Derivations on Metric Measure Spaces

by

Jasun Gong

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Doctoral Committee:

Professor Mario Bonk, Chair
Professor Alexander I. Barvinok
Professor Juha Heinonen (Deceased)
Associate Professor James P. Tappenden
Assistant Professor Pekka J. Pankka

"Or se' tu quel Virgilio e quella fonte
che spandi di parlar si largo fiume?"
rispuos'io lui con vergognosa fronte.
"O de li altri poeti onore e lume,
vagliami 'l lungo studio e 'l grande amore
che m'ha fatto cercar lo tuo volume.
Tu se' lo mio maestro e 'l mio autore,
tu se' solo colui da cu' io tolsi
lo bello stilo che m'ha fatto onore."

[*"And are you then that Virgil, you the fountain
that freely pours so rich a stream of speech?"
I answered him with shame upon my brow.
"O light and honor of all other poets,
may my long study and the intense love
that made me search your volume serve me now.
You are my master and my author, you–
the only one from whom my writing drew
the noble style for which I have been honored."*]

from the Divine Comedy by Dante Alighieri,
as translated by Allen Mandelbaum [Man82].

In memory of Juha Heinonen,
my advisor, teacher, and friend.

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CHAPTER I

Introduction and Main Results

One of the central themes in metric-space analysis is to understand how the choices of metric and measure on a space determine the geometry of the space. When these choices are made in a compatible way, many familiar facts and constructions from analysis extend naturally from the setting of Euclidean spaces to that of general *metric measure spaces*, that is, metric spaces equipped with a measure.

Among many formulations of first-order calculus on such spaces, the following work focuses mainly on objects called *derivations*. The subsequent results extend Weaver's theory of metric derivations and exterior differentiation [Wea00].

1.1 Lipschitz Functions and Rademacher's Theorem.

To begin, let (X, ρ) be a metric space. Recall that a function $f : X \to \mathbb{R}$ is said to be Lipschitz if the quotients $|f(y) - f(x)|/\rho(x, y)$ are uniformly bounded over all pairs (x, y) in $X \times X$, where $x \neq y$. In the case of $\mathbb{R}^n$ with the standard metric, the classical theorem of Rademacher [Rad19] states that every Lipschitz function is almost everywhere (a.e.) differentiable with respect to Lebesgue n-measure.

However, the Lipschitz condition is a purely metric condition. Suppose that we have a metric space which supports some form of differential calculus. It is then natural to inquire whether Lipschitz functions on that space exhibit similar differentiability properties as their counterparts on Euclidean spaces.

For our purposes, this work will only address those cases in which the target is $\mathbb{R}^n$ and the source is a metric measure space. There is a substantial body of literature in which these roles are reversed. Indeed, Kirchheim has proven that given any metric space X, every Lipschitz map $f : \mathbb{R}^n \to X$ is "metrically differentiable" at Lebesgue a.e. point [Kir94, Thm 2]. We will not pursue this direction here, but for further reading on this subject, see [Amb90], [AK00b], [DCP95], [Kir94], and [KS93].

For measure spaces (X, μ), Weaver has developed a theory of first-order calculus in terms of objects called *metric derivations* [Wea00], [Wea99]. To explain the terminology, he first defines a type of distance between μ-measurable subsets of X, called a *measurable metric*. In the case when X admits a metric in the usual sense, the measurable metric incorporates information from both the measure and the metric.

Returning to the general case, Weaver formulates a Lipschitz-type property for functions in terms of measurable metrics. This framework then leads us to consider metric derivations: these consist of a subspace of linear operators from the space of "measurably Lipschitz" functions, as given above, to the space of μ-essentially bounded functions on X, which we denote by $L^\infty(X, \mu)$.

Metric derivations generalize the usual differential operators on $\mathbb{R}^n$. For instance, they satisfy the Leibniz rule for products of Lipschitz functions. They also form a vector space over $\mathbb{R}$, as well as a module over the ring $L^\infty(X, \mu)$. Put another way, Rademacher's theorem is encapsulated in the structure of a metric measure space if there exists a nonzero metric derivation on that space.

The framework of [Wea00] is very general and it admits a large class of examples. These include classical spaces such as Riemannian manifolds, fractal-type spaces such as the Sierpinski carpet and the Laakso spaces [Laa00], and infinite-dimensional spaces such as Banach manifolds and abstract Wiener spaces. To motivate the absence of a metric, Weaver also examines *Dirichlet spaces* on which the structure is determined by a σ-finite measure and by bilinear operators called (L^∞)-*diffusion forms*. In this setting one recovers a measurable metric of the above type [Wea00, Thm 54] but it is unclear whether this reduces to a pointwise metric.

1.2 Motivations and Main Results.

Our work on metric measure spaces follows Weaver, but our perspective takes a more geometric direction. Here we develop a notion of derivation which is similar to [Wea00], but where the choice of metric (in the usual sense) plays a more important role. Like those of Weaver, these derivations form a module over $L^\infty(X, \mu)$. We are then motivated by the following question, which is unsolved even in the case of $\mathbb{R}^n$ with the standard metric. *Given a metric space, what measures on the space induce a nontrivial supply of derivations?*

As a means of measuring the non-triviality of these modules, we use various notions from linear algebra. Our goal will be to detect linearly independent sets of derivations on a metric measure space and to determine the *rank* of the module of derivations. Another goal will be to detect *generating sets* of such modules.

We first answer this question for two Euclidean spaces: the line and the plane. For $k = 1, 2$, we classify all measures μ on $\mathbb{R}^k$ whose associated modules of derivations contain linearly independent sets of cardinality k. Specifically, these are measures which are absolutely continuous to Lebesgue k-measure, and the proof uses new results about the structure of Lebesgue null sets due to Alberti, Csörnyei, and Preiss [ACP05]. For Lebesgue singular measures μ, we also construct a derivation which generates the full module of derivations on $(\mathbb{R}^2, \mu)$. It remains unclear, however, what conditions on μ will ensure a nonzero (generating) derivation.

In this direction, Stefan Wenger has made the following observation [unpublished] which relates our theory of derivations to the theory of currents. On a complete, separable metric space, each 1-dimensional current (in the sense of Ambrosio and Kirchheim [AK00a, Defn 3.1]) induces a derivation, and the underlying measure is the mass of the current [AK00a, Defn 2.6]. Conversely, it follows easily from the definitions that if δ is a derivation on (X, μ) and if B is a ball in X, then the map

$$(f, \pi) \ \mapsto \ \int_B f \cdot \delta\pi \, d\mu$$

determines an (Ambrosio-Kirchheim) 1-dimensional current. From this correspondence, the problem of classifying Lebesgue singular measures that induce nontrivial derivations on $\mathbb{R}^n$ is equivalent to the so-called "Flat Chain Conjecture" about one-dimensional currents on $\mathbb{R}^n$ [AK00a, Sect 11]. We will not discuss currents here, but for further reading, see [AK00a], [Lan], and the forthcoming article [HdP].

For $\alpha \in [0, 2]$, we also consider the class of measures in $\mathbb{R}^n$ which are concentrated on subsets of σ-finite α-dimensional Hausdorff measure. The structure of their modules of derivations is similar to the cases of $\mathbb{R}$ and $\mathbb{R}^2$. To obtain these results, we use the notion of a pushforward derivation, as well as additional facts from geometric measure theory. Among these is the Besicovitch-Federer Projection Theorem [Mat95, Thm 18.1]. Roughly speaking, it reduces such subsets to two cases: sets which exhibit good properties under orthogonal projection onto hyperplanes, and unions of C^1-smooth submanifolds.

Lastly, we turn to a more general setting. In the spirit of Rademacher's theorem, Cheeger has proven a differentiability theorem [Che99, Thm 4.38] for real-valued Lipschitz functions on a large class of metric measure spaces. Such spaces are characterized by two properties: a growth condition on the measure, called the *doubling* *property*, and the validity of a *weak* $(1, p)$-*Poincaré inequality*, which generalizes the classical Poincaré inequality on $\mathbb{R}^n$. For concreteness, examples of such spaces (and references) include Carnot-Carathéodory spaces [Gro96], [Mon02], the spaces of Laakso [Laa00], complete manifolds with non-negative Ricci curvature [Bus82], spaces of strong A_∞-geometry [DS90], and boundaries (at infinity) of certain hyperbolic buildings [BP99].

Cheeger and Weaver have jointly shown that these spaces admit nontrivial modules of metric derivations [Wea00, Thm 43]. Proceeding in this direction, we prove an analogue of their result for our derivations. We also show a special case of a conjecture of Cheeger [Che99, Thm 4.63] which concerns the non-degeneracy of certain Lipschitz images of such spaces. The proofs of these facts use techniques from Sobolev spaces on metric spaces, which we discuss in Chapter 6.

We note that Keith has generalized Cheeger's theorem to a larger class of spaces [Kei04, Thm 2.3.1]. Furthermore, his techniques [Kei04, Sect 2.4] can be adapted to prove the same case of Cheeger's conjecture. The argument uses a new fact about sets of non-differentiability of Lipschitz functions on $\mathbb{R}^2$; see [ACP05, Thm 12].

1.3 Plan of the Thesis.

This paper is organized into eight chapters. The remainder of Chapter I consists of notation and terminology which are used throughout this work. We also recall briefly some familiar notions from measure theory and from functional analysis.

In Chapter II we review basic properties of Lipschitz functions on general metric spaces. These include a normed linear structure on the space of bounded Lipschitz functions and the corresponding weak-$*$ topology. We also review several approximation theorems for Lipschitz functions on Euclidean spaces, which are given in terms of smooth functions and polynomials.

In Chapter III we begin our discussion of derivations on metric measure spaces and their geometric properties. These operators exhibit similar properties as vector fields on smooth manifolds, which include a locality property and the push-forward construction. If the module of derivations is finitely generated, then it also admits a type of generalized vector bundle structure. In the case of $\mathbb{R}^n$, we show additional properties of derivations, such as a variant of the Chain Rule, which are similar to those of Euclidean partial differential operators.

In Chapter IV, we characterize measures on $\mathbb{R}$ which admit nontrivial derivations. More generally, for a measure which is supported on a 'one-dimensional' subset of $\mathbb{R}^n$, its module of derivations is generated by at most one element. To prove the latter fact, we use the concept of rectifiability from geometric measure theory.

In Chapter V, we prove two main facts about modules of derivations with respect to Lebesgue singular measures on $\mathbb{R}^2$: (1) the rank of the module is at most one, and (2) if the module is nonzero, then it is generated by a single derivation. The proofs use several facts due to Alberti, Csörnyei, and Preiss [ACP05] about the geometry of Lebesgue null sets in the plane. As an application of our techniques, we study measures μ on $\mathbb{R}^n$ which are concentrated on 'two-dimensional' sets. This also uses tools from rectifiability.

In Chapter VI, we introduce p-PI spaces. These are the metric measure spaces for which Cheeger's differentiability theorem holds. We show that such spaces also admit a nontrivial module of derivations. The proof will require Cheeger's theorem [Che99, Thm 4.38] as well as techniques from Sobolev spaces on metric spaces. As previously discussed, we also prove a special case of his conjecture.

Each of the remaining two chapters is an appendix of facts. In Chapter VII we compare Lipschitz continuity with Weaver's (measurable) Lipschitz property for functions, and we also show that a class of his metric derivations induce our derivations. Chapter VIII is a collection of assorted facts in functional analysis which cannot be easily found in the literature; for completeness, we give their proofs.

1.4 Notation and Conventions.

The standard basis of unit vectors on $\mathbb{R}^n$ is denoted by $\{\vec{e}_i\}_{i=1}^n$ and for $1 \le i \le n$, the ith Euclidean coordinate function is denoted by x_i.

If S is a set and if f and g are real-valued functions on S, then we denote their pointwise maximum and minimum as $f \vee g$ and $f \wedge g$, respectively.

As before, a triple (X, ρ, μ) is a *metric measure space* if (X, ρ) is a metric space and μ is a measure on X. Here and in the sequel, $\mathcal{H}_X^k$ denotes k-dimensional Hausdorff measure on a metric space X. When the metric is understood, $\dim_{\mathcal{H}}(E)$ denotes the Hausdorff dimension of a subset E in X, which is defined as

$$\dim_{\mathcal{H}}(E) \ := \ \inf\{\alpha \in [0,\infty) \ : \ \mathcal{H}^\alpha(E) = 0\}.$$

If $X = \mathbb{R}^n$, then we write $\mathcal{H}^k = \mathcal{H}_X^k$, and m_n denotes Lebesgue n-measure on $\mathbb{R}^n$.

Let (X, μ) be a measure space. A collection of subsets $\{X_i\}_{i=1}^\infty$ is a *measurable decomposition* of X if $\mu(X \setminus \bigcup_{i=1}^\infty X_i) = 0$ and if $\mu(X_i \cap X_j) = 0$, whenever $i \neq j$.

If Z is a μ-measurable subset of X, then we denote by $\mu\lfloor Z$ the *restriction (measure)* of μ onto Z, which we define as

$$\big(\mu\lfloor Z\big)(W) \ := \ \mu(Z \cap W).$$

If $\mu\lfloor Z = \mu$, then we say that μ is *concentrated* on Z. On a metric space (X, ρ) with a Borel measure μ, the latter notion differs from the *support* of μ, which is defined to be the smallest closed set on which μ is concentrated [Mat95, Defn 1.12], i.e.

$$\begin{aligned}\operatorname{spt}(\mu) &:= X \setminus \{x : \mu(B(x,r)) = 0 \text{ holds for some } r > 0\} \\ &= \{x \in X : \mu(B(x,r)) > 0 \text{ holds for all } r > 0\}.\end{aligned}$$

Recall that a measure μ on X is *absolutely continuous* to another measure ν on X, denoted $\mu \ll \nu$, if every ν-null set is also a μ-null set. Two measures μ and ν on X are *(mutually) singular*, denoted $\mu \perp \nu$, if there exists $A \subset X$ so that μ is concentrated on A and ν is concentrated on $X \setminus A$. By the Lebesgue Decomposition Theorem [Fol99, Thm 3.8], for all σ-finite Radon measures μ and ν on X, there exist σ-finite Radon measures μ_{AC} and μ_S so that the following conditions hold:

$$\mu = \mu_{AC} + \mu_S, \quad \mu_{AC} \ll \nu, \quad \mu_S \perp \nu. \tag{1.4.1}$$

When $X = \mathbb{R}^n$ and $\nu = m_n$, then we call μ_{AC} the Lebesgue absolutely continuous part of μ and μ_S the Lebesgue singular part of μ.

For $p \in [1,\infty)$, $L^p(X,\mu)$ denotes the Banach space of p-integrable, real-valued functions on X with respect to μ, and $L^\infty(X,\mu)$ denotes the space of μ-essentially bounded, real-valued functions. The usual norms on these spaces are defined as

$$\|u\|_{\mu,p} := \begin{cases} \left[\int_X |u|^p \, d\mu\right]^{1/p}, & p \in [1,\infty) \\ \inf\left\{\lambda \geq 0 : \mu\big(\{x : |u(x)| > \lambda\}\big) = 0\right\}, & p = \infty. \end{cases} \tag{1.4.2}$$

If the measure is understood, for $p \in [1,\infty)$ we write $\|u\|_p$ as a shorthand for $\|u\|_{\mu,p}$. We always write $\|u\|_\infty$ for the supremum norm of a function u, whenever it exists.

Given a Banach space V, we denote its dual Banach space by V^*. We write $v_n \rightharpoonup v$ if $\{v_i\}_{i\in I}$ is a net in V which converges *weakly* to v, which means that $\langle v^*, v_n\rangle \to \langle v^*, v\rangle$ holds for all $v^* \in V^*$. For a dual Banach space W with pre-dual V, we write $w_i \overset{*}{\rightharpoonup} w$ if $\{w_i\}_{i\in I}$ is a weak-$*$ convergent sequence in W with weak-$*$ limit w, or equivalently, if $\langle w_i, v\rangle \to \langle w, v\rangle$ holds for all $v \in V$.

CHAPTER II

Preliminaries: Lipschitz Functions

We begin by reviewing basic properties of Lipschitz functions on a metric space. One crucial fact is that the space of bounded Lipschitz functions is a dual Banach space and therefore admits a weak-$*$ topology. In what follows, we will examine convergence of bounded Lipschitz functions in this topology, as well as properties of the pre-dual space. We will also recall several well-known approximation theorems for Lipschitz functions on $\mathbb{R}^n$.

2.1 Lipschitz Functions and Weak-$*$ Topologies.

Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. A map $f : X \to Y$ is *Lipschitz* if

$$L(f) \ := \ \sup\left\{\frac{\rho_Y(f(x), f(y))}{\rho_X(x, y)} \ : \ x, y \in X \text{ and } x \neq y\right\}$$

is finite. If $L(f) \leq C$ holds for some $C \geq 0$, then we say that f is *C-Lipschitz*. If a map $g : X \to Y$ is Lipschitz on every bounded subset of X, then we say that g is *locally Lipschitz*.

We denote the space of Lipschitz maps from X to Y by $\mathrm{Lip}(X, Y)$ and the space of locally Lipschitz maps from X to Y by $\mathrm{Lip}_{loc}(X; Y)$. Several elementary facts are stated below without proof. In particular, Part (2) follows easily from the definition of $\mathcal{H}^\alpha$, for $\alpha \geq 0$; see [Mat95, Thm 7.5].

Lemma 2.1.1. *Let (X, ρ_X), (Y, ρ_Y), and (Z, ρ_Z) be metric spaces, and let $C_1, C_2 \geq 0$.*

1. *If $f : X \to Y$ is C_1-Lipschitz and if $g : Y \to Z$ is C_2-Lipschitz, then the map $g \circ f : X \to Z$ is C_1C_2-Lipschitz.*

2. *Let $f \in \mathrm{Lip}(X;Y)$ and let $\alpha \geq 0$. Then for all $\mathcal{H}^\alpha$-measurable subsets A in X,*

$$\mathcal{H}^\alpha(f(A)) \ \leq \ L(f)^\alpha \cdot \mathcal{H}^\alpha(A).$$

If $Y = \mathbb{R}$, then we write $\mathrm{Lip}(X) := \mathrm{Lip}(X;\mathbb{R})$ and $\mathrm{Lip}_{loc}(X) := \mathrm{Lip}_{loc}(X;\mathbb{R})$ for short. The space of bounded, real-valued Lipschitz functions is denoted by $\mathrm{Lip}_\infty(X)$. These spaces enjoy many additional properties, which we state below.

Lemma 2.1.2. *Let (X, ρ) and (Y, ρ') be metric spaces, and let $C \geq 0$.*

1. *$\mathrm{Lip}(X)$ is a vector space over $\mathbb{R}$, and $\mathrm{Lip}_\infty(X)$ is an algebra over $\mathbb{R}$.*

2. *If f and g are C-Lipschitz functions on X, then so are $f \vee g$ and $f \wedge g$.*

3. *If $\{f_i\}_{i \in I}$ is a family of C-Lipschitz functions on X, then the function*

$$x \mapsto \inf_i f_i(x)$$

   is C-Lipschitz provided that it is finite at one point of X.

4. *Let $A \subset X$. If $f \in \mathrm{Lip}(A)$, then there exists $F \in \mathrm{Lip}(X)$ so that $F|A = f$ and $L(F) = L(f)$. In addition, if f is a bounded function, then we may choose F to be a bounded function that satisfies $\|f\|_\infty = \|F\|_\infty$.*

Parts (1) to (3) of Lemma 2.1.2 are standard facts. Assuming these, we prove Part (4), which we will call the *McShane-Whitney extension.*[1]

[1]Our terminology may be non-standard. The usual notion of McShane-Whitney extension does not preserve boundedness of the function.

Proof of Lemma 2.1.2, Part (4). The lemma is clearly true for the zero function, so assume that $f \neq 0$. Put $L := L(f)$. For each point $a \in A$, consider the function

$$f_a(x) \ := \ f(a) + L \cdot \rho(a, x).$$

Clearly, the family $\{f_a\}_{a \in A}$ is uniformly L-Lipschitz. Observe also that by construction, we have $f(x) \leq f_a(x)$ for all $a, x \in X$, and that equality holds if and only if $x = a$. Now consider the function

$$\tilde{F}(x) \ := \ \inf\{f_a(x) : a \in A\}.$$

By Part (3), $\tilde{F}$ is also L-Lipschitz, and this proves the first assertion of (4).

To see that $\tilde{F}$ extends f, note that $f(x) \leq \tilde{F}(x)$ holds for all $x \in X$. This follows from the previous observation and by taking an infimum over all $a \in A$. In particular we have $f(a) \leq \tilde{F}(a)$, for all $a \in A$. However, $\tilde{F}$ is an infimum, so we always have $\tilde{F}(x) \leq f_a(x)$, for all $x \in X$. In the case $x = a$, we obtain $\tilde{F}(a) \leq f_a(a) = f(a)$, therefore $\tilde{F}(a) = f(a)$ holds for all $a \in A$.

Towards the second assertion, if f is not bounded, then put $F := \tilde{F}$. Otherwise, assume that $f \in \mathrm{Lip}_\infty(A)$ and consider the bounded function

$$F(x) \ := \ \big(\tilde{F}(x) \vee (-\|f\|_\infty)\big) \wedge \|f\|_\infty.$$

Clearly we have $\|F\|_\infty = \|f\|_\infty$. By Part (2), F is also L-Lipschitz. From the identity

$$f(x) \ = \ \big(f(x) \vee (-\|f\|_\infty)\big) \wedge \|f\|_\infty,$$

we see that $F|A = f$ follows from $\tilde{F}|A = f$. □

In addition to a linear structure, $\mathrm{Lip}_\infty(X)$ is a Banach space under the norm

$$\|f\|_{\mathrm{Lip}} \ := \ \max(\|f\|_\infty, L(f)). \tag{2.1.1}$$

However, more is true. The following fact is due to Weaver [Wea99].

Lemma 2.1.3 (Weaver, 1996). *If (X, ρ) is a metric space, then $\mathrm{Lip}_\infty(X)$ is a dual Banach space with respect to the norm in (2.1.1). In addition, on bounded sets of $\mathrm{Lip}_\infty(X)$, its weak-$*$ topology agrees with the topology of pointwise convergence.*

The next corollary clarifies some details of weak-$*$ convergence from Theorem 2.1.3. It gives simpler criteria for detecting weak-$*$ convergent sequences in $\mathrm{Lip}_\infty(X)$.

Corollary 2.1.4. *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, and let f and $\{f_n\}_{n=1}^\infty$ be functions in $\mathrm{Lip}_\infty(X)$.*

1. *The sequence $\{f_n\}_{n=1}^\infty$ converges weak-$*$ to f in $\mathrm{Lip}_\infty(X)$ if and only if f_n converges pointwise to f and $\sup_n \|f_n\|_{\mathrm{Lip}} < \infty$.*
2. *If f_n converges uniformly to f and $\sup_n L(f_n) < \infty$, then $f_n \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(X)$.*
3. *Let $\pi \in \mathrm{Lip}(X, Y)$. If $f_n \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(Y)$, then $f_n \circ \pi \overset{*}{\rightharpoonup} f \circ \pi$ in $\mathrm{Lip}_\infty(X)$.*

To prove Part (1) of Corollary 2.1.4, we will use a fact from functional analysis [Yos95, Thm V.1.9(ii) & V.1.10].

Theorem 2.1.5. *Let $(V, \|\cdot\|)$ be a dual Banach space and suppose that a sequence $\{v_i\}_{i=1}^\infty$ converges weak-$*$ to $v \in V$. Then $\{\|v_i\|\}_{i=1}^\infty$ is uniformly bounded and satisfies*

$$\|v\| \ \leq \ \liminf_{i \to \infty} \|v_i\|.$$

Proof of Corollary 2.1.4. (1) If $f_n \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(X)$, then by Theorem 2.1.5, the sequence $\{f_n\}_{n=1}^\infty$ is a norm-bounded set in $\mathrm{Lip}_\infty(X)$. By Lemma 2.1.3, f_n converges pointwise to f. On the other hand, if $\sup_n \|f_n\|_{\mathrm{Lip}} \leq C$ holds for some $C \geq 0$, then by Lemma 2.1.3, the pointwise convergence $f_n \to f$ implies weak-$*$ convergence.

(2) If f_n converges uniformly to f, then there is a $N \in \mathbb{N}$ so that $\|f_n - f\|_\infty < 1$ holds whenever $n > N$. A straightforward estimate then gives

$$\|f_n\|_\infty \ \leq \ \|f_n - f\|_\infty + \|f\|_\infty \ \leq \ 1 \vee \Big(\max_{1 \leq n \leq N} \|f_n - f\|_\infty \Big) + \|f\|_\infty \ < \ \infty.$$

From the previous estimate and the hypothesis $\sup_n L(f_n) < \infty$, it follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is a bounded subset of $\mathrm{Lip}_\infty(X)$. By Lemma 2.1.3, the pointwise convergence $f_n \to f$ implies weak-$*$ convergence in $\mathrm{Lip}_\infty(X)$.

(3) By hypothesis, we have $f_n \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(Y)$, so there is a $K > 0$ so that $\|f_n\|_{\mathrm{Lip}} < K$ holds for all n. From this we obtain the estimates

$$\begin{aligned}\sup_n \|f_n \circ \pi\|_\infty &\leq \sup_n \|f_n\|_\infty < K, \\ L(f_n \circ \pi) &\leq L(f_n) \cdot L(\pi) \leq K \cdot L(\pi),\end{aligned}$$

and these imply that the sequence $\{f_n \circ \pi\}_{n=1}^{\infty}$ is bounded in $\mathrm{Lip}_\infty(X)$. Clearly, $f_n \circ \pi$ converges pointwise to $f \circ \pi$, so by invoking Theorem 2.1.3 once more, we have $f_n \circ \pi \overset{*}{\rightharpoonup} f \circ \pi$ in $\mathrm{Lip}_\infty(X)$. □

With the same proof, a stronger form of Part (3) of Corollary 2.1.4 holds for a class of homeomorphisms of metric spaces. We define them below.

Definition 2.1.6. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. We say that a homeomorphism $\varphi : X \to X'$ is *bi-Lipschitz* if both φ and φ^{-1} are Lipschitz mappings. Similarly, an embedding $\psi : X \to Y$ is a *bi-Lipschitz embedding* if it is a bi-Lipschitz homeomorphism onto its image.

Corollary 2.1.7. *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, let $\varphi : X \to Y$ be a bi-Lipschitz homeomorphism, and let f and $\{f_n\}_{n=1}^{\infty}$ be functions in $\mathrm{Lip}_\infty(Y)$. Then $f_n \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(Y)$ if and only if $f_n \circ \varphi \overset{*}{\rightharpoonup} f \circ \varphi$ in $\mathrm{Lip}_\infty(X)$.*

For further reading about Lipschitz maps on metric spaces and their properties, see [Hei01, Chap 6], [LV77], [Mat95, Chap 7], and [Wea99]. In the case of real-valued Lipschitz functions, see [EG92, Chap 3] or [Hei05, Sect 3].

2.2 The Arens-Eells Space.

If (X,ρ) is a bounded metric space, then $\mathrm{Lip}(X) = \mathrm{Lip}_\infty(X)$, and by Lemma 2.1.3, $\mathrm{Lip}(X)$ is a dual Banach space. In this case, we also obtain an explicit pre-dual for $\mathrm{Lip}_\infty(X)$. In fact, more is true. Recall first that a metric space is *separable* if it contains a countable dense subset.

Lemma 2.2.1. *If (X,ρ) is a bounded, separable metric space, then the pre-dual of $\mathrm{Lip}_\infty(X)$ is a separable Banach space.*

Later we will use Lemma 2.2.1 in order to invoke facts from functional analysis about dual Banach spaces of this type. Towards the proof, we first give a description of the pre-dual. The following discussion is from [Wea99, Sect 1.1 & 2.2].

Let (X,ρ) be a bounded metric space and without loss of generality, assume that $\mathrm{diam}(X) = 1$. Let X^+ be the set of all points of X as well as one additional point, which we call e. The metric ρ on X extends to a metric ρ^+ on X^+ by the formula

$$\rho^+(x,y) \ := \begin{cases} \rho(x,y), & x,y\in X \\ 1, & x\in X, y=e. \end{cases}$$

As a Banach space, $\mathrm{Lip}_\infty(X)$ is isometrically isomorphic to the space

$$\mathrm{Lip}_0(X^+) \ := \ \{f\in \mathrm{Lip}(X^+) \ : \ f(e)=0\}$$

[Wea00, Thm 1.7.2]. Indeed, $\mathrm{Lip}_0(X^+)$ is a Banach space, and its norm is given by the Lipschitz constant $\|f\|_{\mathrm{Lip}_0} := L(f)$ [Wea00, Thm 1.6.2b].

For any point $x\in X^+$, let δ_x denote the Dirac measure supported on x. If x and y are distinct points in X^+, put $m_{xy} := \delta_x - \delta_y$. The space of signed measures

$$\widetilde{AE}(X^+) \ := \ \operatorname*{span}_{\mathbb{R}}\big\{m_{xy} \ : \ x,y\in X^+, \ x\neq y\big\}.$$

admits a norm [Wea99, Cor 2.2.3], which is given by the formula

$$\|m\|_{AE} \;:=\; \inf\Big\{\sum_{i=1}^{n}|a_i|\cdot\rho(x_i,y_i)\;:\;m\;=\;\sum_{i=1}^{n}a_i\,m_{x_iy_i}\Big\}. \tag{2.2.1}$$

The *Arens-Eells space* $AE(X^+)$ of X^+ is defined as the norm-completion of $\widetilde{AE}(X^+)$ with respect to $\|\cdot\|_{AE}$. By [Wea99, Thm 2.2.2], we have $\big[AE(X^+)\big]^* \cong \mathrm{Lip}_0(X^+)$. Intuitively, the additional point e in X^+ leads to a decomposition of measures

$$m_{xy} \;=\; m_{xe} + m_{ey},$$

for all $x, y \in X$. This leads to the following correspondence: for $\phi \in [AE(X^+)]^*$, we obtain Lipschitz functions f_ϕ by the rule $f_\phi(x) := \phi(m_{xe})$.

Combining our previous conclusions, we observe that $AE(X^+)$ is the pre-dual of $\mathrm{Lip}_\infty(X)$. With this additional information, we now prove Lemma 2.2.1.

Proof of Lemma 2.2.1. Let (X,ρ) be a bounded, separable metric space, let Y be a countable, dense subset of X, and let $\epsilon > 0$ be given. By definition, for all $x, x' \in X$, there exist $y, y' \in Y$ so that $\rho(x,y) < \epsilon/2$ and $\rho(x',y') < \epsilon/2$. We then compute

$$\begin{aligned} m_{xx'} - m_{yy'} &\;=\; (\delta_x - \delta_{x'}) - (\delta_y - \delta_{y'}) \;=\; m_{xy} - m_{x'y'}, \\ \|m_{xx'} - m_{yy'}\|_{AE} &\;=\; \|m_{xy} - m_{x'y'}\|_{AE} \;\le\; \rho(x,y) + \rho(x',y') \;<\; \epsilon. \end{aligned}$$

More generally, if $m = \sum_{i=1}^{n} a_i\, m_{x_i x_i'}$, then for each $i = 1, 2, \ldots, n$, we may choose pairs $y_i, y_i' \in Y$ so that $\rho(x_i,y_i) < \epsilon/b_i$ and $\rho(x_i',y_i') < \epsilon/b_i$, where $b_i := 2n(|a_i| \vee 1)$. Putting $m' := \sum_{i=1}^{n} a_i\, m_{y_i y_i'}$, a similar computation then gives

$$\begin{aligned} \|m - m'\|_{AE} &\;\le\; \sum_{i=1}^{n} \|m_{x_iy_i} - m_{x_i'y_i'}\|_{AE} \\ &\;\le\; \sum_{i=1}^{n} |a_i|\cdot\big(\rho(x_i, y_i + \rho(x_i', y_i')\big) \;<\; n\cdot\Big(\frac{\epsilon}{2n} + \frac{\epsilon}{2n}\Big) \;=\; \epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows that $\widetilde{AE}(X^+)$ contains a countable, dense subset. Recall that the norm completion $AE(X^+)$ is formed as a set of Cauchy sequences of $\widetilde{AE}(X^+)$, so $AE(X^+)$ also contains a countable, dense subset. This proves the lemma. □

2.3 Lipschitz Functions on $\mathbb{R}^n$.

On Euclidean spaces, Lipschitz functions (with respect to the standard metric) enjoy many additional properties. By the Weierstrass approximation theorem [CH53, II.4.2] a continuous, real-valued function on $\mathbb{R}^n$ can be approximated locally uniformly by polynomials. Moreover, by well-known properties of convolution (with respect to smooth mollifiers), continuous functions can also be approximated locally uniformly by smooth functions [EG92, Thm 4.2.1.1].

In a similar spirit, a Lipschitz function on $\mathbb{R}^n$ can be approximated locally by such functions in the weak-$*$ topology of Lemma 2.1.3.

Lemma 2.3.1. *Let $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$.*

1. *For $L = L(f)$, there is a sequence of smooth, bounded L-Lipschitz functions $\{h_j\}_{j=1}^\infty$ so that $h_j \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(\mathbb{R}^n)$.*
2. *There is a sequence of polynomials $\{P_m\}_{m=1}^\infty$ on $\mathbb{R}^n$ so that for every compact subset K, we have $P_m \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(K)$.*

To prove Part (2), we will require the following classical fact [CH53, Thm II.4.3].

Theorem 2.3.2. *Let K be a compact subset of $\mathbb{R}^n$ and let $\varphi \in C^\infty(\mathbb{R}^n)$. Then there is a sequence of polynomials $P_m : \mathbb{R}^n \to \mathbb{R}$ so that on K, the functions P_m converge uniformly to φ and the gradients ∇P_m converge uniformly to $\nabla\varphi$.*

Proof of Lemma 2.3.1. To prove Part (1), consider the convolutions $f_\epsilon := f * \eta_\epsilon$, where η_ϵ is a smooth symmetric mollifier on $\mathbb{R}^n$. In particular, f_ϵ is C^∞-smooth, η_ϵ

has mass 1, and $\operatorname{spt}(\eta_\epsilon) = \bar{B}(0,\epsilon)$. From these properties we obtain

$$\begin{aligned} |f_\epsilon(x) - f_\epsilon(y)| &\leq \int_{\mathbb{R}^n} |f(x-z) - f(y-z)| \cdot \eta_\epsilon(z)\, dz \\ &\leq L(f) \cdot \int_{\mathbb{R}^n} |x-y| \cdot \eta_\epsilon(z)\, dz \\ &= L(f) \cdot |x-y| \end{aligned}$$

whenever x and y are points in $\mathbb{R}^n$. Taking suprema, we obtain $L(f_\epsilon) \leq L(f)$ for each $\epsilon > 0$. A similar estimate also shows that f_ϵ converges uniformly to f:

$$\begin{aligned} |f_\epsilon(x) - f(x)| &\leq \int_{\mathbb{R}^n} |f(z) - f(x)| \cdot \eta_\epsilon(x-z)\, dz \\ &\leq L(f) \cdot \int_{\mathbb{R}^n} |x-z| \cdot \eta_\epsilon(x-z)\, dz \\ &= L(f) \cdot \epsilon \cdot \int_{\mathbb{R}^n} \eta_\epsilon(x-z)\, dz \;=\; L(f) \cdot \epsilon. \end{aligned}$$

Now let $\{\epsilon_j\}_{j=1}^\infty$ be any sequence of positive numbers decreasing to zero, and for each $j \in \mathbb{N}$, put $h_j := f_{\epsilon_j}$. By Part (2) of Corollary 2.1.4, we have $h_j \overset{*}{\rightharpoonup} f$ in $\operatorname{Lip}_\infty(\mathbb{R}^n)$.

The previous estimate also shows that $\{\|h_j\|_\infty\}_{j=1}^\infty$ is uniformly bounded, because

$$|h_j(x)| \;\leq\; |f(x)| + |h_j(x) - f(x)| \;\leq\; \|f\|_\infty + L(f) \cdot \epsilon_j,$$

holds for all $x \in \mathbb{R}^n$. This proves Part (1).

It remains to show Part (2). By Part (1), there are smooth, bounded Lipschitz functions $\{h_m\}_{m=1}^\infty$ so that $h_m \overset{*}{\rightharpoonup} f$ in $\operatorname{Lip}_\infty(\mathbb{R}^n)$. For each $m \in \mathbb{N}$, consider the closed n-cube $Q_m = [-m,m]^n$. By Theorem 2.3.2, there is a polynomial $P_m : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions:

$$\big\|(h_m - P_m)|Q_m\big\|_\infty \;<\; \frac{1}{m}, \tag{2.3.1}$$

$$\big\|(\nabla h_m - \nabla P_m)|Q_m\big\|_\infty \;\leq\; \frac{1}{m}. \tag{2.3.2}$$

For a fixed $m_0 \in \mathbb{N}$, note that P_m converges pointwise to f on Q_{m_0}, as $m \to \infty$. To see this, note that h_m converges uniformly to f, so by inequality (2.3.1),

$$\left\|(f - P_m)|Q_{m_0}\right\|_\infty \;\leq\; \|f - h_m\|_\infty \;+\; \left\|(h_m - P_m)|Q_{m_0}\right\|_\infty \;<\; \epsilon + \epsilon \;=\; 2\epsilon$$

holds for sufficiently large indices $m \in \mathbb{N}$.

So if K is an arbitrary compact subset of $\mathbb{R}^n$, then $K \subset Q_{m_0}$ holds for some $m_0 \in \mathbb{N}$. As a result, from inequalities (2.3.1) and (2.3.2) it follows that $\{P_m\}_{m=1}^\infty$ is a norm-bounded sequence in $\mathrm{Lip}_\infty(K)$. Therefore by Theorem 2.1.3 we have $P_m \xrightarrow{*} f$ in $\mathrm{Lip}_\infty(K)$, and this proves the lemma. $\square$

CHAPTER III

Derivations: Basic Properties

In this chapter we introduce the fundamental object of this paper. Following Weaver [Wea00], a *derivation* on a metric measure space (X, ρ, μ) is a type of generalized differential operator which acts linearly on $\mathrm{Lip}_\infty(X)$. However, there is a good geometric interpretation of derivations as *measurable vector fields*. We will see that derivations have good locality and push-forward properties. Moreover, the space of derivations forms a module. It also leads to a type of vector bundle structure on X.

Here we assume that μ is a Radon measure on X. In other words, μ is a Borel regular measure and bounded subsets of X have finite μ-measure.

3.1 First Notions and a Few Examples.

Let (X, ρ, μ) be a metric measure space. By Lemma 2.1.3, $\mathrm{Lip}_\infty(X)$ is a dual Banach space. In addition, $L^\infty(X, \mu)$ is a dual Banach space under the norm

$$\|u\|_{\mu,\infty} := \inf\Big\{\lambda \geq 0 : \mu\big(\{x : |u(x)| > \lambda\}\big) = 0\Big\},$$

and its pre-dual is $L^1(X, \mu)$.

The following definition is adapted from [Wea00, Defn 21]. Strictly speaking, Weaver's notions of Lipschitz function and metric derivation are different from ours. In Chapter VII we provide a clarification between the definitions here and in [Wea00].

Definition 3.1.1. A *derivation* $\delta : \mathrm{Lip}_\infty(X) \to L^\infty(X,\mu)$ is a linear map which satisfies the following conditions:

1. Weak-$*$ continuity on bounded sets: if $\{f_i\}_{i\in I}$ is a norm-bounded net in $\mathrm{Lip}_\infty(X)$ that converges weak-$*$ to f in $\mathrm{Lip}_\infty(X)$, then the net $\{\delta f_i\}_{i\in I}$ converges weak-$*$ to δf in $L^\infty(X,\mu)$.

2. The Leibniz rule: for all $f, g \in \mathrm{Lip}_\infty(X)$, we have $\delta(f\cdot g) = f\cdot\delta g + g\cdot\delta f$.

The space of derivations on (X,ρ,μ) is denoted by $\Upsilon(X,\mu)$.

Remark 3.1.2. Let δ be a derivation on (X,μ). By the Leibniz rule, we obtain

$$\delta(1) \;=\; 1\cdot\delta(1) + 1\cdot\delta(1) \;=\; 2\cdot\delta(1),$$

so $\delta(1) = 0$. By linearity, $\delta c = 0$ holds whenever c is a constant function on X.

Recall that for dual Banach spaces V and W, a linear map $T : V \to W$ is *weak-$*$ continuous* if it maps weak-$*$ convergent nets in V to weak-$*$ convergent nets in W. It follows that the condition of weak-$*$ continuity is stronger than that of weak-$*$ continuity on bounded sets.

When the context is clear, we refer to the bounded weak-$*$ continuity property of derivations simply as the *continuity property (of derivations)*. In Chapter VIII we recall the definition of a net. In some cases, however, the continuity of a derivation reduces to checking sequences instead of nets. As a convenient terminology, we say that a linear map $T : V \to W$ is *sequentially weak-$*$ continuous* if it maps weak-$*$ convergent sequences in V to weak-$*$ convergent sequences in W.

Lemma 3.1.3. *Let (X,ρ) be a bounded, separable metric space, let μ be a Radon measure on X, and let $T : \mathrm{Lip}_\infty(X) \to L^\infty(X,\mu)$ be a linear map. Then T is weak-$*$ continuous on bounded sets if and only if it is sequentially weak-$*$ continuous.*

Remark 3.1.4. By Theorem 2.1.5, a weak-$*$ convergent sequence must be norm-bounded, so one direction of Lemma 3.1.3 is clear. If a map is weak-$*$ continuous on bounded sets, then by definition it must preserve weak-$*$ convergent sequences.

The other direction of the lemma follows from a more general fact from functional analysis. We postpone its proof to Section 8.2.

We next observe that every $\delta \in \Upsilon(X, \mu)$ is also continuous as a bounded linear operator between the Banach spaces $\mathrm{Lip}_\infty(X)$ and $L^\infty(X, \mu)$. This fact also holds more generally. We state it below.

Lemma 3.1.5. *Let W be a Banach space and let V be a separable Banach space. If $T : V^* \to W^*$ is a sequentially weak-$*$ continuous, linear map, then T is norm-bounded: that is, there is a $C \geq 0$ so that for all $v \in V$, we have*

$$\|Tv\|_{W^*} \;\leq\; C \cdot \|v\|_{V^*}.$$

Here $\|\cdot\|_{W^*}$ and $\|\cdot\|_{V^*}$ denote the norms of W^* and V^*, respectively.

