The ample cone of a morphism

by

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To Sandra, Óscar Henrique and Martinho

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CHAPTER I

Introduction

The main goal of this thesis is to study the geometric structure of relative ample cones for a projective morphism. On the one hand, we work out in detail some foundational results about relative cones that are difficult to find in the literature. We also give some new results and examples.

The seminal paper of Kleiman [9] introduced a systematic way of dealing with invertible sheaves numerically, providing an elegant framing for analyzing their properties by using the language of cones. The duality between curves and divisors given by the intersection pairing has been deeply explored by consideration of several different cones. This approach has proved to be essential for the development of modern algebraic geometry, specially through the work started by Mori on the Cone Theorem [13] that paved the way for the Minimal Model Program. While there has been much emphasis on cones of curves, in this thesis we take a preferential view from the perspective of cones of divisors.

After a brief summary of intersection theory on curves against divisors on projective schemes (Chapter II), we review in detail its counterpart relative to proper maps between quasi-projective schemes (Chapter III).

In order to present what consists new material in the subsequent chapters, we now state theorems that gather the main results from each one, using notation that with will later be introduced in detail.

The first theorem concerns the Néron-Severi group relative to a sequence of blowups with smooth centers. Let Y be a smooth variety of dimension $m \geq 2$. Let $\pi\colon X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Y$ be a sequence of blowups of smooth irreducible subvarieties of codimension ≥ 2. Let $E_1, \ldots, E_n$ be the strict transforms on X of the exceptional locus of each map $\pi_1, \ldots, \pi_n$ respectively.

Theorem A (Chapter IV)**.** *There are rational curves $C_1, \ldots, C_n$ in X being mapped to points in Y such that:*

1. *The $n \times n$ matrix $A = ((C_i \cdot E_j)_{ij})$ has determinant $(-1)^n$ and its inverse $A^{-1} = (d_{ij})$ has non-positive integer coefficients;*
2. *The numerical classes $([E_1], \ldots, [E_n])$ form a basis of $N^1(X/Y)_{\mathbb{Z}}$ with dual basis $(d_{1j}[C_1] + \ldots + d_{nj}[C_n])_{1\leq j\leq n}$ of $N^1(X/Y)_{\mathbb{Z}}$ with respect to the intersection pairing.*

The motivation for Theorem A was an observation in an article by Lipman and Watanabe [12] that states this result in the case Y is a surface.

The second theorem is the generalization of three possible ways of characterizing ampleness in the case of $\mathbb{R}$-divisors in the relative setting. Let $f\colon X \longrightarrow S$ be a projective morphism of quasi-projective schemes. Let $\overline{\mathrm{NE}}(X/S)$ be the closed relative cone of curves.

Theorem B (Chapter V)**.** *An $\mathbb{R}$-divisor $D \in \mathrm{Div}(X)_{\mathbb{R}}$ is f-ample if and only if any of the following conditions hold:*

1. *(Fibre-wise amplitude) D_s is ample for all $s \in S$;*

2. *(Nakai's criterion)* $(D^{\dim V} \cdot V) > 0$ *for every irreducible variety* $V \subseteq X$ *mapped to a point;*

3. *(Kleiman's criterion)* $(D \cdot C) > 0$ *for all* $C \in \overline{\mathrm{NE}}(X/S) \backslash \{0\}$.

The corresponding statement for $\mathbb{Z}$ or $\mathbb{Q}$-divisors is very standard. While this extension for $\mathbb{R}$-divisors was probably understood as a "folk theorem", it does not seem to have been worked out in detail in the literature.

The third theorem is another extension to the relative setting of a known result but requiring a different approach than the previous. It is a theorem by Campana-Peternell [1] stating that the boundary of the nef cone of a projective scheme is locally cut out by polynomials in a dense open subset. We show that the same happens when considering the boundary $\mathcal{B}_{X/S}$ of the relative nef cone $\mathrm{Nef}(X/S)$.

Theorem C (Chapter VI)**.** *There is a dense open set* $U \subseteq \mathcal{B}_{X/S}$ *with the following property:*

For all $D \in U$*, there is a proper irreducible variety* $V \subseteq X$ *mapping to a point in* S *and an open neighborhood* W *of* D *in* $N^1(X/S)_{\mathbb{R}}$ *such that*

$$W \cap \mathcal{B}_{X/S} = W \cap \mathcal{N}_V,$$

where $\mathcal{N}_V$ *denotes the null locus defined by* V*.*

The fourth theorem shows that the relative nef cone can be non-polyhedral.

Theorem D (Chapter VII)**.** *There exists a morphism* $f \colon X \longrightarrow \mathbb{A}^4$*, obtained as sequence of blowups of smooth centers, such that* $\mathrm{Nef}(X/\mathbb{A}^4)$ *is non-polyhedral.*

The example we found is constructed by relating the geometry of the relative nef cone with that of the nef cone of a surface. The construction is as follows.

We start by blowing-up a point in $\mathbb{A}^4$ obtaining an exceptional divisor E_0 isomorphic to $\mathbb{P}^3$ in that first step. We then consider a smooth surface $S \subseteq E_0$ together with two irreducible smooth curves $C_1, C_2 \subseteq S$ meeting transversally which are also ample divisors on S. In practice, the surface S we will have in mind is a particular $K3$ surface with round ample cone. The second step is to blowup the curve C_1 followed by the blowup of the strict transform of C_2. Figure 1.1 describes what happens on E_0 at this stage.

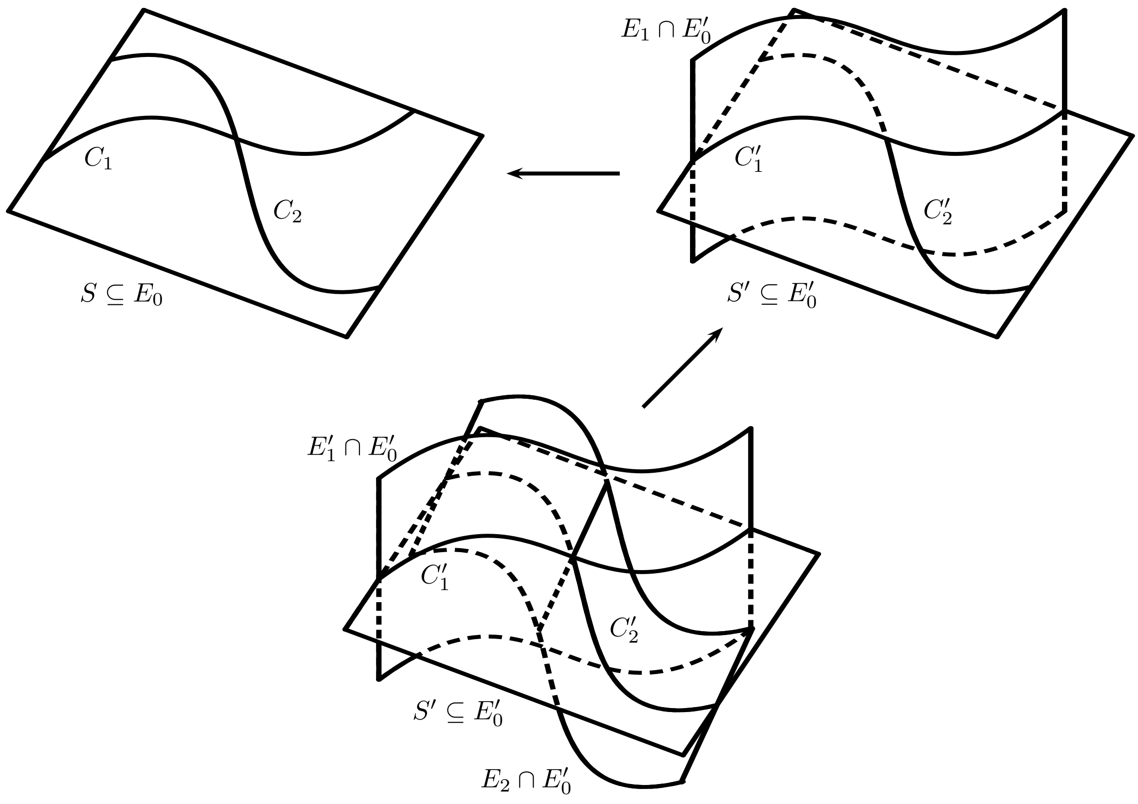


Figure 1.1: Blowup of the two curves

Using the notation introduced in Chapter VII, let $e_1 = dd_1 - \delta_1$ and $e_2 = dd_2 - \delta_2 - \delta$. Assuming that $e_1, e_2 \leq 0$ we show that a numerical class $D = H - xE_1 - yE_2 \in$

$N^1(Y/\mathbb{A}^4)_{\mathbb{R}}$ is nef if and only if,

$$\delta_1 x + \delta y \leq d_1, 0 \leq y \leq x, \delta x + \delta_2 y \leq d_2$$

and

$$\pi_* D|_S = h - xC_1 - yC_2 \in \mathrm{Nef}(S).$$

For the purpose of finding such a morphism where the relative nef cone fails to be polyhedral we use a quartic surface $S \subseteq \mathbb{P}^3$ analyzed by Cutkosky in [2]. The surface S is a $K3$ surface whose Picard group is isomorphic to $\mathbb{Z}^3$ with intersection form,

$$q = 4a^2 - 4b^2 - 4c^2.$$

It has a circular nef cone given by,

$$\mathrm{Nef}(S) = \{(a, b, c) \in \mathbb{R}^3 \mid q(a, b, c) \geq 0, a \geq 0\}.$$

It turns out that one can choose curves C_1 and C_2 representatives of the numerical classes $(5, 1, 0)$ and $(2, 0, 1)$, satisfying the assumptions of Theorem D. In this specific case, the section of the relative nef cone we obtain, is defined by the conditions,

$$0 \leq y \leq x, 1 - 5x - 2y \geq \sqrt{x^2 + y^2}$$

and its shape is described in Figure 1.2.

This is the example given in the proof of Theorem VII.1, yielding the promised case of a non-polyhedral relative nef cone for a sequence of blowups.

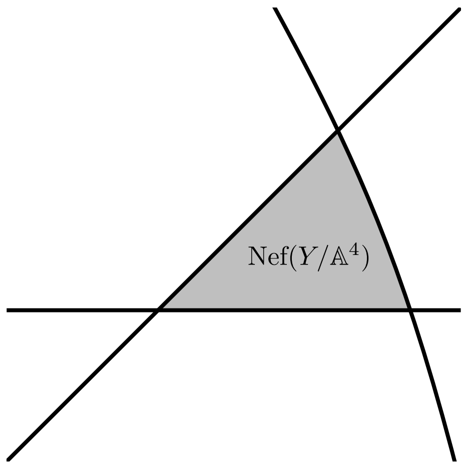


Figure 1.2: Section of the relative nef cone Nef($Y/\mathbb{A}^4$)

CHAPTER II

Intersection theory on projective schemes

We will work over the field of complex numbers $\mathbb{C}$.

In this chapter we will recall and develop the basic facts we need from intersection theory on projective schemes.

Let $f : X \longrightarrow Y$ be a morphism of schemes where X is quasi-projective. Throughout this work, f^*D will denote the divisor class associated to the pullback $f^*\mathcal{O}_Y(D) \in \operatorname{Pic}(X)$ (see Lemma III.1 for a proof that such a divisor class actually exists). In particular, if X is a subscheme of Y we will use the notation $D|_X$ referring to the divisor class defined by $\mathcal{O}_X(D)$.

Throughout this chapter we let X be a projective scheme of dimension n. We denote by $\operatorname{Div}(X)$ the group of Cartier divisors on X.

2.1 Intersection numbers

Definition II.1. Let $D_1, \ldots, D_n$ be Cartier divisors on X. The intersection number

$$(D_1 \cdot \ldots \cdot D_n)$$

is the coefficient of $m_1 \cdots m_n$ in the polynomial

$$\chi(X, m_1 D_1 + \cdots + m_n D_n)$$

where χ denotes the Euler characteristic.

We list the main properties of this intersection number in the next proposition. For proofs we refer to [3] and [9].

Proposition II.2. *Intersection numbers on X have the following properties:*

a) The map defined by

$$(D_1, \ldots, D_n) \longrightarrow (D_1 \cdot \ldots \cdot D_n)$$

is multilinear, symmetric and takes integer values;

b) $(D_1 \cdot \ldots \cdot D_n)$ only depends on the linear equivalence class of each divisor D_i.

c) If $D_1, \ldots, D_n$ are effective and meet transversally at a finite number of smooth points, then $(D_1 \cdot \ldots \cdot D_n)$ is the cardinality of $D_1 \cap \ldots \cap D_n$;

d) (Projection formula) Let $\pi : Y \longrightarrow X$ be a generically finite surjective morphism of projective varieties. Then,

$$(\pi^* D_1 \cdot \ldots \cdot \pi^* D_n) = \deg(\pi) \cdot (D_1 \cdot \ldots \cdot D_n).$$

e) Given a closed subscheme $V \subseteq X$ of dimension k, we denote

$$(D_1 \cdot \ldots \cdot D_k \cdot V) = (D_1|_V \cdot \ldots \cdot D_k|_V).$$

If D_n is an integral effective divisor with associated subscheme $V \subseteq X$, then

$$(D_1 \cdot \ldots \cdot D_n) = (D_1 \cdot \ldots \cdot D_{n-1} \cdot V);$$

2.2 Numerical properties

Intersection theory leads to a natural equivalence relation on the group of Cartier divisors $\mathrm{Div}(X)$.

Definition II.3. Two Cartier divisors $D_1, D_2 \in \mathrm{Div}(X)$ are numerically equivalent, and we write

$$D_1 \equiv_{\mathrm{num}} D_2$$

if, $(D_1 \cdot C) = (D_2 \cdot C)$ for every integral curve $C \subseteq X$. The Néron-Severi group of X is the group

$$N^1(X) = \mathrm{Div}(X)/ \equiv_{\mathrm{num}}$$

of numerical equivalence classes of X.

We now list some relevant facts related to numerical equivalence.

a) The Néron-Severi group $N^1(X)$ is a free abelian group of finite rank. Its rank is called the Picard number of X, denoted by $\rho(X)$;

b) Intersection numbers factor through numerical equivalence in the sense that if we have Cartier divisors $D_1, D_1', \ldots, D_n, D_n' \in \mathrm{Div}(X)$ such that each $D_i \equiv_{\mathrm{num}} D_i'$, then

$$(D_1 \cdot \ldots \cdot D_n) = (D_1' \cdot \ldots \cdot D_n');$$

c) If $f : X \longrightarrow Y$ is a map of projective schemes there is an induced functorial group homomorphism

$$f^* : N^1(Y) \longrightarrow N^1(X).$$

A numerical property of a divisor is a property that holds for any divisor within a numerical class. It is particularly remarkable the existence of such properties with a pure geometric meaning.

As a first example we have ampleness.

Definition II.4. A divisor D is ample if some positive multiple mD defines an embedding $f : X \longrightarrow \mathbb{P}^n$ such that $f^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_X(mD)$.

The numerical nature of ampleness results from Nakai's criterion.

Theorem II.5 (Nakai's criterion)**.** *A divisor D is ample if and only if $(D^{\dim(V)} \cdot V) > 0$ for any irreducible proper subvariety $V \subseteq X$.*

A second example is bigness.

Definition II.6. A divisor D is big if there is a positive number $C > 0$ so that $h^0(mD) \geq C \cdot m^n$ for all $m \gg 0$.

Its numerical nature comes from the following theorem.

Theorem II.7. *The following conditions are equivalent:*

i) D is big;

ii) There is an ample divisor A and an effective divisor N such that mD is linearly equivalent to $A + N$ for some positive integer m;

iii) There is an ample divisor A and an effective divisor N such that mD is numerically equivalent to $A + N$ for some positive integer m.

As an example of a numerical property by definition we have nefness.

Definition II.8. A Cartier divisor $D \in \mathrm{Div}(X)$ is nef if and only if $(D \cdot C) \geq 0$ for all proper irreducible curves $C \subseteq X$.

It turns out to be rather useful to work with divisors having coefficients in a field. We set,

$$\begin{aligned} \mathrm{Div}(X)_{\mathbb{Q}} &:= \mathrm{Div}(X) \otimes \mathbb{Q} \\ \mathrm{Div}(X)_{\mathbb{R}} &:= \mathrm{Div}(X) \otimes \mathbb{R}. \end{aligned}$$

Elements in $\mathrm{Div}(X)_{\mathbb{Q}}$ are called $\mathbb{Q}$-divisors and elements in $\mathrm{Div}(X)_{\mathbb{R}}$ are $\mathbb{R}$-divisors. We write them as formal linear combinations

$$D = \sum r_i D_i$$

where the D_i are Cartier divisors in X and the r_i are either rational numbers if D is a $\mathbb{Q}$-divisor or real numbers if D is an $\mathbb{R}$-divisor. There is an inclusion $\mathrm{Div}(X)_{\mathbb{Q}} \subseteq \mathrm{Div}(X)_{\mathbb{R}}$ and there are maps $\mathrm{Div}(X) \longrightarrow \mathrm{Div}(X)_{\mathbb{Q}}$, $\mathrm{Div}(X) \longrightarrow \mathrm{Div}(X)_{\mathbb{R}}$ whose kernels are the torsion divisors.

In order to extend intersection theory to these divisors it is convenient to work with a suitable more general notion of curve.

Definition II.9. Let $Z_1(X)$ be the free abelian group generated by the integral proper curves of X. The elements of $Z_1(X)$ are called 1-cycles and we write them as a finite formal sum

$$C = \sum a_i C_i$$

where the C_i are integral proper curves of X and the a_i are integers. Allowing the a_i to be rational (real) numbers we define the group of rational (real) 1-cycles $Z_1(X)_{\mathbb{Q}}$ ($Z_1(X)_{\mathbb{R}}$). There is an inclusion $Z_1(X) \subseteq Z_1(X)_{\mathbb{Q}} \subseteq Z_1(X)_{\mathbb{R}}$.

In this thesis we will be particularly interested in working with real coefficients and that is where we will focus our attention.

We extend the definition of intersection numbers to divisors and 1-cycles with real coefficients by linearity. More specifically, given a divisor $D = \sum r_i D_i$ and a 1-cycle $C = \sum a_j C_j$, the intersection number $(D \cdot C)$ is the real number,

$$\sum r_i a_j (D_i \cdot C_j).$$

Definition II.10. Two 1-cycles C_1, C_2 are numerically equivalent, and we write,

$$C_1 \equiv_{\mathrm{num}} C_2$$

if, $(D \cdot C_1) = (D \cdot C_2)$ for any Cartier divisor $D \in \mathrm{Div}(X)$. We define the quotient groups of numerical equivalence classes,

$$\begin{aligned} N_1(X) &:= Z_1(X)/\equiv_{\text{num}} \\ N_1(X)_{\mathbb{Q}} &:= Z_1(X)_{\mathbb{Q}}/\equiv_{\text{num}} \\ N_1(X)_{\mathbb{R}} &:= Z_1(X)_{\mathbb{R}}/\equiv_{\text{num}} \end{aligned}$$

The definition of numerical equivalence for $\mathbb{R}$-divisors is the same as for Cartier divisors. The real Néron-Severi group of X is

$$N^1(X)_{\mathbb{R}} := \text{Div}(X)_{\mathbb{R}}/\equiv_{\text{num}} .$$

One observes that intersection numbers define a bilinear pairing

$$N^1(X) \times N_1(X) \longrightarrow \mathbb{Z}$$

and consequently $N^1(X)$ and $N_1(X)$ are free abelian groups of rank $\rho(X)$. On the other hand, there is an isomorphism

$$N^1(X)_{\mathbb{R}} \cong N^1(X) \otimes \mathbb{R}$$

so $N^1(X)_{\mathbb{R}}$ is a finite-dimensional real vector space of dimension $\rho(X)$. The intersection paring $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$ is in fact a perfect pairing.

