Equilibria in a Hotelling Model: First-Mover Advantage?

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Abstract

We study a generalized Hotelling duopoly in which a consumer's net utility from a product depends on the location of product and consumer in product attribute space, a random utility term that captures idiosyncratic preferences, and the price of the product. Our model allows us to vary the relative impact of product attribute preferences (i.e., location) and idiosyncratic preferences on consumer utility. Since the model is analytically intractable, we computationally study equilibria under both simultaneous location and sequential location by the two firms, with prices decided simultaneously after locations have been chosen.

In our numerical analysis, the simultaneous game admits only symmetric equilibria. If product attribute preferences are weak (i.e., distance costs are unimportant) and idiosyncratic preferences are also weak, both firms locate in the interior of the feasible location space. With strong product attribute preferences and weak idiosyncratic preferences, price competition is at its most intense, leading to maximal differentiation. As idiosyncratic preferences become more important, price competition is mitigated, leading to both firms locating at the market center.

In the sequential game, we find that when product attribute preferences are strong and idiosyncratic preferences are weak, the follower has a strong desire to avoid price competition and the leader experiences a first-mover advantage. However, if idiosyncratic preferences are also strong, price competition is somewhat weaker, and the leader cedes a location and profit advantage to the follower. If product attribute preferences and idiosyncratic preferences are both weak, maximal differentiation results, and entry order is irrelevant. Finally, when idiosyncratic preferences dominate, price competition is a non-issue, and both players locate at the market center. Once again, the order of entry does not matter.

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1 Introduction

Optimal positioning of a product has long been recognized as critical to its success in maintaining market-share and profit. Much of our intuition about product positioning is based on the canonical model of Hotelling (1929) and the vast literature it has generated. In this model, a product's attributes are represented by its location in the unit interval, $[0, 1]$. Consumers are identified by their ideal attributes and are distributed uniformly over the interval. A consumer suffers disutility from consuming a product with an attribute far from her ideal attribute. The disutility is modeled as an increasing function of the distance between the product's attribute and the consumer's ideal attribute.

While many models examine simultaneous location choice by firms in consumer attribute space, in practice entry into a new market is typically sequential rather than simultaneous. It is widely believed that the first mover may have an advantage in this situation, since it can locate at the market center, and thus minimize the overall distance cost its consumers must incur. Golder and Tellis (1993) report that two famous examples of this pioneer advantage are Coca Cola and Crisco Shortening. However, they also point out that many markets have been characterized by a first-mover disadvantage, rather than an advantage, such as disposable diapers (Pampers over Chux) and chocolate (Hershey over Whitman's). Previous explanations for a first-mover disadvantage have typically relied on dynamic frictions that violate the symmetry of payoffs in the game. For example, a follower may use a leader's experience to learn about the consumer distribution (Fershtman et al. 1990), or gain a technological advantage by developing a lower cost product (Tyagi 2000).[1]

Of course, products have many attributes, and modeling consumer preferences along a single dimension represents a stylized simplification. Let the single dimension represent the most important product attribute. For example, if the product is a soft drink such as Coke or Pepsi, taste is arguably the most important attribute. Nevertheless, a consumer's preferences over other attributes, such as brand image or celebrity endorsements, may also influence her decision. One way to represent preferences over other unmodeled or unobserved attributes is via a random utility for the product. The random utility term makes a consumer's choice across the two products probabilistic, rather than deterministic. Anderson et al. (1992) describes some of the basics of probabilistic choice models. In particular, if the random term comes from a doubly exponential distribution, the choice probabilities are given by a multinomial logit model.

We consider a generalization of the Hotelling duopoly model that allows for heterogeneous preferences over unmodeled attributes in this manner. For clarity, we refer to a consumer's preferences over location in the unit interval as "product attribute preferences", and to her preferences over unmodeled attributes as "idiosyncratic preferences." By varying parameter values in the model, we vary the importance of product attribute preferences and idiosyncratic preferences in the consumer's purchase decision.