The proof of Lemma 3.1.5 will use the following fact about compactness in the weak-$*$ topology [Rud91, Thm 3.17].

Theorem 3.1.6 (Banach-Alaoglu). *Let V be a separable Banach space. If $\{v_n^*\}_{n=1}^\infty$ is a bounded sequence in V^*, then it contains a weak-$*$ convergent subsequence $\{v_{n_k}^*\}_{k=1}^\infty$.*

Proof of Lemma 3.1.5. We argue by contradiction, so suppose that for each $n \in \mathbb{N}$, there is a $v_n \in V$ so that $\|v\|_{V^*} \leq 1$ and $\|Tv_n\|_{W^*} > n$. By Theorem 2.1.5 the sequence $\{v_n\}_{n=1}^\infty$ is norm-bounded, so by Theorem 3.1.6 there is a subsequence $\{v_{n_j}\}_{j=1}^\infty$ which converges weak-$*$ to v in V.

Since T is weak-$*$ continuous on bounded sets, it follows from Lemma 3.1.3 that $\{Tv_{n_j}\}_{j=1}^\infty$ is a weak-$*$ convergent sequence in W. By Theorem 2.1.5, it is also a bounded set. On the other hand, by construction we have $\|Tv_{n_j}\|_W > n_j$ for all $j \in \mathbb{N}$. This is a contradiction, so the lemma follows. $\square$

Combining Lemmas 3.1.3 and 3.1.5, we see that the same conclusion follows from more general assumptions.

Corollary 3.1.7. *Let (X, ρ) be a bounded, separable metric space, let μ be a Radon measure on X, and let $T : \mathrm{Lip}_\infty(X) \to L^\infty(X, \mu)$ be a linear map that is weak-$*$ continuous on bounded sets. Then T is norm-bounded.*

In view of Corollary 3.1.7, every derivation $\delta \in \Upsilon(X, \mu)$ is a bounded linear operator and therefore has a well-defined operator norm. We denote it by

$$\|\delta\| \;:=\; \sup\big\{\|\delta f\|_{\mu,\infty} \,:\, \|f\|_{\mathrm{Lip}} \le 1\big\}.$$

Observe also that $\Upsilon(X, \mu)$ has the structure of a *module* over the ring $L^\infty(X, \mu)$. Indeed, the scalar action of a function $\lambda \in L^\infty(X, \mu)$ on $\delta \in \Upsilon(X, \mu)$ is determined by the action of $\lambda \cdot \delta$ on functions $f \in \mathrm{Lip}(X)$. This is then determined by the rule

$$(\lambda \cdot \delta) f(x) \;:=\; \lambda(x) \cdot \delta f(x).$$

Following [Wea00], the dual module to $\Upsilon(X, \mu)$ over $L^\infty(X, \mu)$ is denoted $\Omega(X, \mu)$, and its elements are called *measurable* 1*-forms*. Similarly to differential forms on a smooth manifold, one defines the *exterior differential* $d : \mathrm{Lip}_\infty(X) \to \Omega(X, \mu)$ by duality. Given $f \in \mathrm{Lip}_\infty(X)$, the measurable 1-form df is characterized by the action

$$\langle \delta, df \rangle \;=\; \delta f. \tag{3.1.1}$$

Below we list several examples of derivations on various spaces. For their proofs, see [Wea00, Sect 5B], [Wea00, Thm 37], and [Wea00, Cor 35], respectively.

Example 3.1.8. The module $\Upsilon(\mathbb{R}^n, m_n)$ is generated by the Euclidean partial differential operators $\{\partial_{x_i}\}_{i=1}^n$. In addition, we have the module isomorphism

$$\Upsilon(\mathbb{R}^n, m_n) \;\cong\; \bigoplus_{i=1}^{n} L^\infty(\mathbb{R}^n, m_n).$$

In fact, a more general statement is true. See Corollary 3.5.4.

Example 3.1.9. Let M be a Riemannian manifold and let v be the volume element. Then $\Upsilon(M, v)$ is isomorphic to the $L^\infty(M, v)$-module of bounded measurable sections of TM, the tangent bundle of M.

Example 3.1.10. If μ is any measure on $\mathbb{R}$ which is concentrated on the 'middle-thirds' Cantor set, then $\Upsilon(\mathbb{R}, \mu)$ is the zero module.

3.2 The Locality Property.

On a smooth manifold M, vector fields are *local* objects. In other words, their action on a function $f \in C^\infty(M)$ near a point $x \in M$ depends only on the behavior of f near x. The next lemma shows that derivations have a similar property. This becomes a convenient technical tool in later sections, because often we will use it to reduce to the case of sets of finite measure.

Theorem 3.2.1. *Let A be a μ-measurable subset of X. Then*

$$\Upsilon(A, \mu) \;\cong\; \{\chi_A \delta \,:\, \delta \in \Upsilon(X, \mu)\}. \tag{3.2.1}$$

Remark 3.2.2 (Notation). (1) We follow the conventions of [Wea00]. When the context is clear, for $\delta \in \Upsilon(X, \mu)$ we will write $\chi_A \delta \in \Upsilon(A, \mu)$.

(2) The left-hand side of equation (3.2.1) denotes the module of derivations on the metric measure space $(A, \rho, \mu\lfloor A)$. We will consistently write μ for $\mu\lfloor A$.

Theorem 3.2.1 is known as the *locality property* for derivations. It is a variant of [Wea00, Thm 29] and the proof is similar. For completeness we give a sketch of the argument, and to do this we use three additional facts. The first is an elementary characterization of weak-$*$ convergence of nets (Lemma 8.1.5) and the second is the Banach-Alaoglu Theorem (Theorem 3.1.6). The third is an auxiliary fact due to Weaver [Wea99, Lem 7.2.3] which we state below; see also [Wea00, Lem 27].

Lemma 3.2.3 (Weaver, 1996). *Let $\delta \in \Upsilon(X,\mu)$ and let $A \subset X$ be μ-measurable. If $f, g \in \mathrm{Lip}_\infty(X)$ satisfy $f = g$ μ-a.e. on A, then $\delta f = \delta g$ holds μ-a.e. on A.*

The proofs of Lemma 3.2.3 in [Wea99] and [Wea00] hold in the general setting of *W^*-domain algebras*, which we will not discuss here. For a direct proof in the setting $\mathrm{Lip}_\infty(X)$, see [Hei07, Lem 13.4].

Proof of Theorem 3.2.1. We first show the inclusion ($\supset$) in equation (3.2.1), so let $\delta \in \Upsilon(X,\mu)$ be arbitrary. From $\chi_A\delta$ we define a map $\delta^* : \mathrm{Lip}_\infty(A) \to L^\infty(A,\mu)$ in the following way. Given $f \in \mathrm{Lip}_\infty(A)$, let $F \in \mathrm{Lip}_\infty(X)$ be its McShane-Whitney extension (as in Part (4) of Lemma 2.1.2) and then put

$$\delta^* f(x) \;=\; \begin{cases} (\chi_A\delta)F(x), & x \in A \\ 0, & x \in X \setminus A. \end{cases}$$

By Lemma 3.2.3, $\delta^* f$ is independent of the choice of extension of f.

We now show that $\delta_A \in \Upsilon(A,\mu)$. Clearly, δ_A is linear and satisfies the Leibniz rule. It remains to show that δ^* is continuous, so suppose that $\{f_i\}_{i\in I}$ is a norm-bounded net in $\mathrm{Lip}_\infty(X)$ which converges weak-$*$ to f. To verify that $\delta^* f_i \overset{*}{\rightharpoonup} \delta^* f$ in $L^\infty(X,\mu)$, we invoke Lemma 8.1.5 from Chapter VIII. It then suffices to show that every sub-net of $\{\delta^* f_i\}_{i\in I}$ has a further sub-net which is weak-$*$ convergent and with the same weak-$*$ limit $\delta^* f$.

Since any sub-net $\{f_{\varphi(j)}\}_{j\in J}$ is also bounded, the corresponding net of extensions G_j of $f_{\varphi(j)}$ is also bounded in $\mathrm{Lip}_\infty(X)$. By Theorem 3.1.6 there is a further sub-net $\{G_{\psi(k)}\}_{k\in K}$ and a function $G \in \mathrm{Lip}_\infty(X)$ so that $G_{\psi(k)} \xrightarrow{*} G$. This implies that $G_{\psi(k)}$ converges pointwise to G, so by Part (3.2.3) of Lemma 3.2.3, we have $G|A = f$.

Since $\delta_A f$ is independent of the extension of f, we obtain $\delta^* f = \chi_A \cdot \delta G$. From the continuity of δ, we also obtain $\delta G_{b_c} \xrightarrow{*} \delta G$ in $L^\infty(X,\mu)$, and hence

$$\delta^* f_{\varphi(\psi(k))} \;=\; \chi_A \cdot \delta G_{\psi(k)} \xrightarrow{*} \chi_A \cdot \delta G \;=\; \delta^* f$$

in $L^\infty(A,\mu)$. By the previous reduction, this gives the inclusion ($\supset$).

For the other set inclusion ($\subset$), let $\delta \in \Upsilon(A,\mu)$ be arbitrary, and put

$$\delta|_A F(x) \;:=\; \begin{cases} \delta(F|A)(x), & x \in A \\ 0, & x \in X \setminus A. \end{cases} \tag{3.2.2}$$

The map $\delta|_A$ is well-defined because $F|A \in \mathrm{Lip}_\infty(A)$ whenever $F \in \mathrm{Lip}_\infty(X)$. By similar arguments as before, we obtain $\delta|_A \in \Upsilon(X,\mu)$.

We claim that the map $\delta \mapsto \delta|_A$ is an isomorphism and that $\delta \mapsto \delta^*$ is its inverse. To see this, let $g \in \mathrm{Lip}_\infty(A)$, $f \in \mathrm{Lip}_\infty(X)$, and $x \in A$ be given, and let G_A be the McShane-Whitney extension of g. For $\eta \in \Upsilon(A,\mu)$ and $\delta \in \Upsilon(X,\mu)$, we compute

$$(\eta|A)^* g(x) \;=\; (\eta|A) G_A(x) \;=\; \eta(G_A|A)(x) \;=\; \eta g(x), \tag{3.2.3}$$

$$\big((\delta^*)|A\big) f(x) \;=\; \delta^*(f|A)(x) \;=\; (\chi_A \delta) F_A(x) \;=\; \chi_A(x) \cdot \delta f(x). \tag{3.2.4}$$

This proves the lemma. □

From Theorem 3.2.1 we obtain several additional facts. The first fact states that a subdivision of X into μ-measurable subsets induces a splitting of $\Upsilon(X,\mu)$ into submodules with respect to these subsets.

Lemma 3.2.4. *Let μ be a Radon measure on X, and let $\{X_i\}_{i=1}^N$ be a μ-measurable decomposition of X. Then we have the $L^\infty(X,\mu)$-module isomorphism*

$$\Upsilon(X,\mu) \;\cong\; \bigoplus_{i=1}^N \Upsilon(X_i,\mu).$$

Remark 3.2.5. Recall that $\Upsilon(X_i,\mu)$ refers to derivations on X_i with respect to the *restriction* measure $\mu_i := \mu\lfloor X_i$ and not the measure μ.

However, it remains true that $\Upsilon(X_i,\mu)$ is a module over $L^\infty(X,\mu)$. To see this, note that $\mu_i \ll \mu$ holds for each $i \in \mathbb{N}$, and hence $L^\infty(X,\mu)$ is a linear subspace of $L^\infty(X_i,\mu_i)$. It follows that $\lambda \cdot \delta f \in L^\infty(X_i,\mu_i)$ holds whenever $\delta \in \Upsilon(X,\mu_i)$, $f \in \mathrm{Lip}_\infty(X_i)$, and $\lambda \in L^\infty(X,\mu)$.

Proof. Put $M := \bigoplus_{i=1}^N \Upsilon(X_i,\mu)$. We claim that the map $T : \Upsilon(X,\mu) \to M$ given by

$$T(\delta) \;:=\; (\chi_{X_1}\delta,\ldots,\chi_{X_N}\delta)$$

is an isomorphism. For $\delta_i \in \Upsilon(X_i,\mu)$, let $\delta_i|_{X_i}$ denote its extension to $\Upsilon(X,\mu)$ as given in formula (3.2.2). We further claim that the map $S : M \to \Upsilon(X,\mu)$ given by

$$S(\delta_1,\ldots,\delta_N) \;:=\; \sum_{i=1}^N \delta_i|_{X_i}$$

is the inverse of T. Clearly, both S and T are homomorphisms, so it suffices to show that $S \circ T = \mathrm{id}_{\Upsilon(X,\mu)}$ and that $T \circ S = \mathrm{id}_M$.

Let $\delta \in \Upsilon(X,\mu)$. From equation (3.2.4) in the proof of Theorem 3.2.1, we obtain $(\chi_{X_i}\delta)|_{X_i} = \delta$ for all $1 \le i \le N$. It follows that

$$(S\circ T)(\delta) \;=\; S(\chi_{X_1}\delta,\ldots,\chi_{X_N}\delta) \;=\; \sum_{i=1}^N (\chi_{X_i}\delta)|_{X_i} \;=\; \Big(\sum_{i=1}^N \chi_{X_i}\Big)\delta \;=\; \delta.$$

From the definitions, $\chi_{X_i}(\delta|_{X_j})$ is zero whenever $i \neq j$. So for $(\delta_1,\ldots,\delta_N) \in M$, it follows from equation (3.2.3) that

$$\begin{aligned}(T\circ S)(\delta_1,\ldots,\delta_N) &= T\big(\sum_{i=1}^{N}\delta_i|_{X_i}\big)\\ &= \Big(\chi_{X_1}\big(\sum_{i=1}^{N}\delta_i|_{X_i}\big),\ldots,\chi_{X_N}\big(\sum_{i=1}^{N}\delta_i|_{X_i}\big)\Big) \;=\; (\delta_1,\ldots,\delta_n).\end{aligned}$$

This proves the lemma. □

The next fact gives a method of 'gluing' derivations from separate subsets of X into a derivation on all of X. In Chapters IV and V, we will use it to construct generators of $\Upsilon(\mathbb{R}^n,\mu)$ for certain measures μ.

Lemma 3.2.6. *Let $\{X_i\}_{i=1}^{\infty}$ be a μ-measurable decomposition of X. For each $i\in\mathbb{N}$, let δ_i be a derivation in $\Upsilon(X_i,\mu)$ which satisfies $\|\delta_i\|\le 1$. Then the linear operator $\delta:\mathrm{Lip}_\infty(X)\to L^\infty(X,\mu)$ given by*

$$\delta f \;:=\; \sum_{i=1}^{\infty}\chi_{X_i}\cdot\delta_i f \tag{3.2.5}$$

is a derivation in $\Upsilon(X,\mu)$ and satisfies $\|\delta\|\le 1$.

Remark 3.2.7. In equation (3.2.5), the terms on the right-hand side should be understood as *zero extensions*. More precisely, we have $\big(\chi_{X_i}\cdot\delta_i f\big)(x) = \delta_i f(x)$ whenever $x\in X_i$ and $\big(\chi_{X_i}\cdot\delta_i f\big)(x)=0$ whenever $x\in X\setminus X_i$.

Proof of Lemma 3.2.6. The map δ in formula (3.2.5) is clearly linear and satisfies the Leibniz rule. Since $\|\delta_i\|\le 1$, for each $f\in\mathrm{Lip}_\infty(X)$ with $\|f\|_{\mathrm{Lip}}\le 1$, we have

$$\mu(\{x\in X\,:\,|\delta f(x)|>1\}) \;\le\; \sum_{i=1}^{\infty}\mu(\{x\in X_i\,:\,|\delta_i f(x)|>1\}) \;=\; 0.$$

As a result, we obtain the bound $\|\delta\|\le 1$.

It remains to show that δ is continuous, so let $h\in L^1(X,\mu)$ be arbitrary and let f and $\{f_j\}_{j\in I}$ be functions in $\mathrm{Lip}_\infty(X)$ so that $f_j\xrightarrow{*} f$ and so that $C:=\sup_j\|f_j\|_{\mathrm{Lip}}<\infty$. Let $\epsilon>0$ be given.

By Theorem 2.1.3, f_j converges pointwise to f. Since $\{X_i\}_{i=1}^{\infty}$ is a measurable decomposition of X, there is an $N \in \mathbb{N}$ so that whenever $n \geq N$, we have

$$\sum_{i=n+1}^{\infty} \int_{X_i} |h|\, d\mu \;\leq\; \frac{\epsilon}{4C}.$$

On the other hand, for all $i \in \mathbb{N}$ and $j \in I$ we have $\|f_j|X_i\|_{\mathrm{Lip}} \leq C$, so by Theorem 2.1.3 we also have $f_j|X_i \overset{*}{\rightharpoonup} f|X_i$ in $\mathrm{Lip}_\infty(X_i)$. As a result, there is an $j_0 \in I$ so that

$$\left| \int_{X_i} h \cdot \delta_i f_j\, d\mu \;-\; \int_{X_i} h \cdot \delta_i f\, d\mu \right| \;<\; \frac{\epsilon}{2N}$$

holds, for all $1 \leq i \leq N$ and all $j_0 \prec j$. Combining the estimates above, for the same choices of j we obtain

$$\begin{aligned}
\left| \int_X h \cdot \delta(f_j - f)\, d\mu \right| \;&\leq\; \sum_{i=1}^{N} \left| \int_{X_i} h \cdot \delta(f_j - f)\, d\mu \right| + \sum_{i=N+1}^{\infty} \int_{X_i} |\delta(f_j - f)| \cdot |h|\, d\mu \\
&=\; N \cdot \frac{\epsilon}{2N} \;+\; \|\delta(f_j - f)\|_{\mu,\infty} \cdot \sum_{i=N+1}^{\infty} \int_{X_i} |h|\, d\mu \\
&\leq\; \frac{\epsilon}{2} \;+\; 2C \cdot \|\delta\| \cdot \frac{\epsilon}{4C} \\
&\leq\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon.
\end{aligned}$$

This proves the continuity of δ. □

As a final consequence of Theorem 3.2.1, we observe that the action of a derivation in $\Upsilon(X, \mu)$ can be extended to locally Lipschitz functions on X. The proof is similar to that of the previous lemma, but we will leave aside any issues of continuity.

Theorem 3.2.8. *Let $\delta \in \Upsilon(X, \mu)$.*

1. *There is a linear operator $\bar{\delta} : \mathrm{Lip}_{loc}(X) \to L^\infty_{loc}(X, \mu)$ so that $\bar{\delta}|\,\mathrm{Lip}_\infty(X) = \delta$. In addition, $\bar{\delta}$ is unique in the following sense: for all functions $f \in \mathrm{Lip}_{loc}(X)$ and for all balls B of finite radius in X, we have*

$$\chi_B \cdot \bar{\delta} f \;=\; \chi_B \cdot \delta F_B, \tag{3.2.6}$$

where F_B is any bounded Lipschitz extension of $f|B$ to all of X.

2. If (X,ρ) is separable, then $\bar{\delta}$ also satisfies $\bar{\delta}\big(\mathrm{Lip}(X)\big) \subset L^\infty(X,\mu)$ and

$$\|\bar{\delta} f\|_{\mu,\infty} \;\leq\; \|\delta\| \cdot L(f). \tag{3.2.7}$$

Remark 3.2.9. By the uniqueness of the extension $\bar{\delta}$, it follows that $\bar{\delta}$ satisfies a *local* version of the Leibniz rule. Indeed, for all $f, g \in \mathrm{Lip}_{loc}(X)$ we have

$$\chi_B \cdot \bar{\delta}(f \cdot g) \;=\; \chi_B\big(f \cdot \bar{\delta} g \,+\, g \cdot \bar{\delta} f\big).$$

Proof of Theorem 3.2.8. Fix $x_0 \in X$ and for $k \in \mathbb{N}$, put $A_k := B(x_0,k) \setminus \bar{B}(x_0,k-1)$. The collection of the sets $\{A_k\}_{k=1}^\infty$ is a cover of X, and each annulus A_k is a bounded set. As a result, if $f \in \mathrm{Lip}_{loc}(X)$, then $f|A_k \in \mathrm{Lip}_\infty(A_k)$ holds for each $k \in \mathbb{N}$.

Using the locality property (Theorem 3.2.1), consider the derivations $\delta_k := \chi_{A_k}\delta$ in $\Upsilon(A_k,\mu)$. We then define the operator $\bar{\delta}$ by the rule

$$\bar{\delta} f \;:=\; \sum_{k=1}^\infty \chi_{A_k} \cdot \delta_k(f|A_k), \tag{3.2.8}$$

where once again, the terms on the right-hand side are understood as zero extensions. Indeed, for each $f \in \mathrm{Lip}_{loc}(X)$, the function $\bar{\delta} f$ is well-defined for μ-a.e. $x \in A_k$ and hence for μ-a.e. $x \in X$. Since each map δ_k is linear, so is $\bar{\delta}$.

By its construction, $\bar{\delta} g = \delta g$ holds whenever $g \in \mathrm{Lip}_\infty(X)$. More generally, let $f \in \mathrm{Lip}_{loc}(X)$, let B be any ball in X, and let F_B be the McShane-Whitney extension of $f|B$. If $\delta \in \Upsilon(X,\mu)$, then by formula (3.2.8) and Lemma 3.2.3 we have

$$\begin{aligned} \chi_B \cdot \bar{\delta} f \;&=\; \sum_{k=1}^\infty \chi_{A_k} \cdot \chi_B \cdot \delta_k(f|A_k) \\ &=\; \sum_{k=1}^\infty \chi_{A_k} \cdot \chi_B \cdot \delta_k(F_B|A_k) \;=\; \sum_{k=1}^\infty \chi_{A_k} \cdot \chi_B \cdot \delta F_B \;=\; \chi_B \cdot \delta F_B. \end{aligned}$$

Formula (3.2.6) follows.

Let $f \in \mathrm{Lip}_{loc}(X)$. To see that the function $\bar{\delta} f$ lies in $L^{\infty}_{loc}(X,\mu)$, let K be any compact subset of X. Since K is bounded, there is a $k_0 \in \mathbb{N}$ so that $K \subset B(x_0, k_0)$. Put $\Lambda := \max\{\|\delta_k f\|_{\mu,\infty} : 1 \leq k \leq k_0\}$. We now compute

$$\mu(\{x \in K \,:\, |\bar{\delta} f(x)| > \Lambda\}) \;\leq\; \sum_{k=1}^{k_0} \mu(\{x \in A_k \,:\, |\delta f(x)| > \Lambda\}) \;=\; 0.$$

It follows that $\chi_K \cdot \delta f \in L^{\infty}(X,\mu)$, and this proves the first assertion.

Towards the second assertion, assume now that X is separable. Let $\{x_j\}_{j=1}^{\infty}$ be a countably dense subset of X. Then the collection of balls $B_j := B(x_j, 1/2)$ covers X. For each $f \in \mathrm{Lip}(X)$ and each $j \in \mathbb{N}$, put $g_j := f - \inf_{B_j} f$. We then compute

$$\|g_j|B_j\|_{\infty} \;\leq\; \left|\sup_{B_j} f - \inf_{B_j} f\right| \;\leq\; L(f) \cdot \mathrm{diam}(B_j) \;=\; L(f).$$

Now let $F_j : X \to \mathbb{R}$ be the McShane-Whitney extension of $g_j|B_j$, so $L(F_j) = L(f)$ and $\|F_j\|_{\infty} = \|g_j|B_j\|_{\infty}$. From these bounds and the previous estimate, we obtain

$$\|\delta F_j\|_{\mu,\infty} \;\leq\; \|\delta\| \cdot \|F_j\|_{\mathrm{Lip}} \;=\; \|\delta\| \cdot \big(\|F_j\|_{\infty} \vee L(F_j)\big) \;\leq\; \|\delta\| \cdot L(f).$$

Lastly, consider the pairwise disjoint collection of sets

$$C_1 \;:=\; B_1, \quad C_j \;:=\; B_j \setminus \bigcup_{i=1}^{j} B_i, \text{ for } j = 2, 3, \ldots$$

which form a measurable decomposition of X. Put $\delta_j := \chi_{B_j}\delta$ and $\bar{\delta} := \sum_j \chi_{C_j}\delta_j$.

By the locality property (Theorem 3.2.1), for each $j \in \mathbb{N}$ we have $\delta_j \in \Upsilon(B_j, \mu)$ and by Part (3.2.3) of Lemma 3.2.3, we have $\delta_j g_j = \chi_{B_j} \cdot \delta F_j$. Since every derivation applied to a constant function is zero, we have $\delta_j f = \delta_j g_j$ and hence $\delta_j f = \chi_{B_j} \cdot \delta F_j$. Putting $\lambda := \|\delta\| \cdot L(f)$, we now compute

$$\begin{aligned}
\mu(\{x \in X \,:\, |\bar{\delta} f(x)| > \lambda\}) \;&\leq\; \sum_{j=1}^{\infty} \mu(\{x \in C_j \,:\, |\delta_j f(x)| > \lambda\}) \\
&\leq\; \sum_{j=1}^{\infty} \mu(\{x \in C_j \,:\, |\delta F_j(x)| > \lambda\}) \;=\; 0.
\end{aligned}$$

This gives inequality (3.2.7) and proves the theorem. □

3.3 Derivations, Bundle Structures, and Linear Algebra.

In the previous section, the locality property of derivations was motivated by viewing metric measure spaces as similar to smooth manifolds. Here we follow this analogy further by introducing additional properties of derivations that are reminiscent of smooth vector fields.

The following theorem is due to Weaver, and it is an immediate consequence of [Wea00, Thm 10] and [Wea00, Cor 24]. It states that in certain cases, $\Upsilon(X,\mu)$ can be realized as measurable type of "vector bundle" over X.

Theorem 3.3.1 (Weaver, 1999). *Let (X,ρ,μ) be a metric measure space and suppose that $\Upsilon(X,\mu)$ is a finitely generated module over $L^\infty(X,\mu)$. Then there is a $k \in \mathbb{N}$ and a measurable decomposition $X = \coprod_{n=1}^{k} X^n$ so that for each $x \in X^n$ there is a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ with the following property: for each n, $\Upsilon(X^n,\mu)$ is isometrically and weak-$*$ continuously isomorphic to the set of bounded μ-measurable functions $f : X^n \to \mathbb{R}^n$ with respect to the norm*

$$\|f\| := \mu\text{-ess-sup}_{x \in X^n} \|f(x)\|_x.$$

Such a structure is an example of a *(F) Banach bundle*, which we will not discuss here; for further details, see [DG83]. It is important to note that, unlike the case of vector bundles on manifolds, Banach bundles often do not satisfy a *local triviality* condition [Hir94, Sect 4.1]. In spite of this, Theorem 3.3.1 shows that the case of finitely generated modules $\Upsilon(X,\mu)$ have a clear geometric interpretation.

Later we consider a certain class of metric measure spaces (X,ρ,μ), called p-PI spaces, which admit a similar bundle structure as in Theorem 3.3.1; see Theorem 6.2.1. For such spaces, it is not known whether the associated modules $\Upsilon(X,\mu)$ are finitely generated. However, we show in Chapter VI that these modules do satisfy a related condition in terms of linearly independent sets. We recall the definition from linear algebra [Hun80, Sect IV.2].

Definition 3.3.2. Let $m \in \mathbb{N}$. A subset $\{\delta_i\}_{i=1}^m$ in $\Upsilon(X,\mu)$ is *linearly independent* if the following implication holds: whenever there are functions $\{\lambda_i\}_{i=1}^m$ in $L^\infty(X,\mu)$ so that $\sum_{i=1}^m \lambda_i\delta_i$ is the zero derivation, then each λ_i is the zero function. The set $\{\delta_i\}_{i=1}^m$ is *linearly dependent* if it is not linearly independent.

The *rank* of $\Upsilon(X,\mu)$ is the largest cardinality of linearly independent sets of derivations in $\Upsilon(X,\mu)$.

Remark 3.3.3. In the case of a zero measure, linear independence becomes a degenerate notion. Indeed, if $\mu = 0$, then $L^\infty(X,\mu)$ consists of the zero function only, and $\Upsilon(X,\mu)$ is the zero module. Moreover, if $\sum_{i=1}^N \lambda_i\delta_i = 0$ holds for a collection of functions $\{\lambda_i\}_{i=1}^N$ in $L^\infty(X,\mu)$, then we obtain trivially $\lambda_i = 0$ for each index i. As a result, $\{0\}$ is a linearly independent set in $\Upsilon(X,\mu)$.

To avoid such pathologies, we will discuss linear independence of sets in $\Upsilon(X,\mu)$ only when μ is a nonzero measure and when X has positive μ-measure.

Remark 3.3.4 (Bases and free modules). Our notion of rank may differ from other definitions; as an example, see [Hun80, Sect IV.2]. In that reference, *rank* makes sense only for *free modules* over rings which have the *invariant dimension property* [Hun80, Sect IV.2]. Recall that a module M over R is a *free module* if it admits a *basis*, that is, a linearly independent generating set. We will not define the invariant dimension property here. However, $L^\infty(X,\mu)$ is a commutative ring with identity and therefore has this property [Hun80, Cor IV.2.12].

For rings R with the invariant dimension property, the *rank* of M is then defined as the cardinality of any basis of M. To reiterate, we will not use this notion of rank, but the notion of rank as given in Definition 3.3.2.

Note that there exist modules of derivations which do not have bases and hence are not free modules.

Example 3.3.5. Consider the line $L = \{2\} \times \mathbb{R}$ and the ball $B = B(0,1)$ in $\mathbb{R}^2$, and put $X = B \cup L$ and $\mu = m_2 \lfloor B + \mathcal{H}^1 \lfloor L$. By Theorem 3.2.1, the operators $\delta_1 := \chi_B \partial_{x_1} + \chi_L \partial_{x_2}$ and $\delta_2 := \partial_{x_2}$ are derivations in $\Upsilon(X, \mu)$. In fact, they form a generating set. However, from the identity $\chi_L(\delta_1 - \delta_2) = 0$, we see that the set $\{\delta_1, \delta_2\}$ must be linearly dependent.

As we will see later, for the previous example there are no generating sets for $\Upsilon(X, \mu)$ which are linearly independent; see Corollary 5.2.7.

The next lemma collects a few elementary facts about linearly independent sets of derivations. Their proofs are straightforward and we omit them.

Lemma 3.3.6. *Let $N \in \mathbb{N}$.*

1. *If $\{\delta_i\}_{i=1}^N$ is a linearly independent set in $\Upsilon(X, \mu)$, then so are subsets of $\{\delta_i\}_{i=1}^N$.*
2. *If $\{\delta_i\}_{i=1}^N$ is a linearly independent set in $\Upsilon(X, \mu)$ and if A is a measurable subset of X with $\mu(A) > 0$, then $\{\chi_A \delta_i\}_{i=1}^N$ is a linearly independent set in $\Upsilon(A, \mu)$.*

It is a general fact [HK98, Thm 5.5] that if R is a commutative ring with identity and if M is a free R-module generated by n elements, then the rank of M is n. Under additional hypotheses on X, a weak converse to this fact holds true for the modules $\Upsilon(X, \mu)$. With this in mind, we now give a definition and state a few lemmas.

Definition 3.3.7. Let G be a subset of $\mathrm{Lip}_\infty(X)$, and let $\mathbb{R}[G]$ denote the subalgebra in $\mathrm{Lip}_\infty(X)$ formed from sums and products of functions from G. We say that G is a *generating set* for $\mathrm{Lip}_\infty(X)$ if $\mathrm{Lip}_\infty(X)$ is precisely the closure of $\mathbb{R}[G]$

with respect to the bounded weak-$*$ topology. In such a case, we say that each function $g_i \in G$ is a *generator* of $\mathrm{Lip}_\infty(X)$.

Furthermore, we say that $\mathrm{Lip}_\infty(X)$ is N*-generated* if there is a generating set G for $\mathrm{Lip}_\infty(X)$ with cardinality N. We also say that $\mathrm{Lip}_\infty(X)$ is *finitely generated* (as an algebra) if it is N-generated for some $N \in \mathbb{N}$.

Example 3.3.8. If B is any bounded subset of $\mathbb{R}^n$, then by Lemma 2.3.1, the functions $\{x_i\}_{i=1}^n$ form a generating set for $\mathrm{Lip}_\infty(B)$. Therefore $\mathrm{Lip}_\infty(B)$ is n-generated. More generally, compact Riemannian manifolds M have finitely generated $\mathrm{Lip}_\infty(M)$.

Lemma 3.3.9. *Let $N \in \mathbb{N}$. Suppose that $\mathrm{Lip}_\infty(X)$ is N-generated with generating set $\{g_i\}_{i=1}^N$ and suppose also that $\{\delta_i\}_{i=1}^N$ is a linearly independent set in $\Upsilon(X,\mu)$. Then the matrix $[\delta_i g_j(x)]_{i,j=1}^N$ is non-singular for μ-a.e. $x \in X$.*

In the proof, we use the Laplace expansion formula for square matrices [HK98, Eqn 5.21]. Given a matrix $A = [a_{ij}]_{i,j=1}^n$, recall that the *cofactor* $A(i_1, \ldots, i_m | j_1, \ldots j_m)$ of A is the $(n-m) \times (n-m)$ matrix formed by omitting from A the rows indexed by $i_1, i_2, \ldots, i_m$ and the columns indexed by $j_1, j_2, \ldots j_m$. The Laplace expansion formula then gives, for $i = 1, 2, \ldots, n$,

$$\det A \;=\; \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det A(i|j). \tag{3.3.1}$$

Proof. Put $M := [\delta_i g_j]_{i,j=1}^n$ and suppose that the set $E := \{x \in X : \det M = 0\}$ has positive μ-measure. By formula (3.3.1) we obtain

$$0 \;=\; \chi_E \cdot \Delta_j^k \cdot \det M \;=\; \chi_E \cdot \sum_{i=1}^{n} (-1)^{i+j} \cdot \det M(i|j) \cdot \delta_i g_k,$$

for all j and k, and where Δ_j^k is the Kronecker delta. Next, consider the derivations

$$\delta_j' \;:=\; \sum_{i=1}^{n} \chi_E \cdot (-1)^{i+j} \cdot \det M(i|j)\, \delta_i \tag{3.3.2}$$

By construction, we have $\delta'_j g_k = 0$ for all indices j and k. Since $\{g_k\}_{k=1}^N$ is a generating set for $\mathrm{Lip}_\infty(X)$, it follows from continuity that $\delta'_j f = 0$ holds for all $f \in \mathrm{Lip}_\infty(X)$ and all indices j. However, the set $\{\delta_i\}_{i=1}^N$ is linearly independent by hypothesis. It follows that $\chi_E \cdot \det M(i|j) = 0$ holds, for all indices i and j.

Let $1 \leq k < N$ be the least number with the following property: there is a cofactor sub-matrix A of M which has zero determinant and for some $i, j \in \{1, 2, \ldots, N\}$, there is a cofactor $A(i; j)$ of A which has nonzero determinant. Up to a permutation of indices, let $A := [\delta_i g_j(x)]_{i,j=1}^k$. By the same arguments as above, we see that

$$0 \;=\; \sum_{i=1}^{k} \chi_E \cdot (-1)^{i+j} \cdot \det A(i|j)\, \delta_i.$$

Since one of the determinants $\det A(i|j)$ is nonzero, it follows that the set $\{\chi_E \delta_i\}_{i=1}^k$ is linearly dependent in $\Upsilon(X, \mu)$. By Lemma 3.3.6, the sets $\{\delta_i\}_{i=1}^k$ and $\{\delta_i\}_{i=1}^N$ are also linearly dependent in $\Upsilon(X, \mu)$, which is a contradiction.

It follows that for $1 \leq k \leq N$, the determinant of each $k \times k$ cofactor of M is zero. In the case $k = 1$, we see that $\chi_E \cdot \delta_i g_j = 0$ holds for all indices i and j, and hence $\chi_E \cdot \delta_i f = 0$ holds for all $f \in \mathrm{Lip}_\infty(X)$ and all $1 \leq i \leq N$. Therefore each derivation $\chi_E \delta_i$ is zero, so we have $\chi_E = 0$ and hence $\mu(E) = 0$. $\square$

The next corollary is a type of Gram-Schmidt orthogonalization for linearly independent sets of derivations. It is a direct consequence of Lemma 3.3.9.

Corollary 3.3.10. *Let $N \in \mathbb{N}$. Suppose that $\mathrm{Lip}_\infty(X)$ is N-generated with generating set $\{g_i\}_{i=1}^N$ and suppose also that $\{\delta_i\}_{i=1}^N$ is a linearly independent collection in $\Upsilon(X, \mu)$. Then there is a linearly independent set $\{\delta'_i\}_{i=1}^N$ in $\Upsilon(X, \mu)$ so that*

1. *if $i \neq j$, then $\delta'_i g_j \;=\; 0$;*

2. *for all $1 \leq i \leq n$, the set $\{x \in X \,:\, \delta'_i g_i(x) = 0\}$ has μ-measure zero.*

Recalling the exterior differential map from formula (3.1.1), one may interpret the conclusions of Corollary 3.3.10 in the following way. By selecting a generating set $\{g_j\}_{j=1}^N$ of $\mathrm{Lip}_\infty(X)$, conclusions (1) and (2) become "orthogonality" relations between the derivations $\{\delta_i\}_{i=1}^N$ and the measurable 1-forms $\{dg_j\}_{j=1}^N$.

Proof of Corollary 3.3.10. By hypothesis, there is a linearly independent set $\{\delta_i\}_{i=1}^N$ in $\Upsilon(\mathbb{R}^n, \mu)$. If the above conclusions (1) and (2) are not satisfied for these, then choose scalars $\{\lambda_{ij}\}_{i,j=1}^N$ in $L^\infty(X, \mu)$ as in formula (3.3.2), and put $\delta'_j := \sum_{i=1}^n \lambda_{ij}\delta_i$. Arguing once more by the Laplace expansion formula, it is easy to see that the derivations $\{\delta'_i\}_{i=1}^N$ do satisfy conclusions (1) and (2).

It remains to show that $\{\delta'_i\}_{i=1}^N$ is a linearly independent set, so suppose there are functions $\{\lambda_i\}_{i=1}^N$ in $L^\infty(X, \mu)$ so that $\sum_{i=1}^N \lambda_i\delta'_i$ is the zero derivation. In particular, for each generator g_j we will use conclusion (1) to obtain

$$0 \;=\; \Big(\sum_{i=1}^N \lambda_i\delta'_i\Big) g_j \;=\; \lambda_j \cdot \delta'_j g_j.$$

By conclusion (2), $\delta'_j g_j$ is μ-a.e. nonzero, which means that $\lambda_j = 0$ holds μ-a.e. The linear independence follows. □

Lemma 3.3.11. *Let $N \in \mathbb{N}$ and suppose that $\mathrm{Lip}_\infty(X)$ is N-generated. Then the module $\Upsilon(X, \mu)$ has rank at most N.*

Proof. We argue by contradiction. Suppose that $\{\delta_i\}_{i=1}^{N+1}$ is a linearly independent set in $\Upsilon(X, \mu)$, so by Lemma 3.3.6 the subset $\{\delta_i\}_{i=1}^N$ is also linearly independent. Let $\{g_i\}_{i=1}^N$ be a generating set for $\mathrm{Lip}_\infty(X)$, so there must exist $L^\infty(X, \mu)$-linear combinations $\{\delta'_i\}_{i=1}^N$ of the $\{\delta_i\}_{i=1}^N$ which satisfy the conclusions of Corollary 3.3.10 with respect to the $\{g_i\}_{i=1}^N$. In particular, $\{\delta'_i\}_{i=1}^N$ is a linearly independent set.

Since δ_{N+1} is a nonzero operator, there must be an index j for which $\delta_{N+1} g_j$ is not identically zero. Let J be the set of all such indices, and to simplify notation, put $\delta'_{N+1} := \delta_{N+1}$ and $J' := J \cup \{N+1\}$. Consider the functions $\{\lambda_j\}_{j\in J'}$, given by

$$\lambda_j \;:=\; \begin{cases} \delta'_{N+1} g_j \cdot \prod_{i\in J\setminus\{j\}} \delta'_i g_i, & j \in J \\ \qquad\qquad - \prod_{i\in J} \delta'_i g_i, & j = N+1. \end{cases} \tag{3.3.3}$$

By inspection, λ_j lies in $L^\infty(X,\mu)$, for each $j \in J'$. In addition, note that if $j \in J$, then by conclusion (1) of Corollary 3.3.10, we obtain

$$\sum_{i\in J'} \lambda_i \cdot \delta'_i g_j \;=\; \lambda_j \cdot \delta'_j g_j \;+\; \lambda_{N+1} \cdot \delta_{N+1} g_j \;=\; 0.$$

Otherwise $j \notin J$, and by construction we have $\delta'_i g_j = 0$, for each $i \in J'$. This shows that the set $\{\delta'_i\}_{i\in J'}$ is linearly dependent, which is a contradiction. As a result, the initial set of derivations $\{\delta_i\}_{i=1}^{n+1}$ is linearly dependent. □

We now compare linearly independent sets in $\Upsilon(X,\mu)$ with generating sets of $\Upsilon(X,\mu)$. Roughly speaking, if $\mathrm{Lip}_\infty(X)$ is a finitely generated algebra and if there is a linearly independent set in $\Upsilon(X,\mu)$ of sufficiently large cardinality, then we obtain generating sets of $\Upsilon(X,\mu)$ with the same cardinality.

Theorem 3.3.12. *Let $N \in \mathbb{N}$ and let μ be a Radon measure on X. Suppose that* $\mathrm{Lip}_\infty(X)$ *is N-generated and that the rank of $\Upsilon(X,\mu)$ is N. Then for any $\epsilon > 0$, there is a subset X_ϵ of X so that $\mu(X \setminus X_\epsilon) < \epsilon$ and that $\Upsilon(X_\epsilon,\mu)$ is generated by N derivations.*

Proof. Let $\epsilon > 0$ be arbitrary, and let $\{g_i\}_{i=1}^N$ be a generating set of $\mathrm{Lip}_\infty(X)$. By hypothesis there is a linearly independent set $\{\delta_i\}_{i=1}^N$ in $\Upsilon(X,\mu)$, so there exist derivations $\{\delta'_i\}_{i=1}^N$ in $\Upsilon(X,\mu)$ which satisfy the orthogonality relations of Corollary 3.3.10.