We understand by cone, a subset of a finite-dimensional vector space closed for multiplication by positive scalars. In $N^1(X)_{\mathbb{R}}$ we define the following cones:

$$\begin{aligned} \text{Amp}(X) &:= \text{convex cone spanned by ample Cartier divisors} \\ \text{Big}(X) &:= \text{convex cone spanned by big Cartier divisors} \\ \text{Nef}(X) &:= \text{convex cone spanned by nef Cartier divisors} \\ \overline{\text{Eff}}(X) &:= \text{closure of the convex cone spanned by effective Cartier divisors} \end{aligned}$$

Theorem II.11 (Kleiman)**.** *[11, Theorem 1.4.9] If D is a nef $\mathbb{R}$-divisor on X, then* $(D^{\dim V} \cdot V) \geq 0$ *for every irreducible variety $V \subseteq X$.*

Theorem II.12. *On a projective scheme X we have the following equalities,*

$$\begin{aligned} \mathrm{Amp}(X) &= \mathrm{int}(\mathrm{Nef}(X)) \\ \mathrm{Big}(X) &= \mathrm{int}(\overline{\mathrm{Eff}}(X)). \end{aligned}$$

In $N^1(X)_{\mathbb{R}}$ we define the cone of curves $\mathrm{NE}(X)$ spanned by the effective 1-cycles. Its closure $\overline{\mathrm{NE}}(X)$ is the closed cone of curves.

Campana and Peternell studied in [1] geometric properties of the nef cone and found two interesting results. One is a generalized Nakai's theorem for $\mathbb{R}$-divisors.

Theorem II.13 (Nakai's for $\mathbb{R}$-divisors)**.** *If D is an $\mathbb{R}$-divisor on X, then D is ample if and only if* $(D^{\dim V} \cdot V) > 0$ *for every irreducible variety $V \subseteq X$.*

For a simplified proof of this theorem we refer to [11, Theorem 2.3.18]. The other result is related to the structure of the nef cone and states that the nef boundary is locally cut out by polynomials in a dense open subset.

Theorem II.14 (Campana-Peternell)**.** *Let* $\beta_X = \mathrm{Nef}(X) \backslash \mathrm{Amp}(X)$ *be the nef boundary. There is an open dense subset $U \subseteq \beta_X$ with the following property. For all $\delta \in U$, there is a proper irreducible variety $V \subseteq X$ mapping to a point in S and an open neighborhood W of δ in $N^1(X)_{\mathbb{R}}$ such that,*

$$W \cap \beta_X = W \cap \{\delta \in N^1(X)_{\mathbb{R}} \mid (\delta^{\dim V} \cdot V) = 0\}.$$

CHAPTER III

The relative setting

3.1 Relative intersection theory

Now, assume X, S are quasi-projective schemes and let

$$f : X \longrightarrow S$$

be a projective morphism and let n be the dimension of X.

Intersection theory does not apply directly on X because it is not complete. However one can do intersection theory on X against projective subvarieties of X, in particular subvarieties mapping to a point. We will now explain this in more detail. The main references for this chapter are [8], [7] and [10].

To begin with, it will be convenient to notice that on X the canonical map from Cartier divisors to the Picard group is surjective.

Lemma III.1. *The map* $\begin{array}{c}\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)\\ D \mapsto \mathcal{O}_X(D)\end{array}$ *is surjective.*

Proof. Let $\mathcal{O}_X(1)$ be an ample invertible sheaf on the quasi-projective scheme X. Given a line bundle L on X, there is an integer $m \gg 0$ such that, $L \otimes \mathcal{O}_X(m)$ and $\mathcal{O}_X(m)$ are globally generated. Choosing regular global sections of these invertible sheaves and taking their zero loci, allows us to consider effective divisors $D_1, D_2 \in$

$\mathrm{Div}(X)$ for which, $\mathcal{O}_X(D_1) = L \otimes \mathcal{O}_X(m)$ and $\mathcal{O}_X(D_2) = \mathcal{O}_X(m)$. As such, $\mathcal{O}_X(D_1 - D_2) = L$, showing that the map $\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$ is indeed surjective. □

Definition III.2. For a projective subvariety $V \subseteq X$ of dimension k, and Cartier divisors $D_1, \ldots, D_k \in \mathrm{Div}(X)$, we set

$$(D_1 \cdot \ldots \cdot D_k \cdot V) = (D_1|_V \cdot \ldots \cdot D_k|_V).$$

We may also use the notation $(D_1 \cdot \ldots \cdot D_k)_V$.

Remark III.3. Note that the restriction of each divisor D_i to V is only defined if the support of D_i does not contain V. However, from Lemma III.1, $D_i|_V$ may represent a linear equivalence divisor class corresponding to the line bundle $\mathcal{O}_V(D_i)$ and this is the notation we will use. So, one can just think that each $D_i|_V$ is represented by some divisor $D_i' \in \mathrm{Div}(V)$ and

$$(D_1 \cdot \ldots \cdot D_k \cdot V) = (D_1' \cdot \ldots \cdot D_k').$$

Alternatively, we could replace each D_i with some linear equivalence class D_i'' whose support does not contain V, so that the restriction $D_i''|_V$ might refer to an actual divisor on V and then

$$(D_1 \cdot \ldots \cdot D_k \cdot V) = (D_1''|_V \cdot \ldots \cdot D_k''|_V).$$

Recall that the projectivity hypothesis implies that all fibres of f over points of S are projective and therefore we can, in particular, intersect Cartier divisors on X with any subvariety contained in a fibre.

Example III.4. Let $C \subseteq X$ be a proper integral curve mapping to a point in S and let D be a divisor in $\mathrm{Div}(X)$. The intersection number $(D \cdot C)$ is just the degree of $\mathcal{O}_C(D)$.

Example III.5. Let $\pi : X = \mathrm{Bl}_{\{0\}}\mathbb{A}^2 \longrightarrow \mathbb{A}^2$ be the blowup of the origin of the affine plane. In this situation each Cartier divisor $D \in \mathrm{Div}(X)$ is represented by a linear combination,

$$D = aE + \sum a_i C_i$$

where a and the a_i are integers, E is the exceptional divisor for π and each C_i is the strict transform of an irreducible plane curve by π. Here, $E \simeq \mathbb{P}^1$ is a projective subvariety of X. Even though X is not projective we still may intersect the divisor D against E. The intersection number $(E \cdot E)$ is by definition the degree of the line bundle $\mathcal{O}_E(E)$. So we have,

$$\begin{aligned}(E \cdot E) &= \deg \mathcal{O}_E(E) \\ &= \deg_E N_{E/X} \\ &= \deg \mathcal{O}_E(-1) \\ &= -1.\end{aligned}$$

On the other hand, for each C_i, the intersection number $(E \cdot C_i) = \deg \mathcal{O}_E(C_i)$ is the number of intersection points of C_i with E counted with multiplicities. By linearity, we obtain,

$$(E \cdot D) = -a + \sum a_i (D \cdot C_i).$$

Example III.6. Let $\pi : X = \mathrm{Bl}_{\{0\}}\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{n+1}$ be the blowup of the origin of the n-dimensional affine space. We denote by $E \simeq \mathbb{P}^n$ the exceptional divisor. Any Cartier divisor $D \in \mathrm{Div}(X)$ gives rise to a line bundle $\mathcal{O}_E(D)$ which must be linearly equivalent to $\mathcal{O}_E(m)$ for some integer m. As a result, given $D_1, \ldots, D_n \in \mathrm{Div}(X)$ such that each $\mathcal{O}_E(D_i)$ is linearly equivalent to $\mathcal{O}_E(m_i)$ ($m_i \in \mathbb{Z}$), we get the intersection number

$$(D_1 \cdot \ldots \cdot D_n \cdot E) = m_1 \cdot \ldots \cdot m_n.$$

Intersection number on fibres lead to an equivalence relation of divisors that can be set up with respect to a proper morphism between quasi-projective varieties.

Definition III.7. Two Cartier divisors $D_1, D_2 \in \mathrm{Div}(X)$ are *relatively numerically equivalent over* S, and we write

$$D_1 \equiv_S D_2$$

if, $(D_1 \cdot C) = (D_2 \cdot C)$ for every proper integral curve C mapping to a point. We denote by

$$N^1(X/S) = \mathrm{Div}(X)/\equiv_S$$

the resulting abelian group of relative numerical equivalence classes of divisors. By construction, $N^1(X/S)$ is free abelian and finitely generated as we will show in Theorem III.20. Its rank, denoted by $\rho(X/S)$, is the relative Picard number.

Remark III.8. We also define the analogous equivalence relation in the Picard group $\mathrm{Pic}(X)$. Given $L_1, L_2 \in \mathrm{Pic}(X)$, $L_1 \equiv_S L_2$ if $(L_1 \cdot C) = (L_2 \cdot C)$ for every proper integral curve C mapping to a point. It follows from Definition III.2 that the intersection number of divisors against proper subvarieties is independent of the linear equivalence class of each divisor. In particular, from Lemma III.1, $N^1(X/S) \simeq \mathrm{Pic}(X)/\equiv_S$.

One other point to make here is that intersection numbers factor through relative numerical equivalence. More specifically, for any projective subvariety $V \subseteq X$ of dimension k mapped to a point, we have the following result.

Lemma III.9. *If $D \in \mathrm{Div}(X)$ and $D \equiv_S 0$, then $D|_V \equiv_{\mathrm{num}} 0$. In particular, restriction of divisors induces a map $\theta|_V : N^1(X/S) \longrightarrow N^1(V)$.*

Proof. If $D \equiv_S 0$ then for any proper curve C contained in V we have,

$$(D|_V \cdot C) = (D \cdot C) = 0$$

and as a result $D|_V$ is numerically trivial. □

As a consequence of this lemma, given $\delta_1, \ldots, \delta_k \in N^1(X/S)$ we denote,

$$(\delta_1 \cdot \ldots \cdot \delta_k \cdot V) = (\delta_1|_V \cdot \ldots \cdot \delta_k|_V).$$

We proceed showing examples of relative numerical equivalence groups.

Example III.10. If X is projective and S is a closed point then $N^1(X/S) = N^1(X)$.

Example III.11. If $f : X = \mathrm{Bl}_{\{0\}}\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{n+1} = S$ is the blowup of the origin on the n-dimensional affine space then we can define an isomorphism

$$i : N^1(X/S) \longrightarrow N^1(E) = N^1(\mathbb{P}^n) \ .$$
$$\delta \mapsto \delta|_E$$

For showing that i is injective, let $\delta \in N^1(X/S)$ be a numerical class such that $i(\delta) = 0$. Then $\delta|_E \equiv_{\text{num}} 0$ and for any proper integral curve $C \subseteq E$, we have

$$(\delta \cdot C) = (\delta|_E \cdot C) = 0.$$

Therefore, $\delta = 0$ because all curves contained in fibres are those contained in E and this shows i is injective. Since $E|_E$ is the divisor class associated to $\mathcal{O}_E(-1)$ and its numerical class is a generator of $N^1(E)$, the map i is also surjective.

Example III.12. Generalizing a little further the previous example, we consider the blowup of a finite set of points $P = \{p_1, \ldots, p_m\}$ in $\mathbb{A}^{n+1}$. Then, for $f : X = \mathrm{Bl}_P\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{n+1} = S$ we conclude likewise that $N^1(X/S) \simeq N^1(E_1) \times \ldots \times N^1(E_m) \simeq \mathbb{Z}^m$ where each E_i is the exceptional fibre over the point p_i.

In the case of birational maps we now show that the number of components of the exceptional locus is an upper bound for the rank $\rho(X/S)$ of $N^1(X/S)$. We start with an auxiliary lemma.

Lemma III.13. *If D is a divisor in* $\mathrm{Div}(S)$*, then the pullback f^*D is relatively numerically trivial.*

Proof. For any proper irreducible curve C contained in a fibre, one can find a divisor D' in $\mathrm{Div}(S)$ linearly equivalent to D such that the point $f(C)$ is not in the support of D'. Therefore,

$$(f^*D \cdot C) = (f^*D' \cdot C) = 0$$

and this shows that $f^*D \equiv_S 0$. □

Proposition III.14. *If $\pi : X \longrightarrow S$ is a birational map and S is smooth, then $N^1(X/S)$ is generated by numerical classes of divisors whose support is contained in the exceptional locus* $\mathrm{Exc}(\pi)$*.*

Proof. Let D be an irreducible effective divisor in $\mathrm{Div}(X)$. It will be enough to show that D is relatively numerically equivalent to a divisor whose support is contained in the exceptional locus of π. We suppose D is not contained in $\mathrm{Exc}(\pi)$, otherwise the result would follow immediately. Then, $\pi(D)$ is a divisor in S and we denote its pullback $\pi^*(\pi(D))$ by D'. The divisor D' is linearly equivalent to $D - E$ for some divisor E supported in $\mathrm{Exc}(\pi)$. By Lemma III.13, $D' \equiv_S 0$ because D' is the pullback of a divisor in $\mathrm{Div}(S)$. So, $D \equiv_S E$ as we wanted. □

Corollary III.15. *If $\pi : X \longrightarrow S$ is a birational map to a smooth variety S and its exceptional locus can be expressed as a union* $\mathrm{Exc}(\pi) = E_1 \cup \ldots \cup E_n$ *of irreducible codimension 1 subvarieties $E_i \subseteq X$, then the relative Picard number $\rho(X/S)$ is at most n.*

Proof. From Proposition III.14, the numerical classes $[E_1], \ldots, [E_n]$ are generators of $N^1(X/S)$. □

Example III.16. Let Y be a smooth variety of dimension $m \geq 2$. Let $\pi : X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Y$ be a sequence of blowups of smooth subvarieties of codimension ≥ 2. Let $E_1, \ldots, E_n$ be the strict transforms on X of the exceptional locus of each map $\pi_1, \ldots, \pi_n$ respectively. Then, the exceptional locus

$$\mathrm{Exc}(\pi) = E_1 \cup \ldots \cup E_n$$

is a union of the codimension 1 subvarieties $E_i \subseteq X$. Hence, $\rho(X/S) \leq n$. In Chapter IV, we will see that actually there is an equality $\rho(X/S) = n$.

Example III.17. We will now see a case of a sequence of blowups where the restriction map $\theta|_V : N^1(X/S) \longrightarrow N^1(V)$ referred in Lemma III.9 is not surjective. In fact, in this example there is a strict inequality $\rho(X/S) < \rho(V)$. Let $\pi : X \xrightarrow{\pi_2} \mathrm{Bl}_{\{0\}}\mathbb{A}^3 \xrightarrow{\pi_1} \mathbb{A}^3 = S$ be the composition of maps $\pi_1 \circ \pi_2$ where π_1 is the blowup of the origin in $\mathbb{A}^3$ and π_2 is the blowup of a smooth irreducible curve $C \subseteq \mathrm{Bl}_{\{0\}}\mathbb{A}^3$ intersecting the exceptional locus $\mathrm{Exc}(\pi)$ transversally at 2 distinct points. Let V be the strict transform of $\mathrm{Exc}(\pi)$ by π_2. By Example III.16, $\rho(X/S) \leq 2$. On the other hand, V is isomorphic to the projective plane $\mathbb{P}^2$ blownup at 2 distinct points, whose Néron-Severi group is isomorphic to $\mathbb{Z}^3$. As such $\rho(X/S) < 3 = \rho(V)$ and $\theta|_V$ is not surjective.

The next result illustrates how to obtain functorial homomorphisms between relative Néron-Severi groups.

Proposition III.18. *Let X, X', S, S' be quasi-projective schemes. Consider the following commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\beta} & S \end{array}$$

where f and f' are proper morphisms. Then there is an induced functorial group homomorphism $(\alpha/\beta)^* : N^1(X/S) \longrightarrow N^1(X'/S')$. Moreover $(\alpha/\beta)^*$ is injective if for every proper integral curve $C \subseteq X$ mapping to a point by f, there is a proper integral curve $C' \subseteq X'$ mapping to a point by f' such that $\alpha(C') = C$.

Proof. Let $D \in \mathrm{Div}(X)$ such that $D \equiv_S 0$. Given a proper integral curve $C' \subseteq X'$ mapped to a point by f', from the projection formula, we have

$$(\alpha^* D \cdot C') = \deg(\alpha|_{C'})(D \cdot \alpha(C')) = 0$$

Therefore, $\alpha^* D \equiv_{S'} 0$ which shows that $(\alpha/\beta)^*$ is well-defined.

For the injectivity statement, let $D \in \mathrm{Div}(X)$ such that $\alpha^* D \equiv_{S'} 0$. Given a proper integral curve $C \subseteq X$ mapping to a point by f, let $C' \subseteq X'$ be a proper integral curve mapping to a point by f' such that $\alpha(C') = C$. Once again by the projection formula, we have

$$0 = (\alpha^* D \cdot C') = (D \cdot C)$$

which establishes the result. □

3.2 Theorem of the base

We state a generalized version of the Hodge Index theorem that will be a key ingredient for showing the main theorem in this section. Its proof will appear afterwards.

Proposition III.19. *Let A be an ample divisor on a smooth projective variety V of dimension $n \geq 2$. Let D and B be two divisors on V such that*

$$(B^2 \cdot A^{n-2}) > 0, \quad (D \cdot B \cdot A^{n-2}) = 0, \quad (D^2 \cdot A^{n-2}) \geq 0.$$

Then $D \equiv_{\mathrm{num}} 0$. In particular, if $(D \cdot A^{n-1}) = 0$ and $(D^2 \cdot A^{n-2}) \geq 0$ then $D \equiv_{\mathrm{num}} 0$.

We follow with the fundamental result that shows how properness for mappings allows an important common feature shared between absolute and relative Néron-Severi groups.

Theorem III.20 (Theorem of the base). *$N^1(X/S)$ is a free abelian group of finite rank.*

Proof. [8, Proposition IV.4.3] We will use induction on $\dim S$.

If $\dim S = -1$ then S is the empty set and the result is trivial. Suppose now that $\dim S = n \geq 0$ and that the theorem holds whenever $\dim S < n$.

We will do a series of reductions in several steps.

Step 1. We may assume that X and S are integral schemes.

In fact, let X_i be the irreducible components of X with their induced reduced structures. Let S_i be the scheme-theoretic image of X_i by f. Since X_i is reduced, this means that S_i is just the reduced induced structure on the closure of the image $f(X_i)$. The S_i are irreducible because the X_i also are. This way we get induced projective maps $f_i : X_i \longrightarrow S_i$ of integral schemes.

It follows from Proposition III.18 that $N^1(X/S)$ injects into $N^1(\bigsqcup X_i/S_i) = \bigoplus N^1(X_i/S_i)$.

Step 2. We may assume X is smooth.

Take a resolution of singularities $\mu : X' \longrightarrow X$ where X' is smooth and μ is a projective birational morphism. We obtain an injection $N^1(X/S) \hookrightarrow N^1(X'/S)$ using again Proposition III.18.

Step 3. We may assume $f : X \longrightarrow S$ has connected fibres.

Using a Stein factorization as shown in the diagram

$$\begin{array}{ccc} X & \xrightarrow{f'} & S' \\ {\scriptstyle f}\downarrow & \swarrow_{g} & \\ S & & \end{array}$$

where f' is projective with connected fibres and g is finite, we get $N^1(X/S) \hookrightarrow N^1(X/S')$.

Step 4. We may assume that $f : X \longrightarrow S$ is smooth and that all fibres are irreducible of the same dimension.

Since $f : X \longrightarrow S$ is a morphism of integral schemes over $\mathbb{C}$ and X is smooth, by generic smoothness there is a nonempty open set $U \subseteq S$ such that $f|_{f^{-1}(U)}$ is smooth.

Let $S' = S \setminus U$ and $V' = f^{-1}(S')$. Then,

$$N^1(X/S) \hookrightarrow N^1(f^{-1}(U)/U \sqcup V'/S') = N^1(f^{-1}(U)/U) \oplus N^1(V'/S').$$

Since $\dim S' < \dim S$, by the induction hypothesis $N^1(V'/S')$ is free abelian of finite rank. As a result, we only need to prove the theorem for $f|_{f^{-1}(U)}$. Moreover, from the fact that f has connected fibres we conclude that all fibres are irreducible and have the same dimension applying [15, Corollary I.§8.1].

Step 5. We are assuming that X is a smooth variety, S is an integral scheme and $f : X \longrightarrow S$ is a smooth projective morphism with irreducible fibres of the same dimension. Under these hypothesis, we can now show that the relative Néron-Severi group $N^1(X/S)$ is a subgroup of $N^1(X_s)$ for any $s \in S$, where $X_s = f^{-1}(s)$ is the fibre over the point s.

