprod_{0\leq n\leq k}(X'_n \times \Delta_n)$ in $|X'|$, then the $F_k|X'|$'s form an increasing filtration for $|X'|$ and each $(F_{k+1}|X'|, F_k|X'|)$ is an NDR pair by [30, Lemma 11.3]. By [43, Theorem 9.4] it follows that $(|X'|, F_0|X'|)$ is an NDR pair. But we just showed that the inclusion $i_0 : F_0|X'| = \mathcal{F}(\mathcal{H}_o) \to |X'|$ is a homotopy equivalence. Hence by [14, Corollary 7.4.1] we get that $\mathcal{F}(\mathcal{H}_o) \subset |X'| = B\mathcal{C}(\mathcal{H}_o)$ is a deformation retract.

The fact that $B\mathcal{C}$ is compactly generated and Hausdorff follows from [43, Theorems 9.2 and 9.4]. $\square$

Let us recall now May's definition of properness for a simplicial space X.

Definition 2.7. Let X be a simplicial space X. For $q \geq 0$ let

$$sX_q = \bigcup_{0 \leq j \leq q} s_j X_q \subset X_{q+1}$$

where the $s_j : X_q \to X_{q+1}$ are the degeneracy maps. Then X is said to be a proper simplicial set if for each q, (X_{q+1}, sX_q) is a strong NDR pair.

Lemma 2.8. *The simplicial spaces X, X' of the previous theorem are proper.*

Proof. The simplicial space X is trivially proper, as for every $q \geq 0$ and every $0 \leq j \leq q$ we have that $s_j X_q = X_{q+1}$.

Let us show now that X' is proper. Fix $q \geq 0$ and for $0 \leq j \leq q$ define $A_j = s_{q-j} X'_q$. We are going to show that (X'_{q+1}, A_j) is an NDR pair represented by a pair (h_j, u_j) where $h_j(A_i \times [0,1]) \subset A_j$ for $i < j$ and $u_j(x) < 1$ for every $x \in A_j$. If this is true, then it follows by [30, Lemma A.6] that $(X'_{q+1}, \bigcup_{0 \leq j \leq q} A_j) = (X'_{q+1}, sX'_q)$ is an NDR pair represented by some pair (j, v) as in the cited lemma where the map $v : X'_{q+1} \to [0,1]$ is given by $v(x) = \min(u_1, ..., u_n)$. It follows that $v(x) < 1$ for all $x \in X_{q+1}$, as the same is true for each u_j. Thus each (X'_{q+1}, sX'_q) is a strong NDR pair, and therefore X' is proper.

Take a homotopy $h : \mathcal{U}(\mathcal{H}_o) \times [0,1] \to \mathcal{U}(\mathcal{H}_o)$ rel $\{Id_{\mathcal{H}_o}\}$ from the identity to the constant map $c_{Id_{\mathcal{H}_o}}$ as in the previous theorem. Let $u : \mathcal{U}(\mathcal{H}_o) \to [0, \frac{1}{2})$ be a continuous map such that $u^{-1}(0) = \{Id_{\mathcal{H}_o}\}$. Such a function can be always be found as $\mathcal{U}(\mathcal{H}_o)$ is a metrizable space. Take $x_{q+1} = (f_1, ..., f_{q+1}) \in X'_{q+1}$ a sequence of morphisms in $\mathcal{C}(\mathcal{H}_o)$ such that $t(f_i) = s(f_{i+1})$ for $1 \leq i \leq n-1$. Then $f_{q-j+1} = (\alpha_{q-j+1}, \beta_{q-j+1})$ for some unitary operators $\alpha_{q-j+1}, \beta_{q-j+1}$. Define

$$u_j(x_{q+1}) = u(\alpha_{q-j+1}) + u(\beta_{q-j+1}).$$

Defined this way $u_j : X'_{q+1} \to [0,1]$ is a continuous map and $u_j(x_{q+1}) < 1$ for every $x_{q+1} \in X_{q+1}$ as $u(\alpha) < \frac{1}{2}$ for every $\alpha \in \mathcal{U}(\mathcal{H}_o)$. Also $u_j(x_{q+1}) = 0$ if and only if

$u(\alpha_{q-i+1}) = u(\beta_{q-j+1}) = 0$, which occurs if and only if $\alpha_{q-j+1} = \beta_{q-j+1} = I_{\mathcal{H}_o}$; that is, $f_{q-j+1} = Id_{t(f_{q-j})}$. We conclude that $u_j^{-1}(0) = A_j$.

Let us now define $h_j : X'_{q+1} \times [0,1] \to X'_{q+1}$. Take $x_{q+1} = (f_1, ..., f_{q+1}) \in X'_{q+1}$ composable morphisms in $\mathcal{C}(\mathcal{H}_o)$, and write $s(f_i) = (\mathcal{H}_o, F_{f_i})$, $t(f_i) = (\mathcal{H}_o, G_{f_i})$. Then

$$f_i = (\alpha_i, \beta_i) : (\mathcal{H}_o, F_{f_i}) \to (\mathcal{H}_o, G_{f_i}).$$

Define

$$\begin{aligned} g_{q-j+1}(x_{q+1}, t) = (h(\alpha_{q-j+1}, t), h(\beta_{q-j+1}, t)) : \\ (H_o, F_{f_{q-j+1}}) \to (H_o, h(\beta_{q-j+1}, t) F_{f_{q-j+1}} h(\alpha_{q-j+1}, t)^{-1}) \end{aligned}$$

and for $i > q-j+1$

$$\begin{aligned} g_i(x_{q+1}, t) &= (\alpha_i, \beta_i) : \\ &(\mathcal{H}_o, \beta_{i-1} ... \beta_{q-j+2} h(\beta_{q-j+1}, t) F_{f_{q-j+1}} h(\alpha_{q-j+1}, t)^{-1} \alpha_{q-j+2}^{-1} ... \alpha_{i-1}^{-1}) \\ &\to (\mathcal{H}_o, \beta_i ... \beta_{q-j+2} h(\beta_{q-j+1}, t) F_{f_{q-j+1}} h(\alpha_{q-j+1}, t)^{-1} \alpha_{q-j+2}^{-1} ... \alpha_i^{-1}). \end{aligned}$$

From this definition, it follows that

$$s(g_{q-j+1}(x_{q+1}, t)) = t(f_{q-j})$$

$$s(g_{i+1}(x_{q+1}, t)) = t(g_i(x_{q+1}, t))$$

for all $t \in [0,1]$ and all $q - j + 1 \leq i \leq q+1$, and thus

$$(f_1, ..., f_{q-j}, g_{q-j+1}(x_{q+1}, t), ..., g_{q+1}(x_{q+1}, t)) \in X'_{q+1} \text{for all } t \in [0,1].$$

We have that $g_i(x_{q+1}, 0) = (\alpha_i, \beta_i) = f_i$ for $i \geq q - j + 1$ and $g_{q-j+1}(x_{q+1}, 1) = (h(\alpha_{q-j+1}, 1), h(\beta_{q-j+1}, 1)) = (Id_{\mathcal{H}_o}, Id_{\mathcal{H}_o})$, so

$$(f_1, ..., f_{q-j}, g_{q-j+1}(x_{q+1}, 1), ..., g_{q+1}(x_{q+1}, 1)) \in s_{q-j} X'_q = A_j.$$

Hence if we define $h_j : X'_{q+1} \times [0,1] \to X_{q+1}$ for $x_{q+1} = (f_1, ..., f_{q+1})$ by

$$h_j(x_{q+1}, t) = (f_1, ..., f_{q-j}, g_{q-j+1}(x_{q+1}, t), ..., g_{q+1}(x_{q+1}, t))$$

we have that h_j is a continuous map such that

$$h_j(x_{q+1}, 0) = (f_1, ..., f_{q-j}, g_{q-j+1}(x_{q+1}, 0), ..., g_{q+1}(x_{q+1}, 0)) = (f_1, ..., f_n) = x_{q+1}$$

and

$$h_j(x_{q+1}, 1) = (f_1, ..., f_{q-j}, g_{q-j+1}(x_{q+1}, 1), ..., g_{q+1}(x_{q+1}, 1)) \in A_j.$$

Also if $x_{q+1}, \in A_j$, then the unitary operators of f_{q-j+1} are given by $(\alpha_{q-j}, \beta_{q-j}) = (Id_{\mathcal{H}_o}, Id_{\mathcal{H}_o})$. Because $h(Id_{\mathcal{H}_o}, t) = Id_{\mathcal{H}_o}$ for all $t \in [0,1]$, we get that if $x = (f_1, ..., f_{q+1}) \in A_j$ then $h_j(x,t) = x$.

On the other hand if $i < j$ $x_{q+1} = (f_1, ..., f_{q+1}) \in A_j$, then the unitary operators of the morphism f_{q-j+1} are $(I_{\mathcal{H}_o}, I_{\mathcal{H}_o})$ and so are the components of the morphism in the $(q-j+1)$-th component of $h_j(x,t)$ for all $t \in [0,1]$. This proves the lemma. $\square$

Remark 2.9. Let $\mathcal{V}$ be the category whose object set is $\mathbb{N}$ (we think of an object n as the vector space $\mathbb{C}^n$). There are no morphisms between different objects and for any $n \geq 0$

$$\mathcal{V}(n,n) = U(n), \quad \text{the space of unitary operators in } \mathbb{C}^n.$$

$\mathcal{V}$ is a topological category with the norm topology on $U(n)$. In addition, $\mathcal{V}$ has the structure of a symmetric bimonoidal category under the operations $\oplus$ and $\otimes$. The operation $\oplus$ takes a pair (m,n) to $m+n$ and $U(m) \times U(n)$ to $U(m+n)$ by the block sum of matrices. On the other hand the functor $\otimes$ takes a pair (m,n) to mn and maps $U(m) \times U(n)$ to $U(mn)$ by means of the lexicographic ordering. (See [11, Example 3.1].) The geometric realization

$$B\mathcal{V} = \coprod_{n \geq 0} BU(n)$$

classifies complex vector bundles. On the other hand, the category $\mathcal{V}$ can be completed to get a category $\mathcal{V}^{-1}\mathcal{V}$. (See [45] and [20] for details.) This category is such that $B\mathcal{V}^{-1}\mathcal{V} \simeq BU \times \mathbb{Z}$ and thus classifies complex K-theory. However, as pointed out by Thomason, the category $\mathcal{V}^{-1}\mathcal{V}$ does not admit a structure of a symmetric bimonoidal category coming from that of $\mathcal{V}$. We only have the structure of a symmetric monoidal category coming from the operation $\oplus$ on $\mathcal{V}$. Thus on the space $B\mathcal{V}^{-1}V$ we only have the structure of an H-space corresponding to the Whitney sum of vector bundles. Our construction partially solves this problem in the sense that $B\mathcal{C}$ is already completed, and we have two operations $\oplus$ and $\otimes$, but we do not have coherent units.

Consider now $\mathcal{D}_{\pm}$ (resp. $\mathcal{D}$), the full subcategory of $\mathcal{C}$ whose objects are of the form $(\mathcal{H}, F)$, where Index $F = \pm 1$, (resp. Index $F = 1$). As we prove in the appendix, if $(\mathcal{H}, F)$, $(\mathcal{K}, G)$ are objects in $\mathcal{C}$ with Index $F =$ Index $G = \pm 1$ (resp. 1) then Index $P_{F,G} = \pm 1$ (resp. 1). Therefore we can restrict the functor $\otimes$ to $\mathcal{D}_{\pm}$ and $\mathcal{D}$, and thus we get

$$B\otimes : B\mathcal{D}_{\pm} \times B\mathcal{D}_{\pm} \to B\mathcal{D}_{\pm},$$

$$B\otimes : B\mathcal{D} \times B\mathcal{D} \to B\mathcal{D}.$$

Corollary 2.10. *The classifying space of $\mathcal{D}_{\pm}$ (resp. $\mathcal{D}$) has the same homotopy type as $BU \times \{\pm 1\}$ (resp. BU). Also $B\mathcal{D}_{\pm}$ and $B\mathcal{D}$ are compactly generated Hausdorff. Moreover, $B\mathcal{D}_{\pm} \approx \mathbb{Z}/2 \times B\mathcal{D}$.*

Proof. The proof about the homotopy types follows by the same argument as in the previous proposition. The fact that $B\mathcal{D}_{\pm} \approx \mathbb{Z}/2 \times B\mathcal{D}$ follows from the existence of the involution $* : \mathcal{D}_{\pm} \to \mathcal{D}_{\pm}$. $\square$

We will also make use of the following category. Let us denote by $\mathcal{E}$ the full subcategory of $\mathcal{C}$ whose objects are of the form $(\mathcal{H}, F)$, where F is an isomorphism. For a fixed Hilbert space $\mathcal{H}_o$ the set

$$GL(\mathcal{H}_o) = \{F : \mathcal{H}_o \to \mathcal{H}_o / F \text{ is an isomorphism } \}$$

is contractible under the norm topology. Then using the same argument as in the previous cases we have

Corollary 2.11. *The space $B\mathcal{E}$ is contractible, compactly generated and Hausdorff.*

Notice that for any object $\mathfrak{a} = (\mathcal{H}, F)$ of $\mathcal{E}$ and any object $\mathfrak{b} = (\mathcal{K}, G)$ of $\mathcal{C}$ we have that $\mathfrak{a} \otimes \mathfrak{b} = (\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P_{F,G})$, where $P_{F,G}$ is defined as before. But we show in Lemma 1.9 of the appendix that

$$\text{dimKer } P_{F,G} = (\text{dimKer } F)(\text{dimKer } G) + (\text{dimCoKer } F)(\text{dimCoKer } G) = 0,$$

and similarly for $P^*_{F,G}$. It follows then that $P_{F,G}$ is an isomorphism and thus $\otimes : \mathcal{C} \times \mathcal{E} \to \mathcal{E}$. After applying B we get a map $B\otimes : B(\mathcal{C} \times \mathcal{E}) \to B\mathcal{E}$. But since both $B\mathcal{C}$ and $B\mathcal{E}$ are compactly generated and Hausdorff we get that $B(\mathcal{C} \times \mathcal{E})$ is canonically homeomorphic to $B\mathcal{C} \times B\mathcal{E}$. Thus we get a map $B\mathcal{C} \times B\mathcal{E} \to B\mathcal{E}$ which, by abuse of notation, we call $B\otimes$.

2.3 The Category $\mathcal{C}^s$

In the previous section we constructed a category $\mathcal{C}$ together with functors $\oplus$ and $\otimes$ that are associative, commutative and distributive up to natural isomorphisms that satisfy some coherences. In what follows this will cause some technical difficulties. Thus we are going to replace the category $\mathcal{C}$ by an equivalent category where the corresponding operations are strictly associative and strictly distributive from the left. This is a standard procedure in category theory.

Definition 2.12. Let us construct a category $\mathcal{C}^s$ as follows: The objects of $\mathcal{C}^s$ are formal nonempty strings $\mathfrak{a}$ of the form

$$\mathfrak{a} = \mathfrak{a}_{11} \boxtimes \cdots \boxtimes \mathfrak{a}_{1i_1} \boxplus \mathfrak{a}_{21} \boxtimes \cdots \boxtimes \mathfrak{a}_{2i_2} \cdots \boxplus \mathfrak{a}_{n1} \boxtimes \cdots \boxtimes \mathfrak{a}_{ni_n}$$

where $\mathfrak{a}_{ij}$ is an object in $\mathcal{C}$.

Notation 2.13. Given objects $\mathfrak{a}_1, ..., \mathfrak{a}_n$ of $\mathcal{C}$ we will write

$$\mathfrak{a}_1 a_2 ... \mathfrak{a}_n = \mathfrak{a}_1 \boxtimes \mathfrak{a}_2 \boxtimes \cdots \boxtimes \mathfrak{a}_n.$$

Thus we will write

$$\mathfrak{a} = \mathfrak{a}_{11} ... \mathfrak{a}_{1i_1} \boxplus \mathfrak{a}_{21} ... \mathfrak{a}_{2i_2} \cdots \boxplus \mathfrak{a}_{n1} ... \mathfrak{a}_{ni_n}$$

for a general object in $\mathcal{C}^s$.

If we have two objects in $\mathcal{C}^s$

$$\mathfrak{a} = \mathfrak{a}_{11} ... \mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1} ... \mathfrak{a}_{ni_n},$$
$$\mathfrak{b} = \mathfrak{b}_{11} ... \mathfrak{b}_{1j_1} \boxplus \cdots \boxplus \mathfrak{b}_{m1} ... \mathfrak{b}_{mj_m}$$

then we can define operations $\boxplus$ and $\boxtimes$ by

$$\begin{aligned} \mathfrak{a} \boxplus \mathfrak{b} &:= \mathfrak{a}_{11} ... \mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1} ... \mathfrak{a}_{ni_n} \boxplus \mathfrak{b}_{11} ... \mathfrak{b}_{1j_1} \boxplus \cdots \boxplus \mathfrak{b}_{m1} ... \mathfrak{b}_{mj_m}, \\ \mathfrak{a} \boxtimes \mathfrak{b} &:= \mathfrak{a}_{11} ... \mathfrak{a}_{1i_1} \mathfrak{b}_{11} ... \mathfrak{b}_{1j_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1} ... \mathfrak{a}_{ni_n} \mathfrak{b}_{11} ... \mathfrak{b}_{1j_1} \boxplus \cdots \boxplus \\ &\quad \mathfrak{a}_{11} ... \mathfrak{a}_{1i_1} \mathfrak{b}_{m1} ... \mathfrak{b}_{mj_m} \boxplus \cdots \boxplus \mathfrak{a}_{n1} ... \mathfrak{a}_{ni_n} \mathfrak{b}_{m1} ... \mathfrak{b}_{mj_m}. \end{aligned}$$

To define the morphisms of $\mathcal{C}^s$, for an object

$$\mathfrak{a} = \mathfrak{a}_{11} ... \mathfrak{a}_{1i_1} \boxplus \mathfrak{a}_{21} ... \mathfrak{a}_{2i_2} \boxplus \cdots \boxplus \mathfrak{a}_{n1} ... \mathfrak{a}_{ni_n}$$

of $\mathcal{C}^s$, we define

$$\begin{aligned} \mathcal{T}(\mathfrak{a}) &= ((\cdots(\cdots(\cdots(\mathfrak{a}_{11} \otimes \mathfrak{a}_{12}) \otimes \cdots) \otimes \mathfrak{a}_{1i_1}) \oplus (\cdots(\mathfrak{a}_{21} \otimes \mathfrak{a}_{22}) \otimes \cdots) \otimes \mathfrak{a}_{2i_1})) \oplus \\ &\quad \cdots \oplus (((\cdots(\mathfrak{a}_{n1} \otimes \mathfrak{a}_{n2} \cdots) \otimes \mathfrak{a}_{ni_n})) \cdots). \end{aligned}$$

Thus $\mathcal{T}(\mathfrak{a})$ is the object of $\mathcal{C}$ obtained by inserting parenthesis in $\mathfrak{a}$ in a consistent way and applying the corresponding operations in $\mathcal{C}$. Then for objects $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathcal{C}^s$ we define the morphisms $\mathfrak{a} \to \mathfrak{b}$ to be precisely the morphisms $\mathcal{T}(\mathfrak{a}) \to \mathcal{T}(\mathfrak{b})$ in $\mathcal{C}$. This way we get the category $\mathcal{C}^s$. The operations $\boxplus$ and $\boxtimes$ can be extended to a morphism as follows: if $f : \mathfrak{a} \to \mathfrak{c}$ and $g : \mathfrak{b} \to \mathfrak{d}$ are morphisms in $\mathcal{C}^s$ then we define $f \boxplus g : \mathfrak{a} \boxplus \mathfrak{b} \to \mathfrak{c} \boxplus \mathfrak{d}$ to be the composition in $\mathcal{C}$

$$\mathcal{T}(\mathfrak{a} \boxplus \mathfrak{b}) \to \mathcal{T}(\mathfrak{a}) \oplus \mathcal{T}(\mathfrak{b}) \overset{\mathcal{T}(f)\oplus\mathcal{T}(g)}{\to} \mathcal{T}(\mathfrak{c}) \oplus \mathcal{T}(\mathfrak{d}) \to \mathcal{T}(\mathfrak{c} \boxplus \mathfrak{d}).$$

Here the outer maps are the ones obtained by using the natural isomorphisms in $\mathcal{C}$. By the coherence of the operations in $\mathcal{C}$ this is well defined.

Similarly we define $f \boxtimes g : \mathfrak{a} \boxtimes \mathfrak{b} \to \mathfrak{c} \boxtimes \mathfrak{d}$ to be the composition in $\mathcal{C}$

$$\mathcal{T}(\mathfrak{a} \boxtimes \mathfrak{b}) \to \mathcal{T}(\mathfrak{a}) \otimes \mathcal{T}(\mathfrak{b}) \overset{\mathcal{T}(f)\otimes\mathcal{T}(g)}{\to} \mathcal{T}(\mathfrak{c}) \otimes \mathcal{T}(\mathfrak{d}) \to \mathcal{T}(\mathfrak{c} \boxtimes \mathfrak{d}).$$

This way we obtain a category $\mathcal{C}^s$ with operations $\boxtimes$ and $\boxplus$ that are strictly associative and such that $\mathfrak{a} \boxtimes (\mathfrak{b} \boxplus \mathfrak{c}) = \mathfrak{a} \boxtimes \mathfrak{b} \boxplus \mathfrak{a} \boxtimes \mathfrak{c}$. We can make $\mathcal{C}^s$ into a topological category just by noting that by definition we have

$$\mathcal{C}_0^s = \coprod_{n\geq 1} \coprod_{i_1+\cdots+i_k=n} \mathcal{C}_0^{i_1} \times \cdots \times \mathcal{C}_0^{i_k}$$

so we can give each $\mathcal{C}_0^{i_1} \times \cdots \times \mathcal{C}_0^{i_k}$ the product topology and thus $\mathcal{C}_0^s$ inherits a topology. Similarly we can give a topology on the set $\mathcal{C}_1^s$ via the functor $\mathcal{T}$. Defined this way we have that $\mathcal{C}^s$ is a topological category. The point is that $\mathcal{C}^s$ is equivalent to the category $\mathcal{C}$ via the continuous functors $\mathcal{T} : \mathcal{C}^s \to \mathcal{C}$ and $i : \mathcal{C} \to \mathcal{C}^s$, where i is the inclusion functor. This is precisely our next theorem.

Theorem 2.14. *$\mathcal{C}^s$ is a topological category together with two strictly associative operations that satisfy the strictly left distributivity property. In addition the functors $\mathcal{T} : \mathcal{C}^s \to \mathcal{C}$ and $i : \mathcal{C} \to \mathcal{C}^s$ define an equivalence of topological categories.*

Proof. We have already seen that $\mathcal{C}^s$ is a topological category together with two operations $\boxplus$ and $\boxtimes$. The facts that $\boxplus$ and $\boxtimes$ are strictly associative and satisfy the strictly left distributivity property follow immediately from the definition. We need to check that $\mathcal{T}$ and i define an equivalence of categories. By definition, if $\mathfrak{a}$ is an object of $\mathcal{C}$, then $i(\mathfrak{a}) = \mathfrak{a}$, the formal string with only one object. Therefore $\mathcal{T} \circ i = Id$. On the other hand, if $\mathfrak{a}$ is an object of $\mathcal{C}^s$, then

$$\mathfrak{a} = \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \mathfrak{a}_{21}...\mathfrak{a}_{2i_2} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n}.$$

By definition

$$\begin{aligned} \mathcal{T}(\mathfrak{a}) \quad &= \quad ((\cdots(\cdots(\cdots(\mathfrak{a}_{11} \otimes \mathfrak{a}_{12}) \otimes \cdots) \otimes \mathfrak{a}_{1i_1}) \oplus (\cdots(\mathfrak{a}_{21} \otimes \mathfrak{a}_{22}) \otimes \cdots) \otimes \mathfrak{a}_{2i_1})) \oplus \\ &\quad\quad \cdots \oplus (((\cdots(\mathfrak{a}_{n1} \otimes \mathfrak{a}_{n2} \cdots) \otimes \mathfrak{a}_{ni_n})) \cdots) \end{aligned}$$

and thus $i \circ \mathcal{T}(\mathfrak{a}) = \mathfrak{b}$, where $\mathfrak{b}$ is the object of $\mathcal{C}^s$ consisting of the string of length 1 defined by

$$\begin{aligned} &((\cdots(\cdots(\cdots(\mathfrak{a}_{11} \otimes \mathfrak{a}_{12}) \otimes \cdots) \otimes \mathfrak{a}_{1i_1}) \oplus (\cdots(\mathfrak{a}_{21} \otimes \mathfrak{a}_{22}) \otimes \cdots) \otimes \mathfrak{a}_{2i_1})) \oplus \\ &\cdots \oplus (((\cdots(\mathfrak{a}_{n1} \otimes \mathfrak{a}_{n2} \cdots) \otimes \mathfrak{a}_{ni_n})) \cdots). \end{aligned}$$

However, on $\mathcal{C}^s$ the morphisms $\mathfrak{a} \to i \circ \mathcal{T}(\mathfrak{a})$ are precisely the morphisms in $\mathcal{C}$

$$\mathcal{T}(\mathfrak{a}) \to \mathcal{T}(i\mathcal{T}(\mathfrak{a})).$$

But $\mathcal{T}(\mathfrak{a}) = \mathcal{T}(i\mathcal{T}(\mathfrak{a}))$, therefore we can define a natural transformation

$$\eta : Id \to i\mathcal{T}$$

by simply defining $\eta_{\mathfrak{a}}$ for each object $\mathfrak{a}$ of $\mathcal{C}^s$, as $\eta_{\mathfrak{a}} = Id_{\mathcal{T}(\mathfrak{a})}$. This clearly defines a natural transformation. On the other hand, all the functors and natural transformations in sight are continuous. We conclude then that $\mathcal{C}$ and $\mathcal{C}^s$ are equivalent topological categories. $\square$

Remark 2.15. The previous theorem is a straightforward generalization [28, Theorem XI.3.2]. Also May (see [32, proposition 3.5]) has a proof for this in the case where we have a symmetric bimonoidal category.

Remark 2.16. Because of the previous theorem we will often identify the categories $\mathcal{C}^s$ and $\mathcal{C}$ and the functors $\boxplus$ and $\boxtimes$ with the functors $\oplus$ and $\otimes$ without further comment.

We will denote by $\mathcal{D}^s_\pm$, (resp. $\mathcal{D}^s$) the full subcategory of $\mathcal{C}^s$ whose objects are the elements $\mathfrak{g} = \mathfrak{g}_1 ... \mathfrak{g}_n$, $n \geq 1$, where $\mathfrak{g}_i$ is an object of $\mathcal{D}_\pm$ (resp. $\mathcal{D}$) for $1 \leq i \leq n$. Also denote by $\mathcal{E}^s$ the full subcategory of $\mathcal{C}$ whose objects are elements $\mathfrak{b}$ for which $\mathcal{T}(\mathfrak{b})$ is an object of $\mathcal{E}$. Thus the objects of $\mathcal{E}^s$ are the elements of the form

$$\mathfrak{b} = \mathfrak{b}_{11} ... \mathfrak{b}_{1i_1} \boxplus \cdots \boxplus \mathfrak{b}_{m1} ... \mathfrak{b}_{mi_m}$$

such that for every $1 \leq r \leq m$ there is a $1 \leq j \leq i_r$ such that $\mathfrak{b}_{rj}$ is an object of $\mathcal{E}$. Notice that $\mathcal{D}^s_\pm, \mathcal{D}^s$ and $\mathcal{E}^s$ are categories which are equivalent to $\mathcal{D}_\pm \mathcal{D}$ and $\mathcal{E}$ respectively.

CHAPTER 3

Twistings in K-theory

In this chapter we give a model for the most general twistings in twisted K-theory. The underlying idea is to use Segal's degeneracy free geometric realization to get a space $\mathcal{B}$ which is homotopy equivalent to $K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3) \times BBSU_{\otimes}$, together with a universal fibration over $\mathcal{B}$ with fiber (weakly) homotopy equivalent to $BU \times \mathbb{Z}$. This universal fibration is obtained by applying Quillen's small object argument in a certain category.

3.1 Preliminaries

Let us begin by introducing the notation that is going to be used throughout this chapter and by recalling Segal's degeneracy free geometric realization.

Notation 3.1. If $X = \{X_n\}_{n\geq 0}$ is a simplicial set (space), then following Segal (see [42, Appendix]) we denote by

$$\|X\| = \left(\coprod_{n\geq 0} X_n \times \Delta_n\right) / \sim,$$

where $(\partial_i x, t) \sim (x, \delta_i t), x \in X_n, t \in \Delta_{n-1}$.

Let $\|X\|_k$ be the image of $\coprod_{0\leq n\leq k}(X_n \times \Delta_n)$ in $\|X\|$ with the quotient topology. We give $\|X\|$ the homotopy direct limit of the $\|X\|_k$. We refer to the space $\|X\|$ as Segal's geometric realization of the simplicial set X.

Notice that in this construction we ignore the degeneracy maps, so we can extend this construction to wider a class of objects than simplicial sets (spaces).

Definition 3.2.
- Define Δ^{inj} to be the category whose objects are the sets $\{0, ..., n\}$ and whose morphisms are the *injective* maps $\{0, ..., n\}$ to $\{0, ..., m\}$.
- An injective simplicial set (space) is a contravariant functor $X : \Delta^{inj} \to \textit{sets}$ $(\textit{spaces})$.

Note that having an injective simplicial set (space) is equivalent to having a sequence of sets (spaces) X_n together with maps $\partial_i : X_{n+1} \to X_n$ such that $\partial_i \partial_j = \partial_{j-1}\partial_i$ for $i < j$. Segal's realization makes sense in general for an injective simplicial set.

Definition 3.3. If X is a topological space, we say that X is a (left) $B\mathcal{D}^s_{\pm}$-space if we have a continuous map $\mu : B\mathcal{D}^s_{\pm} \times X \to X$ such that $\mu(g_1 g_2, x) = g_1 \mu(g_2, x)$ for all $x \in X$ and all $g_1, g_2 \in B\mathcal{D}^s_{\pm}$.

Notation 3.4. From now on we will write gx to mean $\mu(g, x)$.

If X is a $B\mathcal{D}^s_{\pm}$-space X, we can construct out of X an injective simplicial set

$$\{B_*(*, B\mathcal{D}^s_{\pm}, X)\}_{n \geq 0},$$

defined by

$$B_n(*, B\mathcal{D}^s_{\pm}, X) = \begin{cases} X & \text{if } n = 0, \\ (B\mathcal{D}^s_{\pm})^n \times X & \text{if } n > 0. \end{cases}$$

The face maps $\partial_i : B_n(*, B\mathcal{D}^s_{\pm}, X) \to B_{n-1}(*, B\mathcal{D}^s_{\pm}, X)$ are given by:

$$\partial_i([g_1, ... g_n]x) = \begin{cases} [g_2, ..., g_n]x & \text{if } i = 0, \\ [g_1, ..., g_i g_{i+1}, ..., g_n]x & \text{if } 0 < i < n, \\ [g_1, ..., g_{n-1}]g_n x & \text{if } i = n. \end{cases}$$

Notation 3.5. If X is a $B\mathcal{D}^s_\pm$-space, then we denote

$$\mathbb{B}(*, B\mathcal{D}^s_\pm, X) = \|B_*(*, B\mathcal{D}^s_\pm, X)\|.$$

In general we will denote by $[g_1, ..., g_n]x$ a general element in $B_n(*, B\mathcal{D}^s_\pm, X)$, where $g_1, ..., g_n \in B\mathcal{D}^s_\pm$ and $x \in X$. In addition, if $t \in \Delta_n$, then we will denote by $|[g_1, ..., g_n]x, t|$ the equivalence class representing the pair $([g_1, ..., g_n]x, t)$ in $\mathbb{B}(*, B\mathcal{D}^s_\pm, X)$.