Assume first that X is a bounded metric space. The subset X_ϵ is constructed as follows: given $c > 0$, first consider the subset

$$X_c^1 \;:=\; \{x \in X \,:\, |\delta'_1 g_1(x)| \;>\; c\}.$$

Clearly $\mu(X \setminus \bigcup_{c>0} X_c^1) = 0$ and if $c < c'$, then $X_{c'}^1 \subset X_c^1$. Since μ is Radon and X is bounded, we see that X has finite μ-measure. It follows that

$$\lim_{c \to 0} \mu(X \setminus X_c^1) \;=\; \mu\Big(\bigcap_{c>0} X \setminus X_c^1\Big) \;=\; \mu\Big(X \setminus \bigcup_{c>0} X_c^1\Big) \;=\; 0.$$

Choose $c > 0$ sufficiently small so that $\mu(X \setminus X_c^1) < \epsilon/N$, and put $X^1 := X_c^1$. Iterating further, for $2 \le i \le N$, let $c_i > 0$ and consider the subsets

$$X_{c_i}^i \;:=\; \{x \in X^{i-1} \,:\, |\delta_i' g_i(x)| > c_i\}.$$

Arguing as before, we obtain subsets X^i of X so that $\mu(X^{i-1} \setminus X^i) < \epsilon/N$. Put $X_\epsilon = X^N$. It follows that

$$\mu(X \setminus X_\epsilon) \;=\; \mu(X \setminus X^N) \;\le\; \sum_{i=1}^N \mu(X^{i-1} \setminus X^i) \;<\; N \cdot \frac{\epsilon}{N} \;=\; \epsilon.$$

To see that the set $\{\delta_i'\}_{i=1}^N$ generates $\Upsilon(X_\epsilon, \mu)$, let $\delta \in \Upsilon(X, \mu)$ be arbitrary. It suffices to show that δ is a $L^\infty(X,\mu)$-linear combination of the $\{\delta_i'\}_{i=1}^N$. Choose $\{\lambda_i\}_{i=1}^{N+1}$ in $L^\infty(X,\mu)$ as in equation (3.3.3), with δ for δ_{N+1}, and put

$$\delta' \;:=\; \lambda_{N+1}\delta + \sum_{i=1}^N \lambda_i \delta_i'. \tag{3.3.4}$$

By construction, $\delta' g_j = 0$ holds for all $1 \le j \le N$. Moreover, $\{g_i\}_{i=1}^N$ is a generating set for $\mathrm{Lip}_\infty(X)$, so every function $f \in \mathrm{Lip}_\infty(X)$ is a weak-$*$ limit of polynomials in $\{g_i\}_{i=1}^N$. From the continuity of each δ_i, it follows that $\delta' f = 0$ holds for every $f \in \mathrm{Lip}_\infty(X)$. Therefore δ' is zero, and equation (3.3.4) becomes

$$\delta \;=\; \frac{-1}{\lambda_{N+1}} \cdot \sum_{i=1}^N \lambda_i \delta_i'.$$

From our previous choices, we have $\lambda_{N+1} = \prod_{i=1}^N \delta_i' g_i$. By definition of X_ϵ, there is a $C > 0$ so that $|\lambda_{N+1}(x)| \ge C$ holds for μ-a.e. $x \in X_\epsilon$. As a result, each function λ_i/λ_{N+1} lies in $L^\infty(X_\epsilon, \mu)$, and hence δ is a linear combination of the $\{\delta_i'\}_{i=1}^N$.

If X is not bounded, then fix a point $x_0 \in X$ and consider the sequence of annuli

$$A^n \;:=\; \{x \in X \,:\, n \leq \rho(x, x_o) < n+1\}.$$

By the previous argument, for all $\epsilon > 0$ and each $n \in \mathbb{N}$ there exists a subset A^n_ϵ of A^n so that $\mu(A^n \setminus A^n_\epsilon) < \epsilon \cdot 2^{-n}$ and so that the set $\{\chi_{A^n}\delta'_i\}_{i=1}^N$ generates $\Upsilon(A^n, \mu)$. Putting $X_\epsilon := \bigcup_{n=1}^\infty A^n_\epsilon$, we see that

$$\mu(X \setminus X_\epsilon) \;\leq\; \sum_{n=1}^{\infty} \mu(A^n \setminus A^n_\epsilon) \;<\; \epsilon.$$

Moreover, by the locality property (Theorem 3.2.1), $\{\delta'_i\}_{i=1}^N$ generates $\Upsilon(X, \mu)$. This proves the theorem. □

3.4 Pushforwards of Derivations.

Following the example of manifolds once again, let M and N be Riemannian manifolds and let $f : M \to N$ be a smooth injective map. To each vector field v in the tangent bundle TM, the derivative map Df induces a pushforward vector field

$$f_\# v(f(x)) \;:=\; Df(x) \cdot v(x)$$

at every point in $f(M)$. In addition, if $\dim M = \dim N$ then for bounded open subsets B of M and functions $g \in C^\infty(N)$, we have the change of variables formula

$$\int_{f(B)} g(y)\, dV_N(y) \;=\; \int_B (g \circ f)(x) \cdot \det Df(x)\, dV_M(x),$$

where V_M and V_N are the volume elements of M and N, respectively. On Euclidean spaces, the Area and Co-Area Formulas [EG92, Thms 3.3.2.1 & 3.4.2.1] generalize the above identity to Lipschitz non-injective maps f and Lipschitz functions g.

In what follows, we return to the setting of metric measure spaces, and we will formulate the notion of a pushforward derivation in terms of similar transformation formulas. To begin, let $\pi \in \mathrm{Lip}(X,Y)$ and let μ be a signed Borel measure on X. Recall that for Borel subsets E of Y, the *pushforward measure* $\pi_\#\mu$ on Y is given by

$$\pi_\#\mu(E) \;:=\; \mu(\pi^{-1}(E)),$$

In addition, the following transformation formula [Mat95, Thm 1.19]

$$\int_Y h\, d(\pi_\#\mu) \;=\; \int_X (h\circ\pi)\, d\mu \tag{3.4.1}$$

is valid[1] for all $h \in L^1(Y,\pi_\#\mu)$. In the next lemma, we show that every derivation in $\Upsilon(X,\mu)$ induces a well-defined pushforward derivation in $\Upsilon(Y,\pi_\#\mu)$.

Lemma 3.4.1. *Let (X,ρ_X) and (Y,ρ_Y) be metric spaces, and let $\pi : X \to Y$ be a Lipschitz map. For each $\delta \in \Upsilon(X,\mu)$, there is a unique derivation $\pi_\#\delta \in \Upsilon(Y,\pi_\#\mu)$, called the pushforward of δ under π, that satisfies*

$$\int_Y h\cdot(\pi_\#\delta)f\, d(\pi_\#\mu) \;=\; \int_X (h\circ\pi)\cdot\delta(f\circ\pi)\, d\mu \tag{3.4.2}$$

for all $f \in \mathrm{Lip}_\infty(Y)$ and all $h \in L^1(Y,\pi_\#\mu)$. In addition, we have

$$\|\pi_\#\delta\| \;\le\; (1 \vee L(\pi))\cdot\|\delta\|. \tag{3.4.3}$$

Proof. Put $\nu := \pi_\#\mu$. By formula (3.4.1), for each $h \in L^1(Y,\nu)$ we have

$$\|h\|_{\nu,1} \;:=\; \int_Y |h|\, d\nu \;=\; \int_X |h\circ\pi|\, d\mu \;<\; \infty,$$

so $h\circ\pi \in L^1(X,\mu)$. Let $f \in \mathrm{Lip}_\infty(Y)$. We now observe that

$$\left|\int_X (h\circ\pi)\cdot\delta(f\circ\pi)\, d\mu\right| \;\le\; \|h\|_{\nu,1}\cdot\|\delta(f\circ\pi)\|_{\mu,\infty}, \tag{3.4.4}$$

so the map $l_f(h) := \int_X (h\circ\pi)\cdot\delta(f\circ\pi)d\mu$ is an element of the dual $[L^1(Y,\nu)]^*$. By duality, there is a unique function $w_{\pi,f} \in L^\infty(Y,\nu)$ so that the action of l_f can be

[1] Strictly speaking, [Mat95, Thm 1.19] holds for positive measures only. However, every signed measure is the difference of positive measures, so equation (3.4.1) follows by invoking [Mat95, Thm 1.19] for the positive measures separately and then taking their difference.

realized by integration against $w_{\pi,f}$. In symbols, we have

$$\int_X (h \circ \pi) \cdot \delta(f \circ \pi)\, d\mu \;=\; l_f(h) \;=\; \int_Y h \cdot w_{\pi,f}\, d\nu \tag{3.4.5}$$

for all $h \in L^1(Y, \nu)$. We now define the operator $\pi_\# \delta : \mathrm{Lip}_\infty(Y) \to L^\infty(Y, \pi_\# \mu)$ by the formula $(\pi_\# \delta) f := w_{\pi,f}$. Clearly $\pi_\# \delta$ is linear and by equation (3.4.5), it also satisfies the transformation formula (3.4.2).

Claim 3.4.2. The operator $\pi_\# \delta$ lies in $\Upsilon(Y, \pi_\# \mu)$.

By inequality (3.4.4) and equation (3.4.5), we obtain the estimate

$$|l_f(h)| \;:=\; \left| \int_Y h \cdot (\pi_\# \delta) f \, d\nu \right| \;\le\; \|h\|_{\nu,1} \cdot \|\delta(f \circ \pi)\|_{\mu,\infty}.$$

Taking suprema over all nonzero h, we see that $\|l_f\|_* \le \|\delta(f \circ \pi)\|_{\mu,\infty}$. Observe that the operator norm of $l_f \in [L^1(Y,\nu)]^*$ agrees with the norm of $(\pi_\# \delta) f$ in $L^\infty(Y, \nu)$. From this and from the boundedness of δ, it follows that there is a $C > 0$ so that

$$\|(\pi_\# \delta) f\|_{\nu,\infty} \;=\; \|l_f\|_* \;\le\; \|\delta(f \circ \pi)\|_{\mu,\infty} \;\le\; C \cdot \|f \circ \pi\|_{\mathrm{Lip}}.$$

Since the norm on $\mathrm{Lip}_\infty(X)$ is defined as a maximum, we begin by estimating

$$\begin{aligned} L(f \circ \pi) \;&\le\; L(\pi) \cdot L(f), \\ \|f \circ \pi\|_\infty \;&=\; \sup_{x \in X} |f\big(\pi(x)\big)| \;\le\; \sup_{y \in Y} |f(y)| \;=\; \|f\|_\infty \end{aligned}$$

from which we obtain, for $C' = C \cdot (1 \vee L(\pi))$,

$$\|(\pi_\# \delta) f\|_{\nu,\infty} \;\le\; C' \cdot \|f\|_{\mathrm{Lip}}.$$

Inequality (3.4.3) follows. The continuity of $\pi_\# \delta$ follows from both formula (3.4.2) and the continuity of δ. To see this, let f and $\{f_i\}_{i \in I}$ be functions in $\mathrm{Lip}_\infty(Y)$ so that $f_i \xrightarrow{*} f$ and that $\sup_i \|f_i\|_{\mathrm{Lip}} < \infty$. By Corollary 2.1.4, we have $f_n \circ \pi \xrightarrow{*} f \circ \pi$

in $\mathrm{Lip}_\infty(X)$. Moreover, for each $h \in L^1(Y,\nu)$ we have $h \circ \pi \in L^1(X,\mu)$ by formula (3.4.1). It follows that

$$\begin{aligned}\int_Y h \cdot (\pi_\# \delta) f_i \, d\nu &= \int_X (h \circ \pi) \cdot \delta(f_i \circ \pi)\, d\mu \\ &\to \int_X (h \circ \pi) \cdot \delta(f \circ \pi)\, d\mu = \int_Y h \cdot (\pi_\# \delta) f \, d\nu.\end{aligned}$$

Since h was arbitrary, we have $(\pi_\#\delta) f_i \xrightharpoonup{*} (\pi_\#\delta) f$ in $L^\infty(X,\mu)$. By similar arguments, the Leibniz rule for $\pi_\#\delta$ follows from the Leibniz rule for δ.

Lastly, suppose that δ' is another derivation in $\Upsilon(Y,\nu)$ which satisfies formula (3.4.2). For any $f \in \mathrm{Lip}_\infty(Y)$, we have $(\delta' - \pi_\#\delta) f = 0$ by linearity. As a result, $\delta' = \pi_\#\delta$ and this gives the desired uniqueness. □

Let $\pi \in \mathrm{Lip}(X;Y)$ be arbitrary. Observe that each function $\lambda \in L^\infty(X,\mu)$ induces a scalar action on each derivation $\delta \in \Upsilon(Y, \pi_\#\mu)$, by the rule

$$\lambda \cdot \delta := (\lambda \circ \pi)\delta. \tag{3.4.6}$$

By the transformation formula (3.4.2), the map $\delta \mapsto \pi_\#\delta$ then determines a homomorphism of $L^\infty(X,\mu)$-modules. We denote it by $\pi_\# : \Upsilon(X,\mu) \to \Upsilon(Y,\pi_\#\mu)$.

Under compatible choices of measures, bi-Lipschitz homeomorphisms give rise to isomorphisms of modules in the above sense. To explain the terminology, on a space X, two measures μ and ν are *mutually absolutely continuous* if $\mu \ll \nu$ and $\nu \ll \mu$.

Theorem 3.4.3. *Let (X,ρ,μ) and (Y,ρ',ν) be metric measure spaces, and let $\varphi : X \to Y$ be a bi-Lipschitz embedding. If $\varphi_\#\mu$ and ν are mutually absolutely continuous Radon measures, then $\Upsilon(X,\mu)$ and $\Upsilon(Y,\nu)$ are isomorphic as $L^\infty(X,\mu)$-modules.*

To this end, we require an additional lemma. Both the proof of Theorem 3.4.3 and the lemma will use pushforward derivations.

Lemma 3.4.4. *Let (X, ρ) be a metric space and let μ and ν be Radon measures on X. If $\mu \ll \nu$, then $\Upsilon(X, \nu)$ is a sub-module of $\Upsilon(X, \mu)$.*

Proof of Lemma 3.4.4. If $g \in L^\infty(X, \nu)$, then the set $\{x : |g(x)| > \|g\|_{\nu,\infty}\}$ has ν-measure zero and hence μ-measure zero. This implies that $\|g\|_{\mu,\infty} \leq \|g\|_{\nu,\infty}$, and hence $L^\infty(X, \nu)$ is a linear subspace of $L^\infty(X, \mu)$. As a result, each $\delta \in \Upsilon(X, \nu)$ is a well-defined map from $\mathrm{Lip}_\infty(X)$ to $L^\infty(X, \mu)$. To avoid confusion, we denote the latter map by δ_μ.

Clearly, δ_μ is linear and satisfies the Leibniz rule. To see that δ_μ is bounded, note that if δ is bounded with constant $C \geq 0$, then the previous estimate shows that

$$\|\delta_\mu f\|_{\mu,\infty} \ \leq \ \|\delta f\|_{\nu,\infty} \ \leq \ C \cdot \|f\|_{\mathrm{Lip}}.$$

Lastly, we show that δ_μ is continuous. Let $w \in L^1_{loc}(X, \nu)$ be the Radon-Nikodym derivative of μ with respect to ν, so $d\mu = w d\nu$. If h is an arbitrary function in $L^1(X, \mu)$, then the product $h \cdot w$ lies in $L^1(X, \nu)$. So if $\{f_i\}_{i \in I}$ is a net in $\mathrm{Lip}_\infty(X)$ which converges weak-$*$ to f, then it follows from the continuity of δ that

$$\int_X h \cdot \delta f_i \, d\mu \ = \ \int_X h \cdot w \cdot \delta f_i \, d\nu \ \to \ \int_X h \cdot w \cdot \delta f \, d\nu \ = \ \int_X h \cdot \delta f \, d\mu.$$

Therefore δ_μ is weak-$*$ continuous, and this proves the lemma. □

By the argument of the previous proof, if two measures μ and ν on a space X are mutually absolutely continuous, then we have $L^\infty(X, \mu) = L^\infty(X, \nu)$. From this we obtain the following fact: a derivation $\delta \in \Upsilon(X, \mu)$ depends only on the metric and the *measure class* of μ, that is, the class of measures which are mutually absolutely continuous to μ.

Corollary 3.4.5. *Let (X, ρ) be a metric space. If μ and ν are two mutually absolutely continuous Radon measures on X, then $\Upsilon(X, \mu) \cong \Upsilon(X, \nu)$.*

Assuming Lemma 3.4.4, we now prove Theorem 3.4.3. In what follows, if μ_1 and μ_2 are mutually absolutely continuous measures on X, then we will no longer distinguish between derivations in $\Upsilon(X,\mu_1)$ or in $\Upsilon(X,\mu_2)$.

Proof of Theorem 3.4.3. Since $\varphi_{\#}\mu$ and ν are mutually absolutely continuous measures, we have $L^\infty(Y,\nu) = L^\infty(Y,\varphi_{\#}\mu)$. As a result, both $\Upsilon(Y,\varphi_{\#}\mu)$ and $\Upsilon(Y,\nu)$ are $L^\infty(X,\mu)$-modules, where the scalar action is defined as in equation (3.4.6). By Lemma 3.4.1, $\Upsilon(Y,\nu)$ and $\Upsilon(Y,\varphi_{\#}\mu)$ are equal as sets and isomorphic as modules.

It then suffices to show that $\varphi_{\#} : \Upsilon(X,\mu) \to \Upsilon(Y,\varphi_{\#}\mu)$ is a module isomorphism. To this end, we claim that its inverse is the map $\varphi_{\#}^{-1} : \Upsilon(Y,\varphi_{\#}\mu) \to \Upsilon(X,\mu)$. Indeed, $\varphi_{\#}^{-1}(\varphi_{\#}\mu) = \mu$ follows from definition. Letting $\delta \in \Upsilon(X,\mu)$, $h \in L^1(X,\mu)$, and $f \in \mathrm{Lip}_\infty(X)$ be arbitrary, the transformation formula (3.4.2) gives

$$\begin{aligned}\int_Y h\cdot\varphi_{\#}(\varphi_{\#}^{-1}\delta)f\,d\mu &= \int_X (h\circ\varphi)\cdot\varphi_{\#}^{-1}\delta\big(f\circ\varphi\big)\,d(\varphi_{\#}\mu)\\ &= \int_X (h\circ\varphi\circ\varphi^{-1})\cdot\delta(f\circ\varphi\circ\varphi^{-1})\,d(\varphi_{\#}^{-1}(\varphi_{\#}\mu))\\ &= \int_Y h\cdot\delta f\,d\mu.\end{aligned}$$

It follows that $\varphi_{\#}\circ\varphi_{\#}^{-1}$ is the identity map on $\Upsilon(X,\mu)$. A similar computation then shows that $\varphi_{\#}^{-1}\circ\varphi_{\#}$ is the identity on $\Upsilon(Y,\nu)$, which gives the theorem. □

3.5 Derivations on $\mathbb{R}^n$.

On Euclidean spaces with a prescribed Radon measure μ, derivations in $\Upsilon(\mathbb{R}^n,\mu)$ exhibit behavior that is similar to that of the partial differential operators $\{\partial_{x_i}\}_{i=1}^n$. For instance, they satisfy a weak form of the Chain Rule for derivatives.

To formulate this fact, first recall that by Theorem 3.2.8, every derivation δ in $\Upsilon(\mathbb{R}^n,\mu)$ extends to a linear operator $\bar{\delta} : \mathrm{Lip}_{loc}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n,\mu)$. So if P is a polynomial on $\mathbb{R}^n$, then the function $\bar{\delta}P$ is well-defined and lies in $L^\infty(\mathbb{R}^n,\mu)$. In addition, from formula (3.2.6) it follows that $\bar{\delta}P$ is determined uniquely by the function values of P. With this in mind, we write δP for $\bar{\delta}P$.

Proposition 3.5.1. *Let μ be a Radon measure on $\mathbb{R}^n$ and let $\delta \in \Upsilon(\mathbb{R}^n, \mu)$. For each $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$, there exists a μ-measurable map $g_f = (g_f^1, \ldots, g_f^n) : \mathbb{R}^n \to \mathbb{R}^n$ with the following properties:*

$$\|g_f^i\|_{\mu,\infty} \leq L(f), \text{ for } i = 1, 2, \ldots, n \tag{3.5.1}$$

$$\delta f = \sum_{i=1}^{n} g_f^i \cdot \delta x_i \ \ \mu\text{-a.e.} \tag{3.5.2}$$

In the case when f is smooth, we may take g_f to be the gradient of f.

Proof. Let $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$ be given. We argue by cases.

Case 1: f is a polynomial. If f is the coordinate function x_i, then f satisfies formula (3.5.2) μ-a.e. with $g_f = \vec{e}_i$. More generally, if f is a polynomial, then by the local Leibniz rule (Remark 3.2.9), f satisfies formula (3.5.2) μ-a.e. with $g_f = \nabla f$.

Case 2: f is smooth. By Part (2) of Lemma 2.3.1, there is a sequence of polynomials $\{P_m\}_{m=1}^{\infty}$ so that for any compact subset K of $\mathbb{R}^n$, we have $P_m \xrightharpoonup{*} f$ in $\mathrm{Lip}(K)$. So by the weak-$*$ continuity of δ, we obtain $\delta P_m \xrightharpoonup{*} \delta f$ in $L^\infty(K, \mu)$. On the other hand, the uniform convergence $\nabla P_m \to \nabla f$ on K also implies the convergence

$$\sum_{i=1}^{n} \frac{\partial P_m}{\partial x_i} \cdot \delta x_i \xrightharpoonup{*} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \delta x_i$$

in $L^\infty(K, \mu)$. This follows from the Dominated Convergence Theorem, because for any $\varphi \in L^1(\mathbb{R}^n, \mu)$ and for sufficiently large m, we have

$$\left| \varphi \cdot \delta x_i \cdot \frac{\partial P_m}{\partial x_i} \right| \leq |\varphi| \cdot |\delta x_i| \cdot \Big(\left| \frac{\partial f}{\partial x_i} \right| + 1 \Big).$$

μ-a.e. on K. From this it follows that, as $m \to \infty$,

$$\int_K \varphi \cdot \Big(\sum_{i=1}^{n} \frac{\partial P_m}{\partial x_i} \cdot \delta x_i \Big)\, d\mu \to \int_K \varphi \cdot \Big(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \delta x_i \Big)\, d\mu.$$

However, each P_m satisfies equation (3.5.2) μ-a.e. on K, so by uniqueness of weak-$*$ limits, f satisfies formula (3.5.2) μ-a.e. on K, with $g_f = \nabla f$. Since K was arbitrary, we may choose it within the collection of cubes

$$Q(a) \;:=\; [a_1, a_1+1] \times \ldots \times [a_n, a_n+1]$$

with indices $a = (a_1, \ldots, a_n)$ varying over the integer lattice $\mathbb{Z}^n$. Such cubes cover all of $\mathbb{R}^n$, so as a result, f satisfies equation (3.5.2) μ-a.e. on $\mathbb{R}^n$, with $g_f = \nabla f$.

Case 3: f is arbitrary. Let $L := L(f)$. By Part (1) of Lemma 2.3.1, there is a sequence of smooth, bounded L-Lipschitz functions $\{f_k\}_{k=1}^\infty$ so that $f_k \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(\mathbb{R}^n)$. By the continuity of δ, we obtain $\delta f_k \overset{*}{\rightharpoonup} \delta f$ in $L^\infty(\mathbb{R}^n, \mu)$.

Since the sequence $\{f_k\}_{k=1}^\infty$ is uniformly L-Lipschitz, for all $1 \leq i \leq n$, the sequence $\{\partial_i f_k\}_{k=1}^\infty$ are norm-bounded in $L^\infty(\mathbb{R}^n, \mu)$ with $\sup_k \|\partial_i f_k\|_{\mu,\infty} \leq L$. For $i = 1$, it follows from weak-$*$ compactness (Theorem 3.1.6) that there is a weak-$*$ convergent subsequence $\{\partial_1 f_{k_j}\}_{j=1}^\infty$ of $\{\partial_1 f_k\}_{k=1}^\infty$ in $L^\infty(\mathbb{R}^n, \mu)$.

Taking further subsequences if necessary, we may assume that for each $1 \leq i \leq n$, the same sequence $\{f_{k_j}\}_{j=1}^\infty$ gives a weak-$*$ convergent subsequence $\{\partial_i f_{k_j}\}_{j=1}^\infty$ in $L^\infty(\mathbb{R}^n, \mu)$ with weak-$*$ limit g_f^i. By lower semi-continuity of norms (Theorem 2.1.5) we have $\|g_f^i\|_{\mu,\infty} \leq L$. Arguing similarly as before, we see that

$$\sum_{i=1}^n \frac{\partial f_{k_j}}{\partial x_i} \cdot \delta x_i \;\overset{*}{\rightharpoonup}\; \sum_{i=1}^n g_f^i \cdot \delta x_i$$

in $L^\infty(\mathbb{R}^n, \mu)$. By uniqueness of weak-$*$ limits once more, formula (3.5.2) holds for each limit function g_f^i. This proves the Proposition. □

Remark 3.5.2. If μ is absolutely continuous to m_n, then by Rademacher's theorem, every Lipschitz function on $\mathbb{R}^n$ is μ-a.e. differentiable. Therefore equation (3.5.2) holds for every $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$, under the choice $g_f := \nabla f$.

This follows from the fact that $\partial_i f_j \overset{*}{\rightharpoonup} \partial_i f$ in $L^\infty(\mathbb{R}^n, \mu)$ holds whenever $f_j \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(\mathbb{R}^n)$; we will show this in the proof of Corollary 3.5.4. As a result, there is no need to appeal to weak-$*$ compactness in the proof of Proposition 3.5.1.

However, Lipschitz functions on $\mathbb{R}^n$ need not be differentiable a.e. with respect to an arbitrary Radon measure μ, so it is unreasonable to expect $g_f = \nabla f$ in general. In fact, there are examples where this is untrue. It has been shown in [PT95] that for each Lebesgue null set N in the real line, there exists a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at no point of N.

The following three facts are consequences of Proposition 3.5.1. The first fact follows directly from equation (3.5.2) and we omit the proof. The second fact generalizes Example 3.1.8, and the argument is similar to [Wea00, Sect 5B]. The third fact is a technical tool for the proof of Theorem 5.3.1, in which we show the convergence of sequence of derivations to a limit derivation.

Corollary 3.5.3. *Let μ be a Radon measure on $\mathbb{R}^n$ and let $\delta \in \Upsilon(\mathbb{R}^n, \mu)$. If $\delta x_i = 0$ holds for every $1 \le i \le n$, then δ is the zero derivation.*

Corollary 3.5.4. *If μ is a Radon measure on $\mathbb{R}^n$ with $\mu \ll m_n$, then the partial differential operators $\{\partial_{x_i}\}_{i=1}^n$ form a generating set for $\Upsilon(\mathbb{R}^n, \mu)$. In addition, we have the following isomorphism of $L^\infty(\mathbb{R}^n, \mu)$-modules:*

$$\Upsilon(\mathbb{R}^n, \mu) \;\cong\; \bigoplus_{i=1}^n L^\infty(\mathbb{R}^n, \mu).$$

Proof. Since $\mu \ll m_n$, then by Rademacher's theorem, each function $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$ is μ-a.e. differentiable. It follows that the operators $\partial_{x_i} : \mathrm{Lip}_\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n, \mu)$ are well-defined. Clearly, each is linear, norm-bounded, and satisfies the Leibniz rule.

Let w be the Radon-Nikodym derivative of μ with respect to m_n, so $d\mu = w dm_n$. To show continuity, suppose that $\{f_j\}_{j\in I}$ is a norm-bounded net in $\mathrm{Lip}_\infty(\mathbb{R}^n)$ that converges weak-$*$ to f. Without loss of generality, we may assume that f is zero, otherwise we would study the net $\{f_j - f\}_{j\in I}$ instead.

Given $h \in L^1(\mathbb{R}^n, \mu)$, observe again that the product $h \cdot w$ lies in $L^1(\mathbb{R}^n, m_n)$. Letting $\epsilon > 0$ be given, there is a $R > 0$ so that

$$\int_{\mathbb{R}^n \setminus B(0,R)} |h| \, d\mu \; < \; \frac{\epsilon}{8C}.$$

Letting η_ϵ denote a smooth, symmetric mollifier on $\mathbb{R}^n$ (see the proof of Lemma 2.3.1), there is a $N \in \mathbb{N}$ so that the convolution $h_N := (\chi_{B(0,R)} \cdot h \cdot w) * \eta_{1/N}$ satisfies

$$\int_{B(0,R)} |h_N - h \cdot w| \, dm_n \; \le \; \frac{\epsilon}{8C}.$$

Combining the estimates above, we obtain

$$\int_{\mathbb{R}^n} |h_N - h \cdot w| \, dm_n \; \le \; \int_{B(0,R)} |h_N - h \cdot w| \, dm_n \; + \; \int_{\mathbb{R}^n \setminus B(0,R)} |h| \, d\mu \; \le \; \frac{\epsilon}{4C}.$$

The net $\{f_j\}_{j\in I}$ is norm-bounded and converges pointwise to zero, so by the Bounded Convergence Theorem, the net $\{\partial_i h_N \cdot f_j\}_{j\in I}$ converges to zero in $L^1(\mathbb{R}^n, m_n)$-norm. In particular, for sufficiently large indices j we have

$$\int_{\mathbb{R}^n} |\partial_i h_N \cdot f_j| \, dm_n \; < \; \frac{\epsilon}{2}.$$

Combining the above estimates, we integrate by parts and further estimate

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} h \cdot w \cdot \partial_i f_j \, dm_n \right| \;&=\; \left| \int_{\mathbb{R}^n} \Big((h \cdot w - h_N) \; + \; h_N \Big) \cdot \partial_i f_j \, dm_n \right| \\
&=\; \left| \int_{\mathbb{R}^n} \Big((h \cdot w - h_N) \cdot \partial_i f_j \; - \; \partial_{x_i} h_N \cdot f_j \Big) \, dm_n \right| \\
&\le\; \|\partial_i f_j\|_\infty \cdot \int_{\mathbb{R}^n} |h \cdot w - h_N| \, dm_n \; + \; \int_{\mathbb{R}^n} |\partial_i h_N \cdot f_j| \, dm_n \\
&<\; 2C \cdot \frac{\epsilon}{4C} + \frac{\epsilon}{2} \; = \; \epsilon.
\end{aligned}$$

This shows the continuity of each operator ∂_i.

To see that $\{\partial_i\}_{i=1}^n$ is a generating set, let $\delta \in \Upsilon(\mathbb{R}^n, \mu)$ be arbitrary. We observe that the derivation $\delta' := \delta - \sum_{i=1}^{\infty} \delta x_i \, \partial_i$ satisfies $\delta' x_i = 0$, for each $1 \leq i \leq n$. By Corollary 3.5.3, δ' must be identically zero.

Lastly, put $M := \bigoplus_{i=1}^n L^\infty(\mathbb{R}^n, \mu)$. Consider the linear maps $T : \Upsilon(\mathbb{R}^n, \mu) \to M$ and $S : M \to \Upsilon(\mathbb{R}^n, \mu)$ given by the formulas

$$\begin{aligned} T(\delta) &:= (\delta x_1, \ldots, \delta x_n), \\ S(\lambda_1, \ldots, \lambda_n) &:= \sum_{i=1}^n \lambda_i \partial_i. \end{aligned}$$

Clearly, $T \circ S = \mathrm{id}_M$, and $S \circ T = \mathrm{id}_{\Upsilon(\mathbb{R}^n,\mu)}$ follows from the observation that $\delta - \sum_{i=1}^n \delta x_i \, \partial_i$ is the zero derivation. This proves the corollary. □

Corollary 3.5.5. *Let μ be a Radon measure on $\mathbb{R}^n$, let K be a compact subset of $\mathbb{R}^n$, and let $p, q \in (1, \infty)$ satisfy $p^{-1} + q^{-1} = 1$. If $\mathcal{F}$ is a uniformly Lipschitz family in* $\mathrm{Lip}(K)$*, then there is a constant $C = C(\mathcal{F}, n) > 0$ so that for all $f \in \mathcal{F}$, for all $h \in L^q(K, \mu)$, and for all $\delta \in \Upsilon(K, \mu)$,*

$$\left| \int_K h \cdot \delta f \, d\mu \right| \leq C \cdot \|h\|_q \cdot \left(\max_{1 \leq k \leq n} \|\delta x_k\|_p \right). \tag{3.5.3}$$

Proof. Choose $L > 0$ so that $L(f) \leq L$ holds, for all $f \in \mathcal{F}$. By Proposition 3.5.1, there are functions $\{g_f^k\}_{k=1}^n$ in $L^\infty(K, \mu)$ so that formula (3.5.2) holds μ-a.e. on K. Hölder's inequality then gives

$$\left| \int_{\mathbb{R}^n} h \cdot \delta f \, d\mu \right| \leq \|h\|_q \cdot \sum_{k=1}^n \|g_f^k\|_{\mu,\infty} \cdot \|\delta x_k\|_p \leq n \cdot L \cdot \|h\|_q \cdot \left(\max_{1 \leq k \leq n} \|\delta x_k\|_p \right).$$

This proves the corollary. □

We close this section by stating several corollaries about the module structure of derivations on $\mathbb{R}^n$. These follow from facts in Sections 2.3 and 3.3.

Corollary 3.5.6. *Let μ be a nonzero Radon measure on $\mathbb{R}^n$. Then any set of $n+1$ derivations in $\Upsilon(\mathbb{R}^n, \mu)$ is linearly dependent.*

Proof. Since μ is a nonzero measure, there is a number $R > 0$ so that $\mu(B(0, R)) > 0$. For convenience, we write $B = B(0, R)$.

Let $\{\delta_i\}_{i=1}^{n+1}$ be an arbitrary subset of $\Upsilon(\mathbb{R}^n, \mu)$ and consider the corresponding subset $\{\chi_B \delta_i\}_{i=1}^{n+1}$ in $\Upsilon(B, \mu)$. By Part (2) of Theorem 2.3.1, $\{x_i\}_{i=1}^{n}$ is a generating set for $\mathrm{Lip}_\infty(B)$, so by Theorem 3.3.11, the set $\{\chi_B \delta_i\}_{i=1}^{n+1}$ is linearly dependent in $\Upsilon(B, \mu)$. So by the contrapositive of Part (2) of Lemma 3.3.6, $\{\delta_i\}_{i=1}^{n+1}$ is a linearly dependent set in $\Upsilon(\mathbb{R}^n, \mu)$. □

The proof of the next corollary is similar to the previous proof. After reducing to the case of bounded sets, one invokes Theorem 3.3.12 in place of Theorem 3.3.11.

Corollary 3.5.7. *Let μ be a Radon measure on $\mathbb{R}^n$, and suppose that $\{\delta_i\}_{i=1}^{n}$ is a linearly independent set in $\Upsilon(\mathbb{R}^n, \mu)$. For any $\epsilon > 0$, there is a subset X_ϵ of $\mathbb{R}^n$ so that $\mu(\mathbb{R}^n \setminus X_\epsilon) < \epsilon$ and that $\Upsilon(X_\epsilon, \mu)$ is generated by n derivations.*

CHAPTER IV

Structure of Derivations on 1-Dimensional Spaces

Adapting the terminology of [Fal86], we say that a subset A in $\mathbb{R}^n$ is a *k-set* if A is $\mathcal{H}^k$-measurable set of σ-finite $\mathcal{H}^k$-measure. We now characterize measures on 1-sets that admit nonzero modules of derivations.

Theorem 4.0.8. *Let μ be a Radon measure on $\mathbb{R}^n$, let A be a 1-set, and suppose that μ is concentrated on A. If $\mu_{\mathcal{H}}$ is the absolutely continuous part of μ with respect to $\mathcal{H}^1\lfloor A$, then the modules $\Upsilon(\mathbb{R}^n, \mu)$ and $\Upsilon(\mathbb{R}^n, \mu_{\mathcal{H}})$ are isomorphic.*

Recall that Theorem 3.3.12 and Corollary 3.5.7 were formulated from the perspective of abstract metric measure spaces, and their proofs relied on measure-theoretic and linear algebraic techniques. In contrast, the setting of Euclidean spaces is more concrete and we can employ techniques which are more geometric in nature. This includes the notion of *rectifiability* from geometric measure theory.

Here and in later sections, we will tacitly invoke Theorem 3.2.8. So if $\pi \in \mathrm{Lip}(\mathbb{R}^n)$ and if $\delta \in \Upsilon(\mathbb{R}^n, \mu)$, then as before, $\delta\pi$ is a well-defined function in $L^\infty(\mathbb{R}^n, \mu)$.

4.1 Derivations on $\mathbb{R}$.

We begin with some terminology. Every Radon measure μ on $\mathbb{R}^n$ is σ-finite, so we may apply formula (1.4.1) to obtain $\mu = \mu_S + \mu_{AC}$, where μ_S is the Lebesgue singular part of μ and μ_{AC} is the Lebesgue absolutely continuous part of μ.

There is a simple characterization of measures μ on $\mathbb{R}$ for which $\Upsilon(\mathbb{R},\mu)$ is nontrivial. The proof follows closely the argument in [AK00a, pp. 15–16].

Theorem 4.1.1. *Let μ be a Radon measure on $\mathbb{R}$, let E be a m_1-null set in $\mathbb{R}$, and suppose that μ_S is concentrated on E. Then for all $\delta \in \Upsilon(\mathbb{R},\mu)$ and all $f \in \mathrm{Lip}_\infty(\mathbb{R})$, we have*

$$\delta f(x) \;=\; \begin{cases} \delta(\mathrm{id})(x)\cdot f'(x), & x \in \mathbb{R}\setminus E \\ 0, & x \in E, \end{cases} \tag{4.1.1}$$

where $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ *is the identity map and $f'(x)$ is the derivative of f at x.*

By definition we have $\mu_{AC} \ll \mu$, so by Lemma 3.4.4 we see that $\Upsilon(\mathbb{R},\mu)$ is a submodule of $\Upsilon(\mathbb{R},\mu_{AC})$. As a consequence of the theorem, it follows that $\Upsilon(\mathbb{R},\mu)$ and $\Upsilon(\mathbb{R},\mu_{AC})$ are isomorphic as $L^\infty(\mathbb{R},\mu)$-modules. For $n = 1$, it is a sharper version of Corollary 3.5.7.

Proof of Theorem 4.1.1. Let $\delta \in \Upsilon(\mathbb{R},\mu)$ be arbitrary. Observe that the collection of sets $E_k := E \cap (k, k+1]$, $k \in \mathbb{Z}$, form a measurable decomposition of E. So to show that $\chi_E\delta$ is the zero derivation, it suffices to show that $\chi_{E_k}\delta$ is zero for each k.

Fix $k \in \mathbb{N}$. Since μ is Borel regular, for each $\epsilon > 0$ there is a countable collection of disjoint open intervals $\{O_j\}_{j=1}^\infty$ so that their union $O := \bigcup_{j=1}^\infty O_j$ contains E_k and so that $m_1(O) < \epsilon$. Now consider the functions

$$g_\epsilon(x) \;:=\; \int_k^x \chi_{\mathbb{R}\setminus O}(t)\,dt.$$

As an indefinite integral of a characteristic function, each g_ϵ is 1-Lipschitz. Moreover, on $(k, k+1]$ the functions g_ϵ converge uniformly to $g(x) := x - k$ as $\epsilon \to 0$, because

$$|g(x) - g_\epsilon(x)| \;=\; \left|\int_k^x 1\,dt - \int_k^x \chi_{\mathbb{R}\setminus O}(t)\,dt\right| \;=\; \int_k^\infty \chi_O(t)\,dt \;\le\; m_1(O) \;<\; \epsilon.$$

As $\epsilon \to 0$, the convergence $g_\epsilon \overset{*}{\rightharpoonup} g$ in $\mathrm{Lip}(E_k)$ follows from Part (2) of Corollary 2.1.4. So by continuity of δ, we also have $\delta g_\epsilon \overset{*}{\rightharpoonup} \delta g$ in $L^\infty(E_k, \mu)$, for all $\delta \in \Upsilon(\mathbb{R}, \mu)$.

On the other hand, note that g_ϵ is constant on each interval O_j. By the locality property (Theorem 3.2.1) we have $\chi_{O_j} \cdot \delta g_\epsilon = 0$ for each j, and hence $\chi_{E_k} \cdot \delta g_\epsilon = 0$. From weak-$*$ continuity it follows that $\chi_{E_k} \cdot \delta g = 0$, and because the derivation δ applied to a constant function is zero, it also follows that $\chi_{E_k} \cdot \delta(\mathrm{id}) = 0$. So by Corollary 3.5.3, $\chi_{E_k}\delta$ must be the zero derivation in $\Upsilon(E_k, \mu)$. However, $k \in \mathbb{Z}$ was arbitrary, so $\chi_E \delta$ must also be the zero derivation in $\Upsilon(\mathbb{R}, \mu)$.

By the previous argument, we have $\delta = \chi_{\mathbb{R}\setminus E}\delta$, so δ determines a derivation in $\Upsilon(\mathbb{R}, \mu_{AC})$ which we will also call δ. From $\mu_{AC} \ll m_1$ and from Rademacher's theorem, it follows that every Lipschitz function is differentiable μ_{AC}-a.e. Now put

$$\delta' \;:=\; \delta \;-\; \delta(\mathrm{id}) \cdot \Big(\chi_{\mathbb{R}\setminus E} \frac{d}{dx}\Big).$$

By inspection we have $\delta'(\mathrm{id}) = 0$, so by Corollary 3.5.3 we also have $\delta' = 0$. This gives formula (4.1.1). □

4.2 Preliminaries: Geometric Measure Theory.

Following [Mat95, Defn 15.3], we now introduce the notions of k-rectifiable sets[1] and purely k-unrectifiable sets in $\mathbb{R}^n$.