The multinomial logit model does not admit closed-form solutions for prices or locations. In fact, even the deterministic choice version of the Hotelling model with prices may be intractable rendering a numerical analysis necessary (Eaton and Lipsey 1976, Prescott and Visscher 1977). Nevertheless, we show analytically that, in equilibrium, the firm that is closer to the market center has a higher

[1] There are countervailing forces as well: For example, Schmalensee (1982) recognizes that consumer loyalty is a factor that typically benefits the first mover in a dynamic setting.

price, greater expected demand, and a higher profit than its competitor. To analyze equilibrium locations, we then turn to a computational analysis of the model.

As a benchmark, we first consider simultaneous location choice by the two firms, followed by simultaneous determination of prices. We find that when product attribute preferences and idiosyncratic preferences are both weak, both firms locate in the interior of the location space, symmetrically around the market center. If product attribute preferences are strong while idiosyncratic preferences remain weak, the intensified price competition drives both firms to differentiate maximally from each other (so they locate at opposite extremes of the location space). As idiosyncratic preferences become dominant, price competition is mitigated leading to both firms locating at the market center.

We then focus on sequential location choice: the leader firm picks a location, the follower firm observes this choice and picks a location, and then both firms simultaneously choose prices. Numerically, we show that the interplay between product attribute preferences and idiosyncratic preferences critically affects whether there is a first-mover advantage:

- When product attribute preferences are strong, but idiosyncratic preferences are weak, a follower's desire to avoid price competition creates a distinct advantage for the first firm to enter the market. However, the leader also prefers to mitigate price competition, and typically does not locate at the market center.

- With strong product attribute preferences, as idiosyncratic preferences become more important, the onus gradually shifts on to the leader to mitigate price competition by ceding locations close to the market center. Equilibrium is characterized first by symmetric interior locations, so neither firm has an advantage, and then by a first-mover disadvantage. That is, the follower is closer to the market center than the leader, and has a higher profit.

- With weak product attribute preferences and strong idiosyncratic preferences, price competition is relatively weak, and both firms locate at the market center.

- Finally, weak product attribute preferences and weak idiosyncratic preferences are characterized by maximal differentiation, with firms locating at opposite ends of the unit interval.

In the Hotelling model without prices, of course, both firms locate at the market center. d'Aspremont et al. (1979) show that, when prices are introduced and transportation costs are quadratic in distance, maximal differentiation in product position results. Maximal differentiation also occurs under sequential entry if firms are restricted to locating in the unit interval, as shown by Tabuchi and Thisse (1995).

There nevertheless appears to be a popular belief that symmetric location models displays a first-mover advantage, and that the leader will inevitably occupy the market center. For example, Golder and Tellis (1993) suggest that firms which enter early and position near the center of the market can receive higher profits. Similar suggestions about the benefit (in terms of profit, market share, or both) of entering first and locating near the center appear in Boulding and Christen (2003), Lieberman and Montgomery (1988), Lilien et al. (1992), and Bohlmann et al. (2002). Theoretically, Tabuchi and Thisse (1995) show that if product positions are unrestricted relative to the demand distribution (i.e., firms can locate anywhere on the real line), a large first-mover advantage exists, with the leader occupying the market center. On the practical side, Song et al. (1999) find that managers to a significant extent believe in a first-mover advantage.

Some recent work, however, has found instances of a first-mover disadvantage, even in games with symmetric payoffs. In pure location games without pricing, the first mover may experience a disadvantage when the product attribute space has two or more dimensions (Chawla et al. (2006) and Wagener (2006)). The paper closest to ours in spirit is Rhee (2006), which computationally demonstrates a first-mover disadvantage for a range of parameter values in a Hotelling model with quadratic transportation costs and random utility. With asymmetric payoffs and the follower having a technological advantage (in the form of a lower production cost), Tyagi (2000) shows that the leader will cede the market center and may have lower profit than the follower.