For a $B\mathcal{D}^s_\pm$-space X, the map $X \to *$ induces a map of injective simplicial sets

$$B_*(*, B\mathcal{D}^s_\pm, X) \to B_*(*, B\mathcal{D}^s_\pm, *).$$

By applying Segal's realization we obtain a map

$$p : \mathbb{B}(*, B\mathcal{D}^s_\pm, X) \to \mathbb{B}(*, B\mathcal{D}^s_\pm, *).$$

As in the case of the two sided bar construction (see [30, Section 9] for definition), this map is a quasifibration. This can be seen directly by adapting May's proof to this situation (see [36, Theorem 7.6]). Thus p is thought of as the universal X-quasifibration. As a particular case we can take $X = B\mathcal{C}^s$ to get a quasifibration $p : \mathbb{B}(*, B\mathcal{D}^s_\pm, B\mathcal{C}^s) \to \mathbb{B}(*, B\mathcal{D}^s_\pm, *)$ whose fibers are homotopy equivalent to $BU \times \mathbb{Z}$. Note that as $B\mathcal{D}^s_\pm$ is a model for $K_\otimes$, then $\mathbb{B}(*, B\mathcal{D}^s_\pm, *)$ is a model for $BK_\otimes$.

The quasifibration p can be used as a first candidate to define twistings in K-theory. However, in order to have the structure of an abelian group on the twistings, we need to modify both the spaces $\mathbb{B}(*, B\mathcal{D}^s_\pm, B\mathcal{C}^s)$ and $\mathbb{B}(*, B\mathcal{D}^s_\pm, *)$ and the map p. The underlying idea is to use Quillen's small object argument in a certain category to get a universal Serre fibration over a space homotopy equivalent to $\mathbb{B}(*, B\mathcal{D}^s_\pm, *)$. This fibration comes equipped with fiberwise operations which induces a "group up to homotopy". We explain precisely what we mean by this in the following section.

3.2 Universal fibration

In this section we replace the spaces $\mathbb{B}(*, B\mathcal{D}_{\pm}^{s}, B\mathcal{C}^{s})$ and $\mathbb{B}(*, B\mathcal{D}_{\pm}^{s}, *)$ and the map p in order to get a "universal" fibration $p_{\mathcal{G}^{\infty}} : \mathcal{G}^{\infty} \to \mathcal{B}$.

To begin, consider the functor

$$* : \mathcal{D}_{\pm}^{s} \times \mathcal{C}^{s} \times \mathcal{E}^{s} \to \mathcal{C}^{s} \times \mathcal{E}^{s},$$

that for objects $\mathfrak{g}$ of $\mathcal{D}_{\pm}^{s}$ and $(\mathfrak{a}, \mathfrak{b})$ of $\mathcal{C}^{s} \times \mathcal{E}^{s}$ is defined by

$$\mathfrak{g} * (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{g} \otimes \mathfrak{a}, \mathfrak{g} \otimes \mathfrak{b}).$$

We define $*$ similarly on morphisms. On the level of classifying spaces we obtain a map

$$B* : B(D_{\pm}^{s} \times \mathcal{C}^{s} \times \mathcal{E}^{s}) \simeq B\mathcal{D}_{\pm}^{s} \times B\mathcal{C}^{s} \times B\mathcal{E}^{s} \to B\mathcal{C}^{s} \times B\mathcal{E}^{s}$$

and thus we can see $B\mathcal{C}^{s} \times B\mathcal{E}^{s}$ as a $B\mathcal{D}_{\pm}^{s}$-space. We also can see $B\mathcal{E}^{s}$ as a $B\mathcal{D}_{\pm}^{s}$-space with action coming from the functor $\otimes$. According to this we have the following definition.

Definition 3.6. Define the spaces

$$\mathcal{G} = \mathbb{B}(*, B\mathcal{D}_{\pm}^{s}, B\mathcal{C}^{s} \times B\mathcal{E}^{s}),$$

$$\mathcal{B} = \mathbb{B}(*, B\mathcal{D}_{\pm}^{s}, B\mathcal{E}^{s}).$$

Notice that the map $\mathcal{B} = \mathbb{B}(*, B\mathcal{D}_{\pm}^{s}, B\mathcal{E}^{s}) \to \mathbb{B}(*, B\mathcal{D}_{\pm}^{s}, *)$ coming from $B\mathcal{E}^{s} \to *$ is a quasifibration with contractible fibers and thus it is a weak homotopy equivalence. Therefore $\mathcal{B}$ is weakly equivalent to $BK_{\otimes}$. On the other hand the second projection map $B\mathcal{C}^{s} \times B\mathcal{E}^{s} \to B\mathcal{E}^{s}$ is clearly $B\mathcal{D}_{\pm}^{s}$ equivariant so it induces a map $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{B}$. This map is a quasifibration with fibers homotopy equivalent $BU \times \mathbb{Z}$. We are going to show that the map $p_{\mathcal{G}}$ has a fiberwise operation ϕ_{G} that is associative, fiberwise commutative up to homotopy and that has a space of units which we will denote by $\mathcal{F}$. (We cannot obtain a strict unit as the operation $\oplus$ does not admit a coherent unit). We will construct these operation and homotopies in what follows.

- **Fiberwise Operation:** We want to constuct an operation on the fibers of p_G. To do so, notice that by [42, Propositions A.1, A.2] and [30, Corollary 11.6] we have

$$\mathcal{G} \times_{\mathcal{B}} \mathcal{G} \simeq \mathbb{B}(*, B\mathcal{D}^s_{\pm}, B\mathcal{C}^s \times B\mathcal{C}^s \times B\mathcal{E}^s).$$

The map

$$B(\oplus, Id) : B\mathcal{C}^s \times B\mathcal{C}^s \times B\mathcal{E}^s \to B\mathcal{C}^s \times B\mathcal{E}^s$$

is $B\mathcal{D}^s_{\pm}$-equivariant as the operations $\oplus$ and $\otimes$ are distributive, where $B\mathcal{D}^s_{\pm}$ acts diagonally on $B\mathcal{C}^s \times B\mathcal{C}^s \times B\mathcal{E}^s$. Thus we have an induced map $\phi_{\mathcal{G}} : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \to \mathcal{G}$ which defines a fiberwise operation; that is, the following diagram is commutative

$$\begin{tikzcd} \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \arrow[r, "\phi_{\mathcal{G}}"] \arrow[dr, "p_{\mathcal{G}} \times_{\mathcal{B}} p_{\mathcal{G}}"'] & \mathcal{G} \arrow[d, "p_{\mathcal{G}}"] \\ & \mathcal{B}. \end{tikzcd} \tag{3.1}$$

- **Associativity:** Since the operation $\oplus$ is strictly associative on $B\mathcal{C}^s$ we have that the fiberwise operation $\phi_{\mathcal{G}}$ is associative; that is, the following diagram is commutative

$$\begin{tikzcd} (\mathcal{G} \times_{\mathcal{B}} \mathcal{G}) \times_{\mathcal{B}} \mathcal{G} \arrow[r, "\simeq"] \arrow[d, "\phi_{\mathcal{G}} \times_{\mathcal{B}} Id"'] & \mathcal{G} \times_{\mathcal{B}} (\mathcal{G} \times_{\mathcal{B}} \mathcal{G}) \arrow[d, "Id \times_{\mathcal{B}} \phi_{\mathcal{G}}"] \\ \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \arrow[dr, "\phi_{\mathcal{G}}"] & \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \arrow[d, "\phi_{\mathcal{G}}"] \\ & \mathcal{G}. \end{tikzcd} \tag{3.2}$$

Here the isomorphism on the top row is the canonical isomorphism

$$(\mathcal{G} \times_{\mathcal{B}} \mathcal{G}) \times_{\mathcal{B}} \mathcal{G} \xrightarrow{\simeq} \mathcal{G} \times_{\mathcal{B}} (\mathcal{G} \times_{\mathcal{B}} \mathcal{G})\,.$$

Therefore on the fibers of $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{B}$ we have an associative operation.

- **Commutativity up to fiberwise homotopy:** Let us show now that this operation is commutative up to fiberwise homotopy. To do so consider the functors $\oplus : \mathcal{C}^s \times \mathcal{C}^s \to \mathcal{C}^s$ and $\oplus\tau : \mathcal{C}^s \times \mathcal{C}^s \to \mathcal{C}^s$, where $\tau : \mathcal{C}^s \times \mathcal{C}^s \to \mathcal{C}^s \times \mathcal{C}^s$ is the functor that interchanges the copies of $\mathcal{C}^s$. As noted earlier the operation $\oplus$ in our original category $\mathcal{C}$ is commutative up to coherent isomorphism. Thus in the category $\mathcal{C}^s$ we have a natural transformation from $\oplus$ to $\oplus\tau$; indeed, if $\mathfrak{a}$ and $\mathfrak{b}$ are two objects of $\mathcal{C}^s$ then we can define $\rho^s_{\mathfrak{a},\mathfrak{b}} : \mathfrak{a} \oplus \mathfrak{b} \to \mathfrak{b} \oplus \mathfrak{a}$ to be the morphism in $\mathcal{C}^s$ given by the composition in $\mathcal{C}$

$$\mathcal{T}(\mathfrak{a} \oplus \mathfrak{b}) \to \mathcal{T}(\mathfrak{a}) \oplus \mathcal{T}(\mathfrak{b}) \overset{\rho_{\mathcal{T}(\mathfrak{a}),\mathcal{T}(\mathfrak{b})}}{\to} \mathcal{T}(\mathfrak{b}) \oplus \mathcal{T}(\mathfrak{a}) \to \mathcal{T}(\mathfrak{b} \oplus \mathfrak{a}),$$

where the outer maps are the canonical isomorphisms in $\mathcal{C}$ and the middle map is the natural transformation $\rho : \oplus \to \oplus \circ \tau$ described in Chapter 2. This gives a natural transformation and in fact commutes with left multiplication by elements of $\mathcal{D}^s_{\pm}$, as can easily be seen. By taking geometric realization, we obtain a homotopy

$$h_1 : B\mathcal{C}^s \times B\mathcal{C}^s \times I \to B\mathcal{C}^s$$

which is $B\mathcal{D}^s_{\pm}$-equivariant, where $B\mathcal{D}^s_{\pm}$ acts trivially on I. Hence the map

$$h_1 \times Id : B\mathcal{C}^s \times B\mathcal{C}^s \times B\mathcal{E}^s \times I \to B\mathcal{C}^s \times B\mathcal{E}^s$$

is also $B\mathcal{D}^s_{\pm}$-equivariant. This map induces a simplicial map between the injective simplicial sets, and by taking Segal's realization we get a map

$$H_{\mathcal{G}} : \|\{B_*(*, B\mathcal{D}^s_{\pm}, B\mathcal{C}^s \times B\mathcal{E}^s)\} \times I_*\| \to \mathcal{G}.$$

Here by I_* we mean the simplicial space generated by I; that is, $I_n = I$ for all n and the degeneracy and face maps are the identity. We have an identification

$$\|\{B_*(*, B\mathcal{D}^s_\pm, B\mathcal{C}^s \times B\mathcal{E}^s)\} \times I_*\| \simeq \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \times I$$

that comes from parts iii). and iv). of [42, Proposition A.1.] and the fact that $|I_*| \simeq I$. We therefore obtain a homotopy

$$H_{\mathcal{G}} : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \times I \to \mathcal{G}$$

between the maps

$$\begin{aligned} \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \times I &\to \mathcal{G} \\ (x, y) &\mapsto \phi_{\mathcal{G}}(x, y), \\ (x, y) &\mapsto \phi_{\mathcal{G}}(y, x). \end{aligned}$$

The homotopy $H_{\mathcal{G}}$ is such that if $\pi_{\mathcal{G}} : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \times I \to \mathcal{B}$ is defined by $\pi_{\mathcal{G}}(x, y, t) = p \times_{\mathcal{B}} p(x, y) = p(x) = p(y)$ for all $t \in [0, 1]$, then the following diagram commutes

$$\begin{tikzcd} \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \times I \arrow[r, "H_{\mathcal{G}}"] \arrow[dr, "\pi_{\mathcal{G}}"'] & \mathcal{G} \arrow[d, "p_{\mathcal{G}}"] \\ & \mathcal{B}. \end{tikzcd} \tag{3.3}$$

- **Space of units:** Because of the lack of a unit for the operation $\oplus$ on $B\mathcal{C}$, we do not have a strict unit for the operation $\phi_{\mathcal{G}}$. However, if we let $B\mathcal{D}^s_\pm$ act on $B\mathcal{E}^s \times B\mathcal{E}^s$ by the diagonal action then we can consider the space of units

$$\mathcal{F} = \mathbb{B}(*, B\mathcal{D}^s_\pm, B\mathcal{E}^s \times B\mathcal{E}^s) \subset \mathcal{G}.$$

We denote by $i_{\mathcal{G}} : \mathcal{F} \to \mathcal{G}$ the inclusion map. Notice that all the operations defined on $\mathcal{G}$ restrict to $\mathcal{F}$; that is, $\phi_{\mathcal{G}}(\mathcal{F} \times_{\mathcal{B}} \mathcal{F}) \subset \mathcal{F}$ and $H_{\mathcal{G}}(\mathcal{F} \times_{\mathcal{B}} \mathcal{F} \times I) \subset \mathcal{F}$.

On the other hand we have a functor $\mathcal{A} : \mathcal{C} \to \mathcal{C}$; namely the adjoint functor, defined on objects $\mathfrak{a} = (\mathcal{H}, F)$ by $\mathcal{A}(\mathfrak{a}) = (\mathcal{H}, F^*)$, where F^* is the adjoint operator of F. If $f = (\alpha, \beta)$ is a morphism from $\mathfrak{a} = (\mathcal{H}, F)$ to $\mathfrak{b} = (\mathcal{K}, G)$ then since the operators α and β are unitary it follows that $\mathcal{A}(f) := (\beta, \alpha)$ is a morphism from $\mathcal{A}(\mathfrak{a})$ to $\mathcal{A}(\mathfrak{b})$. The functor $\mathcal{A}$ is such that it commutes with left multiplication on $\mathcal{C}$; that is, if $\mathfrak{a}, \mathfrak{b}$ are two objects of $\mathcal{C}$, then

$$\begin{aligned}\mathcal{A}(\mathfrak{a} \otimes \mathfrak{b}) &= \mathcal{A}((\mathcal{H}, F) \otimes (\mathcal{K}, G)) \\ &= \mathcal{A}(\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P_{F,G}) \\ &= (\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P^*_{F,G}).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\mathfrak{a} \otimes \mathcal{A}(\mathfrak{b}) &= (\mathcal{H}, F) \otimes (\mathcal{K}, G^*) \\ &= (\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, R).\end{aligned}$$

By direct computation we have that $P_{F,G^*} = P^*_{F,G}$, thus

$$\mathcal{A}(\mathfrak{a} \otimes \mathfrak{b}) = \mathfrak{a} \otimes \mathcal{A}(\mathfrak{b}).$$

Similarly we see that if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and $\mathfrak{d}$ are objects in $\mathcal{C}$ and $f : \mathfrak{a} \to \mathfrak{b}$ and $g : \mathfrak{c} \to \mathfrak{d}$ are morphisms then $\mathcal{A}(f \otimes g) = f \otimes \mathcal{A}(g)$. Also we have that $\mathcal{A}(\mathfrak{a} \oplus \mathfrak{b}) = \mathcal{A}(\mathfrak{a}) \oplus \mathcal{A}(\mathfrak{b})$ since

$$\begin{aligned}\mathcal{A}(\mathfrak{a} \oplus \mathfrak{b}) = \mathcal{A}(\mathcal{H} \oplus \mathcal{K}, F \oplus G) &= (\mathcal{H} \oplus \mathcal{K}, F^* \oplus G^*) \\ &= (\mathcal{H}, F^*) \oplus (\mathcal{K}, G^*) = \mathcal{A}(\mathfrak{a}) \oplus \mathcal{A}(\mathfrak{b}),\end{aligned}$$

and similarly $\mathcal{A}(f \oplus g) = \mathcal{A}(f) \oplus \mathcal{A}(g)$ for morphisms f and g.

The functor $\mathcal{A}$ gives rise to a functor $\mathcal{A}^s : \mathcal{C}^s \to \mathcal{C}^s$ in the following way: take an object $\mathfrak{a} = \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n}$ of $\mathcal{C}^s$ and define

$$\mathcal{A}^s(\mathfrak{a}) = \mathfrak{a}_{11}...\mathcal{A}(\mathfrak{a}_{1i_1}) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathcal{A}(\mathfrak{a}_{ni_n}).$$

Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are two objects of $\mathcal{C}^s$ and $f : \mathfrak{a} \to \mathfrak{b}$ is a morphism. Thus f is a morphism in $\mathcal{C}$, $f : \mathcal{T}(\mathfrak{a}) \to \mathcal{T}(\mathfrak{b})$. But since we have that $\mathcal{A}$ commutes with $\oplus$ and left multiplication we have that $\mathcal{T}(\mathcal{A}^s(\mathfrak{a})) = \mathcal{A}(\mathcal{T}(\mathfrak{a}))$ and $\mathcal{T}(\mathcal{A}^s(\mathfrak{b})) = \mathcal{A}(\mathcal{T}(\mathfrak{b}))$. Thus we can define $\mathcal{A}^s(f) = \mathcal{A}(f) : \mathcal{T}(\mathcal{A}^s(\mathfrak{a})) \to \mathcal{T}(\mathcal{A}^s(\mathfrak{b}))$.

On the level of classifying spaces the functor $\mathcal{A}^s$ induces a map $B\mathcal{A}^s : B\mathcal{C}^s \to B\mathcal{C}^s$. As pointed out before this map is $B\mathcal{C}^s$-equivariant and, in particular, $B\mathcal{D}^s_{\pm}$-equivariant. Thus the map

$$B\mathcal{A}^s \times Id : B\mathcal{C}^s \times B\mathcal{E}^s \to B\mathcal{C}^s \times B\mathcal{E}^s$$

is $B\mathcal{D}^s_{\pm}$-equivariant and induces a map $j_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ over $\mathcal{B}$. Let us see now that this map is an "inverse" for the operation $\phi_{\mathcal{G}}$. More precisely, the map $g \mapsto \phi_{\mathcal{G}}(g, j_{\mathcal{G}}(g))$ factors, up to homotopy, through a map that lies in $\mathcal{F}$; that is, we can find a map $u_{\mathcal{G}} : \mathcal{G} \to \mathcal{F}$ and a homotopy $K_{\mathcal{G}} : \mathcal{G} \times I \to \mathcal{G}$ over $\mathcal{B}$ such that

$$K_{\mathcal{G}}(g, 0) = \phi_{\mathcal{G}}(g, j_{\mathcal{G}}(g)),$$

$$K_{\mathcal{G}}(g, 1) = i_{\mathcal{G}} \circ u_{\mathcal{G}}(g).$$

To construct $K_{\mathcal{G}}$ consider the functor $\mathcal{R} : \mathcal{C}^s \to \mathcal{C}^s$ defined as follows.

For an object $\mathfrak{a} = \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n}$ of $\mathcal{C}^s$

$$\mathcal{R}(\mathfrak{a}) = \mathfrak{a}_{11}...\mathfrak{a}_{1(i_1-1)}(\mathfrak{a}_{1i_1} \oplus \mathcal{A}(\mathfrak{a}_{1i_1})) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{n(i_n-1)}(\mathfrak{a}_{ni_n} \oplus \mathcal{A}(\mathfrak{a}_{ni_n}))$$

and for a morphism $f : \mathfrak{a} \to \mathfrak{b}$ in $\mathcal{C}^s$, $\mathcal{R}(f)$ is the composition in $\mathcal{C}$

$$\begin{aligned}
& \mathcal{T}(\mathcal{R}(\mathfrak{a})) \\
= \;& \mathcal{T}(\mathfrak{a}_{11}...\mathfrak{a}_{1(i_1-1)}(\mathfrak{a}_{1i_1} \oplus \mathcal{A}(\mathfrak{a}_{1i_1})) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{n(i_n-1)}(\mathfrak{a}_{ni_n} \oplus \mathcal{A}(\mathfrak{a}_{ni_n}))) \\
\overset{\simeq}{\to} \;& \mathcal{T}(\mathfrak{a}) \oplus \mathcal{T}(\mathcal{A}^s(\mathfrak{a})) \overset{f \oplus \mathcal{A}^s(f)}{\to} \mathcal{T}(\mathfrak{b}) \oplus \mathcal{T}(\mathcal{A}^s(\mathfrak{b})) \\
\overset{\simeq}{\to} \;& \mathcal{T}(\mathfrak{b}_{11}...\mathfrak{b}_{1(i_1-1)}(y_{1i_1} \oplus \mathcal{A}(\mathfrak{b}_{1i_1})) \boxplus \cdots \boxplus \mathfrak{b}_{n1}...\mathfrak{b}_{n(i_n-1)}(\mathfrak{b}_{ni_n} \oplus \mathcal{A}(\mathfrak{b}_{ni_n}))) \\
= \;& \mathcal{T}(\mathcal{R}(\mathfrak{b})),
\end{aligned}$$

where the outer maps are the ones obtained by using the natural isomorphisms in $\mathcal{C}$.

On the other hand we can consider the functor $\mathcal{R}_1 : \mathcal{C}^s \to \mathcal{C}^s$ given by the composition

$$\mathcal{C}^s \overset{\Delta}{\to} \mathcal{C}^s \times \mathcal{C}^s \overset{Id \times \mathcal{A}^s}{\to} \mathcal{C}^s \times \mathcal{C}^s \overset{\oplus}{\to} \mathcal{C}^s.$$

Here $\mathcal{C}^s \overset{\Delta}{\to} \mathcal{C}^s \times \mathcal{C}^s$ is the diagonal functor.

Thus for an object $\mathfrak{a} = \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n}$ of $\mathcal{C}^s$ we have that

$$\begin{aligned} \mathcal{R}_1(\mathfrak{a}) &= \mathfrak{a} \boxplus \mathcal{A}^s(\mathfrak{a}) \\ &= \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n} \boxplus \mathfrak{a}_{11}...\mathcal{A}(\mathfrak{a}_{1i_1}) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathcal{A}(\mathfrak{a}_{ni_n}). \end{aligned}$$

We have a natural transformation $\zeta : \mathcal{R}_1 \to \mathcal{R}$. For an object $\mathfrak{a}$ of $\mathcal{C}^s$ $\zeta_x : \mathcal{R}_1(\mathfrak{a}) \to \mathcal{R}(\mathfrak{a})$ is the morphism in $\mathcal{C}^s$ given by the composition in $\mathcal{C}$

$$\begin{aligned} &\mathcal{T}(\mathcal{R}_1(\mathfrak{a})) \\ &= \mathcal{T}(\mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n} \boxplus \mathfrak{a}_{11}...\mathcal{A}(\mathfrak{a}_{1i_1}) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathcal{A}(\mathfrak{a}_{ni_n})) \\ &\overset{\cong}{\to} \mathcal{T}(\mathfrak{a}_{11}...\mathfrak{a}_{1(i_1-1)} \boxtimes (\mathfrak{a}_{1i_1} \boxplus \mathcal{A}(\mathfrak{a}_{1i_1})) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{n(i_n-1)} \boxtimes (\mathfrak{a}_{ni_n} \boxplus \mathcal{A}(\mathfrak{a}_{ni_n}))) \\ &\overset{\cong}{\to} \mathcal{T}(\mathfrak{a}_{11}...\mathfrak{a}_{1(i_1-1)}(\mathfrak{a}_{1i_1} \oplus \mathcal{A}(\mathfrak{a}_{1i_1})) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{n(i_n-1)}(\mathfrak{a}_{ni_n} \oplus \mathcal{A}(\mathfrak{a}_{ni_n}))) \\ &= \mathcal{T}(\mathcal{R}(\mathfrak{a})). \end{aligned}$$

This is clearly a natural transformation. Thus on the classifying spaces ζ gives rise to a homotopy $h_2 : B\mathcal{C}^s \times I \to B\mathcal{C}^s$. As before it follows by direct computation that h_2 is $B\mathcal{D}^s_{\pm}$-equivariant. Notice that by definition

$$\begin{aligned} h_2(x, 0) &= B\mathcal{R}_1(x) = x \oplus B\mathcal{A}^s(x), \\ h_2(x, 1) &= B\mathcal{R}(x). \end{aligned}$$

Let us show now that we can find a homotopy h_3 from $B\mathcal{R}$ to a map whose image lies in $B\mathcal{E}^s \subset B\mathcal{C}^s$. To define h_3 we will find a simplicial map

$$k_* : N(\mathcal{C}^s) \times I_* \to N(\mathcal{C}^s)$$

where $N(\mathcal{C}^s)$ is the nerve of $\mathcal{C}^s$ (seen as a topological category).

Consider the map

$$\mathcal{O} : \mathcal{C}_0 \times I \to \mathcal{C}_0$$

defined as follows: for an object $\mathfrak{a} = (\mathcal{H}, F)$ and $t \in [0, 1]$ define

$$\mathcal{O}(A, t) = (\mathcal{H} \oplus \mathcal{H}, R_t),$$

where R_t is given by the matrix

$$\begin{bmatrix} (1-t)F & -tI_{\mathcal{H}} \\ tI_{\mathcal{H}} & (1-t)F^* \end{bmatrix}.$$

In the same way we showed that $P_{F,G}$ are Fredholm operators whenever F, G are Fredholm, we can show that R_t is a Fredholm operator. Moreover, note that for $t = 0$, $R_0 = F \oplus F^*$ and we claim that for $t > 0$, R_t is an isomorphism. To see this, note that if $h \oplus k \in \mathcal{H} \oplus \mathcal{H}$ is such that $R_t(h \oplus k) = 0$, then we get that

$$\begin{aligned} (1-t)F(h) &= tk, \\ (1-t)k &= tF^*(h). \end{aligned}$$

We conclude that $k = \frac{1-t}{t}F(h)$, which implies that $h = -\lambda^2 F^* F(h)$, where $\lambda = \frac{1-t}{t}$ is a real number. Hence

$$\begin{aligned} 0 &\leq \|h\|^2 = \left\langle -\lambda^2 F^* F(h), h \right\rangle = -\lambda^2 \left\langle F^* F(h), h \right\rangle = -\lambda^2 \left\langle F(h), F(h) \right\rangle \\ &= -\lambda^2 \|F(h)\|^2 \leq 0 \end{aligned}$$

and thus $h = 0$. Similarly we see that $k = 0$, thus R_t is injective. The same argument shows that R_t^* is injective and thus R_t is an isomorphism for $t > 0$. In particular, R_1 is an isomorphism.

The map $\mathcal{O}$ is a homotopy in $\mathcal{C}_0$ from the map

$$(\mathcal{H}, F) \mapsto (\mathcal{H} \oplus \mathcal{H}, F \oplus F^*)$$

to a map whose image lies in $\mathcal{C}_0$. Notice that the Hilbert spaces of $\mathcal{O}((\mathcal{H}, F), t)$ do not change as t varies.

Using $\mathcal{O}$ we can define

$$k_0 : \mathcal{C}_0^s \times I \to \mathcal{C}_0^s$$

as follows: take $\mathfrak{a} = \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n}$ an object of $\mathcal{C}^s$ and define

$$k_0(\mathfrak{a}, t) = \mathfrak{a}_{11}...\mathcal{O}(\mathfrak{a}_{1i_1}, t) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathcal{O}(\mathfrak{a}_{ni_n}, t).$$

By definition we have that $k_0(\mathfrak{a}, 0) = \mathcal{R}(\mathfrak{a})$ and the Hilbert spaces of $\mathcal{T}(k_0(\mathfrak{a}, t))$ do not change as t varies.

Suppose now that $f : \mathfrak{a} \to \mathfrak{b}$ is a morphism in $\mathcal{C}^s$. Then $\mathcal{R}(f) : \mathcal{R}(\mathfrak{a}) \to \mathcal{R}(\mathfrak{b})$ is a morphism in $\mathcal{C}^s$; that is,

$$\mathcal{R}(f) = (\alpha, \beta) : \mathcal{T}(\mathcal{R}(\mathfrak{a})) \to \mathcal{T}(\mathcal{R}(\mathfrak{b}))$$

for some unitary operators α and β. As mentioned before the Hilbert space components of $\mathcal{T}(\mathcal{O}(\mathfrak{a}, t)), \mathcal{T}(\mathcal{O}(\mathfrak{b}, t))$ are the same as those of $\mathcal{T}(\mathcal{O}(\mathfrak{a}, 0)), \mathcal{T}(\mathcal{O}(\mathfrak{b}, 0))$, respectively. Moreover, by a direct computation it follows that the operators α and β induce morphisms

$$f_t = (\alpha, \beta) : \mathcal{T}(\mathcal{O}(\mathfrak{a}, t)) \to \mathcal{T}(\mathcal{O}(\mathfrak{b}, t))$$

in $\mathcal{C}$, so we get a morphism

$$\mathcal{O}(f, t) = (\alpha, \beta) : \mathcal{O}(\mathfrak{a}, t) \to \mathcal{O}(\mathfrak{b}, t).$$

Thus for a composable sequence of morphisms $(f_1, ..., f_n)$ in $\mathcal{C}^s$, we define

$$k_*(f_1, ..., f_n, t) = (\mathcal{O}(f_1, t), ..., \mathcal{O}(f_n, t)).$$

It is clear from this definition that we get a simplicial map

$$k_* : N(\mathcal{C}^s) \times I_* \to N(\mathcal{C}^s).$$

On the level of classifying spaces this map induces a homotopy

$$h_3 = Bk_* : B\mathcal{C}^s \times I \to B\mathcal{C}^s.$$

We claim that this map is $B\mathcal{D}^s_\pm$-equivariant. The action of $B\mathcal{D}^s_\pm$ on $B\mathcal{C}^s \times I$ comes from consideration of the simplicial space $N(\mathcal{D}^s_\pm) \times N(\mathcal{C}^s) \times I_*$ and the simplicial map

$$N(\mathcal{D}^s_\pm) \times N(\mathcal{C}^s) \times I_* \to N(\mathcal{C}^s) \times I_*$$

induced by the the functor $\otimes : \mathcal{D}^s_\pm \times \mathcal{C}^s \to \mathcal{C}^s$.

Suppose then that $\mathfrak{g} = \mathfrak{g}_1 ... \mathfrak{g}_m$ is an object in $\mathcal{D}^s_\pm$, $\mathfrak{a} = \mathfrak{a}_{11}...\mathfrak{a}_{1i_1} \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{ni_n}$ is an object in $\mathcal{C}^s$ and $t \in [0,1]$. Then we get that

$$\begin{aligned} &\mathfrak{g} \otimes k_0(\mathfrak{a}, t) \\ =&\mathfrak{g} \otimes (\mathfrak{a}_{11}...\mathfrak{a}_{1(i_1-1)}(\mathcal{O}(\mathfrak{a}_{1i_1}, t)) \boxplus \cdots \boxplus \mathfrak{a}_{n1}...\mathfrak{a}_{n(i_n-1)}\mathcal{O}(\mathfrak{a}_{ni_n}, t)) \\ =&\mathfrak{g}_1...\mathfrak{g}_n\mathfrak{a}_{11}...\mathfrak{a}_{1(i_1-1)}(\mathcal{O}(\mathfrak{a}_{1i_1}, t)) \boxplus \cdots \boxplus \mathfrak{g}_1...\mathfrak{g}_n\mathfrak{a}_{n1}...\mathfrak{a}_{n(i_n-1)}\mathcal{O}(\mathfrak{a}_{ni_n}, t)) \\ =&k_0(\mathfrak{g} \otimes \mathfrak{a}, t), \end{aligned}$$

and similarly for k_n, $n > 0$.

Notice that $h_3(\mathfrak{a}, 0) = B\mathcal{R}(\mathfrak{a})$ and that $h_3(\mathfrak{a}, 1)$ lies in $B\mathcal{E}^s$.