Definition 4.2.1. Let $k \in \mathbb{N}$. A $\mathcal{H}^k$-measurable subset E of $\mathbb{R}^n$ is *k-rectifiable* if

$$E \;=\; N \cup \Big(\bigcup_{i=1}^{\infty} f_i(A_i) \Big), \tag{4.2.1}$$

holds, where N is a $\mathcal{H}^k$-null set and for each $i \in \mathbb{N}$, A_i is a subset of $\mathbb{R}^k$ with $m_k(A_i) > 0$ and $f_i : A_i \to \mathbb{R}^n$ is a Lipschitz map. A $\mathcal{H}^k$-measurable subset F of $\mathbb{R}^n$ is *purely k-unrectifiable* if $\mathcal{H}^k(F \cap E) = 0$ holds for all k-rectifiable subsets E of $\mathbb{R}^n$.

[1] The terminology here differs from that in [Fed69, Sect 3.2.14]; such sets are also called *countable $(\mathcal{H}^k, k)$-rectifiable sets*. A similar difference in terminology occurs for purely k-unrectifiable sets.

Next we list several properties of k-rectifiable sets. The first result is a structure theorem for k-sets on $\mathbb{R}^n$ [Mat95, Thm 15.6]. The second result states that for a k-rectifiable set E, the regularity of the images $f_i(A_i)$ from equation (4.2.1) can be substantially improved [Fed69, Lem 3.2.18 & Thm 3.2.29].

Theorem 4.2.2. *Let $k \in \mathbb{N}$ and let A be a k-set. Then $A = E \cup F$, where E is a k-rectifiable subset of $\mathbb{R}^n$ and F is a purely k-unrectifiable subset of $\mathbb{R}^n$.*

Theorem 4.2.3 (Federer). *The following are equivalent.*

1. *E is a k-rectifiable subset of $\mathbb{R}^n$.*
2. *There exists a collection of C^1-smooth k-submanifolds $\{M_i\}_{i=1}^{\infty}$ in $\mathbb{R}^n$ so that*

$$\mathcal{H}^k(E \setminus (\bigcup_{i=1}^{\infty} M_i)) \;=\; 0.$$

3. *For all $L > 1$, there exists a collection of compact subsets $\{K_i\}_{i=1}^{\infty}$ of $\mathbb{R}^k$ and countably many L-bi-Lipschitz maps $\varphi_i : \mathbb{R}^k \to \mathbb{R}^n$ so that $\{\varphi_i(K_i)\}_{i=1}^{\infty}$ is a pairwise-disjoint collection of subsets of E and so that*

$$\mathcal{H}^k\big(E \setminus \bigcup_{i=1}^{\infty} \varphi_i(K_i)\big) \;=\; 0. \tag{4.2.2}$$

Example 4.2.4. By Property (2) of Theorem 4.2.3, every smooth k-dimensional sub-manifold M of $\mathbb{R}^n$ is a k-rectifiable set.

Similarly to smooth manifolds, every k-rectifiable set E in $\mathbb{R}^n$ admits a type of differentiable structure. As in Property (3) of Theorem 4.2.3, assume that E can be written in the form of equation (4.2.2). For $x \in \varphi_i(K_i)$, the *approximate tangent space of E at x* [Fed69, Thm 3.2.19] is defined to be the k-dimensional vector space

$$\mathrm{Tan}^k(E, x) \;:=\; \operatorname*{span}_{\mathbb{R}}\big\{D\varphi_i\big(\varphi_i^{-1}(x)\big) \cdot \vec{e}_j\big\}_{j=1}^{k}.$$

Since $\mathcal{H}^k(N) = 0$, we see that the collection $\{\varphi_i(K_i)\}_{i=1}^\infty$ forms a $\mathcal{H}^k$-measurable decomposition of E. As a result, the space $\operatorname{Tan}^k(E, x)$ is well-defined for $\mathcal{H}^k$-a.e. $x \in E$. With this ambiguity understood, we now define the *approximate tangent bundle* of a k-rectifiable set E to be the set of pairs

$$\operatorname{Tan}^k(E) \ := \ \{(x, v) \, : \, x \in E, \, v \in \operatorname{Tan}^k(E, x)\}.$$

There is a natural projection map $p : \operatorname{Tan}^k(E) \to E$ by the formula $p(x, v) = x$.

As in Riemannian geometry, there is a natural inner product on approximate tangent spaces. Indeed, for $\mathcal{H}^k$-a.e. $x \in \varphi_i(K_i)$ and for all $v_1, v_2 \in \operatorname{Tan}^k(E, x)$, put

$$\langle v_1, v_2 \rangle_x \ := \ \langle \vec{u}_1, \vec{u}_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$ and where, for $i = 1, 2$, we have

$$v_i \ := \ D\varphi_i\big(\varphi_i^{-1}(x)\big) \cdot \vec{u}_i.$$

From this, we obtain a norm on $\operatorname{Tan}^k(E, x)$ by the formula $\|v\|_x := \sqrt{\langle v, v \rangle_x}$. Since each map φ_i is bi-Lipschitz, it follows that the norm $\| \cdot \|_x$ is comparable to the usual Euclidean norm on $\mathbb{R}^k$.

Definition 4.2.5. A *section* of the approximate tangent bundle $\operatorname{Tan}^k(E)$ is a map $s : E \to \operatorname{Tan}^k(E)$ which satisfies $s \circ p = \mathrm{id}_E$.

If μ is a measure on $\mathbb{R}^n$, then a *μ-measurable section* $s : E \to \operatorname{Tan}^k(E)$ is a section of $\operatorname{Tan}^k(E)$ which is also a μ-measurable map.

A *bounded μ-measurable section* $s : E \to \operatorname{Tan}^k(E)$ is a μ-measurable section of $\operatorname{Tan}^k(E)$ with the following property: there is a constant $C \geq 0$ so that $\|s(x)\|_x \leq C$ holds for μ-a.e. $x \in E$.

Just as derivatives are maps between tangent spaces, there is also the notion of an approximate differential of a function. It is a map between approximate tangent spaces and often it is defined in terms of *approximate limits* [EG92, Sect 1.7.2]. Here we give an equivalent definition on k-rectifiable sets [Fed69, Thm 3.2.19] by using the additional structure of Theorem 4.2.3.

Following [Fed69, Sect 3.1.22], let $f : \mathbb{R}^m \to \mathbb{R}^n$ be continuous and let $S \subset \mathbb{R}^m$. Given $a \in \bar{S}$, we say that *f is differentiable relative to S at a* if and only if there exists a linear map $\zeta_a : \mathbb{R}^m \to \mathbb{R}^n$ so that

$$\lim_{n\to\infty} \frac{|f(x_n) - f(a) - \zeta_a(x_n - a)|}{|x_n - a|} \;=\; 0$$

holds for all sequences $\{x_n\}_{n=1}^{\infty}$ in S which converge to a. If it exists, then we write $D[f|S](a) := \zeta_a$. The next result follows from [Fed69, Lem 3.2.17]; if E is a k-rectifiable set, then differentiation relative to E is well-defined for Lipschitz functions.

Lemma 4.2.6 (Federer). *Let $K \subset \mathbb{R}^k$ with $m_k(K) > 0$ and let $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ be a bi-Lipschitz embedding. If $f : \varphi(K) \to \mathbb{R}^n$ is a Lipschitz map, then for $\mathcal{H}^k$-a.e. $a \in \varphi(K)$, $D[f|\varphi(K)](a)$ exists and satisfies the identity*

$$D[f|\varphi(K)](a) \circ D\varphi(\varphi^{-1}(a)) \;=\; D[f \circ \varphi](\varphi^{-1}(a)). \tag{4.2.3}$$

Remark 4.2.7 (Uniqueness). If $D[f|\varphi(K)](a)$ exists, then it is uniquely determined up to a $\mathcal{H}^k$-null set. Indeed, since φ is bi-Lipschitz, if φ is differentiable at z then $D\varphi(z)$ is invertible. As a result, equation (4.2.3) can be rewritten as

$$D[f|\varphi(K)](a) \;=\; D[f \circ \varphi](\varphi^{-1}(a))[\circ D\varphi(\varphi^{-1}(a))]^{-1}.$$

Since the right-hand side of the above equation is defined $\mathcal{H}^k$-a.e. on $\varphi(K)$, then so is the left-hand side.

Definition 4.2.8. Let E be a k-rectifiable subset of $\mathbb{R}^n$ and as in Theorem 4.2.3, let $E = N \cup (\bigcup_i \varphi_i(K_i))$. If $x \in \varphi_i(K_i)$, then for each $f \in \mathrm{Lip}(E)$ the *approximate differential of f at x* is the linear map $D^A f(x) := D[f|\varphi_i(K_i)](x)$.

In [Wea00, Thm 38], Weaver has verified that for k-rectifiable sets E in $\mathbb{R}^n$, the module $\Upsilon(E, \mathcal{H}^k)$ is isomorphic to the module of $\mathcal{H}^k$-essentially bounded sections of $\mathrm{Tan}^k(E)$. Moreover, his proof shows that for every $\delta \in \Upsilon(E, \mathcal{H}^k)$, there is a section $v : E \to \mathrm{Tan}^k(E)$ so that $D^A f \cdot v = \delta f$ holds for all $f \in \mathrm{Lip}_\infty(E)$.

Recall that by Corollary 3.5.4, for each $1 \le i \le n$, the partial differential operator ∂_i is a derivation in $\Upsilon(\mathbb{R}^n, \mu)$, whenever $\mu \ll m_n$. So if $u = (u_1, \ldots, u_n)$ is a vector in $\mathbb{R}^n$, then the differential operator in the direction of u, i.e.

$$D_u := \sum_{i=1}^{n} u_i \partial_i \tag{4.2.4}$$

is also a derivation in $\Upsilon(\mathbb{R}^n, \mu)$. With these facts in mind, the next lemma relates directional differentiation on $\mathbb{R}^k$ and approximate differentiation on k-rectifiable sets in terms of pushforward derivations.

Lemma 4.2.9. *Let $K \subset \mathbb{R}^k$ with $m_k(K) > 0$, let $\varphi : K \to \mathbb{R}^n$ be a bi-Lipschitz embedding, and let $u \in \mathbb{R}^k$. Then for all $f \in \mathrm{Lip}_\infty(\varphi(K))$, we have*

$$D^A f \cdot (\varphi_\# u) = \big(\varphi_\# D_u\big) f, \tag{4.2.5}$$

where $\varphi_\# u(x) := D\varphi\big(\varphi^{-1}(x)\big) \cdot u$ is an approximate tangent vector in $\mathrm{Tan}^1(E, x)$, and where D_u is the map given in formula (4.2.4).

Proof. Let $E = \varphi(K)$ and let $\mu := \varphi^{-1}_\# \mathcal{H}^k$. For all functions $h \in L^1(E, \mathcal{H}^k)$ and all $f \in \mathrm{Lip}_\infty(E)$, we invoke formulas (3.4.2) and (4.2.3) in order to obtain

$$\begin{aligned}
\int_E h(x) \cdot \big(\varphi_\# D_u\big) f(x)\, d\mathcal{H}^k(x) &= \int_K (h \circ \varphi)(y) \cdot D(f \circ \varphi)(y) \cdot u\, d\mu(y) \\
&= \int_K (h \circ \varphi)(y) \cdot D^A f\big(\varphi(y)\big) \cdot \big(D\varphi(y) \cdot u\big)\, d\mu(y) \\
&= \int_E h(x) \cdot D^A f(x) \cdot \big(D\varphi(\varphi^{-1}(x)) \cdot u\big)\, d\mathcal{H}^k(x).
\end{aligned}$$

The lemma follows. □

Lastly, we also require a characterization of purely k-unrectifiable subsets of $\mathbb{R}^n$. The following theorem [Mat95, Thm 18.1] gives one in terms of orthogonal projections of linear subspaces of $\mathbb{R}^n$.

Below, "almost every subspace" is to be understood in terms of Haar measure on $\mathcal{G}(n,k)$, the space of k-dimensional subspaces of $\mathbb{R}^n$; see [Mat95, Sect 3.9]. For our purposes here, only the case $k = 1$ is relevant. By identifying each 1-dimensional subspace with a pair of antipodal points on the sphere $\mathbb{S}^{n-1}$, the Haar measure on $\mathcal{G}(n,1)$ then reduces to the (normalized) surface measure $\mathcal{H}^{n-1}\lfloor\mathbb{S}^{n-1}$.

Theorem 4.2.10 (Besicovitch, Federer). *Let F be a k-set. Then F is purely k-unrectifiable if and only if for almost every subspace $V \in \mathcal{G}(n,k)$, the orthogonal projection of F onto V has $\mathcal{H}^k$-measure zero.*

4.3 Derivations on 1-Sets.

Using theorems from the previous section, every 1-set admits a decomposition into two parts: a set which projects to a null set in a.e. direction, and a union of bi-Lipschitz images of compacta from $\mathbb{R}$. So to prove Theorem 4.0.8, it then suffices to check derivations on each subset separately. The next result shows that sets of the first kind cannot support any nonzero derivations.

Lemma 4.3.1. *Let μ be a Radon measure on $\mathbb{R}^n$. Suppose that F is a 1-set and that μ is concentrated on F. If F is purely 1-unrectifiable, then $\Upsilon(\mathbb{R}^n,\mu) = 0$.*

Proof. We argue by contradiction, so suppose δ is a nonzero derivation in $\Upsilon(\mathbb{R}^n,\mu)$. As a first case, assume that F is a bounded subset of $\mathbb{R}^n$.

By Theorem 4.2.10, there is a collection of 1-dimensional subspaces $\{V_i\}_{i=1}^n$ of $\mathbb{R}^n$ whose linear span is $\mathbb{R}^n$ and whose associated orthogonal projections $\pi^i : \mathbb{R}^n \to V_i$ satisfy $\mathcal{H}^1(\pi^i(F)) = 0$, for each $1 \le i \le n$. This means that the $\mathbb{R}$-linear span of the functions $\{\pi_i\}_{i=1}^n$ agrees with the $\mathbb{R}$-linear span of $\{x_i\}_{i=1}^n$, so the set $\{\pi^i\}_{i=1}^n$ must generate the coordinate functions $\{x_i\}_{i=1}^n$ on $\mathbb{R}^n$.

By hypothesis, δ is nonzero. By Corollary 3.5.3 one of the functions $\{\delta x_i\}_{i=1}^n$ is nonzero, and therefore one of the functions $\{\delta\pi^i\}_{i=1}^n$ is also nonzero. Suppose that $\delta\pi^i$ is such a function, and consider the sets $F_i^- := \{x \in F : \delta\pi^i(x) < 0\}$, $F_i^+ := \{x \in F : \delta\pi^i(x) > 0\}$, and $F_i := F_i^+ \cup F_u^+$. As a result, the derivation

$$\delta' \;:=\; (\chi_{F_i^+} - \chi_{F_i^-})\delta$$

satisfies $\delta'\pi^i > 0$ for μ-a.e. point in F_i. By Lemma 3.4.1, there is a unique derivation $\pi^i_\#\delta'$ in $\Upsilon(\mathbb{R}, \pi^i_\#\mu)$ which satisfies the transformation formula (3.4.2). From $\mathcal{H}^1(p^i(F)) = 0$ and from Lemma 2.1.1, we have $m_1(\pi^i(F)) = 0$. As constructed, the measure $\pi^i_\#\mu$ is concentrated on $\pi^i(F)$, so by Theorem 4.1.1 we also have $\pi^i_\#\delta' = 0$.

Let I be any bounded interval in $\mathbb{R}$. Let I_i be its preimage under π^i, which is an unbounded subset of $\mathbb{R}^n$. By hypothesis, F is bounded and μ is Radon and concentrated on F. So from formula (3.4.1) we obtain

$$\int_I 1\, d(\pi^i_\#\mu) \;=\; \int_{I_i} 1\, d\mu \;=\; \int_{I_i\cap F} 1\, d\mu \;=\; \mu(F\cap I_i) \;<\; \infty.$$

It follows that $1 \in L^1(\mathbb{R}, \pi^i_\#\mu)$.

Since $\pi^i_\#\delta'$ is the zero derivation in $\Upsilon(\mathbb{R}, \pi^i_\#\mu)$, we see that $\chi_I(\pi_\#\delta')$ is the zero derivation in $\Upsilon(I, \pi^i_\#\mu)$. Using formula (3.4.2) with $h = \chi_I$ and $f = \mathrm{id}_{\mathbb{R}}|I$, we have

$$\begin{aligned} 0 \;=\; \int_I 1\cdot(\pi^i_\#\delta')f\, d(\pi^i_\#\mu) \;&=\; \int_{F\cap I_i} 1\cdot\delta'(f\circ\pi^i)\, d\mu \\ &=\; \int_{F_i\cap I_i} \delta'\pi^i\, d\mu \\ &=\; \int_{F_i^+\cap I_i} \delta\pi^i\, d\mu + \int_{F_i^-\cap I_i} (-\delta\pi^i)\, d\mu \;>\; 0. \end{aligned}$$

This is a contradiction, so we must have $\delta = 0$.

In the case when F is unbounded, let $\{A_k\}_{k=1}^{\infty}$ be a μ-measurable decomposition of bounded subsets of $\mathbb{R}^n$. By the previous case, the derivation $\chi_{A_k \cap F}\delta$ is zero for each $k \in \mathbb{N}$, from which it follows that $\delta = 0$. This proves the lemma. $\square$

Corollary 4.3.2. *Let μ be a measure on $\mathbb{R}^n$, and suppose it is concentrated on a set of Hausdorff dimension less than one. Then $\Upsilon(\mathbb{R}^n, \mu) = 0$.*

Proof. Let A be a subset of $\mathbb{R}^n$ on which μ is concentrated. If $\dim_{\mathcal{H}}(A) < 1$, then $\mathcal{H}^1(A) = 0$, so A is purely 1-unrectifiable. By Lemma 4.3.1, $\Upsilon(\mathbb{R}^n, \mu) = 0$. $\square$

It remains to consider the case of 1-rectifiable subsets of $\mathbb{R}^n$. The next lemma characterizes measures μ on 1-rectifiable sets that admit nonzero modules of derivations. The idea is to pass to subsets of $\mathbb{R}$ by taking pushforward derivations. Using the "pullback data" from the structure of derivations on $\mathbb{R}$, one then constructs an explicit generator for $\Upsilon(\mathbb{R}^n, \mu)$.

Lemma 4.3.3. *Let μ be a Radon measure on $\mathbb{R}^n$, let E be a 1-rectifiable subset in $\mathbb{R}^n$, and suppose that μ is concentrated on E.*

1. *If μ is singular to $\mathcal{H}^1\lfloor E$, then the module $\Upsilon(\mathbb{R}^n, \mu)$ is zero.*

2. *If $\mu_{\mathcal{H}}$ is the absolutely continuous part of μ with respect to $\mathcal{H}^1\lfloor E$, then $\Upsilon(\mathbb{R}^n, \mu)$ is isomorphic to the $L^\infty(\mathbb{R}^n, \mu)$-module of bounded $\mu_{\mathcal{H}}$-measurable sections of* $\mathrm{Tan}^1(E)$.

Proof. By Part (3) of Theorem 4.2.3, E is the union of a $\mathcal{H}^1$-null set N and a further union of pairwise-disjoint image sets $\varphi_i(K_i)$, $i \in \mathbb{N}$, where each map $\varphi_i : \mathbb{R} \to \mathbb{R}^n$ is C-bi-Lipschitz, for some $C \geq 1$. Let μ_S be the Lebesgue singular part of μ, let E' be an $\mathcal{H}^1$-null set on which μ_S is concentrated, and put $E'' := E \setminus E'$.

Recall once more that a $\mathcal{H}^1$-null set is a purely 1-unrectifiable set. So by Theorem 4.3.1, we have $\chi_N \delta = 0$ for every $\delta \in \Upsilon(E, \mu)$. In particular, if $\mu = \mu_S$, then μ is concentrated on E' and hence $\chi_{E'}\delta = 0$. From the locality property (Theorem 3.2.1), it follows that $\delta = 0$. This proves Part (1).

To prove Part (2), we first observe the following fact.

Claim 4.3.4. Every $L^\infty(E,\mu)$-section $v : E \to \mathrm{Tan}^1(E)$ determines a derivation $\delta_v \in \Upsilon(\mathbb{R}^n,\mu)$ by the rule $\delta_v f := \chi_{E''} \cdot (D^A f \cdot v)$.

To prove the claim, for $i \in \mathbb{N}$, suppose that $m_1(K_1) > 0$. By definition we have $\mu\lfloor E'' = \mu_{\mathcal{H}}$, and therefore $(\varphi_i^{-1})_\#(\mu\lfloor E'')$ is a measure on $\mathbb{R}$ which is absolutely continuous to m_1. By Rademacher's Theorem, $D\varphi_i(y)$ exists for $(\varphi_i^{-1})_\#(\mu\lfloor E'')$-a.e. $y \in K_i$, and hence $D\varphi_i(\varphi_i^{-1}(x))$ exists for μ-a.e. $x \in \varphi(K_i) \cap E''$.

Since v is a section of $\mathrm{Tan}^1(E)$, there is a function $\lambda \in L^\infty(\mathbb{R}^n,\mu)$ so that for each $i \in \mathbb{N}$ and for μ-a.e. point $x \in \varphi_i(K_i) \cap E''$, we have

$$v(x) \;=\; \lambda(x) \cdot D\varphi_i(\varphi_i^{-1}(x)) \cdot \vec{e}_1.$$

So from formula (4.2.5) and the above equation, we obtain, for all $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$,

$$\begin{aligned} D^A f(x) \cdot v(x) \;&=\; D^A f(x) \cdot \big[\lambda(x) \cdot D\varphi_i(\varphi_i^{-1}(x)) \cdot \vec{e}_1\big] \\ &=\; \lambda(x) \cdot \big((\varphi_i)_\#\partial_1\big) f(x). \end{aligned}$$

It follows that $\delta_v = (\chi_{E''} \cdot \lambda) \cdot (\varphi_i)_\#\partial_1$. By inspection, we have $\delta_v \in \Upsilon(\mathbb{R}^n,\mu)$, and this proves Claim 4.3.4.

For the other direction, let F be a m_1-null set in $\mathbb{R}$ on which the Lebesgue singular part of $(\varphi_i^{-1})_\#\mu$ is concentrated, and put $G := \mathbb{R} \setminus F$. If ∂_1 denotes the Euclidean differential operator on $\mathbb{R}$, then by Theorem 4.1.1, $\chi_G\partial_1$ generates $\Upsilon(K_i, (\varphi_i^{-1})_\#\mu)$.

Since each φ_i is bi-Lipschitz, then by Theorem 3.4.3, the modules $\Upsilon(\varphi_i(K_i),\mu)$ and $\Upsilon(K_i, (\varphi_i^{-1})_\#\mu)$ are isomorphic, so the pushforward $\delta_i := (\varphi_i)_\#(\chi_G\partial_1)$ generates $\Upsilon(\varphi_i(K_i),\mu)$. By Part (1) of Lemma 3.4.1, for each $i \in \mathbb{N}$, the derivation δ_i satisfies

$$\|\delta_i\| \;\leq\; \big(1 \vee L(\varphi_i)\big) \cdot \|\partial_1\| \;\leq\; C \cdot 1.$$

To summarize, there is a measurable decomposition $\bigcup_i \varphi_i(K_i)$ of E and on each subset $\varphi_i(K_i)$, there is a derivation δ_i in $\Upsilon(\varphi_i(K_i), \mu)$ which satisfies $\|\delta_i\| \leq C$. We now invoke Lemma 3.2.6 (with $C^{-1}\delta_i$ for δ_i) from which we obtain the derivation

$$\delta_E \;:=\; \sum_{i=1}^{\infty} \chi_{\varphi_i(K_i)} \delta_i.$$

By construction, δ_E generates $\Upsilon(\mathbb{R}^n, \mu)$. In addition, by Lemma 4.2.9 the action of each δ_i agrees with approximate differentiation in the direction of the vectorfield $v_i := D\varphi_i(\varphi^{-1}(x)) \cdot \vec{e}_1$. We then see that each $\delta \in \Upsilon(\mathbb{R}^n, \mu)$ is determined by an $L^\infty(\mathbb{R}^n, \mu)$-multiple of the section $v := \sum_i \chi_{\varphi_i(K_i)} \cdot v_i$. □

In the case $k = 1$, note that the approximate tangent bundle $\mathrm{Tan}^1(E)$ of a 1-rectifiable set E is generated by the single vector-field $\tau_E : E \to \mathbb{R}^n$, given by

$$\tau_E(x) \;:=\; D\varphi_i\big(\varphi_i^{-1}(x)\big) \cdot \vec{e}_1.$$

Combined with the previous proof, this observation implies the following corollary.

Corollary 4.3.5. *Let μ be a Radon measure on $\mathbb{R}^n$, let E be a 1-rectifiable subset in $\mathbb{R}^n$, and suppose that μ is concentrated on E. If the absolutely continuous part $\mu_{\mathcal{H}}$ of μ is nonzero and is concentrated on a subset E' of E, then the derivation*

$$\delta_E f(x) \;:=\; \chi_{E'}(x) \cdot D^A f(x) \cdot \tau_E(x)$$

generates the module $\Upsilon(\mathbb{R}^n, \mu)$.

The proof of Theorem 4.0.8 now follows easily from the previous facts about 1-rectifiable and purely 1-unrectifiable sets.

Proof of Theorem 4.0.8. Since A is a 1-set, by Theorem 4.2.2 there is a 1-rectifiable set E and a purely 1-unrectifiable set F so that $A = E \cup F$. If μ is singular to $\mathcal{H}^1 \lfloor A$, then μ is concentrated on a $\mathcal{H}^1$-null set and hence on a purely 1-unrectifiable set. By Lemma 4.3.1, we have $\Upsilon(\mathbb{R}^n, \mu) = 0$.

This proves that if μ is singular to $\mathcal{H}^1\lfloor A$, then $\Upsilon(\mathbb{R}^n, \mu) = 0$. By the locality property, it also shows that $\Upsilon(\mathbb{R}^n, \mu\lfloor F) = 0$. Let $v : E \to \mathrm{Tan}^1(E)$ be the section as in the proof of Lemma 4.3.3. Then the derivation $f \mapsto \chi_E \cdot \langle D^A f, v\rangle$ generates $\Upsilon(\mathbb{R}^n, \mu\lfloor E)$. Part (2) then follows from the locality property. $\square$

The following corollary is a restatement of Theorem 4.0.8. It specifies further the structure of the module of derivations on 1-sets in $\mathbb{R}^n$, by collecting various facts from this chapter.

Corollary 4.3.6. *Let μ be a Radon measure on $\mathbb{R}^n$, let A be a 1-set, and suppose that μ is concentrated on A.*

1. *If A is purely 1-unrectifiable, then $\Upsilon(\mathbb{R}^n, \mu) = 0$.*

2. *If A is not purely 1-unrectifiable, then $\Upsilon(\mathbb{R}^n, \mu)$ is isomorphic to the $L^\infty(\mathbb{R}^n, \mu)$-module of bounded, $\mu_{\mathcal{H}}$-measurable sections of the approximate tangent bundle $\mathrm{Tan}^1(E)$. Here E is the 1-rectifiable part of A, as given in Theorem 4.2.2, and $\mu_{\mathcal{H}}$ is the part of μ which is absolutely continuous to $\mathcal{H}^1\lfloor E$.*

To explain the proof, the first assertion of the corollary follows directly from Lemma 4.3.1. For the second assertion, one first decomposes A into a purely 1-unrectifiable subset and a 1-rectifiable subset (Theorem 4.2.2). It then suffices to handle the case of the 1-rectifiable subset, and for that one argues similarly as in the proof of Lemma 4.3.3.

CHAPTER V

Structure of Derivations on 2-Dimensional Spaces

In the last chapter we proved Theorem 4.1.1 by the following argument: if a Radon measure μ on $\mathbb{R}$ is concentrated on a m_1-null set E, then one covers E by open sets of arbitrarily small m_1-measure. From these covers, one forms a sequence of uniformly Lipschitz functions that violates the continuity of any nonzero derivation in $\Upsilon(\mathbb{R}, \mu)$.

For $n > 1$, the difficulty in extending the previous proof to $\mathbb{R}^n$ lies in choosing covers with a suitable geometry. To this end, we first recall some recent results of Alberti, Csörnyei, and Preiss about the structure of Lebesgue null sets in $\mathbb{R}^2$ [ACP05]. We then adapt these results to prove the following fact.

Theorem 5.0.7. *Let μ be a Radon measure on $\mathbb{R}^2$. If μ is singular to Lebesgue 2-measure, then any two derivations in $\Upsilon(\mathbb{R}^2, \mu)$ form a linearly dependent set.*

To prove this, one proceeds as in the 1-dimensional case. From appropriate covers of m_2-null sets, one constructs sequences of uniformly Lipschitz functions which converge pointwise to x_1 and x_2. By using appropriate limits of these sequences, one then shows that any two derivations in $\Upsilon(\mathbb{R}^2, \mu)$ form a linearly dependent set.

Remark 5.0.8. Note that in the setting of $\mathbb{R}^2$, Theorem 5.0.7 implies a sharper version of Corollary 3.5.6. Indeed, if $\mu \perp m_2$, then any two derivations in $\Upsilon(\mathbb{R}^2, \mu)$ form a linearly dependent set. So by Part (1) of Lemma 3.3.6, any three derivations in $\Upsilon(\mathbb{R}^2,\mu)$ also form a linearly dependent set. If instead $\mu \ll m_2$, then by Corollary 3.5.4 the set $\{\partial_{x_i}\}_{i=1}^2$ generates $\Upsilon(\mathbb{R}^2,\mu)$, and this clearly implies that any three derivations in $\Upsilon(\mathbb{R}^2,\mu)$ also form a linearly dependent set. Corollary 3.5.6 then follows from the Lebesgue decomposition of measures in $\mathbb{R}^2$.

In fact, more is true. We also give a sharper form of Corollary 3.5.7, which concerns the cardinality of generating sets for $\Upsilon(\mathbb{R}^2,\mu)$. As another application, we obtain an analogue of Theorem 4.0.8 for 2-sets in $\mathbb{R}^n$.

In what follows, we refer to Lebesgue null sets in $\mathbb{R}^2$ as null sets, Lebesgue singular measures as singular measures, etc. As before, we consider only Radon measures μ.

5.1 Preliminaries: Null Sets in $\mathbb{R}^2$.

Towards a new covering theorem, we begin with a few definitions from [ACP05].

Definition 5.1.1. Let $f:\mathbb{R}\to\mathbb{R}$ be 1-Lipschitz. An x_1-*curve*[1] is a graph of the form

$$\gamma_1(f) \;:=\; \{(x_1, f(x_1)) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\},$$

and we will refer to f as the *(Lipschitz) parametrization* of $\gamma_i(f)$. An x_1-*stripe of thickness* δ is a subset of the form

$$\mathcal{N}_1(f;\delta) \;:=\; \{(x_1,x_2) \,:\, |x_2 - f(x_1)| \,\le\, \delta/2\}.$$

A x_2-*curve* $\gamma_2(f)$ and a x_2-*stripe* $\mathcal{N}_2(f;\delta)$ are similarly defined.

Theorem 5.1.2 (Alberti-Csörnyei-Preiss, 2005). *Let E be a null set in $\mathbb{R}^2$. Then E can be written as $E = E^1 \cup E^2$ such that each set E^i satisfies the following property: for each $\epsilon > 0$, E^i can be covered by countably many x_i-stripes of thickness δ_j, where $\sum_j \delta_j < \epsilon$.*

[1] This notation differs from that of [ACP05]; in their work, such curves are called x-curves and y-curves.

Remark 5.1.3. Theorem 5.1.2 can be improved in the case when E is a *compact* null set. From the proof in [ACP05, pp. 4-5] one observes that each set E^i satisfies the following stronger properties.

1. The set E^i can be covered by *finitely* many x_i-stripes. In addition, given $\epsilon > 0$ we may choose a uniform thickness $\delta > 0$ for each x_i-stripe, so that the total thickness of all the x_i-stripes remains less than ϵ.
2. The covering x_i-stripes can be chosen in the following way: the Lipschitz parametrizations of the associated x_i-curves are piecewise-linear functions with finitely many corner points.[2]

In the next theorem we show that the covering x_i-stripes for each E^i can be chosen so that intersections occur only along their boundaries. This will be a technical convenience in the proof of Theorem 5.0.7.

Theorem 5.1.4. *Let E be a compact null set in $\mathbb{R}^2$. Then $E = E^1 \cup E^2$, where for $i = 1, 2$, each sets E^i has the properties given in Remark 5.1.3. In addition, the covering x_i-stripes for E^i can be chosen to have pairwise-disjoint interiors.*

Before proving the theorem, we first require a lemma. It assures that the x_i-curves associated to the covering x_i-stripes of E^i can be chosen without crossings. So if $\gamma_i(f)$ and $\gamma_i(g)$ are such x_i-curves, then either $f \leq g$ holds on all of $\mathbb{R}$, or $f \geq g$ holds on all of $\mathbb{R}$.

Lemma 5.1.5. *Let $i \in \{1, 2\}$ and let $\{\alpha^j\}_{j=1}^N$ be a collection of x_i-curves. Then there is a collection of x_i-curves $\{\eta^j\}_{i=1}^N$, with $\eta^j = \gamma_i(f_j)$, so that*

$$\bigcup_{j=1}^{N} \alpha^j = \bigcup_{j=1}^{N} \eta^j \tag{5.1.1}$$

[2] That is, points of non-differentiability.

and for each $t \in \mathbb{R}$ and each $1 < i \leq N$, the following holds:

$$f_{i-1}(t) \ \leq \ f_i(t). \tag{5.1.2}$$

Moreover, if the curves $\{\alpha^j\}_{j=1}^N$ are piecewise linear, then so are the curves $\{\eta^j\}_{j=1}^N$.

Proof of Lemma 5.1.5. Given the collection of x_i-curves $\{\alpha^j\}_{j=1}^N$, let $\{g_j\}_{j=1}^N$ denote their Lipschitz parametrizations. That is, each function $g_j : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz and satisfies $\alpha^j = \gamma_i(g_j)$. We now argue inductively, by selecting collections of x_i-curves of increasing cardinality and whose parametrizations satisfy conditions (5.1.1) and (5.1.2). To simplify notation, we assume that $i = 1$ and we use the variables x, y in place of x_1, x_2, respectively.

The case $N = 1$ is vacuous, so by the induction hypothesis, we assume that

$$g_1 \ \leq \ g_2 \ \leq \ \ldots \ \leq \ g_{N-1}. \tag{5.1.3}$$

Put $h_0 := g_N$. We define inductively the functions h_j and f_j by

$$\begin{aligned} 1 \leq j < N: \quad f_j \ &:= \ g_j \wedge h_{j-1}, \quad h_j \ := \ g_j \vee h_{j-1}, \\ j = N: \quad f_j \ &:= \ h_{N-1}. \end{aligned}$$

By Part (2) of Lemma 2.1.2, f_j and h_j are 1-Lipschitz. This shows that we obtain well-defined x_i-curves $\eta^j = \gamma_i(f_j)$. Moreover, if each g_j is piecewise-linear, then by construction, f_j and h_j are also piecewise-linear.

Observe next that for $j < N - 1$, the estimate $g_j \leq h_j$ holds by definition and $g_j \leq g_{j+1}$ holds by assumption. So by inequality (5.1.3) it follows that

$$f_j \ \leq \ g_j \wedge h_{j-1} \ \leq \ g_j \ \leq \ g_{j+1} \wedge h_j \ = \ f_{j+1}.$$

Lastly, $f_{N-1} \leq f_N$ holds by construction, so this gives condition (5.1.2).

For the other condition, observe that for $0 \leq j < N$, we have

$$\{y = h_j(x)\} \cup \{y = g_{j+1}(x)\} \ = \ \{y = f_{j+1}(x)\} \cup \{y = h_{j+1}(x)\}$$

and the union of sets $\bigcup_{j=1}^{N}\{y = g_j(x)\}$ can then be transformed as follows:

$$\begin{aligned}
\bigcup_{j=1}^{N}\{y = g_j(x)\} &= \{y = h_0(x)\} \cup \{y = g_1(x)\} \cup \bigcup_{j=2}^{N-1}\{y = g_j(x)\} \\
&= \{y = f_1(x)\} \cup \{y = h_1(x)\} \cup \bigcup_{j=2}^{N-1}\{y = g_j(x)\} \\
&= \ldots = \bigcup_{j=1}^{N}\{y = f_j(x)\}.
\end{aligned}$$

This means that the x_i-curves $\{\eta^j\}_{j=1}^N$ satisfy condition (5.1.1). □

We now prove Theorem 5.1.4. The argument is an iterative procedure. At each stage, one chooses new stripes which satisfy the following properties: they cover the null set, they preserve the order of the previous stripes, but the interior of the "bottom-most" stripe does not meet the others.

Proof of Theorem 5.1.4. Let $\epsilon > 0$ be given. By Remark 5.1.3, we have $E = E^1 \cup E^2$ and there are collections of x_1-stripes $\{\mathcal{N}_1^j\}_{j=1}^{N_1}$ and x_2-stripes $\{\mathcal{N}_2^j\}_{j=1}^{N_2}$ whose unions cover E^1 and E^2, respectively. Since the argument is symmetric, we assume that $E = E^1$. As a simpler notation, we also write N for N_1 and $\mathcal{N}^j$ for $\mathcal{N}_1^j$.

By Lemma 5.1.5, we also assume that the parametrizations $\{f_j\}_{j=1}^N$ of the x_1-stripes $\{\mathcal{N}^j\}_{j=1}^N$ satisfy conditions (5.1.2). As a first iteration, put

$$f_j^1(x) := \begin{cases} f_1, & i = 1 \\ f_j(x) \vee \big(f_1(x) + \delta\big), & 1 < j \leq N \end{cases} \tag{5.1.4}$$

and consider the stripes $\mathcal{M}^j := \mathcal{N}(f_j^1; \delta)$, for $1 \leq j \leq N$.

Claim 5.1.6. The collection of stripes $\{\mathcal{M}^j\}_{j=1}^N$ satisfies the following properties:

(i) the stripes $\{\mathcal{M}^j\}_{j=1}^N$ also cover E;

(ii) none of the stripes $\{\mathcal{M}^j\}_{j=2}^N$ meet the interior of $\mathcal{M}_1$.

For (i), it suffices to show $\mathcal{N}^j \setminus \mathcal{N}^1 \subset \mathcal{M}^j$ for each $j > 1$, from which we obtain

$$(5.1.5) \qquad E \subset \bigcup_{i=1}^{N} \mathcal{N}^j = \mathcal{N}^1 \cup \Big(\bigcup_{i=2}^{N} \mathcal{N}^j \setminus \mathcal{N}^1 \Big) \subset \mathcal{M}^1 \cup \Big(\bigcup_{i=2}^{N} \mathcal{M}^j \Big) = \bigcup_{i=1}^{N} \mathcal{M}^j.$$

Let $p \in \mathcal{N}^j \setminus \mathcal{N}^1$ with $p = (p_1, p_2)$, and we will argue by cases. If $f_j^1(p_1) = f_j(p_1)$, then from the definition of $\mathcal{N}^j$ we obtain

$$|p_2 - f_j^1(p_1)| = |p_2 - f_j(p_1)| \leq \delta/2,$$

which gives the inclusion (5.1.5). If instead $f_j^1(p_1) = f_1(p_1) + \delta$, then

$$p_2 \leq f_j(p_1) + \delta/2 \leq f_j^1(p_1) + \delta/2.$$

In addition, we know that $p \notin \mathcal{N}^1$ and $f_1 \leq f_j$, so we further obtain

$$p_2 \geq f_1(p_1) + \delta/2 = f_j^1(p_1) - \delta/2.$$

Combining the two estimates above, we obtain inclusion (5.1.5). This proves (i).

To show (ii), from the definitions of f_1^1 and f_j^1 we see that whenever $j \neq N$,

$$f_1^1(p_1) + \delta/2 = f_1(p_1) + \delta/2 \leq f_j^1(p_1) - \delta/2.$$

So if p lies in the interior of $\mathcal{M}_1$, then by formula (5.1.4) we obtain

$$p_2 < f_1(p_1) + \delta/2 \leq f_j^1(p_1) - \delta/2.$$

As a result, $p \notin \mathcal{M}_j$. This proves (ii) and Claim 5.1.6.

We now iterate the same argument with the new collection $\{\gamma_1(f_j^1)\}_{j=2}^N$ in place of the old collection $\{\gamma_1(f_j)\}_{j=2}^N$. Note that the new curve parametrizations $\{f_j^1\}_{j=1}^N$ will also satisfy condition (5.1.2), and this follows from the definition in formula (5.1.4). More explicitly, given an index $1 \leq j < N$, we have

$$f_j^1(p_1) = f_j(p_1) \vee (f_1(p_1) + \delta) \leq f_{j+1}(p_1) \vee (f_1(p_1) + \delta) = f_{j+1}^1(p_1)$$

for each $p \in \mathbb{R}$. For $2 < j \leq N$ we define an analogous function $f_j^2 : \mathbb{R} \to \mathbb{R}$ by

$$f_j^2(p_1) \;:=\; f_j^1(p_1) \vee \big(f_2^1(x_1) + \delta\big)$$

and by similar arguments, none of the stripes $\{\mathcal{N}_i(f_j^2;\delta)\}_{j=3}^N$ meets the interior of either $\mathcal{N}_i(f_1^1;\delta)$ or $\mathcal{N}_i(f_2^2;\delta)$.

Iterating further, we obtain a 1-Lipschitz function $f_j^j : \mathbb{R} \to \mathbb{R}$ and an x_1-curve $\gamma^j := \gamma_1(f_j^j)$ for each $1 \leq j \leq N$. Putting $\mathcal{N}^j := \mathcal{N}(f_j^j;\delta)$, it follows that $\{\mathcal{N}^j\}_{j=1}^N$ is the desired collection of x_1-stripes of thickness δ. □

To complete this discussion of null sets, we recall a fact [ACP05, Rmk 3(ii)] about the geometry of the subsets E^1 and E^2. Roughly speaking, it states that E^1 is purely 1-unrectifiable in close-to-vertical directions, in the sense that E^1 intersects x_2-curves in sets of $\mathcal{H}^1$-measure zero. In a similar sense, E^2 is purely 1-unrectifiable in close-to-horizontal directions.