Set $D_s = D|_{X_s}$ for any $D \in \mathrm{Div}(X)$ and any $s \in S$. Let n be the dimension of each fibre. Let D be a divisor on X and suppose that $D_{s_0} \equiv_{\mathrm{num}} 0$ for some $s_0 \in S$. We claim that $D \equiv_S 0$.

Since f is flat, the Euler characteristic $\chi(\mathcal{F}_s)$ is independent of $s \in S$. As such, for any Cartier divisors $D_1, \dots, D_r \in \mathrm{Div}(X)$, the intersection number $(D_1 \cdot \dots \cdot D_r \cdot X_s)$ is also independent of s.

If $n = 0$, then $D \equiv_S 0$ by definition. If $n = 1$, then $(D \cdot X_s) = 0$ for all $s \in S$ and consequently $D \equiv_S 0$.

If $n = 2$, by the projectivity of f we can consider a divisor $A \in \mathrm{Div}(X)$ whose divisor class A_s is ample for all $s \in S$. By virtue of the independency of intersection numbers along fibres and D_{s_0} being numerically trivial, we have

$$(D \cdot A^{n-1} \cdot X_s) = (D^2 \cdot A^{n-2} \cdot X_s) = 0 \quad \text{for all} \;\; s \in S,$$

which means that

$$(D_s \cdot A_s^{n-1}) = (D_s^2 \cdot A_s^{n-2}) = 0 \quad \text{for all} \;\; s \in S.$$

With this equality, we want to apply Proposition III.19 to each divisor class D_s. We can actually do it because for any $s \in S$, A_s is ample and X_s is a smooth projective variety, as a consequence of f being a smooth and projective map. We conclude that $D_s \equiv_{\mathrm{num}} 0$ for all $s \in S$ and therefore $D \equiv_S 0$ as required.

The theorem is proved since $N^1(X_s)$ is a free abelian group. □

We now turn to the proof of Proposition III.19 for which we will need some auxiliary results.

Theorem III.21 (**Hodge Index Theorem**). *[See [5], Theorem V.1.9] Let H be an ample divisor on a smooth surface X. Suppose that D is a divisor such that $(D \cdot H) = 0$ and $(D^2) \geq 0$. Then $D \equiv_{\mathrm{num}} 0$.*

Corollary III.22. *Suppose B and D are divisors on a smooth surface X such that,*

$$(B^2) > 0, \quad (B \cdot D) = 0, \quad (D^2) \geq 0.$$

Then $D \equiv_{\text{num}} 0$.

Proof. Let H be an ample divisor on X. By Hodge Index, $(B \cdot H) \neq 0$ otherwise B would have to be numerically trivial. So, let r be a real number such that,

$$((D + rB) \cdot H) = 0.$$

Note that $((D+rB)^2) = (D^2)+r^2.(B^2) \geq 0$ and therefore $D+rB \equiv_{\text{num}} 0$, by Hodge Index once again. But,

$$0 = (B \cdot (D + rB)) = r.(B^2)$$

implies $r = 0$, meaning that $D \equiv_{\text{num}} 0$. □

Lemma III.23. *Let* $f : X \longrightarrow S$ *be a projective morphism of quasi-projective schemes and let* $C \subseteq S$ *be a proper integral curve. Then, there is a proper integral curve* $C' \subseteq X$ *such that* $f(C') = C$.

Proof. We can replace S by C and X by $f^{-1}(C)$. We can also assume X is a projective variety. Indeed, X is projective by the projectivity of C and the morphism f. We may assume X is integral replacing it by the reduced scheme of an irreducible component surjecting onto C.

We now use induction on $n = \dim(X)$.

If $n = 1$ we can just take $C' = X$.

For $n > 1$, let $W \subseteq X$ be a codimension 1 subscheme defined by a very ample divisor $H \in \text{Div}(X)$. Assuming $f(W) \neq C$, let $p \in C$ be a point not contained in $f(W)$ and let $F = f^{-1}(p)$ be the fibre over p. Then $W \cap F$ is empty and as a result $H|_F \equiv_{\text{num}} 0$. Hence, $H|_F$ is not an ample divisor class, contradicting the fact that H is ample. So $f(C) = W$ and the result follows by the induction hypothesis taking an irreducible component of W surjecting onto C. □

Proof of Proposition III.19. We start by fixing an integral proper curve C in V and we want to show that under the stated assumptions, $(D \cdot C) = 0$.

Let $\pi : V' \longrightarrow V$ be the blowing-up of V along C with exceptional locus $E \subseteq V'$. Let m be a positive integer such that $H = m\pi^*A - E$ is a very ample divisor on V'. Since V' is a projective variety over an algebraically closed field of characteristic 0, by Bertini's Theorem [6, Corollary 6.11], we can find a proper subvariety $W' \subseteq V'$ of dimension 2 which is a complete intersection of $n-2$ linear sections defined by H. By construction, the variety W' is the strict transform under π of a 2-dimensional subscheme $W \subseteq V$, which is a complete intersection of $n-2$ effective divisors of the linear system $|mA - C|$. As such, the curve C is contained in W.

Moreover, W is in fact a projective variety. It is projective because V is projective. It is irreducible for being topologically the image under π of W'. Also, W is a Cohen-Macaulay scheme as it is a complete intersection on a smooth variety. In order to show that W is reduced, we use the fact that any Cohen-Macaulay scheme whose singular locus has codimension ≥ 1 is reduced [4, Theorem 18.15]. But this is certainly the case, by virtue of W being birational to the variety W'.

Hence, replacing A by mA and consequently A^{n-2} by W, our assumptions become,

$$(B^2 \cdot W) > 0, \quad (B \cdot D \cdot W) = 0, \quad (D^2 \cdot W) \geq 0.$$

Since $C \subseteq W$, we have

$$\begin{aligned} (D \cdot C) &= (D|_C)_C \\ &= (D|_W|_C)_C \\ &= (D|_W \cdot C)_W \end{aligned}$$

and therefore we are left with having to prove that $(D|_W \cdot C)_W = 0$.

Let $\mu : W' \longrightarrow W$ be a resolution of singularities. By projection formula, we get that

$$(\mu^* B|_W)^2 > 0, \quad (\mu^* D|_W \cdot \mu^* B|_W) = 0, \quad (\mu^* D|_W)^2 \geq 0.$$

From Corollary III.22, we obtain $\mu^* D|_W \equiv_{\text{num}} 0$. We may assume the map μ is projective because W is projective, and therefore use Lemma III.23. So, let C' be an integral proper curve in $\mu^{-1}(C)$ that maps onto C. Then, $\mu|_{C'} : C' \longrightarrow C$ is a finite morphism of degree $d > 0$. Using the projection formula and the fact that $\mu^* D|_W$ is numerically trivial, one obtains

$$d.(D|_W \cdot C)_W = (\mu^* D|_W \cdot C')_{W'} = 0.$$

Thus, $(D|_W \cdot C)_W = 0$ as we wanted. □

CHAPTER IV

Relative Néron-Severi group of a sequence of blowups

Let $\pi : X \longrightarrow Y$ be a sequence of blowups with smooth centers starting from a smooth variety Y. In this chapter we present a result showing the intersection pairing

$$N^1(X/Y) \times N_1(X/Y) \longrightarrow \mathbb{Z}$$

defines a duality of $\mathbb{Z}$-modules. This will allow to prove that the numerical classes of the strict transforms of the blownup smooth centers form a basis for $N^1(X/Y)$. Additionally, we will provide an explicit method for obtaining a dual basis formed by anti-effective 1-cycles.

Because we deal in smooth varieties in this chapter, all Cartier divisors will be seen as Weil divisors and represented by formal linear combinations of codimension 1 subvarieties.

We start with a lemma that will introduce the generic setting we will work with.

Lemma IV.1. *Let Y be a smooth variety of dimension $m \geq 2$. Let*

$$\pi : X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Y$$

be a sequence of blowups of smooth subvarieties $V_i \subseteq X_i$ of codimension ≥ 2. For each $k \geq i \geq 1$, let $E_i^{(k)} \subseteq X_k$ be the strict transform on X_k of the exceptional divisor of $X_i \xrightarrow{\pi_i} X_{i-1}$. There exist rational curves $C_i^{(k)} \subseteq E_i^{(k)}$ such that:

a) $\pi_i(C_i^{(i)})$ *is a point;*

b) $C_i^{(k)}$ *is the strict transform of* $C_i^{(i)}$ *under the composition map of blowups* $X_k \longrightarrow X_i$*;*

c) $C_i^{(k)} \not\subseteq E_j^{(k)}$*, for any* $j \neq i$*.*

Proof. We claim we only need to find a point $p_i \in E_i^{(i)}$ that is not contained in any divisor

$$E_j^{(i)}, \quad j < i$$

nor in the image of the exceptional locus

$$f_i(\mathrm{Exc}(f_i)),$$

where $f_i : X \longrightarrow X_i$ is the composite map of blowups.

Suppose p_i is such a point. Since $\pi_i(p_i) \in V_{i-1}$, the fibre $\pi_i^{-1}(\pi_i(p_i))$ is isomorphic to a projective space $\mathbb{P}^{r_i-1}$, where r_i is the codimension of V_{i-1} in X_{i-1}. We define the curve

$$C_i^{(i)} \subseteq \pi_i^{-1}(\pi_i(p_i)) \subseteq E_i^{(i)}$$

to be a line passing through the point p_i and let

$$C_i^{(k)} \subseteq E_i^{(k)}, \quad k > i$$

be the strict transform of $C_i^{(i)}$ on X_k. The fact that these strict transforms are well defined is a direct consequence of assuming the point $p_i \in C_i^{(i)}$ is not in $f_i(\mathrm{Exc}(f_i))$. Conditions a) and b) are automatically satisfied by definition. Condition c), for $j < i$, follows from $E_j^{(i)}$ not containg p_i and, for $j > i$, comes from $f_i(\mathrm{Exc}(f_i))$ not containing p_i. This proves the claim.

We are left with having to show the existence of such a point $p_i \in E_i^{(i)}$. We do this establishing that the closed set

$$((\cup_{j<i}E_j^{(i)}) \cup \overline{f_i(\mathrm{Exc}(f_i))}) \cap E_i^{(i)}$$

is a codimension ≥ 1 algebraic subset of $E_i^{(i)}$.

For $j < i$, each $E_j^{(i)}$ is the strict transform of the divisor $E_j^{(i-1)}$, hence

$$E_j^{(i)} \cap E_i^{(i)}$$

has codimension ≥ 1 in $E_i^{(i)}$ and so does $(\cup_{j<i}E_j^{(i)}) \cap E_i^{(i)}$.

On the other hand,

$$\begin{aligned} \overline{f_i(\mathrm{Exc}(f_i))} &= \overline{f_i(\cup_{j>i}E_j^{(n)})} \\ &= \cup_{j>i}\overline{f_i(E_j^{(n)})}. \end{aligned}$$

Since, for $j > i$, each set $f_i(E_j^{(n)})$ factors through the codimension ≥ 2 variety V_{j-1} in X_{j-1}, then $\overline{f_i(E_j^{(n)})}$ has codimension ≥ 2 in X_i implying that $(\cup_{j>i}\overline{f_i(E_j^{(n)})}) \cap E_i^{(i)}$ is codimension ≥ 1 in $E_i^{(i)}$.

This way we conclude that

$$((\cup_{j<i}E_j^{(i)}) \cup \overline{f_i(\mathrm{Exc}(f_i))}) \cap E_i^{(i)}$$

is a codimension ≥ 1 algebraic subset of $E_i^{(i)}$, as we wanted. □

All results in this chapter assume the setting of Lemma IV.1 together with its notation. We shall omit superscripts whenever implicit from context.

Proposition IV.2. *The $n \times n$ matrix $A = ((E_i \cdot C_j)_{ij})$ has determinant $(-1)^n$.*

Proof. We use induction on the number of blowups n.

If $n = 1$, we have $\mathcal{O}_{E_1}(-E_1) = \mathcal{O}_{E_1}(1)$ and this implies that $\deg \mathcal{O}_{C_1}(-E_1) = 1$ because C_1 is a line in a fibre. So, $(E_1 \cdot C_1) = -1$ and the result follows.

For $n > 1$, denote $E'_i = E_i^{(n-1)}$ and $C'_i = C_i^{(n-1)}$ for each $1 \leq i < n$. Set the $(n-1) \times (n-1)$ matrix

$$A' = ((E'_i \cdot C'_j)_{i,j<n}).$$

By construction and from the induction hypothesis we have $\det(A') = (-1)^{n-1}$. We claim that for all $1 \leq i, j < n$,

$$(E_i \cdot C_j) = (E'_i \cdot C'_j) - (E_i \cdot C_n)(E_n \cdot C_j).$$

For that purpose, it is convenient to notice the intersection number $(E_n \cdot C_n)$ is -1, as in the case $n = 1$. This is so because C_n is a line in a fibre and therefore $\mathcal{O}_{E_n}(-E_n) = \mathcal{O}_{E_n}(1)$ implies $\deg_{C_n} \mathcal{O}(-E_n) = 1$. On the other hand, there is an integer a such that,

$$\pi_n^* E'_i = E_i + aE_n.$$

Since $\pi_n(C_n)$ is a point, by projection formula,

$$(\pi^* E'_i \cdot C_n) = 0.$$

But,

$$\begin{aligned}(\pi_n^* E'_i \cdot C_n) &= ((E_i + aE_n) \cdot C_n) \\ &= (E_i \cdot C_n) + a(E_n \cdot C_n) \\ &= (E_i \cdot C_n) - a\end{aligned}$$

yielding

$$a = (E_i \cdot C_n)$$

and in particular,

$$\pi_n^* E'_i = E_i + (E_i \cdot C_n)E_n.$$

Moreover, $\pi_n(C_j) = C'_j$ and using projection formula once again, we get

$$\begin{aligned}(E'_i \cdot C'_j) &= (\pi_n^* E'_i \cdot C_j) \\ &= (E_i \cdot C_j) + (E_i \cdot C_n)(E_n \cdot C_j),\end{aligned}$$

showing

$$(E_i \cdot C_j) = (E'_i \cdot C'_j) - (E_i \cdot C_n)(E_n \cdot C_j)$$

as we claimed.

This equation allows a simple description of how to obtain A from A'. We set the column vector

$$s = \begin{pmatrix} (E_1 \cdot C_n) & \cdots & (E_{n-1} \cdot C_n) \end{pmatrix}^{\mathrm{T}}$$

and the row vector

$$b = \begin{pmatrix} (E_n \cdot C_1) & \cdots & (E_n \cdot C_{n-1}) \end{pmatrix}$$

whose product is the $(n-1) \times (n-1)$ matrix

$$sb = \begin{pmatrix} (E_i \cdot C_n)(E_n \cdot C_j) \end{pmatrix}_{i,j<n}.$$

With this notation the matrices A and A' are related by the following formula,

$$A = \begin{pmatrix} A' - sb & s \\ b & -1 \end{pmatrix}.$$

In order to compute $\det(A)$ we point out that all rows of the $n \times n$ matrix

$$\begin{pmatrix} sb & -s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s \\ 0 \end{pmatrix} \begin{pmatrix} b & -1 \end{pmatrix}$$

are multiples of the row vector

$$\begin{pmatrix} b & -1 \end{pmatrix}.$$

As a consequence, the determinant

$$\det(A) = \begin{vmatrix} A' - sb & s \\ b & -1 \end{vmatrix} = \left| \begin{pmatrix} A' - sb & s \\ b & -1 \end{pmatrix} + \begin{pmatrix} sb & -s \\ 0 & 0 \end{pmatrix} \right|$$

$$= \begin{vmatrix} A' & 0 \\ b & -1 \end{vmatrix}$$

$$= -\det(A')$$

$$= -(-1)^{n-1}$$

$$= (-1)^n$$

as required. □

Analyzing further the nature of the matrix $((E_i \cdot C_j)_{ij})$, we are able to conclude that the numerical classes $[E_1], \ldots, [E_n]$ form a basis for $N^1(X/Y)$, by finding a dual basis with respect to the intersection pairing, consisting of linear combinations of the numerical classes $[C_1], \ldots, [C_n] \in N_1(X/Y)$ with non-positive integer coefficients.

Proposition IV.3. *The $n \times n$ matrix $A = ((E_i \cdot C_j)_{ij})$ is invertible and $A^{-1} = (d_{ij})$ has non-positive integer coefficients. For all $1 \leq i, j \leq n$,*

$$(E_i \cdot (d_{1j}C_1 + \ldots + d_{nj}C_n)) = \delta_{ij}.$$

Proof. From Proposition IV.2, the matrix A has determinant $(-1)^n$, hence A is invertible and A^{-1} has integer coefficients.

We use induction on the order of the square matrix A to show that A^{-1} has non-positive integer coefficients.

If $n = 1$ the result follows from the fact that $A = (-1) = A^{-1}$.

For $n > 1$, we use the same construction and notation of Proposition IV.2. Starting with the formula

$$A = \begin{pmatrix} A' - sb & s \\ b & -1 \end{pmatrix}$$

we are able to find the inverse matrix A^{-1} explicitly.

Using row equivalence and noticing A' is invertible by induction hypothesis, we obtain

$$\begin{aligned}
\left(A \,\vdots\, \mathrm{Id}_{n+1} \right) &\Leftrightarrow \begin{pmatrix} \mathrm{Id}_n & s \\ 0 & 1 \end{pmatrix} \left(\begin{matrix} A' - sb & s \\ b & -1 \end{matrix} \,\vdots\, \mathrm{Id}_{n+1} \right) \\
&\Leftrightarrow \begin{pmatrix} A'^{-1} & 0 \\ 0 & -1 \end{pmatrix} \left(\begin{matrix} A' & 0 \\ b & -1 \end{matrix} \,\vdots\, \begin{matrix} \mathrm{Id}_n & s \\ 0 & 1 \end{matrix} \right) \\
&\Leftrightarrow \begin{pmatrix} \mathrm{Id}_n & 0 \\ b & 1 \end{pmatrix} \left(\begin{matrix} \mathrm{Id}_n & 0 \\ -b & 1 \end{matrix} \,\vdots\, \begin{matrix} A'^{-1} & A'^{-1}s \\ 0 & -1 \end{matrix} \right) \\
&\Leftrightarrow \left(\mathrm{Id}_{n+1} \,\vdots\, \begin{matrix} A'^{-1} & A'^{-1}s \\ bA'^{-1} & bA'^{-1}s - 1 \end{matrix} \right).
\end{aligned}$$

As a result,

$$A^{-1} = \begin{pmatrix} A'^{-1} & A'^{-1}s \\ bA'^{-1} & bA'^{-1}s - 1 \end{pmatrix}.$$

By construction, C_j is irreducible and is not contained in E_i whenever $i \neq j$. This means $(E_i \cdot C_j)$ counts intersection multiplicities for $i \neq j$, and consequently, vectors b and s both have non-negative entries. By induction hypothesis A'^{-1} has non-positive entries and we conclude that A^{-1} has non-positive entries.

Let $A = (a_{ij})$ and $A^{-1} = (d_{ij})$. Then for all $1 \leq i, j \leq n$,

$$\begin{aligned} \delta_{ij} &\Leftrightarrow a_{i1}d_{1j} + \ldots + a_{in}d_{nj} \\ &\Leftrightarrow (E_i \cdot C_1)d_{1j} + \ldots + (E_i \cdot C_n)d_{nj} \\ &\Leftrightarrow (E_i \cdot (d_{1j}C_1 + \ldots + d_{nj}C_n)) \end{aligned}$$

as wanted. □

Corollary IV.4. *The numerical classes* $([E_1], \ldots, [E_n])$ *form a basis of* $N^1(X/Y)$ *with dual basis* $(d_{1j}[C_1] + \ldots + d_{nj}[C_n])_{1 \leq j \leq n}$ *of* $N_1(X/Y)$ *with respect to the intersection pairing.*

Proof. By Corollary III.15, $[E_1], \ldots, [E_n]$ generate $N^1(X/Y)$. They are linearly independent because if

$$\sum a_i[E_i] = 0$$

then, by Proposition IV.3,

$$0 = ((\sum a_i E_i) \cdot (d_{1i}C_1 + \ldots + d_{ni}C_n)) = a_i$$

for all $1 \leq i \leq n$. So, $([E_1], \ldots, [E_n])$ is a basis of $N^1(X/Y)$.