Let $h_4 : B\mathcal{C}^s \times I \to B\mathcal{C}^s$ be the concatenation of h_2 and h_3. Then h_4 is $B\mathcal{D}^s_\pm$-equivariant and thus gives rise to a map which we call $K_\mathcal{G} : \mathcal{G} \times I \to \mathcal{G}$ over $\mathcal{B}$. The map $K_\mathcal{G}$ is a homotopy over $\mathcal{B}$; that is, the diagram

(3.4)
$$\begin{tikzcd} \mathcal{G} \times I \arrow[r, "K_\mathcal{G}"] \arrow[dr, "p_{\mathcal{G}\times I}"'] & \mathcal{G} \arrow[d, "p_\mathcal{G}"] \\ & \mathcal{B} \end{tikzcd}$$

is commutative, where $p_{\mathcal{G}\times I}(g, t) = p_\mathcal{G}(g)$.

$K_{\mathcal{G}}$ is a homotopy between the maps

$$g \mapsto \phi_{\mathcal{G}}(g, j_{\mathcal{G}}(g)),$$

$$g \mapsto i_{\mathcal{G}} \circ u_{\mathcal{G}},$$

where $u_{\mathcal{G}}(g)$ is defined to be $K_{\mathcal{G}}(g,1) \in \mathcal{F} \subset \mathcal{G}$. Notice that the homotopy $K_{\mathcal{G}}$ and the maps $j_{\mathcal{G}}$ and $u_{\mathcal{G}}$ restrict to $\mathcal{F}$; that is,

$$K_{\mathcal{G}}(\mathcal{F} \times I) \subset \mathcal{F}, \quad j_{\mathcal{G}}(\mathcal{F}) \subset \mathcal{F} \quad \text{and} \quad u_{\mathcal{G}}(\mathcal{F}) \subset \mathcal{F}.$$

These properties for $\mathcal{G}$ and $\mathcal{F}$ motivate the following definition.

Definition 3.7. Let $\mathcal{I}$ be the category whose objects are pair of spaces over $\mathcal{B}$; that is, an object of $\mathcal{I}$ is a pair of spaces $Y \subset X$ together with a continuous function $p_X : X \to \mathcal{B}$. We also require the existence of an associative operation on the fibers of X; that is, we have a map $\phi_X : X \times_{\mathcal{B}} X \to X$ with $\phi_X(Y \times_{\mathcal{B}} Y) \subset Y$ such that the following diagrams are commutative:

(3.5)
$$\begin{tikzcd} X \times_{\mathcal{B}} X \arrow[r, "\phi_X"] \arrow[dr, "p_X \times_{\mathcal{B}} p_X"'] & X \arrow[d, "p_X"] \\ & \mathcal{B}, \end{tikzcd}$$

(3.6)
$$\begin{tikzcd} (X \times_{\mathcal{B}} X) \times_{\mathcal{B}} X \arrow[r, "\simeq"] \arrow[d, "\phi_X \times_{\mathcal{B}} Id"'] & X \times_{\mathcal{B}} (X \times_{\mathcal{B}} X) \arrow[d, "Id \times_{\mathcal{B}} \phi_X"] \\ X \times_{\mathcal{B}} X \arrow[dr, "\phi_X"] & X \times_{\mathcal{B}} X \arrow[d, "\phi_X"] \\ & X. \end{tikzcd}$$

Here the isomorphisms on the top rows are the canonical isomorphism

$$(X \times_{\mathcal{B}} X) \times_{\mathcal{B}} X \xrightarrow{\simeq} X \times_{\mathcal{B}} (X \times_{\mathcal{B}} X).$$

In addition we require that the operation ϕ_X is commutative up to homotopy; that is, there is a homotopy $H_X : X \times_{\mathcal{B}} X \times I \to X$ between

$$(x_1, x_2) \mapsto \phi_X(x_1, x_2),$$

$$(x_1, x_2) \mapsto \phi_X(x_2, x_1),$$

such that the following diagram commutes:

$$\begin{tikzcd} X \times_{\mathcal{B}} X \times I \arrow[r, "H_X"] \arrow[dr, "p_{X \times_{\mathcal{B}} X \times I}"'] & X \arrow[d, "p_X"] \\ & \mathcal{B}. \end{tikzcd} \tag{3.7}$$

Here $p_{X \times_{\mathcal{B}} X \times I}(x_1, x_2, t) = p_X(x_1) = p_X(x_2)$. The homotopy H_X is required to satisfy the property $H_X(Y \times_{\mathcal{B}} Y \times I) \subset Y$.

We also require the existence of maps $j_X : X \to X$, $u_X : X \to Y$ and a homotopy $K_X : X \times I \to X$ between the maps

$$x \mapsto \phi_X(x, j_X(x))$$

and

$$x \mapsto i_X \circ u_X(x),$$

where $i_X : Y \to X$ is the inclusion map. The homotopy K_X is required to be a homotopy over $\mathcal{B}$; that is, we require that the following diagram commutes:

$$\begin{tikzcd} X \times I \arrow[r, "K_X"] \arrow[dr, "p_{X \times I}"'] & X \arrow[d, "p_X"] \\ & \mathcal{B}. \end{tikzcd} \tag{3.8}$$

The maps K_X, j_X are required to satisfy $K_X(Y \times I) \subset Y$, $j_X(Y) \subset Y$.

Thus an object in $\mathcal{I}$ is determined by the data $(X, Y, p_X, i_X, \phi_X, H_X, K_X, j_X, u_X)$. For simplicity we will denote an object of $\mathcal{I}$ by (X, Y, p_X).

If (X, Y, p_X), $(X', Y', p_{X'})$ are two objects in $\mathcal{I}$, then a morphism in $\mathcal{I}$ is a map $f : X \to X'$ such that $f(Y) \subset Y'$ and f respects all the structure in sight; that is, the following diagrams are commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p_X"'] & X' \arrow[d, "p_{X'}"] \\ & \mathcal{B}, \end{tikzcd} \qquad \begin{tikzcd} X \times_{\mathcal{B}} X \arrow[r, "f \times_{\mathcal{B}} f"] \arrow[d, "\phi_X"'] & X' \times_{\mathcal{B}} X' \arrow[d, "\phi_{X'}"] \\ X \arrow[r, "f"'] & X', \end{tikzcd}$$

$$\begin{tikzcd} X \times_{\mathcal{B}} X \times I \arrow[r, "f \times_{\mathcal{B}} f \times id"] \arrow[d, "H_X"'] & X' \times_{\mathcal{B}} X' \times I \arrow[d, "H_{X'}"] \\ X \arrow[r, "f"'] & X', \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_X"'] & X' \arrow[d, "j_{X'}"] \\ X \arrow[r, "f"'] & X', \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u_X"'] & X' \arrow[d, "u_{X'}"] \\ Y \arrow[r, "f"'] & Y', \end{tikzcd} \qquad \begin{tikzcd} X \times I \arrow[r, "f \times id"] \arrow[d, "K_X"'] & X' \times I \arrow[d, "K_{X'}"] \\ X \arrow[r, "f"'] & X'. \end{tikzcd}$$

The category $\mathcal{I}$ is defined so that $(\mathcal{G}, \mathcal{F}, p_{\mathcal{G}})$ is an object in $\mathcal{I}$. $\mathcal{I}$ has a terminal object $(\mathcal{B}, \mathcal{B}, id_{\mathcal{B}})$ and we can see $p_{\mathcal{G}}$ as a morphism in $\mathcal{I}$ from $(\mathcal{G}, \mathcal{F}, p_{\mathcal{G}})$ to $(\mathcal{B}, \mathcal{B}, id_{\mathcal{B}})$. We want to replace $(\mathcal{G}, \mathcal{F}, p_{\mathcal{G}})$ by an object of $\mathcal{I}$, $(\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty})$ such that the map $p_{\mathcal{G}^\infty}$ is a Serre fibration, where $\mathcal{G}^\infty$ is as close to $\mathcal{G}$ as possible. For doing this we use the small object argument in the category $\mathcal{I}$. We prove this below in Theorem 3.8.

Let us denote by $\mathcal{J}/\mathcal{B}$ the category of triples (X, Y, p_X) where $Y \subset X$ are spaces and $p_X : X \to \mathcal{B}$ is a map.

A morphism in $\mathcal{J}/\mathcal{B}$ from (X, Y, p_X) to $(X', Y', p_{X'})$ is a map $f : X \to X'$ such that $f(Y) \subset Y'$ and the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p_X"'] & X' \arrow[d, "p_{X'}"] \\ & \mathcal{B}. \end{tikzcd}$$

Notice that we have a forgetful functor $U : \mathcal{I} \to \mathcal{J}/\mathcal{B}$. This functor has a left adjoint $F : \mathcal{J}/\mathcal{B} \to \mathcal{I}$ (whose existence is guaranteed by Freyd's adjoint functor Theorem

[19]) and the category $\mathcal{I}$ has arbitrary coproducts and pushouts. Also it has all the limits of the form $x_1 \to x_2 \to \cdots$.

We now prove that we can replace the map $p_{\mathcal{G}}$ by a Serre fibration using the small object argument.

Theorem 3.8. *There exists an object $(\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty})$ of $\mathcal{I}$, with $p_{\mathcal{G}^\infty}$ a Serre fibration, and a morphism $f_\infty : (\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \to (\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty})$ in $\mathcal{I}$ such that, as map of spaces, $f_\infty : \mathcal{G} \to \mathcal{G}^\infty$ is a weak homotopy equivalence.*

Proof. For each $n \geq 0$ consider $j_n : \mathbb{D}^n \to \mathbb{D}^n \times I$ defined by $j_n(x) = (x, 0)$. Let $S^0(n)$ be the set of pairs (α, β) of maps of spaces such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D}^n \arrow[r, "\alpha"] \arrow[d, "j_n"'] & \mathcal{G} \arrow[d, "p_{\mathcal{G}}"] \\ \mathbb{D}^n \times I \arrow[r, "\beta"'] & \mathcal{B}. \end{tikzcd}$$

Suppose that $(\alpha, \beta) \in S^0(n)$. Then we can make $\mathbb{D}^n$ into a space over $\mathcal{B}$ by defining $p_{\mathbb{D}^n,\alpha} = p_{\mathcal{G}} \circ \alpha : \mathbb{D}^n \to \mathcal{B}$. Similarly, we can make $\mathbb{D}^n \times I$ into a space over $\mathcal{B}$ by $p_{\mathbb{D}^n \times I,\beta} = \beta : \mathbb{D}^n \times I \to \mathcal{B}$. Consider now the disjoint union $\mathbb{D}^n_{\mathcal{F}} := \mathcal{F} \coprod \mathbb{D}^n$. The maps $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{B}$ and $p_{\mathbb{D}^n,\alpha} : \mathbb{D}^n \to \mathcal{B}$ give rise to a map $p_{\mathbb{D}^n_{\mathcal{F}}} : \mathbb{D}^n_F \to \mathcal{B}$. On the other hand we have the inclusion map $i_{\mathbb{D}^n_{\mathcal{F}}} : \mathcal{F} \to \mathbb{D}^n_{\mathcal{F}}$. Then $(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}})$ is an object in $\mathcal{J}/\mathcal{B}$. Similarly, by considering $\mathbb{D}^n \times I_{\mathcal{F}} := \mathcal{F} \coprod (\mathbb{D}^n \times I)$ we get an object $(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}})$ in $\mathcal{J}/\mathcal{B}$. Notice that $j'_n = Id_{\mathcal{F}} \coprod j_n : \mathbb{D}^n_{\mathcal{F}} \to \mathbb{D}^n \times I_{\mathcal{F}}$ is a morphism in $\mathcal{J}/\mathcal{B}$. Also we get maps $\alpha' : \mathbb{D}^n_{\mathcal{F}} \to \mathcal{G}$ and $\beta' : \mathbb{D}^n \times I_{\mathcal{F}} \to \mathcal{B}$ such that the following diagram in $\mathcal{J}/\mathcal{B}$ commutes:

$$\begin{tikzcd} (\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\alpha'"] \arrow[d, "j'_n"'] & (\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \arrow[d, "p_{\mathcal{G}}"] \\ (\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r, "\beta'"'] & (\mathcal{B}, \mathcal{B}, Id_{\mathcal{B}}). \end{tikzcd}$$

By adjunction, the previous diagram gives rise to a commutative diagram in $\mathcal{I}$:

$$\begin{tikzcd}
F(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\widetilde{\alpha'}"] \arrow[d, "F(j'_n)"'] & (\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \arrow[d, "p_{\mathcal{G}}"] \\
F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r, "\widetilde{\beta'}"'] & (\mathcal{B}, \mathcal{B}, Id_{\mathcal{B}}).
\end{tikzcd}$$

Consider $(\mathcal{G}^1, \mathcal{F}^1, p_{\mathcal{G}^1})$ the pushout in $\mathcal{I}$ of the following diagram:

$$\begin{tikzcd}
\coprod_{n\geq 0} \coprod_{(\alpha,\beta)\in S^1(n)} F(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\amalg_{n\geq 0}\amalg_{(\alpha,\beta)\in S^1(n)} \widetilde{\alpha'}"] \arrow[d, "\amalg_{n\geq 0}\amalg_{(\alpha,\beta)\in S^1(n)} F(j'_n)"'] & (\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \arrow[d, "f_1"] \\
\coprod_{n\geq 0} \coprod_{(\alpha,\beta)\in S^1(n)} F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r] & (\mathcal{G}^1, \mathcal{F}^1, p_{\mathcal{G}^1}).
\end{tikzcd}$$

This way we get a morphism $f_1 : (\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \to (\mathcal{G}^1, \mathcal{F}^1, p_{\mathcal{G}^1})$ in $\mathcal{I}$. We can repeat this process and thus for each n inductively we get an object $(\mathcal{G}^n, \mathcal{F}^n, p_{\mathcal{G}^n})$ in $\mathcal{I}$ and a morphism $f_n : (\mathcal{G}^{n-1}, \mathcal{F}^{n-1}, p_{\mathcal{G}^{n-1}}) \to (\mathcal{G}^n, \mathcal{F}^n, p_{\mathcal{G}^n})$:

$$\begin{tikzcd}
(\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \arrow[r, "f_1"] \arrow[d, "p_{\mathcal{G}}"'] & (\mathcal{G}^1, \mathcal{F}^1, p_{\mathcal{G}^1}) \arrow[r, "f_1"] \arrow[d, "p_{\mathcal{G}}"'] & \cdots \\
(\mathcal{B}, \mathcal{B}, Id_{\mathcal{B}}) \arrow[r, "="'] & (\mathcal{B}, \mathcal{B}, Id_{\mathcal{B}}) \arrow[r, "="'] & \cdots.
\end{tikzcd}$$

Consider now the diagram

$$(\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \xrightarrow{f_1} (\mathcal{G}^1, \mathcal{F}^1, p_{\mathcal{G}^1}) \xrightarrow{f_2} \cdots .$$

Let $(\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty})$ be the colimit of this diagram in $\mathcal{I}$. Then we get a commutative diagram

$$\begin{tikzcd}
(\mathcal{G}, \mathcal{F}, p_{\mathcal{G}}) \arrow[r, "f_\infty"] \arrow[dr, "p_{\mathcal{G}}"'] & (\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty}) \arrow[d, "p_{\mathcal{G}^\infty}"] \\
& (\mathcal{B}, \mathcal{B}, id_{\mathcal{B}}).
\end{tikzcd}$$

The space $\mathcal{G}^\infty$ can be taken to be the colimit in the category of spaces of the diagrams

$$\mathcal{G} \xrightarrow{f_1} \mathcal{G}^1 \xrightarrow{f_2} \mathcal{G}^2 \to \ldots \to \mathcal{G}^n \to \cdots.$$

Let us show now that the map $p_{\mathcal{G}^\infty} : \mathcal{G}^\infty \to \mathcal{B}$ is a Serre fibration. To check this it suffices to show that for every commutative diagram of spaces

$$\begin{tikzcd} \mathbb{D}^n \arrow[r, "\alpha"] \arrow[d, "j_m"'] & \mathcal{G}^\infty \arrow[d, "p_{\mathcal{G}^\infty}"] \\ \mathbb{D}^n \times I \arrow[r, "\beta"'] & \mathcal{B}, \end{tikzcd} \tag{3.9}$$

we have a lifting $H : \mathbb{D}^n \times I \to \mathcal{G}^\infty$.

Suppose then that we are given such a diagram. As before we can see $\mathbb{D}^n$ and $\mathbb{D}^n \times I$ as spaces over $\mathcal{B}$ by defining $p_{\mathbb{D}^n,\alpha} = p_{\mathcal{G}^\infty} \circ \alpha : \mathbb{D}^n \to \mathcal{B}$ and $p_{\mathbb{D}^n \times I,\beta} = p_{\mathcal{G}^\infty} \circ \beta : \mathbb{D}^n \times I \to \mathcal{B}$. Define $\mathbb{D}^n_{\mathcal{F}} := \mathcal{F} \coprod \mathbb{D}^n$ and $\mathbb{D}^n \times I_{\mathcal{F}} := \mathcal{F} \coprod (\mathbb{D}^n \times I)$. Then we get objects in $\mathcal{J}/\mathcal{B}$, $(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}})$ and $(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}})$, a morphism

$$j'_n : (\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \to (\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}})$$

and a corresponding commutative diagram in the category $\mathcal{J}/\mathcal{B}$:

$$\begin{tikzcd} (\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\alpha'"] \arrow[d, "j'_n"'] & (\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty}) \arrow[d, "p_{\mathcal{G}}"] \\ (\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r, "\beta'"'] & (\mathcal{B}, \mathcal{B}, Id_{\mathcal{B}}). \end{tikzcd}$$

By adjunction we get a corresponding diagram in $\mathcal{I}$:

$$\begin{tikzcd} F(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\widetilde{\alpha'}"] \arrow[d, "F(j'_n)"'] & (\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty}) \arrow[d, "p_{\mathcal{G}}"] \\ F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r, "\widetilde{\beta'}"'] & (\mathcal{B}, \mathcal{B}, Id_{\mathcal{B}}). \end{tikzcd} \tag{3.10}$$

Since $\mathcal{G}^\infty$ is also the colimit in the category spaces of the $\mathcal{G}^i$'s and $\mathbb{D}^n$ is a small object with respect to these kinds of diagrams we have that the diagram (3.9) factors as

$$\begin{tikzcd} \mathbb{D}^n \arrow[r, "\alpha'_k"] \arrow[d, "j_n"'] & \mathcal{G}^k \arrow[r, "f'_{k,\infty}"] \arrow[d, "p_{\mathcal{G}^k}"] & \mathcal{G}^\infty \arrow[d, "p_{\mathcal{G}^\infty}"] \\ \mathbb{D}^n \times I \arrow[r, "\beta'_k"'] & \mathcal{B} \arrow[r, "="'] & \mathcal{B} \end{tikzcd}$$

for some k big enough, and thus $(\alpha', \beta') \in S^k(n)$. This implies that the diagram (3.10) factors in $\mathcal{I}$ as

$$\begin{tikzcd}
F(\mathbb{D}^n_{\mathcal{F}}, E, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\widetilde{\alpha'_k}"] \arrow[d, "F(j'_n)"'] & (\mathcal{G}^k, \mathcal{F}^k, p_{\mathcal{G}^k}) \arrow[r, "f'_{k+1,\infty}"] \arrow[d, "p_{\mathcal{G}^k}"'] & (\mathcal{G}^\infty, \mathcal{F}^\infty, p_{\mathcal{G}^\infty}) \arrow[d, "p_{\mathcal{G}^\infty}"] \\
F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r, "\widetilde{\beta'_k}"'] & (\mathcal{B}, \mathcal{B}, id_{\mathcal{B}}) \arrow[r, "="'] & (\mathcal{B}, \mathcal{B}, id_{\mathcal{B}}).
\end{tikzcd}$$

Notice that the outer square in

$$\begin{tikzcd}
F(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}}) \arrow[r, "\widetilde{\alpha'_k}"] \arrow[d, "F(j'_n)"'] & (\mathcal{G}^k, \mathcal{F}^k, p_{\mathcal{G}^k}) \arrow[r, "f'_{k+1}"] \arrow[d, "p_{\mathcal{G}^k}"'] & (\mathcal{G}^{k+1}, \mathcal{F}^{k+1}, p_{\mathcal{G}^{k+1}}) \arrow[d, "p_{\mathcal{G}^{k+1}}"] \\
F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \arrow[r, "\widetilde{\beta'_k}"'] & (\mathcal{B}, \mathcal{B}, id_{\mathcal{B}}) \arrow[r, "="'] & (\mathcal{B}, \mathcal{B}, id_{\mathcal{B}})
\end{tikzcd}$$

has a lifting $H' : F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}}) \to (\mathcal{G}^k, \mathcal{F}^k, p_{\mathcal{G}^k})$ in $\mathcal{I}$ by construction. Composing this lifting with $f_{k,\infty}$ we get a lifting for the diagram (3.10), and by adjunction we get a corresponding lifting for (3.9). This shows that $p_{\mathcal{G}^\infty} : \mathcal{G}^\infty \to \mathcal{B}$ is a Serre fibration. To finish we show now that $f_\infty : \mathcal{G} \to \mathcal{G}^\infty$ is a weak equivalence. Since the space $\mathcal{G}^\infty$ is the colimit (in the category of spaces) of the $\mathcal{G}^i$'s, to see that f_∞ is a weak equivalence all we have to show is that each $f_k : \mathcal{G}^{k-1} \to \mathcal{G}^k$ is a weak homotopy equivalence. But this follows easily from the adjunction between (F, U) and the fact that $\mathcal{G}^k$ is obtained from $\mathcal{G}^{k-1}$ by attaching copies of $F(\mathbb{D}^n \times I_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n \times I_{\mathcal{F}}})$ along $F(\mathbb{D}^n_{\mathcal{F}}, \mathcal{F}, p_{\mathbb{D}^n_{\mathcal{F}}})$, which gives rise to homotopy equivalences since the map $j_n : \mathbb{D}^n \to \mathbb{D}^n \times I$ is a homotopy equivalence. $\square$

3.3 Definition

We are now ready to give a definition for the most general twistings in K-theory. Consider X a CW-complex and $f : X \to \mathcal{B}$ any continuous map. Then we can consider the pullback $f^*(p_{\mathcal{G}^\infty}) : f^*(\mathcal{G}^\infty) \to X$. Any section $\sigma : X \to f^*(\mathcal{G}^\infty)$ of $f^*(p_{\mathcal{G}^\infty})$ is of the form $\sigma(x) = (x, \sigma_{\mathcal{G}^\infty}(x))$ for some map $\sigma_{\mathcal{G}} : X \to \mathcal{G}^\infty$ such that

$p_{\mathcal{G}^\infty} \circ \sigma_{\mathcal{G}^\infty} = f$. Thus if σ and τ are two sections of $f^*(p_{\mathcal{G}^\infty})$ and if we write them as $\sigma(x) = (x, \sigma_{\mathcal{G}^\infty}(x))$, $\tau(x) = (x, \tau_{\mathcal{G}^\infty}(x))$ as before, then we can get a new section of $f^*(p_{\mathcal{G}^\infty})$, which we denote by $\sigma+\tau$, defined by $(\sigma+\tau)(x) = (x, \phi_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x), \tau_{\mathcal{G}^\infty}(x)))$.

As mentioned before, $p_{\mathcal{G}|\mathcal{F}} : \mathcal{F} \to \mathcal{B}$ is a quasifibration with contractible fibers. By the long exact sequence of homotopy groups we get that $p_{\mathcal{G}|\mathcal{F}}$ is a weak equivalence. Since f_∞ is also a weak homotopy equivalence it follows that $p_{\mathcal{G}^\infty|\mathcal{F}^\infty} : \mathcal{F}^\infty \to \mathcal{B}$ is both a weak homotopy equivalence and a Serre fibration. The same is true for $f^*(p_{\mathcal{G}|\mathcal{F}})$. The class of maps that are both fibrations and weak equivalences satisfy the left lifting property with respect to relative CW-pairs, and therefore any two sections of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty}) : f^*(\mathcal{F}^\infty) \to X$ are homotopic.

With this in mind we can give the following definition.

Definition 3.9. For a CW-complex X and a continuous map $f : X \to \mathcal{B}$ we define $K^0_f(X)$ to be the set of homotopy classes of the sections of $f^*(p_{\mathcal{G}^\infty}) : f^*(\mathcal{G}^\infty) \to X$ of the form $\sigma + \tau$, where σ is a section of $f^*(p_{\mathcal{G}^\infty}) : f^*(\mathcal{G}^\infty) \to X$ and τ is a section of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty}) : f^*(\mathcal{F}^\infty) \to X$.

Theorem 3.10. *$K^0_f(X)$ has the structure of an abelian group.*

Proof. Take $a, b \in K^0_f(X)$ and let $\sigma + \tau$ and $\sigma' + \tau'$ be representatives of a and b respectively, where σ and σ' are sections of $f^*(p_{\mathcal{G}^\infty})$ and τ and τ' are sections of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty})$. Then we define $a+b$ as the element in $K^0_f(X)$ represented by $(\sigma+\sigma')+(\tau+\tau')$. Since $\tau+\tau'$ is still a section of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty})$, we get that $a+b$ is an element of $K^0_f(X)$. Clearly this definition does not depend on the representatives we picked, so we get a well defined element $a + b \in K^0_f(X)$.

As the operation $\phi_{\mathcal{G}^\infty}$ is strictly associative and commutative up to homotopy we get at once that $+$ defines an associative and commutative binary operation on

$K^0_f(X)$. On the other hand, given any sections τ_1 and τ_2 of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty})$ consider $e = [\tau_1 + \tau_2] \in K^0_f(X)$. This does not depend on the sections τ_1 and τ_2; as any sections of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty})$ are homotopic. The claim is that e is a unit for the operation $+$ on $K^0_f(X)$. To see this take $a \in K^0_f(X)$ and take $\sigma + \tau$ as a representative for a. Then by definition we have that $a + e = e + a$ is the homotopy class of the section $(\sigma + \tau_1) + (\tau + \tau_2)$. But this section is homotopic to a section $\sigma + (\tau + \tau_1 + \tau_2)$. Since all the sections of $f^*(p_{\mathcal{G}^\infty|\mathcal{F}^\infty})$ are homotopic we get that $\tau + \tau_1 + \tau_2$ is homotopic to τ and thus $(\sigma + \tau_1) + (\tau + \tau_2)$ is homotopic to $\sigma + \tau$; that is, $a + e = e + a = a$.

Let us see now that we have inverses. By the definition of the category $\mathcal{I}$ we have maps $j_{\mathcal{G}^\infty}, u_{\mathcal{G}^\infty}$ and a homotopy $K_{\mathcal{G}^\infty} : \mathcal{G}^\infty \times I \to \mathcal{G}^\infty$ from the map $x \mapsto \phi_{\mathcal{G}^\infty}(x, j_{\mathcal{G}^\infty}(x))$ to the map $x \mapsto i_{\mathcal{G}^\infty} \circ u_{\mathcal{G}^\infty}(x)$. Thus take $a = [\sigma + \tau] \in K^0_f(X)$. Then we can write σ in the form $\sigma(x) = (x, \sigma_{\mathcal{G}^\infty}(x))$ for some map $\sigma_{\mathcal{G}^\infty} : X \to \mathcal{G}^\infty$ such that $p_{\mathcal{G}^\infty} \circ \sigma_{\mathcal{G}^\infty} = f$. Consider $\sigma'_{\mathcal{G}^\infty} = j_{\mathcal{G}^\infty} \circ \sigma_{\mathcal{G}^\infty}$ and $\sigma'(x) = (x, \sigma'_{\mathcal{G}^\infty}(x))$. Then σ' is a section of $f^*(p_{\mathcal{G}^\infty})$ and thus we can take the element $a' \in K^0_f(X)$ represented by $a' = [\sigma' + \tau]$. Let us show that this is the inverse for a.

By definition, $a + a'$ is represented by $(\sigma + \sigma') + (\tau + \tau)$. But

$$(\sigma + \sigma')(x) = (x, \phi_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x), \sigma'_{\mathcal{G}^\infty}(x))) = (x, \phi_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x), j_{\mathcal{G}^\infty} \circ \sigma_{\mathcal{G}^\infty}(x))).$$

The homotopy $K_{\mathcal{G}^\infty}$ gives a homotopy from $\phi_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x), j_{\mathcal{G}^\infty} \circ \sigma_{\mathcal{G}^\infty}(x))$ to the map $i_{\mathcal{G}^\infty} \circ u_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x))$. This shows that the section $\sigma + \sigma'$ is homotopic to the section $x \mapsto (x, i_{\mathcal{G}^\infty} \circ l_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x)))$. Let $\eta' = l_{\mathcal{G}^\infty}(\sigma_{\mathcal{G}^\infty}(x))$. Then $\eta' : X \to \mathcal{F}$ and $\sigma + \sigma'$ is homotopic to the section $x \mapsto (x, i_{\mathcal{G}^\infty} \circ \eta'(x))$. If we call $\eta(x) = (x, i_{\mathcal{G}^\infty} \circ \eta'(x))$, which is a section of $f^*(p_{\mathcal{G}^\infty|F^\infty})$, then the section $\sigma + \sigma'$ is homotopic to the section η. Thus $a + a'$ is represented by $\eta + \tau$, which also represents e. Thus $a + a' = e = a' + a$.

This proves that $K^0_f(X)$ is an abelian group under the operation $+$. $\square$

CHAPTER 4

The construction in the equivariant case

In this chapter we generalize the constructions of Chapter 2 in the case that we have actions of a group in sight. In this chapter we only deal with actions of a compact Lie group and thus G will denote a compact Lie group throughout this chapter.

Following [9] we have the following definition.

Definition 4.1. A Hilbert space $\mathcal{H}_G$ is called a G-stable Hilbert space if $\mathcal{H}_G$ is a unitary representation of G in which each irreducible representation of G occurs infinitely many times.

As pointed out in [9], the space of Fredholm operators in $\mathcal{H}_G$ with norm topology is not a classifying space for equivariant K-theory as G does not necessarily act continuously on this space. However, if we consider the closed subspace

$$\mathcal{F}_G(\mathcal{H}_G) = \{F \in \mathcal{F}(\mathcal{H}_G)/g \mapsto gFg^{-1} \text{ is continuous}\}$$

then G acts continuously on $\mathcal{F}_G(\mathcal{H}_G)$ and this is a classifying space for K^0_G.

Once and for all fix a set $\mathfrak{U}_G$ of G-stable Hilbert spaces that is closed under direct sum and tensor products. When we speak of a G-stable Hilbert space we will assume that it is in $\mathfrak{U}_G$.

For $\mathcal{H}_G, K_G \in \mathfrak{U}_G$ we define

$$\mathcal{U}_G(\mathcal{H}_G, \mathcal{K}_G) = \{\alpha : \mathcal{H}_G \to \mathcal{K}_G / \alpha \text{ is unitary and } g \mapsto g\alpha g^{-1} \text{ is continuous }\}.$$

In [9, Appendix 3] it is proved that $\mathcal{U}_G(\mathcal{H}_G, \mathcal{H}_G)$ is equivariantly contractible and thus the same is true for $\mathcal{U}_G(\mathcal{H}_G, \mathcal{K}_G)$ for any $\mathcal{H}_G, \mathcal{K}_G$.

Definition 4.2. If $\mathcal{Q}$ is a topological category, then $\mathcal{Q}$ will be called a G-category if the spaces $\mathcal{Q}_0$ and $\mathcal{Q}_1$ have the structure of left G-spaces and the structural maps are G-equivariant. Thus we have maps

$$s : \mathcal{Q}_1 \to \mathcal{Q}_0,$$

$$t : \mathcal{Q}_1 \to \mathcal{Q}_0,$$

$$u : \mathcal{Q}_0 \to \mathcal{Q}_1,$$

and

$$\circ : \mathcal{Q}_1 \times_{\mathcal{Q}_0} \mathcal{Q}_1 \to \mathcal{Q}_1$$

which are G-equivariant, where $\mathcal{Q}_1 \times_{\mathcal{Q}_0} \mathcal{Q}_1$ is given the diagonal G-action.