In the proof of Theorem 5.0.7, this property of "directional pure unrectifiability" will ensure that such subsets are negligible to derivations. See Remark 5.2.4.

Lemma 5.1.7. *Let E be a null set in $\mathbb{R}^2$ and let $L \in (0,1)$. For $\{i,j\} = \{1,2\}$, if γ is an x_j-curve whose parametrization is L-Lipschitz, then $\mathcal{H}^1(\gamma \cap E^i) = 0$, where E^i is the subset from Theorem 5.1.2.*

Proof. Since the argument is symmetric in i and j, we assume that $i = 1$ and $j = 2$. Let γ be a x_2-curve in $\mathbb{R}^2$ as above, and put $F^1 := E^1 \cap \gamma$. By Theorem 5.1.4, for any $\epsilon > 0$, the set E^1 can be covered by a collection of x_1-stripes $\{\mathcal{N}^k\}_{k=1}^\infty$, each of thickness δ_k, so that $\sum_{k=1}^\infty \delta_k < \epsilon$. Clearly, the same union also covers F^1.

Let p_k be the point in $\gamma \cap \mathcal{N}^k$ with least x_2-coordinate. Note that $\gamma \cap \mathcal{N}^k$ can be covered by a set of the form $K(p_k) \cap \mathcal{N}^k$, where $K(p_k)$ is a one-sided cone with vertex

p_k, direction $\vec{e}_2$, and opening angle $2\arctan(1/L)$. In particular, the set $K(p_k) \cap \mathcal{N}^k$ has diameter at most $C \cdot \delta_k$, where C is a positive constant depending only on L.

In this way we cover $F^1 \cap \gamma$ with open sets $\{O_k\}_{k=1}^{\infty}$, each of which has diameter at most $2C \cdot \delta_k$ and hence at most $2C \cdot \epsilon$. We now estimate:

$$\mathcal{H}^1(F^1 \cap \gamma) \;\leq\; \limsup_{\epsilon \to 0} \sum_{k=1}^{\infty} \operatorname{diam}(O_k) \;\leq\; \limsup_{\epsilon \to 0} \sum_{k=1}^{\infty} 2C \cdot \delta_k \;\leq\; 2C \cdot \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the lemma follows. □

5.2 Linearly Independent Derivations on $\mathbb{R}^2$.

Towards proving Theorem 5.0.7, we first show a special case.

Lemma 5.2.1. *Let μ be a Radon measure on $\mathbb{R}^2$. If μ is concentrated on a compact null set E of $\mathbb{R}^2$, then any two derivations of $\Upsilon(\mathbb{R}^2, \mu)$ are linearly dependent.*

The proof can be divided into two steps. As stated before, the first step is to cover the null set by unions of x_1- and x_2-stripes of decreasing total thickness. From these covers, one constructs two sequences of Lipschitz functions which approximate the coordinate functions x_1 and x_2 in the weak-$*$ topology of $\operatorname{Lip}(E)$.

Using these sequences, the second step is to show a linear relation between δx_1 and δx_2 that holds true for any $\delta \in \Upsilon(\mathbb{R}^2, \mu)$. The linear dependence then follows from this relation. To simplify the discussion, we state and prove the first step as a separate lemma. It is a construction similar to that of Theorem 4.1.1.

Lemma 5.2.2. *Let E be a compact null set in $\mathbb{R}^2$, and let $E = E^1 \cup E^2$ be the decomposition as given in Theorem 5.1.4. Then for $\{i, l\} = \{1, 2\}$, there is a sequence of uniformly Lipschitz functions $\{\varphi_{i,k}\}_{k=1}^{\infty}$ on $\mathbb{R}^2$ so that*

1. *$\varphi_{i,k} \overset{*}{\rightharpoonup} x_l$ in $\operatorname{Lip}(E)$;*

2. *locally, the restriction $\varphi_{i,k}|E^i$ depends only on the variable x_i.*

We will formulate property (2) more precisely in the proof. Put simply, each $\varphi_{i,k}$ is constructed from a cover of E^i by x_i-stripes. The behavior of $\varphi_{i,k}$ near a point $p \in E^i$ is then determined by the geometry of the x_i-stripe which contains p.

Proof of Lemma 5.2.2. Let $\epsilon > 0$ be given. Since E is bounded, we may assume that E lies in the cube $[0,1]^2$. By Theorem 5.1.4, for $i = 1, 2$ each of the sets E^i can be covered by N x_i-stripes of thickness δ, where $N \cdot \delta < \epsilon$ and where the interiors of the stripes are pairwise disjoint. The argument is symmetric in x_1 and x_2, so for simplicity we study the case $i = 1$ and $l = 2$.

For $1 \le j \le N$, let $\mathcal{N}^j := \mathcal{N}_1(f_j; \delta)$ be an x_1-stripe as described above. Emphasizing the dependence on ϵ, we also put $\mathcal{N}(\epsilon) := \mathbb{R}^2 \setminus \bigcup_j \mathcal{N}^j$. Now consider the family of functions $\varphi_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ that are given by the formula

$$\varphi_\epsilon(p) \;:=\; \int_{\{p_1\}\times[0,p_2]} \chi_{\mathcal{N}(\epsilon)}\, d\mathcal{H}^1, \tag{5.2.1}$$

where $p = (p_1, p_2) \in \mathbb{R}^2$. Indeed, if $p \in E$, then we obtain the estimates

$$0 \;\le\; \varphi_\epsilon(p) \;\le\; p_2 \;\le\; 1.$$

It follows that the sequence $\{\varphi_\epsilon|E\}_{\epsilon>0}$ is bounded in sup-norm. In addition, any subsequence of $\{\varphi_\epsilon\}_{\epsilon>0}$ converges pointwise to the function x_2. To see this, note that

$$0 \;\le\; p_2 - \varphi_\epsilon(p) \;\le\; \sum_{j=1}^{N} \int_{\{p_1\}\times\mathbb{R}} \chi_{\mathcal{N}^j}\, d\mathcal{H}^1 \;=\; N\cdot\delta \;<\; \epsilon.$$

Claim 5.2.3. The family of functions $\{\varphi_\epsilon\}_{\epsilon>0}$ is uniformly 3-Lipschitz.

To begin, let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be points in $\mathbb{R}^2$. We argue by cases.

Case 0*: p and q lie on the same vertical line.* Since φ_ϵ is the indefinite integral of a bounded function with sup-norm 1, it is 1-Lipschitz in the variable x_2. As a result,

$$|\varphi_\epsilon(p) - \varphi_\epsilon(q)| \;\le\; |p_2 - q_2| \;=\; |p - q|. \tag{5.2.2}$$

Case 1: p and q lie in the same stripe $\mathcal{N}^j$. The line segment $\{p_1\}\times[f_j(p_1)-\delta/2, p_2]$ lies entirely in $\mathcal{N}^j$, so consider its lower endpoint $p' := (p_1, f_j(p_1) - \delta/2)$. From integration we then obtain the identity

$$\varphi_\epsilon(p) \;=\; \int_{\{p_1\}\times[0,\,p_2]} \chi_{\mathcal{N}(\epsilon)}\, d\mathcal{H}^1 \;=\; \int_{\{p_1\}\times[0,\,f_j(p_1)-\delta/2]} \chi_{\mathcal{N}(\epsilon)}\, d\mathcal{H}^1 \;=\; \varphi_\epsilon(p').$$

Similarly, we see that the point $q' := (q_1, f_j(q_1) - \delta/2)$ satisfies $\varphi_\epsilon(q) = \varphi_\epsilon(q')$. Recall that the interiors of the $\{\mathcal{N}^j\}_{j=1}^\infty$ are pairwise disjoint, so a ray with initial point p' and direction $-\vec{e}_2$ will cross through $j-1$ stripes of thickness δ. It follows that

$$\begin{aligned} \varphi_\epsilon(p') &= f_j(p_1) - \delta/2 - (j-1)\cdot\delta \\ \varphi_\epsilon(q') &= f_j(q_1) - \delta/2 - (j-1)\cdot\delta \end{aligned}$$

and because f_j is 1-Lipschitz, $\varphi_\epsilon|\mathcal{N}^j$ is also 1-Lipschitz:

$$|\varphi_\epsilon(p) - \varphi_\epsilon(q)| \;=\; |f_j(p_1) - f_j(q_1)| \;=\; |p_1 - q_1| \;\leq\; |p-q|. \tag{5.2.3}$$

From the previous equations, we also see that whenever $p \in \mathcal{N}^j$, we have

$$\varphi_\epsilon(p) \;=\; f_j(p_1) - \delta/2 - (j-1)\cdot\delta. \tag{5.2.4}$$

Case 2: $p, q \notin \mathcal{N}(\epsilon)$, and both points lie between the same pair of stripes[3]. The argument is similar to Case 1. If p and q lie below every x_1-stripe, then we have $\varphi_\epsilon(p) = p_2$ and $\varphi_\epsilon(q) = q_2$.

Otherwise let j_0 be the largest index so that $f_{j_0}(p_1) \leq p_2$. Since the $\{\mathcal{N}^j\}_{j=1}^N$ have pairwise-disjoint interiors, from integration we obtain $\varphi_\epsilon(p) = p_2 - j_0\cdot\delta$ and $\varphi_\epsilon(q) = q_2 - j_0\cdot\delta$. In either case, we find that

$$\begin{aligned} \varphi_\epsilon(p) - \varphi_\epsilon(q) &= p_2 - q_2 \\ |\varphi_\epsilon(p) - \varphi_\epsilon(q)| &= |p_2 - q_2| \;\leq\; |p-q|. \end{aligned}$$

[3]It does not follow, of course, that p and q lie in the same connected component of $\mathcal{N}(\epsilon)$. Consider, for example, the case where the boundaries of two stripes meet.

Case 3: p and q are arbitrary. By Case 2 we may assume that p and q are separated by a boundary curve of some stripe $\mathcal{N}^j$. Without loss of generality, it is an upper boundary curve, i.e. the graph

$$\Gamma := \{(x_1, x_2) : x_2 = f_j(x_1) + \delta/2\}$$

and moreover, assume that p lies above the curve and q lies below the curve:

$$p_2' := f_j(p_1) + \delta/2 \leq p_2, \quad q_2' := f_j(q_1) + \delta/2 \geq q_2.$$

Observe that the points $p' = (p_1, p_2')$ and $q' = (q_1, q_2')$ lie on the same vertical lines as p and q, respectively. Moreover, p' and q' also lie on Γ. Using the Triangle Inequality and inequalities (5.2.2) and (5.2.3) from the previous cases, we now estimate

$$(5.2.5) \quad \begin{cases} |\varphi_\epsilon(p) - \varphi_\epsilon(q)| \leq |\varphi_\epsilon(p) - \varphi_\epsilon(p')| + |\varphi_\epsilon(p') - \varphi_\epsilon(q')| + |\varphi_\epsilon(q') - \varphi_\epsilon(q)| \\ \qquad \leq |p_2 - p_2'| + |f_j(p_1) - f_j(q_1)| + |q_2' - q_2|. \end{cases}$$

We claim further that the following inequality

$$(5.2.6) \qquad |p_2 - p_2'| + |q_2' - q_2| \leq |p_2 - q_2| + |p_2' - q_2'|$$

is true for all choices of $p_2' \leq p_2$ and $q_2 \leq q_2'$, and this can be shown geometrically. By studying the intervals $I_p := [p_2', p_2]$ and $I_q := [q_2, q_2']$ in $\mathbb{R}$, one first observes that $m_1(I_p) + m_1(I_q)$ is the left-hand side of inequality (5.2.6). Arguing further by cases,

1. Suppose that I_p and I_q are disjoint. Depending on the relative positions of p_2 and q_2, the union $I_p \cup I_q$ lies in one of the intervals $[p_2, q_2]$ or $[q_2', p_2']$. Inequality (5.2.6) then follows from the estimate

$$\begin{aligned} m_1(I_p) + m_1(I_q) &\leq m_1([p_2, q_2]) \vee m_1([p_2', q_2']) \\ &\leq m_1([p_2, q_2]) + m_1([p_2', q_2']) = |p_2 - q_2| + |p_2' - q_2'|. \end{aligned}$$

2. Suppose that $I_q \subset I_p$. Then for the intervals $I' := [p_2', q_2']$ and $I = [q_2, p_2]$, we have $I_p = I \cup I'$ and $I_q = I \cap I'$. Inequality (5.2.6) then follows from

$$m_1(I_p) + m_1(I_q) \;=\; m_1(I) + m_1(I') \;=\; |p_2 - q_2| + |p_2' - q_2'|.$$

If instead $I_p \subset I_q$, then the roles are reversed: we have $I_q = I \cup I'$ and $I_p = I \cap I'$. However, the same identity holds, which also gives inequality (5.2.6).

3. As a final case, suppose that $I_p \cap I_q \neq \emptyset$, $I_p \not\subset I_q$, and $I_q \not\subset I_p$. Of the intervals $[q_2, p_2]$ and $[p_2', q_2']$, one is $I_p \cup I_q$ and the other is $I_p \cap I_q$. Inequality (5.2.6) then follows from the estimate

$$m_1(I_p) + m_1(I_q) \;=\; m_1(I_p \cup I_q) + m_1(I_p \cap I_q) \;=\; |p_2 - q_2| + |p_2' - q_2'|.$$

This proves inequality (5.2.6). From this and inequality (5.2.5), we then obtain

$$\begin{aligned} |\varphi_\epsilon(p) - \varphi_\epsilon(q)| &\leq |f_j(p_1) - f_j(q_1)| + |p_2 - p_2'| + |q_2' - q_2| \\ &\leq 1 \cdot |p_1 - q_1| + |p_2 - q_2| + |p_2' - q_2'| \\ &\leq |p_1 - q_1| + |p_2 - q_2| + |p_1 - q_1| \\ &\leq 3 \cdot |p - q|, \end{aligned}$$

which gives Claim 5.2.3. Lastly, this claim and the bound $\|\varphi_\epsilon|E\|_\infty \leq 1$ imply that $\{\varphi_\epsilon\}_{\epsilon>0}$ is a norm-bounded net in $\mathrm{Lip}_\infty(E)$.

Letting $\{\epsilon_k\}_{k=1}^\infty$ be any decreasing sequence in $(0,1]$ with $\epsilon_k \searrow 0$, the functions $\varphi_{1,k} := \varphi_{\epsilon_k}$ converge pointwise to x_2. By Lemma 2.1.3, this is equivalent to the convergence $\varphi_{1,k} \xrightarrow{*} x_2$ in $\mathrm{Lip}_\infty(E)$, which is property (1) in the lemma.

We now discuss property (2) of the lemma. By equation (5.2.4), we see that the restriction $\varphi_{1,k}|\mathcal{N}^j$ agrees with the function

$$F_j(p_1) \;:=\; f_j(p_1) - \delta/2 - (j-1) \cdot \delta. \tag{5.2.7}$$

This dictates the local behavior of $\varphi_{i,k}$ in the following way. Given any point $p \in E^1$ and any neighborhood O of p, there is a stripe $\mathcal{N}^j$ which contains p and where the function $\varphi_{1,k}|(O \cap \mathcal{N}^j)$ agrees with a univariate function in the variable x_1. With this understood, property (2) follows. □

Remark 5.2.4. (1) When restricted to the bounded null set E^i, observe that the Lipschitz functions $\{\varphi_{i,k}\}_{k=1}^{\infty}$ from Lemma 5.2.2 are piecewise-linear in the x_i-coordinate and constant in the other coordinate. So for each index k, there is a finite union of line segments $\{\ell_{jk}\}$ which are orthogonal to $\vec{e}_i$ and on which $\varphi_{i,k}$ is non-differentiable.

However, the sets $\ell^k := \bigcup_j \ell_{jk}$ are negligible under the action of derivations in $\Upsilon(\mathbb{R}^2, \mu)$. To see this, recall that by Lemma 5.1.7, each set $E^i \cap \ell_{jk}$ has zero $\mathcal{H}^1$-measure, so in particular the set $E^{ik} := E^i \cap \ell^k$ is purely 1-unrectifiable. By Lemma 4.3.1, we see that $\chi_{E^{ik}}\delta = 0$ holds for every $\delta \in \Upsilon(\mathbb{R}^2, \mu)$.

(2) Observe that the points in ℓ^k are not the only points of non-differentiability for the function $\varphi_{i,k}$. By construction, $\varphi_{i,k}$ is piecewise linear on every line in the direction $\vec{e}_i$. As a result, boundary points on each x_i-stripe $\mathcal{N}_i^j$ are also points of non-differentiability. In the proof of Lemma 5.2.1, we will treat these sets separately.

Proof of Lemma 5.2.1. By Theorem 5.1.4, E can be written in the form $E = E^1 \cup E^2$, where for each $k \in \mathbb{N}$, the set E^i can be covered by N_k x_i-stripes, each of thickness δ, and so that $N_k \cdot \delta < 2^{-k}$. For simplicity, we also assume that E^1 and E^2 are disjoint sets, otherwise we may study the sets E^2 and $E^1 \setminus E^2$ instead.

We also partition each E^i further into two subsets. For each $i = 1, 2$, let $\{\mathcal{N}_i^j\}_{j=1}^{N_k}$ be the collection of x_i-stripes whose union covers E^i. We then consider the union of all boundaries of x_i-stripes, for $1 \leq j \leq N_k$ and for all $k \in \mathbb{N}$:

$$\Gamma^i := \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{N_k} \partial \mathcal{N}^j.$$

Put $\Gamma := \Gamma^1 \cup \Gamma^2$. We now argue by cases.

Case 1: $\mu(\Gamma) > 0$. By construction, Γ is a 1-rectifiable set. If $\Upsilon(\Gamma, \mu) = 0$, then we would have $\chi_\Gamma(\delta_1 - \delta_2) = 0$ for all derivations $\delta_1, \delta_2 \in \Upsilon(\mathbb{R}^2, \mu)$, and this would prove the lemma. Therefore we assume that the module $\Upsilon(\Gamma, \mu)$ is nonzero. It then follows from Corollary 4.3.5 that $\Upsilon(\Gamma, \mu)$ is generated by the derivation δ_Γ.

This implies that for any two nonzero derivations δ_1 and δ_2 in $\Upsilon(\mathbb{R}^2, \mu)$, there exist nonzero functions $\Lambda_1, \Lambda_2 \in L^\infty(\mathbb{R}^2, \mu)$ so that $\delta_1 = \Lambda_1 \delta_E$ and $\delta_2 = \Lambda_2 \delta_\Gamma$. By inspection, the linear combination $\Lambda_2\delta_1 - \Lambda_1\delta_2$ is precisely the zero derivation. It follows that $\{\delta_1, \delta_2\}$ is a linearly dependent set in $\Upsilon(\mathbb{R}^2, \mu)$, as desired.

Case 2: $\mu(\Gamma) = 0$. The argument will be symmetric in x_1 and x_2, so assume that $i = 1$. Let $\{f_j\}_{j=1}^N$ be the parametrizations of the x_1-stripes $\{\mathcal{N}_1^j\}_{j=1}^{N_k}$, and let $\{\varphi_{1,k}\}_{k=1}^\infty$ be the sequence of uniformly Lipschitz functions from Lemma 5.2.2.

Claim 5.2.5. There exists $g_1 \in L^\infty(\mathbb{R}^2, \mu)$ so that for each $\delta \in \Upsilon(\mathbb{R}^2, \mu)$, we have

$$\chi_{E^1} \cdot \delta x_2 \;=\; g_1 \cdot \chi_{E^1} \cdot \delta x_1 \;\; \mu\text{-a.e.} \tag{5.2.8}$$

By Theorem 5.1.4, recall that the x_1-curves $\{\gamma_1(f_j)\}_{j=1}^N$ are piecewise linear. As given in Item (1) of Remark 5.2.4, let ℓ^k be the union of vertical line segments on which $\varphi_{1,k}$ is non-differentiable, and put $E^{1k} = E^1 \cap \ell^k$.

Let f_j' denote the derivative of f_j. The image $A_j := f_j'(\mathbb{R} \setminus \mathrm{proj}_{\vec{e}_2}(\ell^k))$ is clearly a finite set in $[-1, 1]$. So up to a finite union of vertical lines, we may partition the interior of $\mathcal{N}^j$ into finitely many subsets of the form

$$\mathcal{N}^j(c) \;:=\; \{x \in \mathrm{int}(\mathcal{N}^j) \;:\; f_j'(x_1) = c\}.$$

Next, recall that $\varphi_{1,k}$ is piecewise-linear on vertical lines L and constant on subsets of the form $L \cap \mathcal{N}_1^j$. It follows that for every point p in the interior of $\mathcal{N}_1^j$, the partial derivative $\partial_2\varphi_{1,k}(p)$ exists and is zero.

Let $\delta \in \Upsilon(\mathbb{R}^2, \mu)$ be arbitrary. By the previous observation, the locality property (Theorem 3.2.1), the Chain Rule (Proposition 3.5.1), and formula (5.2.7) we obtain

$$\begin{aligned} \chi_{\mathcal{N}^j(c)} \cdot \delta\varphi_{1,k} &= \chi_{\mathcal{N}^j(c)} \cdot (\partial_1\varphi_{1,k} \cdot \delta x_1 \, + \, \partial_2\varphi_{1,k} \cdot \delta x_2) \\ &= \chi_{\mathcal{N}^j(c)} \cdot (f_j' \cdot \delta x_1 \, + \, 0 \cdot \delta x_2) \\ &= \chi_{\mathcal{N}^j(c)} \cdot (c \cdot \delta x_1). \end{aligned}$$

We now define an auxiliary function $h_k : \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$h_k \; := \; \sum_{j=1}^{N} \sum_{c \in A_j} c \cdot \chi_{\mathcal{N}^j(c)}.$$

By hypothesis, Γ has μ-measure zero, so the set $E^1 \cap \partial\mathcal{N}_1^j$ also has μ-measure zero. As a result, the collection of subsets $\{\mathcal{N}^j(c) : c \in A_j\}$ is a measurable partition of $(E^1 \setminus E^{1k}) \cap \mathcal{N}_1^j$ and hence the collection of subsets $\{\mathcal{N}^j(c) : c \in A_j,\, 1 \leq j \leq N_k\}$ is a measurable partition of $E^1 \setminus E^{1k}$. It follows that the identity

$$\delta\varphi_{1,k} \; = \; h_k \cdot \delta x_1 \tag{5.2.9}$$

holds μ-a.e. on the set $E^1 \setminus E^{1k}$.

By item (1) of Remark 5.2.4, we have $\chi_{E^{1k}}\delta = 0$. This means that we also have $\delta x_1(p) = 0$ and $\delta\varphi_{1,k}(p) = 0$ whenever $p \in E^{1k}$. It follows that equation (5.2.9) holds more generally; it is valid for μ-a.e. point in E^1.

Since each f_j is the parametrization of an x_1-curve, it follows that $|f_j'| \leq 1$ holds on $(E^1 \cap \mathcal{N}_1^j) \setminus E^{1j}$ for each $1 \leq j \leq N_k$. This implies that $\{h_k\}_{k=1}^{\infty}$ is a subset of the closed unit ball of $L^\infty(\mathbb{R}^2, \mu)$. By weak-$*$ compactness (Theorem 3.1.6), it contains a weak-$*$ convergent subsequence $\{h_{k_m}\}_{m=1}^{\infty}$. Let g_1 denote the weak-$*$ limit. By the lower-semicontinuity of norms (Theorem 2.1.5), we see that $\|g_1\|_{\mu,\infty} \leq 1$. A straightforward argument also gives $h_{k_m} \cdot \delta x_1 \overset{*}{\rightharpoonup} g_1 \cdot \delta x_1$ in $L^\infty(E^1, \mu)$.

By Lemma 5.2.2 we have $\varphi_{1,k_m} \overset{*}{\rightharpoonup} x_2$ in $\mathrm{Lip}_\infty(E)$. By equation (5.2.9) and by the continuity of δ, we obtain the convergence

$$h_{k_m} \cdot \delta x_1 \;=\; \delta\varphi_{1,k_m} \overset{*}{\rightharpoonup} \delta x_2 \text{ in } L^\infty(E^1,\mu).$$

By uniqueness of weak-$*$ limits, we must have $g_1 \cdot \delta x_1 = \delta x_2$ for μ-a.e. point in E^1. Observe that this is precisely equation (5.2.8), which proves Claim 5.2.5.

By the symmetry of the argument, there also exists a function $g_2 \in L^\infty(\mathbb{R}^2,\mu)$ so that $\|g_2\|_{\mu,\infty} \leq 1$ and so that, for every $\delta \in \Upsilon(\mathbb{R}^2,\mu)$, we have

$$\delta x_1 \;=\; g_2 \cdot \delta x_2 \;\; \mu\text{-a.e. on } E^2. \tag{5.2.10}$$

We now show that any two derivations δ_1 and δ_2 in $\Upsilon(\mathbb{R}^2,\mu)$ form a linearly dependent set. Without loss of generality, neither δ_1 nor δ_2 is zero, so consider the functions

$$\begin{aligned} \lambda_1 \;&:=\; \chi_{E^1} \cdot \delta_2 x_1 + \chi_{E^2} \cdot \delta_2 x_2 \\ \lambda_2 \;&:=\; -\chi_{E^1} \cdot \delta_1 x_1 - \chi_{E^2} \cdot \delta_1 x_2. \end{aligned}$$

One easily sees that λ_1 and λ_2 both lie in $L^\infty(\mathbb{R}^2,\mu)$. Moreover, the linear combination $\delta := \lambda_1\delta_1 + \lambda_2\delta_2$ annihilates the coordinate functions x_1 and x_2, because for points in E^1 the linear relation (5.2.8) implies the identities

$$\begin{aligned} \delta x_1 \;&=\; (\delta_2 x_1) \cdot \delta_1 x_1 - (\delta_1 x_1) \cdot \delta_2 x_1 \;=\; 0 \\ \delta x_2 \;&=\; (\delta_2 x_1) \cdot \delta_1 x_2 - (\delta_1 x_1) \cdot \delta_2 x_2 \;=\; (\delta_2 x_1) \cdot g_1 \cdot \delta_1 x_1 - (\delta_1 x_1) \cdot g_1 \cdot \delta_2 x_1 \;=\; 0. \end{aligned}$$

For μ-a.e. point in E^2, a similar argument shows that $\delta x_1 = 0$ and $\delta x_2 = 0$. By Corollary 3.5.3, it follows that $\delta = 0$.

Suppose now that $\lambda_1 = \lambda_2 = 0$. From formulas (5.2.8) and (5.2.10), we obtain

$$\delta_1 x_1 \;=\; \delta_2 x_1 = 0 \text{ on } E^1 \;\text{ and }\; \delta_1 x_2 \;=\; \delta_2 x_2 = 0 \text{ on } E^2.$$

By Corollary 3.5.3, this implies that δ_1 and δ_2 are both zero, which is a contradiction. As a result, the derivations $\chi_{\mathbb{R}^2\setminus\Gamma}\delta_1$ and $\chi_{\mathbb{R}^2\setminus\Gamma}\delta_2$ form a linearly dependent set. $\square$

By using the Borel regularity of μ, Theorem 5.0.7 follows readily from Lemma 5.2.1.

Proof of Theorem 5.0.7. We argue by contradiction, so suppose that μ is a nonzero measure that μ is concentrated on a null set E in $\mathbb{R}^2$, and that there is a linearly independent set $\{\delta_1, \delta_2\}$ in $\Upsilon(\mathbb{R}^2, \mu)$.

We also assume that E is a bounded set. Indeed, since μ is Radon, the square $Q_{ab} := [a, a+1) \times [b, b+1)$ is μ-measurable for each pair $(a, b) \in \mathbb{Z}^2$. Moreover, there must be a pair (a', b') for which $\mu(Q_{a'b'}) > 0$, otherwise we would have

$$\mu(\mathbb{R}^2) \;\leq\; \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \mu(Q_{ab}) \;=\; 0$$

and hence μ would be the zero measure. As a shorthand, put $Q := Q_{a'b'}$.

Since the set $\{\delta_1, \delta_2\}$ is linearly independent in $\Upsilon(\mathbb{R}^2, \mu)$, by Lemma 3.3.6, the set $\{\chi_Q\delta_1, \chi_Q\delta_2\}$ must also be linearly independent in $\Upsilon(Q, \mu)$. We then arrive at the desired contradiction if we show that any two derivations in $\Upsilon(Q, \mu)$ form a linearly dependent set. To simplify the argument, we put $E = Q$.

Since μ is Radon, there is compact exhaustion $\{E_k\}_{k=1}^{\infty}$ of E, which means that $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$ and $\mu(E \setminus E_k) \to 0$ as $k \to \infty$. In particular, each E_k is a compact null set. So by Theorem 5.1.4, each has the form $E_k = E_k^1 \cup E_k^2$, where we again assume that $E_k^1 \cap E_k^2 = \emptyset$.

By Lemma 5.1.4, the sets E_k^1 and E_k^2 can be covered by unions of finitely many x_1- and x_2-stripes with total thicknesses 2^{-k}, respectively. For $i = 1, 2$, let Γ_k^i be the union of the boundaries of the x_i-stripes, and put $\Gamma = \bigcup_{k=1}^{\infty}(\Gamma_k^1 \cup \Gamma_k^2)$. Clearly, Γ is a 1-rectifiable subset of $\mathbb{R}^2$. We now argue by cases.

Case 1: $\mu(\Gamma) > 0$. Arguing as in Case 1 in the proof of Lemma 5.2.1, we see that each of the derivations $\chi_\Gamma \delta_1$ and $\chi_\Gamma \delta_2$ is a $L^\infty(\mathbb{R}^2, \mu)$-multiple of the derivation δ_Γ, as given in Corollary 4.3.5. It follows that the set $\{\chi_\Gamma \delta_1, \chi_\Gamma \delta_2\}$ is linearly dependent in $\Upsilon(\mathbb{R}^2, \mu)$, which proves the theorem.

Case 2: $\mu(\Gamma) = 0$. For each $k \in \mathbb{N}$ there exist g_k^1 and g_k^2 in $L^\infty(\mathbb{R}^2, \mu)$ which satisfy the linear relations (5.2.8) and (5.2.10) μ-a.e. on E_k^1 and E_k^2, respectively. In particular, the same linear relations also hold on the smaller sets $F_k^1 := E_k^1 \setminus E_{k-1}$ and $F_k^2 := E_k^2 \setminus E_{k-1}$, respectively.

For $i = 1, 2$ put $F^i := \bigcup_{k=1}^\infty F_k^i$, and hence $E = F^1 \cup F^2$. Consider the functions

$$h_i \;:=\; \sum_{k=1}^{\infty} \chi_{F_k^i} \cdot g_k^i.$$

By construction, h_1 and h_2 lie in $L^\infty(\mathbb{R}^2, \mu)$ and they satisfy the linear relations

$$\chi_{F^1} \cdot \delta x_2 \;=\; \chi_{F^1} \cdot h_1 \cdot \delta x_1 \tag{5.2.11}$$

$$\chi_{F^2} \cdot \delta x_1 \;=\; \chi_{F^2} \cdot h_2 \cdot \delta x_2 \tag{5.2.12}$$

for every $\delta \in \Upsilon(\mathbb{R}^2, \mu)$. From these relations, consider the scalars

$$\begin{aligned} \lambda_1 &:= \chi_{F^1} \cdot \delta_2 x_1 + \chi_{F^2} \cdot \delta_2 x_2 \\ \lambda_2 &:= -\chi_{F^1} \cdot \delta_1 x_1 - \chi_{F^2} \cdot \delta_1 x_2 \end{aligned}$$

and put $\delta' := \lambda_1 \delta_1 + \lambda_2 \delta_2$. For $i = 1, 2$, we see that $\chi_{F^i} \cdot \delta' x_i = 0$, because

$$\chi_{F^i} \cdot \delta' x_i \;=\; (\chi_{F^i} \cdot \delta_2 x_i) \cdot \delta_1 x_i - (\chi_{F^i} \cdot \delta_1 x_i) \cdot \delta_2 x_i \;=\; 0.$$

For $i \neq j$, we have $\chi_{F^i} \cdot \delta' x_j = 0$, and this follows from equations (5.2.11) and (5.2.12):

$$\begin{aligned} \chi_{F^i} \cdot \delta' x_j &= (\chi_{F^i} \cdot \delta_2 x_i) \cdot \delta_1 x_j - (\chi_{F^i} \cdot \delta_1 x_i) \cdot \delta_2 x_j \\ &= \chi_{F^i} \cdot \big[\delta_2 x_i \cdot (h_i \cdot \delta_1 x_i) - \delta_1 x_i \cdot (h_i \cdot \delta_2 x_i)\big] \;=\; 0. \end{aligned}$$

Therefore δ' is the zero derivation, by Corollary 3.5.3.

Lastly, suppose that λ_1 and λ_2 are both identically zero. Arguing as in the proof of Lemma 5.2.1, this implies that δ_1 and δ_2 are both zero, which is a contradiction. □

In the previous proof, the argument reduced to two cases: (1) a 1-rectifiable set Γ consisting of boundaries of covering stripes for the null set E, and (2) subsets of E on which the linear relations (5.2.11) and (5.2.12) hold. The next corollary states that similar linear relations are also valid μ-a.e. on the exceptional 1-rectifiable set Γ. This will be a technical convenience in the proof of Theorem 5.3.1.

Corollary 5.2.6. *Let μ be a singular Radon measure on $\mathbb{R}^2$ and suppose that E is a bounded null set in $\mathbb{R}^2$ on which μ is concentrated. Then there exist μ-measurable subsets F^1 and F^2 in $\mathbb{R}^2$ and functions $g_1, g_2 \in L^\infty(\mathbb{R}^2, \mu)$ so that $E = F^1 \cup F^2$ and that for all derivations $\delta \in \Upsilon(\mathbb{R}^2, \mu)$, we have*

$$\chi_{F^1} \cdot \delta x_2 = \chi_{F^1} \cdot g_1 \cdot \delta x_1 \tag{5.2.13}$$

$$\chi_{F^2} \cdot \delta x_1 = \chi_{F^2} \cdot g_2 \cdot \delta x_2. \tag{5.2.14}$$

In what follows, we assume all the notation from the proof of Theorem 5.0.7.

Proof. From Case 2 of the proof of Theorem 5.0.7 (where $\mu(\Gamma) = 0$), we see that the same arguments remain valid for the subsets $E^1 \setminus \Gamma^1$ and $E^2 \setminus \Gamma^2$. As a result, we have the following identities for the functions h_1 and h_2:

$$\chi_{F^1\setminus\Gamma^1} \cdot \delta x_2 = \chi_{F^1\setminus\Gamma^1} \cdot h_1 \cdot \delta x_1,$$

$$\chi_{F^2\setminus\Gamma^2} \cdot \delta x_1 = \chi_{F^2\setminus\Gamma^2} \cdot h_2 \cdot \delta x_2.$$

We show next that Γ^1 and Γ^2 exhibit similar linear relations. The argument is symmetric, so we will assume that $i = 1$.

Let $l \in \mathbb{N}$, $k \in \mathbb{N}$, and $j \in \{1, 2, \ldots, N_k\}$ be arbitrary integers, and consider the upper and lower boundary curves

$$\begin{aligned} A^+_{jkl} &:= \left\{ \left(t, f_j(t) + \frac{\delta}{2}\right) : t \in [l, l+1] \right\} \subset \partial \mathcal{N}^{kj}_1, \\ A^-_{jkl} &:= \left\{ \left(t, f_j(t) - \frac{\delta}{2}\right) : t \in [l, l+1] \right\} \subset \partial \mathcal{N}^{kj}_1. \end{aligned}$$

By the symmetry of the argument, we consider only the case of A^+_{jkl}, and the other case is similar.

For $p = (p_1, p_2)$, the map $\varphi(p) := p_1 - l$ is clearly a bi-Lipschitz homeomorphism of A^+_{jkl} onto $[0, 1]$. As a shorthand, we write $A := A^+_{jkl}$, $\psi := \varphi^{-1}$, and $\nu := \varphi_{\#}\mu$. If the singular part ν_S of ν is nonzero, then let E be a null set in $\mathbb{R}$ on which ν_S is concentrated and put $w := \chi_{\mathbb{R}\setminus E}$.

By Rademacher's theorem, every Lipschitz function is ν-a.e. differentiable on the set $[0, 1]\setminus E$. Note that the coordinate function x_2 is a bounded Lipschitz function on A, and therefore $x_2 \circ \varphi$ is a bounded Lipschitz function on $[0, 1]$. Letting $h \in L^1(\mathbb{R}^2, \mu)$ and $\delta \in \Upsilon(\mathbb{R}^2, \mu)$ be arbitrary, we then invoke the transformation formula (3.4.2) and the 1-dimensional formula (4.1.1) to obtain

$$\begin{aligned} \int_A h \cdot \delta x_2 \, d\mu &= \int_A (h \circ \psi \circ \varphi) \cdot \delta(x_2 \circ \psi \circ \varphi) \, d\mu = \int_0^1 (h \circ \psi) \cdot (\varphi_{\#}\delta)(x_2 \circ \psi) \, d\nu \\ &= \int_0^1 (h \circ \psi) \cdot (x_2 \circ \psi)' \cdot (\varphi_{\#}\delta)(\mathrm{id}_{\mathbb{R}}) \cdot w \, d\nu \\ &= \int_A h \cdot \big((x_2 \circ \psi)' \circ \varphi\big) \cdot \delta\varphi \cdot (w \circ \varphi) \, d\mu. \end{aligned}$$

Put $\lambda_A := ((x_2 \circ \psi)' \circ \varphi) \cdot (w \circ \varphi)$. Since h was arbitrary, it follows that

$$\text{(5.2.15)} \qquad \chi_A \cdot \delta x_2 = \lambda^+_A \cdot \chi_A \cdot \delta\varphi = \lambda^+_A \cdot \chi_A \cdot \delta(x_1 - l) = \lambda^+_A \cdot \chi_A \cdot \delta x_1,$$

and a straightforward estimate shows that $\lambda^+_A \in L^\infty(A, \mu)$:

$$\begin{aligned} |\lambda^+_A| &\le |(x_2 \circ \psi)' \circ \varphi| \cdot |w \circ \varphi| \\ &\le |(x_2 \circ \psi)'| \cdot 1 \le L(x_2 \circ \psi) \le L(x_2) \cdot L(\psi) \le 1 \cdot 2. \end{aligned}$$

By a similar argument, for $A' := A^-_{jkl}$ there exists $\lambda_{A'} \in L^\infty(A', \mu)$ which satisfies

$$\chi_{A'} \cdot \delta x_2 \;=\; \lambda_{A'} \cdot \chi_{A'} \cdot \delta x_1. \tag{5.2.16}$$

Combining equations (5.2.15) and (5.2.16) and suppressing the indices j, k, l, we put

$$\lambda_1 \;:=\; \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} (\chi_A \cdot \lambda_A + \chi_{A'} \cdot \lambda_{A'}),$$

and summing over j, k, l, we then obtain the identity

$$\chi_{\Gamma^1} \cdot \delta x_2 \;=\; \lambda_1 \cdot \chi_{\Gamma^1} \cdot \delta x_1.$$

By the symmetry of the argument, we run a similar construction by using boundaries of x_2-stripes and invoking the transformation formula (3.4.2) again. From it we obtain a function $\lambda_2 \in L^\infty(\mathbb{R}^2, \mu)$ that satisfies

$$\chi_{\Gamma^2} \cdot \delta x_2 \;=\; \lambda_2 \cdot \chi_{\Gamma^2} \cdot \delta x_1.$$

Lastly, for $i = 1, 2$ we define $g_i := \chi_{F^i \setminus \Gamma^i} \cdot h_i + \chi_{\Gamma^i} \cdot \lambda_i$. By the above identities, it follows that the functions g_1 and g_2 satisfy formulas (5.2.13) and (5.2.14) on F^1 and F^2, respectively. This proves the corollary. $\square$

The next corollary settles Example 3.3.5. It shows that for many measures μ on $\mathbb{R}^2$, the module $\Upsilon(\mathbb{R}^2, \mu)$ is not necessarily a free module over $L^\infty(\mathbb{R}^2, \mu)$.

Corollary 5.2.7. *Let μ be a Radon measure on $\mathbb{R}^2$, and suppose that $\{\delta_1, \delta_2\}$ is a linearly independent set in $\Upsilon(\mathbb{R}^2, \mu)$. Then $\mu \ll m_2$.*

Proof. We argue by contradiction, so suppose that the singular part of μ is nonzero. Let A be a null set in $\mathbb{R}^2$ on which the singular part is concentrated, so $\mu(A) > 0$. By Part (2) of Lemma 3.3.6, the set $\{\chi_A \delta_i\}_{i=1}^2$ is linearly independent in $\Upsilon(A, \mu)$.

By inspection, for each $\lambda \in L^\infty(X, \mu\lfloor A)$ the restriction $\lambda|A$ lies in $L^\infty(A, \mu)$. As $\mu\lfloor A \ll \mu$, it follows by Lemma 3.4.4 that each $\chi_A \delta_i$ is a derivation in $\Upsilon(\mathbb{R}^2, \mu\lfloor A)$. So suppose there are scalars λ_1 and λ_2 in $L^\infty(\mathbb{R}^2, \mu\lfloor A)$ so that

$$\lambda_1 \cdot (\chi_A \delta_1) f \;+\; \lambda_2 \cdot (\chi_A \delta_2) f \;=\; 0$$

holds for all $f \in \mathrm{Lip}_\infty(\mathbb{R}^2)$. Since the set $\{\chi_A \delta_i\}_{i=1}^2$ is linearly independent in $\Upsilon(A, \mu)$, both $\lambda_1|A$ and $\lambda_2|A$ must be the zero function in $L^\infty(A, \mu)$. This means that λ_1 and λ_2 are the zero function in $L^\infty(X, \mu\lfloor A)$, and hence the set of derivations $\{\chi_A \delta_i\}_{i=1}^2$ is also linearly independent in the module $\Upsilon(\mathbb{R}^2, \mu\lfloor A)$.

On the other hand, $\mu\lfloor A$ is singular. By Theorem 5.0.7, the set $\{\chi_A \delta_i\}_{i=1}^2$ is linearly dependent in $\Upsilon(\mathbb{R}^2, \mu\lfloor A)$, which is a contradiction. The corollary follows. □

5.3 Singular Measures on $\mathbb{R}^2$.

In the previous section we proved that for a singular measure μ on $\mathbb{R}^2$, the rank of the module $\Upsilon(\mathbb{R}^2, \mu)$ is at most one. The next theorem discusses the number of generators of the module. In particular, it is a sharper version of Corollary 3.3.12 and Theorem 5.0.7, which discusses only a proper subset X_ϵ of $X = \mathbb{R}^2$ and the number of generators of $\Upsilon(X_\epsilon, \mu)$.