Besides, the intersection pairing implies that $N_1(X/Y)$ is isomorphic to a subgroup of $\mathrm{Hom}(N^1(X/Y), \mathbb{Z})$. By Proposition IV.3, $(d_{1j}[C_1] + \ldots + d_{nj}[C_n])_{1 \leq j \leq n}$ defines a dual basis of $([E_1], \ldots, [E_n])$ showing that the intersection pairing actually defines an isomorphism

$$N_1(X/Y) \cong \mathrm{Hom}(N^1(X/Y), \mathbb{Z}).$$

□

Corollary IV.5. *When* Y *is a 2-dimensional variety, the intersection pairing matrix* $A = ((E_i \cdot E_j)_{ij})$ *has determinant* $(-1)^n$*. Its inverse matrix* $A^{-1} = (d_{ij})$ *has non-*

positive entries and

$$(d_{11}[E_1] + \ldots + d_{n1}[E_n], d_{12}[E_1] + \ldots + d_{n2}[E_n], \ldots, d_{1n}[E_1] + \ldots + d_{nn}[E_n])$$

forms a dual basis to $([E_1], \ldots, [E_n])$.

Proof. Since each E_i is an integral curve, the only integral curve it contains is E_i itself, meaning that $C_i = E_i$. As such, the result follows immediately from Proposition IV.2 and Corollary IV.4. □

Example IV.6. Let $\pi : X = X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Y$ be a sequence of blowups where $Y = \mathbb{A}^2$ is the affine plane. We describe the maps involved:

- π_1 is the blowup of the origin;
- π_2 is the blowup of a point in E_1;
- π_3 is the blowup of the intersection point $E_1 \cap E_2$.

We exhibit the intersection matrix $A = ((E_i \cdot E_j)_{ij})$ and its inverse A^{-1} after each blowup.

n	1	2	3
A	$\begin{pmatrix} -1 \end{pmatrix}$	$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$
A^{-1}	$\begin{pmatrix} -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 & -2 \\ -1 & -2 & -3 \\ -2 & -3 & -6 \end{pmatrix}$

In this case the dual basis to $([E_1], [E_2], [E_3])$ with respect to the intersection pairing is

$$(-[E_1] - [E_2] - 2[E_3], -[E_1] - 2[E_2] - 3[E_3], -2[E_1] - 3[E_2] - 6[E_3]).$$

Example IV.7. Let $\pi : X = X_3 \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Y$ be a sequence of blowups where $Y = \mathbb{A}^3$ is the affine space. We describe the maps involved:

– π_1 is the blowup of the origin;

– π_2 is the blowup of a smooth conic in $E_1 \cong \mathbb{P}^2$;

– π_3 is the blowup of the curve $E_1 \cap E_2$.

We assume C_2 does not meet C_1 and C_3 The matrices $A = ((E_i \cdot C_j)_{ij})$ and A^{-1} we obtain after each blowup, as long as we pick up rational curves according to Lemma IV.1, are the following.

n	1	2	3
A	$\begin{pmatrix} -1 \end{pmatrix}$	$\begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$	$\begin{pmatrix} -5 & 0 & 1 \\ 0 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}$
A^{-1}	$\begin{pmatrix} -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ -2 & -3 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 & -2 \\ -2 & -3 & -5 \\ -4 & -5 & -10 \end{pmatrix}$

Notice that this time the matrices are no longer symmetric. The dual basis to $([E_1], [E_2], [E_3])$ with respect to the intersection pairing is

$$(-[C_1] - 2[C_2] - 4[C_3], -[C_1] - 3[C_2] - 5[C_3], -2[C_1] - 5[C_2] - 10[C_3]).$$

CHAPTER V

Relative real Nakai's criterion

The primary goal of this chapter is to extend Nakai's criterion for $\mathbb{R}$-divisors to the relative setting. We present here a detailed discussion leading to the proof of that result. We start by introducing the relative notion of amplitude for $\mathbb{Q}$-divisors and show its numerical nature. Then, we carefully do the same for $\mathbb{R}$-divisors and enhance the differences arising from this viewpoint. We also give examples illustrating how to define relative ample cones using Nakai's criterion for a mapping.

The results of this chapter are mostly based on the exposition in [11].

5.1 Relative amplitude for $\mathbb{Q}$-divisors

Throughout this chapter we let $f : X \longrightarrow S$ be a projective morphism of quasi-projective schemes, unless otherwise stated.

Definition V.1. A line bundle L on X is f-very ample if the canonical map

$$f^*f_*L \longrightarrow L$$

is surjective and defines an embedding $X \hookrightarrow \mathbb{P}(f_*L)$ of schemes over S. A line bundle L on X is f-ample if mL is f-very ample for some positive integer $m > 0$. A $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X)_{\mathbb{Q}}$ is f-ample (or f-very ample) if there is an integer $n > 0$ such that the line bundle $\mathcal{O}_X(nD)$ is f-ample (or f-very ample).

Remark V.2. Note that f-amplitude is local on the base. Given a divisor $D \in \text{Div}(X)_{\mathbb{Q}}$, if there is a covering of S by affine open sets U_i such that $D|_{f^{-1}(U_i)}$ is $f|_{f^{-1}(U_i)}$-ample we can show that D is f-ample. By quasi-compactness of S we may assume that the covering is finite. Taking positive integers n_i for which each divisor class $n_i D|_{f^{-1}(U_i)}$ is $f|_{f^{-1}(U_i)}$-very ample and letting n to be a common multiple of every n_i, we conclude that nD is f-very ample.

We proceed with the cohomological characterization of f-amplitude.

Theorem V.3. *The following conditions are equivalent:*

a) D is f-ample;

b) For any coherent sheaf $\mathcal{F}$ on X and $m \gg 0$, $R^i f_*(\mathcal{F}(mD)) = 0$ for all $i > 0$;

c) For any coherent sheaf $\mathcal{F}$ on X and $m \gg 0$ the canonical map

$$f^* f_*(\mathcal{F}(mD)) \longrightarrow \mathcal{F}(mD)$$

is surjective.

Proof. All these conditions are local on S and as a result we may assume that S is an affine scheme. Taking this into account, in order to show $a) \Rightarrow b)$ let $n > 0$ be an integer such that nD is very ample. Let $\mathcal{F}$ be a coherent sheaf on X. Applying [5, Proposition III.5.2(b)] to $\mathcal{O}_X(nD)$ and the sheaves $\mathcal{F}, \mathcal{F}(D), \ldots, \mathcal{F}((n-1)D)$ one obtains that $R^i(X, \mathcal{F}(mD) = 0$ for all $n \gg 0$. For the remaining we just observe that $b) \Leftrightarrow c)$ is the content of [5, Proposition III.5.3] and that $c) \Rightarrow a)$ follows from the proof of[5, Theorem II.7.6], applied in both cases to the invertible sheaf $\mathcal{O}_X(D)$. $\square$

The following result shows that the notion of f-amplitude can be reduced to the absolute setting in the case of Cartier $\mathbb{Q}$-divisors.

Theorem V.4 (Fibre-wise amplitude)**.** *[See [11], Theorem 1.2.17] Let $D \in \mathrm{Div}(X)_{\mathbb{Q}}$ and for $s \in S$ set $X_s = f^{-1}(s)$, $D_s = D|_{X_s}$. Then D is f-ample if and only if D_s is ample on X_s for all $s \in S$.*

In order to prove this theorem we will need the following proposition showing that ampleness is an open property in families.

Proposition V.5. *Let $f : X \longrightarrow S$ be a proper morphism of schemes and $D \in \mathrm{Div}(X)_{\mathbb{Q}}$. Let $s \in S$ be a point. If D_s is ample in X_s then $D_{s'}$ is ample for all s' in a neighborhood $U \subseteq S$ of s.*

Proof. First we claim that for any coherent sheaf $\mathcal{F}$ on X, there is a neighborhood U' of s where

$$R^i f_*(\mathcal{F}(mD)) = 0$$

for $i > 0$ and $m \gg 0$.

We prove the claim by descending induction on i. The statement is true for large i so assume that it holds for some $i > 1$ and all $\mathcal{F}$. We want to show that it holds also for $i - 1$.

Let $u_1, \ldots, u_p$ be generators of the maximal ideal $\mathfrak{m}_s$ in an affine neighborhood $\mathrm{Spec}(A)$ of s. This gives rise to a presentation

$$A^{\oplus p} \xrightarrow{\alpha} A \longrightarrow A/\mathfrak{m}_s \longrightarrow 0$$

of $A/\mathfrak{m}_s$, where α is defined by mapping each element $(a_1, \ldots, a_p)$ to $\Sigma a_i u_i$. Then we get an exact diagram:

$$\begin{array}{ccccccccccc}
0 \longrightarrow \mathrm{Ker}(f^*\alpha \otimes 1) \longrightarrow & \mathcal{O}_X^p \otimes \mathcal{F} & & \xrightarrow{f^*\alpha\otimes 1} & & \mathcal{F} \longrightarrow \mathcal{O}_{X_s} \otimes \mathcal{F} \longrightarrow 0 \\
& & \searrow & & \nearrow & \\
& & & \mathrm{Im}(f^*\alpha \otimes 1) & & \\
& & \nearrow & & \searrow & \\
& 0 & & & & 0
\end{array}$$

By the inductive hypothesis, we can find a neighborhood U' of s where for $m \gg 0$,

$$R^i f_*(\operatorname{Ker}(f^*\alpha \otimes 1)(mD)) = 0.$$

This implies the surjectivity on U' of the map,

$$R^{i-1} f_*(\mathcal{O}_X^p \otimes \mathcal{F}(mD)) \longrightarrow R^{i-1} f_*(\operatorname{Im}(f^*\alpha \otimes 1)(mD)).$$

Moreover, since $\mathcal{O}_{X_s}(mD)$ is ample, the higher direct images of $\mathcal{O}_{X_s} \otimes \mathcal{F}(mD)$ will vanish for sufficiently large m. So, it is also surjective the map

$$R^{i-1} f_*(\operatorname{Im}(f^*\alpha \otimes 1)(mD)) \twoheadrightarrow R^{i-1} f_*(\mathcal{F}(mD))$$

and therefore the composition

$$R^{i-1} f_*(\mathcal{O}_X^p \otimes \mathcal{F}(mD)) \longrightarrow R^{i-1} f_*(\mathcal{F}(mD))$$

is surjective on U' for $m \gg 0$.

From the projection formula,

$$R^{i-1} f_*(\mathcal{O}_X^p \otimes \mathcal{F}(mD)) \cong \mathcal{O}_X^p \otimes R^{i-1} f_*(\mathcal{F}(mD))$$

and the surjective map

$$\mathcal{O}_X^p \otimes R^{i-1} f_*(\mathcal{F}(mD)) \twoheadrightarrow R^{i-1} f_*(\mathcal{F}(mD))$$

is just $\alpha \otimes 1$ by construction. Hence,

$$R^{i-1} f_*(\mathcal{F}(mD)) = \mathfrak{m}_s R^{i-1} f_*(\mathcal{F}(mD))$$

and by Nakayama's Lemma $R^{i-1} f_*(\mathcal{F}(mD)) = 0$ on U' as wanted. This proves the claim.

Applying this result to the ideal sheaf $\mathcal{I}_{X_s/X}$ we obtain a surjective map

$$f_*\mathcal{O}_X(mD) \longrightarrow f_*(\mathcal{O}_{X_s}(mD))$$

on a neighborhood U' of s for $m \gg 0$. We now form a commutative diagram,

$$\begin{array}{ccc} f^*f_*(\mathcal{O}_X(mD)) & \twoheadrightarrow & f^*f_*(\mathcal{O}_{X_s}(mD)) \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho_{X_s}} \\ \mathcal{O}_X(mD) & \longrightarrow & \mathcal{O}_{X_s}(mD) \end{array}$$

where ρ_X and ρ_{X_s} are the canonical maps.

Since $\mathcal{O}_{X_s}(mD)$ is ample, the map

$$f^*f_*(\mathcal{O}_{X_s}(mD)) = H^0(X_s, \mathcal{O}_X(mD)) \otimes \mathcal{O}_{X_s} \xrightarrow{\rho_{X_s}} \mathcal{O}_{X_s}(mD)$$

is also surjective for sufficiently large m. Looking back at the diagram, it follows that ρ_X is surjective on X_s and consequently surjective near X_s because $\mathrm{Coker}(\rho_X)$ has closed support for being coherent. By virtue of f being a closed map, we can take an open affine neighborhood $U = \mathrm{Spec} B \subseteq U'$ of s, such that ρ_X is surjective on $f^{-1}(U)$. Picking a finite number of sections generating $f_*(\mathcal{O}_X(mD))$ and pulling them back to X, we reach a surjective morphism of sheaves on $f^{-1}(U)$,

$$f^*\mathcal{O}_U^{n+1} \twoheadrightarrow \mathcal{O}_{f^{-1}(U)}(mD)$$

which defines a map

$$\varphi : f^{-1}(U) \longrightarrow \mathbb{P}^n(\mathcal{O}_U^{n+1}) = \mathbb{P}^n \times U$$

such that $\varphi^*_{\mathbb{P}^n \times U}(1) = \mathcal{O}_{f^{-1}(U)}(mD)$. The invertible sheaf $\mathcal{O}_{f^{-1}(U)}(mD)|_{X_s}$ is ample and therefore $\varphi|_{X_s}$ is finite. From upper semicontinuity of fibre dimension and the properness of f we conclude that $\varphi|_{X_{s'}}$ is finite for all s' on a neighborhood of s. □

We now turn to the proof of the theorem.

Proof of Theorem V.4. If D is f-ample then let s be a point in S. Let $j : X_s \hookrightarrow X$ be the inclusion mapping of the fibre over s in X. For any coherent sheaf $\mathcal{F}$ on X_s

and any integers $i, m > 0$ we have that,

$$\begin{aligned} H^i(X_s, \mathcal{F} \otimes j^*\mathcal{O}_X(mD)) &= H^i(X, j_*(\mathcal{F} \otimes j^*\mathcal{O}_X(mD))) && \text{because } j \text{ is finite} \\ &= H^i(X, j_*\mathcal{F} \otimes \mathcal{O}_X(mD)) && \text{by projection formula.} \end{aligned}$$

For $m \gg 0$, $R^i f_*(j_*\mathcal{F} \otimes \mathcal{O}_X(mD)) = 0$ thanks to the f-ampleness of D and this implies that,

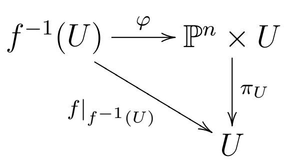
$$\begin{aligned} H^i(X, j_*\mathcal{F} \otimes \mathcal{O}_X(mD)) &= H^i(S, f_*(j_*(\mathcal{F}) \otimes \mathcal{O}_X(mD))) \\ &= H^i(S, (f|_{X_s})_*(\mathcal{F} \otimes j^*\mathcal{O}_X(mD))). \end{aligned}$$

Since $(f|_{X_s})_*(\mathcal{F} \otimes j^*\mathcal{O}_X(mD))$ is supported at a point, we obtain

$$H^i(X_s, \mathcal{F} \otimes j^*\mathcal{O}_X(mD)) = 0$$

which shows that D_s is ample.

For the converse, suppose that D_s is ample for all $s \in S$. From the fact that f-amplitude is a local condition on S, we just need to show that for each $s \in S$ has a neighborhood $U \subseteq S$ such that $D|_{f^{-1}(U)}$ is $f|_{f^{-1}(U)}$-ample. The proof of Proposition V.5 shows we can find for each $s \in S$ a neighborhood $U \subseteq S$ such that $f|_U$ factors though $\mathbb{P}^n \times U$ as shown in the diagram



where φ is a finite map, π_U is the projection on U and $\mathcal{O}_{f^{-1}(U)}(mD) = \varphi^*\mathcal{O}_{\mathbb{P}^n \times U}(1)$. Since we only need to prove that mD is $f|_{f^{-1}(U)}$-ample we can replace D by mD and assume $\mathcal{O}_{f^{-1}(U)}(D) = \varphi^*\mathcal{O}_{\mathbb{P}^n \times U}(1)$.

Let $\mathcal{F}$ be a coherent sheaf on $f^{-1}(U)$. For all integers $i, r > 0$ we have

$$\begin{aligned} R^i(f|_{f^{-1}(U)})_*(\mathcal{F}(rD)) &= H^i(f^{-1}(U), \mathcal{F}(rD))^{\sim} && \text{because } U \text{ is affine} \\ &= H^i(\mathbb{P}^n \times U, \varphi_*\mathcal{F}(rD))^{\sim} && \text{because } \varphi \text{ is finite} \\ &= H^i(\mathbb{P}^n \times U, \varphi_*(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n \times U}(1))^{\sim} && \text{by projection formula} \\ &= R^i(\pi_U)_*(\varphi_*(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}^n \times U}(1)). \end{aligned}$$

From the π_U-amplitude of $\mathcal{O}_{\mathbb{P}^n \times U}(1)$ we get that for $r \gg 0$, $R^i f_*(\mathcal{F}(rD)) = 0$, and therefore $D|_{f^{-1}(U)}$ is $f|_{f^{-1}(U)}$-ample as required. □

As a corollary, one obtains a relative version of the classical Nakai-Moishezon Criterion.

Corollary V.6 (Nakai-Moishezon criterion for a mapping)**.** *A divisor $D \in \mathrm{Div}(X)_{\mathbb{Q}}$ is f-ample if and only if $D^{\dim V}.V > 0$ for every irreducible subvariety $V \subseteq X$ of positive dimension that maps to a closed point in S.*

Proof. This follows directly form Theorem V.4 combined with Theorem II.13. □

Corollary V.7. *f-ampleness is a numerical property of $\mathbb{Q}$-divisors.*

Example V.8. Let Y be a smooth surface and let

$$\pi : X = X_n \xrightarrow{\pi_n} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Y$$

be a sequence of blowups of points $p_i \in X_i$. Let

$$\mathrm{Exc}(\pi) = E_1 \cup \ldots \cup E_n$$

where each E_i is the strict transform in X of the exceptional divisor of $X_i \xrightarrow{\pi_i} X_{i-1}$. Then, by Corollary IV.5, the intersection matrix $A = ((E_i \cdot E_j)_{ij})$ is invertible and letting $A^{-1} = (d_{ij})$, the divisor classes

$$\xi_j = d_{1j}[E_1] + \ldots + d_{nj}[E_n], \quad 1 \leq j \leq n$$

form a dual basis to $([E_1], \ldots, [E_n])$. Therefore, by Corollary V.6, a divisor $D \in \mathrm{Div}(X)_{\mathbb{Q}}$ is π-ample if and only if its relative numerical equivalence class is a linear combination

$$\sum a_i \xi_i$$

for some rational coefficients $a_i \geq 0$.

5.2 Relative amplitude for $\mathbb{R}$-divisors

We recall that an $\mathbb{R}$-divisor is an element in $\mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes \mathbb{R}$ written as a finite sum

$$\sum r_i D_i$$

where each r_i is a real number and each D_i is a Cartier divisor in $\mathrm{Div}(X)$. A divisor is effective if it can be written as a sum where each r_i is positive and each D_i is effective.

One can easily extend intersection theory for $\mathbb{R}$-divisors just by setting

$$(\sum r_i D_i \cdot C) = \sum r_i (D_i \cdot C)$$

for any given proper integral curve $C \subseteq X$ mapping to a point in S.

Definition V.9. The group of numerically trivial $\mathbb{R}$-divisors over S, denoted by $\mathrm{Num}(X/S)_{\mathbb{R}}$, is formed by the $\mathbb{R}$-divisors D such that

$$(D \cdot C) = 0$$

for any proper integral curve $C \subseteq X$ mapping to a point in S. We define the relative Néron-Severi group of $\mathbb{R}$-divisors over S as being the quotient

$$N^1(X/S)_{\mathbb{R}} := \mathrm{Div}(X)_{\mathbb{R}} / \mathrm{Num}(X/S)_{\mathbb{R}}.$$

We start by giving a useful characterization of $\mathrm{Num}(X/S)_{\mathbb{R}}$.