Given a G-category $\mathcal{Q}$ for a subgroup $H \subset G$ we denote by $\mathcal{Q}^H$ the fixed category of $\mathcal{Q}$; that is, $\mathcal{Q}^H$ is the subcategory of $\mathcal{Q}$ whose objects are the objects of $\mathcal{Q}$ fixed by H and whose morphisms are the morphisms of $\mathcal{Q}$ which are fixed by H. As a subcategory of $\mathcal{Q}$, $\mathcal{Q}^H$ is a topological category.

We show now that all the constructions done in the non-equivariant case have a parallel treatment in the equivariant case. We start by constructing a G-category with operations $\oplus$ and $\otimes$ and then compute the (weak) homotopy type of its classifying space.

Definition 4.3. Let $\mathcal{C}_G$ be the category whose objects are tuples of the form $\mathfrak{a} = (\mathcal{H}_G, F)$, where $\mathcal{H}_G \in \mathfrak{U}_G$, $F \in \mathcal{F}_G(\mathcal{H}_G)$.

For objects $\mathfrak{a} = (\mathcal{H}_G, F)$ and $\mathfrak{b} = (\mathcal{K}_G, R)$ of $\mathcal{C}_G$, a morphism $f : \mathfrak{a} \to \mathfrak{b}$ is a pair of unitary operators (α, β) , where $\alpha, \beta \in \mathcal{U}_G(\mathcal{H}_G, \mathcal{K}_G)$ are such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{H}_G \arrow[r, "F"] \arrow[d, "\alpha"'] & \mathcal{H}_G \arrow[d, "\beta"] \\ \mathcal{K}_G \arrow[r, "R"'] & \mathcal{K}_G. \end{tikzcd}$$

Composition in $\mathcal{C}_G$ is given by composition of the respective unitary operators.

This defines a category $\mathcal{C}_G$ for any compact Lie group G. Just as in the non-equivariant case we can make $\mathcal{C}_G$ into a topological category by giving a topology to the object and morphism sets of $\mathcal{C}$. Moreover, the spaces $(\mathcal{C}_G)_0$ and $(\mathcal{C}_G)_1$ can be given the structure of G-spaces by conjugation (which in this case gives rise to a continuous action, as we are working with the subspace of operators for which this action is continuous). The structural maps are easily seen to be equivariant maps. Hence the category $\mathcal{C}_G$ is a G-category.

Just as in the non-equivariant case we want to define functors $\oplus$ and $\otimes$. We use the same definition but this time we need to be careful as we are working with a subspace of the space of Fredholm operators, and thus we need to show that the functors $\oplus$ and $\otimes$ preserve the continuity of the G-action. Thus suppose that $\mathfrak{a} = (\mathcal{H}_G, F)$ and $\mathfrak{b} = (\mathcal{K}_G, R)$ are two objects in $\mathcal{C}_G$. Since for any $g \in G$ we have that

$$g(F \oplus R)g^{-1} = (gFg^{-1}) \oplus (gRg^{-1}),$$

it follows that $F \oplus R \in \mathcal{F}_G(\mathcal{H}_G \oplus \mathcal{K}_G)$. Similarly if $f_1 = (\alpha, \beta)$ and $f_2 = (\delta, \eta)$ are two morphisms in $\mathcal{C}_G$, then $f_1 \oplus f_2 = (\alpha \oplus \delta, \beta \oplus \eta)$ is also a morphism in $\mathcal{C}_G$. This means that the functor $\oplus : \mathcal{C}_G \times \mathcal{C}_G \to \mathcal{C}_G$ is well defined. Moreover, the identities

$$g(F \oplus R)g^{-1} = (gFg^{-1}) \oplus (gRg^{-1})$$

and

$$g(f_1 \oplus f_2)g^{-1} = (gf_1g^{-1}) \oplus (gf_2g^{-1})$$

say that $\oplus$ is a G-equivariant functor, in the sense that when $\oplus$ is restricted to the object space and morphism space we get a continuous G-equivariant map.

Similarly, as G acts unitarily on $\mathcal{H}_G$, for $g \in G$ we have that $(gFg^{-1})^* = gF^*g^{-1}$. This implies that $F^* \in \mathcal{F}_G(\mathcal{H}_G)$. Similarly, $R^* \in \mathcal{F}_G(\mathcal{K}_G)$. Hence, if we define

$$P_{F,G} : \mathcal{H}_G \otimes \mathcal{K}_G \oplus \mathcal{H}_G \otimes \mathcal{H}_G \to \mathcal{H}_G \otimes \mathcal{K}_G \oplus \mathcal{H}_G \otimes \mathcal{H}_G$$

as in the non-equivariant case by the matrix

$$\begin{bmatrix} -I \otimes R & F^* \otimes I \\ F \otimes I & I \otimes R^* \end{bmatrix}$$

then $P_{F,R} \in \mathcal{F}_G(\mathcal{H}_G \otimes \mathcal{K}_G \oplus \mathcal{H}_G \otimes \mathcal{K}_G)$. An analog statement is true on the level of morphisms. Thus the functor $\otimes : \mathcal{C}_G \times \mathcal{C}_G \to \mathcal{C}_G$ is well defined, continuous and G-equivariant.

In the non-equivariant case we showed that the functors $\oplus$ and $\otimes$ are associative, commutative and distributive up to natural coherent isomorphisms. To do so we showed that the canonical unitary isomorphisms between the corresponding Hilbert spaces give rise to such isomorphisms in $\mathcal{C}$. All of the natural isomorphisms taking place in this constructions are G-continuous and thus the functors $\oplus$ and $\otimes$ in the equivariant case are commutative, associative and distributive up to natural coherent G-equivariant isomorphisms.

Let us show now that as in the non-equivariant case, the classifying space of $\mathcal{C}_G$ has the correct (weak) homotopy type.

Theorem 4.4. *As G-spaces, $B\mathcal{C}_G$ and $\mathcal{F}_G(\mathcal{H}_G)$ have the same weak homotopy type.*

Proof. We follow the same steps as in Theorem 2.6. Let us fix a G-stable Hilbert space $\mathcal{H}_G$ and denote by $\mathcal{C}_G(\mathcal{H}_G)$ the full subcategory of $\mathcal{C}_G$ whose objects are the objects of the form $(\mathcal{H}_G, F)$, where $F \in \mathcal{F}_G(\mathcal{H}_G)$.

We want to show that $\mathcal{C}_G$ and $\mathcal{C}_G(\mathcal{H}_G)$ are equivalent categories through equivariant equivalences (meaning that the respective maps of objects and morphisms are equivariant). These equivalences are constructed in the same way as in Theorem 2.6 and are easily checked to be equivariant. It follows that $B\mathcal{C}_G$ and $B\mathcal{C}_G(\mathcal{H}_G)$ are homotopy equivalent G-spaces.

As a second step we will show that $B\mathcal{C}_G(\mathcal{H}_G)$ has the same weak homotopy type as $\mathcal{F}_G(\mathcal{H}_G)$. So as in Theorem 2.6 we denote by $\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G))$ the topological category whose object space is $\mathcal{F}_G(\mathcal{H}_G)$ and which has no nonidentity morphisms. This is a G-category and we have as G-spaces $B\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G)) \cong \mathcal{F}_G(\mathcal{H}_G)$.

Let $i : \mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G)) \to \mathcal{C}_G(\mathcal{H}_G)$ be the inclusion functor. The induced map

$$Bi : B\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G)) \cong \mathcal{F}_G(\mathcal{H}_G) \to B\mathcal{C}_G(\mathcal{H}_G)$$

is an equivariant map which we now show to be a weak equivalence. Thus we have to show that for every subgroup $H \subset G$ we have that $(Bi)^H : (B\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G)))^H \to (B\mathcal{C}_G(\mathcal{H}_G))^H$ is a weak homotopy equivalence. Take $H \subset G$ a subgroup. By the lemma below we have natural homeomorphisms

$$(B\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G)))^H \cong B(\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G))^H) \text{ and } (B\mathcal{C}_G(\mathcal{H}_G))^H \cong B(\mathcal{C}_G(\mathcal{H}_G)^H).$$

Therefore we only need to check that the map

$$Bi^H : B(\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G))^H) \to B(\mathcal{C}_G(\mathcal{H}_G)^H)$$

induced by the inclusion of categories $i^H : \mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G))^H \to \mathcal{C}_G(\mathcal{H}_G)^H$ is a weak homotopy equivalence. But we can prove this exactly as we proved Theorem 2.6. We see that on the level of simplicial spaces i^H induces a levelwise homotopy equivalence, which follows from the fact that $\mathcal{U}_G(\mathcal{H}_G)$ is equivariantly contractible as proved in [9]. So we can find a G-equivariant homotopy $h' : \mathcal{U}_G(\mathcal{H}_G) \times I \to \mathcal{U}_G(\mathcal{H}_G)$ from the identity to the constant map $F \mapsto Id_{\mathcal{H}_G}$. Let $f(t) = h'(Id, t)$. Then $f(t)$ is fixed by every element in G and we define $h(F, t) = h'(F, t)f(t)^{-1}$. Then h is an equivariant homotopy from the identity to the map $F \mapsto Id_{\mathcal{H}_G} : \mathcal{H}_G \to \mathcal{H}_G$ with the additional property that $h(Id, t) = Id$ for all $0 \leq t \leq 1$. Using h we can see, just as in the non-equivariant case, that i^H induces a levelwise weak homotopy equivalence and that the nerves of the categories $\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G))^H, \mathcal{C}_G(\mathcal{H}_G)^H$ are proper simplicial spaces. Thus by [31, Theorem $A.4$] we have that $(Bi)^H : (B\mathfrak{F}(\mathcal{C}_G(\mathcal{H}_G)))^H \to (B\mathcal{C}_G(\mathcal{H}_G))^H$ is a weak homotopy equivalence. □

Lemma 4.5. *Let X be a simplicial G-space. Then for each subgroup $H \subset G$ we have a natural homeomorphism $|X|^H \cong |X^H|$, where X^H is the fixed point simplicial space of X by H.*

□

Let us now consider now $\mathcal{D}_{\pm,G}$ (resp. $\mathcal{D}_G$) to be the full subcategory of $\mathcal{C}_G$ whose objects are the pairs $(\mathcal{H}_G, F)$ where F is a Fredholm operator of Index ± 1 (resp. 1). Just as in the previous theorem we get:

Theorem 4.6. *As a G-spaces, $B\mathcal{D}_{\pm,G}$ (resp. $B\mathcal{D}_G$) is of the same weak homotopy type as the G-subspace of $\mathcal{F}_G(\mathcal{H}_G)$ of Fredholm operators of index ± 1 (Resp. 1).*

□

As before we have $\otimes : \mathcal{D}_{\pm,G} \times \mathcal{D}_{\pm,G} \to \mathcal{D}_{\pm,G}$ and $\otimes : \mathcal{D}_G \times \mathcal{D}_G \to \mathcal{D}_G$, so on the level of classifying spaces we get maps $B\otimes : B\mathcal{D}_{\pm,G} \times B\mathcal{D}_{\pm,G} \to B\mathcal{D}_{\pm,G}$ and $B\otimes : B\mathcal{D}_G \times B\mathcal{D}_G \to B\mathcal{D}_G$. As in the equivariant case we can replace the category

$\mathcal{C}_G$ by a category $\mathcal{C}^s_G$ on which the operations $\oplus$ and $\otimes$ are strictly associative and are distributive from the left. Within this category there correspond full subcategories $\mathcal{D}_{\pm,G}$ and $\mathcal{D}^s_G$ equivalent to $\mathcal{D}_{\pm,G}$ and $\mathcal{D}_G$ respectively. Thus on the level of classifying spaces we have a semigroup $B\mathcal{D}^s_{\pm,G} \approx \mathbb{Z}/2 \times B\mathcal{D}_G$ acting on $B\mathcal{C}^s_G$. Also we obtain a functor $\otimes : \mathcal{D}_\pm \times \mathcal{C}_G \to \mathcal{C}_G$ by letting G act trivially on the Hilbert spaces defining $\mathcal{D}_\pm$. By passing to the level of classifying spaces we get an action of $B\mathcal{D}^s_\pm$ into $B\mathcal{C}^s_G$. We study such actions of topological semigroups in the next chapter.

CHAPTER 5

Constructions on semigroups

In this chapter we prove general results on how to replace actions of topological semigroups by actions of topological groups of the same weak homotopy type. More concretely the goal of this chapter is to prove the following theorem.

Theorem 5.1. *Let $\mathcal{M}$ be a topological semigroup that is connected and of the homotopy type of a CW-complex that satisfies the unit condition. Then we can find a topological group H and topological semigroup $\mathcal{M}'$, together with semigroup homomorphisms $i_1 : \mathcal{M}' \to \mathcal{M}$ and $i_2 : \mathcal{M}' \to H$ that are weak homotopy equivalences. Moreover, if $\mathcal{M}$ acts on a space X then the action of $\mathcal{M}$ can be extended to an actions of $\mathcal{M}$ and H on X that are compatible with i_1 and i_2.*

The unit condition that we refer to in the previous theorem is given in the following definition.

Definition 5.2. Let $\mathcal{M}$ be a topological semigroup. We say that $\mathcal{M}$ satisfies the unit condition if $\mathcal{M}$ has both left and right units up to homotopy and that the left unit is nondegenerate.

Throughout this chapter $\mathcal{M}$ will always denote a topological semigroup that is connected and of the homotopy type of a CW-complex that satisfies the unit condition.

The application that we have in mind is the action of the semigroup $B\mathcal{D}^s_{\pm}$ on $B\mathcal{C}^s$. We want to replace this action by the action of a topological group of the same weak homotopy type as $B\mathcal{D}^s_{\pm}$. Since $B\mathcal{D}^s_{\pm} \simeq \mathbb{Z}/2 \times B\mathcal{D}^s$ we only need to concentrate on the action of $B\mathcal{D}^s$ on $B\mathcal{C}^s$. Notice that since the operation $\otimes$ corresponds to the tensor product, as is mentioned in the appendix, and the trivial line bundle is a unit up to isomorphism with respect to this multiplication, the semigroup $B\mathcal{D}^s$ is connected and has both a left and right unit up to homotopy and it is also of the homotopy type of a CW-complex. In fact it can be seen directly that any element of the form $(\mathcal{H}, F)$ where F is a Fredholm operator that has Index 1 and is surjective is a unit up to homotopy. In addition, after the adjunction of a whisker space if necessary, we can assume that the left unit is nondegenerate.

We will prove Theorem 5.1 by following these steps:

(1). We show that under the given circumstances we can find a contractible semigroup mapping homomorphically into $\mathcal{M}$. This is done by using obstruction theory.

(2). We show that the action of the topological semigroup $\mathcal{M}$ can be replaced by the action of a topological monoid $\hat{\mathcal{M}}$ of the same weak homotopy type.

(3). Finally, we show that the action of a topological monoid $\hat{\mathcal{M}}$ can be replaced by the action of a topological group H of the same weak homotopy type.

Throughout this chapter we will denote by $\mathcal{U}$ the category of compactly generated topological spaces.

5.1 Contractible subgroups

In this section we complete step (1) of our program. We begin with a definition.

Definition 5.3. Let $(\mathbb{T}, \mu, \eta)$ be the monad on $\mathcal{U}$ that for a topological space X corresponds to the free semigroup $\mathbb{T}(X)$ generated by X. The structural maps $\mu : \mathbb{T}^2 \to \mathbb{T}$ and $\eta : Id \to \mathbb{T}$ denote the multiplication and unit transformations respectively. If X is an algebra for this monad we will always denote by $\xi_X : \mathbb{T}(X) \to X$ to the algebra structural map.

If we denote by $\mathcal{SS}$ the category of simplicial objects in $\mathcal{U}$, then the monad $\mathbb{T}$ on $\mathcal{U}$ gives rise to a monad $\mathbb{T}_*$ on $\mathcal{SS}$. Notice that if X is a semigroup then so is $B(\mathbb{T}, \mathbb{T}, X)$, as geometric realization preserves products on $\mathcal{SS}$, and thus we have a naturally induced binary operation on $B(\mathbb{T}, \mathbb{T}, X)$. As usual, we have a map $\epsilon_X : B(\mathbb{T}, \mathbb{T}, X) \to X$ which is a homomorphism of semigroups and, as map of topological spaces is a homotopy equivalence.

To rectify the semigroup $\mathcal{M}$ we will get a tower of semigroups that have higher connectivity. As a first step toward this tower we have the following lemma:

Lemma 5.4. *Suppose that $\mathcal{M}$ is a topological semigroup that is connected and of the homotopy type of a CW-complex. Assume that $\mathcal{M}$ satisfies the unit condition. If $p : \tilde{\mathcal{M}} \to \mathcal{M}$ is the universal cover of $\mathcal{M}$ then we can find a strictly associative binary operation $\tilde{\nu} : \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ with both left and right unit up to homotopy, and such that the following diagram commutes*

$$\begin{tikzcd}
\tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \arrow[r, "\tilde{\nu}"] \arrow[d, "p\times p"'] & \tilde{\mathcal{M}} \arrow[d, "p"] \\
\mathcal{M} \times \mathcal{M} \arrow[r, "\nu"'] & \mathcal{M}.
\end{tikzcd}$$

Proof. Let us denote by e_1 the right unit of $\mathcal{M}$ and by ν the multiplication function on $\mathcal{M}$. By considering a path from e_1 to a left unit we can assume that e_1 is also a left unit.

$$(*)\qquad \begin{tikzcd} & & \tilde{\mathcal{M}} \arrow[d, "p"] \\ \tilde{\mathcal{M}} \times \tilde{\mathcal{M}} \arrow[r, "p \times p"'] \arrow[urr, dashed, "\tilde{\nu}"] & \mathcal{M} \times \mathcal{M} \arrow[r, "\nu"'] & \mathcal{M}. \end{tikzcd}$$

Take $e_2 \in p^{-1}(e_1)$. We want to find a lifting $\tilde{\nu}$ of the diagram $(*)$, with $\tilde{\nu}(e_2, e_2) = x$ for some $x \in p^{-1}(\nu(e_1, e_1))$. To find such a lifting we will show that

$$(5.1)\qquad (\nu \circ (p \times p))_*(\pi_1(\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}, (e_2, e_2))) \subset p_*(\pi_1(\tilde{\mathcal{M}}, x)).$$

As e_1 is a right unit up to homotopy we can find a homotopy $H_r : \mathcal{M} \times I \to \mathcal{M}$ from $m \mapsto \nu(m, e_1)$ to the identity. Let $\gamma : [0,1] \to \mathcal{M}$ be defined by $\gamma(t) = H_r(e_1, t)$. The path γ is a path in $\mathcal{M}$ from $\nu(e_1, e_1)$ to e_1. We can find a lifting $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(1) = e_1$. Let $x = \tilde{\gamma}(0) \in p^{-1}(\nu(e_1, e_1))$.

For loops $f_1, f_2 : [0,1] \to \mathcal{M}$ at e_1, define $f_1 \bullet f_2$ to be the loop at e_1 defined by

$$(f_1 \bullet f_2)(t) = \begin{cases} \overline{\gamma}(3t) \text{ if } t \in [0, \frac{1}{3}], \\ \nu(f_1(3t-1), f_2(3t-1)) \text{ if } t \in [\frac{1}{3}, \frac{2}{3}], \\ \gamma(3t-2) \text{ if } t \in [\frac{2}{3}, 1]. \end{cases}$$

Here $\overline{\gamma}(t) = \gamma(1-t)$. From this definition it follows at once that if $f_1, f_2, g_1, g_2 : [0,1] \to \mathcal{M}$ are loops at e_1 then we have the following associativity

$$(5.2)\qquad (f_1 \bullet g_1) * (f_2 \bullet g_2) \cong (f_1 * f_2) \bullet (g_1 * g_2)$$

through a homotopy fixing the base point e_1. On the other hand, given a loop $f : [0,1] \to \mathcal{M}$ at e_1 we consider the map $G : [0,1] \times [0,1] \to \mathcal{M}$ defined by $G(s,t) = H_r(f(s), t)$. Notice that $G(0,t) = H_r(e_1, t) = \gamma(t)$, $G(1,t) = H_r(e_1, t) = \gamma(t)$, $G(s, 0) = H_r(f(s), 0) = \nu(f(s), e_1)$ and $G(s, 1) = f(s)$. Thus if we denote by

c_{e_1} the constant loop based at e_1, then the map G gives rise to a homotopy fixing the base point e_1 from the loop $f \bullet c_{e_1}$ to f; that is,

$$f \bullet c_{e_1} \cong f. \tag{5.3}$$

On the other hand, as we are assuming that e_2 is also a left unit, we can find a homotopy $H_l : \mathcal{M} \times I \to \mathcal{M}$ from $m \mapsto \nu(e_1, m)$ to the identity. Since we are assuming that e_1 is nondegenerate, by using the homotopy lifting extension property we can assume without of loss of generality that the map $t \mapsto H_l(e_1, t) = \gamma(t) = H_r(e_1, t)$. Using the homotopy H_l we can show in the same way that we showed (5.3) that

$$c_{e_1} \bullet f \cong f. \tag{5.4}$$

If we take $f_2, g_1 = c_{e_1}$ to be the constant loops at e_1, by (5.2), (5.3), (5.4) it follows that

$$f_1 * g_2 \cong f_1 \bullet g_2. \tag{5.5}$$

Suppose then that $\alpha, \beta : [0,1] \to \tilde{\mathcal{M}}$ are any loops at e_2. Then by definition and by (5.5) we get

$$(\nu \circ (p \times p))_*([\alpha \times \beta]) = [\nu(p \circ \alpha, p \circ \beta)] = \phi_\gamma([p \circ \alpha] \bullet [p \circ \beta]) \tag{5.6}$$

$$= \phi_\gamma([p \circ \alpha] + [p \circ \beta]) = \phi_\gamma([p \circ \alpha]) + \phi_\gamma([p \circ \beta]) \tag{5.7}$$

$$= p_*(\phi_{\tilde{\gamma}}([\alpha] + [\beta])). \tag{5.8}$$

Here

$$\begin{array}{llll} \phi_\gamma : & \pi_1(\mathcal{M}, e_1) \to \pi_1(\mathcal{M}, \nu(e_1, e_1)), & \phi_{\tilde{\gamma}} : & \pi_1(\tilde{\mathcal{M}}, e_2) \to \pi_1(\tilde{\mathcal{M}}, \nu(e_2, e_2)) \\ & [\alpha] \mapsto [\gamma * \alpha * \overline{\gamma}] & & [\alpha] \mapsto [\tilde{\gamma} * \alpha * \overline{\tilde{\gamma}}] \end{array}$$

are the usual homomorphisms of change of base point. Notice that (5.6) implies that (5.1) is true and thus we have a unique lifting $\tilde{\nu}$ for $(*)$ such that $\nu(e_2, e_2) = x$. Similarly, we can see that this operation is associative, and that e_2 is both a left and right unit up to homotopy. $\square$

Remark 5.5. The hypothesis that $\mathcal{M}$ has both left and right unit is used to show that $c_{e_1} \bullet f \cong f \bullet c_{e_1}$. In general, the existence of a left unit up to homotopy does not guarantee the existence of a right unit. This can be seen by considering the discrete semigroup Λ that only contains three elements e, x, xe. The multiplication in this semigroup is as follows: $a \cdot x = x$ for all $a \in \Lambda$. $e \cdot e = e$ and $a \cdot e = xe$ for $x \neq e$. Finally $a \cdot xe = xe$ for all a. By a direct computation it follows that this operation on Λ is associative. Also e is a left unit for Λ but there is no right unit.

Let $\mathcal{M}_1 := \mathcal{M}$ and $\mathcal{M}_2 := \tilde{\mathcal{M}}$. As shown in the previous lemma, the semigroup structure of $\mathcal{M}$ gives rise to a structure of a semigroup on $\mathcal{M}_2$, with both left and right units up to homotopy. The idea for finding a contractible semigroup mapping into $\mathcal{M}$ is to construct a sequence of semigroups with both left and right units up to homotopy $(\mathcal{M}, \nu) = (\mathcal{M}_1, \nu_2)$, $(\mathcal{M}_2, \nu_2)$,...,$(\mathcal{M}_n, \nu_n)$, ... such that each $\mathcal{M}_n$ is $(n-1)$-connected and such that we have continuous homomorphisms of semigroups $f_n : \mathcal{M}_{n+1} \to \mathcal{M}_n$. We prove this in the following lemma.

Lemma 5.6. *Let $\mathcal{M}$ be a connected topological semigroup of the homotopy type of a CW-complex that satisfies the unit condition. Then we can find a sequence of semigroups $\{\mathcal{M}_n\}_{n\geq 1}$ such that $\mathcal{M}_n$ is $(n-1)$-connected, and a sequence of continuous homomorphisms $f_n : \mathcal{M}_{n+1} \to \mathcal{M}_n$. Also, each $\mathcal{M}_n$ is of the homotopy type of a CW-complex and satisfies the unit condition.*

Proof. We will construct this sequence inductively. We start with $(\mathcal{M}_1, \nu_1) = (\mathcal{M}, \nu)$

and $(\mathcal{M}_2, \nu_2) = (\tilde{\mathcal{M}}, \tilde{\nu})$, as before. As shown in the previous lemma we have the structure of a semigroup with both left and right unit up to homotopy on $\mathcal{M}_2$ such that $f_1 = p : \mathcal{M}_2 \to \mathcal{M}_1$ is a homomorphism of semigroups. Also, $\mathcal{M}_2$ has the homotopy type of a CW-complex. Notice that $\mathcal{M}_1$ is 0-connected and $\mathcal{M}_2$ is 1-connected. Suppose then that we have constructed an $(i-1)$-connected semigroup $(\mathcal{M}_i, \nu_i)$ and homomorphisms of semigroups $f_{i-1} : \mathcal{M}_i \to \mathcal{M}_{i-1}$ for $2 \leq i \leq n$ such that $\mathcal{M}_i$ has the homotopy type of a CW-complex and has both left and right unit.

Let us denote by $\pi_n = \pi_n(\mathcal{M}_n)$. We can find a model for $K(\pi_n, n)$ that is a topological group. This is achieved, for example, by taking $B^n(\pi_n)$ where we see π_n as a discrete group. From now on when we speak of $K(\pi_n, n)$ we will always mean a group model of it and we will denote by m_n the multiplication map and by 1 its unit.

To construct $\mathcal{M}_{n+1}$ we will first obtain a continuous homomorphism of semigroups $G_n : B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n) \to K(\pi_n, n)$. The idea for finding this map is to inductively lift maps in each stage of the natural filtration of $B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n)$. Notice that the simplicial space $B_*(\mathbb{T}, \mathbb{T}, \mathcal{M}_n)$ is proper, hence the geometric realization and Segal's fat realization give rise to equivalent spaces. Here, we will work with Segal's fat realization.

To begin, notice that homotopy classes of maps $\mathcal{M}_n \to K(\pi_n, n)$ are in one to one correspondence with elements in $H^n(\mathcal{M}_n, \pi_n)$. By the universal coefficient Theorem we get a natural isomorphism $H^n(\mathcal{M}_n, \pi_n) \cong \mathrm{Hom}(H_n(\mathcal{M}_n, \mathbb{Z}), \pi_n)$. (There is no Ext term, as $H_{n-1}(\mathcal{M}_n, \mathbb{Z}) = 0$ by the $(n-1)$-connectivity of $\mathcal{M}_n$ and the Hurewicz Theorem.) On the other hand, as $\mathcal{M}_n$ is $(n-1)$-connected, by the Hurewicz Theorem we get a natural isomorphism $H_n(\mathcal{M}_n, \mathbb{Z}) \xrightarrow{\approx} \pi_n(\mathcal{M}_n) = \pi_n$. Thus, we can conclude that there is a natural isomorphism $H^n(\mathcal{M}_n, \pi_n) \cong \mathrm{Hom}(\pi_n, \pi_n)$. Let

$g_0 : \mathcal{M}_n \to K(\pi_n, n)$ be a representative of the cohomology class corresponding to the identity map $Id : \pi_n \to \pi_n$ (under this isomorphism). We can pick the representative g_o to be such that $g_0(e_n) = 1$. (Here, e_n is a unit up to homotopy for ν_n.) Indeed, if $\tilde{g}_0$ is any representative in this class, then as $K(\pi_n, n)$ is path connected we can take $\alpha : [0, 1] \to K(\pi_n, n)$ to be a path from $\tilde{g}_0(e_n)$ to 1. Then the map $H(x, t) = m_n(\tilde{g}_0(x), \alpha(t)^{-1})$ gives a homotopy from the desired map g_0 to $\tilde{g}_0$.

We will show next that the map g_0 is a homomorphism of semigroups up to homotopy; that is, we will see that the following diagram commutes up to homotopy

$$\begin{tikzcd} \mathcal{M}_n \times \mathcal{M}_n \arrow[r, "\nu_n"] \arrow[d, "g_0 \times g_0"'] & \mathcal{M}_n \arrow[d, "g_0"] \\ K(\pi_n, n) \times K(\pi_n, n) \arrow[r, "m_n"'] & K(\pi_n, n). \end{tikzcd} \tag{5.9}$$

To see this, we will use the following lemma.

Lemma 5.7. *The semigroup operation ν_n of $\mathcal{M}_n$ induces the usual addition on $\pi_n(\mathcal{M}_n)$.*

Proof. The proof is similar to that of Lemma 5.4. □

To show that (5.9) commutes up to homotopy notice that homotopy classes of maps $\mathcal{M}_n \times \mathcal{M}_n \to K(\pi_n, n)$ are in one-to-one correspondence with cohomology classes in $H^n(\mathcal{M}_n \times \mathcal{M}_n, \pi_n)$. By the Kunneth Theorem in cohomology we have

$$H^n(\mathcal{M}_n \times \mathcal{M}_n, \pi_n) \cong H^n(\mathcal{M}_n, \pi_n) \oplus H^n(\mathcal{M}_n, \pi_n).$$

Here there is no Tor term by the connectivity assumptions. As before, we can identify $H^n(\mathcal{M}_n, \pi_n)$ with $\mathrm{Hom}(\pi_n, \pi_n)$. Thus, under these identifications we have that the homotopy class $[g_0] \in [\mathcal{M}_n, K(\pi_n, n)]$ corresponds to $Id \in \mathrm{Hom}(\pi_n, \pi_n)$. Using the previous lemma we conclude that

$$[g_0 \circ \nu_n] \mapsto Id \oplus Id \in \mathrm{Hom}(\pi_n, \pi_n) \oplus \mathrm{Hom}(\pi_n, \pi_n) \cong H^n(\mathcal{M}_n \times \mathcal{M}_n, \pi_n).$$

On the other hand, we can also identify

$$H^n(K(\pi_n, n) \times K(\pi_n, n), \pi_n) \approx \mathrm{Hom}(\pi_n, \pi_n) \oplus \mathrm{Hom}(\pi_n, \pi_n)$$

by using the Hurewicz and Kunneth Theorems. Under these identifications the homotopy class $[m_n]$ corresponds to $Id \oplus Id \in \mathrm{Hom}(\pi_n, \pi_n) \oplus \mathrm{Hom}(\pi_n, \pi_n)$. (This follows from the previous lemma, applied to $(K(\pi_n, n), m_n)$ instead of $(\mathcal{M}_n, \nu_n)$.) Therefore, $[(g_0 \times g_0) \circ m_n]$ also corresponds to $Id \oplus Id \in Hom(\pi_n, \pi_n) \oplus Hom(\pi_n, \pi_n)$. We conclude that (5.9) commutes up to homotopy.