Our contribution to this literature is to highlight the relative importance of product attribute preferences and idiosyncratic preferences in generating a first-mover advantage or disadvantage in the Hotelling setting. When distance costs are large, product attribute preferences are strong. In such a setting, weak idiosyncratic preferences (i.e., a low variance for the random utility term) generate a first-mover advantage. However, as idiosyncratic preferences become more important (i.e., the variance of the random term increases), a first-mover disadvantage occurs. Conversely, with weak product attribute preferences, equilibrium is symmetric, with firms locating either at the market center or at opposite ends of the unit interval.

The rest of this paper is organized as follows. We present the model in detail in §2. In §3, we provide some analytic results. The computational analysis of location equilibria and their features is provided in §4. We conclude in §5.

2 Model

We consider a generalization of the Hotelling location model. There are two firms, Leader ($\mathcal{L}$) and Follower ($\mathcal{F}$), competing in a market. There is a continuum of consumers with mass 1, located in the space $\mathcal{S}$ = [0,1], with uniform consumer density in the space. The space $\mathcal{S}$ reflects the possible values of the modeled product attribute, and a consumer is said to be "located at" $y \in \mathcal{S}$ if, for that consumer, the ideal level of the product attribute is y. In stage one of the game, the two firms each position their product in the attribute space, by choosing a location in $\mathcal{S}$. We consider both simultaneous and sequential location in the first stage. In the sequential move game, firm $\mathcal{L}$ chooses its location first, and firm $\mathcal{F}$ chooses its location after observing $\mathcal{L}$'s location choice. In stage two, after observing each other's locations, the firms simultaneously choose prices.[2]

Both firms are assumed to have zero marginal costs, allowing us to focus on the mechanics of the location and pricing game. In addition, we assume there are no fixed costs. These assumptions simplify the analysis, but do not affect our results in any qualitative manner.

Each consumer buys one unit of the good from either $\mathcal{L}$ or $\mathcal{F}$. As described below, given the prices and locations of both firms, each consumer's decision is probabilistic. Both firms maximize expected profit, which is the product of their price and overall expected demand.

Suppose a consumer located at $y \in \mathcal{S}$ purchases the good at price p_j from firm j located at

[2]Some of the previous literature allows firms to locate outside the consumer space $\mathcal{S}$. See, for example, d'Aspremont et al. (1979) and the references in Anderson et al. (1992), Chapter 8.

$x_j \in \mathcal{S}$. In addition to the price p_j, the consumer incurs a distance cost $|y - x_j|^\alpha$, where $|y - x_j|$ is the distance between x_j and y and $\alpha \geq 1$ is a parameter that affects the relative importance of distance compared to price. As can be seen, this cost is increasing in distance, reflecting the fact that consumers incur greater disutility from purchasing a product that is further away from their desired level in the attribute space. For example, if $\alpha = 1$, the distance cost is linear in distance, whereas it is quadratic if $\alpha = 2$. The restriction of α to be at least 1 ensures convexity of the distance cost. The quantity $p_j + |y - x_j|^\alpha$ may be interpreted as the full price paid by the consumer at y. The consumption value of the good, assumed to be the same across the two firms, is denoted by $v_0 > 0$. Then, the consumer's deterministic utility in this scenario is

$$u(y, x_j, p_j) = v_0 - |y - x_j|^\alpha - p_j. \tag{1}$$

In addition, the consumer obtains a random utility ϵ_{yj} from the good of each firm j. This random utility term models the impact of unmodeled, idiosyncratic factors on consumer utility. By controlling the variance of ϵ_{yj}, we will be able to control the importance of the idiosyncratic factors. Thus, his overall utility in the above scenario is

$$\tilde{u}(y, x_j, p_j) = v_0 - |y - x_j|^\alpha - p_j + \epsilon_{yj}. \tag{2}$$

Each consumer purchases one unit of the good from the firm that provides him the highest overall utility. We assume the consumer does not have an outside option, and must purchase one of the two goods.[3] Since v_0 is common across the two goods, it does not affect the relative choice between the goods. Hence, for the rest of the paper, we assume $v_0 = 0$.