Lemma V.10. *A numerically trivial* $\mathbb{R}$*-divisor over* S *is an* $\mathbb{R}$*-linear combination of numerically trivial integral divisors over* S*, meaning that,*

$$\mathrm{Num}(X/S)_{\mathbb{R}} = \mathrm{Num}(X/S) \otimes \mathbb{R}.$$

Proof. Let $D_0 = \Sigma r_i D_i \in \mathrm{Div}(X)_{\mathbb{R}}$ be a numerically trivial $\mathbb{R}$-divisor over S. Let

$$V =< D_i >_{\mathrm{Div}(X)_{\mathbb{R}}}$$

be the finite dimensional real vector subspace of $\mathrm{Div}(X)_{\mathbb{R}}$ generated by the D_i. Then

$$V' = V \cap \mathrm{Num}(X/S)_{\mathbb{R}}$$

is also a finite dimensional vector space which contains D_0. As such, we can find a finite number of integral proper curves $C_1, \ldots, C_n \subseteq X$ mapped to points so that,

$$V' = \{D \in V | (D \cdot C_i) = 0 \text{ for all } 1 \leq i \leq n\}.$$

Consider now the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Ker}\alpha \longrightarrow < D_i > \xrightarrow{\alpha} \mathbb{Z}^n$$

where $< D_i >$ is the subgroup of $\mathrm{Div}(X)$ generated by the D_i and α is the homomorphism defined by mapping each $D \in < D_i >$ to $((D \cdot C_1), \ldots, (D \cdot C_n))$. By tensoring with $\mathbb{R}$, one gets an exact sequence

$$0 \longrightarrow \mathrm{Ker}\alpha \otimes \mathbb{R} \longrightarrow V \xrightarrow{\alpha \otimes 1} \mathbb{R}^n$$

because $\mathbb{R}$ is a flat $\mathbb{Z}$-module. But the homomorphism $\alpha \otimes 1$ is defined by intersection with the curves $C_1, \ldots, C_n$ and as a result,

$$V' = \mathrm{Ker}(\alpha \otimes 1) = \mathrm{Ker}\alpha \otimes \mathbb{R},$$

by construction. Since $D_0 \in V'$ and $\text{Ker}\alpha \subseteq \text{Num}(X/S)$, we conclude that D_0 can be written as an $\mathbb{R}$-linear combination of numerically trivial integral divisors over S, as required. $\square$

Corollary V.11. *There is an isomorphism of finite-dimensional real vector spaces,*

$$N^1(X/S)_{\mathbb{R}} \cong N^1(X/S) \otimes \mathbb{R}.$$

Proof. Tensoring the short exact sequence

$$0 \longrightarrow \text{Num}(X/S) \longrightarrow \text{Div}(X) \longrightarrow N^1(X/S) \longrightarrow 0$$

with $\mathbb{R}$ yields another short exact sequence

$$0 \longrightarrow \text{Num}(X/S) \otimes \mathbb{R} \longrightarrow \text{Div}(X)_{\mathbb{R}} \longrightarrow N^1(X/S) \otimes \mathbb{R} \longrightarrow 0$$

due to $\mathbb{R}$ being a flat $\mathbb{Z}$-module. Therefore,

$$\begin{aligned} N^1(X/S) \otimes \mathbb{R} &\cong \text{Div}(X)_{\mathbb{R}}/\text{Num}(X/S) \otimes \mathbb{R} \\ &\cong \text{Div}(X)_{\mathbb{R}}/\text{Num}(X/S)_{\mathbb{R}} && \text{by Lemma V.10} \\ &= N^1(X/S)_{\mathbb{R}}. \end{aligned}$$

Both vector spaces are finite-dimensional because $N^1(X/S)$ is finitely generated by Theorem III.20. $\square$

Definition V.12. A divisor in $\text{Div}(X)_{\mathbb{R}}$ is f-ample if it can be written as a finite sum

$$\sum r_i D_i$$

where each r_i is a positive real number and each D_i is an f-ample integral Cartier divisor.

Proposition V.13. *f-ampleness is a numerical property of $\mathbb{R}$-divisors.*

Proof. Let A be an f-ample divisor in $\mathrm{Div}(X)_{\mathbb{R}}$ and B a divisor in $\mathrm{Num}(X/S)_{\mathbb{R}}$. We want to show that $A+B$ is f-ample. We can assume that $A = rD$ for some positive real number r and integral Cartier divisor in $\mathrm{Div}(X)$ because f-ampleness is stable under addition. Let $r_1, r_2 \in \mathbb{Q}^+$ such that $r_1 < r^{-1} < r_2$. Then there is a $t \in (0,1)$ such that $r^{-1} = tr_1 + (1-t)r_2$. Consequently,

$$A + B = rt(D + r_1 B) + r(1-t)(D + r_2 B).$$

Since $D + r_1 B$ and $D + r_2 B$ are both f-ample, so is $A + B$. □

The relative notion of nefness is the following.

Definition V.14. A divisor $D \in \mathrm{Div}(X)_{\mathbb{R}}$ is f-nef if $(D \cdot C) \geq 0$ for any integral proper curve C mapped to a point. A numerical class in $N^1(X/S)_{\mathbb{R}}$ is f-nef if it is represented by an f-nef divisor.

Theorem V.15 (Kleiman for a mapping)**.** *An $\mathbb{R}$-divisor D on X is f-nef if and only if $(D^{\dim V} \cdot V) \geq 0$ for every irreducible variety $V \subseteq X$ mapped to a point.*

Proof. This is a restatement of Theorem II.11 in the relative setting. □

We are particularly interested in understanding the structure of the f-ample numerical classes and their relation with the f-nef numerical classes inside the finite-dimensional vector space $N^1(X/S)_{\mathbb{R}}$. One easily observes that f-ampleness and f-nefness are both stable under addition and positive scalar multiplication straight from their definition. This motivates the use of the respective associated cones which we denote by,

$$\mathrm{Amp}(X/S) := \text{convex cone of } f\text{-ample classes in } N^1(X/S)_{\mathbb{R}}$$

$$\mathrm{Nef}(X/S) := \text{convex cone of } f\text{-nef classes in } N^1(X/S)_{\mathbb{R}}.$$

Proposition V.16. *There are f-ample integral divisors $A_1, \ldots, A_n \in \mathrm{Div}(X)$ whose classes form a finite basis for $N^1(X/S)_{\mathbb{R}}$.*

Proof. By Corollary V.11 we may pick a a basis for $N^1(X/S)_{\mathbb{R}}$ consisting of a finite number of classes of integral divisors $D_1, \ldots, D_n \in \mathrm{Div}(X)$. Let A be an f-ample integral divisor. We can find a sufficiently large integer $m > 0$ such that the divisors $mA + D_i$ are f-ample and their numerical classes are linearly independent in $N^1(X/S)_{\mathbb{R}}$. Letting

$$A_i = mA + D_i$$

for all i, we obtain a desired basis of f-ample integral divisors for $N^1(X/S)_{\mathbb{R}}$. □

The following lemma will be the key ingredient for showing that f-ampleness is an open property in $N^1(X/S)_{\mathbb{R}}$.

Lemma V.17. *Let $A \in \mathrm{Amp}(X/S)$ and $D \in \mathrm{Div}_{\mathbb{R}}(X)$. Then, $mA+D \in \mathrm{Amp}(X/S)$ for all real $m \gg 0$.*

Proof. If $D = rD'$ for some real number $r \in \mathbb{R}$ and integral divisor $D' \in \mathrm{Div}(X)$, let $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r < r_2$. We can pick $q \in \mathbb{Q}$ so that each $\mathbb{Q}$-divisor $qA + r_i D$ is f-ample. Therefore,

$$mA + r_i D = (m - q)A + (qA + r_i D)$$

is an f-ample $\mathbb{R}$-divisor for all real $m \gg 0$. Letting $t \in (0, 1)$ such that $r = tr_1 + (1 - t)r_2$ we obtain,

$$mA + D = t(mA + r_1 D) + (1 - t)(mA + r_2 D)$$

which shows $mA + D$ is f-ample for all sufficiently large m.

In the case of a general $\mathbb{R}$-divisor $D = r_1D_1 + \ldots + r_nD_n$ with $r_i \in \mathbb{R}$ and $D_i \in \mathrm{Div}(X)$, we have that the $\mathbb{R}$-divisors

$$mn^{-1}A + r_iD_i$$

are simultaneously f-ample for all $m \gg 0$. Hence,

$$mA + D = \Sigma(mn^{-1}A + r_iD)$$

will also be f-ample. □

Corollary V.18. *The f-ample cone* $\mathrm{Amp}(X/S)$ *is an open subset of* $N^1(X/S)_{\mathbb{R}}$.

Proof. Let $H \in \mathrm{Amp}(X/S)$ be an f-ample $\mathbb{R}$-divisor. Let $A_1, \ldots A_n \in \mathrm{Amp}(X/S)$ be a finite set of divisors whose classes form a basis for $N^1(X/S)_{\mathbb{R}}$. By Corollary V.17, consider a sufficiently large $m > 0$ such that,

$$mH - A_1 - \ldots - A_n$$

is f-ample. Then, the $\mathbb{R}$-divisor

$$H - m^{-1}A_1 - \ldots - m^{-1}A_n$$

is also f-ample. As such, for any real numbers $r_1, \ldots r_n \geq -m^{-1}$, the $\mathbb{R}$-divisor

$$H + r_1A_1 + \ldots + r_nA_n$$

is f-ample, proving that all $\mathbb{R}$-divisors are f-ample in a neighborhood of H. □

We can now extend fibre-wise amplitude to $\mathbb{R}$-divisors.

Lemma V.19. *Let $D \in \mathrm{Div}(X)_{\mathbb{R}}$ and let $s \in S$ be a point. If D_s is ample then there is a neighborhood U of s such that $D|_{f^{-1}(U)}$ is $f|_{f^{-1}(U)}$-ample.*

Proof. By Proposition V.16 we can write $D = \sum \alpha_i A_i$ where each $\alpha_i \in \mathbb{R}$ and the A_i are f-ample integral divisors. By Corollary V.18, let $0 < r \ll 1$ such that $(D - \sum rA_i)_s$ is an ample $\mathbb{Q}$-divisor class. From Proposition V.5, we can find a neighborhood U of s where $(D - \sum rA_i)|_{f^{-1}(U)}$ is $f|_{f^{-1}(U)}$-ample. Thus,

$$D|_{f^{-1}(U)} = (D - \sum rA_i)|_{f^{-1}(U)} + \sum rA_i|_{f^{-1}(U)}$$

is $f|_{f^{-1}(U)}$-ample. □

Theorem V.20 (Fibre-wise amplitude for $\mathbb{R}$-divisors)**.** *Let $D \in \mathrm{Div}(X)_{\mathbb{R}}$. Then D is f-ample if and only if D_s is ample for all $s \in S$.*

Proof. We only need to show that if D_s is ample for all $s \in S$ then D is f-ample. Applying Lemma V.19, for each $s \in S$ there is an open neighborhood U_s of s such that $D|_{f^{-1}(U)}$ is $f|_{f^{-1}(U)}$-ample. By quasi-compactness we can find a finite subcover $\{U_{s_0}, \ldots, U_{s_m}\}$ of S. Let $A_1, \ldots, A_n$ be a basis of f-ample integral divisors for $N^1(X/S)_{\mathbb{R}}$. Corollary V.18 allows us to consider a real number $0 < r \ll 1$ such that each $(D - \sum rA_i)$ is a $\mathbb{Q}$-divisor and $(D - \sum rA_i)|_{f^{-1}(U)}$ is $f|_{f^{-1}(U_{s_i})}$-ample for all i. Then, $(D - \sum rA_i)$ is f-ample by Theorem V.4. So,

$$D = (D - \sum rA_i) + \sum rA_i$$

is f-ample. □

Corollary V.21 (Relative Nakai's criterion for $\mathbb{R}$-divisors)**.** *If D is an $\mathbb{R}$-divisor on X, then D is f-ample if and only if $(D^{\dim V} \cdot V) > 0$ for every irreducible variety $V \subseteq X$ mapped to a point.*

Proof. This comes directly from Theorem V.20 together with Theorem II.13. □

Lemma V.22. *Let D be an f-nef $\mathbb{R}$-divisor on X. Then, for any f-ample $\mathbb{R}$-divisor A on X, $D + \varepsilon A$ is f-ample for every $\varepsilon > 0$.*

Proof. By fibre-wise amplitude we only need to show $(D+\varepsilon A)$ is ample on each fibre for any $\varepsilon > 0$. Since D_s is nef and A_s is ample, it follows from [11, Corollary 1.4.10] that $(D+\varepsilon A)$ is ample as wanted. □

Theorem V.23. *The following equalities hold,*

$$\mathrm{Amp}(X/S) = \mathrm{int}(\mathrm{Nef}(X/S)), \quad \mathrm{Nef}(X/S) = \overline{\mathrm{Amp}(X/S)}.$$

Proof. Let D be an $\mathbb{R}$-divisor on X whose relative numerical class is in $\mathrm{int}(\mathrm{Nef}(X/S))$. Then, let A be an f-ample $\mathbb{R}$-divisor such that $D - A$ is f-nef. By Lemma V.22,

$$D = (D - A) + A$$

is f-ample meaning that $\mathrm{Amp}(X/S) \supseteq \mathrm{int}(\mathrm{Nef}(X/S))$. By virtue of the f-ample cone being an open set contained in the f-nef cone it follows that $\mathrm{Amp}(X/S) = \mathrm{int}(\mathrm{Nef}(X/S))$.

On the other hand, let $\xi \in \overline{\mathrm{Amp}(X/S)}$. Let (ξ_n) be a sequence of f-ample classes converging to ξ. Then, for any curve $C \subseteq X$ mapped to a point,

$$(\xi_n \cdot C) > 0.$$

As intersection against C defines a linear functional on $N^1(X/S)_{\mathbb{R}}$, which is in particular an $\mathbb{R}$-valued continuous function, we obtain

$$(\xi \cdot C \geq 0).$$

Hence ξ is f-nef and we conclude that $\mathrm{Nef}(X/S) = \overline{\mathrm{Amp}(X/S)}$. □

One can also consider cones inside $N_1(X/S)_{\mathbb{R}}$. We define,

$$\mathrm{NE}(X/S) := \text{convex cone of effective classes in } N_1(X/S)_{\mathbb{R}}.$$

We let the relative closed cone of curves be $\overline{\mathrm{NE}}(X/S)$, the closure of $\mathrm{NE}(X/S)$ in $N_1(X/S)_{\mathbb{R}}$. We state an alternative characterization of f-ampleness in terms of intersection against this cone.

Theorem V.24. *(Kleiman's criterion) If D is an $\mathbb{R}$-divisor on X, then D is f-ample if and only if $(D \cdot C) > 0$ for all $C \in \overline{\mathrm{NE}}(X/S)\backslash\{0\}$.*

Theorem V.24 is a straight generalization of the result in the absolute case and we omit its proof. For the proof in the absolute case we refer to [11, Theorem 1.4.29] and [9, Proposition II.4.8].

In the following examples we describe several different relative nef cones of a mapping. In all the analyzed cases these cones will be polyhedral although in general this does not always happen as we will see in Chapter VII.

Example V.25. Going back to Example V.8 the cone $\mathrm{Nef}(X/Y)$ is generated by n extremal rays spanned by the ξ_i, while the n numerical classes $[E_i]$ are distinct extremal rays spanning $\overline{\mathrm{NE}}(X/Y)$ and defining the faces of the f-nef cone.

Example V.26. Let $\pi : X = X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} \mathbb{A}^3 = Y$ be a sequence of blowups with the following description:

– π_1 is the blowup of the origin;

– π_2 is the blowup of 3 distinct non-collinear points $p_1, p_2, p_3 \in \mathrm{Exc}(\pi_1) \cong \mathbb{P}^2$.

The exceptional divisor

$$\mathrm{Exc}(\pi) = E \cup E_1 \cup E_2 \cup E_3$$

has 4 components where E is the strict transform of $\mathrm{Exc}(\pi_1)$ under π_2 and each $E_i = \pi_2^{-1}(p_i)$ for $1 \leq i \leq 3$. Then $N^1(X/Y)_{\mathbb{R}} =< E, E_1, E_2, E_3 >\cong \mathbb{R}^4$. We want to find the relative nef cone $\mathrm{Nef}(X/Y)$.

Let $H = -E - E_1 - E_2 - E_3$. We will use the notation,

e_i := divisor class of E_i

h := divisor class of H

L := strict transform of line in $\mathrm{Exc}(\pi_1)$ not containing any p_i

L_j := strict transform of line in $\mathrm{Exc}(\pi_1)$ through p_j not containing any p_i $(i \neq j)$

L_{jk} := strict transform of line in $\mathrm{Exc}(\pi_1)$ through p_j and p_k

F_j := $E \cap E_j$.

We set the multiplication table,

$$(e_i \cdot L) = 0 \qquad (e_i \cdot L_{jk}) = \delta_{ij} + \delta_{ik} \qquad (e_i \cdot F_j) = -\delta_{ij}$$

$$(h \cdot L) = 1 \qquad (h \cdot L_{jk}) = 1 \qquad (h \cdot F_j) = 0.$$

We claim $\mathrm{Nef}(X/Y)$ is defined by 6 linear inequalities imposed by intersection with the curves $L_{12}, L_{13}, L_{23}, F_1, F_2, F_3$.

In order to prove the claim, consider the hyperplane section of the relative nef cone,

$$V = \mathrm{Nef}(X/Y) \cap \{\xi \in N^1(X/Y) \mid (\xi \cdot L) = 1\}.$$

So, we write a divisor class $\xi \in V$ as $\xi = h - ae_1 - be_2 - ce_3$ and obtain a system of inequalities,

$$(\xi \cdot F_1) = a \geq 0 \qquad (\xi \cdot L_{12}) = 1 - a - b \geq 0$$

$$(\xi \cdot F_2) = b \geq 0 \qquad (\xi \cdot L_{13}) = 1 - b - c \geq 0$$

$$(\xi \cdot F_3) = c \geq 0 \qquad (\xi \cdot L_{23}) = 1 - a - c \geq 0$$

that define the polyhedron pictured in Figure 5.1.

It is now enough to show that the vertices of this polyhedron are π-nef. By Nakai's criterion, a divisor class is π-nef if the restrictions $\xi|_E$ and $\xi|_{E_i}$ are nef. Since each $E_i \cong \mathbb{P}^2$, we have that $\xi|_{E_i}$ is nef for all $1 \leq i \leq 3$ if and only if $(\xi \cdot F_i) \geq 0$ for

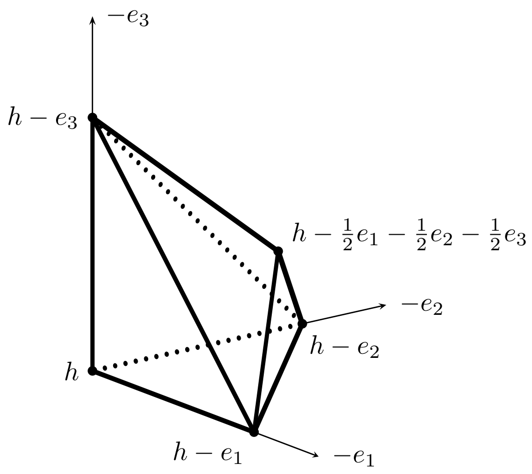


Figure 5.1: Section of Nef(X/Y) for p_1, p_2, p_3 collinear

all $1 \leq i \leq 3$. But this comes directly from the initial setup. On the other hand, each vertex restricted to E is a positive multiple of a divisor class represented by an irreducible curve. More specifically,

$$h|_E = [L] \quad (h - e_i)|_E = [L_i] \quad (h - \tfrac{1}{2}e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3)|_E = \tfrac{1}{2}[C],$$

where C is the strict transform of a conic in Exc(π_1) through the points p_1, p_2, p_3. Thus, we only need to check that the self-intersection of these classes is non-negative. Indeed,

$$([L])^2 = 1 \quad ([L_i])^2 = 0 \quad ([C])^2 = 1,$$

showing Figure 5.1 is a hyperplane section of Nef(X/Y). We also conclude that Nef(X/Y) has 5 extremal rays spanned by

$$h, \quad h - e_1, \quad h - e_2, \quad h - e_3, \quad h - \frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3$$

and $\overline{\text{NE}}(X/Y)$ has 6 extremal rays spanned by the classes of

$$F_1, F_2, F_3, L_{12}, L_{13}, L_{23}.$$

Example V.27. Consider the same setting of Example V.26 assuming p_1, p_2, p_3 are collinear. We maintain the notation except for L_{jk} since a line through 2 points will contain the third. Instead, we denote

$$L_{123} := \text{strict transform of line in } \mathrm{Exc}(\pi_1) \text{ through } p_1, p_2, p_3.$$

We obtain a multiplication table

$$(e_i \cdot L) = 0 \qquad (e_i \cdot L_{123}) = 1 \qquad (e_i \cdot F_j) = -\delta_{ij}$$
$$(h \cdot L) = 1 \qquad (h \cdot L_{123}) = 1 \qquad (h \cdot F_j) = 0.$$

This time, $\mathrm{Nef}(X/Y)$ will be defined by the 4 inequalities imposed by F_1, F_2, F_3, L_{123}. In fact, for $\xi = h - ae_1 - be_2 - ce_3$, these curves yield a system of conditions

$$a, b, c \geq 0$$
$$(\xi \cdot L_{123}) = 1 - a - b - c \geq 0$$

describing the polyhedron in Figure 5.2.