Let us denote by F_k the image of the natural map

$$\coprod_{0 \leq j \leq k} (\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_k) \to B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n).$$

The spaces $\{F_k\}_{k \geq 0}$ form an increasing filtration for $B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n)$ with $F_0 = \mathbb{T}(\mathcal{M}_n)$. We have a continuous map $g_0 : \mathcal{M}_n \to K(\pi_n, n)$. This map induces a homomorphism of semigroups

$$h_0 = \mathbb{T}(g_0) : F_0 = \mathbb{T}(\mathcal{M}_n) \to K(\pi_n, n).$$

We wish to extend this map inductively to each F_k. For $k = 1$ we need to find an extension $h_1 : F_1 \to K(\pi_n, n)$ of h_0. To find h_1, all we have to do is extend the map $j_1 : \mathbb{T}^2(\mathcal{M}_n) \times \partial\Delta_1 \to K(\pi_n, n)$ to $\mathbb{T}^2(\mathcal{M}_n) \times \Delta_1$ determined by the face maps in such a way that the extended map is a homomorphism of semigroups when restricted to $\mathbb{T}^2(\mathcal{M}_n) \times \{t\}$ for all $t \in \Delta_1$. Since $\mathbb{T}^2(\mathcal{M}_n)$ is free on $\mathbb{T}(\mathcal{M}_n)$, this amounts to finding an extension of the the corresponding map $\tilde{j_1} : \mathbb{T}^1(\mathcal{M}_n) \times \Delta_1 \to K(\pi_n, n)$. This map is such that, for $x_1 ... x_r \in \mathbb{T}(\mathcal{M}_n)$,

$$\tilde{j_1}(x_1 ... x_r, 0) = h_0(x_1 ... x_r) = m_n(g_0(x_1), m_n(..., m_n(g_0(x_{r-1}), g_0(x_r)))),$$

$$\tilde{j_1}(x_1 ... x_r, 1) = h_0(\nu_n(x_1, (\nu_n(..., \nu_n(x_{r-1}, x_r))))) = g_0(\nu_n(x_1, \nu_n(..., \nu_n(x_{r-1}, x_r)))).$$

Thus, we need to find a homotopy between the maps

$$\tilde{j}_{1|\mathbb{T}^1(\mathcal{M}_n)\times\{0\}} \quad \text{and} \quad \tilde{j}_{1|\mathbb{T}^1(\mathcal{M}_n)\times\{1\}}.$$

But such a homotopy exists because (5.9) commutes up to homotopy.

By the above argument we can find an extension $h_1 : F_1 \to K(\pi_n, n)$ of h_0. Now we will use induction and obstruction theory to obtain extensions $h_k : F_k \to K(\pi_n, n)$. Suppose we have constructed a map $h_k : F_k \to K(\pi_n, n)$ extending h_0 for $k \geq 1$ such that when restricted to $\mathbb{T}^k(\mathcal{M}_n) \times \{t\}$ we get a homomorphism of semigroups for all $t \in \Delta_k$. To get the desired extension $h_{k+1} : F_{k+1} \to K(\pi_n, n)$, all we only need to find is an extension of the map $j_{k+1} : \mathbb{T}^{k+2}(\mathcal{M}_n) \times \partial\Delta_{k+1} \to K(\pi_n, n)$ determined by the face maps and h_k. Since $\mathbb{T}^{k+2}(\mathcal{M}_n)$ is free on $\mathbb{T}^{k+1}(\mathcal{M}_n)$, this map determines and is determined by a map $\tilde{j}_{k+1} : \mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1} \to K(\pi_n, n)$. We want to extend the latter to $\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}$. As this space has the homotopy type of a CW-complex we can apply obstruction theory to get such a lifting. To do so we need to show that the associated obstructions

$$\omega_i \in H^{i+1}(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_i(K(\pi_n, n)))$$

vanish. This is automatic for $i \neq n$, as $\pi_i(K(\pi_n, n)) = 0$. So we only need to deal with the case $i = n$. By the Kunneth Theorem,

$$\begin{aligned} &H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n) \\ \cong &H^n(\mathbb{T}^{k+1}(\mathcal{M}_n), \pi_n) \oplus H^{n-k}(\mathbb{T}^{k+1}(\mathcal{M}_n), \pi_n). \end{aligned}$$

For $k \geq 1$ we have $H^{n-k}(\mathbb{T}^{k+1}(\mathcal{M}_n), \pi_n) = 0$. This is because $\mathbb{T}^{k+1}(\mathcal{M}_n)$ is built out of iterated products of $\mathcal{M}_n$ and $\mathcal{M}_n$ is $(n-1)$-connected so by the Hurewicz Theorem and the universal coefficient Theorem, $H^j(\mathcal{M}_n, \pi_n) = 0$ for $j < n$. Thus, we can conclude that the map

$$i^* : H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \pi_n) \to H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n)$$

induced by the inclusion map $i : \mathbb{T}^k(\mathcal{M}_n) \times \partial\Delta_k \to \mathbb{T}^k(\mathcal{M}_n) \times \Delta_k$ is an isomorphism. On the other hand, by the Kunneth Theorem in cohomology, we have that the map

$$i^* : H^{n+1}(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \pi_n) \to H^{n+1}(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n)$$

is injective. Therefore, by looking at the pair $(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1})$ we obtain the following long exact sequence

$$\begin{aligned}
\cdots \to & H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \pi_n) \to H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n) \\
\xrightarrow{\delta} & H^{n+1}(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n) \\
\to & H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \pi_n) \to H^n(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n) \to \cdots,
\end{aligned}$$

we see that

$$H^{n+1}(\mathbb{T}^{k+1}(\mathcal{M}_n) \times \Delta_{k+1}, \mathbb{T}^{k+1}(\mathcal{M}_n) \times \partial\Delta_{k+1}, \pi_n) = 0,$$

and thus the class ω_n vanishes.

This way, we can get coherent maps $h_k : F_k \to K(\pi_n, n)$. Now the topology on $B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n)$ is given the direct limit topology of the filtration $\{F_k\}_{k\geq 0}$. Therefore we get out of these successive extensions a continuous map $G_n : B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n) \to K(\pi_n, n)$. By construction, this map is a homomorphism of semigroups.

Let us define now $\mathcal{M}_{n+1}$ to be the homotopy fiber of

$$G_n : B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n) \to K(\pi_n, n)$$

($K(\pi_n, n)$ is given the unit element as base point). This way, we naturally get the structure of a semigroup with unit up to homotopy on $\mathcal{M}_n$. Indeed, given elements

$(x_i, \gamma_i) \in \mathcal{M}_{n+1}$ for $i = 1, 2$, we define $\nu_{n+1}((x_0, \gamma_0), (x_1, \gamma_1)) := (x_0 x_1, m_n(\gamma_1, \gamma_2))$. Clearly, $\mathcal{M}_{n+1}$ has the homotopy type of a CW-complex, and by the long exact sequence on homotopy groups, we get that $\mathcal{M}_{n+1}$ is n-connected.

The first projection map $p_n : \mathcal{M}_{n+1} \to B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n)$ is a semigroup homomorphism. Composing this with the semigroup homomorphism

$$B(\mathbb{T}, \mathbb{T}, \mathcal{M}_n) \to \mathcal{M}_n$$

we get the desired semigroup homomorphism $f_n : \mathcal{M}_{n+1} \to \mathcal{M}_n$. □

Lemma 5.8. *Let $\mathcal{M}$ be a connected topological semigroup of the homotopy type of a CW-complex that satisfies the unit condition. Then we can find a contractible semigroup $\mathcal{M}_\infty$ with both left and right unit up to homotopy, together with a homomorphism of semigroups $\mathcal{M}_\infty \to \mathcal{M}$.*

Proof: Define $\mathcal{M}_\infty$ to be the inverse limit of the sequence

$$\cdots \mathcal{M}_{n+1} \xrightarrow{f_n} \mathcal{M}_n \to \cdots \to \mathcal{M}_2 \xrightarrow{f_1} \mathcal{M}_1.$$

Then by the $\lim^1$ exact sequence on homotopy we get that $\mathcal{M}_\infty$ is a contractible semigroup with both left and right units up to homotopy together with a map of semigroups $f_\infty : \mathcal{M}_\infty \to \mathcal{M}$. □

This completes step (1) of our program.

Remark 5.9. In the special case of $\mathcal{M} = B\mathcal{D}^s$, which is a semigroup with unit up to homotopy, as we pointed out before, we can find such a contractible subgroup in the following way: Fix a complex separable infinite dimensional Hilbert space and call it $\mathcal{H}_o$. On $\mathcal{H}_o$, fix a nonzero element $h_0 \in \mathcal{H}_o$. Consider a universe $\mathfrak{U}$ that consists of iterated direct sums and tensor copies of $\mathcal{H}_o$ that is closed under tensor products and direct sums. On each Hilbert space $\mathcal{K}$ in $\mathfrak{U}$ we have an identified nonzero element $z_{\mathcal{K}}$ in the following way. For $\mathcal{K} = \mathcal{H}_o$, we have that $z_{\mathcal{H}_o} = h_0$. If we have selected elements $z_{\mathcal{K}_1} \in \mathcal{K}_1$, $z_{\mathcal{K}_2} \in \mathcal{K}_2$, for Hilbert spaces $\mathcal{K}_1, \mathcal{K}_2 \in \mathfrak{U}$ define

$$z_{\mathcal{K}_1 \oplus \mathcal{K}_2} = z_{\mathcal{K}_1} \oplus 0 \in \mathcal{K}_1 \oplus \mathcal{K}_2, \tag{5.10}$$

$$z_{\mathcal{K}_1 \otimes \mathcal{K}_2} = z_{\mathcal{K}_1} \otimes z_{\mathcal{K}_2} \in \mathcal{K}_1 \otimes \mathcal{K}_2. \tag{5.11}$$

By induction on the number of copies attached by means of tensor products and direct sums used to define any element in $\mathfrak{U}$ we can select a nonzero element in each Hilbert space in $\mathfrak{U}$ in a consistent way satisfying properties described in (5.10).

Based on this selection, we can consider a subcategory $\mathcal{V}$ of $\mathcal{C}$ (here we work with the chosen universe $\mathfrak{U}$) as follows: The objects of $\mathcal{V}$ are the pairs $(\mathcal{K}, F)$ where $\mathcal{K} \in \mathfrak{U}$ and $F : \mathcal{K} \to \mathcal{K}$ is a Fredholm operator for which there exists an isomorphism $\gamma : \mathcal{L} = \mathbb{C}\{z_{\mathcal{K}}\}^{\perp} \to \mathcal{K}$ such that

$$F(h) = \begin{cases} \gamma(h) \ \text{ if } h \in \mathcal{L} \\ 0 \text{ if } h \in \mathbb{C}\{z_{\mathcal{K}}\}. \end{cases} \tag{5.12}$$

If $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ are two objects of $\mathcal{V}$, then a morphism $(\alpha, \beta) : \mathfrak{a} \to \mathfrak{b}$ is a pair of unitary operators $\alpha, \beta : \mathcal{H} \to \mathcal{K}$ such that, as before, the following diagram is commutative

$$\begin{tikzcd} \mathcal{H} \arrow[r, "F"] \arrow[d, "\alpha"'] & \mathcal{H} \arrow[d, "\beta"] \\ \mathcal{K} \arrow[r, "G"] & \mathcal{K}. \end{tikzcd}$$

But in addition we require that $\alpha(z_H) = z_{\mathcal{K}}$. It follows that in this case the set of morphisms with source a fixed object $\mathfrak{a}$ of $\mathcal{V}$ is also contractible. Also note that for a fixed $\mathcal{K}$ the space of Fredholm operators of the form (5.12) is contractible. This is a consequence of Kuiper's Theorem. Therefore, as in Theorem 2.6 we have that $B\mathcal{V}$ is contractible.

On the other hand, we claim that $\mathcal{V}$ is closed under the operation $\otimes$. To see this, suppose that $\mathfrak{a} = (\mathcal{H}, F)$ and $\mathfrak{b} = (\mathcal{K}, G)$ are two objects in $\mathcal{V}$. Then, by definition, we have that

$$\mathfrak{a} \otimes \mathfrak{b} = (\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P_{F,G})$$

where $P_{F,G}$ is given by the matrix

$$\begin{bmatrix} -I \otimes G & F^* \otimes I \\ F \otimes I & I \otimes G^* \end{bmatrix}.$$

Note that in this case $\mathrm{Ker}\ P_{F,G} = \mathbb{C}\{z_{\mathcal{H}} \otimes z_{\mathcal{K}} \oplus 0 \otimes 0\}$. Also, we know that Index $P_{F,G} =$ (Index F)(Index G) $= 1$, thus we have that $P_{F,G}$ is surjective. Hence, if we call

$$\gamma = P_{F,G|\mathbb{C}\{z_{\mathcal{H}} \otimes z_{\mathcal{K}} \oplus 0 \otimes 0\}^{\perp}} : \mathbb{C}\{z_{\mathcal{H}} \otimes z_{\mathcal{K}} \oplus 0 \otimes 0\}^{\perp} \to \mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K},$$

we see that γ is an isomorphism and that $P_{F,G}$ has the form

$$P_{F,G}(h) = \begin{cases} \gamma(z) & \text{if } h \in \mathbb{C}\{z_{\mathcal{H}} \otimes z_{\mathcal{K}} \oplus 0 \otimes 0\}^{\perp}, \\ 0 & \text{if } z \in \mathbb{C}\{z_{\mathcal{H}} \otimes z_{\mathcal{K}} \oplus 0 \otimes 0\}. \end{cases}$$

So we see that $(\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P_{F,G})$ is also an object of $\mathcal{V}$. Also, if $f : \mathfrak{a} \to \mathfrak{b}$ and $g : \mathfrak{c} \to \mathfrak{d}$ are morphisms in $\mathcal{V}$, then by a direct computation we have that $f \otimes g : \mathfrak{a} \otimes \mathfrak{c} \to \mathfrak{b} \otimes \mathfrak{d}$ is also a morphism in $\mathcal{V}$.

As before, we can consider the strict version of the category $\mathcal{V}$, say $\mathcal{V}^s$ that consists of formal strings $\mathfrak{a}_1 ... \mathfrak{a}_n$ for objects $\mathfrak{a}_1, ..., \mathfrak{a}_n$ and $n \geq 1$. The category $\mathcal{V}^s$ is equivalent to the category $\mathcal{V}$ and thus we have that $B\mathcal{V}^s$ is contractible. On the other hand, note that $B\mathcal{V}^s$ is a semigroup and that we have a homomorphism $B\mathcal{V}^s \to B\mathcal{D}^s$ induced by the inclusion functor $i : \mathcal{V} \to \mathcal{D}$.

5.2 Monoid replacement

In this section we complete step (2) of our program. For this we fix a contractible semigroup $\mathcal{M}_\infty$ and a homomorphism of semigroups $f_\infty : \mathcal{M}_\infty \to \mathcal{M}$. Here, $\mathcal{M}$ is a semigroup that satisfies the same properties listed above. We will show that we can replace the semigroup action of $\mathcal{M}$ on a space X by the action of a topological monoid $\hat{\mathcal{M}}$ weakly equivalent to $\mathcal{M}$. In more concrete terms we prove the following theorem.

Theorem 5.10. *Let $\mathcal{M}$ be a topological semigroup of the homotopy type of a CW complex and assume that $\mathcal{M}$ satisfies the unit condition. Then we can find a topological semigroup $\mathcal{M}'$, a topological monoid $\hat{\mathcal{M}}$, and homomorphisms of semigroups $i_1 : \mathcal{M}' \to \mathcal{M}$ and $i_2 : \mathcal{M}' \to \hat{\mathcal{M}}$. Moreover, if $\mathcal{M}$ acts on a space X, then this action induces actions of $\mathcal{M}'$ and $\hat{\mathcal{M}}$ that are compatible with i_1 and i_2.*

Before proving this theorem we need some definitions.

Definition 5.11.
- Define a monad $(\mathbb{D}, \mu_{\mathbb{D}}, \eta_{\mathbb{D}})$ in the following way. For a space X, $\mathbb{D}X$ is the smallest semigroup containing both X and $\mathcal{M}_\infty$. This monad comes from the adjunction of the functors $\mathbb{D} : \mathcal{U} \to \mathit{Top.Semigroups}$, $U : \mathit{Top.Semigroup} \to \mathcal{U}$, where U is the forgetful functor.
- Also, define the monad $(\mathbb{C}, \mu_{\mathbb{C}}, \eta_{\mathbb{C}})$ that for a space assigns the smallest monoid containing X and a disjoint unit.

We can give an explicit description for the monads $\mathbb{C}$ and $\mathbb{D}$. If X is a space in

$\mathcal{U}$, then

$$\mathbb{D}X = \coprod_{k\geq 0} \coprod_{n_1,\ldots,n_k\geq 0, j_1,\ldots,j_k\in\{0,1\}} \mathcal{M}_\infty^{j_1} \times X^{n_1} \ldots \times \mathcal{M}_\infty^{j_k} \times X^{n_k}$$

$$\mathbb{C}X = \coprod_{k\geq 0} (X_+)^k$$

As usual, X_+ is the disjoint union of X and a point.

We have a morphism of monads $\mathbb{D} \to \mathbb{C}$ given by the map $\mathcal{M}_\infty \to *$, and the homomorphism $f_\infty : \mathcal{M}_\infty \to \mathcal{M}$ determines a structure of a $\mathbb{D}$-algebra on $\mathcal{M}$. Thus, we get a morphism of semigroups

$$j : B(\mathbb{D}, \mathbb{D}, \mathcal{M}) \to B(\mathbb{C}, \mathbb{D}, \mathcal{M}).$$

In the following lemma we show that j is a homotopy equivalence.

Lemma 5.12. *The map $j : B(\mathbb{D}, \mathbb{D}, \mathcal{M}) \to B(\mathbb{C}, \mathbb{D}, \mathcal{M})$ is a homotopy equivalence.*

Proof. As $\mathcal{M}_\infty$ is contractible, it follows that each map

$$j_n : B_n(\mathbb{D}, \mathbb{D}, \mathcal{M}) \to B_n(\mathbb{C}, \mathbb{D}, \mathcal{M})$$

is a homotopy equivalence. It follows easily that $B_*(\mathbb{D}, \mathbb{D}, \mathcal{M})$ and $B_*(\mathbb{C}, \mathbb{D}, \mathcal{M})$ are proper simplicial spaces. Then the lemma follows by [31, Theorem A.4]. □

We are now ready to prove the following theorem.

Theorem 5.13. *Let $\mathcal{M}$ be a topological semigroup of the homotopy type of a CW-complex and assume that $\mathcal{M}$ satisfies the unit condition. Then we can find a topological semigroup $\mathcal{M}'$, a topological monoid $\hat{\mathcal{M}}$, and homomorphisms of semigroups $i_1 : \mathcal{M}' \to \mathcal{M}$ and $i_2 : \mathcal{M}' \to \hat{\mathcal{M}}$ that are weak equivalences. Moreover, if $\mathcal{M}$ acts on a space X, then this action induces actions of $\mathcal{M}'$ and $\hat{\mathcal{M}}$ that are compatible with i_1 and i_2.*

Proof. Consider $\mathcal{M}' = B_n(\mathbb{D}, \mathbb{D}, \mathcal{M})$ and $\hat{\mathcal{M}} = B_n(\mathbb{C}, \mathbb{D}, \mathcal{M})$. It follows that $\mathcal{M}'$ is a semigroup, $\hat{\mathcal{M}}$ is a monoid, and that we have homomorphisms $i_1 : \mathcal{M}' \to \mathcal{M}$ and $i_2 = j : \mathcal{M}' \to \hat{\mathcal{M}}$. By [30, Proposition 9.8] we have that i_1 is a homotopy equivalence, and by the previous lemma, so is $i_2 = j$. Now suppose that $\mathcal{M}$ acts on a space X; that is, we have a continuous map $\mu : \mathcal{M} \times X \to X$ which we denote by $\cdot$, satisfying $m_1 \cdot (m_2 \cdot x) = (m_1 m_2) \cdot x$. Notice that this action defines an action of $\mathbb{D}^k\mathcal{M}$ on X for $k \geq 0$. To see this, notice that the action of $\mathcal{M}$ on X induces actions of $\mathcal{M}_\infty$ and $\mathcal{M}^n$ on X. Since

$$\mathbb{D}\mathcal{M} = \coprod_{k\geq 0} \coprod_{n_1,\ldots,n_k\geq 0, j_1,\ldots,j_k\in\{0,1\}} \mathcal{M}_\infty^{j_1} \times \mathcal{M}^{n_1} \times \cdots \times \mathcal{M}_\infty^{j_k} \times \mathcal{M}^{n_k},$$

if $(m_1, ..., m_k) \in \mathbb{D}\mathcal{M}$ then we can define $(m_1, ..., m_k) \cdot x = m_1 \cdot (\cdots (m_k \cdot x))$. Having defined an action of $\mathbb{D}^k\mathcal{M}$ on X, we define an action of $\mathbb{D}^{k+1}\mathcal{M}$ by simply noting that $\mathbb{D}^{k+1} = \mathbb{D}(\mathbb{D}^k\mathcal{M})$. Having done this we can define an action of $B(\mathbb{D}, \mathbb{D}, \mathcal{M})$ on X in the following way: Take $y \in B(\mathbb{D}, \mathbb{D}, \mathcal{M})$. Then we can write $y = [m, t]$ where $m \in D^{k+1}$, $t \in \Delta_k$. Then simply define $y \cdot x := m \cdot x$. It is follows directly from the definition that this is well defined and that for each $y \in B(\mathbb{D}, \mathbb{D}, \mathcal{M}) = \mathcal{M}'$ and $x \in X$ we have $y \cdot x = i_1(y) \cdot x$. A similar statement follows for $B(\mathbb{C}, \mathbb{D}, \mathcal{M})$. $\square$

The previous theorem says that we can replace the semigroup $\mathcal{M}$ by the monoid $\hat{\mathcal{M}}$. This completes step (2) of our program.

5.3 Group replacement

In this section we complete the final step of our program of replacing actions of topological semigroups by actions of topological groups of the same weak homotopy type. We want to show that any connected topological monoid $\hat{\mathcal{M}}$ can be replaced in a similar way by a topological group H. As a space, $\hat{\mathcal{M}}$ is a based space with base point the unit of $\hat{\mathcal{M}}$. For technical reasons, we will need that the base point of $\hat{\mathcal{M}}$ is non-degenerate. If this is not the case, we can replace $\hat{\mathcal{M}}$ by the whiskered space $\mathcal{M} \vee I$. This is also a topological monoid that is homotopy equivalent to $\mathcal{M}$ with non-degenerate base point 1.

We want to prove the following theorem.

Theorem 5.14. *Let $\hat{\mathcal{M}}$ be a connected topological monoid of the homotopy type of a CW-complex with non-degenerate unit. Then we can find a topological group H, and a topological monoid H', together with homomorphisms $j_1 : H' \to \hat{\mathcal{M}}$ and $j_2 : H' \to H$ that are weak homotopy equivalences. Moreover, if $\hat{\mathcal{M}}$ acts on a space X, then the action of $\hat{\mathcal{M}}$ can be extended to actions of H' and H on X that are compatible with j_1 and j_2.*

As before, we need some definitions before we prove the theorem.

Definition 5.15.

- Define the monad $(\mathbb{D}_{Mo}, \mu_{\mathbb{D}_{Mo}}, \eta_{\mathbb{D}_{Mo}})$ to be the monad that for a based space (X, x_0) corresponds to the free monoid on X with unit the base point x_0.

- Define the monad $(\mathbb{D}_{Gr}, \mu_{\mathbb{D}_{Gr}}, \eta_{\mathbb{D}_{Gr}})$ to be the monad that for a based space (X, x_0) corresponds to the free group on X with unit the base point x_0.

As before, these monads are constructed from the adjunction of these functors and the respective forgetful functor. Since $\hat{\mathcal{M}}$ is a monoid, it is an algebra for the monad $\mathbb{D}_{Mo}$, and we have a morphism of monads $\mathbb{D}_{Mo} \to \mathbb{D}_{Gr}$. Thus, we can consider the simplicial spaces $B_*(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$ and $B_*(\mathbb{D}_{Gr}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$. As the unit of $\hat{\mathcal{M}}$ is non-degenerate we have that the simplicial space $B_*(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$ is proper. By [42, Proposition A.1 (iv).] the natural map $\mathbb{B}(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}) \to B(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$ is a weak homotopy equivalence. Also, we have that the natural map $B(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}) \to$

$\hat{\mathcal{M}}$ is a homotopy equivalence. These maps are homomorphisms of monoids. In conclusion, we have a homomorphism of monoids $\mathbb{B}(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}) \to \hat{\mathcal{M}}$ that is a weak homotopy equivalence.

On the other hand, the morphism of monads $\mathbb{D}_{Mo} \to \mathbb{D}_{Gr}$ induces a simplicial map

$$g_* : B_*(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}) \to B_*(\mathbb{D}_{Gr}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}).$$

This in turns induces a homomorphism of monoids

$$g : \mathbb{B}(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}) \to \mathbb{B}(\mathbb{D}_{Gr}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}).$$

We will show in the next lemma that g is a weak equivalence.

Lemma 5.16. *The map $g : \mathbb{B}(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}}) \to \mathbb{B}(\mathbb{D}_{Gr}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$ induced by the morphism of monads $\mathbb{D}_{Mo} \to \mathbb{D}_{Gr}$ is a weak homotopy equivalence.*

Proof. Let us denote by $X_* = B_*(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \mathcal{B})$ and $Y_* = B_*(\mathbb{D}_{Gr}, \mathbb{D}_{Mo}, \mathcal{B})$. We are going to show that each $g_n : X_n \to Y_n$ is a weak homotopy equivalence. If this is true, then by [42, Proposition A.1.(ii).], it follows that g is a weak homotopy equivalence.

By definition we have that $X_n = \mathbb{D}_{Mo}^{n+1}\hat{\mathcal{M}}$ and $Y_n = \mathbb{D}_{Gr}\mathbb{D}_{Mo}^{n}\hat{\mathcal{M}}$. Let $W = \mathbb{D}_{Mo}^{n}(\hat{\mathcal{M}})$. Since $\hat{\mathcal{M}}$ is connected, so is $\mathbb{D}_{Mo}\hat{\mathcal{M}}$, and, thus, by induction, it follows that W is a connected monoid. For similar reasons, we also have that each X_n and Y_n is connected. Since X_n and Y_n are connected H-spaces, to show that g_n is a weak equivalence we only need to show that g_n induces isomorphisms in homology.

Notice that X_n and Y_n are the free monoid and group with unit the base point of W. As W is connected, by [12, Theorem 2.3.3] we get that g_n is a weak homotopy equivalence. (Barratt and Priddy work on the category of based simplicial sets, but we can achieve this by taking the singular complex.) $\square$

Proof. of Theorem 5.14. Let $H' = \mathbb{B}(\mathbb{D}_{Mo}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$ and $H = \mathbb{B}(\mathbb{D}_{Gr}, \mathbb{D}_{Mo}, \hat{\mathcal{M}})$. Then H' is a topological monoid, H is a topological group and we have homomorphisms $j_1 : H' \to \hat{\mathcal{M}}$ and $j_2 = g : H' \to H$ which we have shown are weak homotopy equivalences. As in the case of the semigroup replacement, we can see that the action of $\hat{\mathcal{M}}$ on a space X gives rise to actions of H' and H that are compatible with the maps j_1 and j_2. $\square$

Finally, Theorem 5.1 follows from Theorems 5.10 and 5.14.

We can apply this result to the particular case of $\mathcal{M} = B\mathcal{D}^s$, which is a topological semigroup, of the homotopy type of a CW-complex and with both left and right units (which we can assume are non-degenerate) acting on $B\mathcal{C}^s$. This way, we obtain a topological group of the same weak homotopy type which we will denote by S', together with an action of S' on $B\mathcal{C}^s$. Therefore, $S := \mathbb{Z}/2 \times S' \simeq K_{\otimes}$ is a topological group acting on $B\mathcal{C}^s$.

CHAPTER 6

A spectrum representing equivariant K-theory

The goal of this chapter is to construct a G-equivariant spectrum E_G that represents G-equivariant K-theory together with an action of the topological group $S \simeq K_{\otimes}$ that was constructed in the previous chapter. The action of S on E that we will obtain is an action by maps of spectra.

Throughout this chapter, it will be more convenient to work with $\mathbb{Z}/2$-graded Hilbert spaces. We include in the appendix a description of these spaces together with an analogue of the space $\mathcal{F}(\mathcal{H})$ for a $\mathbb{Z}/2$-graded Hilbert space. We start by recalling some of the definitions and the analogues for the equivariant situation.

Definition 6.1. A complex separable infinite dimensional $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}_*$ is a Hilbert space $\mathcal{H}_*$ that can be decomposed in the form $\mathcal{H}_* = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ are two complex separable infinite dimensional Hilbert spaces. In the case that both $\mathcal{H}_0$ and $\mathcal{H}_1$ are G-stable Hilbert spaces, then we say that $\mathcal{H}_*$ is a $\mathbb{Z}/2$-graded G-stable Hilbert space.

Definition 6.2. For a $\mathbb{Z}/2$-graded G-stable Hilbert space $\mathcal{H}_G$ we denote by $\mathcal{F}^1_G(\mathcal{H}_G)$ the space of self-adjoint degree 1 Fredholm operators on $\mathcal{H}_G$ such that $g \mapsto gFg^{-1}$ is continuous.

If we give $\mathcal{F}^1_G(\mathcal{H}_G)$ the norm topology then we also get a classifying space for G-

equivariant K-theory. As in the non-equivariant situation we have a G-equivariant homeomorphism

$$\mathfrak{r} : \mathcal{F}_G(\mathcal{H}_G) \to \mathcal{F}^1_G(\mathcal{H}_{G,0} \oplus \mathcal{H}_{G,1})$$
$$F \to \mathfrak{r}(F) = \begin{bmatrix} & F \\ F^* & \end{bmatrix}$$

where $\mathcal{H}_{G,0} = \mathcal{H}_{G,1} = \mathcal{H}_G$.

All the constructions in the previous chapters work if, instead of working with Fredholm operators, we work with self-adjoint degree 1 Fredholm operators. To be more precise, given $\mathfrak{U}_G$ as before, we can construct a universe $\mathfrak{U}^1_G$ of $\mathbb{Z}/2$-graded Hilbert spaces closed under direct sums and tensor products, and such that $\mathcal{H}_G \oplus \mathcal{H}_G$ is in $\mathfrak{U}^1_G$ for all $\mathcal{H}_G \in \mathfrak{U}_G$. Then, if G is a compact Lie group, we can construct a category $\mathcal{C}^1_G$ whose objects are tuples $(\mathcal{H}_G, F)$, where $\mathcal{H}_G$ is a $\mathbb{Z}/2$-graded Hilbert space in $\mathfrak{U}^1_G$ and F is a self-adjoint, degree 1 Fredholm operator such that $g \mapsto gFg^{-1}$ is continuous. If $(\mathcal{H}_G, F)$ and $(\mathcal{K}_G, R)$ are two objects of $\mathcal{C}^1_G$, then a morphism

$$f : (\mathcal{H}_G, F) \to (\mathcal{K}_G, R)$$

is a pair of degree zero, unitary operators (α, β) such that $g \mapsto g\alpha g^{-1}$ and $g \mapsto g\beta g^{-1}$ are continuous functions and the following diagram is commutative

$$\begin{tikzcd} \mathcal{H}_G \arrow[r, "F"] \arrow[d, "\alpha"'] & \mathcal{H}_G \arrow[d, "\beta"] \\ \mathcal{K}_G \arrow[r, "R"'] & \mathcal{K}_G. \end{tikzcd}$$

We have a continuous functor $\mathcal{R} : \mathcal{C}_G \to \mathcal{C}^1_G$ defined on objects by $(\mathcal{H}, F) \mapsto (\mathcal{H} \oplus \mathcal{H}, \mathfrak{r}(F))$ and on morphisms by $\mathcal{R}(\alpha, \beta) = (\alpha \oplus \alpha, \beta \oplus \beta)$. This functor gives us an equivariant map $B\mathcal{R} : B\mathcal{C}_G \to B\mathcal{C}^1_G$ which is a G-homotopy equivalence, since the simplicial spaces $B_*\mathcal{C}_G^H \to B_*\mathcal{C}_G^{1H}$ are proper simplicial spaces, and for each subgroup H of G, $B\mathcal{R}^H : B\mathcal{C}_G^H \to B\mathcal{C}_G^{1H}$ is a homotopy equivalence.