Theorem 5.3.1. *Let μ be a Radon measure on $\mathbb{R}^2$ and suppose that the module $\Upsilon(\mathbb{R}^2, \mu)$ is nontrivial. If μ is singular, then $\Upsilon(\mathbb{R}^2, \mu)$ is generated by one element.*

Example 5.3.2. The module $\Upsilon(\mathbb{R}^2, \mu)$ can be zero, even when $\mu \neq 0$. Weaver [Wea00, Thm 41] has shown that if S is the 'middle-thirds' Sierpinski carpet and if $\alpha = \log 8 / \log 3$, then $\Upsilon(\mathbb{R}^2, \mathcal{H}^\alpha\lfloor S) = 0$.

To explain the proof, the idea is to choose a derivation which acts "maximally" on the functions x_1 and x_2. To this end, we use standard facts from functional analysis in order to obtain a maximal linear operator of the above type. An additional argument ensures that such an operator is a derivation, from which Theorem 5.3.1 follows.

To begin, let E be a null set in $\mathbb{R}^2$ and suppose that μ is concentrated on E. Let $\mathcal{L}$ be the space of bounded linear operators from $\mathrm{Lip}_\infty(\mathbb{R}^2)$ to $L^\infty(\mathbb{R}^2,\mu)$. Clearly, $\mathcal{L}$ is a vector space, and it becomes a Banach space under the operator norm

$$\|T\| \;:=\; \sup\big\{\|Tf\|_{\mu,\infty} \,:\, f\in \mathrm{Lip}_\infty(\mathbb{R}^2),\ \|f\|\le 1\big\}.$$

By Corollary 5.2.6, E has the form $E = F^1 \cup F^2$ and there exist g_1 and g_2 in $L^\infty(\mathbb{R}^2,\mu)$ for which the linear relations (5.2.13) and (5.2.14) both hold on F^1 and F^2, respectively. For $i = 1, 2$, consider the sets

$$V_i \;:=\; \big\{\delta x_i \,:\, \delta\in \Upsilon(\mathbb{R}^2, \mu\lfloor F^i),\ \|\delta\|\le 1\big\}.$$

By Theorem 3.2.8, the action of each $\delta\in\Upsilon(\mathbb{R}^2,\mu)$ extends to Lipschitz functions on $\mathbb{R}^2$. In particular, we have $\delta x_i \in L^\infty(\mathbb{R}^2,\mu)$, and from inequality (3.2.7) we obtain

$$\|\delta x_i\|_{\mu\infty} \;\le\; \|\delta\|\cdot L(x_i) \;\le\; 1.$$

Therefore V_i is a subset of the closed unit ball in $L^\infty(\mathbb{R}^2,\mu)$.

Lemma 5.3.3. *Let E, F^1, F^2, μ, V_1, V_2 be as above. There exist derivations δ_1^* and δ_2^* in $\Upsilon(\mathbb{R}^2,\mu)$ so that for $i = 1, 2$ and for μ-a.e. $p\in F^i$, we have*

$$\delta_i^* x_i(p) \;=\; \sup\{v(p) \,:\, v\in V_i\}. \tag{5.3.1}$$

Before proving the lemma, we require a few additional facts. The first fact is the lower semi-continuity of the norm under weak-$*$ convergence (Theorem 2.1.5) and the second fact is Mazur's Lemma [Rud91, Thm 3.13], of which one version (Lemma 8.2.8) is stated in Chapter VIII. The third fact is a compactness theorem on $\mathcal{L}(Y;Z)$, the space of bounded linear operators between Banach spaces Y and Z. Specifically, Z will be the dual of a Banach space W, so $Z = W^*$.

Recall that a net $\{S_j\}_{j\in J}$ in $\mathcal{L}(Y;Z)$ converges to an operator S in the *weak-$*$ operator topology* if $\langle w, S_j y\rangle \to \langle w, Sy\rangle$ holds, for all $y \in Y$ and all $w \in W$.

Recall also that the space $\mathcal{L}(Y;Z)$ admits an *operator norm*, which is given by

$$\|T\| \;:=\; \sup\{\|Ty\|_Z \;:\; y \in Y,\; \|y\|_Y \leq 1\},$$

and where $\|\cdot\|_Y$ and $\|\cdot\|_Z$ are the norms on Y and Z, respectively. As a result, the closed unit ball in $\mathcal{L}(Y;Z)$ is well-defined.

Theorem 5.3.4. *Let Y be a Banach space and let Z be a dual Banach space. If B is the closed unit ball in $\mathcal{L}(Y;Z)$, then B is compact in the weak-$*$ operator topology.*

If Y and Z are Hilbert spaces with $Y = Z$, then Theorem 5.3.4 is a standard fact from the theory of operator algebras. The same proof is equally valid in our setting; see [KR97, Thm 5.1.3]. For completeness, however, we will discuss this topology on $\mathcal{L}(Y;Z)$ and prove Theorem 5.3.4 in Chapter VIII.

Proof of Lemma 5.3.3. As a first case, we assume that E is a bounded subset of $\mathbb{R}^2$. The argument proceeds in three stages, which are stated below as claims. As before, the argument is symmetric, so without loss of generality we assume that $i = 2$. We also write V for V_2, F for F_2, and μ for $\mu\lfloor F^2$.

Claim 5.3.5. The supremum in equation (5.3.1) is attained by w, for some function $w \in L^\infty(\mathbb{R}^2, \mu)$.

We first define a relation $(\prec)$ on V by the following rule:

$$\delta x_2 \;\prec\; \delta' x_2 \;\iff\; \delta x_2 \;\leq\; \delta' x_2 \;\;\mu\text{-a.e.}$$

Note that $(V, \prec)$ is a *directed set* (see Section 8.1), which means that:

1. we have $v \prec v$, for each $v \in V$;

2. if $v, v', v'' \in V$ satisfy $v \prec v'$ and $v' \prec v''$, then $v \prec v''$;

3. for all $v, v' \in V$, there is a $v'' \in V$ so that $v \prec v''$ and $v' \prec v''$.

The first two properties are clear from the definition of $\prec$. For the third property, let δ and δ' be derivations in $\Upsilon(\mathbb{R}^2, \mu)$ and put

$$\begin{aligned} K &:= \{p \in \mathbb{R}^2 : \delta x_2(p) \leq \delta' x_2(p)\}, \\ \delta'' &:= \chi_K \delta' + \chi_{\mathbb{R}^2 \setminus K} \delta. \end{aligned}$$

By the locality property (Theorem 3.2.1), δ'' is a well-defined derivation in $\Upsilon(\mathbb{R}^2, \mu)$. From its construction, it is immediate that $\delta x_2 \prec \delta'' x_2$ and $\delta' x_2 \prec \delta'' x_2$.

From this relation, we see that V is a net which is indexed by its own elements. In other words, $v \in V$ has index v, so formally $v_v := v$ and $V = \{v_v : v \in V\}$.

By Theorem 3.1.6, the closed unit ball in $L^\infty(\mathbb{R}^2, \mu)$ is weak-$*$ compact. Therefore there is an index set I, an element $w \in V$, and a sub-net $W = \{w_i\}_{i \in I}$ of V so that $w_i \overset{*}{\rightharpoonup} w$ in $L^\infty(\mathbb{R}^2, \mu)$. By Definition 8.1.1, there is a map $\varphi : I \to V$ so that

(N1) for each $i \in I$, there is a $v_{\varphi(i)} \in V$ so that $w_i \; = \; v_{\varphi(i)}$;

(N2) for all $v \in V$, there is a $i_0 \in I$ so that if $i_0 \prec i$, then $v_v \prec w_i$.

Because $L^\infty(\mathbb{R}^2, \mu)$ is the dual of a separable Banach space, we may assume that the sub-net W is in fact a sequence; for a proof, see Lemma 8.1.7.

We now show that $v \prec w$ holds for all $v \in V$. Supposing otherwise, there exists $v \in V$ with $v \neq w$, and there is a subset G of F so that $\mu(G) > 0$ and $w(x) < v(x)$, for all $x \in G$. By (N2), there is a $i_0 \in I$ so that whenever $i_0 \prec i$, the inequality

$$v(x) \;\leq\; v_v(x) \;\leq\; w_i(x)$$

holds for all $x \in G$. Letting $g : \mathbb{R}^2 \to [0,\infty)$ be an arbitrary function in $L^1(\mathbb{R}^2,\mu)$, we then obtain the following estimates:

$$\int_G v \cdot g \, d\mu \;\leq\; \int_G w_i \cdot g \, d\mu,$$

$$\int_G v \cdot g \, d\mu \;=\; \lim_i \int_G v \cdot g \, d\mu \;\leq\; \lim_i \int_G w_i \cdot g \, d\mu \;=\; \int_G w \cdot g \, d\mu.$$

This contradicts our hypothesis, which gives the claim.

Claim 5.3.6. The supremum in equation (5.3.1) is attained by Tx_2, for some $T \in \mathcal{L}$.

Since each $w_i \in W$ has the form $w_i = \delta_{\varphi(i)} x_2$, consider the subset $D := \{\delta_{\varphi(i)}\}_{i=1}^{\infty}$ of the closed unit ball of $\Upsilon(\mathbb{R}^2,\mu)$. By Theorem 5.3.4, there is a further subsequence D' of D which converges in the weak-$*$ operator topology. Writing $D' := \{\delta_k\}_{k=1}^{\infty}$, let $\delta_2^* : \mathrm{Lip}_\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2,\mu)$ be the limit operator. For all $h \in L^1(\mathbb{R}^2,\mu)$, we have

$$\int_{\mathbb{R}^2} h \cdot \delta_k x_2 \, d\mu \;\to\; \int_{\mathbb{R}^2} h \cdot \delta_2^* x_2 \, d\mu.$$

Equivalently, we have $\delta_k x_2 \overset{*}{\rightharpoonup} \delta_2^* x_2$ in $L^\infty(\mathbb{R}^2,\mu)$. By uniqueness of weak-$*$ limits, we obtain $\delta_2^* x_2 = w$, as desired.

Claim 5.3.7. The map δ_2^* is a derivation in $\Upsilon(\mathbb{R}^2,\mu)$.

Clearly, δ_2^* is linear. Since D' is a convergent sequence in the weak-$*$ operator topology, for each $f \in \mathrm{Lip}_\infty(\mathbb{R}^2)$ we have $\delta_k f \overset{*}{\rightharpoonup} \delta_2^* f$ in $L^\infty(\mathbb{R}^2,\mu)$. By lower semicontinuity of the $L^\infty(\mathbb{R}^2,\mu)$-norm (Theorem 2.1.5), we obtain the estimate

$$\|\delta_2^* f\|_{\mu,\infty} \;\leq\; \liminf_{k\to\infty} \|\delta_k f\|_{\mu,\infty} \;\leq\; 1 \cdot \|f\|_{\mathrm{Lip}}. \tag{5.3.2}$$

Therefore δ_2^* is bounded. Similarly, for all pairs f_1 and f_2 in $\mathrm{Lip}_\infty(\mathbb{R}^2)$, we obtain $\delta_k f_1 \overset{*}{\rightharpoonup} \delta_2^* f_1$ and $\delta_k f_2 \overset{*}{\rightharpoonup} \delta_2^* f_2$ in $L^\infty(\mathbb{R}^2,\mu)$. In particular, both functions f_1 and f_2 are bounded, so we further obtain the weak-$*$ convergence

$$f_1 \cdot \delta_k f_2 + f_2 \cdot \delta_k f_1 \;\overset{*}{\rightharpoonup}\; f_1 \cdot \delta_2^* f_2 + f_2 \cdot \delta_2^* f_1$$

in $L^\infty(\mathbb{R}^2,\mu)$. The Leibniz Rule for δ_2^* then follows from uniqueness of weak-$*$ limits. It remains to show continuity. For this, we use a stronger mode of convergence.

Since μ is Radon and F is bounded, it follows that μ is a finite measure and for each $1 \le q < \infty$, $L^q(\mathbb{R}^2,\mu)$ is a dense subset of $L^1(\mathbb{R}^2,\mu)$. This shows that for $p > 1$ with $p^{-1}+q^{-1}=1$, the functions $\delta_k x_2$ converge weakly to $\delta_2^* x_2$ in $L^p(\mathbb{R}^2,\mu)$. By Lemma 8.2.8, there exist (finite) convex combinations of the form

$$\tilde{\delta}_i \;:=\; \sum_{k=1}^{\infty} c_{ik}\cdot \delta_k x_2$$

which converge in norm to $\delta_2^* x_2$ in $L^p(X,\mu)$. Since each $\tilde{\delta}_i$ is a finite sum, we have $\tilde{\delta}_i \in \Upsilon(\mathbb{R}^2,\mu)$ for each $i \in \mathbb{N}$. In addition, $\tilde{\delta}_i x_1$ also converges in norm to $\delta_2^* x_1$ in $L^p(\mathbb{R}^2,\mu)$, because by the linear relation (5.2.12), we may estimate as follows:

$$\begin{aligned}\int_{\mathbb{R}^2} |\tilde{\delta}_i x_1 - \delta_2^* x_1|^p \, d\mu \;&\le\; \int_{\mathbb{R}^2} |g_2|^p \cdot |\tilde{\delta}_i x_2 - \delta_2^* x_2|^p \, d\mu \\ &\le\; \|g_2\|_{\mu,\infty}^p \cdot \|\tilde{\delta}_i x_2 - \delta_2^* x_2\|_{\mu,p}^p \;\to\; 0.\end{aligned}$$

The subset $L^q(\mathbb{R}^2,\mu)$ is dense in $L^1(\mathbb{R}^2,\mu)$, so for $j=1,2$ we have $\tilde{\delta}_i x_j \overset{*}{\rightharpoonup} \delta_2^* x_j$ in $L^\infty(\mathbb{R}^2,\mu)$. In fact, more is true.

Subclaim 5.3.8. The sequence $\{\tilde{\delta}_i\}_{i=1}^\infty$ converges to δ_2^* in the weak-$*$ operator topology.

Let $f \in \mathrm{Lip}(\mathbb{R}^2)$ be arbitrary. By the Chain Rule (Proposition 3.5.1), there exist functions g_f^1 and g_f^2 in $L^\infty(\mathbb{R}^2,\mu)$ so that the following identity holds:

$$\tilde{\delta}_i f - \delta_i f \;=\; g_f^1\cdot(\tilde{\delta}_i x_1 - \delta_i x_1) \;+\; g_f^2\cdot(\tilde{\delta}_i x_2 - \delta_i x_2).$$

Both sequences $\{\tilde{\delta}_i x_j\}_{j=1}^\infty$ and $\{\delta_i x_j\}_{j=1}^\infty$ have the same weak-$*$ limit $\delta_2^* x_j$, and as a result, $\tilde{\delta}_i x_j - \delta_i x_j \overset{*}{\rightharpoonup} 0$ in $L^\infty(\mathbb{R}^2,\mu)$. So from the previous identity, we obtain $\tilde{\delta}_i f - \delta_i f \overset{*}{\rightharpoonup} 0$. On the other hand, by Claim 5.3.6 we have $\delta_i f \overset{*}{\rightharpoonup} \delta_2^* f$ in $L^\infty(\mathbb{R}^2,\mu)$, and hence $\tilde{\delta}_i f \overset{*}{\rightharpoonup} \delta_2^* f$ in $L^\infty(\mathbb{R}^2,\mu)$. This gives Subclaim 5.3.8.

We now prove Claim 5.3.7. Suppose $\{f_m\}_{m=1}^{\infty}$ is a sequence in $\mathrm{Lip}_\infty(\mathbb{R}^2)$ such that f_m converges pointwise to zero and such that $\|f_m\|_{\mathrm{Lip}} \leq 1$. Let $\epsilon > 0$ and $h \in L^1(\mathbb{R}^2, \mu)$ be arbitrary. As before, for $1 < q < \infty$, $L^q(\mathbb{R}^2, \mu)$ is a dense subset of $L^1(\mathbb{R}^2, \mu)$, so there is a function $h' \in L^q(\mathbb{R}^2, \mu)$ which satisfies

$$\int_{\mathbb{R}^2} |h' - h| \, d\mu \; < \; \frac{\epsilon}{4}.$$

Since $\tilde{\delta}_i$ converges to δ_2^* in the weak-$*$ operator topology, we see that for each $m \in \mathbb{N}$, the sequence $\{\tilde{\delta}_i f_m\}_{i=1}^{\infty}$ converges weak-$*$ to $\delta_2^* f_m$ in $L^\infty(\mathbb{R}^2, \mu)$. By the lower semi-continuity of the L^∞-norm (Theorem 2.1.5) and the boundedness of each $\tilde{\delta}_i$, we obtain

$$\|\delta_2^* f_m\|_{\mu,\infty} \;\leq\; \liminf_{i\to\infty} \|\tilde{\delta}_i f_m\|_{\mu,\infty} \;\leq\; \liminf_{i\to\infty} \|\tilde{\delta}_i\| \cdot \|f_m\|_{\mathrm{Lip}} \;\leq\; 1 \cdot 1,$$

for all $m \in \mathbb{N}$. From the above estimates, it follows by the Triangle Inequality that

$$\text{(5.3.3)} \qquad \begin{cases} \left| \int_{\mathbb{R}^2} h \cdot \delta_2^* f_m \, d\mu \right| \leq \int_{\mathbb{R}^2} |h - h'| \cdot |\delta_2^* f_m| \, d\mu + \left| \int_{\mathbb{R}^2} h' \cdot \delta_2^* f_m \, d\mu \right| \\ \qquad \leq 1 \cdot \frac{\epsilon}{4} + \left| \int_{\mathbb{R}^2} h' \cdot \delta_2^* f_m \, d\mu \right| \end{cases}$$

By the Triangle Inequality, we further obtain

$$\text{(5.3.4)} \qquad \begin{cases} \left| \int_{\mathbb{R}^2} h' \cdot \delta_2^* f_m \, d\mu \right| = \left| \int_{\mathbb{R}^2} h' \cdot \left(\delta_2^* f_m - \tilde{\delta}_l f_m + \tilde{\delta}_l f_m - \tilde{\delta}_k f_m + \tilde{\delta}_k f_m \right) d\mu \right| \\ \qquad \leq \left| \int_{\mathbb{R}^2} h' \cdot (\delta_2^* - \tilde{\delta}_l) f_m d\mu \right| + \left| \int_{\mathbb{R}^2} h' \cdot (\tilde{\delta}_l - \tilde{\delta}_k) f_m d\mu \right| \\ \qquad + \left| \int_{\mathbb{R}^2} h' \cdot \tilde{\delta}_k f_m d\mu \right| \end{cases}$$

For $j = 1, 2$, we have $\tilde{\delta}_k x_j \longrightarrow \delta_2^* x_j$ in $L^p(\mathbb{R}^2, \mu)$. As a result, there is a $N \in \mathbb{N}$ so that whenever $k, l \geq N$, we have the inequality

$$\|(\tilde{\delta}_l - \tilde{\delta}_k) x_j\|_p \;<\; \frac{\epsilon}{4 \cdot 2 \cdot \|h'\|_q}.$$

Since the sequence $\{f_m\}_{m=1}^{\infty}$ is uniformly 1-Lipschitz, we now invoke Corollary 3.5.5 for $n = 2$ and $C = 1$. From this and the estimate above, it follows that

$$\left| \int_{\mathbb{R}^2} h' \cdot (\tilde{\delta}_l - \tilde{\delta}_k) f_m \, d\mu \right| \;\leq\; 2 \cdot \|h'\|_q \cdot \left(\max_{j=1,2} \|(\tilde{\delta}_l - \tilde{\delta}_k) x_j\|_p \right) \;\leq\; \frac{\epsilon}{4}. \tag{5.3.5}$$

With $k \geq N$ chosen as above, we may choose a sufficiently large $m \in \mathbb{N}$ so that, by the continuity of $\tilde{\delta}_k$, we obtain the estimate

$$\left| \int_{\mathbb{R}^2} h' \cdot \tilde{\delta}_k f_m \, d\mu \right| \;\leq\; \frac{\epsilon}{4}. \tag{5.3.6}$$

Lastly, recall that by Subclaim 5.3.8, the sequence $\{\tilde{\delta}_l\}_{l=1}^{\infty}$ converges to δ_2^* in the weak-$*$ operator topology. With $m \in \mathbb{N}$ as above, we may choose $l \geq N$ so that

$$\left| \int_{\mathbb{R}^2} h' \cdot (\tilde{\delta} - \tilde{\delta}_l) f_m \, d\mu \right| \;\leq\; \frac{\epsilon}{4}. \tag{5.3.7}$$

Combining inequalities (5.3.3) through (5.3.7), we obtain $\left| \int_{\mathbb{R}^2} h \cdot \delta_2^* f_m \, d\mu \right| < \epsilon$, from which Claim 5.3.7 follows.

To complete the proof, suppose that E is unbounded. For each $(j,k) \in \mathbb{Z}^2$, put

$$E_{jk} \;:=\; E \cap \big((j, j+1] \times (k, k+1] \big).$$

By similar arguments, there is a $\delta_{2,jk}^*$ in $\Upsilon(E_{jk}, \mu)$ which satisfies $\|\delta_{2,jk}^*\| \leq 1$ and

$$\delta_{2,jk}^* x_2(p) \;=\; \sup \Big\{ \delta x_2(p) \;:\; \delta \in \Upsilon\big(\mathbb{R}^2, \mu \lfloor (F^2 \cap E_{jk})\big),\; \|\delta\| \leq 1 \Big\}$$

for μ-a.e. $p \in E_{jk}$. By Lemma 3.2.6, the linear operator $\delta_2^* := \sum_{j,k} \chi_{E_{jk}} \delta_{2,jk}^*$ is a derivation in $\Upsilon(\mathbb{R}^2, \mu)$ that satisfies $\|\delta_2^*\| \leq 1$. By construction, it also satisfies equation (5.3.1). This proves the lemma. □

We now prove Theorem 5.3.1. It suffices to show that the derivations given in Lemma 5.3.3 produce a generator for $\Upsilon(\mathbb{R}^2, \mu)$.

Proof of Theorem 5.3.1. Let E be a null set, and suppose that μ is concentrated on E. By Corollary 5.2.6, E can be written as a union $E = F^1 \cup F^2$, where F^1 and F^2 are disjoint. In addition, there exist functions g_1 and g_2 in $L^\infty(\mathbb{R}^2, \mu)$ for which the linear relations (5.2.13) and (5.2.14) both hold.

By Lemma 5.3.3 there are derivations $\delta_1^* \in \Upsilon(\mathbb{R}^2, \mu\lfloor F^1)$ and $\delta_2^* \in \Upsilon(\mathbb{R}^2, \mu\lfloor F^2)$ which satisfy equation (5.3.1) for $i = 1, 2$. Using the locality property (Theorem 3.2.1), we now define a derivation $\delta^* \in \Upsilon(\mathbb{R}^2, \mu)$ by the formula

$$\delta^* \;:=\; \chi_{F^1}\delta_1^* \,+\, \chi_{F^2}\delta_2^*. \tag{5.3.8}$$

Claim 5.3.9. The derivation δ^* generates $\Upsilon(E, \mu)$.

Indeed, let $\delta \in \Upsilon(E, \mu)$ be arbitrary and consider the function

$$\lambda(p) \;:=\; \begin{cases} \delta x_1(p) \,/\, \delta_1^* x_1(p), & p \in F^1,\ \delta_1^* x_1(p) \neq 0 \\ \delta x_2(p) \,/\, \delta_2^* x_2(p), & p \in F^2,\ \delta_2^* x_2(p) \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5.3.9}$$

Clearly, the derivation $\|\delta\|^{-1}\delta$ has norm 1. By the definition of δ^* and by Lemma 5.3.3, we see that the derivation δ^* satisfies

$$\delta^* x_i(p) \;=\; \chi_{F^i}(p)\cdot \delta_i^* x_i(p) \;\geq\; \chi_{F^i}(p)\cdot \|\delta\|^{-1}\cdot \delta x_i(p)$$

for μ-a.e. $p \in F^i$ and hence, for μ-a.e. $p \in E$ and $i \in \{1, 2\}$, we have

$$\delta^* x_i(p) \;\geq\; \|\delta\|^{-1}\cdot \delta x_i(p).$$

By the symmetry of the argument, we see that δ_1^* and δ_2^* are also minimal derivations in $\Upsilon(\mathbb{R}^2, \mu\lfloor F^1)$ and in $\Upsilon(\mathbb{R}^2, \mu\lfloor F^2)$, respectively, in the sense that

$$-\delta^* x_i(p) \;\leq\; -\|\delta\|^{-1}\cdot \delta x_i(p)$$

holds for μ a.e. $p \in E$. So from equation (5.3.1), we obtain the inequality

$$\|\delta\|^{-1} \cdot |\delta x_i| \;\leq\; |\delta_i^* x_i|$$

for μ-a.e. point in E and for $i \in \{1,2\}$. In particular, if $\delta_i^* x_i = 0$ then $\delta x_i = 0$. It follows that $|\lambda| \leq \|\delta\|$ holds μ-a.e., and hence $\lambda \in L^\infty(\mathbb{R}^2, \mu)$. To prove the claim, it suffices to show that $\delta - \lambda\delta^*$ is zero. The argument is symmetric in F^1 and F^2, so without loss, let $p \in F^1$. Observe that if $\delta_1^* x_1(p) = 0$, then $\delta x_1(p) = 0$ and

$$\delta x_1(p) - \lambda(p) \cdot \delta^* x_1(p) \;=\; 0.$$

Moreover, by the linear relation (5.2.13) we obtain

$$\delta x_2(p) - \lambda(p) \cdot \delta^* x_2(p) \;=\; g_1(p) \cdot \delta x_1(p) - \lambda(p) \cdot g_1(p) \cdot \delta^* x_1(p) \;=\; 0.$$

Hence we may assume that $\delta_1^* x_1(p) \neq 0$. Computing further, we have the identities

$$\begin{aligned} \delta x_1(p) - \lambda(p) \cdot \delta^* x_1(p) \;&=\; \delta x_1(p) - \frac{\delta x_1(p)}{\delta_1^* x_1(p)} \cdot \delta_1^* x_1(p) \;=\; 0, \\ \delta x_2(p) - \lambda(p) \cdot \delta^* x_2(p) \;&=\; g_1(p) \cdot \delta x_1(p) - \frac{\delta x_1(p)}{\delta_1^* x_1(p)} \cdot g_1(p) \cdot \delta_1^* x_1(p) \;=\; 0. \end{aligned}$$

Therefore $\chi_{F^1} \cdot (\delta - \lambda\delta^*) = 0$. By the symmetry of the argument, this gives both Claim 5.3.9 and the theorem. □

We now summarize our results by stating a structure theorem for $\Upsilon(\mathbb{R}^2, \mu)$. To formulate it, ($\cong$) will denote an isomorphism of modules over $L^\infty(\mathbb{R}^2, \mu)$. As before, let μ_S and μ_{AC} be the Lebesgue singular and absolutely continuous parts of μ, respectively. By Remark 3.2.5, $L^\infty(\mathbb{R}^2, \mu_{AC})$ is a module over $L^\infty(\mathbb{R}^2, \mu)$.

Theorem 5.3.10. *Let μ be a Radon measure on $\mathbb{R}^2$.*

1. *If $\Upsilon(\mathbb{R}^2, \mu_S) \neq 0$, then $\Upsilon(\mathbb{R}^2, \mu) \cong [L^\infty(\mathbb{R}^2, \mu_{AC})]^2 \oplus L^\infty(\mathbb{R}^2, \mu_S)$.*

2. *If $\Upsilon(\mathbb{R}^2, \mu_S) = 0$, then $\Upsilon(\mathbb{R}^2, \mu) \cong L^\infty(\mathbb{R}^2, \mu_{AC}) \oplus L^\infty(\mathbb{R}^2, \mu_{AC})$.*

Proof. We will prove Part (1). The proof of Part (2) is similar. Let E be a null set on which μ_S is concentrated, and put $D := \mathbb{R}^2 \setminus E$. Since we have $\mu \lfloor E = \mu_S$ and $\delta = \chi_E \delta$, for all $\delta \in \Upsilon(\mathbb{R}^2, \mu_S)$, it follows from the locality property that

$$\Upsilon(E, \mu) \;=\; \Upsilon(E, \mu_S) \;\cong\; \chi_E \cdot \Upsilon(\mathbb{R}^2, \mu_S) \;=\; \Upsilon(\mathbb{R}^2, \mu_S).$$

Similarly, we also have $\Upsilon(D, \mu) \cong \Upsilon(\mathbb{R}^2, \mu_{AC})$.

By hypothesis $\Upsilon(\mathbb{R}^2, \mu_S)$ is nonzero, so from the proof of Theorem 5.3.1 it is generated by the derivation δ^*, as defined in formula (5.3.8). It follows that $\Upsilon(\mathbb{R}^2, \mu_S)$ and $L^\infty(\mathbb{R}^2, \mu_S)$ are isomorphic as $L^\infty(\mathbb{R}^2, \mu)$-modules. We now invoke Lemma 3.2.4 and Corollary 3.5.4 to obtain

$$\begin{aligned} \Upsilon(\mathbb{R}^2, \mu) \;&\cong\; \Upsilon(D, \mu) \oplus \Upsilon(E, \mu) \;\cong\; \Upsilon(\mathbb{R}^2, \mu_{AC}) \oplus \Upsilon(\mathbb{R}^2, \mu_S) \\ &\cong\; \big[L^\infty(\mathbb{R}^2, \mu_{AC}) \oplus L^\infty(\mathbb{R}^2, \mu_{AC})\big] \oplus L^\infty(\mathbb{R}^2, \mu_S). \end{aligned}$$

This gives the desired isomorphism of modules. □

5.4 Derivations on 2-Sets.

To close the discussion of derivations on $\mathbb{R}^2$, we now give a second application of our results from previous sections. We begin by recalling Theorem 4.0.8. For measures μ that are concentrated on 1-sets in $\mathbb{R}^n$, the module $\Upsilon(\mathbb{R}^n, \mu)$ has a similar structure to a module of derivations on $\mathbb{R}$. To prove this, one shows that non-degeneracy of derivations is preserved under the pushforward procedure.

Now consider measures that are concentrated on 2-sets in $\mathbb{R}^n$. With small modifications in the proof, a similar fact holds true: linear independence of derivations is also preserved under the pushforward procedure.

Theorem 5.4.1. *Let μ be a Radon measure on $\mathbb{R}^n$, let A be a 2-set in $\mathbb{R}^n$, and suppose that μ is concentrated on A.*

1. *Any three derivations in* $\Upsilon(\mathbb{R}^n,\mu)$ *form a linearly dependent set.*

2. *If* $\Upsilon(\mathbb{R}^n,\mu)$ *contains a linearly independent set of two derivations, then there is a 2-rectifiable subset* E *of* A *so that* $\mu \ll \mathcal{H}^2\lfloor E$*. Moreover,* $\Upsilon(\mathbb{R}^n,\mu)$ *is isomorphic to the module of* $L^\infty(\mathbb{R}^n,\mu)$*-sections of* $\mathrm{Tan}^2(E)$*.*

As before, the proof of Theorem 5.4.1 reduces to the separate cases of 2-rectifiable sets and purely 2-unrectifiable sets. The next lemma addresses the latter case, and the ideas in its proof are borrowed from Lemmas 3.3.9 and 4.3.1.

Stated briefly, one considers pushforward derivations and applies the Besicovitch-Federer projection theorem to obtain a contradiction. In order to implement this strategy, however, one must find the right coordinate functions on the image of the projection. This is the contribution of Lemma 3.3.9.

Lemma 5.4.2. *Let* μ *be a measure on* $\mathbb{R}^n$*, let* F *be a 2-set in* $\mathbb{R}^n$*, and suppose that* μ *is concentrated on* F*. If* F *is purely 2-unrectifiable, then any two derivations in* $\Upsilon(\mathbb{R}^n,\mu)$ *form a linearly dependent set.*

Proof of Lemma 5.4.2. If $n = 2$ then the lemma reduces to Theorem 5.0.7, so we assume that $n \geq 3$. We argue by contradiction, so suppose that there is a linearly independent set $\{\delta_1, \delta_2\}$ in $\Upsilon(\mathbb{R}^n,\mu)$.

Since F is purely 2-unrectifiable, by Theorem 4.2.10 there are n spanning directions $\{\vec{v}_i\}_{i=1}^n$ in $\mathbb{R}^n$ so that the orthogonal projections of F onto each of the 2-planes

$$V_{ij} \;:=\; \mathrm{span}_{\mathbb{R}}\{\vec{v}_i, \vec{v}_j\}, \;\; 1 \leq i < j \leq n$$

are $\mathcal{H}^2$-null sets. For simplicity, we assume that $\vec{v}_i = \vec{e}_i$ holds for all $1 \leq i \leq n$ and that the projections above consist of pairs of Euclidean coordinate functions

$$\mathrm{proj}_{V_{ij}} \;=\; (x_i, x_j) : \mathbb{R}^n \to V_{ij}.$$

The argument continues in several stages, which we formulate below as claims.

Claim 5.4.3. For $1 \leq i \leq 2$ and $1 \leq j \leq n$, the $n \times 2$ matrix-valued function $[\delta_i x_j]$ has rank two μ-a.e. on F.

Supposing otherwise, there is a subset $G \subset F$ with $\mu(G) > 0$ and so that the matrix $[\chi_G \cdot \delta_i x_j]$ has rank at most one μ-a.e. This implies that any two of the rows $\{(\delta_1 x_j, \delta_2 x_j)\}_{j=1}^{\infty}$ are parallel vectors μ-a.e. on G. As a result, the functions $\lambda_1 := \chi_G \cdot \delta_2 x_1$ and $\lambda_2 := -\delta_1 x_1$ satisfy

$$\lambda_1 \cdot \delta_1 x_j + \lambda_2 \cdot \delta_2 x_j \;=\; 0 \;\; \mu\text{-a.e.} \tag{5.4.1}$$

for all j. By Corollary 3.5.3, this contradicts the linear independence of $\{\delta_1, \delta_2\}$, which proves the claim.

Claim 5.4.4. There is a measurable decomposition $\bigcup_{i<j} F_{ij} = \mathbb{R}^n$ so that on F_{ij}, the derivations $\chi_{F_{ij}}\delta_1$ and $\chi_{F_{ij}}\delta_2$ are determined by their action on x_i and x_j.

From the previous claim, for μ-a.e. $p \in F$ there are two columns of the matrix $[\delta_i x_j(p)]$ which are linearly independent vectors. As a first case, assume that the first and second columns of $[\delta_i x_j]$ form a nonsingular 2×2 matrix. This implies that, for each $3 \leq j \leq n$, there exist μ-measurable functions $\{\lambda_{ij}\}_{i=1}^{3}$ on $\mathbb{R}^n$ so that

$$\lambda_{1j} \begin{bmatrix} \delta_1 x_1 \\ \delta_2 x_1 \end{bmatrix} + \lambda_{2j} \begin{bmatrix} \delta_1 x_2 \\ \delta_2 x_2 \end{bmatrix} + \lambda_{3j} \begin{bmatrix} \delta_1 x_j \\ \delta_2 x_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

holds μ-a.e. on F. Now consider the subsets

$$A_{ij} \;:=\; \Big\{ x \in \mathbb{R}^n \,:\, |\lambda_{ij}(x)| = \max_k |\lambda_{kj}(x)| \Big\}.$$

Clearly, the sets $\{A_{ij}\}_{i=1}^{3}$ partition $\mathbb{R}^n$ and at every point $x \in A_{3j}$, we have

$$\delta_i x_j \;=\; \frac{\lambda_{1j}}{\lambda_{3j}} \cdot \delta_i x_1 + \frac{\lambda_{2j}}{\lambda_{3j}} \cdot \delta_i x_2$$

for $i = 1, 2$. Putting $F_{12} := \bigcap_{j=3}^{n} A_{3j}$, it follows from the previous equation and from Lemma 2.3.1 that $\chi_{F_{12}}\delta_1$ and $\chi_{F_{12}}\delta_2$ are determined by their actions on x_1 and x_2.

From other pairs of linearly independent columns of $[\delta_k x_l]$ we may take similar $(n-2)$-fold intersections F_{ij}. Arguing as before, these sets also have the property that the derivations $\chi_{F_{ij}}\delta_1$ and $\chi_{F_{ij}}\delta_2$ are determined by their action on the coordinate functions x_i and x_j. This proves Claim 5.4.4.

Claim 5.4.5. For all pairs of indices $i < j$, there are two linearly independent derivations in $\Upsilon(\mathbb{R}^2, \nu_{ij})$, where $\nu_{ij} := (\mathrm{proj}_{V_{ij}})_{\#}(\mu\lfloor F_{ij})$.

We first construct linearly independent sets in $\Upsilon(\mathbb{R}^n, \mu\lfloor F_{ij})$ which satisfy similar conclusions to those of Corollary 3.3.10. The claim then follows by taking pushforward derivations. To simplify the discussion, we assume that μ is concentrated on $F \cap F_{12}$. Now consider the derivations

$$\begin{aligned} \delta_1^* &:= \chi_{F_{12}} \cdot \big[(\delta_1 x_1)\delta_2 - (\delta_2 x_1)\delta_1\big] \\ \delta_2^* &:= \chi_{F_{12}} \cdot \big[(\delta_2 x_2)\delta_1 - (\delta_1 x_2)\delta_2\big]. \end{aligned}$$

By Claim 5.4.3, the matrix $[\delta_i x_j]_{i,j=1}^2$ is nonsingular μ-a.e. on F_{12}, so the set $\{\delta_1^*, \delta_2^*\}$ is linearly independent in $\Upsilon(\mathbb{R}^n, \mu\lfloor F_{12})$. If $i \neq j$, then we have $\delta_i^* x_j = 0$, as well as

$$\delta_1^* x_1 \ = \ \delta_2^* x_2 \ = \ \det[\delta_i x_j] \ \neq \ 0 \ \ \mu\text{-a.e.}$$

For $i = 1, 2$, without loss of generality we may assume that $\delta_i^* x_i > 0$ holds μ-a.e. Otherwise we would consider the sets $A_i := \{\delta_i^* x_i > 0\}$ and $B_i := \{\delta_i^* x_i < 0\}$ and study $(\chi_{A_i} - \chi_{B_i})\delta_i$ in place of δ_i.

Put $p = \mathrm{proj}_{V_{12}}$. By Theorem 3.4.1, the pushforward derivations $p_{\#}\delta_i$ satisfy

$$(5.4.2) \qquad \int_{\mathbb{R}^n} h \cdot (p_{\#}\delta_i^*) x_i \, d\nu_{ij} \ = \ \int_F (h \circ p) \cdot \delta_i^*(x_i \circ p) \, d\mu \ = \ \int_F (h \circ p) \cdot \delta_i^* x_i \, d\mu,$$

for all $h \in L^1(\mathbb{R}^2, \nu_{ij})$. By inspection, we have $(p_\# \delta_i^*) x_j = 0$ whenever $i \neq j$. Under the choices $Z_i := F_{12} \cap \{(p_\# \delta_i^*) x_i = 0\}$ and $h = \chi_{Z_i}$, equation (5.4.2) becomes

$$0 \;=\; \int_{p^{-1}(Z_i) \cap F} \delta_i^* x_i \, d\mu.$$

This further implies that $\mu(p^{-1}(Z_i)) = 0$, and hence $p_\# \delta_i^* x_i \neq 0$ holds ν_{ij}-a.e. So by testing against the coordinate functions x_1 and x_2, it is then easy to see that $p_\# \delta_1^*$ and $p_\# \delta_2^*$ form a linearly independent set. This proves Claim 5.4.5.

On the other hand, the image set $\mathrm{proj}_{V_{ij}}(F \cap F_{ij})$ is a Lebesgue 2-null set, so ν_{ij} must be a Lebesgue singular measure on $\mathbb{R}^2$. By Theorem 4.1.1, any two derivations in $\Upsilon(\mathbb{R}^2, \nu_{ij})$ must form a linearly dependent set, and this contradicts Claim 5.4.5. The lemma follows. □

Proof of Theorem 5.4.1. Since A is a 2-set, by Theorem 4.2.2 it has the form $A = E \cup F$, where E is a 2-rectifiable set and F is a purely 2-unrectifiable set.

Suppose that $\mu(F) > 0$. By Lemma 5.4.2, any two derivations in $\Upsilon(F, \mu)$ form a linearly dependent set. By the locality property (Theorem 3.2.1), for all derivations δ_1 and δ_2 in $\Upsilon(\mathbb{R}^n, \mu)$ there exist λ_1 and λ_2 in $L^\infty(F, \mu)$, not both zero, so that

$$\lambda_1 \cdot (\chi_F \delta_1) \;+\; \lambda_2 \cdot (\chi_F \delta_2) \;=\; 0.$$

As in Remark 3.2.7, for $i = 1, 2$, let $\Lambda_i : \mathbb{R}^n \to \mathbb{R}$ be the zero extension of $\chi_F \cdot \lambda_i$, which is a nonzero function in $L^\infty(\mathbb{R}^n, \mu)$. For all $f \in \mathrm{Lip}_\infty(\mathbb{R}^n)$, we have

$$\lambda_i \cdot (\chi_F \delta_i) f \;=\; \Lambda_i \cdot \delta_i f,$$

and hence $\{\delta_1, \delta_2\}$ is a linearly dependent set in $\Upsilon(\mathbb{R}^n, \mu)$, as desired.

Without loss of generality, we now assume that $\mathcal{H}^2(F) = 0$ and that $A = E$. Let $C > 1$. By Theorem 4.2.3 there are compact sets K_i in $\mathbb{R}^2$ and C-bi-Lipschitz embeddings $\varphi_i : K_i \to \mathbb{R}^n$ so that $\mu(E \setminus \bigcup_{i=1}^{\infty} \varphi_i(K_i)) = 0$ and so that the collection $\{\varphi_i(K_i)\}_{i=1}^{\infty}$ is pairwise disjoint. Put $\psi_i := (\varphi_i|K_i)^{-1}$.