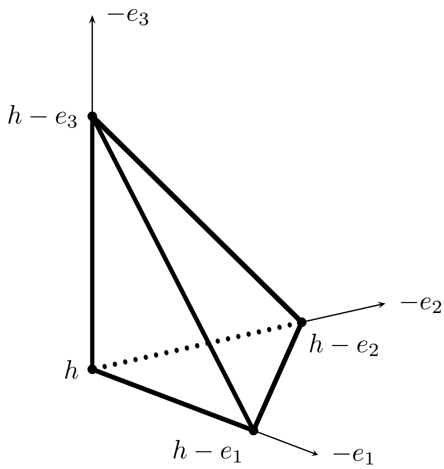


Figure 5.2: Section of $\mathrm{Nef}(X/Y)$ for p_1, p_2, p_3 non-collinear

The argument used in Example V.26 to show h and $h - e_i$ are nef classes works here again exactly the same way. Hence, they span 4 extremal rays of $\mathrm{Nef}(X/Y)$. Moreover $\overline{\mathrm{NE}}(X/Y)$ has 4 extermal rays spanned by the classes of F_1, F_2, F_3, L_{123}.

CHAPTER VI

Relative Campana-Peternell theorem

In this chapter we will explore the geometrical properties of the boundary of the relative nef cone with respect to a mapping. We start by introducing some notation.

We lef $f : X \longrightarrow S$ be a projective morphism of quasi-projective schemes. Let $\mathcal{B}_{X/S}$ be the f-nef boundary $\mathrm{Nef}(X/S) \backslash \mathrm{Amp}(X/S)$. We denote by $\mathcal{V}$ the set of all proper irreducible varieties on X mapping to a point in S. If $V \in \mathcal{V}$ then there is an associated function

$$\begin{aligned} \varphi_V \colon N^1(X/S)_{\mathbb{R}} &\longrightarrow \mathbb{R} \\ D &\longmapsto (D^{\dim V} \cdot V). \end{aligned}$$

We also define the null locus

$$\mathcal{N}_V := \{D \in N^1(X/S)_{\mathbb{R}} \mid \varphi_V(D) = 0\}.$$

Remark VI.1. By considering a finite basis of integral divisors on $N^1(X/S)_{\mathbb{R}}$, the function φ_V is a homogeneous polynomial with rational coefficients on the respective basis coordinates. In particular, the family of null loci $(\mathcal{N}_V)_{V \in \mathcal{V}}$ has at most countable many distinct members.

With this in mind we can extend the Theorem II.14 of Campana-Peternell characterizing the boundary of the nef cone, to the relative setting. This result will tell that the boundary of the f-nef cone is also locally cut out by polynomials in a dense open substet.

Theorem VI.2. *There is a dense open set $U \subseteq \mathcal{B}_{X/S}$ with the following property. For all $\xi \in U$, there is a proper irreducible variety $V \subseteq X$ mapping to a point in S and an open neighborhood W of ξ in $N^1(X/S)_{\mathbb{R}}$ such that, $W \cap \mathcal{B}_{X/S} = W \cap \mathcal{N}_V$.*

Proof: For each proper irreducible variety $V \subseteq X$ mapped to a point, let $\mathcal{B}_V = \mathcal{B}_{X/S} \cap \mathcal{N}_V$ and let O_V be the interior of $\mathcal{B}_V$ in $\mathcal{B}_{X/S}$. By the relative Nakai's criterion,

$$\mathcal{B}_{X/S} = \bigcup_{V \in \mathcal{V}} \mathcal{B}_V. \tag{6.1}$$

We claim that

$$U' = \bigcup_{V \in \mathcal{V}} O_V$$

is dense in $\mathcal{B}_{X/S}$.

Suppose U' is not dense in $\mathcal{B}_{X/S}$. Then there is a point $\xi \in \mathcal{B}_{X/S}$ with an open neighborhood W such that its closure $\overline{W}$ in $\mathcal{B}_{X/S}$ is compact and $W \cap O_V = \emptyset$ for all $V \in \mathcal{V}$. This implies that fixing some $V \in \mathcal{V}$, the interior of $\overline{W} \cap \mathcal{B}_V$ in $\overline{W}$ does not intersect W and consequently is empty. Hence,

$$(\overline{W} \cap \mathcal{B}_V)_{V \in \mathcal{V}} = (\overline{W} \cap \mathcal{N}_V)_{V \in \mathcal{V}}$$

is a family of closed subsets of $\overline{W}$ having empty interior with at most countable many distinct members by Remark VI.1. By virtue of $\overline{W}$ being a complete topological space, we can apply Baire's theorem to conclude that the set

$$\bigcup_{V \in \mathcal{V}} (\overline{W} \cap \mathcal{B}_V) = \overline{W} \cap (\bigcup_{V \in \mathcal{V}} \mathcal{B}_V) = \overline{W} \qquad \text{by (6.1)}$$

has empty interior in $\overline{W}$. This is absurd and the claim follows.

It is clear that the set $U \subseteq \mathcal{B}_{X/S}$ satisfying the conditions of the theorem is open. We now prove that every point in U' is a limit of points in U. Due to U' being dense in $\mathcal{B}_{X/S}$ this will show that so is U as required.

For this purpose we first note that any point $\xi \in O_V$ such that

$$d\varphi_V(\xi; H) > 0$$

for some $V \in \mathcal{V}$ and $H \in \mathrm{Amp}(X/S)$ must be in U. Indeed, this set up assures us that near ξ, O_V is a piece of a regular hypersurface in $N^1(X/S)_{\mathbb{R}}$. On the other hand $\mathcal{B}_{X/S}$ is a topological manifold of codimension 1 in $N^1(X/S)_{\mathbb{R}}$ for being the boundary of a convex open set. Since $O_V \subseteq \mathcal{B}_{X/S}$ and both are topological manifolds of the same dimension near ξ, then there is an open neighborhood W of ξ in $N^1(X/S)_{\mathbb{R}}$ such that $W \cap \mathcal{B}_{X/S} = W \cap \mathcal{N}_V$. This means that $\xi \in U$ as wanted.

The final step will be establishing that for a fixed point $\xi \in U'$ we can find a variety $V \in \mathcal{V}$ and an f-ample divisor H such that ξ is a limit of points $\xi' \in O_V$ satisfying

$$d\varphi_V(\xi'; H) > 0.$$

To this end, consider a variety V of positive minimal dimension mapping to a point $s \in S$ such that $\xi \in O_V$. Let H be an f-ample divisor whose restriction H_s to X_s is an integral very ample divisor class. Then for any $\xi' \in O_V$,

$$\begin{aligned} d\varphi_V(\xi'; H) &= \frac{d}{dt}|_{t=0}[\varphi_V(\xi' + tH)] \\ &= \frac{d}{dt}|_{t=0}[((\xi' + tH)^{\dim V} \cdot V)] \\ &= \frac{d}{dt}|_{t=0}[(\xi'^{\dim V} \cdot V) + t\dim V(\xi'^{\dim V-1} \cdot H \cdot V) + t^2(\ldots)] \\ &= \dim V(\xi'^{\dim V-1} \cdot H \cdot V). \end{aligned}$$

When V is a curve, $d\varphi_V(\xi'; H) = (H \cdot V)$ is positive because H is f-ample. In case $\dim V > 1$, using Bertini's Theorem, the divisor class H_s can be assumed to be represented by a section meeting V properly at an irreducible variety that we denote by $V \cap H$. By minimality of V, ξ is a limit of points $\xi' \in O_V$ such that,

$$\varphi_{V \cap H}(\xi') > 0$$

and therefore satisfying

$$d\varphi_V(\xi'; H) = \dim V \varphi_{V \cap H}(\xi') > 0,$$

as required.

CHAPTER VII

Non-polyhedral relative nef cone for a sequence of blowups

Our primary goal in this chapter is finding examples of relative nef cones for a sequence of blowups that fail to be polyhedral. Since there are plenty of well know surfaces with non-polyhedral nef cones, we develop an approach based on making a connection between relative nefness for a given morphism and nefness on a surface contained in the exceptional locus of that morphism. This will allow us to prove the existence of such non-polyhedral relative nef cones.

7.1 Main theorem: construction and notation

The statement of the main theorem is the following.

Theorem VII.1. *There exists a morphism $f\colon X \longrightarrow \mathbb{A}^4$, obtained as sequence of blowups of smooth centers, such that* $\mathrm{Nef}(X/\mathbb{A}^4)$ *is non-polyhedral.*

For the purpose of proving Theorem VII.1, we now introduce the construction we will use throughout this chapter together with its notation.

We define a sequence of blowups

$$\pi\colon X = X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 \xrightarrow{\pi_0} \mathbb{A}^4$$

the following way. The map $X_0 \longrightarrow \mathbb{A}^4$ is the blowup of the origin with exceptional divisor $E_0 \cong \mathbb{P}^3$. Then, let S be a smooth surface on E_0. In practice, the surface

S we will have in mind is a particular $K3$ surface with round ample cone. Let C_1 and C_2 be two irreducible smooth curves on S intersecting transversally that are also ample divisors on S. The map $X_1 \longrightarrow X_0$ is the blowup of C_1 and $X_2 \longrightarrow X_1$ is the blowup of the strict transform of C_2.

On E_0 there is an invertible sheaf $\mathcal{O}_{E_0}(1) = \mathcal{O}_{E_0}(-E_0)$ and we set,

$$
\begin{aligned}
d &= \deg S, & d_1 &= \deg C_1, & d_2 &= \deg C_2, \\
\delta &= (C_1 \cdot C_2), & \delta_1 &= (C_1 \cdot C_1), & \delta_2 &= (C_2 \cdot C_2),
\end{aligned}
$$

considering degrees on E_0 and intersection numbers on S. We let E_i be the exceptional divisor for each map π_i and set:

$$
\begin{aligned}
E_i^{(j)} &:= \text{strict transform of } E_i \text{ on } X_j \text{ for } j \geq i; \\
E_{10}^{(i)} &:= E_1^{(i)} \cap E_0^{(i)} \text{ on } E_0^{(i)} \subseteq X_i \text{ for } i \geq 1; \\
E_{20}^{(i)} &:= E_2^{(i)} \cap E_0^{(i)} \text{ on } E_0^{(i)} \subseteq X_i \text{ for } i = 2; \\
S^{(i)} &:= \text{strict transform of } S \text{ on } E_0^{(i)} \subseteq X_i \text{ for } i \geq 0; \\
C_1^{(0)} &:= C_1 \text{ on } S^{(1)} \subseteq X_0; \\
C_1^{(1)} &:= E_1^{(1)} \cap S^{(1)} \text{ on } S^{(1)} \subseteq X_0; \\
C_1^{(2)} &:= \text{strict transform of } C_1^{(1)} \text{ on } S^{(2)} \subseteq X_2; \\
C_2^{(0)} &:= C_2 \text{ on } S^{(1)} \subseteq X_0; \\
C_2^{(1)} &:= \text{strict transform of } C_2^{(0)} \text{ on } S^{(1)} \subseteq X_1; \\
C_2^{(2)} &:= E_2^{(2)} \cap S^{(2)} \text{ on } S^{(2)} \subseteq X_2; \\
H^{(i)} &:= \text{pullback of } -E_0 \text{ on } X_i \text{ for } i \geq 0; \\
H_0^{(i)} &:= \text{divisor class } H|_{E_0^{(i)}} \text{ in } N^1(E_0^{(i)})_{\mathbb{R}} \text{ for } i \geq 0; \\
h^{(i)} &:= \text{divisor class } h|_{S^{(i)}} \text{ in } N^1(S^{(i)})_{\mathbb{R}} \text{ for } i \geq 0.
\end{aligned}
$$

Note that for $i > j$, the maps $S^{(i)} \longrightarrow S^{(j)}$, $C_1^{(i)} \longrightarrow C_1^{(j)}$ and $C_2^{(i)} \longrightarrow C_2^{(j)}$ are isomorphisms and the divisor class h is invariant under the pullback isomorphism

$N^1(S^{(i)}) \longrightarrow N^1(S^{(j)})$. From now on we shall omit superscripts an we do in figure 7.1 illustrating the blowups on E_0.

We now state the proposition that will allow us to prove the main theorem.

Proposition VII.2. *Let* $e_1 = dd_1 - \delta_1$ *and* $e_2 = dd_2 - \delta_2 - \delta$. *Assuming that* $e_1, e_2 \leq 0$ *we have that a numerical class* $D = H - xE_1 - yE_2 \in N^1(X/\mathbb{A}^4)_{\mathbb{R}}$ *is nef if and only if,*

$$\delta_1 x + \delta y \leq d_1, 0 \leq y \leq x, \delta x + \delta_2 y \leq d_2$$

and

$$D|_S = h - xC_1 - yC_2 \in \mathrm{Nef}(S).$$

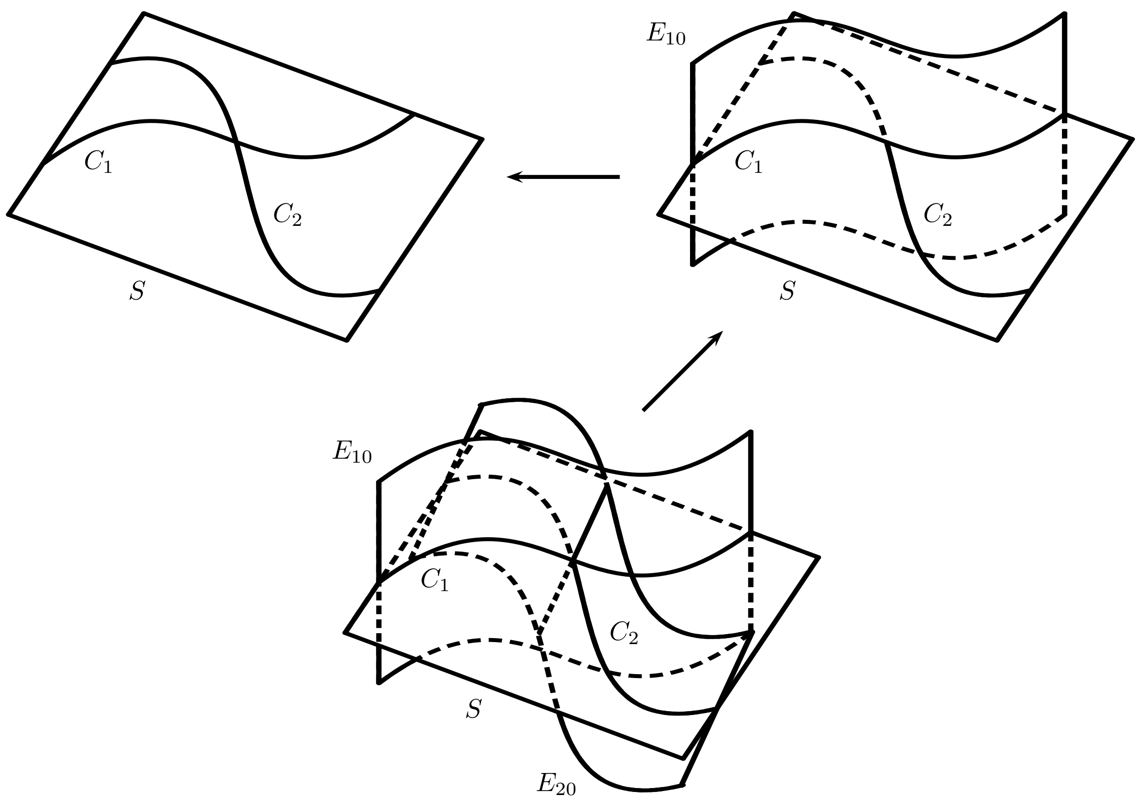


Figure 7.1: Blowup of the curves C_1 and C_2 on E_0

Remark VII.3. From Corollary IV.4, the divisor classes E_0, E_1, E_2 form a basis for the relative Néron-Severi group $N^1(X/\mathbb{A}^4)_{\mathbb{R}}$. Since $H = -E_0 - E_1 - E_2$, the divisor classes $H, -E_1, -E_2$ also define a basis for $N^1(X/\mathbb{A}^4)_{\mathbb{R}}$. We observe that in order for a numerical class $D = tH - xE_1 - yE_2 \in N^1(X/\mathbb{A}^4)_{\mathbb{R}}\backslash\{0\}$ to be π-nef we must have $t > 0$ and $x, y \geq 0$. Indeed, if $f_i \subseteq E_i$ is the fibre over a point in $C_i\backslash C_j$ $(i \neq j)$ then $(H \cdot f_i) = (E_j \cdot f_i) = 0$ and $(E_i \cdot f_i) = -1$ which implies $x, y \leq 0$. It is clear that $t \geq 0$ because for a curve $C \subseteq X$ not meeting $E_1 \cup E_2$ one obtains a positive intersection number $(D \cdot C) = x(H \cdot C) > 0$. Moreover, if $t = 0$ and $x, y \leq 0$ then by considering a curve $C \subseteq X$ meeting both E_1 and E_2 but not contained in $E_1 \cup E_2$ we get $(D \cdot C) < 0$. Consequently, $t > 0$ for any non-trivial π-nef divisor D. As such the relative nef cone $\mathrm{Nef}(X/\mathbb{A}^4)$ is totally described by the section $t = 1$ in the sense that any π-nef divisor is a multiple of one in that section.

Using the notation from [5, Chapter V.§2], the numbers e_1, e_2 denote the e invariant of the ruled surfaces $E_1^{(1)}$ and $E_2^{(2)}$ respectively. The assumption $e_1, e_2 \leq 0$ is used for technical convenience. Geometrically, this theorem says that under these circumstances the cone $\mathrm{Nef}(X/\mathbb{A}^4)$ is obtained by intersecting a polyhedral cone defined by linear conditions with an affine transformation of a part of the cone $\mathrm{Nef}(S)$. In particular, by considering a surface S where $\mathrm{Nef}(S)$ is non-polyhedral we will be able to construct an example where $\mathrm{Nef}(X/\mathbb{A}^4)$ is non-polyhedral.

7.2 $\mathbb{Q}$-twists

For the proof of Proposition VII.2 we will use the formalism of $\mathbb{Q}$-twisted bundles that we now shortly introduce.

Definition VII.4. A $\mathbb{Q}$-twisted bundle on a projective variety X,

$$E{<}\gamma{>}$$

is an ordered pair consisting of a vector bundle E on X and a numerical class $\gamma \in$

$N^1(X)_{\mathbb{Q}}$ defined up to isomorphism. If D is a $\mathbb{Q}$-divisor on X, we write $E{<}D{>}$ for the twist of E with the numerical class of D.