On $\mathcal{C}_G^1$ we also have operations $\oplus$ and $\otimes$, but in this case the operation $\otimes$ is given as follows. Supppose $(\mathcal{H}, F)$and $(\mathcal{K}, R)$ are two objects in $\mathcal{C}_G^1$. Then

$$(\mathcal{H}, F) \otimes (\mathcal{K}, R) = (\mathcal{H} \otimes \mathcal{K}, F \otimes I + I \otimes R).$$

From the definition, we have that the following diagrams commute

$$\begin{tikzcd}
\mathcal{C}_G \times \mathcal{C}_G \arrow[r, "\otimes"] \arrow[d, "\mathcal{R}\times\mathcal{R}"'] & \mathcal{C}_G \arrow[d, "\mathcal{R}"] \\
\mathcal{C}_G^1 \times \mathcal{C}_G^1 \arrow[r, "\otimes"'] & \mathcal{C}_G^1,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_G \times \mathcal{C}_G \arrow[r, "\oplus"] \arrow[d, "\mathcal{R}\times\mathcal{R}"'] & \mathcal{C}_G \arrow[d, "\mathcal{R}"] \\
\mathcal{C}_G^1 \times \mathcal{C}_G^1 \arrow[r, "\oplus"'] & \mathcal{C}_G^1.
\end{tikzcd}$$

We show the commutativity of these diagrams for the non-equivariant situation in the appendix. The equivariant case is handled in a similar way.

Convention 6.3. Whenever a, b, c and d are graded objects, we use the common convention

$$(a \otimes b) \circ (c \otimes d) = (-1)^{(\deg b)(\deg c)}(a \circ c) \otimes (b \circ d).$$

Remark 6.4. Whenever we take the tensor product of two $\mathbb{Z}/2$-graded Hilbert spaces we mean the graded tensor product; that is, if $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, then $\mathcal{H} \otimes \mathcal{K} = (\mathcal{H}_0 \otimes \mathcal{K}_0 \oplus \mathcal{H}_1 \otimes \mathcal{K}_1) \oplus (\mathcal{H}_0 \otimes \mathcal{K}_1 \oplus \mathcal{H}_1 \otimes \mathcal{K}_0)$.

On $\mathcal{C}_G^1$ we have a copy of $\mathcal{D}_G$, namely, we can take $\mathcal{D}_G^1$ to be the full subcategory whose objects are of the form $\mathcal{R}(A)$, where A is an object of $\mathcal{D}_G$. Also, we have a copy of $\mathcal{D}$, namely, the category $\mathcal{D}^1$ which is the full subcategory of $\mathcal{C}_G^1$ whose objects are the objects of $\mathcal{D}_G^1$ where G acts trivially on the corresponding Hilbert space.

By abuse of notation, from now on we will denote $\mathcal{C}_G^1, \mathcal{D}_G^1$ and $\mathcal{D}^1$ by $\mathcal{C}_G, \mathcal{D}_G$ and $\mathcal{D}$ respectively, since it will be clear in each case if we are working in the graded or ungraded case, and these are equivalent categories.

Consider now a complex finite dimensional G-representation V and denote by n the dimension of V as a real vector space. Since G is a compact Lie group we can find a positive definite Hermitian form $\langle\ ,\ \rangle$ which is invariant under the G-action; that is, $\langle v, w\rangle = \langle g \cdot v, g \cdot w\rangle$ for all $v, w \in V$ and all $g \in G$. Let us denote by C_V to the Clifford algebra associated to the norm $\|v\|^2 = \langle v, v\rangle$. The action of G on V extends naturally to an action of G on C_V. Take K_V an irreducible G-equivariant C_V-module; that is, we have a G-equivariant homomorphism $\rho : C_V \to Aut(K_V)$, where G acts on $Aut(K_V)$ by conjugation. If we pick an orthogonal basis $e_1, ..., e_n$ for V, then each e_i acts on K_V as a skew-adjoint unitary operator. We will denote by J_i the operator $i\rho(e_k)$ on K_V. Then $J_1, ..., J_n$ are unitary, self-adjoint degree 1 operators such that $J_iJ_j = -J_jJ_i$ for $i \neq j$.

If V is as above then we will denote by $\Omega^V\mathcal{C}_G$ the topological category whose object and morphism spaces are $\Omega^V(\mathcal{C}_G)_0$ and $\Omega^V(\mathcal{C}_G)_1$, respectively. An object ω of $\Omega^V\mathcal{C}_G$ is then a map $\omega : S^V \to (\mathcal{C}_G)_0$. As $(\mathcal{C}_G)_0 = \coprod_{\mathcal{H}_G\in\mathcal{U}_G} \mathcal{F}^1_G(\mathcal{H}_G)$, the image of an object ω will lie in $\mathcal{F}^1_G(\mathcal{H}_G)$ for some $\mathcal{H}_G \in \mathcal{U}_G$. Thus, we will denote an object of $\Omega^V(\mathcal{C}_G)_0$ as a pair $(\mathcal{H}_G, \omega)$, where $\omega : S^V \to \mathcal{F}^1_G(\mathcal{H}_G)$.

The functor $\otimes : \mathcal{D} \times \mathcal{C}_G \to \mathcal{C}_G$ can be extended to an operation $\otimes : \mathcal{D} \times \Omega^V\mathcal{C}_G \to \Omega^V\mathcal{C}_G$ by defining $(s \otimes \omega)(v) = s \otimes \omega(v)$. Here, s is an object in $\mathcal{D}$ and ω is an object in $\Omega^V\mathcal{C}_G$. Similarly, we define $\otimes$ on morphisms.

In the non-equivariant case, after a change of variables, we have that the Bott periodicity map, as described in [7], is given by

$$\mathcal{F}^1(\mathcal{H}) \to \Omega^n\mathcal{F}^1(K_n \otimes \mathcal{H})$$

$$F \mapsto \omega_F$$

where $\omega_F : S^n \to \mathcal{F}^1(K_n \otimes \mathcal{H})$ is given by

$$\omega_F(a_1, ..., a_n) = \sum_{1\le k\le n} a_k J_k \otimes I + (\sqrt{1 - a_1^2 - \cdots - a_n^2}) I \otimes F \text{ if } a_1^2 + \cdots + a_n^2 \le 1,$$
$$\omega_F(a_1, ..., a_n) = \sum_{1\le k\le n} b_k J_k \otimes I + (\sqrt{1 - b_1^2 - \cdots - b_n^2}) I \otimes I \text{ if } a_1^2 + \cdots + a_n^2 \ge 1.$$

Here S^n is the one point compactification of $\mathbb{R}^n$ and $(b_1, ..., b_n) = \frac{1}{a_1^2+...+a_n^2}(a_1, ..., a_n)$ is the reflection of the point $(a_1, ..., a_n)$ with respect to the unit circle.

We can generalize the Bott map to get a functor that is G-equivariant on objects and morphisms in the following way. Define the functor $\mathcal{P} : \mathcal{C}_G \to \Omega^V \mathcal{C}_G$ given as follows: if $(\mathcal{H}, F)$ is an object of $\mathcal{C}_G$, then

$$\mathcal{P}(\mathcal{H}, F) = (K_V \otimes \mathcal{H}, \omega_F),$$

where if $v = \sum_{1\le k\le n} a_k e_k$ and $\frac{v}{\|v\|^2} = \sum_{1\le k\le n} b_k e_k$, then

$$\omega_F(v) = \sum_{1\le k\le n} a_k J_k \otimes I + (\sqrt{1 - \|v\|^2}) I \otimes F \text{ if } \|v\| \le 1,$$
$$\omega_F(v) = \sum_{1\le k\le n} b_k J_k \otimes I + (\sqrt{1 - \frac{1}{\|v\|^2}}) I \otimes I \text{ if } \|v\| \ge 1,$$

and if (α, β) is a morphism in $\mathcal{C}_G$, then

$$\mathcal{P}(\alpha, \beta) = (\Omega^V(I \otimes \alpha), \Omega^V(I \otimes \beta)).$$

It follows at once that $\mathcal{P}$ is G-equivariant on objects and morphisms. By construction, the functor $\mathcal{P}$ induces a functor $\mathcal{P}^s : \mathcal{C}_G^s \to \Omega^V \mathcal{C}_G^s$ and, by passing to classifying spaces, we get a map

$$f := B\mathcal{P}^s : B\mathcal{C}_G^s \to B\Omega^V \mathcal{C}_G^s \xrightarrow{\simeq} \Omega^V B\mathcal{C}_G^s.$$

Remark 6.5. The natural map $B\Omega^V \mathcal{C}_G^s \to \Omega^V B\mathcal{C}_G^s$ is a weak homotopy equivalence. To see this, note first that the restriction of the natural map $B\Omega^V \mathcal{C}_G^s \to \Omega^V B\mathcal{C}_G^s$

to the respective connected components containing the base points is a weak homotopy equivalence by [30, Theorem 12.3]. Also note that both $B\Omega^V\mathcal{C}_G^s$ and $\Omega^V B\mathcal{C}_G^s$ are double loop spaces. This implies that $B\Omega^V\mathcal{C}_G^s \to \Omega^V B\mathcal{C}_G^s$ is a weak homotopy equivalence.

The construction of the desired G-equivariant spectrum E_G that represents G-equivariant K-theory together with an action of S by maps of spectra relies on the following theorem.

Theorem 6.6. *There exists a map*

$$F : \mathbb{B}(S, S, B\mathcal{C}_G^s) \to \Omega^V B\mathcal{C}_G^s$$

such that F is S-equivariant and $F([s,c]) = s \cdot f(c)$ for $s \in S_G$ and $c \in B\mathcal{C}_G^s$.

Recall that here $\mathbb{B}(S, S, B\mathcal{C}_G^s)$ means the geometric realization without degeneracies of the simplicial space $B_*(S, S, B\mathcal{C}_G^s)$.

To prove the Theorem 6.6 we will prove the following lemma first.

Lemma 6.7. *We can find a $B\mathcal{D}_\pm^s$-equivariant map*

$$F_1 : \mathbb{B}(B\mathcal{D}_\pm^s, B\mathcal{D}_\pm^s, B\mathcal{C}_G^s) \to \Omega^V B\mathcal{C}_G^s.$$

Proof. We will construct successive extensions of the map

$$g_0 : B\mathcal{D}_\pm^s \times B\mathcal{C}_G^s \to \Omega^V B\mathcal{C}_G^s$$
$$(s, c) \mapsto s \cdot f(c)$$

to the different stages of the filtration of $\mathbb{B}(B\mathcal{D}_\pm^s, B\mathcal{D}_\pm^s, B\mathcal{C}_G^s)$. Getting such extensions is equivalent to finding a sequence of maps

$$g_n : (B\mathcal{D}_\pm^s)^{n+1} \times B\mathcal{C}_G^s \times \Delta_n \to \Omega^V B\mathcal{C}_G^s$$

such that $g_n(\partial_i(s_0, ..., s_n, c), t) = g_{n+1}((s_0, ..., s_n, c), \delta_i(t))$ for all $0 \leq i \leq n$ and all $s_0, ..., s_n \in B\mathcal{D}^s_\pm$, $c \in B\mathcal{C}^s_G$, and with $g_0(s, c) = s \cdot f(c)$. We will construct the maps g_n inductively. To complete the induction we are going to show that in the original category $\mathcal{C}_G$ we can extend the desired map on a general simplex by taking the linear combination of the restrictions to the boundary (after changing by unitary operators the underlying Hilbert spaces so we can take linear combinations). To be more precise, for $k \geq 1$ and $0 \leq i \leq k$, define functors

$$\mathcal{G}_{k,i} : \mathcal{D}^k_\pm \times \mathcal{C}_G \to \Omega^V \mathcal{C}_G$$

on objects as follows. Consider objects $s_i = (\mathcal{L}_i, R_i)$ of $\mathcal{D}_\pm$ for $0 \leq i \leq k$, and $c = (\mathcal{H}, F)$ an object of $\mathcal{C}_G$. Define

$$\mathcal{G}_{k,i}(s_1, ..., s_k, c) = s_1 \otimes (\cdots \otimes (s_{k-i} \otimes (\mathcal{P}(s_{k-i+1} \otimes \cdots \otimes ((s_k \otimes c)))))).$$

Similarly, we define $\mathcal{G}_{k,i}$ on morphisms. According to this definition,l we have that $\mathcal{G}_{k,i}(s_1, ..., s_k, c)$ is given by the pair

$$(\mathcal{K}_{s_1,...,s_k,c,i}, \omega_{s_1,...,s_k,c,i}),$$

where $\mathcal{K}_{s_1,...,s_k,c,i}$ is the $\mathbb{Z}/2$-graded Hilbert space

$$\mathcal{K}_{s_1,...,s_k,c,i} = \mathcal{L}_1 \otimes (\mathcal{L}_2 \otimes (\cdots \otimes (\mathcal{L}_{k-i} \otimes (K_V \otimes (\mathcal{L}_{k-i+1} \otimes \cdots \otimes (\mathcal{L}_k \otimes \mathcal{H})))))).$$

And $\omega_{s_1,...,s_k,c,i}$ is the operator

$$\omega_{s_1,...,s_k,c,i} : S^V \to \mathcal{F}^1(\mathcal{K}_{s_1,...,s_k,c,i})$$

whose value at $v = \sum_{1\leq k\leq n} a_k e_k \in S^V$, with $\|v\| \leq 1$ is

(6.1)

$$\omega_{s_1,...,s_k,c,i}(v) = \sum_{1\leq r\leq n} I \otimes \cdots \otimes (a_r J_r) \otimes \cdots \otimes I + t_v(\sum_{k-i+1\leq r\leq k} (I \otimes \cdots \otimes R_r \otimes \cdots \otimes I))$$

$$+ t_v I \otimes \cdots \otimes F + \sum_{1\leq r\leq k-i} I \otimes \cdots \otimes R_r \otimes \cdots \otimes I, \tag{6.2}$$

where

$$t_v = \sqrt{1 - a_1^2 - ... - a_n^2}.$$

A similar formula holds in the case $\|v\| \geq 1$.

Fix $v = \sum_{1 \leq k \leq n} a_k e_k \in S^V$. From now on we will assume that $\|v\| \leq 1$, as everything that will come is similar for the case $\|v\| \geq 1$. We will begin by proving the following lemma. Define

$$\mathcal{M}_{s_1,...,s_k,c} = \mathcal{K}_V \otimes (\mathcal{L}_1 \otimes \cdots \otimes (\mathcal{L}_k \otimes \mathcal{H})).$$

Lemma 6.8. *There are degree 0 unitary operators*

$$\alpha_{s_1,...,s_k,c,i} : \mathcal{K}_{s_1,...,s_k,c,i} \to \mathcal{M}_{s_1,...,s_k,c}$$

such that if

$$\sigma_{s_1,...,s_k,c,i} = \alpha_{s_1,...,s_k,c,i} \omega_{s_1,...,s_k,c,i}(v) \alpha^{-1}_{s_1,...,s_k,c,i},$$

then for $t_0, ..., t_k \geq 0$ *with* $\sum_{i=0}^k t_i = 1$, *the operator*

$$\sum_{i=0}^k t_i \sigma_{s_1,...,s_k,c,i}$$

is a Fredholm operator.

Proof. Using the associativity and commutativity of the tensor product up to natural isomorphism, we can find a degree 0 unitary operator

$$\alpha_{s_1,...,s_k,c,i} : \mathcal{K}_{s_1,...,s_k,c,i} \to \mathcal{M}_{s_1,...,s_k,c}.$$

According to (6.1) we have that

$$\sigma_{s_1,...,s_k,c,i} = \alpha_{s_1,...,s_k,c,i} \omega_{s_1,...,s_k,c,i}(v) \alpha^{-1}_{s_1,...,s_k,c,i}$$

is given by

$$\sum_{1\le r\le n}(a_rJ_r)\otimes I\otimes\cdots\otimes I+\sum_{1\le r\le k-i} I\otimes\cdots\otimes R_r\otimes\cdots\otimes I$$
$$+\,t_vI\otimes\cdots\otimes F+t_v(\sum_{k-i+1\le r\le k}(I\otimes\cdots\otimes R_r\otimes\cdots\otimes I)).$$

Thus if $t_0,...,t_k\ge 0$ with $\sum_{i=0}^k t_i=1$, then

$$\sum_{i=0}^k t_i\sigma_{s_1,\ldots,s_k,c,i}=\sum_{1\le r\le n}(a_rJ_r)\otimes I\otimes\cdots\otimes I+t_vI\otimes\cdots\otimes F$$
$$+\sum_{r=1}^k(\sum_{i=1}^{k-r}t_i+\sum_{r=k-r+1}^k t_it_v)I\otimes\cdots\otimes R_r\otimes\cdots I.$$

To see that this is a Fredholm operator we consider two cases.

Case 1: $a_1=\cdots=a_n=0$. In this case we have that $t_v=1$ and thus

$$\sum_{i=0}^k t_i\sigma_{s_1,\ldots,s_k,c,i}=I\otimes\cdots\otimes F+\sum_{r=1}^k I\otimes\cdots\otimes R_r\otimes\cdots I.$$

We know that if A and B are two degree 1 self-adjoint Fredholm operators, then $A\otimes I+I\otimes B$ is also a Fredholm operator. Thus, by induction on k we have that $I\otimes\cdots\otimes F+\sum_{r=1}^k I\otimes\cdots\otimes R_r\otimes\cdots I$ is a degree 1 self-adjoint Fredholm operator.

Case 2: $a_i\ne 0$ for some i. Without loss of generality, we can assume that $a_1\ne 0$. Then

$$\sum_{i=0}^k t_i\sigma_{s_1,\ldots,s_k,c,i}=a_1J_1\otimes I+K$$

where

$$K=\sum_{2\le r\le n}(a_rJ_r)\otimes I\otimes\cdots\otimes I+t_vI\otimes\cdots\otimes F$$
$$+\sum_{r=1}^k(\sum_{i=1}^{k-r}t_i+\sum_{r=k-r+1}^k t_it_v)I\otimes\cdots\otimes R_r\otimes\cdots I.$$

Notice that K is a self-adjoint degree 1 operator. Moreover, $J_1\otimes I$ and K anticommute. This follows from the fact that J_1 and J_r anticommute and that $J_1\otimes I$ and

$I \otimes T$ anticommute if T is a degree 1 operator. Thus, we can factor $a_1 J_1 \otimes I + K$ as $(a_1 J_1)^{-1} \circ (I + K')$. But, since K is self-adjoint, and $a_1 J_1$ and K anticommute, then K' is skew-adjoint, and hence has purely imaginary spectrum. Thus $I + K'$ is an isomorphism and in particular, $\sum_{i=0}^{k} t_i \sigma_{s_1,\ldots,s_k,c,i} = (a_1 J_1)^{-1} \circ (I + K')$ is Fredholm. $\square$

Using the previous lemma, we can finish the proof of Lemma 6.7 easily as follows. Consider first

$$\begin{aligned} g_0 : B\mathcal{D}^s_{\pm} \times B\mathcal{C}^s_G &\to \Omega^V B\mathcal{C}^s_G \\ (s, c) &\mapsto s \cdot f(c). \end{aligned}$$

We need to find a map

$$g_1 : B\mathcal{D}^s_{\pm} \times B\mathcal{D}^s_{\pm} \times B\mathcal{C}^s_G \times \Delta_n \to \Omega^V B\mathcal{C}^s_G$$

such that $g_1((s_0, s_1, c), 0) = s_0 \cdot s_1 f(c)$ and $g_1((s_0, s_1, c), 1) = s_0 \cdot f(s_1 \cdot c)$. The previous lemma says that we can find a linear homotopy on the first half composed with the homotopy obtained by using the change the unitary operators. In general, the previous lemma and the coherence of $\oplus$ and $\otimes$ give us the desired extensions in the higher dimensions. This proves Lemma 6.7. $\square$

Proof of Theorem 6.6. Recall that to construct S we first replace the semigroup $\mathcal{M} = B\mathcal{D}^s$ by a monoid $\hat{\mathcal{M}}$, in such a way that we can find a semigruop $\mathcal{M}'$ and homomorphisms of semigroups $i_1 : \mathcal{M}' \to B\mathcal{D}^s$ and $i_2 : \mathcal{M}' \to \hat{\mathcal{M}}$. Using the homomorphism $i_1 : \mathcal{M}' \to B\mathcal{D}^s$ and the fact that $B\mathcal{D}^s_{\pm} \approx \mathbb{Z}/2 \times B\mathcal{D}^s$, we can see F_1 as a map

$$F_1 : \mathbb{B}(\mathbb{Z}/2 \times B\mathcal{D}^s, \mathbb{Z}/2 \times B\mathcal{D}^s, B\mathcal{C}^s_G) \to \Omega^V B\mathcal{C}^s_G.$$

Thus, we can get can get a map

$$F_2 : \mathbb{B}(\mathbb{Z}/2 \times \mathcal{M}', \mathbb{Z}/2 \times \mathcal{M}, B\mathcal{C}^s_G) \to \Omega^V B\mathcal{C}^s_G$$

that is $\mathcal{M}'$-equivariant, such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{B}(\mathbb{Z}/2 \times \mathcal{M}', \mathbb{Z}/2 \times \mathcal{M}', B\mathcal{C}^s_G) \arrow[r, "F_2"] \arrow[d, "\mathbb{B}(i_1)"'] & \Omega^V B\mathcal{C}^s_G \\
\mathbb{B}(\mathbb{Z}/2 \times B\mathcal{D}^s, \mathbb{Z}/2 \times B\mathcal{D}^s, B\mathcal{C}^s_G). \arrow[ur, "F_1"'] &
\end{tikzcd}$$

Since the map $i_1 : \mathcal{M}' \to B\mathcal{D}^s$ is a weak homotopy equivalence, for each k we have that $\mathbb{Z}/2 \times \mathcal{M}'^{k+1} \times B\mathcal{C}^s_G \to (\mathbb{Z}/2 \times B\mathcal{D}^s)^{k+1} \times B\mathcal{C}^s_G$ is a weak homotopy equivalence. Then by [42, proposition A.1] we have that the map

$$\mathbb{B}(i_1) : \mathbb{B}(\mathbb{Z}/2 \times \mathcal{M}', \mathbb{Z}/2 \times \mathcal{M}', B\mathcal{C}^s_G) \to \mathbb{B}(\mathbb{Z}/2 \times B\mathcal{D}^s, \mathbb{Z}/2 \times B\mathcal{D}^s, B\mathcal{C}^s_G)$$

is a weak homotopy equivalence. By the commutativity of the previous diagram and the fact that F_1 is a weak homotopy equivalence, F_2 is a weak homotopy equivalence. On the other hand, by the way we defined the action of $\hat{\mathcal{M}}$ on $B\mathcal{C}^s$, we see that the map F_2 extends to a map $F_3 : \mathbb{B}(\mathbb{Z}/2 \times \hat{\mathcal{M}}, \mathbb{Z}/2 \times \hat{\mathcal{M}}, B\mathcal{C}^s_G) \to \Omega^V B\mathcal{C}^s_G$ that is $\hat{\mathcal{M}}$-equivatiant and fits into a commutative diagram

$$\begin{tikzcd}
\mathbb{B}(\mathbb{Z}/2 \times \mathcal{M}', \mathbb{Z}/2 \times \mathcal{M}', B\mathcal{C}^s_G) \arrow[r, "F_2"] \arrow[d, "\mathbb{B}(i_2)"'] & \Omega^V B\mathcal{C}^s_G \\
\mathbb{B}(\mathbb{Z}/2 \times \hat{\mathcal{M}}, \mathbb{Z}/2 \times \hat{\mathcal{M}}, B\mathcal{C}^s_G). \arrow[ur, "F_3"'] &
\end{tikzcd}$$

By the same argument as before, we have that F_3 is a weak homotopy equivalence. Finally, S was constructed as $\mathbb{Z}/2$ times the group replacement of the monoid $\mathcal{M}$. Thus, we have a monoid S' and homomorphisms $j_1 : \mathbb{Z}/2 \times S' \to \mathbb{Z}/2 \times \hat{\mathcal{M}}$ and $j_2 : \mathbb{Z}/2 \times S' \to S$ that are weak equivalences. In a similar fashion as in the monoid replacement, we see that the map $F_3 : \mathbb{B}(\mathbb{Z}/2 \times \hat{\mathcal{M}}, \mathbb{Z}/2 \times \hat{\mathcal{M}}, B\mathcal{C}^s_G) \to \Omega^V B\mathcal{C}^s_G$ extends to a map

$$F_3 : \mathbb{B}(S, S, B\mathcal{C}^s_G) \to \Omega^V B\mathcal{C}^s_G$$

that is S-equivariant and that is a weak homotopy equivalence. This proves Theorem 6.6. $\square$

The goal of this chapter is to prove Theorem 6.9. We have enough tools to do this.

Theorem 6.9. *There is a G-equivariant spectrum E_G that represents G-equivariant K-theory and that admits an action of S by maps of spectra.*

Proof. What we need to do is construct a G-equivariant spectrum E_G representing G-equivariant K-theory such that S acts on E_G by maps of spectra; that is, for every representation V of G we need to get a G-space $E_G(V)$ together with an action of S on $E_G(V)$ and maps $\sigma : E_G(V) \to \Omega^W E_G(V \oplus W)$ that are both G and S equivariant.

By Theorem 6.6, we have an S, G-equivariant map $F : \mathbb{B}(S, S, B\mathcal{C}^s_G) \to \Omega^V B\mathcal{C}^s_G$ (that is a weak homotopy equivalence) for every complex representation V of G.

Let us denote by U a fixed G-universe consisting of complex G representations. (See [27, I.2] for definition). Restricting to complex representations is not a problem as one can extend our construction to real representations by a standard procedure, see for example [33, XII.2].

For a complex G-representation V, define

$$D(V) = B\mathcal{C}^s_G$$

$$D'(V) = \mathbb{B}(S, S, B\mathcal{C}^s_G).$$

Note that for each complex representations V and W we have S-equivariant maps

$$f_{V,W} : D'(V) \to \Omega^W D(V \oplus W) \tag{6.3}$$

and

$$g_V : D'(V) \to D(V) \tag{6.4}$$

that are G-equivariant weak equivalences. Moreover, if we ignore the S-action, we have a section $s_V : D(V) \to D'(V)$ that is a homotopy inverse of g_V. In addition, we can choose s_V so that the composite $f_{V,W} \circ s_V : D(V) \to \Omega^W D(V \oplus W)$ is the Bott periodicity map and thus gives the sequence of spaces $\{D(V)\}$ the structure of a G-prespectrum.

The prespectrum D classifies G-equivariant K-theory. However, as the map s_V is not S-equivariant, we do not have an action of S on the spectrum D. Thus we need to replace D by an equivalent spectrum that comes with an action of S. We do this next.

For every complex representation W in U, we have a shift desuspension functor Σ^∞_W. This functor is left adjoint to the W-th space evaluation functor Ω^∞_W. (See [33, Chapter XII.6] for definition.) Applying Σ^∞_W to the maps in (6.3) and (6.4) we get maps of spectra as in the following diagram

$$\begin{tikzcd}
\Sigma^\infty_V D'(V) \arrow[r] \arrow[d] & \Sigma^\infty_V \Omega^W D(V \oplus W) \\
\Sigma^\infty_V D(V). &
\end{tikzcd}$$

Note that S acts on the spectra $\Sigma^\infty_V D(V)$ and $\Sigma^\infty_V D(V)$ by maps of spectra, as S acts on $D(V)$ and $D'(V)$ respectively. Also, we have a natural isomorphism

$$\Sigma^\infty_V \Omega^W D(V \oplus W) \cong \Sigma^\infty_{V\oplus W} D(V \oplus W).$$

Thus, we obtain a diagram of spectra for every triple of complex representations V, W, and Z as shown below.

Let E_G be the colimit of this diagram for all complex representations. Then, as all the maps in sight are S-equivariant, we get that S acts on E_G by maps of spectra.

$$\begin{tikzcd}
 & & \Sigma^\infty_{V\oplus W} D'(V\oplus W) \arrow[r] \arrow[d] & \Sigma^\infty_{V\oplus W}\Omega^Z D(V\oplus W) \\
\Sigma^\infty_V D'(V) \arrow[r] \arrow[d] & \Sigma^\infty_V \Omega^W D(V\oplus W) \arrow[r, "\cong"] & \Sigma^\infty_{V\oplus W} D(V\oplus W) & \\
\Sigma^\infty_V D(V) & & &
\end{tikzcd}$$

On the other hand, ignoring the S-action we see that since s_V is a homotopy inverse of the map g_V, the previous diagram reduces to the diagram

$$\Sigma^\infty_V D(V) \to \Sigma^\infty_V \Omega^W D(V\oplus W) \cong \Sigma^\infty_{V\oplus W} D(V\oplus W).$$

But by [27, Theorem $I.4.7$], we have that

$$LD \cong \mathrm{colim} \Sigma^\infty_V D(V).$$

Here L means the spectrification functor left adjoint to the forgetful functor l from the category of G-spectra to G-prespectra. Note that LD is a G-spectrum that represents G-equivariant K-theory. This proves the theorem. $\square$

CHAPTER 7

Equivariant twistings over a point

In this chapter we define the twistings of the completed version of equivariant K-theory over a point, first for compact Lie groups and later for a general topological group. We show that in the compact Lie case the higher twistings for the completion of equivariant K-theory with respect to the augmentation ideal and over a point vanish. Thus in this case the only possible twistings are the ones coming from projective representations. Also we show that in contrast to the compact Lie case there are examples of topological groups for which we can find nonzero higher twistings for the completion of K-theory over a point.

7.1 The compact Lie case

We are going to begin this section by defining the most general twistings for the completed equivariant K-theory with respect to the augmentation ideal over a point for the case a compact Lie group.

Recall that for a compact Lie group G we have constructed a G-equivariant spectrum E_G representing G-equivariant complex K-theory together with a topological group $S \simeq \mathbb{Z}/2 \times BU_{\otimes} = K_{\otimes}$ acting by maps of spectra. The twistings over a point that we will define turn out to be classified by the group

$$K^1_{\otimes}(BG) = [BG, BK_{\otimes}] = [BG, BS].$$

To define twistings of completed G-equivariant K-theory we will need the following lemma about topological groups.

Lemma 7.1. *Let G and H be topological groups. Then there exists a topological group G' and a homomorphism of groups $i : G' \to G$ which is a weak equivalence such that given any continuous map $f : BG \to BH$ there exists a continuous homomorphism $\phi : G' \to H$ such that the following diagram commutes up to homotopy*

$$\begin{tikzcd} BG' \arrow[d, "Bi"'] \arrow[dr, "B\phi"] & \\ BG \arrow[r, "f"'] & BH. \end{tikzcd} \tag{7.1}$$

Proof. After applying the singular functor we can work in the category of simplicial groups, which we denote by SG_*. On SG_* we have a closed model structure described in [40, Section 3.2]. The weak equivalences for this closed model structure are the maps that induce isomorphisms of the homotopy groups. On the other hand, let us denote by Top_* the category of pointed connected CW complexes. On Top_* we have a closed model structure such that the functor

$$B : SG_* \to Top_*$$

$$G \mapsto BG$$

gives an equivalence of the homotopy categories. This follows from [24].

Consider G' a cofibrant aproximation of G. Thus we have that G' is cofibrant and there is a weak equivalence $i : G' \to G$.

We will show that G' is the desired group. To show this we only have to note that

$$\mathrm{Hom}_{Ho(Top_*)}(BG, BH) = \mathrm{Hom}_{Ho(SG_*)}(G', H) = [G', H].$$

The last equality follows the fact that G' is cofibrant and that on SG_* any object is fibrant, so $\mathrm{Hom}_{Ho(SG_*)}(G', H) = [G', H]$. (Here $[G', H]$ means the set of homotopy classes of simplicial group morphisms.) Therefore, given $f : BG \to BH$ we can find a morphism of simplicial groups $\phi : G' \to H$ such that the diagram (7.1) commutes up to homotopy . □

We are now ready to define the twistings of the completion of equivariant K-theory with respect to the augmentation ideal over a point.