As is usual in location models, the location space $\mathcal{S}$ represents product attributes, and the distance cost is the disutility to a consumer of not being able to obtain a product with his most preferred attributes. In the traditional Hotelling model, a location represents a single product attribute. In the model of Lane (1980) and the defender model of Hauser and Shugan (1983), two product attributes are valued, with the angle between them representing a consumer's relative preference across the two attributes. Further, firm location is restricted to lie on a particular arc in the positive quadrant. The angle is a scalar that varies between zero and 90 degrees, and thus can be mapped on to the unit interval. The specific functional form for utility is different in Lane (1980) and Hauser and Shugan (1983) (in particular, the net utility from a product is the consumption utility divided by the price), but nevertheless the broad features of our model can also be embedded in these other settings.

The random utility term represents an idiosyncratic preference for a product, or preferences over an attribute not explicitly considered in the location space $\mathcal{S}$. As mentioned in the introduction, such attributes may include marketing considerations such as branding.

Following Anderson et al. (1992) (see Chapter 2 in particular), we consider a multinomial logit model of random utility. That is, we assume that for each consumer location $y \in \mathcal{S}$ and each firm j, the random utility term ϵ_{yj} is independent and identically distributed with distribution $F(x) = e^{e^{-x/\mu+\gamma}}$, where $\mu > 0$ is a parameter of the logit distribution and $\gamma \approx 0.5772$ is Euler's constant. The double exponential distribution for the ϵ_{yj} terms has mean zero and variance $\mu^2\pi^2/6$. That is, μ is proportional to the standard deviation of the random utility terms.

[3]It is straightforward to extend both the model and our numerical results to the case of an outside option, which may be thought of as a third good that provides the consumer with a deterministic reservation utility.

For each consumer y and each firm j, let $u_j(y) = u(y, x_j, p_j)$ denote the deterministic utility to the consumer from purchasing and consuming the good of firm j. Then, the consumer buys the good from $\mathcal{L}$ if $u_l(y) + \epsilon_l > u_f(y) + \epsilon_f$, or $\epsilon_l - \epsilon_f > u_f(y) - u_l(y)$. Thus, the probability that the consumer at y buys from $\mathcal{L}$ is given by $q_l(y, x_l, x_f, p_l, p_f) = \text{Prob}(\epsilon_l - \epsilon_f > u_f(y) - u_l(y))$, and the probability he buys from $\mathcal{F}$ is $q_f(y, x_l, x_f, p_l, p_f) = 1 - q_l(y, x_l, x_f, p_l, p_f)$. Going forward, for brevity we often write these probabilities as $q_l(y)$ and $q_f(y)$, suppressing the locations and prices. Since $q_l(y) + q_f(y) = 1$ by definition, $q_j(y)$ may be interpreted as the market share of firm j from location y. We also refer to $q_l(y)$ as the expected demand of the consumer at y for firm $\mathcal{L}$.

The expected demand of each firm j is then given by $d_j = \int_{y \in \mathcal{S}} q_j(y) dy$, since the consumer density is uniform over $\mathcal{S}$. Note that $d_l + d_f = 1$, so that d_j may be interpreted as the overall market share of firm j. The expected profit of firm j is $\Pi_j = p_j d_j$.

Theorem 2.2 from Anderson et al. (1992) (on page 39; attributed to an unpublished document by Holman and Marley) is the key to using the multinomial logit model in our analysis:

Theorem 1 *Given deterministic utilities u_l and u_f and the logit parameter μ, the choice probabilities for a consumer located at y are given as follows:*

$$q_l(y) = \frac{e^{u_l(y)/\mu}}{e^{u_l(y)/\mu} + e^{u_f(y)/\mu}}, \qquad q_f(y) = \frac{e^{u_f(y)/\mu}}{e^{u_l(y)/\mu} + e^{u_f(y)/\mu}}.$$

We now consider the effects of μ, which is related to the variation in idiosyncratic preferences, and α, the distance cost parameter, on consumer choice.