Definition VII.5. The isomorphism relation on $\mathbb{Q}$-twisted vector bundles on X is generated by declaring,

$$E{<}A + D{>} = E \otimes \mathcal{O}_X(A){<}D{>}$$

for any vector bundle E on X, any integral Cartier divisor A on X and $D \in \mathrm{Div}(X)_{\mathbb{Q}}$.

Definition VII.6. Let $E{<}\gamma{>}$ be a $\mathbb{Q}$-twisted vector bundle on X. Let ξ and F be the divisor classes on $\mathbb{P}(E)$ corresponding to the invertible sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ and the pullback of γ by the projection $\mathbb{P}(E) \longrightarrow X$ respectively. Then $E{<}\gamma{>}$ is nef (or ample) if and only if $\xi + F$ in nef (or ample).

Proposition VII.7. *Let L, E, M be vector bundles on X and $\gamma \in \mathrm{Div}(X)_{\mathbb{Q}}$.*

a) If $L \longrightarrow M \longrightarrow 0$ is exact and $L{<}\gamma{>}$ is nef (or ample) then so is $M{<}\gamma{>}$;

b) If $0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0$ is exact and both $L{<}\gamma{>}$ and $M{<}\gamma{>}$ are nef (or ample) then so is $E{<}\gamma{>}$.

Proof. See [11], Lemma 6.2.8 and Theorem 6.2.12. □

Definition VII.8. Let E be a vector bundle and A an ample divisor class on X. We define the Barton invariant of E with respect to A as the real number,

$$\beta(X, E, A) = \sup\{t \in \mathbb{Q} \mid E{<}{-tA}{>} \text{ is nef}\}.$$

Corollary VII.9. *Let L, E, M be vector bundles on a smooth projective curve C. Suppose there is a short exact sequence,*

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0.$$

If A is an ample line bundle on C and $L<-\beta(C,M,A)>$ is nef (or equivalently $\beta(C,L,A) \geq \beta(C,M,A)$) then

$$\beta(C,E,A) = \beta(C,M,A).$$

In particular, if ξ and F are the divisor classes on $\mathbb{P}(E)$ of $\mathcal{O}_{\mathbb{P}(E)}(1)$ and the pullback of A by the projection $\mathbb{P}(E) \longrightarrow X$ respectively, then for any real number s,

$$\xi + sF \text{ is nef if and only if } s \geq -\beta(C,M,A).$$

Proof. We claim that for any $t \in \mathbb{Q}$ the twisted bundle $E<-tA>$ is nef if and only if $M<-tA>$ is also so. Since M is a quotient of E, by Proposition VII.7 a) the nefness of $E<-tA>$ implies that $M<-tA>$ is nef. On the other hand, if $E<-tA>$ is nef then we have,

$$t \leq \beta(C,M,A) \leq \beta(C,L,A).$$

Hence, $L<-tA>$ is nef due to the nef cone of $\mathbb{P}(E)$ being closed. As such, we can apply Proposition VII.7 b) and deduce that $E<-tA>$ is nef as claimed.

Assuming the claim then the equality

$$\beta(C,E,A) \leq \beta(C,M,A)$$

follows immediately from the definition of Barton invariant.

The second assertion of the corollary is a direct consequence of the definition of nefness for $\mathbb{Q}$-twisted bundles, at least when s is a rational number. However, we can extend this result for real values of s using again the fact that nefness is a closed condition. □

7.3 Proof of Proposition VII.2

We now start showing a series of lemmas that will lead to the proof of Proposition VII.2.

Lemma VII.10. *Let $D = H - xE_1 - yE_2$. The following conditions are equivalent:*

a) $D|_{E_{20}}$ *is nef;*

b) $D|_{E_2}$ *is nef;*

c) $\delta x + \delta_2 y \leq d_2$ *and* $0 \leq y$.

Proof. We start by showing $a) \Leftrightarrow c)$. Since $E_{20} = \mathbb{P}(N^*_{C_2/E_0^{(1)}})$, then $\mathrm{Pic}(E_{20}) = \mathbb{Z}\mathcal{O}_{E_{20}}(1) \oplus \mathbb{Z}(\pi_2|_{E_{20}})^*\mathrm{Pic}(C_2)$ where $\mathcal{O}_{E_{20}}(1)$ is the line bundle $\mathcal{O}_{E_{20}}(-E_2)$. Besides, any two fibres of $\pi_2|_{E_{20}}\colon E_{20} \longrightarrow C_2$ are numerically equivalent and therefore the divisor classes of $-E_2$ and a fibre F of $\pi_2|_{E_{20}}$ form a basis of $N^1(E_{20})$.

The conormal exact sequence

$$0 \longrightarrow N^*_{S/E_0^{(1)}}|_{C_2} \longrightarrow N^*_{C_2/E_0^{(1)}} \longrightarrow N^*_{C_2/S} \longrightarrow 0$$

defines a section $C_2^{(1)} \longrightarrow C_2^{(2)} \subseteq E_{20}^{(2)}$ for which

$$\mathcal{O}_{E_{20}}(-C_2) \otimes \mathcal{O}_{E_{20}}(1) = (\pi_2|_{E_{20}})^*(N^*_{S/E_0^{(1)}}|_{C_2})$$

together with the restriction morphism $\mathcal{O}_{E_{20}}(1) \longrightarrow \mathcal{O}_{C_2}(1)$. We now want to apply Corollary VII.9 to this exact sequence and the Barton invariants associated with the divisor class P of a closed point on C_2. Straight computations yield

$$\begin{aligned}
\beta(C_2, N^*_{S/E_0^{(1)}}|_{C_2}, P) &= \beta(C_2, \mathcal{O}_{E_0^{(1)}}(-S)|_{C_2}, P) \\
&= \beta(C_2, \mathcal{O}_{E_0^{(1)}}(E_1 - dH)|_{C_2}, P) \\
&= \beta(C_2, \mathcal{O}_{C_2}(E_1 - dH), P) \\
&= \deg_{C_2} \mathcal{O}_{C_2}(E_1 - dH) \\
&= \delta - dd_2
\end{aligned}$$

and

$$\begin{aligned}\beta(C_2, N^*_{C_2/S}, P) &= \beta(C_2, \mathcal{O}_S(-C_2)|_{C_2}, P)\\ &= \deg_{C_2} \mathcal{O}_S(-C_2)\\ &= -\delta_2.\end{aligned}$$

Since

$$\begin{aligned}\beta(C_2, N^*_{S/E_0^{(1)}}|_{C_2}, P) \geq \beta(C_2, N^*_{C_2/S}, P) &\Leftrightarrow \delta - dd_2 \geq -\delta_2\\ &\Leftrightarrow dd_2 + \delta_2 - \delta \leq 0\\ &\Leftrightarrow e_2 \leq 0,\end{aligned}$$

the corollary tells us that

$$\beta(C_2, N^*_{C_2/E_0^{(1)}}, P) = -\delta_2$$

and

(7.1) $$\mathcal{O}_{E_{20}}(-E_2) + s(\pi_2|_{E_{20}})^* P \text{ is nef if and only if } s \geq \delta_2.$$

For determining when the divisor numerical class $D|_{E_{20}}$ is nef we rewrite it as

$$\begin{aligned}D|_{E_{20}} &= (H - xE_1 - yE_2)|_{E_{20}}\\ &= y\mathcal{O}_{E_{20}}(1) + (d_2 - \delta x)\pi_2|^*_{E_{20}} P.\end{aligned}$$

The coefficient y must be non-negative by virtue of $\mathcal{O}_{E_{20}}(1)$ intersecting positively any fibre of $\pi_2|_{E_{20}}$. If $y = 0$ then $D|_{E_{20}}$ is nef if and only if $d_2 - \delta x \geq 0$. If $y > 0$ then $D|_{E_{20}}$ is nef if and only if $(d_2 - \delta x)/y \geq \delta_2$. Therefore, $D|_{E_{20}}$ is nef if and only if

$$d_2 - \delta x - \delta_2 y \geq 0 \text{ and } y \geq 0$$

establishing $a) \Leftrightarrow c)$.

We now show $b) \Leftrightarrow a)$. We use our knowledge regarding nefness on E_{20} to determine nefness on E_2. This is done using the conormal exact sequence

$$0 \longrightarrow N^*_{E_0/X_1}|_{C_2} \longrightarrow N^*_{C_2/X_1} \longrightarrow N^*_{C_2/E_0^{(1)}} \longrightarrow 0$$

which defines the embedding $E_{20} \longrightarrow E_2 = \mathbb{P}(N^*_{C_2/X_1}) \subseteq X_2$ over $C_2^{(1)}$ so that

$$\mathcal{O}_{E_2}(-E_{20}) \otimes \mathcal{O}_{E_2}(1) = (\pi_2|_{E_2})^*(N^*_{E_0/X_1}|_{C_2})$$

together with the restriction morphism $\mathcal{O}_{E_2}(1) \longrightarrow \mathcal{O}_{E_{20}}(1)$. We can apply Corollary VII.9 to this exact sequence as we did before. This time, the relevant Barton invariants are

$$\begin{aligned}
\beta(C_2, N^*_{E_0/X_1}|_{C_2}, P) &= \beta(C_2, \mathcal{O}_{X_1}(-E_0)|_{C_2}, P) \\
&= \beta(C_2, \mathcal{O}_{X_1}(E_1 - H)|_{C_2}, P) \\
&= \beta(C_2, \mathcal{O}_{C_2}(E_1 - H), P) \\
&= \delta - d_2
\end{aligned}$$

and the already computed

$$\beta(C_2, N^*_{C_2/E_0^{(1)}}, P) = -\delta_2.$$

The inequality

$$\begin{aligned}
\beta(C_2, N^*_{E_0/X_1}|_{C_2}, P) \geq \beta(C_2, N^*_{E_0/X_1}|_{C_2}, P) &\Leftrightarrow \delta - d_2 \geq -\delta_2 \\
&\Leftrightarrow d_2 - \delta - \delta_2 \leq 0
\end{aligned}$$

is valid because $d_2 - \delta - \delta_2 \leq dd_2 - \delta - \delta_2 = e_2 \leq 0$. As a result, the corollary yields the condition

(7.2) $\mathcal{O}_{E_2}(-E_2) + s(\pi_2|_{E_2})^*P$ is nef if and only if $s \geq \delta_2$.

We now claim that the restriction morphism $N^1(E_2) \longrightarrow N^1(E_{20})$ is an isomorphism and a class $\theta \in N^1(E_2)$ is nef if and only if $\theta|_{E_{20}}$ is nef. The isomorphism is easily seen since $(\mathcal{O}_{E_2}(-E_2), (\pi_2|_{E_2})^*P)$ forms a basis of $N^1(E_2)$ being mapped to the basis $(\mathcal{O}_{E_{20}}(-E_2), (\pi_2|_{E_{20}})^*P)$. Restriction of nef classes are certainly nef so we are left with having to show that if $\theta|_{E_{20}}$ is nef then θ is also nef. Conditions 7.1 and 7.2 show that if $\theta = -tE_1 + s\pi^*P$ the claim holds for $t > 0$. If $t = 0$ then it is also clear that θ nef $\Leftrightarrow \theta|_{E_{20}}$ nef $\Leftrightarrow s \geq 0$.

If $t < 0$ then neither θ nor $\theta|_{E_{20}}$ are nef since they both intersect any curve mapped to a point by $E_{20} \longrightarrow C_2$ non-negatively. Therefore the claim holds and consequently *a)* $\Leftrightarrow$ *b)*. □

For analyzing nefness on E_1 we will need first to determine what happens on $E_1^{(1)}$.

Lemma VII.11. *Let $D = H - xE_1$. The following conditions are equivalent:*

a) $D|_{E_{10}^{(1)}}$ *is nef;*

b) $D|_{E_1^{(1)}}$ *is nef;*

c) $\delta_1 x \leq d_1$ *and* $0 \leq x$.

Proof. We start by showing *a)* $\Leftrightarrow$ *c)*. Since $E_{10}^{(1)} = \mathbb{P}(N^*_{C_1/E_0^{(0)}})$, then $\mathrm{Pic}(E_{10}^{(1)}) = \mathbb{Z}\mathcal{O}_{E_{10}^{(1)}}(1) \oplus \mathbb{Z}(\pi_1|_{E_{10}^{(1)}})^*\mathrm{Pic}(C_1)$ where $\mathcal{O}_{E_{10}^{(1)}}(1)$ is the line bundle $\mathcal{O}_{E_{10}^{(1)}}(-E_1)$. Besides, any two fibres of $\pi_1|_{E_{10}^{(1)}} : E_{10}^{(1)} \longrightarrow C_1$ are numerically equivalent and therefore the divisor classes of $-E_1$ and a fibre F of $\pi_1|E_{10}^{(1)}$ form a basis of $N^1(E_{10}^{(1)})$.

The conormal exact sequence

$$0 \longrightarrow N^*_{S/E_0^{(0)}}|_{C_1} \longrightarrow N^*_{C_1/E_0^{(0)}} \longrightarrow N^*_{C_1/S} \longrightarrow 0$$

defines a section $C_1^{(0)} \longrightarrow C_1^{(1)} \subseteq E_{10}^{(1)}$ for which

$$\mathcal{O}_{E_{10}^{(1)}}(-C_1) \otimes \mathcal{O}_{E_{10}^{(1)}}(1) = (\pi_1|_{E_{10}^{(1)}})^*(N^*_{S/E_0^{(0)}}|_{C_1})$$

together with the restriction morphism $\mathcal{O}_{E_{10}^{(1)}}(1) \longrightarrow \mathcal{O}_{C_1}(1)$. We now want to apply Corollary VII.9 to this exact sequence and the Barton invariants associated with the divisor class P of a closed point on C_1. Straight computations yield

$$\begin{aligned}
\beta(C_1, N^*_{S/E_0^{(0)}}|_{C_1}, P) &= \beta(C_1, \mathcal{O}_{E_0^{(0)}}(-S)|_{C_1}, P) \\
&= \beta(C_1, \mathcal{O}_{E_0^{(0)}}(-dH)|_{C_1}, P) \\
&= \beta(C_1, \mathcal{O}_{C_1}(-dH), P) \\
&= \deg_{C_1} \mathcal{O}_{C_1}(-dH) \\
&= -dd_1
\end{aligned}$$

and

$$\begin{aligned}
\beta(C_1, N^*_{C_1/S}, P) &= \beta(C_1, \mathcal{O}_S(-C_1)|_{C_1}, P) \\
&= \deg_{C_1} \mathcal{O}_S(-C_1)|_{C_1} \\
&= -\delta_1.
\end{aligned}$$

Since

$$\begin{aligned}
\beta(C_1, N^*_{S/E_0^{(0)}}|_{C_1}, P) \geq \beta(C_1, N^*_{C_1/S}, P) &\Leftrightarrow dd_1 \geq -\delta_1 \\
&\Leftrightarrow dd_1 - \delta_1 \leq 0 \\
&\Leftrightarrow e_1 \leq 0,
\end{aligned}$$

the corollary tells us that

$$\beta(C_1, N^*_{C_1/E_0^{(0)}}, P) = -\delta_1$$

and

$$\mathcal{O}_{E_{10}^{(1)}}(-E_2) + s(\pi_1|_{E_{10}})^* P \text{ is nef if and only if } s \geq \delta_1. \tag{7.3}$$

For determining when the divisor numerical class $D|_{E_{10}^{(1)}}$ is nef we rewrite it as

$$\begin{aligned} D|_{E_{10}^{(1)}} &= (H - xE_1)|_{E_{10}^{(1)}} \\ &= x\mathcal{O}_{E_{10}^{(1)}}(1) + d_1(\pi_1|_{E_{10}})^*P. \end{aligned}$$

The coefficient x must be non-negative by virtue of $\mathcal{O}_{E_{10}^{(1)}}(1)$ intersecting positively any fibre of $\pi_1|_{E_{10}}$. If $x = 0$ then $D|_{E_{10}^{(1)}}$ is nef because $d_1 \geq 0$. If $x > 0$ then $D|_{E_{10}^{(1)}}$ is nef if and only if $d_1/x \geq \delta_1$. Therefore, $D|_{E_{10}^{(1)}}$ is nef if and only if

$$d_1 - \delta_1 x \geq 0 \text{ and } x \geq 0$$

establishing $a) \Leftrightarrow c)$.

We now show $b) \Leftrightarrow a)$. We use our knowledge regarding nefness on $E_{10}^{(1)}$ to determine nefness on $E_1^{(1)}$. This is done using the conormal exact sequence

$$0 \longrightarrow N^*_{E_0/X_0}|_{C_1} \longrightarrow N^*_{C_1/X_0} \longrightarrow N^*_{C_1/E_0^{(0)}} \longrightarrow 0$$

which defines the embedding $E_{10}^{(1)} \longrightarrow E_1^{(1)} = \mathbb{P}(N^*_{C_1/X_0}) \subseteq X_1$ over $C_1^{(0)}$ so that

$$\mathcal{O}_{E_1^{(1)}}(-E_{10}) \otimes \mathcal{O}_{E_1^{(1)}}(1) = (\pi_1|_{E_1})^*(N^*_{E_0/X_0}|_{C_1})$$

together with the restriction morphism $\mathcal{O}_{E_1^{(1)}}(1) \longrightarrow \mathcal{O}_{E_{10}^{(1)}}(1)$. We can apply Corollary VII.9 to this exact sequence as we did before. This time, the relevant Barton invariants are

$$\begin{aligned} \beta(C_1, N^*_{E_0/X_0}|_{C_1}, P) &= \beta(C_1, \mathcal{O}_{X_0}(-E_0)|_{C_1}, P) \\ &= \beta(C_1, \mathcal{O}_{X_0}(-H)|_{C_1}, P) \\ &= \beta(C_1, \mathcal{O}_{C_1}(-H), P) \\ &= -d_1 \end{aligned}$$

and the already computed

$$\beta(C_1, N^*_{C_1/E_0^{(0)}}, P) = -\delta_1.$$

The inequality

$$\beta(C_1, N^*_{E_0/X_0}|_{C_1}, P) \geq \beta(C_1, N^*_{E_0/X_0}|_{C_1}, P) \;\Leftrightarrow\; -d_1 \geq -\delta_1$$
$$\Leftrightarrow\; d_1 - \delta_1 \leq 0$$

is valid because $d_1 - \delta_1 \leq dd_1 - \delta_1 = e_1 \leq 0$. As a result, the corollary yields the condition

(7.4) $\mathcal{O}_{E_1^{(1)}}(-E_1) + s(\pi_1|_{E_1})^* P$ is nef if and only if $s \geq \delta_1$.

We now claim that the restriction morphism $N^1(E_1^{(1)}) \longrightarrow N^1(E_{10}^{(1)})$ is an isomorphism and a class $\theta \in N^1(E_1^{(1)})$ is nef if and only if $\theta|_{E_{10}^{(1)}}$ is nef. The isomorphism is easily seen since $(\mathcal{O}_{E_1^{(1)}}(-E_1), (\pi_1|_{E_1})^* P)$ forms a basis of $N^1(E_1)$ being mapped to the basis $(\mathcal{O}_{E_{10}^{(1)}}(-E_1), (\pi_1|_{E_{10}})^* P)$. Restriction of nef classes are certainly nef so we are left with having to show that if $\theta|_{E_{10}}$ is nef then θ is also nef. Conditions 7.3 and 7.4 show that if $\theta = -tE_1 + s(\pi_1|_{E_1})^* P$ the claim holds for $t > 0$. If $t = 0$ then it is also clear that θ nef $\Leftrightarrow \theta|_{E_{10}}$ nef $\Leftrightarrow s \geq 0$. If $t < 0$ then neither θ nor $\theta|_{E_{10}^{(1)}}$ are nef since they both intersect any curve mapped to a point by $E_{10}^{(1)} \longrightarrow C_1^{(1)}$ non-negatively. Therefore the claim holds and consequently $a) \Leftrightarrow b)$. □

Lemma VII.12. *Let $D = H - xE_1 - yE_2$. The following conditions are equivalent:*

a) $D|_{E_{10}}$ is nef;

b) $D|_{E_1}$ is nef;

c) $\delta_1 x + \delta y \leq d_1$ and $0 \leq y \leq x$.

Proof. The implication $b) \Leftrightarrow a)$ is trivial since nefness is preserved under restriction.