Definition 7.2. Take $\eta \in K^1_{\otimes}(BG) = [BG, BS]$ and represent it by a map $f : BG \to BS$. Take $i : G' \to G$ a homomorphism of groups that is a weak equivalence, $\gamma : G' \to S$ as in Lemma 7.1. Then the G-spectrum E_G can be seen as a G'-spectrum via γ. We will denote this G'-spectrum by E^{γ}_G. We can form the spectrum $F_{G'}(EG'_+, E^{\gamma}_G)$ and then define

$$K^{i,\eta}_{G,Borel}(*) := [S^{-i}, F(EG'_+, E^{\gamma}_G)]_{G'} = \pi_{-i}(F(EG'_+, E^{\gamma}_G)^{G'}).$$

A priori, this definition depends on the approximation G' of G, the representative $f : BG \to BG$ chosen for η, and the $\gamma : G' \to S$ chosen. However, we will see that any choice of approximations gives rise to the same group up to isomorphism (but not in a canonical way).

Let us denote by E the underlying non-equivariant spectrum of E_G. As E_G represents G-equivariant K-theory we have that E represents non-equivariant K-theory. Let us fix an approximation $i : G' \to G$ of G and $\gamma : G' \to S$ as before. The action of G' gives rise to an action of $\pi_0(G') = \pi_0(G)$ on $\pi_n E = E^{-n} = E_n$. We can find an increasing filtration $\{EG'_p\}_{p\geq 0}$ of EG' such that $EG'_{p-1} \to EG'_p$ is a cofibration and EG'_p/EG'_{p-1} is equivalent to $G'_+ \wedge T^p$, where T^p is a wedge of copies of the sphere S^p. Using this filtration we can get a spectral sequence (as in [21, Theorem 10.3]) whose $E^{p,q}_1$ term is given by

$$E^{p,q}_1 = \mathrm{Hom}_{\pi_0(G')}(H_p(G'_+ \wedge T^p), E^q) \Longrightarrow \pi_{-p-q}(F(EG'_+, E^{\gamma}_G)^{G'}) = K^{p+q,\eta}_{G,Borel}(*).$$

The differential d^1 is induced by the map

$$\partial : EG'_p/EG'_{p-1} \to C(EG'_{p-1} \to EG'_p) \to \Sigma EG'_{p-1} \to \Sigma(EG'_{p-1}/EG'_{p-2}).$$

The $E_2^{p,q}$-term of this sequence is isomorphic to

$$E_2^{p,q} = \tilde{H}^p_{G'}(S^0, E^q) := \pi_{-p}(F(EG', HM^q)^{G'}),$$

where HM^q is the Eilenberg-Maclane spectrum corresponding to the Mackey functor M^q defined by $M^q = \underline{\pi}_{-q}(E^\gamma_G)$.

Remark 7.3. This spectral sequence only depends on the action of $\pi_0(G)$ on E^q and this is independent of the approximation G' of G. It follows then that the twistings are well defined up to a non canonical isomorphism.

7.2 Higher Twistings

We are going to show now that there are no higher twistings for the completion of equivariant K-theory with respect to the augmentation ideal and over a point for a compact Lie group G. Thus in this case, there are only lower twistings; that is, those twistings classified by $H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z})$ as in [9].

Definition 7.4. We say that a topological group G satisfies the Atiyah-Segal Completion Theorem if we have that $K^0(BG) = R(G)^\wedge_I$ and $K^1(BG) = 0$, where I is the augmentation ideal of the representation ring $R(G)$.

Note that by [6] it follows that this is true for any compact Lie group. We will denote by k the connective complex K-theory spectrum and by K the spectrum representing complex K-theory. For a prime p we will denote by $\mathbb{Z}_p$ the ring of p-adic integers. Given a spectrum F and an abelian group G we can introduce G coefficients on F by considering the spectrum $F_G = F \wedge MG$, where MG is a Moore spectrum for the group G. We will use this construction for the cases $G = \mathbb{Z}_p$ and $G = \mathbb{Z}/(p^k)$ for a prime number p.

Lemma 7.5. *Let p be a prime number. If G satisfies the Atiyah-Segal Completion Theorem then $K^5_{\mathbb{Z}_p}(BG) = 0$.*

Proof. Let us define $X_k = K \wedge M\mathbb{Z}/(p^k)$ and $X_\infty = \mathrm{holim}_{k\to\infty} X_k$. We will start by showing

$$K \wedge M\mathbb{Z}_p = X_\infty.$$

We have a map $K \wedge M\mathbb{Z}_p \to X_\infty$. Let us show that it induces an isomorphism on homotopy groups. By [1, Proposition 6.6] there is a short exact sequence

$$0 \to \pi_n(K) \otimes \mathbb{Z}_p \to \pi_n(K \wedge M\mathbb{Z}_p) \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(K), \mathbb{Z}_p) \to 0.$$

The group $\mathrm{Tor}_1^{\mathbb{Z}}(\pi_{n-1}(K), \mathbb{Z}_p)$ vanishes, as $\mathbb{Z}_p$ is flat as a $\mathbb{Z}$-module. Thus we have

$$\pi_n(K \wedge M\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p \text{ if } n \text{ is even,} \\ 0 \text{ otherwise.} \end{cases}$$

On the other hand to compute $\pi_n(X_\infty)$ we have a short exact sequence

$$0 \to \lim{}^1 \pi_{n+1}(X_k) \to \pi_n(X_\infty) \to \lim_{k\to\infty} \pi_n(X_k) \to 0.$$

Since $\pi_n(K) = \mathbb{Z}$ or 0 according to whether n is even or odd, then by [1, Proposition 6.6] $\pi_n(X_k) = \mathbb{Z}/(p^k)$ or 0 depending on the parity of n. In any case we have that the map

$$\pi_{n+1}(X_{k+1}) \to \pi_{n+1}(X_k)$$

is onto, so the $\lim^1$ vanishes. Therefore

$$\pi_n(X_\infty) = \begin{cases} \mathbb{Z}_p \text{ if } n \text{ is even,} \\ 0 \text{ otherwise.} \end{cases}$$

and the map $K \wedge M\mathbb{Z}_p \to \mathrm{holim}_{k\to\infty} K \wedge M\mathbb{Z}/(p^k) = X_\infty$ induces isomorphism on π_*.

Let us show now that

$$K^5_{\mathbb{Z}_p}(BG) = X^5_\infty(BG) = 0.$$

To compute this cohomology we consider the short exact sequence

$$0 \to \lim{}^1 X^4_k(BG) \to X^5_\infty(BG) \to \lim_{k\to\infty} X^5_k(BG) \to 0.$$

On the other hand, since $X_k = K \wedge M\mathbb{Z}/(p^k)$, by [1, Proosition 6.6] we have a short exact sequence

$$0 \to K^5(BG) \otimes \mathbb{Z}/(p^k) \to X^5_k(BG) \to \mathrm{Tor}^{\mathbb{Z}}_1(K^6(BG), \mathbb{Z}/(p^k)) \to 0.$$

By assumption, on G we have that $K^5(BG) = K^1(BG) = 0$ and $K^6(BG) = K^0(BG) = R(G)^\wedge_I$. We know that $R(G)$ is a free, and hence flat, $\mathbb{Z}$-module and $R(G)^\wedge_I$ is a flat $R(G)$-module. By change of basis it follows that $R(G)^\wedge_I$ is a flat $\mathbb{Z}$-module. Therefore from the previous short exact sequence we get that $X^5_k(BG) = 0$.

We also have the exact sequence

$$0 \to K^4(BG) \otimes \mathbb{Z}/(p^k) \to X^4_k(BG) \to \mathrm{Tor}^{\mathbb{Z}}_1(K^5(BG), \mathbb{Z}/(p^k)) \to 0.$$

Since $K^5(BG) = 0$, we conclude that $X^4_k(BG) = K^4(BG) \otimes \mathbb{Z}/(p^k)$. From here we can see that the maps $X^4_{k+1}(BG) \to X^4_k(BG)$ are surjective and thus the $\lim^1$ term in the sequence computing $X^5_\infty(BG)$ vanishes. Since the outer terms in that sequence are zero we see that $K^5_{\mathbb{Z}_p}(BG) = X^5_\infty(BG) = 0$. $\square$

Proposition 7.6. *If G satisfies the Atiyah-Segal Completion Theorem then $k^5(BG) = 0$. Also $k^5_{\mathbb{Z}_p}(BG) = 0$ for every prime p .*

Proof. The proofs of both $k^5(BG) = 0$ and $k^5_{\mathbb{Z}_p}(BG) = 0$ are similar but with obvious modifications. Thus we will show in detail that $k^5_{\mathbb{Z}_p}(BG) = 0$.

By the previous lemma we have that $K^5_{\mathbb{Z}_p}(BG) = 0$. In general for a spectrum F we have the Atiyah-Hirzebruch spectral sequence.

$$E_2^{r,s} = H^r(BG, F^s(*)) \Longrightarrow F^{r+s}(BG).$$

Let us apply this for the cases $F = k_{\mathbb{Z}_p}$ and $F = K_{\mathbb{Z}_p}$. This way we obtain two spectral sequences $\{E_n^{r,s}\}$ and $\{{}^1E_n^{r,s}\}$, respectively.

$$E_2^{r,s} = H^r(BG, k^s_{\mathbb{Z}_p}(*)) \Longrightarrow k^{r+s}_{\mathbb{Z}_p}(BG),$$
$${}^1E_2^{r,s} = H^r(BG, K^s_{\mathbb{Z}_p}(*)) \Longrightarrow K^{r+s}_{\mathbb{Z}_p}(BG).$$

For the spectrum $k_{\mathbb{Z}_p}$ we know by [1, Proposition 6.6], that $k^n_{\mathbb{Z}_p}(*) = \pi_{-n}(k) \otimes \mathbb{Z}_p = \mathbb{Z}_p$ if $n \leq 0$ and even, and $k^n_{\mathbb{Z}_p}(*) = \pi_{-n}(ku) \otimes \mathbb{Z}_p = 0$ otherwise. For $K_{\mathbb{Z}_p}$ we know that $K^n_{\mathbb{Z}_p}(*) = \pi_{-n}(K_{\mathbb{Z}_p}) = \mathbb{Z}_p$ if n is even and $\pi_n(K_{\mathbb{Z}_p}) = 0$ otherwise. Thus we have that $E_2^{r,s}$ is a fourth quadrant spectral sequence with $E_2^{r,2s} = H^r(BG, \mathbb{Z}_p)$ for $s \leq 0$ and zero otherwise. Similarly, ${}^1E_2^{r,2s} = H^r(BG, \mathbb{Z}_p)$ for $s \in \mathbb{Z}$, and zero otherwise. See Figure 7.1 below.

The spectrum k comes equipped with a map of spectra $k \to K$ inducing an isomorphism on π_n for $n \geq 0$. By smashing with MZ_p we get a map $k_{\mathbb{Z}_p} \to K_{\mathbb{Z}_p}$ also inducing an isomorphism on π_n for $n \geq 0$. This map induces a map of spectral sequences $\{E_n^{r,s}\} \to \{{}^1E_n^{r,s}\}$ as shown in Figure 7.1.

We will argue by contradiction. So let us assume that $k^5_{\mathbb{Z}_p}(BG) \neq 0$. We know that $K^5_{\mathbb{Z}_p}(BG) = 0$, and we have a map of spectral sequences $\{E_n^{r,s}\} \to \{{}^1E_n^{r,s}\}$. Thus the only way that $k^5_{\mathbb{Z}_p}(BG) \neq 0$ is that one of the differentials that kills elements in total degree 5 in the case $K_{\mathbb{Z}_p}$ fails to do so in the case of $k_{\mathbb{Z}_p}$. Differentials killing elements in total degree 5 must have source of total degree 4. From Figure 7.1 we can see at once that the only sources from the $K_{\mathbb{Z}_p}$ case of total degree 4 missing in the $k_{\mathbb{Z}_p}$ case are $H^0(BG, K^4_{\mathbb{Z}_p}(*))$ and $H^2(BG, K^2_{\mathbb{Z}_p}(*))$.

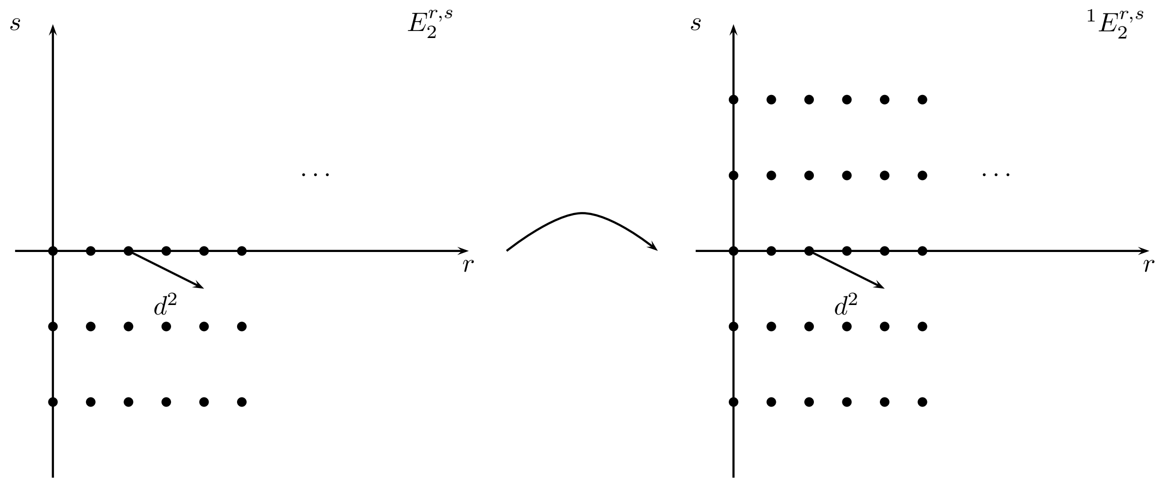

Figure 7.1: Spectral sequences E, ${}^1E$.

We will show that none of these differentials with these sources kill elements of total degree 5 in the case of K, from which we deduce that $k^5_{\mathbb{Z}_p}(BG) = 0$.

Let $*$ be the basepoint of BG and consider the sequence of maps $* \to BG \to *$ factoring the identity $* \to *$. Let us consider now the Atiyah-Hirzebruch spectral sequence applied to the spaces $*$ and BG and the spectrum $K_{\mathbb{Z}_p}$. Then we get a spectral sequence $\{{}^2E_n^{r,s}\}$

$$ {}^2E_2^{r,s} = H^r(*, K^s_{\mathbb{Z}_p}(*)) \Longrightarrow K^{r+s}_{\mathbb{Z}_p}(*) $$

and maps $h_n^{r,s} : {}^1E_n^{r,s} \to {}^2E_n^{r,s}$ and $g_n^{r,s} : {}^2E_n^{r,s} \to {}^1E_n^{r,s}$ of spectral sequences such that $h_n^{r,s} \circ g_n^{r,s} = Id$. (See Figure 7.2.)

The maps $h_n^{r,s}$ and $g_n^{r,s}$ and the identity $h_n^{r,s} \circ g_n^{r,s} = Id$ tell us that all the differentials with source $H^0(BG, K^4_{\mathbb{Z}_p}(*))$ for the spectral sequence $\{{}^1E_n^{r,s}\}$ must vanish, as they do for the spectral sequence $\{{}^2E_n^{r,s}\}$.

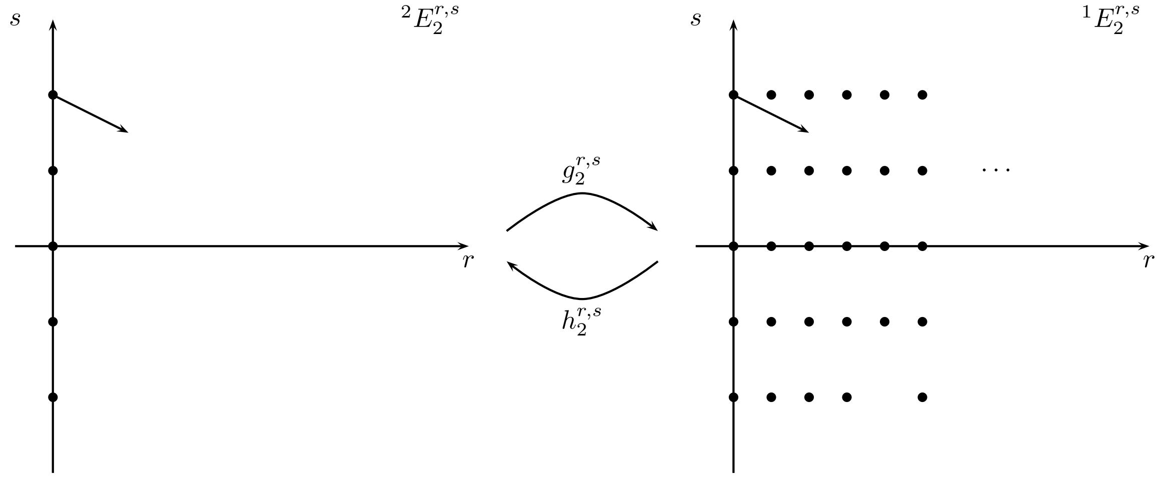

Figure 7.2: Spectral sequences ^1E, ^2E.

Now let us study the case of differentials with source $H^2(BG, K^2_{\mathbb{Z}_p}(*))$ for the spectral sequence $\{^1E_n^{r,s}\}$. We are going to show that all such differentials are trivial. This is a contradiction and hence the proposition follows.

To investigate these differentials we will first study the differentials for the Atiyah-Hirzebruch spectral sequence for the spectrum K. So we have a spectral sequence $\{^3E_n^{r,s}\}$ given by

$$^3E_2^{r,s} = H^r(BG, K^s(*)) \Longrightarrow K^{r+s}(BG).$$

We are going to show first that all the differentials with source $H^2(BG, K^2(*))$ vanish. To show this, notice that $H^2(BG, K^2(*)) = H^2(BG, \mathbb{Z}) = [BG, K(\mathbb{Z}, 2)]$, and the latter is in a one to one correspondence with isomorphism classes of complex line bundles over BG, so every element in $H^2(BG, K^2(*))$ is the first Chern class of a complex line bundle over BG. Let $\alpha \in H^2(BG, K^2(*))$. Then we can find a map $f : BG \to K(\mathbb{Z}, 2)$ such that $\alpha = f^*(c_1(\gamma_1)) = c_1(f^*\gamma_1)$, where γ_1 is universal line bundle over $K(\mathbb{Z}, 2)$. Let $^4E_n^{p,q}$ be the Atiyah-Hirzebruch spectral sequence of the space $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$ corresponding to the spectrum K, so that

$$ {}^4E_2^{r,s} = H^r(K(\mathbb{Z},2), K^s(*)) \Longrightarrow K^{r+s}(K(\mathbb{Z},2)). $$

The ${}^4E_2$-term of this spectral sequence only has terms in the even components and hence the sequence collapse and all the higher differentials are zero. The map f gives a map of spectral sequences $f_n^{r,s} : {}^4E_n^{r,s} \to {}^3E_n^{r,s}$.

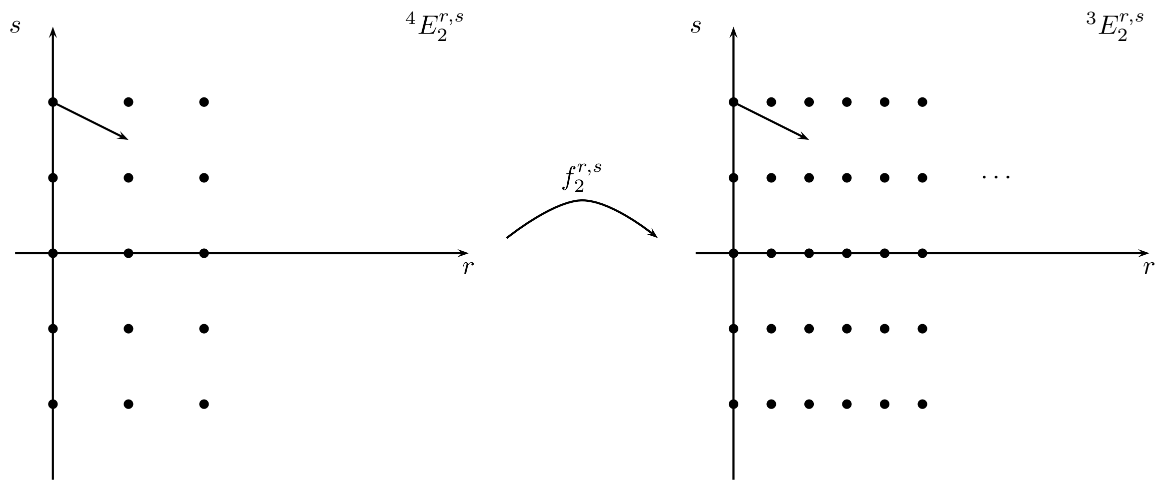

Figure 7.3: Spectral sequences ${}^3E, {}^4E$.

By construction we have that $f_2^{2,2}(c_1(\gamma_1)) = \alpha$. Since all the differentials on $\{{}^4E_n^{r,s}\}$ are zero it follows that α vanishes on all the differentials. Since α was arbitrary we see that all the differentials on the spectral sequence $\{{}^3E_n^{p,q}\}$ with source $H^2(BG, K^2(*))$ must vanish.

Take $i : S \to M\mathbb{Z}_p$ a map representing the unit of $\pi_0 M\mathbb{Z}_p$. This induces a map of spectra $K = K \wedge S \stackrel{1\wedge i}{\to} K \wedge M\mathbb{Z}_p$. This map induces a map of spectral sequences $j_n^{r,s} : {}^3E_n^{r,s} \to {}^1E_n^{r,s}$. Since each term of the spectral sequence ${}^1E$ is a $\mathbb{Z}_p$-module, by tensoring with $\mathbb{Z}_p$ we get a map of spectral sequences $\tilde{j}_n^{r,s} : {}^3E_n^{r,s} \otimes \mathbb{Z}_p \to {}^1E_n^{r,s}$. Notice that already on the E_2-level this map is an isomorphism because $H_n(BG)$ is finitely generated, and thus by [38, Corollary 56.4] we have a short exact sequence

$$0 \to H^r(BG,\mathbb{Z}) \otimes \mathbb{Z}_p \to H^r(BG,\mathbb{Z}_p) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{r+1}(BG),\mathbb{Z}_p) \to 0.$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module it follows that $H^r(BG,\mathbb{Z}) \otimes \mathbb{Z}_p \approx H^r(BG,\mathbb{Z}_p)$, and this isomorphism is precisely the $\tilde{j}$ map. Because the differentials with source $H^2(BG,K^2(*))$ in the spectral sequence $\{{}^1E_n^{r,s}\}$ are all trivial it follows that all the differentials with source $H^2(BG,K^2_{\mathbb{Z}_p}(*))$ are also trivial. □

Definition 7.7. In general for a spectrum F and any integer n we can find the $(n-1)$-connected cover of F, which we denote by $F\langle n\rangle$. This is a spectrum together with a map $F\langle n\rangle \to F$ that induces an isomorphism $\pi_k(F\langle n\rangle) \xrightarrow{\cong} \pi_k F$ for $k \geq n$ and such that $\pi_k(F\langle n\rangle) = 0$ for $k < n$.

According to our notation we have $K\langle 0\rangle = ku$. The periodicity of K implies the following lemma.

Lemma 7.8. *We have an equivalence* $\Sigma^4 k \xrightarrow{\simeq} K\langle 4\rangle$.

Proof. The spectrum k is a spectrum together with a map of spectra $k \to K$ inducing isomorphisms on π_n for $n \geq 0$ and such that $\pi_n(k) = 0$ for $n < 0$. The coefficient ring is $k_* = \mathbb{Z}[\nu]$, where $\nu \in k^{-2} = k_2$. Thus we have a map

$$\Sigma^2 k \xrightarrow{\nu} k.$$

Iterating this map and composing with $k \to K$, we get a map

$$\Sigma^4 k \to K.$$

This map induces an isomorphism on π_n for $n \geq 4$. Also since $\pi_n k = 0$ for $n < 0$ it follows that $\pi_n \Sigma^4 k = 0$ for $n < 4$. Therefore we get an equivalence $\Sigma^4 k \simeq K\langle 4\rangle$. □

Definition 7.9. If $\{G_n\} = \cdots \to G_{n+1} \cdots \to G_2 \to G_1$ is a system of groups, we say that $\{G_n\}$ satisfies the Mittag-Leffler condition if for every i we can find a $j > i$ such that for every $k > j$

$$\mathrm{Im}(G_k \to G_i) = \mathrm{Im}(G_j \to G_i).$$

It is well known that if $\{G_n\}$ satisfies the Mittag-Leffler condition then $\lim^1 G_k = 0$. On the other hand, if each G_k is a countable group, then by [37, Theorem 2] we have that the system $\{G_n\}$ must satisfy the Mittag-Leffler condition.

Suppose that G is a compact Lie group. According to our construction twistings of the completion of equivariant K-theory with respect to the augmentation ideal and over a point are classified by homotopy classes of maps $BG \to BK_\otimes$. We know that the spectrum $K_\otimes$ factors into spectra as

$$K_\otimes \simeq \{\mathbb{Z}/2\} \times BU_\otimes \simeq \{\mathbb{Z}/2\} \times K(\mathbb{Z}, 2) \times BSU_\otimes.$$

Thus twistings of the completion of equivariant K-theory over a point are classified by

$$H^1(BG, \mathbb{Z}/2) \times H^3(BG, \mathbb{Z}) \times [BG, BBSU_\otimes].$$

The higher twistings correspond to the group $[BG, BBSU_\otimes]$. We are now able to show that for a compact Lie group the latter vanishes and so there are no higher twistings for the completion of equivariant K-theory over a point.

Theorem 7.10. *For any compact Lie group G, $bsu^1_\otimes(BG) = [BG, BBSU_\otimes] = 0$.*

Proof. For every $k \geq 0$ denote by F_k the image of $\coprod_{0 \leq n \leq k} (G^n \times \Delta_n)$ in BG. The F_k's form an increasing filtration of BG and since G is compact Lie each F_k is of the homotopy type of a finite CW-complex. Let us denote $A_k = k^4(F_k)$ and

$B_k = bsu^0_{\otimes}(F_k)$. Using the filtration $\{F_k\}$ we get a short exact sequence

$$0 \to \lim{}^1 A_k \to k^5(BG) \to \lim_{k\to\infty} k^5(F_k) \to 0.$$

By Theorem 7.6 we have that the term in the middle vanishes and thus we see that $\lim^1 A_k = 0$. By looking at the Atiyah-Hirzebruch spectral sequence, since F_k is of the homotopy type of a finite CW-complex, we see that each A_k and B_k is finitely generated, in particular countable. Therefore the system $\{A_k\}$ satisfies the Mittag-Leffler condition.

On the other hand, by [2, Corollary 1.4] we have that after localization or completion at any prime p, the spectrum $bsu_{\otimes}$ is unique up to equivalence. Therefore we have $K\langle 4\rangle \wedge M\mathbb{Z}_p \simeq bsu_{\otimes} \wedge M\mathbb{Z}_p$. By Lemma 7.8 we have that $bsu_{\otimes} \wedge M\mathbb{Z}_p \simeq \Sigma^4 k \wedge M\mathbb{Z}_p$. Thus for each k we have that

$$A_k \otimes \mathbb{Z}_p = k^4_{\mathbb{Z}_p} \xrightarrow{\simeq} (bsu_{\otimes} \wedge M\mathbb{Z}_p)^0(F_k) = B_k \otimes \mathbb{Z}_p.$$

The outer equalities follow by [1, Proposition 6.6].

Therefore we have a commutative diagram in which the vertical arrows are isomorphisms

$$\begin{array}{ccccccc} \to & A_n \otimes \mathbb{Z}_p & \cdots & \to & A_2 \otimes \mathbb{Z}_p & \to & A_1 \otimes \mathbb{Z}_p \\ & \downarrow & & & \downarrow & & \downarrow \\ \to & B_n \otimes \mathbb{Z}_p & \cdots & \to & B_2 \otimes \mathbb{Z}_p & \to & B_1 \otimes \mathbb{Z}_p. \end{array} \tag{7.2}$$

Let $i > 0$ be fixed. Since the system $\{A_k\}$ satisfies the Mittag-Leffler property we can find a $j > i$ such that for each $k > j$

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_j \to A_i).$$

The following is a short exact sequence:

$$0 \to \mathrm{Ker}(A_k \to A_i) \to A_k \to \mathrm{Im}(A_k \to A_i) \to 0.$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module we have that

$$0 \to \mathrm{Ker}(A_k \to A_i) \otimes \mathbb{Z}_p \to A_k \otimes \mathbb{Z}_p \to \mathrm{Im}(A_k \to A_i) \otimes \mathbb{Z}_p \to 0$$

is also exact. This shows that $\mathrm{Im}(A_k \to A_i) \otimes \mathbb{Z}_p = \mathrm{Im}(A_k \otimes \mathbb{Z}_p \to A_i \otimes \mathbb{Z}_p)$ and thus we see that for every $k > j$ and every prime p we have

$$\mathrm{Im}(A_k \otimes \mathbb{Z}_p \to A_i \otimes \mathbb{Z}_p) = \mathrm{Im}(A_j \otimes \mathbb{Z}_p \to A_i \otimes \mathbb{Z}_p).$$

By the diagram 7.2 we conclude that for every p

$$\mathrm{Im}(B_k \to B_i) \otimes \mathbb{Z}_p = \mathrm{Im}(B_k \otimes \mathbb{Z}_p \to B_i \otimes \mathbb{Z}_p) =$$

$$\mathrm{Im}(B_j \otimes \mathbb{Z}_p \to B_i \otimes \mathbb{Z}_p) = \mathrm{Im}(B_j \to B_i) \otimes \mathbb{Z}_p.$$

Thus the groups $\mathrm{Im}(B_k \to B_i), \mathrm{Im}(B_j \to B_i)$ are two finitely generated groups that are equal after tensoring with $\mathbb{Z}_p$. By Lemma 7.11 below we see that

$$\mathrm{Im}(B_k \to B_i) = \mathrm{Im}(B_j \to B_i).$$

We have proved that the system $\{B_k\}$ satisfies the Mittag-Leffler condition and thus

$$\lim{}^1 B_k = \lim{}^1 bsu^0_{\otimes}(F_k) = 0.$$

Using the filtration $\{F_k\}$ for the spectrum $bsu_{\otimes}$ we get a short exact sequence

$$0 \to \lim{}^1 B_k \to bsu^1_{\otimes}(BG) \to \lim_{k\to\infty} bsu^1_{\otimes}(F_k) \to 0.$$

Since the $\lim^1$ part vanishes we get that

$$bsu^1_{\otimes}(BG) = \lim_{k\to\infty} bsu^1_{\otimes}(F_k).$$

We show now that the latter vanishes. To see this, note that for every prime p we have a short exact sequence

$$0 \to \lim{}^1 (bsu_{\otimes} \wedge M\mathbb{Z}_p)^0(F_k) \to (bsu_{\otimes} \wedge M\mathbb{Z}_p)^1(BG) \to \lim_{k\to\infty} (bsu_{\otimes} \wedge M\mathbb{Z}_p)^1(F_k) \to 0.$$

The term in the middle vanishes and hence we see that

$$\lim_{k\to\infty}(bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(F_k)=0.$$

But by [1, Proposition 6.6] we have that $(bsu_{\otimes}\wedge M\mathbb{Z}_p)^1(F_k)=bsu^1_{\otimes}(F_k)\otimes\mathbb{Z}_p$. Thus for every prime p the map

$$\lim_{k\to\infty} bsu^1_{\otimes}(F_k)\otimes\mathbb{Z}_p=0.$$

The proof finishes by using Lemma 7.12 to see that

$$\lim_{k\to\infty} bsu^1_{\otimes}(F_k)=0.$$

□

Lemma 7.11. *Suppose that A and B are two finitely generated abelian groups with $A\subset B$ and that for every prime p, $A\otimes\mathbb{Z}_p=B\otimes\mathbb{Z}_p$. Then $A=B$.*

Proof. We have a short exact sequence

$$0\to A\to B\to B/A\to 0.$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module we see that

$$0\to A\otimes\mathbb{Z}_p\to B\otimes\mathbb{Z}_p\to B/A\otimes\mathbb{Z}_p\to 0$$

is also exact. As $A\otimes\mathbb{Z}_p=B\otimes\mathbb{Z}_p$ we see that $B/A\otimes\mathbb{Z}_p=0$. This is true for every p. This implies that $B/A=0$ □

Lemma 7.12. *Let $\cdots\xrightarrow{f_k}G_k\xrightarrow{f_{k-1}}\cdots\xrightarrow{f_2}G_2\xrightarrow{f_1}G_1$ be a system of finitely generated abelian groups such that $\lim_{k\to\infty}G_k\otimes\mathbb{Z}_p=0$ for all primes p. Then $\lim_{k\to\infty}G_k=0$.*

Proof. Let $f:\prod_{i\geq 1}G_i\to\prod_{i\geq 1}G_i$ be defined by

$$f(x_1,x_2,...)=(x_1-f_1(x_2),x_2-f_2(x_3),...).$$

We want to show that f is injective, as $\lim_{k\to\infty} G_k = \mathrm{Ker}(f)$. Suppose

$$x = (x_1, x_2, ...) \in \mathrm{Ker}(f).$$

Then we have that $i_p(x) \in \mathrm{Ker}(f_p) = 0$. Here $i_p : \prod_{i\geq 1} G_i \to \prod_{i\geq 1} G_i \otimes \mathbb{Z}_p$. Thus for each i we have that $x_i \in \mathrm{Ker}(G_k \to G_k \otimes \mathbb{Z}_p)$ for each prime p. Since G_k is finitely generated we have that

$$\bigcap_{p \quad prime} \mathrm{Ker}(G_k \to G_k \otimes \mathbb{Z}_p) = 0.$$

Thus $x = 0$. $\square$

7.3 The noncompact case

In this section we give a definition for the twistings of the completion of equivariant K-theory over a point for topological groups more general than compact Lie groups. The general case has to be handled in a different way as our definition for the classifying space of complex equivariant K-theory was based on compact Lie groups and does not generalize to general topological groups. We also show that under this definition there exist higher twistings over a point contrary to what we showed happens for compact Lie groups.