2.1 Effect of μ: Variation in idiosyncratic preferences

Recall that the standard deviation of the random utility term is $\frac{\mu\pi}{\sqrt{6}}$. Thus, as μ increases, idiosyncratic preferences become more important, or stronger. When $\mu = 0$, this standard deviation is zero, so the model reduces to a deterministic choice model in which consumers' utilities are dictated entirely by locations and prices. Conversely, as $\mu \to \infty$, the random utility term dominates (in magnitude) the effect of location and price. As can be seen from Theorem 1, at each consumer location y, $q_l(y)$ and $q_f(y)$ each approach $\frac{1}{2}$ as $\mu \to \infty$. For intermediate values of μ, the relative magnitude of idiosyncratic tastes in overall utility depends on the locations of consumer and firm, and on the price offered by the firm.

As an illustration, suppose $\alpha = 1$ (the linear distance cost model), firm $\mathcal{L}$ is located at 0.4 with a price of 1, and firm $\mathcal{F}$ is located at 0.9, with a price of 0.8. The deterministic utility of a consumer at y for firm j's product is $u_j(y) = -|y - x_j| - p_j$, so that the expected demand for $\mathcal{L}$ and $\mathcal{F}$ of consumer y are

$$q_l(y) = \frac{e^{(-|y-x_l|-p_l)/\mu}}{e^{(-|y-x_l|-p_l)/\mu} + e^{(-|y-x_f|-p_f)/\mu}}; \qquad q_f(y) = \frac{e^{(-|y-x_f|-p_f)/\mu}}{e^{(-|y-x_l|-p_l)/\mu} + e^{(-|y-x_f|-p_f)/\mu}} \tag{3}$$

Figure 1 shows the expected demands of consumers at different locations for firm $\mathcal{L}$ for some different values of μ. The solid line shows the expected demand when $\mu = 0.001$. The expected demand at each location for such a low value of μ is close to the demand in a deterministic choice model. Suppose μ were zero, so there were no random term. While the equations in Theorem 1 cannot be used to determine the expected demand at each location, it is straightforward to directly

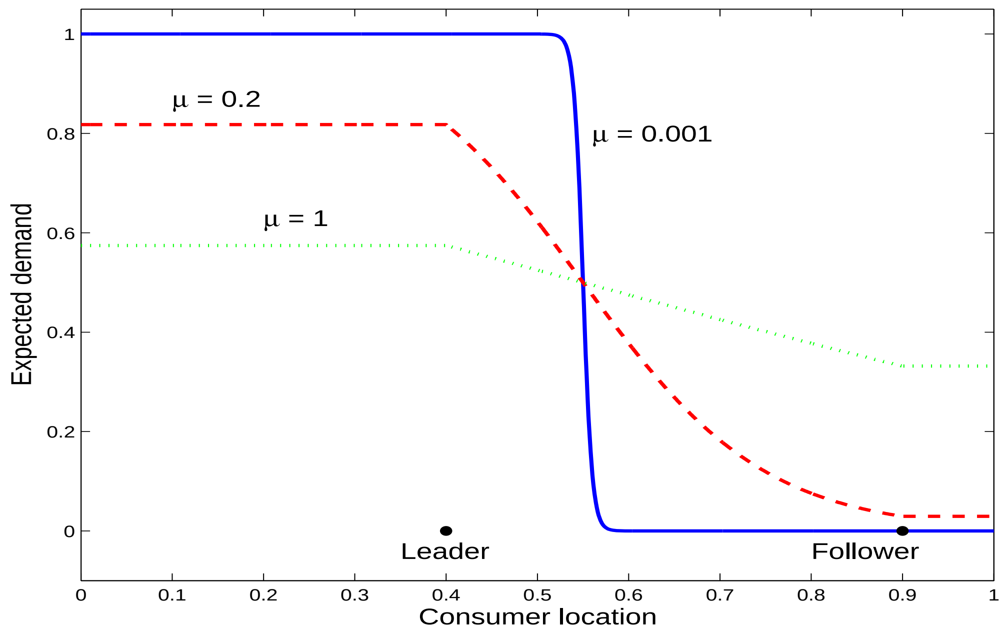

Figure 1: Expected demands for firm $\mathcal{L}$ from consumers at different locations, with $\alpha = 1$, $p_f = 1$, and $p_l = 0.8$