For showing $a) \Leftrightarrow c)$ we find curves defining the given inequalities. The intersection $C_1^{(1)} \cap C_2^{(1)}$ is non-empty because C_1 and C_2 are ample divisors on S. So, let Q

be a point where $C_1^{(1)}$ and $C_2^{(1)}$ meet the fibre $\pi_1|_{E_{10}}^{-1}(\pi_1(Q))$. Let $f_1^Q \subseteq E_1^{(1)}$ be the strict transform of the fibre $\pi_1^{-1}(\pi_1(Q)) \subseteq E_1^{(1)}$ and let $f_2^Q \subseteq E_1^{(1)}$ be the fibre over Q.

The curves we are interested in are f_1^Q, f_2^Q and C_1. In order to calculate their intersection numbers against D it is convenient to point out that the divisor $H - xE_1$ is the pullback $\pi_2^*(H - xE_1)$ which allows the application of projection formula. It is also useful to compute $(E_1^{(1)} \cdot C_1^{(1)})$ beforehand. So,

$$\begin{aligned}(E_1^{(1)} \cdot C_1^{(1)}) &= \deg_{C_1} \mathcal{O}_{X_1}(E_1)|_{C_1} \\ &= \deg_{C_1} \mathcal{O}_S(E_1)|_{C_1} \\ &= \deg_{C_1} \mathcal{O}_S(C_1)|_{C_1} \\ &= \delta_1.\end{aligned}$$

Taking this into account we obtain the following intersections numbers.

$$\begin{aligned}(D \cdot f_1^Q) &= ((H - xE_1 - yE_2)|_{E_{10}} \cdot f_1^Q) \\ &= ((H - xE_1)|_{E_{10}^{(1)}} \cdot (\pi_2|_{E_{10}})_* f_1^Q) - y(E_2|_{E_{10}} \cdot f_1^Q) \\ &= x - y\end{aligned}$$

$$\begin{aligned}(D \cdot f_2^Q) &= ((H - xE_1 - yE_2)|_{E_{10}} \cdot f_2^Q) \\ &= ((H - xE_1)|_{E_{10}^{(1)}} \cdot (\pi_2|_{E_{10}})_* f_2^Q) - y(E_2|_{E_{10}} \cdot f_1^Q) \\ &= y\end{aligned}$$

$$\begin{aligned}(D \cdot C_1) &= ((H - xE_1 - yE_2)|_{E_{10}} \cdot C_1) \\ &= ((H - xE_1)|_{E_{10}^{(1)}} \cdot C_1^{(1)}) - y(E_2|_{E_{10}} \cdot C_1) \\ &= d_1 - x\delta_1 - y\delta.\end{aligned} \tag{7.5}$$

Since all these must be non-negative we conclude that

$$0 \leq y \leq x \quad \text{and} \quad d_1 - x\delta_1 - y\delta \geq 0$$

which yields $a) \Rightarrow c)$ as wanted.

We are left to show the implication $c) \Rightarrow b)$. We want to show that a divisor $D = (H - xE_1 - yE_2)|_{E_1}$ on E_1 satisfying the conditions of $c)$ is nef. For that purpose we are going to check that the intersection number $(D \cdot C)$ is non-negative for any irreducible curve on E_1. We consider all possible curves splitting them into cases according to their geometric nature.

Case 1. The curve C is contained in E_2.

There is a fibre F of $\pi_2|_{E_1} : E_1^{(2)} \longrightarrow E_1^{(1)}$ containing C that is isomorphic to $\mathbb{P}^2$ and such that $-E_2|_F$ is the divisor class of $\mathcal{O}_F(1)$. So,

$$\begin{aligned}
(D \cdot C) &= ((H - xE_1 - yE_2) \cdot C) \\
&= ((H - xE_1) \cdot (\pi_2|_F)_*(C)) + y(-E_2|_F \cdot C) \\
&= y \deg_C \mathcal{O}_F(1)|_C \\
&\geq 0.
\end{aligned}$$

Case 2. The curve C is C_1.

Then, as we saw in 7.5,

$$(D \cdot C) = d_1 - \delta_1 x - \delta y \geq 0.$$

Case 3. The curve $C \not\subseteq E_2$ is contained in E_{10} and distinct from C_1.

Then,

$$\begin{aligned}
(D \cdot C) &= ((H - xE_1 - yE_2)|_{E_{10}} \cdot C) \\
&= ((H - xE_1)|_{E_{10}} \cdot \pi_2(C)) - y(E_2|_{E_{10}} \cdot C).
\end{aligned} \tag{7.6}$$

The key argument here is to find an upper bound for $(E_2|_{E_{10}} \cdot C)$. Since

$$\pi_2|_{E_{10}} \colon E_{10}^{(2)} \longrightarrow E_{10}^{(1)}$$

is a blowup of the intersection set $C_1^{(1)} \cap C_2^{(1)}$ consisting of δ distinct points and $E_2|_{E_{10}}$ is the exceptional divisor for this map, then $(\pi_2|_{E_{10}})^*(C_1) = C_1 + E_2|_{E_{10}^{(2)}}$ by virtue of $C_1^{(1)}$ meeting $C_2^{(1)}$ transversally. As such, from projection formula we get,

$$\begin{aligned}(C_1 \cdot \pi_2(C))_{E_{10}^{(1)}} &= ((C_1 + E_2|_{E_{10}}) \cdot C) \\ &= (C_1 \cdot C)_{E_{10}} + (E_2|_{E_{10}} \cdot C)\end{aligned}$$

hence,

$$(C_1 \cdot \pi_2(C))_{E_{10}^{(1)}} \geq (E_2|_{E_{10}^{(2)}} \cdot C)$$

because $(C_1 \cdot C)_{E_{10}^{(2)}} \geq 0$. This yields the upper bound we wanted and as a result, from 7.6 we have,

$$(D \cdot C) \geq (((H - xE_1)|_{E_{10}^{(1)}} - yC_1^{(1)}) \cdot \pi_2(C))_{E_{10}{}^{(1)}}.$$

We now claim that the divisor class $(H - xE_1)|_{E_{10}^{(1)}} - yC_1^{(1)}$ on $E_{10}^{(1)}$ is nef which is enough to show $(D \cdot C) \geq 0$. Note that,

$$\begin{aligned}\mathcal{O}_{E_{10}^{(1)}}(C_1) &= \mathcal{O}_{E_0}(S)|_{E_{10}^{(1)}} \\ &= \mathcal{O}_{E_0}(dH - E_1)|_{E_{10}^{(1)}}\end{aligned}$$

and therefore $C_1^{(1)}$ is numerically equivalent to $dH - E_1$ on $E_{10}^{(1)}$. Thus the divisor class $(H - xE_1)|_{E_{10}^{(1)}} - yC_1^{(1)}$ is numerically equivalent to

$$((1 - dy)H - (x - y)E_1)|_{E_{10}^{(1)}}$$

and we just need to show its nefness in order to establish the claim. We point out that $1 - dy > 0$. In fact,

$$\begin{aligned} e_1 < 0 &\Leftrightarrow dd_1 - \delta_1 \leq 0 \\ &\Leftrightarrow dd_1 \leq \delta_1 \\ &\Leftrightarrow d_1/\delta_1 \leq 1/d \end{aligned}$$

and consequently

$$\begin{aligned} \delta_1 x + \delta y \leq d_1 &\Leftrightarrow \delta_1 y + \delta y \leq d_1 \\ &\Leftrightarrow y \leq d_1/(\delta_1 + \delta) \\ &\Rightarrow y < d_1/\delta_1 \\ &\Rightarrow y < 1/d \\ &\Leftrightarrow 1 - dy > 0. \end{aligned}$$

So, by Lemma VII.12, the divisor class $((1-dy)H - (x-y)E_1)|_{E_{10}^{(1)}}$ is nef when

$$\begin{aligned} & 0 \leq (x-y)/(1-dy) \leq d_1/\delta_1 \\ \Leftrightarrow\ & 0 \leq \delta_1(x-y) \leq d_1 - dd_1 y \\ \Leftrightarrow\ & 0 \leq x - y \quad \text{and} \quad \delta_1 x + (dd_1 - \delta_1)y \leq d_1 \\ \Leftarrow\ & y \leq x \quad \text{and} \quad \delta_1 x + \delta y \leq d_1 \qquad (\text{because } dd_1 - \delta_1 = e_1 \leq 0 < \delta). \end{aligned}$$

By hypothesis these inequalities hold and we prove the claim as required.

Case 4. The curve $C \subseteq E_1$ is not contained in $E_{10} \cup E_2$.

Then,

$$\begin{aligned} (D \cdot C) &= ((H - xE_1 - yE_2)|_{E_1} \cdot C) \\ &= ((H - xE_1)|_{E_1} \cdot \pi_2(C)) - y(E_2|_{E_1} \cdot C). \end{aligned} \tag{7.7}$$

The key argument here is to find an upper bound for $(E_2|_{E_1} \cdot C)$. Since $\pi_2|_{E_1} : E_1^{(2)} \longrightarrow E_1^{(1)}$ is a blowup of the intersection set $C_1^{(1)} \cap C_2^{(1)}$ consisting of δ distinct points and

$E_2|_{E_1}$ is the exceptional divisor for this map, then $(\pi_2|_{E_1})^*(E_{10}) = E_{10} + E_2|_{E_1}$ by virtue of $C_2^{(1)}$ meeting $E_{10}^{(1)}$ transversally. As such, from projection formula we get,

$$\begin{aligned}(E_{10} \cdot \pi_2(C))_{E_{10}} &= ((E_{10} + E_2|_{E_1}) \cdot C) \\ &= (E_{10} \cdot C)_{E_1} + (E_2|_{E_1} \cdot C)\end{aligned}$$

hence,

$$(E_{10} \cdot \pi_2(C))_{E_1} \geq (E_2|_{E_1} \cdot C)$$

because $(E_1 \cdot C)_{E_1} \geq 0$. This yields the upper bound we wanted and as a result, from 7.7 we have,

$$(D \cdot C) \geq (((H - xE_1)|_{E_1^{(1)}} - yE_{10}^{(1)}) \cdot \pi_2(C))_{E_1{}^{(1)}}.$$

We now claim that the divisor class $(H - xE_1)|_{E_1^{(1)}} - yE_{10}^{(1)}$ on $E_1^{(1)}$ is nef which is enough to show $(D \cdot C) \geq 0$. Note that,

$$\begin{aligned}\mathcal{O}_{E_1^{(1)}}(E_{10}) &= \mathcal{O}_{X_1}(E_0)|_{E_1} \\ &= \mathcal{O}_{X_1}(-H - E_1)|_{E_1}\end{aligned}$$

and therefore $E_{10}^{(1)}$ is numerically equivalent to $-H - E_1$ on $E_1^{(1)}$. Thus the divisor class $(H - xE_1)|_{E_1^{(1)}} - yE_{10}^{(1)}$ is numerically equivalent to

$$((1 + y)H - (x - y)E_{10})|_{E_1^{(1)}}$$

and we just need to show its nefness in order to establish the claim. By Lemma VII.12, this divisor class is nef when

$$\begin{aligned}& 0 \leq (x - y)/(1 + y) \leq d_1/\delta_1 \\ \Leftrightarrow \quad & 0 \leq \delta_1(x - y) \leq d_1(1 + y) \\ \Leftrightarrow \quad & 0 \leq x - y \quad \text{and} \quad \delta_1 x - (d_1 + \delta_1)y \leq d_1 \\ \Leftarrow \quad & y \leq x \quad \text{and} \quad \delta_1 x + \delta y \leq d_1.\end{aligned}$$

By hypothesis these inequalities hold and we prove the claim as required.

□

Lemma VII.13. *Let $D = H - xE_1 - yE_2$. The following conditions are equivalent:*

a) $D|_{E_0}$ is nef;

b) $\delta_1 x + \delta y \leq d_1, 0 \leq y \leq x, \delta x + \delta_2 y \leq d_2$ and $D|_S = h - xC_1 - yC_2 \in \mathrm{Nef}(S)$.

Proof. Since the restriction of a nef divisor is nef, if $D|_{E_0}$ is nef then $D|_{E_1}$, $D|_{E_0}$ and $D|_S$ must be nef. Using Lemmas VII.10 and VII.12 we conclude that all the conditions in b) are necessary.

On the other hand suppose b) is satisfied and let C be an irreducible curve in E_0. Then, if $C \subseteq S \cup E_1 \cup E_2$ then $(D \cdot C) \geq 0$ by Lemmas VII.10 and VII.12. If $C \not\subseteq S \cup E_1 \cup E_2$, then note that $x, y \leq 1/d$. If $C \not\subseteq S \bigcup E_1 \bigcup E_2$, then note that $x, y \leq d_1/\delta_1$ and by assumption $dd_1 - \delta_1 \leq 0$ means $d_1/\delta_1 \leq 1/d$. Hence, $x, y \leq 1/d$. Moreover, $S|_{E_0}$ is linearly equivalent to $(dH - E_1 - E_2)|_{E_0}$ and therefore

$$H - (1/d)E_1 - (1/d)E_2 \equiv (1/d)S.$$

So, $D \equiv (1/d)S + (1/d - x)E_1 + (1/d - y)E_2$ is an effective class on E_0 and then $(D.C) \geq 0$. As a result $D|_{E_0}$ is nef. □

Proof of Proposition VII.2. The divisor D is relatively nef if and only if $D|_{X_2}$ is nef. Since $X_2 = E_0 \cup E_1 \cup E_2$ then by Lemmas VII.10, VII.12 and VII.13 we conclude that $D|_{X_2}$ is nef if and only if all the conditions stated in the theorem are satisfied. □

7.4 Proof of main theorem

Using the construction described in this chapter we are now ready to create an example where the relative nef cone obtained is non-polyhedral. Having Proposition VII.2, it is a matter of picking a smooth projective surface with a couple of smooth ample curves meeting transversally so that the respective relative nef cone has some part defined ba a non-linear condition arising from restriction of divisors to that surface.

Proof of Theorem VII.1. We will use a surface whose existence is guaranteed by a theorem of Morrison.

Theorem VII.14. *[14, Theorem 2.9] For $\rho \leq 11$, every lattice of signature $(1, \rho-1)$ occurs as the Picard group of a smooth projective $K3$ surface.*

By Theorem VII.14, let S be a smooth projective $K3$ surface with intersection form

$$q = 4a^2 - 4b^2 - 4c^2.$$

Cutkosky studied in [2] the properties of divisor on this surface and his results are summarized in the following theorem.

Theorem VII.15. *Let $D = (a, b, c) \in \mathbb{Z}^3 \cong \mathrm{Pic}(S)$ be an ample line bundle and let $h = (1, 0, 0)$ be a divisor on S such that $h = (1, 0, 0)$. Then,*

a) $|h|$ embeds S as a quartic surface on $\mathbb{P}^3$;

b) $|D|$ is base point free;

c) There exists a smooth curve C on S such that $C \sim D$;

d) The nef cone of S is

$$\mathrm{Nef}(S) = \{(a, b, c) \in \mathbb{R}^3 \mid q(a, b, c) \geq 0, a \geq 0\}.$$

Let $D_1, D_2 \in \mathrm{Pic}(S)$ such that

$$D_1 = (5, 1, 0) \qquad \text{and} \qquad D_2 = (2, 0, 1).$$

First note that D_1 is very ample because we can write it as a sum

$$D_1 = h + (4, 1, 0)$$

where h is very ample and $(4, 1, 0)$ is globally generated. Indeed, since $(4, 1, 0)$ is in the interior of $\text{Nef}(S)$ then it is ample and globally generated by Theorem VII.15 *b*). Also, D_2 is ample and by Theorem VII.15 *c*) it can be represented by a smooth curve C_2. Since D_1 is very ample we can pick a smooth curve $C_1 \sim D_1$ meeting C_2 transversally.

We can now apply Proposition VII.2 considering the quartic surface $S \subseteq \mathbb{P}^3$ together with the curves C_1 and C_2. For that purpose we just need to check that $e_1, e_2 \leq 0$, which is the case once we observe the computed parameters for the construction,

$$\begin{array}{lll} d = \deg S = 4, & d_1 = \deg C_1 = 20, & d_2 = \deg C_2 = 18, \\ \delta = (C_1 \cdot C_2) = 36, & \delta_1 = (C_1 \cdot C_1) = 96, & \delta_2 = (C_2 \cdot C_2) = 12, \\ e_1 = dd_1 - \delta_1 = -16, & e_2 = dd_2 - \delta_2 - \delta = -16. & \end{array}$$

Therefore $D = H - xE_1 - E_2$ is nef if and only if

$$\delta_1 x + \delta y \leq d_1, 0 \leq y \leq x, \delta x + \delta_2 y \leq d_2$$

and

$$D|_S = h - xC_1 - yC_2 \in \text{Nef}(S).$$

The first 3 conditions are easily simplified to,

$$24x + 5y \leq 5, 0 \leq y \leq x, 9x + 3y \leq 2.$$

On the other hand,

$$D|_S = (1,0,0) - x(5,1,0) - y(2,0,1)$$
$$= (1 - 5x - 2y, -x, -y)$$

and

$$D|_S \in \text{Nef}(S) \Leftrightarrow (1 - 5x - 2y)^2 - x^2 - y^2 \geq 0 \quad \text{and} \quad 1 - 5x - 2y \geq 0$$
$$\Leftrightarrow 1 - 5x - 2y \geq \sqrt{x^2 + y^2}$$

describes a region of the plane bounded by a non-degenerate conic. Putting together all conditions we obtain 5 curves on the plane making up the respective boundaries as in Figure 7.2.

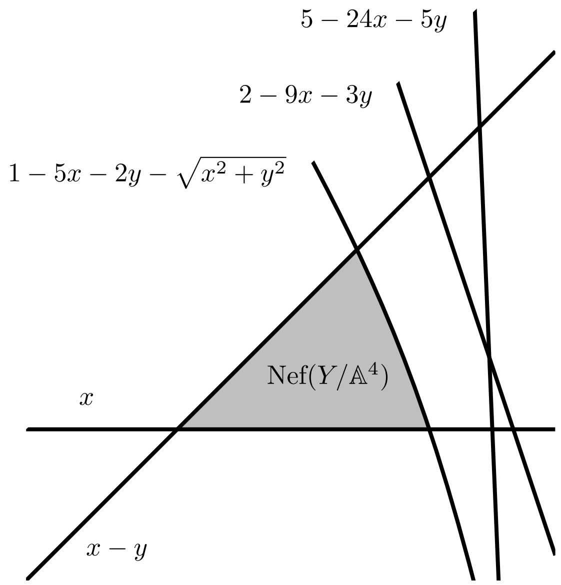


Figure 7.2: Section of $\text{Nef}(Y/\mathbb{A}^4)$ and the 5 conditions

In fact, the relative nef cone is defined by the inequalities,

$$0 \leq y \leq x, 1 - 5x - 2y \geq \sqrt{x^2 + y^2}.$$

As such $\text{Nef}(Y/\mathbb{A}^4)$ is non-polyhedral. $\square$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Frédéric Campana and Thomas Peternell. Algebraicity of the ample cone of projective varieties. *J. Reine Angew. Math.*, 407:160–166, 1990.

[2] S. Dale Cutkosky. Irrational asymptotic behaviour of Castelnuovo-Mumford regularity. *J. Reine Angew. Math.*, 522:93–103, 2000.

[3] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[4] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[5] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1977.

[6] Jean-Pierre Jouanolou. *Théorèmes de Bertini et applications*, volume 42 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1983.

[7] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

[8] Steven L. Kleiman. Toward a numerical theory of ampleness. *Annals of Mathematics. Second Series*, 84:293–344, 1966.

[9] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 1996.

[10] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[11] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[12] Joseph Lipman and Kei-ichi Watanabe. Integrally closed ideals in two-dimensional regular local rings are multiplier ideals. *Math. Res. Lett.*, 10(4):423–434, 2003.

[13] Shigefumi Mori. Threefolds whose canonical bundles are not numerically effective. In *Algebraic threefolds (Varenna, 1981)*, volume 947 of *Lecture Notes in Math.*, pages 155–189. Springer, Berlin, 1982.

[14] D. R. Morrison. On $K3$ surfaces with large Picard number. *Invent. Math.*, 75(1):105–121, 1984.

[15] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999.