Take G to be a topological group. By considering $U_{trivial}$ a G-universe where G acts trivially, we get a spectrum E representing K-theory together with an action of S by maps of spectra. We are going to define the completion of equivariant K-theory over a point.

Definition 7.13. Take $\eta \in K^1_{\otimes}(BG) = [BG, BS]$ and represent it by a map $f : BG \to BS$. Take $i : G' \to G$ a homomorphism of groups that is a weak equivalence and $\gamma : G' \to S$ as in Lemma 7.1. Then the G-spectrum E can be seen as a G'-

spectrum via γ. We write E^γ for this spectrum. Then we define

$$K^{i,\alpha}_{G,Borel}(*) := \pi_{-i}(F(EG'_+, E^\gamma)^{G'}).$$

Thus according to our definition the more general twistings are classified by $[BG, BK_\otimes]$.

As in the compact case, we have a spectral sequence

$$E_2^{p,q} = \tilde{H}^p_{G'}(S^0, E^q) \Longrightarrow \pi_{-p-q}(F(EG'_+, E^\gamma)^{G'}) = K^{p+q,\alpha}_{G,Borel}(*).$$

For compact Lie groups we have given two possible definitions. However, since we have a G'-map $i_* E^\gamma \to E^\gamma_G$ that is a non-equivariant equivalence as ignoring the G' actions these two spectra represent K-theory.

$$\pi_i(F(EG'_+, E^\gamma)^{G'}) \approx \pi_i(F(EG'_+, E^\gamma_G)^{G'})$$

and thus the two given definitions agree.

We are going to show that in general, under this definition, unlike in the compact Lie case, there exist higher twistings.

Proposition 7.14. *There is a topological group G for which there are higher twistings for the completed equivariant twisted K-theory over a point.*

Proof. Let us consider an odd dimensional sphere S^{2n+1} with $n \geq 2$. By the Kan-Thurston Theorem (see [23]) we know that there is a discrete group G_n and a map $f : BG_n \to S^{2n+1}$ that is a homology equivalence. (This map is far from being a weak homotopy equivalence as the homotopy groups of these spaces are very different.) Since f is a homology equivalence, it follows that $bsu^1_\otimes(BG_n) = bsu^1_\otimes(S^{2n+1})$. (This follows as we get isomorphism in the E_2-term and onward in the Atiyah-Hirzebruch spectral sequence.) Let us show now that $bsu^1_\otimes(S^{2n+1}) \neq 0$. This will prove the proposition.

Let p be a prime number. We know that $bsu_{\otimes} \wedge M\mathbb{Z}_p \simeq bsu_{\oplus} \wedge M\mathbb{Z}_p \simeq \Sigma^4 k \wedge M\mathbb{Z}_p$, and thus

$$bsu^1_{\otimes}(S^{2n+1}) \otimes \mathbb{Z}_p = (bsu_{\otimes} \wedge M\mathbb{Z}_p)^1(S^{2n+1}) = k^5_{\mathbb{Z}_p}(S^{2n+1}) = k^5(S^{2n+1}) \otimes \mathbb{Z}_p.$$

Here we used [1, Proposition 6.6] as S^{2n+1} is finite, and also the fact that $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module. Notice that both $bsu^1_{\otimes}(S^{2n+1})$ and $k^5(S^{2n+1})$ are finitely generated abelian groups. In general, if A is a finitely generated abelian group, $A = 0$ if and only if $A \otimes \mathbb{Z}_p = 0$ for every prime number p. Thus, to show that $bsu^1_{\otimes}(S^{2n+1}) \neq 0$, we only need to show that $k^5(S^{2n+1}) \neq 0$. To do so we use the Atiyah-Hirzebruch spectral sequence

$$H^r(S^{2n+1}, k^s(*)) \Longrightarrow k^{r+s}(S^{2n+1}).$$

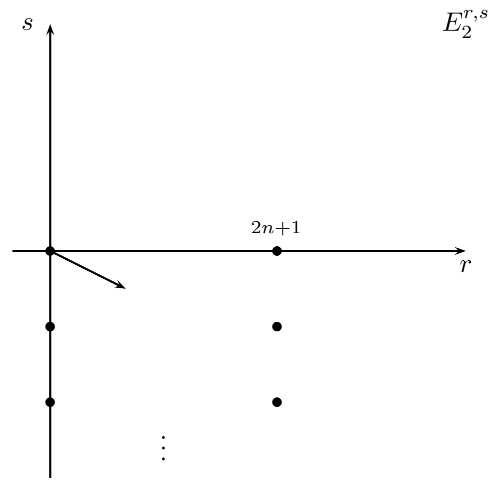

Figure 7.4: Atiyah-Hirzebruch spectral sequences for $k^{r+s}(S^{2n+1})$.

We claim that this spectral sequence collapses on the E_2-term. To see this, we only need to note that the corresponding spectral sequence collapses in the case of K. Since we have a map of spectra $k \to K$ inducing an isomorphism on π_n for $n \geq 0$, the spectral sequence in the case of k also collapses. Since $n \geq 2$, we see

$k^5(S^{2n+1}) = \mathbb{Z} \neq 0.$ $\square$

Example 7.15. Take G_n to be a discrete group as in the previous lemma; that is, such that BG_n is homologically equivalent to S^{2n+1} for $n \geq 2$. It follows that $bsu^1_{\otimes}(BG_n) = bsu_{\otimes}(S^{2n+1})$. But as we showed, for every prime number p,

$$bsu^1_{\otimes}(S^{2n+1}) \otimes \mathbb{Z}_p = k^5(S^{2n+1}) \otimes \mathbb{Z}_p = \mathbb{Z}_p.$$

Since $bsu^1_{\otimes}(S^{2n+1})$ is a finitely generated abelian group, we conclude that

$$bsu^1_{\otimes}(BG_n) = \mathbb{Z}.$$

Thus, the completion of the higher G_n-equivariant twistings over a point are classified by $\mathbb{Z}$. For $m \in \mathbb{Z}$, pick $\alpha_m : BG_n \to BBSU_{\otimes}$ corresponding to the integer m under this identification.

Let us compute the group $K^{i,\alpha_m}_{G_n,Borel}(*)$. To do so, note that, if $G'_n \to G_n$ is an approximation as in Lemma 7.1, then we have a spectral sequence as before whose $E_2^{r,s}$-term is isomorphic to

$$E_2^{r,s} = \tilde{H}^r_{G'_n}(S^0, E^s) := \pi_{-r}(F(EG'_n, HM^s)^{G'_n}).$$

In this case, we have that $\pi_0(G'_n) = \pi_1(BG'_n)$ and, as BG'_n is homologically equivalent to S^{2n+1} with $n \geq 2$, we have $\pi_1(BG'_n)/[\pi_1(BG'_n), \pi_1(BG'_n)] = H_1(BG'_n, \mathbb{Z}) = H_1(S^{2n+1}) = 0$. Thus, the abelianization of $G'_n = \pi_1(BG'_n)$ is trivial. We know that $G'_n = \pi_0(G'_n)$ acts on E^s, but as $E^s \simeq \mathbb{Z}$ or 0, we conclude that $Aut(E^s)$ is abelian. Therefore, the action of G'_n must factor through its abelianization and thus this action is trivial. It follows that the $E_2^{r,s}$-term is isomorphic to

$$H^r(BG'_n, E^s) \Longrightarrow K^{r+s,\alpha_m}_{G_n,Borel}(*).$$

But BG_n, and hence BG'_n, is homologically equivalent to S^{2n+1}, thus this spectral sequence reduces to

$$H^r(S^{2n+1}, K^s) \Longrightarrow K^{r+s,\alpha_m}_{G_n,Borel}(*).$$

This spectral sequence is illustrated in the figure 7.5

In figure 7.5 each dot represents a copy of $\mathbb{Z}$. From here, we see that the only possible non trivial differential is d_{2n+1}. Let us show that it is given by multiplication by $-m$.

If this is the case, then it follows that for $m \neq 0$

$$K^{i,\alpha_m}_{G_n,Borel}(*) = \mathbb{Z}/(m\mathbb{Z}) \text{ if } i \text{ is odd}$$

and

$$K^{i,\alpha_m}_{G_n,Borel}(*) = 0 \text{ if } i \text{ is even}$$

and for $m = 0$

$$K^{i,\alpha_0}_{G_n,Borel}(*) = \mathbb{Z} \text{ for all } i \in \mathbb{Z}.$$

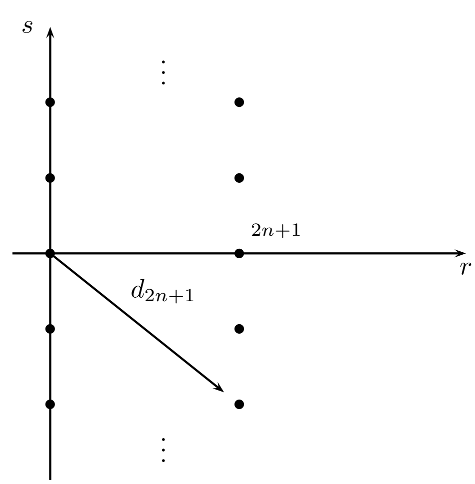

Figure 7.5: Spectral sequence for G_n.

To see that d_{2n+1} is given by multiplication by $-m$, notice that this spectral sequence is the Atiyah-Hirzebruch spectral sequence for $K^*_{\gamma_m}(S^{2n+1})$, where $\gamma_m : S^{2n+1} \to BBSU_\otimes$ corresponds to m under the identification

$$bsu^1_\otimes(S^{2n+1}) = bsu^0_\otimes(S^{2n}) = \mathbb{Z}.$$

$K^*_{\gamma_m}(S^{2n+1})$ is the set of homotopy classes of sections of a fibration $F \to S^{2n+1}$ with fiber type $BU \times \mathbb{Z}$ or $\Omega(BU \times \mathbb{Z}) \simeq U$, depending on whether $*$ is even or odd. Homotopically, this fibration is determined by the map $S^{2n} \to BSU_\otimes$, which is adjoint to the map $\gamma_m : S^{2n+1} \to BBSU_\otimes$. To determine the differential d_{2n+1}, we will follow the same method used in [10]. We have a filtration $* = X_0 = ... = X_{2n} \subset X_{2n+1} = S^{2n+1}$. In this case, the Atiyah-Hirzebruch spectral sequence of such a filtration reduces to the long exact sequence of the pair (X_{2n+1}, X_0) and the only possible non trivial differential d_{2n+1} is the map $K^0_{\gamma_m}(X_0) \to K^1_{\gamma_m}(X_{2n+1}, X_0)$. We write S^{2n+1} as the union of two hemispheres D^{2n}_+ and D^{2n}_- intersecting on the equator S^{2n}. Notice that the pairs $(S^{2n+1}, *)$ and (S^{2n+1}, D^{2n}_-) are equivalent, hence finding the map $K^0_{\gamma_m}(X_0) \to K^1_{\gamma_m}(X_{2n+1}, X_0)$ is equivalent to finding the map $K^0_{\gamma_m}(D^{2n}_-) \to K^1_{\gamma_m}(S^{2n+1}, D^{2n}_-)$. On the other hand, we have an inclusion of pairs $i : (D^{2n}_-, S^2) \to (S^{2n+1}, D^{2n}_+)$. These pairs are equivalent, so we have an isomorphism $i_1^* : K^1_{\gamma_m}(S^{2n+1}, D^{2n}_+) \xrightarrow{\approx} K^1_{\gamma_m}(D^{2n}_-, S^2)$. Also, we have a commutative diagram

$$\begin{tikzcd}
K^0_{\gamma_m}(D^{2n}_+) \arrow[r, "d"] \arrow[d, "i_0^*"'] & K^1_{\gamma_m}(S^{2n+1}, D^{2n}_+) \arrow[d, "i_1^*"] \\
K^0_{\gamma_m}(S^{2n}) \arrow[r, "d'"'] & K^1_{\gamma_m}(D^{2n}_-, S^{2n}).
\end{tikzcd}$$

Since D^{2n}_+ is contractible, we see that $K^0_{\gamma_m}(D^{2n}_+) = K^0(D^{2n}_+) = \mathbb{Z}$, and $[S^{2n}, BBSU_\otimes] =$

$bsu^0_\otimes(S^{2n-1})$. For every prime number p, we have that

$$bsu^0_\otimes(S^{2n-1}) \otimes \mathbb{Z}_p = bsu^0_\oplus(S^{2n-1}) \otimes \mathbb{Z}_p =$$
$$(\Sigma^4 k)^0(S^{2n-1}) \otimes \mathbb{Z}_p = \pi_{2n-5}(k) \otimes \mathbb{Z}_p = 0.$$

Since this is true for all prime p, and $bsu^0_\otimes(S^{2n-1})$ is finitely generated, we see that $[S^{2n}, BBSU_\otimes] = bsu^0_\otimes(S^{2n-1}) = 0$. It follows that $K^0_{\gamma_m}(S^{2n}) = K^0(S^{2n}) \approx \mathbb{Z} \oplus \mathbb{Z}$, with generators 1 and $[L']$, where $[L']$ is the image of $[L]$ under the isomorphism $K^0(S^{2n}) \approx K^0(S^2)$, and L is the tautological line bundle over $S^2 \approx \mathbb{C}P^1$ whose first Chern class generates $H^2(S^2, \mathbb{Z})$.

The map d' takes 1 to 0 and $[L']$ to 1 when we trivialize the fibration F over D^{2n}_- for both groups. On the other hand, when $m = 1$, the map i_0^* takes 1 to $[L']^{-1}$, as we are using the trivialization of F over D^{2n}_- to compute $K^0_{\gamma_m}(S^{2n})$. In general, i_0^* takes 1 to $[L']^{-m}$. But we know that $[L]^{-1} = 2 - [L]$, hence,

$$\begin{aligned}[L]^{-m} &= (2 - [L])^m = (1 + (1 - [L]))^m \\ &= 1 + m(1 - [L]) + \sum_{i=2}^{m} \binom{m}{i} (1 - [L])^i = 1 + m - m[L],\end{aligned}$$

as $(1 - [L])^2 = 0$. Thus $d' i_0^*(1) = -m$. Since i_1^* is an isomorphism, this proves the assertion.

APPENDICES

APPENDIX A

Fredholm operators

This appendix is dedicated to the study of Fredholm operators. We start by recalling some conventions and definitions that will be used throughout.

Convention 1.1. Whenever we speak of a Hilbert space $\mathcal{H}$, we will assume that $\mathcal{H}$ is a complex, separable, infinite dimensional Hilbert space.

Definition 1.2.

- If $\mathcal{H}$ is a Hilbert space we denote by $B(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$ with the norm topology.
- If $\mathcal{H}$ is a Hilbert space we define $\mathcal{F}(\mathcal{H})$ to be the subspace of $B(\mathcal{H})$ consisting of Fredholm operators on $\mathcal{H}$; the space of bounded operators $F : \mathcal{H} \to \mathcal{H}$ with closed image and such that both Ker$F$ and Coker$F$ are finite dimensional with norm topology.

Remark 1.3. It can be seen that if $F : \mathcal{H} \to \mathcal{H}$ is a bounded operator with both Ker$F$ and Coker$F$ finite dimensional subspaces, then the image of F is a closed subspace of $\mathcal{H}$. Hence the classical definition of a Fredholm operator is redundant.

We can use Fredholm operators to give a classifying space for K-theory. More precisely, Atiyah and Janich (see [5, appendix] and [22]) proved that for every compact space X there is a natural isomorphism $I : K^0(X) \xrightarrow{\approx} [X, \mathcal{F}(\mathcal{H})]$.

Definition 1.4. If $F : \mathcal{H} \to \mathcal{H}$ is a Fredholm operator, we define

$$\text{Index } F = \text{dimKer } F - \text{dimCoker } F.$$

In the natural isomorphism $I : K^0(X) \xrightarrow{\approx} [X, \mathcal{F}(\mathcal{H})]$, if we take $X = *$, then I reduces to $I(F) = \text{Index } F$. Therefore, the Index characterizes the connected components of $\mathcal{F}(\mathcal{H})$. We can find an analogue of $\mathcal{F}(\mathcal{H})$ where we allow $\mathcal{H}$ to be graded. More explicitly, we have the following definitions.

Definition 1.5.

- A $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}$ is a Hilbert space $\mathcal{H}$ that can be decomposed as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$, $\mathcal{H}_1$ are two Hilbert spaces.
- Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ be two $\mathbb{Z}/2$-graded Hilbert spaces. A bounded operator $F : \mathcal{H} \to \mathcal{K}$ is said to be degree 0 if $F(\mathcal{H}_i) \subset \mathcal{K}_i$ for $i = 0, 1$. Similarly, we say that F is a degree 1 operator if $F(\mathcal{H}_0) \subset \mathcal{K}_1$ and $F(\mathcal{H}_1) \subset \mathcal{K}_0$
- If $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a $\mathbb{Z}/2$-graded Hilbert space, then we denote by $\mathcal{F}^1(\mathcal{H})$ the space of self-adjoint degree 1 Fredholm operators in $\mathcal{H}$ with norm topology.

Convention 1.6. Whenever a, b, c, d are graded objects, we use the common convention

$$(a \otimes b) \circ (c \otimes d) = (-1)^{(\text{deg} b)(\text{deg} c)}(a \circ c) \otimes (b \circ d).$$

Note that if $\mathcal{H}$ is a Hilbert space, then we may regard $\mathcal{H}_* = \mathcal{H} \oplus \mathcal{H}$ as a $\mathbb{Z}/2$-graded Hilbert space.

Definition 1.7. Define $B^1(\mathcal{H}_*)$ to the space of self-adjoint degree 1 bounded operators.

The spaces $B(\mathcal{H})$, $B^1(\mathcal{H}_*)$ are homeomorphic. The homeomorphism is given by the following map.

Definition 1.8. Let $\mathfrak{r} : B(\mathcal{H}) \to B^1(\mathcal{H}_*)$ be the map defined by

$$\mathfrak{r}(F) = \begin{bmatrix} & F \\ F^* & \end{bmatrix}.$$

Notice that $\mathfrak{r}(\mathcal{F}(\mathcal{H})) \subset \mathcal{F}^1(\mathcal{H}_*)$. In fact, $\mathfrak{r}_{|\mathcal{F}(\mathcal{H})}$ defines a homeomorphism between $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}^1(\mathcal{H}_*)$ and the inverse of $\mathfrak{r}$ is the map $\mathfrak{t} : \mathcal{F}^1(\mathcal{H}_*) \to \mathcal{F}(\mathcal{H})$ defined by

$$\mathfrak{t}(F) = F \circ \tau_{|\mathcal{H}\oplus 0},$$

where $\tau : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is the natural isomorphism that switches the $\mathcal{H}$ factors.

In the graded situation, we have operations that correspond under $\mathfrak{r}$ to the operations $\oplus, \otimes$ described in chapter 2. To see this, consider

$$s : B(\mathcal{H}) \times B(\mathcal{K}) \to B(\mathcal{H} \oplus \mathcal{K})$$
$$(F, G) \mapsto F \oplus G.$$

Note that $s(\mathcal{F}(\mathcal{H}) \times \mathcal{F}(\mathcal{K})) \subset \mathcal{F}(\mathcal{H} \oplus \mathcal{K})$. On the other hand, we can also define

$$s^1 : B^1(\mathcal{H}_*) \times B^1(\mathcal{K}_*) \to B((\mathcal{H} \oplus \mathcal{K})_*)$$
$$(F, G) \mapsto F \oplus G.$$

It is also clear that $s^1(\mathcal{F}^1(\mathcal{H}_*) \times \mathcal{F}^1(\mathcal{K}_*)) \subset \mathcal{F}^1((\mathcal{H} \oplus \mathcal{K})_*)$. In addition, by the definition we have that the following diagram is commutative.

$$\begin{tikzcd} B(\mathcal{H}) \times B(\mathcal{K}) \arrow[r, "s"] \arrow[d, "\mathfrak{r}\times\mathfrak{r}"'] & B(\mathcal{H} \oplus \mathcal{K}) \arrow[d, "\mathfrak{r}"] \\ B^1(\mathcal{H}_*) \times B^1(\mathcal{K}_*) \arrow[r, "s^1"'] & B^1((\mathcal{H} \oplus \mathcal{K})_*). \end{tikzcd}$$

A similar phenomenon occurs for the product. We have maps

$$p : B(\mathcal{H}) \times B(\mathcal{K}) \to B(\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K})$$

$$(F, G) \mapsto P_{F,G}$$

$$p^1 : B^1(\mathcal{H}_*) \times B^1(\mathcal{K}_*) \to B^1(\mathcal{H}_* \otimes \mathcal{K}_*)$$

$$(F, G) \mapsto F \otimes I + I \otimes G.$$

We want to show that these maps satisfy

$$p(\mathcal{F}(\mathcal{H}) \times \mathcal{F}(\mathcal{K})) \subset \mathcal{F}(\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K})$$

$$p^1(\mathcal{F}^1(\mathcal{H}_*) \times \mathcal{F}^1(\mathcal{K}_*)) \subset \mathcal{F}^1(\mathcal{H}_* \otimes \mathcal{K}_*).$$

We will show this first in the graded case and use the result to establish it in the ungraded case. First, note that if F, G are two self-adjoint, degree 1 Fredholm operators, then clearly $R = p^1(F, G) = F \otimes I + I \otimes G$ is a self-adjoint bounded degree 1 operator. To see that R is a Fredholm operator note that $R^2 = F^2 \otimes I + I \otimes G^2$ since $F \otimes I \circ I \otimes G = -I \otimes G \circ F \otimes I$. It follows at once that the positive self-adjoint operator R^2 obviously has finite dimensional kernel. In particular, it has dimension (dimKer F)(dimKer G). Therefore R is a Fredholm operator by section 4 [9]. We conclude that

$$p^1(\mathcal{F}^1(\mathcal{H}_*) \times \mathcal{F}^1(\mathcal{K}_*)) \subset \mathcal{F}^1((\mathcal{H}_*) \otimes (\mathcal{K}_*)). \tag{1}$$

On the other hand, we have a commutative diagram

$$\begin{tikzcd} B(\mathcal{H}) \times B(\mathcal{K}) \arrow[r, "p"] \arrow[d, "\mathfrak{r}\times\mathfrak{r}"'] & B(\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}) \arrow[d, "\mathfrak{r}"] \\ B^1(\mathcal{H}_*) \times B^1(\mathcal{K}_*) \arrow[r, "p^1"'] & B^1(\mathcal{H}_* \otimes \mathcal{K}_*). \end{tikzcd} \tag{2}$$

By (1), (2) and the fact that $\mathfrak{r}$ maps $\mathcal{F}(\mathcal{H})$ homeomorphically onto $\mathcal{F}^1(\mathcal{H}_*)$, we obtain

$$p(\mathcal{F}(\mathcal{H}) \times \mathcal{F}(\mathcal{K})) \subset \mathcal{F}(\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}).$$

That is, if F and G are Fredholm operators, so is $P_{F,G}$. Diagram (2) shows that under our conventions, the product formula in the graded situation is given by the simpler formula $p(F,G) = F \otimes I + I \otimes G$.

In addition, we have that the index of $P_{F,G}$ behaves multiplicatively. We prove this in the following lemma.

Lemma 1.9. *We have*

$$\begin{aligned} dimKer\ P_{F,G} &= (dimKer\ F)(dimKer\ G) + (dimCoKer\ F)(dimCoKer\ G) \\ Index\ P_{F,G} &= (Index\ F)(Index\ G). \end{aligned}$$

Proof. Notice that $P^*_{F,G} = P_{F,G^*}$ and that $P_{F,G^*} \circ P_{F,G}$ is given by the matrix

$$\begin{bmatrix} I \otimes G^*G + F^*F \otimes I & \\ & FF^* \otimes I + I \otimes GG^* \end{bmatrix}.$$

From here we easily see that $\mathrm{Ker}(P_{F,G^*} \circ P_{F,G}) = \mathrm{Ker} P_{F,G}$ and has dimension

$$(\mathrm{dimKer}\ F)(\mathrm{dimKer}\ G) + (\mathrm{dimCoKer}\ F)(\mathrm{dimCoKer}\ G).$$

On the other hand, we have

$$\begin{aligned} \mathrm{Index}\ P_{F,G} &= \mathrm{dimKer}\ P_{F,G} - \mathrm{dimKer}\ P^*_{F,G} = \mathrm{dimKer}\ P_{F,G} - \mathrm{dimKer}\ P_{F,G^*} \\ &= (\mathrm{Index}\ F)(\mathrm{Index}\ G) \end{aligned}$$

□

As mentioned before, we have a natural isomorphism $I : K^0(X) \xrightarrow{\approx} [X, \mathcal{F}(\mathcal{H})]$. We know that $K^0(X)$ has the structure of a ring corresponding to the Whitney sum and tensor product of vector bundles. In the level of Fredholm operators we have the operations s, p that correspond to the functors $\oplus, \otimes$ as defined in chapter 2. We can see s and p as determining H-space structures on $\mathcal{F}(\mathcal{H})$. To do so we need to compose s and p with isomorphisms $\mathcal{H} \approx \mathcal{H} \oplus \mathcal{H}$ and $\mathcal{H} \approx \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H}$. (Any pair of isomorphisms will induce the same operation as the space of such isomorphisms is contractible by Kuiper's Theorem see [25].) This way we obtain a ring structure on the set $[X, \mathcal{F}(\mathcal{H})]$ for a compact space. This ring structure on $[X, \mathcal{F}(\mathcal{H})]$ coincides with the ring structure of $K^0(X)$. More precisely we have the following theorem.

Theorem 1.10. *The natural map* $I : K^0(X) \xrightarrow{\approx} [X, \mathcal{F}(\mathcal{H})]$ *gives an isomorphism of rings.*

Proof. This was proved by Janich in [22]. $\square$

Observation 1.11. We can use the simple looking formula for the product in the graded situation and the commutativity of diagram (2) to see the why the operations $\oplus, \otimes$ are associative, commutative and distributive up to natural isomorphisms that satisfy certain coherences. As mentioned in section 2, to see this we only need to show that the natural isomorphisms between the corresponding underlying Hilbert spaces are morphisms in the category $\mathcal{C}$. Let us show this in detail for the case of

$$\gamma^{\otimes}_{\mathfrak{a},\mathfrak{b}} : \mathfrak{a} \otimes \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{a}$$

the other cases will be similar. Here $\mathfrak{a} = (\mathcal{H}, F)$, $\mathfrak{b} = (\mathcal{K}, G)$ are two objects of $\mathcal{C}$. By definition we have that

$$\mathfrak{a} \otimes \mathfrak{b} = (\mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K}, P_{F,G}),$$

$$\mathfrak{b} \otimes \mathfrak{a} = (\mathcal{K} \otimes \mathcal{H} \oplus \mathcal{K} \otimes \mathcal{H}, P_{G,F}).$$

The tensor products of two Hilbert spaces is commutative up to natural isomorphism. Using these we define $\gamma^{\otimes}_{\mathfrak{a},\mathfrak{b}} := (\tau, \tau) : \mathfrak{a} \otimes \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{a}$, where $\tau : \mathcal{H} \otimes \mathcal{K} \oplus \mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H} \oplus \mathcal{K} \otimes \mathcal{H}$ is the natural isomorphism induced by the natural isomorphism

$\mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$. We need to show that indeed this defines a morphism in the category $\mathcal{C}$. Thus we need to show that the following diagram is commutative.

$$
\begin{tikzcd}
\mathcal{H}\otimes\mathcal{K}\oplus\mathcal{H}\otimes\mathcal{K} \arrow[r, "P_{F,G}"] \arrow[d, "\tau"'] & \mathcal{H}\otimes\mathcal{K}\oplus\mathcal{H}\otimes\mathcal{K} \arrow[d, "\tau"] \\
\mathcal{K}\otimes\mathcal{H}\oplus\mathcal{K}\otimes\mathcal{H} \arrow[r, "P_{G,F}"'] & \mathcal{K}\otimes\mathcal{H}\oplus\mathcal{K}\otimes\mathcal{H}.
\end{tikzcd} \tag{3}
$$

By a direct computation this diagram commutes. However, using (1) there is a better way to see that this diagram commutes. Because of (1), the diagram (3) commutes if and only if the following diagram is commutative

$$
\begin{tikzcd}
\mathcal{H}_*\otimes\mathcal{K}_* \arrow[r, "\mathfrak{r}(F)\otimes I+I\otimes\mathfrak{r}(G)"] \arrow[d, "\tau_*"'] & \mathcal{H}_*\otimes\mathcal{K}_* \arrow[d, "\tau_*"] \\
\mathcal{K}_*\otimes\mathcal{H}_* \arrow[r, "\mathfrak{r}(G)\otimes I+I\otimes\mathfrak{r}(F)"'] & \mathcal{K}_*\otimes\mathcal{H}_*.
\end{tikzcd} \tag{4}
$$

Here $\mathcal{H}_* = \mathcal{H} \oplus \mathcal{H}$, $\mathcal{K}_* = \mathcal{K} \oplus \mathcal{K}$ seen as $\mathbb{Z}/2$-graded Hilbert spaces and $\tau_* : \mathcal{H}_* \otimes \mathcal{K}_* \to \mathcal{K}_* \otimes \mathcal{H}_*$ is the canonical isomorphism. But the commutativity of the diagram is immediate and thus $\gamma^{\otimes}_{\mathfrak{a},\mathfrak{b}}$ is a morphism of the category $\mathcal{C}$

The same situation applies for the other transformations involving commutativity, associativity and distributivity.

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