If μ is singular to $\mathcal{H}^2\lfloor E$, then for each $i \in \mathbb{N}$, the measure $\mu\lfloor\varphi_i(K_i)$ is singular to $\mathcal{H}^2\lfloor\varphi_i(K_i)$ and the measure $(\psi_i)_\#\mu$ is singular to $m_2\lfloor K_i$. By Theorem 5.3.1, the module $\Upsilon(\mathbb{R}^2, (\psi_i)_\#\mu)$ is generated by the derivation δ_i^*, as given in formula (5.3.8). Since φ_i is bi-Lipschitz, it follows from Theorem 3.4.3 that $(\varphi_i)_\#\delta_i^*$ generates $\Upsilon(\varphi_i(K_i), \mu)$. By Part (1) of Lemma 3.4.1, the derivation $(\varphi_i)_\#\delta_i^*$ further satisfies

$$\|(\varphi_i)_\#\delta_i^*\| \;\leq\; (1 \vee L(\varphi_i)) \cdot \|\delta_i^*\| \;\leq\; C \cdot 1.$$

This shows that $\{\|(\varphi_i)_\#\delta_i\|\}_{i=1}^{\infty}$ is uniformly bounded in $\mathbb{R}$. By Theorem 3.2.6, we obtain a derivation in $\Upsilon(\mathbb{R}^n, \mu)$ by the formula

$$\delta^* \;:=\; \sum_{i=1}^{\infty} \chi_{\varphi_i(K_i)}(\varphi_i)_\#\delta_i^*.$$

By construction, δ^* generates $\Upsilon(E, \mu)$. So for all nonzero pairs δ_1 and δ_2 in $\Upsilon(E, \mu)$, there are nonzero functions λ_1 and λ_2 in $L^\infty(E, \mu)$ so that $\delta_1 = \lambda_1\delta^*$ and $\delta_2 = \lambda_2\delta^*$. We then observe that $\lambda_1\delta_1 - \lambda_2\delta_2$ is zero, from which Part (2) follows.

Without loss of generality, assume that $\mu \ll \mathcal{H}^2\lfloor E$. So for each $i \in \mathbb{N}$, we have

$$(\psi_i)_\#\mu\lfloor K_i \;\ll\; (\psi_i)_\#\mathcal{H}^2\lfloor K_i \;\ll\; m_2.$$

By Corollary 3.5.4, $\{\partial_j\}_{j=1}^2$ is a generating set for $\Upsilon(K_i, (\psi_i)_\#\mu)$, so $\{(\varphi_i)_\#\partial_j\}_{j=1}^2$ is a generating set for $\Upsilon(\varphi_i(K_i), \mu)$. By equation (4.2.5), each $(\varphi_i)_\#\partial_j$ is precisely approximate differentiation in the direction of the tangent vector $(\varphi_i)_\#\vec{e}_j$.

Taking sums over $i \in \mathbb{N}$ and invoking Theorem 3.2.6, a similar argument shows that every $\delta \in \Upsilon(E, \mu)$ is generated by bounded measurable sections of the approximate tangent bundle $\mathrm{Tan}^2(E)$. This proves Part (1) and the theorem. $\square$

CHAPTER VI

Derivations on p-PI Spaces

For the spaces $\mathbb{R}$ and $\mathbb{R}^2$, we learned that the existence of linearly independent sets of derivations imposes restrictions on the underlying measures. In this section we show that this principle holds for a general class of metric measure spaces, called p-PI spaces, which we describe in further detail.

It is known that such spaces (X, ρ, μ) possess good geometric properties. In addition, they support a rich theory of Sobolev spaces which generalize the usual function spaces $W^{1,p}(\mathbb{R}^n)$, for $p \in [1, \infty)$. Using techniques from this theory, Cheeger has proven a differentiability theorem for Lipschitz functions on X [Che99].

In turn, from his techniques we obtain derivations on these spaces, with respect to the underlying measure. In the opposite direction, we will use these derivations to address Cheeger's conjecture, which concerns the structure of such measures.

6.1 Preliminaries: Calculus on Metric Spaces.

As in Chapter II, (X, ρ, μ) denotes a metric space (X, ρ) endowed with a Borel regular measure μ. Here and in the remainder of this section, we assume that the measure μ is a *doubling* measure: that is, every ball has finite and positive μ-measure, and there is a constant $\kappa \geq 1$ so that for all balls B in X, we have

(6.1.1) $$\mu(2B) \;\leq\; \kappa \cdot \mu(B),$$

where $2B$ is the ball with same center as B and twice the radius of B.

Remark 6.1.1. Recall that if a metric space X admits a doubling measure, then X is in fact a *doubling metric space*. This means the following: there is a constant $N \in \mathbb{N}$ so that every ball B in X can be covered by N balls of half the radius of B.

Let B be a ball in a doubling metric space (X, ρ). By iterating the doubling property, we see that (B, ρ) is a separable metric space. It follows from Lemma 3.1.3 that a linear map $\delta : \mathrm{Lip}_\infty(B) \to L^\infty(B, \mu)$ is weak-$*$ continuous on bounded sets if and only if δ is sequentially weak-$*$ continuous.

Following [HK98], we now introduce the notion of an upper gradient of a function.

Definition 6.1.2. Let (X, ρ) be a metric space, and let $u : X \to \mathbb{R}$ be a function. A Borel function $g : X \to [0, \infty]$ is an *upper gradient* for u if the inequality

$$|u(y) - u(x)| \;\leq\; \int_a^b g(\gamma(t))\, dt \tag{6.1.2}$$

holds for all rectifiable curves $\gamma : [a, b] \subset \mathbb{R} \to X$ which are parametrized by arc-length and which satisfy $x = \gamma(a)$ and $y = \gamma(b)$.

Example 6.1.3. In the case of $\mathbb{R}^n$, if $f \in \mathrm{Lip}(\mathbb{R}^n)$ then $|\nabla f|$ is an upper gradient of f. This follows from the Fundamental Theorem of Calculus. Indeed, for every rectifiable curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ parametrized by arc-length, we have

$$f(\gamma(b)) - f(\gamma(a)) \;=\; \int_a^b \nabla f(\gamma(t)) \cdot \dot{\gamma}(t)\, dt$$

By the Triangle inequality, it follows that

$$|f(\gamma(b)) - f(\gamma(a))| \;\leq\; \int_a^b |\nabla f(\gamma(t))|\, dt,$$

where $\dot{\gamma}(t)$ is the tangent vector of γ at the point $\gamma(t)$.

Similarly we also consider weak formulations of the classical Poincaré inequality [EG92, Thm 4.5.2.2] in the metric space setting. To fix notation, for $u \in L^1(X,\mu)$ and for a ball B in X, we write the average value of u over B as

$$u_B \ := \ \fint_B u \, d\mu \ := \ \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Definition 6.1.4. A metric measure space (X,ρ,μ) admits a *weak* $(1,p)$*-Poincaré inequality* if there exist constants $\Lambda \geq 1$ and $C > 0$ so that for all balls B in X,

$$\fint_B |u - u_B| \, d\mu \ \leq \ C \cdot \operatorname{diam}(B) \left[\fint_{\Lambda B} g^p \, d\mu \right]^{1/p} \tag{6.1.3}$$

holds for all $u \in L^1_{loc}(X,\mu)$ and where g is an upper gradient of u. The space (X,ρ,μ) is a p*-PI space* if μ is doubling and X admits a weak $(1,p)$-Poincaré inequality.

Remark 6.1.5. The assumptions of a doubling measure μ and a weak $(1,p)$-Poincaré inequality on (X,ρ) imply nontrivial geometric properties on X. For example, David and Semmes have shown that complete p-PI spaces are λ*-quasiconvex* [DS90]. This means that for all $x, y \in X$, there is a curve $\gamma : [a,b] \subset \mathbb{R} \to X$ joining x to y so that

$$\operatorname{length}_X(\gamma) \ \leq \ \lambda \cdot \rho(x,y).$$

In addition, the constant λ depends only on the constants κ and C in inequalities (6.1.1) and (6.1.3), respectively. For a proof, see [DS90] or [Che99, Sect 17].

Let $p \in [1,\infty)$. Recall that the Sobolev space $W^{1,p}(\mathbb{R}^n)$ can be identified as the completion of the space of smooth, Lebesgue p-integrable functions on $\mathbb{R}^n$ with p-integrable weak partial derivatives [EG92, Thm 4.2.1.2], with respect to the norm

$$f \ \mapsto \ \|f\|_{m_n,p} + \sum_{i=1}^n \|\partial_i f\|_{m_n,p} \, .$$

A similar construction is also possible on metric measure spaces, by means of upper gradients. Following [Che99, Sect 2], for $u \in L^p(X,\mu)$ we define

$$\|u\|_{1,p} \ := \ \|u\|_p \ + \ \inf_{\{g_i\}} \liminf_{i \to \infty} \|g_i\|_p \tag{6.1.4}$$

where the infimum is taken over all sequences $\{u_i\}_{i=1}^{\infty}$ in $L^p(X,\mu)$ so that $u_i \to u$ in $L^p(X,\mu)$-norm and so that g_i is an upper gradient of u_i, for each $i \in \mathbb{N}$.

The Sobolev space $H^{1,p}(X,\mu)$ is then defined as the subspace of functions $u \in L^p(X,\mu)$ for which $\|u\|_{1,p} < \infty$. The function $\|\cdot\|_{1,p}$ becomes a norm on $H^{1,p}(X,\mu)$, but more is true [Che99, Thms 2.7 & 4.48].

Theorem 6.1.6 (Cheeger, 1999). *The space $(H^{1,p}(X,\mu), \|\cdot\|_{1,p})$ is a Banach space. If X is a p-PI space and if $p > 1$, then $H^{1,p}(X,\mu)$ is a reflexive Banach space.*

In addition, for each $u \in H^{1,p}(X,\mu)$, the infimum $\|u\|_{1,p}$ in formula (6.1.4) is realized by a unique function $g_u \in L^p(X,\mu)$ [Che99, Thms 2.10 & 2.18]. We call it the *minimal generalized upper gradient* of u.

Theorem 6.1.7 (Cheeger). *For all $p \in (1,\infty)$ and all $f \in H^{1,p}(X,\mu)$, there is a function $g_f \in L^p(X,\mu)$ so that $\|f\|_{1,p} = \|f\|_p + \|g_f\|_p$. In addition, if $g \in L^p(X,\mu)$ is an upper gradient of f, then $g_f \leq g$ holds μ-a.e.*

Remark 6.1.8 (Other constructions). Shanmugalingam [Sha00] has constructed *Newtonian spaces* $N^{1,p}(X,\mu)$ that are equivalent to the spaces $H^{1,p}(X,\mu)$ and that also generalize the classical Sobolev spaces on $\mathbb{R}^n$. For $p \in (1,\infty)$, it follows that

$$H^{1,p}(\mathbb{R}^n, m_n) \;\cong\; N^{1,p}(\mathbb{R}^n, m_n) \;\cong\; W^{1,p}(\mathbb{R}^n).$$

In particular, using the notion of *p-modulus* (an outer measure on families of curves) one defines *weak upper gradients* as functions which satisfy (6.1.2) for mod_p-a.e. curve γ in X. With a similar norm as in formula (6.1.4), the spaces $N^{1,p}(X,\mu)$ are norm completions of functions in $L^p(X,\mu)$ which admit weak upper gradients in $L^p(X,\mu)$.

Preceding these two constructions, Hajłasz [Haj96] has also formulated a notion of Sobolev space $M^{1,p}(X,\mu)$ on a metric space X. Here the role of the gradient of u

is replaced by a "Lipschitz modulus of continuity" $M[u] : X \to [0, \infty]$, which satisfies

$$|u(x) - u(y)| \;\leq\; \big(M[u](x) + M[u](y)\big) \cdot \rho(x, y)$$

for all $x, y \in X$. In particular, a weak $(1, p)$-Poincaré inequality always holds for such functions u, and for all balls B in $\mathbb{R}^n$ (of possibly infinite radius), we have

$$M^{1,p}(B, m_n) \;\cong\; W^{1,p}(B).$$

For further reading about Sobolev spaces, see [Hei01, Chap 5-6] and [Hei07].

Recall from Example 6.1.3 that in $\mathbb{R}^n$, upper gradients generalize the norm of the gradient of a Lipschitz function. We now present a framework [Che99, Sect 1] which extends this analogy. In this case, the forthcoming upper gradients will rely on the behavior of Lipschitz functions on small scales.

Definition 6.1.9. Let $f \in \mathrm{Lip}(X)$. If x is a (non-isolated) point in X, the *pointwise upper and lower Lipschitz constants*[1] of f at x are defined, respectively, as

$$\begin{aligned}
\mathrm{Lip}[f](x) \;&:=\; \limsup_{y \to x} \frac{|f(y) - f(x)|}{\rho(x, y)} \\
\mathrm{lip}[f](x) \;&:=\; \liminf_{r \to 0} \sup_{\rho(x,y) \leq r} \frac{|f(y) - f(x)|}{r}.
\end{aligned}$$

In the case where x is isolated, put $\mathrm{lip}[f](x) = \mathrm{Lip}[f](x) = 0$.

The proof of the next lemma is straightforward, so we omit it.

Lemma 6.1.10. *Let $f \in \mathrm{Lip}(X)$. Then for all $x \in X$, we have*

$$\mathrm{lip}[f](x) \;\leq\; \mathrm{Lip}[f](x) \;\leq\; L(f). \tag{6.1.5}$$

The next lemma [Che99, Prop 1.11] (see also [Sem95, Lem 1.20]) states that for any Lipschitz function, its pointwise lower Lipschitz constant is an upper gradient. By the previous lemma, this also holds true for the pointwise upper Lipschitz constant.

[1] In [Che99], the pointwise lower Lipschitz constant is denoted $Lip[f]$ instead of $\mathrm{lip}[f]$.

Lemma 6.1.11 (Semmes, 1996). *Let* $f \in \mathrm{Lip}(X)$*. Then the functions* $\mathrm{lip}[f]$ *and* $\mathrm{Lip}[f]$ *are upper gradients of* f*.*

6.2 Differentiability Induces Derivations.

Recall that Rademacher's theorem states that every Lipschitz function on $\mathbb{R}^n$ is m_n-a.e. differentiable. As discussed before in Chapter I, Cheeger has proven a similar differentiability theorem [Che99, Thm 4.38] for p-PI spaces, of which one version is stated below. Keith has also extended this result to a larger class of metric measure spaces; for details, see [Kei04, Thm 2.3.1].

To fix notation, for vectors $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ and vectorfields $f : X \to \mathbb{R}^k$ with components $f = (f_1, \ldots, f_k)$, put $a * f := \sum_i a_i f_i$.

Theorem 6.2.1 (Cheeger, 1999). *Let* (X, ρ, μ) *be a p-PI space. Then there exist* $N \in \mathbb{N}$ *and a measurable decomposition* $X = \coprod_{n=1}^{\infty} X^n$ *where for each* $n \in \mathbb{N}$*, we have* $\mu(X^n) > 0$ *and there is an integer* $k = k(n) \leq N$ *and a map* $\xi^n \in \mathrm{Lip}(X; \mathbb{R}^k)$ *with the following properties:*

1. *There is a constant* $K = K(n) > 0$ *so that for all* $x \in X^n$*,*

$$K \;\leq\; \inf\Big\{ \mathrm{Lip}[a * \xi^n](x) \,:\, a \in \mathbb{R}^{k(n)},\ |a| = 1 \Big\}. \tag{6.2.1}$$

2. *For each* $f \in \mathrm{Lip}(X)$*, there is a unique map* $d^n f : X^n \to \mathbb{R}^k$*, with components in* $L^\infty(X, \mu)$*, so that for* μ*-a.e.* $x \in X^n$*,*

$$\limsup_{y \to x} \frac{f(y) - f(x) - \langle d^n f(x), \xi^n(y) - \xi^n(x) \rangle}{\rho(x, y)} \;=\; 0. \tag{6.2.2}$$

Let $\xi^n = (\xi^n_1, \ldots, \xi^n_k)$. To mimic the terminology of manifolds, we refer to the functions $\xi^n_i : X \to \mathbb{R}$ as *coordinate functions* on X^n, the triples $(\xi^n, X^n, \xi^n(X^n))$ as *coordinate charts* on X^n, and the map $d^n f$ as the *(Cheeger) differential* of f on X^n.

Remark 6.2.2. The bound N in Theorem 6.2.1 depends only on the constants from the doubling condition (6.1.1) and the Poincaré inequality (6.1.3).

Remark 6.2.3. Property (1) of Theorem 6.2.1 is a tacit consequence of Cheeger's proof [Che99, pp. 457]. In that proof, one chooses an initial measurable decomposition $X = \coprod_{m=1}^{\infty} Y_m$, where each set Y^m satisfies the properties below. As a shortand, if $a \in \mathbb{R}^k$ then we write g_a for the minimal generalized upper gradient of $a * \xi^n$.

1a. For μ-a.e. $x \in Y^m$ and all $a \in \mathbb{R}^k$, we have $g_a(x) \leq \mathrm{Lip}[a * \xi^m](x)$. In addition, the function $a \mapsto g_a(x)$ is L-Lipschitz on $\mathbb{R}^k$, where $L = \max_i L(\xi_i^m)$.

1b. For μ-a.e. $x \in Y^m$ and all nonzero $a \in \mathbb{R}^k$, we have $g_a(x) > 0$.

From (1a) and (1b), one shows that for all $x \in Y^m$, there exists $K_n(x) > 0$ so that

$$K_n(x) \ \leq \ g_a(x) \ \leq \ \mathrm{Lip}[a * \xi^n](x)$$

holds for all $a \in \mathbb{R}^k$ with $|a| = 1$. One further divides each Y^m into subsets $\{X_{(m)}^n\}_{n=1}^{\infty}$ so that K_n is strictly positive on $X_{(m)}^n$, from which we obtain inequality (6.2.1) for $X_{(m)}^n$ in place of X^n. By relabeling indices, this gives Property (1).

In $\mathbb{R}^n$, for each $1 \leq i \leq n$, the function x_i is precisely the Lipschitz function whose gradient is $\vec{e}_i$. The next corollary is an analogue of this fact for p-PI spaces.

Corollary 6.2.4. *Under the assumptions of Theorem 6.2.1, we have* $d^n \xi_i^n = \vec{e}_i$ *for each* $n \in \mathbb{N}$ *and each* $1 \leq i \leq k$.

Proof. For each $1 \leq i \leq k$, we first observe that the identity

$$\xi_i^n(y) - \xi_i^n(x) - \langle \vec{e}_i, \xi^n(y) - \xi^n(x) \rangle \ = \ 0$$

holds for all $x, y \in X$. If ξ_i^n is nonconstant on every neighborhood of x, we see that the constant vectorfield $\vec{e}_i$ on X satisfies equation (6.2.2). By uniqueness of the

Cheeger differential, we then obtain $d^n\xi_i^n = \vec{e}_i$. So to prove the corollary, it then suffices to show that each ξ_i^n is nonconstant on every ball B in X.

We now argue by contradiction. To simplify notation, assume that $i = 1$. Suppose that there is a ball B in X on which ξ_1^n is constant. Then by equation (6.2.2),

$$\limsup_{y\to x} \sum_{i=2}^{k} (d^n\xi_1^n)_i \cdot \frac{\xi_i^n(y) - \xi_i^n(x)}{\rho(y,x)} \;=\; 0$$

holds for each $x \in B$, and where $(d^n\xi_1^n)_i$ is the ith component of the vectorfield $d^n\xi_1^n$. As a result, for each $r \in \mathbb{R}$, the vectorfield given by

$$x \;\mapsto\; (r, (d^n\xi_i^n)_2, \ldots, (d^n\xi_i^n)_n)$$

also satisfies equation (6.2.2). Because μ is doubling, we have $\mu(B) > 0$ and this contradicts the uniqueness of the differential $d^n\xi^n$ from Theorem 6.2.1. Therefore ξ_1^n cannot be constant on any ball in X. □

For a p-PI space (X, ρ, μ), the module $\Upsilon(X, \mu)$ is nontrivial, and the proof is due to Cheeger and Weaver [Wea00, Thm 43]. In fact, more is true. It is known that such spaces X admit a *measurable co-tangent bundle* T^*X [Che99, pp. 458]. It is constructed from the differentials of the coordinate maps $\{\xi^n\}_{n=1}^{\infty}$ over each X^n. The proof of [Wea00, Thm 43] then shows that T^*X is isomorphic to the dual module $\Omega(X, \mu)$ of measurable 1-forms.

However, as stated in [Wea00] the theorem holds only for metric derivations, and the proof in [Wea00] is non-constructive. The next theorem states that on p-PI spaces, there is a simple formula for derivations in the sense of Chapter III.

Theorem 6.2.5. *Let (X, ρ, μ) be a p-PI space. For $f \in \mathrm{Lip}(X)$, let $d^n f : X^n \to \mathbb{R}^k$ be the Cheeger differential of f. For each $n \in \mathbb{N}$, there are derivations $\{\delta_i^n\}_{i=1}^k$ in $\Upsilon(X^n, \mu)$, where each δ_i^n is given by the formula*

$$\delta_i^n f \;:=\; \langle d^n f, \vec{e}_i \rangle. \tag{6.2.3}$$

The proof of Theorem 6.2.5 requires several steps. The lemma below is taken from Cheeger's proof of Theorem 6.2.1. For $f \in \mathrm{Lip}(X)$, a similar argument from [Che99, pp.457] shows that the components of $d^n f$ lie in $L^\infty(X, \mu)$.

Lemma 6.2.6. *Let (X, ρ, μ) be a p-PI space. For each $n \in \mathbb{N}$, there is a constant $C = C(n) > 0$ so that for all $f \in \mathrm{Lip}(X)$,*

$$|\delta_i^n f| \;\leq\; C \cdot \mathrm{Lip}[f] \;\; \mu\text{-a.e. on } X^n.$$

Proof of Lemma 6.2.6. By inequality (6.2.1), there is a constant $K = K(n) > 0$ so that for μ-a.e. $x \in X^n$, we have $K \leq \mathrm{Lip}[a * \xi^n](x)$ for all $|a| = 1$.

In particular, this inequality also holds for the vector $a = d^n f(x)/|d^n f(x)|$, so by Part (2) of Theorem 6.2.1, we compute

$$\begin{aligned} \mathrm{Lip}\left[\frac{d^n f(x)}{|d^n f(x)|} * \xi^n\right](x) &= \frac{1}{|d^n f(x)|} \cdot \limsup_{y \to x} \frac{|d^n f(x) \cdot (\xi^n(y) - \xi^n(x))|}{\rho(x,y)} \\ &= \frac{1}{|d^n f(x)|} \cdot \limsup_{y \to x} \frac{|f(y) - f(x)|}{\rho(x,y)} \;=\; \frac{\mathrm{Lip}[f](x)}{|d^n f(x)|}. \end{aligned}$$

Using this identity and inequality (6.2.1), we obtain the lemma with $C = 1/K$. □

We now prove Theorem 6.2.5 using Lemma 6.2.6, the conclusions of Theorem 6.2.1, and properties of the Sobolev space $H^{1,p}(X, \mu)$.

Proof of Theorem 6.2.5. We first show that δ_i^n is a derivation; it is clearly linear. To show that δ_i^n is bounded, we invoke Lemmas 6.1.10 and 6.2.6 to obtain

$$|\delta_i^n f(x)| \;\leq\; C \cdot \mathrm{Lip}[f](x) \;\leq\; C \cdot L(f)$$

for μ-a.e. $x \in X^n$. The Leibniz rule comes from the uniqueness of Cheeger differentials, in the following way. Let f and g be arbitrary functions in $\mathrm{Lip}_\infty(X)$. By Theorem 6.2.1, Cheeger differentials are unique, so it suffices to show that the map

$f \cdot d^n g + g \cdot d^n f$ satisfies equation (6.2.2) for the function $f \cdot g$. As a temporary notation, for any function $h : X \to \mathbb{R}$, put

$$Q_h(y,x) \;:=\; [h(y) - h(x)] \,/\, \rho(x,y).$$

From the elementary identity

$$Q_{fg}(y,x) \;=\; \frac{f(y)\cdot g(y) - f(x)\cdot g(x)}{\rho(x,y)} \;=\; f(y)\cdot Q_g(y,x) \;+\; g(x)\cdot Q_f(y,x),$$

we apply equation (6.2.2) to obtain

$$0 \;=\; \limsup_{y\to x}\left| g(x)\cdot Q_f(y,x) - \frac{\langle g(x)\cdot d^n f(x), \xi^n(y) - \xi^n(x)\rangle}{\rho(x,y)}\right|.$$

From the continuity of f, we have

$$\Big|\big(f(y) - f(x)\big)\cdot Q_g(y,x)\Big| \;\le\; |f(y) - f(x)|\cdot L(g) \;\to\; 0$$

as $y \to x$. It follows again from equation (6.2.2) that

$$\begin{aligned} 0 \;&=\; \limsup_{y\to x}\left| f(x)\cdot Q_g(y,x) - \frac{\langle f(x)\cdot d^n g(x), \xi^n(y) - \xi^n(x)\rangle}{\rho(x,y)}\right| \\ &=\; \limsup_{y\to x}\left| f(y)\cdot Q_g(y,x) - \frac{\langle f(x)\cdot d^n g(x), \xi^n(y) - \xi^n(x)\rangle}{\rho(x,y)}\right|. \end{aligned}$$

As a result, the map $f \cdot d^n g + g \cdot d^n f$ is the Cheeger differential of $f \cdot g$.

Claim 6.2.7. The map δ_i^n is weak-$*$ continuous on bounded sets.

Let $x \in X^n$ and let $B = B(x,r)$ be a ball in X. We first show that $\chi_B \delta_i^n$ is continuous, and by Remark 6.1.1 it suffices to show that $\chi_B \delta_i^n$ maps weak-$*$ convergent sequences in $\mathrm{Lip}_\infty(B)$ to to weak-$*$ convergent sequences in $L^\infty(B,\mu)$.

Let f and $\{f_a\}_{a=1}^\infty$ be functions in $\mathrm{Lip}_\infty(X)$ so that $f_a \overset{*}{\rightharpoonup} f$. In particular, f_a converges pointwise to f and on B, the sequence $\{f_a\}_{a=1}^\infty$ is uniformly bounded. So given a point $x_0 \in B$, for sufficiently large a we have

$$|f_a(x_0)| \;\le\; 1 + |f(x_0)|.$$

From this bound and the uniform Lipschitz continuity of the $\{f_a\}_{a=1}^{\infty}$, we obtain

$$\begin{aligned} |f_a(x)| &\leq |f_a(x) - f_a(x_0)| + |f(x_0)| + 1 \\ &\leq L(f_a) \cdot \rho(x, x_0) + |f(x_0)| + 1 \\ &\leq \sup_a L(f_a) \cdot \operatorname{diam}(B) + |f(x_0)| + 1 \ =: \ K \ < \ \infty. \end{aligned}$$

From the estimate above, we further obtain

$$\begin{aligned} \int_B \operatorname{Lip}[f_a]^p \, d\mu \ &\leq \ \int_B L(f_a)^p \, d\mu \ \leq \ \big[\sup_a L(f_a)\big]^p \cdot \mu(B) \\ \int_B |f_a|^p \, d\mu \ &\leq \ K^p \cdot \mu(B). \end{aligned}$$

So for each $p \in (1, \infty)$, the sequence $\{f_a\}_{a=1}^{\infty}$ is a bounded subset of $H^{1,p}(B, \mu)$.

By Theorem 6.1.6, for $p > 1$ the function space $H^{1,p}(B, \mu)$ is reflexive, so there exists a subsequence $\{f_{a_b}\}_{b=1}^{\infty}$ of $\{f_a\}$ and a function $h \in H^{1,p}(B, \mu)$ so that f_{a_b} converges weakly to h in $H^{1,p}(X, \mu)$. By a variant of Mazur's Lemma (Lemma 8.2.8), there is a sequence of convex combinations $\{h_b\}_{b=1}^{\infty}$ in $H^{1,p}(B, \mu)$ of the $\{f_{a_b}\}$ which converge in Sobolev norm to h. It follows that a further subsequence $\{h_{b_c}\}_{c=1}^{\infty}$ of the $\{h_b\}$ converges pointwise to h.

However, by hypothesis $\{f_a\}$ converges pointwise to f, as does the subsequence $\{f_{a_b}\}$. By Lemma 8.2.8, $\{h_b\}$ also converges pointwise to f, as does the subsequence $\{h_{b_c}\}$. It follows that $h = f$ and by Lemma 8.1.4, that $f_a \rightharpoonup f$ in $H^{1,p}(B, \mu)$.

For each $\psi \in L^q(X, \mu)$, we now define a functional on $\operatorname{Lip}_\infty(X)$ by

$$T_\psi(h) \ := \ \int_B \psi \cdot \delta_i^n h \, d\mu.$$

The action of T_ψ on $\operatorname{Lip}_\infty(X)$ is clearly linear. For $m \in \mathbb{N}$, let $C = C(n) > 0$ be the constant as given in Lemma 6.2.6. By Hölder's inequality and Lemma 6.2.6, it is also bounded with respect to the $H^{1,p}$-norm on B. Below, $\|\cdot\|_q$ and $\|\cdot\|_p$ are to be understood as the norms on $L^q(B,\mu)$ and $L^p(B,\mu)$, respectively:

$$\left|\int_B \psi \cdot \delta_i^n h \, d\mu\right| \;\leq\; \|\psi\|_q \cdot \|\delta_i^n h\|_p \;\leq\; C \cdot \|\psi\|_q \cdot \|\operatorname{Lip}[h]\|_p \;\leq\; C \cdot \|\psi\|_q \cdot \|h\|_{1,p}\,.$$

Therefore T_ψ is a bounded linear functional on a linear subspace of $H^{1,p}(B,\mu)$. By the Hahn-Banach Theorem, it extends to an element in the dual space $[H^{1,p}(B,\mu)]^*$ and we also write T_ψ for the extension. Since $f_\alpha \rightharpoonup f$ in $H^{1,p}(B,\mu)$, then by continuity we have $T_\psi(f_\alpha) \to T_\psi(f)$ and hence $T_\psi(f_\alpha - f) \to 0$.

To finish the claim, let $u \in L^1(X,\mu)$ and $\epsilon > 0$ both be given, and put $h_\alpha := f_\alpha - f$ and $C := \sup_{a\in\mathbb{N}} L(h_a)$. Observe that there is a ball B on which

$$\int_{X\setminus B} |u| \, d\mu \;<\; \frac{\epsilon}{3}.$$

In addition, $L^q(B,\mu)$ is a dense subset of $L^1(B,\mu)$, so there is a $\psi \in L^q(B,\mu)$ so that

$$\int_B |u - \psi| \, d\mu \;<\; \frac{\epsilon}{3C}.$$

By the previous case, we know that for $\psi \in L^q(B,\mu)$, there is a $N \in \mathbb{N}$ so that

$$\left|\int_B \psi \cdot \delta_i^n h_a \, d\mu\right| \;<\; \frac{\epsilon}{3}$$

holds whenever $a \geq N$. So from the previous estimates, it follows that

$$\begin{aligned}
\left|\int_X u \cdot \delta_i^n h_a \, d\mu\right| \;&\leq\; \left|\int_B u \cdot \delta_i^n h_a \, d\mu\right| + \frac{\epsilon}{3} \\
&\leq\; \left|\int_B (u-\psi) \cdot \delta_i^n h_a \, d\mu\right| + \left|\int_B \psi \cdot \delta_i^n h_a \, d\mu\right| + \frac{\epsilon}{3} \\
&\leq\; C \cdot \int_B |u-\psi| \, d\mu + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq\; C \cdot \frac{\epsilon}{3C} + \frac{2\epsilon}{3} \;=\; \epsilon.
\end{aligned}$$

This proves the claim and the theorem. □

The next corollary follows directly from Theorem 6.2.5. However, it plays a key part in Section 6.3.

Corollary 6.2.8. *Let (X, ρ, μ) be a p-PI space. Then as defined by formula* (6.2.3)*, the set of derivations $\{\delta_i^n\}_{i=1}^k$ is linearly independent in $\Upsilon(X^n, \mu)$.*

Proof. Suppose there are functions $\{\lambda_i\}_{i=1}^k$ in $L^\infty(X, \mu)$ so that $\sum_i \lambda_i \delta_i^n$ is the zero derivation. This implies that for every ball B in X and every function $f \in \mathrm{Lip}(B)$, the function $\sum_j \lambda_j \cdot \chi_B \cdot \delta_j^n f$ is identically zero. In particular, let $1 \leq i \leq k$ be arbitrary and put $f = \xi_i^n | B$. By Corollary 6.2.4 we obtain

$$0 \;=\; \sum_{j=1}^{k} \chi_B \cdot \lambda_j \cdot \delta_j^n \xi_i^n \;=\; \chi_B \cdot \lambda_i \cdot 1.$$

As a result, λ_i is μ-a.e. zero on every ball B, and hence it is the zero function in $L^\infty(X^n, \mu)$. This gives the desired linear independence. □

6.3 Cheeger's Measure Conjecture.

Following the discussion of Theorem 6.2.1, Cheeger posed a conjecture [Che99, Conj 4.63] of which one version is stated below. As mentioned before in Chapter I, it concerns the non-degeneracy of the images of coordinate charts.

Conjecture 6.3.1 (Cheeger, 1999). *Let (X, ρ, μ) be a p-PI space. Following the notation of Theorem 6.2.1, let $X = \coprod_{n=1}^\infty X^n$ and for each $n \in \mathbb{N}$, let $k = k(n)$ and let $\xi^n : X^n \to \mathbb{R}^k$. Then the image set $\xi^n(X^n)$ has positive m_k-measure.*

Remark 6.3.2. The conjecture remains open in general, but some special cases are known. We list them below.

1. Cheeger has proved Conjecture 6.3.1 in the case when the measure μ is *lower Ahlfors k-regular* [Che99, Thm 13.12]. This means that there exist constants $C > 0$ and $R > 0$ so that, for all $x \in X^n$ and all $0 < r < R$, we have

$$C \cdot r^k \;\leq\; \mu(B(x, r)).$$

2. Keith has proven Conjecture 6.3.1 in the case $k = 1$ [Kei04], but his proof is also valid for $k = 2$. Specifically, his argument reduces to the following question [Kei04, Ques II]: *Does there exist a Radon measure μ in a Euclidean space, singular with respect to Lebesgue measure, such that every Lipschitz function is classically differentiable a.e. with respect to μ?*

   In the case of $\mathbb{R}$, the answer is negative [PT95] and from it, Keith's theorem follows. In the case of $\mathbb{R}^2$, the question has also been answered negatively in [ACP05, Thm 12]. This implies the case $k = 2$ [unpublished].

We now prove Conjecture 6.3.1 for $k = 2$, and our methods are independent from those in [Kei04]. In fact, the case of $k = 2$ is a consequence of the next lemma, which is in turn a direct consequence of Theorem 5.0.7 and Corollary 6.2.8.

Lemma 6.3.3. *Let (X, ρ, μ) be a p-PI space. Under the assumptions of Theorem 6.2.1, let $X = \coprod_{n=1}^{\infty} X^n$ and for each $n \in \mathbb{N}$, let $\xi^n : X^n \to \mathbb{R}^k$. If $k = 2$, then $\xi^n_\# \mu$ is absolutely continuous with respect to Lebesgue k-measure.*

We note that the first part of the proof below holds for all $k \in \mathbb{N}$. The hypothesis $k = 2$ is used only in the second part, where we invoke Theorem 5.0.7.

Proof of Lemma 6.3.3. By Theorem 6.2.5, μ admits a linearly independent set $\{\delta_i^n\}_{i=1}^k$ in $\Upsilon(X^n, \mu)$, as defined by formula (6.2.3). Put $\nu := (\xi^n)_\# \mu$, which is a measure concentrated on the set $\xi^n(X^n)$.

For $i = 1, 2$, consider the pushforward derivations $\xi^n_\# \delta_i^n$ in $\Upsilon(\mathbb{R}^k, \nu)$. We claim that $\{\xi^n_\# \delta_i^n\}_{i=1}^k$ is a linearly independent set in $\Upsilon(\mathbb{R}^k, \xi^n_\# \mu)$, and it suffices to show orthogonality relations similar to those in Corollary 3.3.10. Indeed, by the transformation formula (3.4.2) we have, for all $u \in L^1(\mathbb{R}^k, \nu)$ and all $f \in \mathrm{Lip}(\mathbb{R}^k)$,

$$\int_{\mathbb{R}^k} u \cdot (\xi^n_\# \delta_i^n) f \, d\nu \;=\; \int_{X^n} (u \circ \xi^n) \cdot \delta_i^n (f \circ \xi^n) \, d\mu.$$

Now let $j \in \{1,2\}$ and put $f = x_j$. If $i \neq j$, then by Corollary 6.2.4 we have

$$\int_{\mathbb{R}^k} u \cdot (\xi^n_\# \delta^n_i) x_j \, d\nu \;=\; 0$$

for all $u \in L^1(\mathbb{R}^k, \nu)$, hence $\xi^n_\# \delta^n_i x_j = 0$. On the other hand, for $i \in \{1,2\}$ we have

$$\int_{\mathbb{R}^k} u \cdot (\xi^n_\# \delta^n_i) x_i \, d\nu \;=\; \int_{X^n} (u \circ \xi^n) \cdot \delta^n_i \xi^n_i \, d\mu \;=\; \int_{X^n} u \circ \xi^n \, d\mu \;=\; \int_{\mathbb{R}^k} u \, d\nu$$

for all $u \in L^1(\mathbb{R}^k, \nu)$, so $(\xi^n_\# \delta^n_i) x_i = 1$. As a result, if $\lambda_1(\xi^n_\# \delta^n_1) + \lambda_2(\xi^n_\# \delta^n_2) = 0$, then in particular we obtain the identity

$$0 \;=\; \big(\lambda_1(\xi^n_\# \delta^n_1) + \lambda_2(\xi^n_\# \delta^n_2)\big) x_i \;=\; \lambda_i \cdot (\xi^n_\# \delta^n_i) x_i \;=\; \lambda_i \cdot 1.$$

Hence $\lambda_i = 0$ for each i, and this proves the claim.

Now suppose that ν has a nonzero singular part ν_S, and let Ω be a subset of $\xi^n(X^n)$ on which ν_S is concentrated. By Part (2) of Lemma 3.3.6, the set $\{\chi_\Omega(\xi^n_\# \delta^n_i)\}_{i=1}^k$ is linearly independent in $\Upsilon(\mathbb{R}^2, \nu_S)$. However, if ν_S admits a linearly independent set of two derivations, then by Theorem 5.0.7, it cannot be singular to m_k. This is a contradiction, which proves the lemma. □

Theorem 6.3.4. *Conjecture 6.3.1 is true for $k = 2$.*

Proof. The measure μ is nonzero by hypothesis. In turn, the measure $\xi^n_\# \mu$ is also nonzero and it is concentrated on the image $\xi^n(X^n)$, hence $\xi^n_\# \mu(\xi^n(X^n)) > 0$. By Lemma 6.3.3, $\xi^n_\# \mu$ is absolutely continuous to m_2, so $m_2(\xi^n(X^n)) > 0$. □

CHAPTER VII

Derivations from Measurable Metrics: Appendix A

In this section we introduce the notion of a measurable metric and measurably Lipschitz functions on separable metric measure spaces (X, ρ, μ). From them we will show that Weaver's notion of a metric derivation agrees with Definition 3.1.1.

7.1 Measurable Metrics and Measurably Lipschitz Functions.

Recall that if A and B are subsets of X, then their symmetric difference is the set

$$A \Delta B \ := \ (A \setminus B) \cup (B \setminus A).$$

To fix notation, let $(P_\mu(X), \cong)$ denote the collection of subsets of X with positive μ-measure, under the following equivalence relation. We say that two subsets A and A' are *equivalent* if their symmetric difference has zero μ-measure. In symbols,

$$A' \cong A \iff \mu(A' \Delta A') \ = \ 0. \tag{7.1.1}$$

The following fact will be useful in choosing good equivalent sets.

Lemma 7.1.1. *Let $A \in P_\mu(X)$.*

1. *If $A' \cong A$, then $A' \cap A \cong A$.*

2. *$A \cap \mathrm{spt}(\mu) \cong A$.*

To prove the lemma, we now recall a general covering theorem which is valid on separable metric spaces. For a proof, see [Hei01, Thm 1.2].

Theorem 7.1.2. *Let (X, ρ) be a separable metric space and let $\mathcal{F}$ be a collection of balls with uniformly bounded radius. Then there is a countable, pairwise-disjoint sub-collection $\mathcal{F}'$ of $\mathcal{F}$ so that*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{F}'} 5B$$

where $5B$ is the ball with same center as B but with five times the radius.

Proof of Lemma 7.1.1. Since $A \cap A' \subset A$, the symmetric difference between $A \cap A'$ and A is precisely the set $A \setminus A'$. In symbols,

$$\begin{aligned} A \,\Delta\, (A \cap A') &= \big(A \setminus (A \cap A')\big) \cup \big((A \cap A') \setminus A\big) \\ &= (A \setminus A') \cup \emptyset \;=\; A \setminus A'. \end{aligned}$$

By hypothesis, $A \cong A'$, so $\mu(A \Delta A') = 0$. The set inclusion $A \setminus A' \subset A \Delta A'$ follows from definitions, and from this it follows that

$$\mu\big(A \,\Delta\, (A \cap A')\big) \;=\; \mu(A \setminus A') \;\leq\; \mu(A \,\Delta A') \;=\; 0.$$

This gives Part (1). Towards Part (2), we note that the symmetric difference of A and $A \cap \mathrm{spt}(\mu)$ is precisely $A' := A \setminus \mathrm{spt}(\mu)$, so it suffices to show that $\mu(A') = 0$. By the definition of $\mathrm{spt}(\mu)$, for each $a \in A'$, there is a $r_a > 0$ so that

$$\mu(B(a, 5r)) \;=\; 0$$

holds, whenever $r \in (0, r_a)$. Without loss of generality, assume that $r_a \leq 1$. The collection of balls $\mathcal{F} := \{B(a, r_a)\}_{a \in A'}$ clearly covers A'. By Theorem 7.1.2, there is a countable, pairwise-disjoint sub-collection $\mathcal{F}' := \{B(a_i, r_i)\}_{i=1}^{\infty}$ of $\mathcal{F}$ so that

$$\bigcup_{B \in \mathcal{F}} B \;\subset\; \bigcup_{i=1}^{\infty} B(a_i, 5r_i).$$

By the sub-additivity property of measures, we then obtain

$$\mu(A') \;\le\; \mu\Big(\bigcup_{i=1}^{\infty} B(a_i, 5r_i)\Big) \;\le\; \sum_{i=1}^{\infty} \mu(B(a_i, 5r_i)) \;=\; 0,$$

which gives Part (2). □

We now introduce the notion of a measurable metric ρ_μ as given in [Wea99, Ex 6.1.5]. For a metric space (X, ρ), recall that the distance between nonempty subsets A and B in X is defined by the formula

$$\operatorname{dist}(A, B) \;:=\; \inf\{\rho(a, b) \,:\, a \in A,\, b \in B\}.$$

Intuitively, ρ_μ measures the distance between subsets of positive μ-measure in X, up to the equivalence relation in equation (7.1.1).