compute the demand. A consumer at y buys from firm $\mathcal{L}$ if $u(y, x_l, p_l) > u(y, x_f, p_f)$, or $-|y-0.4|-1 > -|y-0.9|-0.8$. Thus, all consumers located at $y < 0.55$ have a demand of 1 for firm $\mathcal{L}$, and all consumers at $y > 0.55$ have a demand of zero for firm $\mathcal{L}$. The consumer at $y = 0.55$ is exactly indifferent between the two firms.

The dashed line corresponds to $\mu = 0.2$, an intermediate value for the idiosyncratic preferences. For this value of μ, all consumers located to the left of firm $\mathcal{L}$ buy from $\mathcal{L}$ with probability approximately 0.82, so that with probability 0.18 they buy the good of firm $\mathcal{F}$. Conversely, all consumers located to the left of $\mathcal{F}$ buy from $\mathcal{L}$ with probability 0.03, and from $\mathcal{F}$ with probability 0.97.

Finally, the dotted line indicates the expected demands for a high level of idiosyncratic preferences, $\mu = 1$. For each consumer location y, the expected demand for $\mathcal{L}$ is closer to 0.5, varying from 0.57 at locations to the left of $\mathcal{L}$ to 0.33 at locations to the right of $\mathcal{F}$. In general, as μ increases, at a given location y, the expected demands for each of leader and follower $q_l(y), q_f(y)$ approach 0.5.

2.2 Effect of distance cost parameter α

For a fixed consumer at y and firm j at x_j, let $\delta_j = |y - x_j|$ denote the distance between them. Then, the deterministic utility of consumer y for the good of firm j is $u_j(y) = -\delta_j^\alpha - p_j$. Thus, the parameter α affects only the impact of distance (along the product attribute) on consumer utility, and changing α results in a change in the relative importance of product attribute and price on consumer utility. Since δ_j is typically strictly less than 1, increasing α results in a weakening of the importance of distance; i.e., a weakening of product attribute preferences.

The sensitivity of the deterministic utility of consumer y with respect to distance may be determined as follows:

$$\frac{\partial u_j(y)}{\partial \delta_j} = -\alpha \delta_j^{\alpha-1}. \tag{4}$$

In the linear distance model, with $\alpha = 1$, $\frac{\partial u_j(y)}{\partial \delta_j} = -1$, a constant, for all $y \in \mathcal{S}$. As α increases, $\frac{\partial u_j(y)}{\partial \delta_j}$ increases for low values of δ_j, but decreases for high values of δ_j. In the limit, for all $\delta_j < 1$, as $\alpha \to \infty$, $\frac{\partial u_j(y)}{\partial \delta_j} \to 0$, and distance costs, or product attribute preferences, are irrelevant to a consumer's purchase decision.

3 Solving the game

We solve the game by backward induction. Consider stage 2, where firms choose their prices, having chosen locations at stage 1. Given locations x_l, x_f and prices p_l, p_f, the expected profit of firm j, for $j = l, f$, is

$$\Pi_j(x_l, x_f, p_l, p_f) = p_j \int_0^1 q_j(y)dy, \tag{5}$$

where $q_j(y)$, given in equation (3), is a function of x_l, x_f, p_l, p_f, which are suppressed for brevity.

The prices are chosen in a Nash equilibrium of the game at this stage. Each firm chooses its price to maximize its expected profit, holding fixed the price of the other firm. The best response function of firm j is given by the first-order condition $\frac{\partial \Pi_j}{\partial p_j} = 0$, or

$$\int_0^1 q_j(y)\Big[1 - \frac{p_j}{\mu}\Big(1 - q_j(y)\Big)\Big]dy = 0. \tag{6}$$

For now, we assume that the second-order condition $\frac{\partial^2 \Pi_j}{\partial p_j^2} < 0$ is satisfied. We show this analytically when locations are symmetric about the market center in Theorem 6 below, and confirm it computationally in our numeric analysis.