Definition 7.1.3. Let μ be a σ-finite measure on X. *A measurable metric* $\rho_\mu : P_\mu(X) \times P_\mu(X) \to [0, \infty]$*, as induced from a metric* ρ *on* X, is a function of the form

$$\rho_\mu(A, B) \;:=\; \sup\Big\{ \operatorname{dist}(A', B') \,:\, A' \cong A,\; B' \cong B \Big\}.$$

To motivate the terminology, measurable metrics ρ_μ on (X, μ) do share similar qualities with pointwise metrics. For example, it satisfies a weak version of the triangle inequality [Wea99, pp.164]. Indeed, for all $A, B, C \in P_\mu(X)$, we have

$$\rho_\mu(A, B) \;\le\; \sup_{C' \subset C} \rho_\mu(A, C') + \rho_\mu(C', B). \tag{7.1.2}$$

Example 7.1.4. For $X = \mathbb{R}^2$ and $\mu = m_2$, let A be the union of the closed unit ball and the x_2-axis, and let A' be the ball with center $(0, 3)$ and radius 1. Then

$$\rho_\mu(A, A') \;=\; \rho_\mu(B(0, 1), A') \;=\; 1.$$

In other words, the measurable metric ignores the x_2-axis, which is a μ-null set.

Remark 7.1.5. By Lemma 7.1.1, $A' \cong A$ implies $A' \cap A \cong A$. Hence we obtain an equivalent formula for $\rho_\mu(A, B)$ if we infimize instead over subsets $A \cap A'$ and $B \cap B'$ in place of A' and B', respectively:

$$\rho_\mu(A, B) \;\equiv\; \inf \left\{ \operatorname{dist}(A \cap A', B \cap B') \;:\; A' \cong A,\, B' \cong B \right\}.$$

The notion of a measurable metric (and pseudometric) is more general than stated above[1]; for a reference, see [Wea99, Chap 6]. However, we are motivated by metric spaces (X, ρ) which are paired with geometrically compatible measures μ and therefore admit geometrically compatible measurable metrics. Thus we have not provided the most general definition here. However, in the next section we relate such measurable metrics to their respective (pointwise) metrics.

Definition 7.1.6. Given a μ-measurable function $f : X \to \mathbb{R}$, the *essential range* of f is the set

$$\mathcal{R}(f) \;:=\; \{a \in f(X) \;:\; \mu(f^{-1}(U)) \;>\; 0 \text{ for every neighborhood } U \text{ of } a\}.$$

In what follows, we consistently use the distance between the essential ranges of two functions. As a shorthand, we write $\rho_{\mu,f}(A, B) := \operatorname{dist}\big(\mathcal{R}(f|A), \mathcal{R}(f|B)\big)$.

Definition 7.1.7. Let $f \in L^\infty(X, \mu)$. The *μ-Lipschitz constant* of f is the number

$$L_\mu(f) \;:=\; \sup \left\{ \frac{\rho_{\mu,f}(A, B)}{\rho_\mu(A, B)} \;:\; A, B \in P_\mu(X) \text{ and } \rho_\mu(A, B) > 0 \right\}$$

If $L := L_\mu(f) < \infty$, then we say that f is *μ-measurably L-Lipschitz*. The space of such functions will be denoted $\mathrm{Lip}_\mu(X)$. If the constant L is understood, then we say that f is *μ-measurably Lipschitz* whenever $f \in \mathrm{Lip}_\mu(X)$.

Recall that $\mathrm{Lip}_\infty(X)$ is a dual Banach space by Theorem 2.1.3, but this fact holds more generally. By [Wea99, Cor 6.3.3], the space $\mathrm{Lip}_\mu(X)$ also enjoys this property.

[1] In fact, the weak Triangle Inequality (7.1.2) is one of the axioms of a measurable pseudo-metric.

Theorem 7.1.8 (Weaver, 1996). $\mathrm{Lip}_\mu(X)$ *is a dual Banach space under the norm*

$$\|f\|_{\mu,\mathrm{Lip}} \;:=\; \max\big(\|f\|_{\mu,\infty}, L_\mu(f)\big),$$

and on bounded sets of $\mathrm{Lip}_\mu(X)$*, the weak-$*$ topology agrees with the restriction of the weak-$*$ topology of* $L^\infty(X,\mu)$ *to the subspace* $\mathrm{Lip}_\mu(X)$*.*

Remark 7.1.9. In fact, one can prove Theorem 2.1.3 from Theorem 7.1.8 [Wea99, Ex 6.2.2]. To see this, let μ be the *counting measure* on X, that is, every one-point set in X has μ-measure 1. If $A \cong A'$, then A and A' must be the same set, and hence

$$\rho_\mu(\{a\},\{b\}) \;=\; \rho(a,b)$$

holds for all $a, b \in X$ with $a \neq b$. Observe also that each $f \in L^\infty(X,\mu)$ must be *everywhere* bounded. Similarly, the essential range $\mathcal{R}(f)$ is the image set $f(X)$, so

$$\rho_{\mu,f}(\{a\},\{b\}) \;=\; |f(b) - f(a)|$$

holds for all $a, b \in X$. This shows that $L(f) \leq L_\mu(f)$, so every μ-measurably Lipschitz function is a bounded Lipschitz function in the usual sense. In addition, for each $x \in X$, the characteristic function $\chi_{\{x\}}$ is μ-integrable. So if f and $\{f_\alpha\}_{\alpha=1}^\infty$ are functions in $L^\infty_\mu(X,\mu)$ so that $f_\alpha \overset{*}{\rightharpoonup} f$, then as $\alpha \to \infty$,

$$f_\alpha(x) \;=\; \int_X \chi_{\{x\}} f_\alpha \, d\mu \;\to\; \int_X \chi_{\{x\}} f \, d\mu \;=\; f(x).$$

Hence f_α converges *pointwise* to f, and in particular, $f_\alpha \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\infty(X)$.

Note that several facts from Remark 7.1.9 hold in greater generality.

Lemma 7.1.10. *If* $f \in \mathrm{Lip}_\infty(X)$*, then we have the estimates*

$$L_\mu(f) \;\leq\; 2 \cdot L(f) \tag{7.1.3}$$

$$L\big(f|\,\mathrm{spt}(\mu)\big) \;\leq\; 2 \cdot L_\mu(f). \tag{7.1.4}$$

Derivations on metric measure spaces obey the locality property (Theorem 3.2.1), which is restriction to subsets of positive μ-measure in X. So in light of Lemma 7.1.1, it is reasonable to restrict the setting from X to $\operatorname{spt}(\mu)$.

Proof of Lemma 7.1.10. Let A and B be distinct sets in $P_\mu(X)$ so that $\rho_\mu(A,B) > 0$, and up to the equivalence relation (7.1.1) we may assume that $\operatorname{dist}(A,B) > 0$. Note that for all $a \in A \cap \operatorname{spt}(\mu)$, we have $f(a) \in \mathcal{R}(f|A)$. Similarly, $f(b) \in \mathcal{R}(f|B)$ holds whenever $b \in B \cap \operatorname{spt}(\mu)$. So for such points a and b, we obtain

$$\rho_{\mu,f}(A,B) \;\leq\; |f(b) - f(a)|.$$

We now choose points $a \in A \cap \operatorname{spt}(\mu)$ and $b \in B \cap \operatorname{spt}(\mu)$ so that

$$\rho(a,b) \;\leq\; 2 \cdot \operatorname{dist}\big(A \cap \operatorname{spt}(\mu), B \cap \operatorname{spt}(\mu)\big) \;\leq\; 2 \cdot \rho_\mu(A,B).$$

Combining the above estimates, we obtain

$$\frac{1}{2} \cdot \frac{\rho_{\mu,f}(A,B)}{\rho_\mu(A,B)} \;\leq\; \frac{|f(b) - f(a)|}{\rho(a,b)}$$

and by taking suprema over all such points a and b and over all subsets A and B in $P_\mu(X)$, we obtain inequality (7.1.3), as desired.

To show inequality (7.1.4), let $a, b \in \operatorname{spt}(\mu)$ and let $\delta \in (0, \rho(a,b)/2)$ be arbitrary. By definition, the closed balls $A_\delta := \bar{B}(a,\delta)$ and $B_\delta = \bar{B}(b,\delta)$ are sets of positive μ-measure. If A' and B' are subsets of X that are equivalent to A_δ and B_δ, respectively, then the Triangle Inequality gives

$$\operatorname{dist}(A' \cap A_\delta, B' \cap B_\delta) \;\leq\; \rho(a,b) + 2\delta.$$

By Remark 7.1.5 and the previous estimate, we infimize over A' and B' to obtain

$$\rho_\mu(A_\delta, B_\delta) \;\leq\; \operatorname{dist}(A' \cap A_\delta, B' \cap B_\delta) \;\leq\; \rho(a,b) + 2\delta \;\leq\; 2 \cdot \rho(a,b). \tag{7.1.5}$$

Let $f \in \mathrm{Lip}_\infty(X)$ and let $\epsilon > 0$ be given. By continuity, we may choose δ sufficiently small so that the lengths $|f(a) - f(a')|$ and $|f(b) - f(b')|$ are at most ϵ, whenever $a' \in A_\delta$ and $b' \in B_\delta$. Applying the Triangle Inequality once more, we see that

$$|f(b) - f(a)| - 2\epsilon \;\leq\; |f(b') - f(a')|$$

holds for all $a' \in A_\delta$ and all $b' \in B_\delta$. It follows that

$$|f(b) - f(a)| - 2\epsilon \;\leq\; \mathrm{dist}\big(f(A_\delta), f(B_\delta)\big) \;\leq\; \rho_{\mu,f}(A_\delta, B_\delta). \tag{7.1.6}$$

Combining estimates (7.1.5) and (7.1.6), we obtain

$$\begin{aligned} |f(b) - f(a)| - 2\epsilon \;&\leq\; \rho_{\mu,f}(A_\delta, B_\delta) \\ &\leq\; L_\mu(f) \cdot \rho_\mu(A_\delta, B_\delta) \;\leq\; 2 \cdot L_\mu(f) \cdot \rho(a,b) \end{aligned}$$

for all $\epsilon > 0$. Taking the limit as $\epsilon \to 0$ and taking suprema over all points a and b, we obtain inequality (7.1.4). □

Corollary 7.1.11. *Let (X, ρ, μ) be a metric measure space, and let $C_b(X)$ denote the space of bounded, continuous functions from X to $\mathbb{R}$. Then we have the inclusions*

$$\begin{aligned} \mathrm{Lip}_\infty(X) \;&\subset\; \mathrm{Lip}_\mu(X), \\ C_b(X) \cap \mathrm{Lip}_\mu(X) \;&\subset\; \mathrm{Lip}_\infty\big(\mathrm{spt}(\mu)\big). \end{aligned}$$

Proof. The first set inclusion follows from inequality (7.1.3). For the second set inclusion, note that only continuity was needed in the proof of inequality (7.1.4). The argument then generalizes to functions in $C_b(X) \cap \mathrm{Lip}_\mu(X)$. □

7.2 Two Notions of Derivations.

We now introduce Weaver's *metric derivations* [Wea00, Defn 21] and compare the definition with Definition 3.1.1. It uses the notion of *bounded weak-$*$ continuity*, which we define in Section 8.2.

Definition 7.2.1 (Weaver, 1999). Let X be a measurable metric space. A *metric derivation* $\delta : \mathrm{Lip}_\mu(X) \to L^\infty(X,\mu)$ is a boundedly weak-$*$ continuous, linear map that satisfies the Leibniz rule.

Remark 7.2.2. It is worth noting that the setting of [Wea00] is quite general. In particular, the target space $L^\infty(X,\mu)$ in Definition 7.2.1 can be replaced by any *abelian W^*-module* over the ring $L^\infty(X,\mu)$, but we will not discuss such constructions here. For further reading about operator modules over $L^\infty(X,\mu)$, as well as over other function rings and algebras, see [Wea00, Sect 2] and [Wea96, Sect II].

Proposition 7.2.3. *Let μ be a Borel measure on X. If $\delta : \mathrm{Lip}_\mu(X) \to L^\infty(X,\mu)$ is a metric derivation, then the restriction of δ to the linear subspace $\mathrm{Lip}_\infty(X)$ is a derivation in the sense of Definition 3.1.1.*

Proof. To simplify notation, put $\delta' = \delta|\,\mathrm{Lip}_\infty(X)$. It is clear that δ' is linear and satisfies the Leibniz rule. By the definition of the L^∞-norm, we have $\|f\|_{\mu,\infty} \le \|f\|_\infty$ for all $f \in \mathrm{Lip}_\infty(X)$. From this and Lemma 7.1.10 we obtain

$$\|\delta f\|_{\mu,\infty} \;\le\; \max\big(\|f\|_{\mu,\infty}, L_\mu(f)\big) \;\le\; 2\cdot\max\big(\|f\|_\infty, L(f)\big) \;=\; 2\cdot\|f\|_{\mathrm{Lip}},$$

which shows that the operator δ' is bounded.

Lastly, we show that δ' is continuous. By hypothesis, δ is boundedly weak-$*$ continuous, so by Lemma 8.2.4, δ is weak-$*$ continuous on bounded sets.

To this end, let f and $\{f_\alpha\}_{\alpha=1}^\infty$ be functions in $\mathrm{Lip}_\infty(X)$ so that $f_\alpha \xrightarrow{*} f$, and suppose that $L = \|f\|_{\mathrm{Lip}} \vee \sup_\alpha \|f_\alpha\|_{\mathrm{Lip}}$ is a finite number. Next, fix a base point x_0 in X. Without loss, we may assume that for all $\alpha \in \mathbb{N}$ we have

$$f_\alpha(x_0) \;=\; f(x_0), \tag{7.2.1}$$

otherwise we consider $f_\alpha - f_\alpha(x_0)$ and $f - f(x_0)$ in place of f_α and f, respectively. Now let $\epsilon > 0$ be given. For any $\varphi \in L^1(X,\mu)$, the measure $d\mu_\varphi = |\varphi| d\mu$ is finite and

Borel. So by regularity there is an $R > 0$ so that, for $B = \bar{B}(x_0, R)$, we have

$$\mu_\varphi\big(X \setminus B\big) \ < \ \frac{\epsilon}{4L}. \tag{7.2.2}$$

On the other hand, $\{f_\alpha\}_{\alpha=1}^{\infty}$ is uniformly L-Lipschitz, so by equation (7.2.1) we have

$$\begin{aligned} |f_\alpha(x)| \ &\leq \ |f_\alpha(x_0)| + |f_\alpha(x) - f_\alpha(x_0)| \\ &\leq \ |f(x_0)| + L(f_\alpha) \cdot \rho(x - x_0) \\ &\leq \ |f(x_0)| + L \cdot R, \end{aligned}$$

whenever $x \in B$. By the Dominated Convergence Theorem, for any $\psi \in L^1(X, \mu)$ we have $\psi \cdot f_\alpha \to \psi \cdot f$ in $L^1(B, \mu)$; it follows that $f_\alpha \overset{*}{\rightharpoonup} f$ in $L^\infty(B, \mu)$. From Lemma 7.1.10 once more, we have $L_\mu(f_\alpha) \leq 2L$ for all α, and hence $f_\alpha \overset{*}{\rightharpoonup} f$ in $\mathrm{Lip}_\mu(B)$.

Moreover, since δ is weak-$*$ continuous on bounded sets, by Theorem 3.2.1 we obtain $\delta f_\alpha \overset{*}{\rightharpoonup} \delta f$ in $L^\infty(B, \mu)$, so there is a $\alpha \in \mathbb{N}$ so that

$$\left| \int_B \varphi \cdot \delta(f_\alpha - f)\, d\mu \right| \ < \ \frac{\epsilon}{2}. \tag{7.2.3}$$

Combining estimates (7.2.2) and (7.2.3), we further obtain

$$\begin{aligned} \left| \int_X \varphi \cdot \delta(f_\alpha - f)\, d\mu \right| \ &\leq \ \big(\|\delta f_\alpha\|_{\mu,\infty} + \|\delta f\|_{\mu,\infty}\big) \cdot \mu_\varphi(X \setminus B) + \left| \int_B \varphi \cdot \delta(f_\alpha - f)\, d\mu \right| \\ &\leq \ 2L \cdot \frac{\epsilon}{4L} + \frac{\epsilon}{2} \ = \ \epsilon. \end{aligned}$$

This gives the continuity of δ' and proves the proposition. □

CHAPTER VIII

Facts from Functional Analysis: Appendix B

In previous sections we have used many facts from functional analysis, some of which include variants of standard results in the literature. We now list these variants and for completeness, we provide their proofs.

8.1 Nets vs. Sequences.

We begin by recalling the definition of a net.

Definition 8.1.1. A set I is a *directed set*[1] if there is a relation $\prec$ on I which satisfies the following three properties:

- *reflexivity*: for each $i \in I$, we have $i \prec i$;
- *transitivity*: for all $i, j, k \in I$, if $i \prec j$ and $j \prec k$, then $i \prec k$;
- *a successor property*: for all $i, j \in I$, there is a $k \in I$ so that $i \prec k$ and $j \prec k$.

Let X be a topological space. A *net* $Y = \{y_i\}_{i\in I}$ in X is a set of points in X which is indexed by a directed set $I = (I, \prec)$. A net $Z = \{z_j\}_{j\in J}$ is a *sub-net* of Y if there is a map $\varphi : J = (J, \prec') \to I$ so that

- for each $j \in J$, we have $z_j = y_{\varphi(j)}$;

[1] We follow the convention that a directed net is not necessarily a *partially ordered set*. Specifically, we do not require the following condition: if $i \prec j$ and $j \prec i$, then $i = j$.

- for each $i \in I$, there is a $j_o \in J$ so that $i \prec \varphi(j)$ whenever $j_o \prec' j$.

Lastly, we say that a net $Y = \{y_i\}_{i\in I}$ *converges to* $x \in X$ if, for each neighborhood O of x, there is an $i_o \in I$ so that $y_i \in O$ whenever $i_o \prec i$.

Example 8.1.2. Note that $\mathbb{N}$ determines a directed set under the relation $\leq$. As a result, every sequence in a topological space is a net.

It is clear that if a net $\{v_i\}_{i\in I}$ converges in a topological space, then so does every sub-net of $\{v_i\}_{i\in I}$. The following two lemmas are elementary but useful for detecting when a net (or a sequence) converges.

Lemma 8.1.3. *In a topological space X, a net $\{x_i\}_{i\in I}$ converges to a point x if and only if the following property holds: every sub-net of $\{x_i\}_{i\in I}$ contains a further sub-net which converges to x.*

Proof. If $\{x_i\}_{i\in I}$ converges to x, then by the previous observation, every sub-net of $\{x_i\}_{i\in I}$ also converges to x. For the other direction, suppose $\{x_i\}_{i\in I}$ is a net which does not converge to x but has the sub-net property with common sub-limit x. It follows that there is a neighborhood O of x so that for each $i \in I$, there is an $i' \in I$ so that $i \prec i'$ and $x_{i'} \notin O$.

Let I' be the subset of all such indices i'. Observe that for each $i \in I$, there is a $i'_o \in I'$ so that $i \prec i'_0$, and if $i' \in I'$ satisfies $i'_0 \prec i'$, then by transitivity of the relation $\prec$, we also have $i \prec i'$. Therefore the inclusion map $I' \hookrightarrow I$ determines a sub-net $\{x_{i'}\}_{i'\in I'}$ of $\{x_i\}_{i\in I}$.

By construction, the sets $X' := \{x_{i'}\}_{i'\in I'}$ and O are disjoint, so the sub-net X' does not converge to x, and neither does any further sub-net of X'. This is a contradiction, which proves the lemma. $\square$

Arguing similarly, we obtain an analogue of Lemma 8.1.5 about convergent sequences in metric spaces.

Lemma 8.1.4. *In a metric space (X, ρ), a sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point x if and only if the following property holds: every subsequence of $\{x_n\}_{n=1}^{\infty}$ contains a further subsequence which converges to x.*

The next lemma follows easily from Lemma 8.1.3. It is specific to the setting of dual Banach spaces.

Lemma 8.1.5. *In a dual Banach space V, a net $\{v_i\}_{i\in I}$ converges to v in the weak-$*$ topology if and only if the following property holds: every sub-net of $\{v_i\}_{i\in I}$ contains a further sub-net which converges to v in the weak-$*$ topology.*

Proof. Again, one direction is clear: any sub-net of a weak-$*$ convergent net is also weak-$*$ convergent with the same limit. So suppose that $\{v_i\}_{i\in I}$ is a net in V which is not weak-$*$ convergent to v but has the above property with weak-$*$ sub-limit v.

Let W denote the pre-dual of V, and let $w \in W$ be arbitrary. If $\{v_{i_k}\}_{k\in K}$ is any sub-net of $\{v_i\}_{i\in I}$, then by the sub-limit property, there is a further sub-net $\{v_{i_{k_l}}\}_{l\in L}$ which is weak-$*$ convergent to v. So by definition of weak-$*$ convergence, we have $\langle v_{i_{k_l}}, w\rangle \to \langle v, w\rangle$.

This shows that for the net of real numbers $\{\langle v_{i_{k_l}}, w\rangle\}_{l\in L}$, every sub-net has a further convergent sub-net to the same limit $\langle v, w\rangle$. By Lemma 8.1.3, we have $\langle v_i, w\rangle \to \langle v, w\rangle$. Since w was arbitrary, we conclude that $v_i \overset{*}{\rightharpoonup} v$ in V. □

In general, $L^\infty(X, \mu)$ is not separable with respect to the norm topology. So given an uncountable subset S in the closed unit ball of $L^\infty(X, \mu)$, weak compactness only guarantees a convergent sub-net of S. In contrast, weak-$*$ compactness on the closed unit ball of $L^\infty(X, \mu)$ does produce convergent sequences. This follows from the fact that its pre-dual $L^1(X,\mu)$ is a separable Banach space. More generally, we have the following fact [Rud91, Thm 3.16].

Theorem 8.1.6. *Let V be a separable topological vector space. If K is a weak-$*$ compact subset of the dual space V^*, then K is metrizable in the weak-$*$ topology.*

We next discuss a procedure to extract a weak-$*$ convergent subsequence from a weak-$*$ convergent net. This fact is folklore and it holds in the general case of dual Banach spaces with separable pre-dual. For completeness, we include a proof.

Lemma 8.1.7. *Let V be a separable Banach space and let V^* be its Banach dual. Then every norm-bounded, weak-$*$ convergent net in V^* contains a weak-$*$ convergent subsequence with the same weak-$*$ limit.*

Proof. For each $r > 0$ and each $v^* \in V^*$, let $\bar{B}(v^*, r)$ be the closed ball with center v^* and radius r. By Theorem 3.1.6, $\bar{B}(v^*, r)$ is weak-$*$ compact, and by Theorem 8.1.6, the weak-$*$ topology restricted to $\bar{B}^*(v^*, r)$ is metrizable. As a result, v^* has a countable basis of neighborhoods in the weak-$*$ topology of $\bar{B}(v^*, r)$.

Now suppose that $\{v_i^*\}_{i\in I}$ is a net which converges weak-$*$ to v^* in V^* and suppose there is a constant $C \geq 0$ so that $\sup_i \|v_i^*\| \leq C$. By the previous argument, there is a countable basis of neighborhoods $\{U_j\}_{j=1}^{\infty}$ for v^* in the weak-$*$ topology of $\bar{B}(v^*, C)$.

Since $v_i^* \overset{*}{\rightharpoonup} v^*$ in V^*, for each $j \in \mathbb{N}$ there is an $i_j \in I$ so that $v_i^* \in U_j$ whenever $i_j \prec i$. In particular, we may choose indices $\{i_j\}_{j=1}^{\infty} \subset I$ so that $i_j \prec i_{j+1}$ holds for all $j \in \mathbb{N}$. Putting $w_j^* := v_{i_j}^*$, the sequence $\{w_j^*\}_{j=1}^{\infty}$ also converges weak-$*$ to v^*. □

8.2 Modes of Convergence.

Following [DS90, Defn V.5.3], one defines the *bounded weak-$*$ topology*[2] on a dual Banach space V^* in the following way: for each $r > 0$, it is the strongest topology

[2] Our terminology differs from that in [DS90]. On a Banach space X, the bounded weak-$*$ topology on X^* is known as *the bounded X topology on X^**.

which agrees with the weak-$*$ topology on the set $V^* \cap \bar{B}(0,r)$. Thus a subset U in V^* is open in the bounded weak-$*$ topology if and only if for every $r > 0$, the set $U \cap \bar{B}(0,r)$ is a relatively weak-$*$ open set in $\bar{B}(0,r)$.

To avoid confusion, we will denote the bounded weak-$*$ topology on V^* by τ_b and the weak-$*$ topology on V^* by τ_*. We will also say that a subset O in V is τ_b-open (resp. τ_*-open) if it is open with respect to the topology τ_b (resp. τ_*).

Remark 8.2.1. Observe that if $O \subset V$ is τ_*-open, then it is also τ_b-open. This follows because for each $r > 0$, the set $O \cap \bar{B}(0,r)$ is relatively τ_*-open in $\bar{B}(0,r)$.

The following lemma [DS90, Lem V.5.3] gives a concrete characterization of neighborhood bases for the bounded weak-$*$ topology.

Lemma 8.2.2. *Let V be a Banach space and let V^* be its dual. A neighborhood basis for the point $0 \in V^*$ for the topology τ_b consists of the sets*

$$\{v^* \in V^* \,:\, |v^*(v)| < 1 \text{ for all } v \in A\}$$

where $A = \{v_m\}_{m=1}^{\infty}$ is a sequence of elements in V that converge to 0.

We now consider linear operators between dual Banach spaces that are endowed with bounded weak-$*$ topologies.

Definition 8.2.3. Let V^* and W^* be dual Banach spaces. We say that a map $T : V^* \to W^*$ is *boundedly weak-$*$ continuous* if it is continuous with respect to the bounded weak-$*$ topologies on V^* and W^*.

Lemma 8.2.4. *Let $(V^*, \|\cdot\|_{V^*})$ and $(W^*, \|\cdot\|_{W^*})$ be dual Banach spaces, and let $T : V^* \to W^*$ be a linear map. If T is boundedly weak-$*$ continuous, then T is weak-$*$ continuous on bounded sets; that is, if $\{v_i\}_{i\in I}$ is a norm-bounded net so that $v_i \overset{*}{\rightharpoonup} v$ in V^*, then $Tv_i \overset{*}{\rightharpoonup} Tv$ in W^*.*

Proof. Let $T : V^* \to W^*$ be a linear, boundedly weak-$*$ continuous map, let $r > 0$, and let $\{v_i\}_{i\in I}$ be a net in V so that $v_i \overset{*}{\rightharpoonup} v$ in V^* and so that $\sup_i \|v_i\|_{V^*} \leq r$.

We argue by contradiction, so suppose that Tv_i does not converge weak-$*$ to Tv. By definition, there is a τ_*-open neighborhood U of Tv in W^* so that for each $i_0 \in I$, there is a $i \in I$ so that $i_0 \prec i$ and $Tv_i \notin U$.

By Remark 8.2.1, the set U is also τ_b-open in W^*. Since T is boundedly weak-$*$ continuous, the preimage set $T^{-1}(U)$ is τ_b-open in V^*. From the definition of the bounded weak-$*$ topology, it follows that there is a τ_*-open set O in V^* so that

$$T^{-1}(U) \cap \bar{B}(0,r) \;=\; O \cap \bar{B}(0,r). \tag{8.2.1}$$

Since v is a preimage of Tv, we have $v \in T^{-1}(U)$ and by lower semi-continuity of norms, we have $v \in \bar{B}(0,r)$. From equation (8.2.1) it follows that

$$v \;\in\; T^{-1}(U) \cap \bar{B}(0,r) \;=\; O \cap \bar{B}(0,r) \;\subset\; O.$$

By hypothesis, we have $v_i \overset{*}{\rightharpoonup} v$, so there is an $i_1 \in I$ so that $v_i \in O$ whenever $i_1 \prec i$. For such indices i, we invoke equation (8.2.1) again and obtain the inclusions

$$\begin{aligned} v_i \;&\in\; O \cap \bar{B}(0,r) \;=\; T^{-1}(U) \cap \bar{B}(0,r) \;\subset\; T^{-1}(U), \\ Tv_i \;&\in\; T(T^{-1}(U)) \;=\; U. \end{aligned}$$

Letting $i_0 = i_1$, we obtain a contradiction. □

Towards a proof of Lemma 3.1.3, we first show a more general fact. For dual Banach spaces $(V^*, \|\cdot\|_{V^*})$ and $(W^*, \|\cdot\|_{W^*})$ with separable pre-duals X and Y, we show that a linear map $T : V^* \to W^*$ is weak-$*$ continuous on bounded sets if and only if it is sequentially weak-$*$ continuous.

Lemma 8.2.5. *Let W be a Banach space, let ${}^*X$ be a separable Banach space, and let $T : V^* \to W^*$ be a linear map. The following propeties are equivalent:*

1. *T is weak-$*$ continuous on bounded sets;*

2. *T is sequentially weak-$*$ continuous;*

3. *For all $r > 0$, the map $T|\bar{B}(0, r)$ is weak-$*$ continuous.*

Remark 8.2.6. Property (3) means that T is continuous with respect to the weak-$*$ topology on W^* and the *relative* weak-$*$ topology on $\bar{B}(0, r)$ as induced by the weak-$*$ topology on V^*.

It is a straightforward argument to show that (3) implies (1). From the definitions in Section 3.1, it is also clear that (1) implies (2). For completeness, we prove the remaining implication.

Proof of (2) $\Rightarrow$ (3). By hypothesis, T is a sequentially continuous map between V^* and W^* with respect to their weak-$*$ topologies. Letting $r > 0$ be arbitrary, the restriction $T|\bar{B}(0, r)$ is a sequentially continuous map with respect to the relative weak-$*$ topology on $\bar{B}(0, r)$ and the weak-$*$ topology on W^*.

Since V is separable, by Theorem 8.1.6 the weak-$*$ topology on $\bar{B}(0, r)$ is metrizable. It is a fact from topology [Mun75, Thm 21.3] that on metrizable spaces, continuity and sequential continuity are equivalent. From this it follows that $T|\bar{B}(0, r)$ is continuous with respect to the relative weak-$*$ topology on $\bar{B}(0, r)$ and the weak-$*$ topology on W^*. Therefore $T|\bar{B}(0, r)$ is weak-$*$ continuous. □

In light of the previous facts, we see that Lemma 3.1.3 follows easily from the structure of the Arens-Eells space on a metric space X (see Section 2.2).

Proof of Lemma 3.1.3. Let X be a bounded, separable metric space. By Lemma 2.2.1, $AE(X^+)$ is a separable Banach space. Since $\mathrm{Lip}_\infty(X) = [AE(X^+)]^*$, the lemma follows by invoking Lemma 8.2.5. □

Lastly, the following lemma is due to Mazur [Rud91, Thm 3.13]. For previous applications we have used a corollary to the lemma, which is stated below it.

Lemma 8.2.7. *Let $(V, \|\cdot\|)$ be a Banach space. If $\{v_n\}_{n=1}^{\infty}$ is a sequence in V which converges weakly to v, then there are numbers $\{\lambda_{mn}\}_{n,m=1}^{\infty} \subset (0,\infty)$ so that*

1. $\sum_{m=1}^{\infty} \lambda_{mn} = 1$;
2. *for each $n \in \mathbb{N}$, all but finitely many terms of the sequence $\{\lambda_{mn}\}_{m=1}^{\infty}$ are zero.*
3. *the sequence $\left\{ \sum_{m=1}^{\infty} \lambda_{mn} v_m \right\}_{n=1}^{\infty}$ in V converges in norm to v.*

Lemma 8.2.8. *In Lemma 8.2.7, the coefficients can be further chosen so that*

$$\{\lambda_{mn}\}_{m=1}^{n} \;=\; \{0\}.$$

Proof of Lemma 8.2.8. Suppose that $\{v_n\}_{n=1}^{\infty}$ is a sequence in V which converges weakly to a point v in V. For $m \in \mathbb{N}$, the subsequence $S_m := \{v_n\}_{n=m}^{\infty}$ also converges weakly to v. Applying Lemma 8.2.7 to each S_m, we obtain sequences

$$v_n^m \;:=\; \sum_{k=1}^{\infty} \lambda_{kn}^m v_k, \quad n \in \mathbb{N}$$

in V so that each sequence of coefficients $\{\lambda_{kn}^m\}_{n=1}^{\infty}$ satisfy conclusions (1), (2), and (3). For each $m \in \mathbb{N}$, we have $v_n^m \to v$ as $n \to \infty$, so there is an $n_m \in \mathbb{N}$ so that $\|v - v_{n_m}^m\| < 2^{-m}$. By construction, the sequence $\{v_{n_m}^m\}_{m=1}^{\infty}$ converges in norm to v and satisfies the conclusions of Lemma 8.2.7. In addition, for each $m \in \mathbb{N}$ we have $\{\lambda_{kn_m}^m\}_{k=1}^{m} = \{0\}$. □

8.3 Operator Topologies and Compactness.

Recall that by Theorem 5.3.1, if μ is singular to Lebesgue 2-measure, then the module $\Upsilon(\mathbb{R}^2, \mu)$ is generated by at most one derivation. The proof uses Theorem

5.3.4, which concerns compactness on the space of linear operators $\mathcal{L}(Y; Z)$ from a Banach space Y to a dual Banach space Z. As a convention, we always write W for the pre-dual of Z, so $W^* = Z$. We also write $\mathcal{L} = \mathcal{L}(Y; Z)$ as a shorthand.

In this section we prove Theorem 5.3.4. To do this, we first study the relevant topology on $\mathcal{L}$, which we call the *weak-$*$ operator topology*. The discussion below follows closely the exposition in [KR97, Chap 5]. In the case where Y and Z are Hilbert spaces with $Y = Z$, then this topology is the usual *weak operator topology*.

Let $\mathcal{F}_{Y,W}$ denote the family of linear functionals $l_{y,w} : \mathcal{L} \to \mathbb{R}$ of the form

$$l_{y,w}(T) \ := \ \langle w, Ty \rangle$$

where $y \in Y$, $w \in W$, and $T \in \mathcal{L}$. Next, let $|l_{w,y}|(T) := |l_{w,y}(T)|$ and put

$$|\mathcal{F}|_{W,Y} \ := \ \{|l_{w,y}| : l_{w,y} \in \mathcal{F}_{W,Y}\}.$$

Observe that each functional $|l_{w,y}|$ satisfies the properties of a *semi-norm*, that is,

1. homogeneity: for all $T \in \mathcal{L}$ and $r \in \mathbb{R}$, we have

$$|l_{w,y}|(rT) \ = \ |r| \cdot |l_{w,y}|(T);$$

2. sub-additivity: for all $S, T \in \mathcal{L}$, we have

$$|l_{w,y}|(S + T) \ \leq \ |l_{w,y}|(S) + |l_{w,y}|(T).$$

We also note that the family $|\mathcal{F}|_{W,Y}$ is *separating* in the following sense: if T is a nonzero element in $\mathcal{L}$, then there is a $|l_{w,y}| \in |\mathcal{F}|_{W,Y}$ so that $|l_{w,y}|(T) \neq 0$.

We next state a fundamental fact from functional analysis [Rud91, Thm 1.37]. To this end, recall that on a topological vector space V,

- a subset B of V is *balanced* if $B = \{-b : b \in B\}$;

- a subset B of V is *bounded* if for every neighborhood N of 0 in V, there is a number $s > 0$ so that $E \subset tN$ whenever $s < t$.

Here tN refers to the set of vectors $\{tn : n \in N\}$.

Theorem 8.3.1. *Let V be a vector space, and suppose that $\mathcal{F}$ is a family of semi-norms on V that is separating. To each $f \in \mathcal{F}$ and each $n \in \mathbb{N}$, put*

$$V(f;n) \;:=\; \{v \in V \,:\, f(v) \;<\; 2^{-n}\}$$

and let $\mathcal{B}$ be the collection of all finite intersections of the sets $V(f;n)$. Then $\mathcal{B}$ is a convex, balanced, local basis of neighborhoods for a topology τ on V. In addition, V is a locally convex (topological vector) space such that

- *every $f \in \mathcal{F}$ is continuous;*
- *a set E in V is bounded if and only if every $f \in \mathcal{F}$ is bounded on E.*

Definition 8.3.2. The *weak-$*$ operator topology* τ_* on $\mathcal{L}$ is the locally convex topology induced by the family of semi-norms $|\mathcal{F}|_{W,Y}$. A local basis of neighborhoods for τ_* consists of all finite intersections of the sets

$$V(w,y;n) \;:=\; \{T \in \mathcal{L} \,:\, |l_{w,y}|(T) \;<\; 2^{-n}\}.$$

Lastly, let $\{X_i\}_{i=1}^{\infty}$ be a collection of topological spaces. Following [Mun75, Section 2.19] the *product topology* τ_π on $\prod_{i=1}^{\infty} X_i$ is formed by taking the sets $\mathrm{proj}_{X_i}^{-1}(U_i)$ as a sub-basis, where U_i is an open set in X_i and for each $i \in \mathbb{N}$, $\mathrm{proj}_{X_i} : \prod_{i=1}^{\infty} X_i \to X_i$ is the projection map onto X_i.

We are now ready to prove Theorem 5.3.4. It states that the closed unit ball in $\mathcal{L}$ is compact in the weak-$*$ operator topology.

Proof of Theorem 5.3.4. For each $w \in W$ and $y \in Y$, put $r_{w,y} := \|w\|_W \cdot \|y\|_Y$ and consider the intervals $B_{w,y} := [-r_{w,y}, r_{w,y}]$ in $\mathbb{R}$. We now define a map h from the unit ball of $\mathcal{L}$ to the product space $\Pi := \prod_{w\in W,\, y\in Y} B_{w,y}$ by the formula

$$h_{w,y}(T) \ := \ \langle w, Ty \rangle, \ \ h(T) \ := \ \big\{h_{w,y}(T) \ : \ w \in W, y \in Y\big\}.$$

Note that h is injective. To see this, let S and T be operators in $\mathcal{L}$ which satisfy $h(S) = h(T)$. Then for each $y \in Y$, the bounded linear functionals $w \mapsto \langle w, Sy\rangle$ and $w \mapsto \langle w, Ty\rangle$ are equal. From the duality $W^* = Z$, we have $Sy = Ty$, and because y was arbitrary, we obtain $S = T$.

From the definition of the weak-$*$ operator topology on $\mathcal{L}$ and the product topology on Π, it is clear that the map h is a homeomorphism from $(\mathcal{L}, \tau_*)$ onto its image in (Π, τ_π). Since each set $B_{w,y}$ is compact and Hausdorff in $\mathbb{R}$, by Tychonoff's Theorem [Mun75, Thm 37.3], the space Π is also compact and Hausdorff in the product topology. So if $h(\mathcal{L})$ is closed, then $h(\mathcal{L})$ is compact, and by the continuity of h, $\mathcal{L}$ is compact with respect to τ_*.

Claim 8.3.3. The set $h(\mathcal{L})$ is closed in Π.

To this end, let b be a point in the closure of $h(\mathcal{L})$. Letting $a \in \mathbb{R}$, $w_1, w_2 \in W$, and $y_1, y_2 \in Y$ all be arbitrary, for all $\epsilon > 0$ there is a $T \in \mathcal{L}$ so that each of

$$\begin{array}{ll} |b_{w_j,y_k} - \langle w_j, Ty_j\rangle|, & |a \cdot b_{w_j,y_k} - a \cdot \langle w_j, Ty_j\rangle|, \\ |b_{aw_1+w_2,y_k} - \langle aw_1 + w_2, Ty_j\rangle|, & |b_{w_j,ay_1+y_2} - \langle w_j, ay_1 + y_2\rangle| \end{array}$$

is at most $\epsilon/3$, for $\{i, j\} = \{1, 2\}$. By the Triangle Inequality, we then obtain

$$|b_{aw_1+w_2,y_1} - a \cdot b_{w_1,y_1} - b_{w_2,y_1}| \ < \ \epsilon,$$

$$|b_{w_1,ay_1+y_2} - a \cdot b_{w_1,y_1} - b_{w_1,y_2}| \ < \ \epsilon.$$

It follows that b is linear in each index. Because $b_{w,y} \in B_{w,y}$, we have $|b_{w,y}| \leq r_{w,y}$. As a result, b is a bounded bi-linear functional on $W \times Y$. In particular, for each

$y \in Y$ the map $w \mapsto b_{w,y}$ lies in W^* and can be identified with an element $z_y \in Z$. Clearly, the map $y \mapsto z_y$ is bounded because b is bounded. However, observe that the map is also linear because, for all $a \in \mathbb{R}$ and $y_1, y_2 \in Y$, we have

$$\begin{aligned}\langle w, z_{ay_1+y_2}\rangle &= b_{w,ay_1+y_2} = a \cdot b_{w,y_1} + b_{w,y_2} \\ &= a \cdot \langle w, z_{y_1}\rangle + \langle w, z_{y_2}\rangle = \langle w, a \cdot z_{y_1} + z_{y_2}\rangle.\end{aligned}$$

As a result, there is an operator $S \in \mathcal{L}$ so that

$$\langle w, Sy\rangle = \langle w, z_y\rangle = b_{w,y}$$

holds for all $w \in W$ and all $y \in Y$. Moreover, because b is bounded, so is S. This proves the claim and the theorem. $\square$

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