Noting that $q_f(y) + q_l(y) = 1$, the best response conditions can be equivalently written as:

$$\int_0^1 q_j(y)dy - \frac{p_j}{\mu}\int_0^1 q_l(y)q_f(y)dy \quad = \quad 0, \qquad \text{for } j = l, f. \tag{7}$$

The Nash equilibrium prices at stage 2, p_l^*, p_f^*, simultaneously satisfy equations (7). Considering the expressions for $q_l(y)$ and $q_f(y)$ in Theorem 1, and noting that $u_l(y) = -|y - x_l|^\alpha - p_l$ and $u_f(y) = -|y - x_f|^\alpha - p_f$, it is immediate to see that, in general, the best response conditions that determine prices at stage 2 do not admit a closed-form solution.

However, in the special case that firms co-locate (i.e., $x_l = x_f$), equilibrium prices are straightforwardly determined. In this case, the choice probability of any consumer depends only on prices:

$$q_l(y, x, x, p_l, p_f) = \frac{e^{(-|y-x|^\alpha - p_l)/\mu}}{e^{(-|y-x|^\alpha - p_l)/\mu} + e^{(-|y-x|^\alpha - p_f)/\mu}} = \frac{e^{-p_l/\mu}}{e^{-p_l/\mu} + e^{-p_f/\mu}}.$$

Further, since firms have co-located, it must be that $p_l = p_f$, so that $\frac{e^{-p_l/\mu}}{e^{-p_l/\mu} + e^{-p_f/\mu}} = \frac{1}{2}$. Thus, the equilibrium prices are determined to be $p_l^* = p_f^* = 2\mu$.

Given equilibrium prices p_l^*, p_f^* at stage 2, we work back to stage 1. Let $\hat{\Pi}_l(x_l, x_f) = \Pi_l(x_l, x_f, p_l^*(x_l, x_f), p_f^*(x_l, x_f))$. That is, $\hat{\Pi}_l(x_l, x_f)$ is the profit accruing to $\mathcal{L}$ if the two firms locate at x_l

and x_f respectively, and choose equilibrium prices as described above. Let $\hat{\Pi}_f(x_l, x_f)$ be similarly defined.

3.0.1 Simultaneous move game

In the simultaneous move game, both firms simultaneously choose locations to maximize their own profit. Again, the optimum for each player is described by a first-order condition. In a pure strategy equilibrium, the equilibrium locations x_l^* and x_f^* simultaneously satisfy the following equations:

$$\frac{\partial}{\partial x_l}\hat{\Pi}_l(x_l^*, x_f^*) = 0; \qquad \frac{\partial}{\partial x_f}\hat{\Pi}_f(x_l^*, x_f^*) = 0 \tag{8}$$

For the simultaneous move game, we restrict attention to pure-strategy equilibria that are symmetric (that is, $x_l^* + x_f^* = 1$). Based on our numeric findings, we conjecture that the only pure-strategy equilibria in the simultaneous move game are indeed symmetric.

3.0.2 Sequential move game

In the sequential move game, given a location x_l selected by $\mathcal{L}$, firm $\mathcal{F}$ chooses a location x_l which maximizes its profit once both firms set their equilibrium prices:

$$x_f^*(x_l) = \arg\max_{x_f \in \mathcal{S}} \hat{\Pi}_f(x_l, x_f) \tag{9}$$

The corresponding first-order condition for $\mathcal{F}$'s maximization problem is $\frac{\partial \hat{\Pi}_f(x_l, x_f)}{\partial x_f} = 0$.

Continuing backwards, $\mathcal{L}$'s problem is to select a location x_l^* that maximizes profit Π_l assuming that $\mathcal{F}$ will respond with the optimal location choice $x_f^* = x_f^*(x_l^*)$, with both firms choosing equilibrium prices, $p_l^* = p_l^*(x_l^*, x_f^*)$ and $p_f^* = p_f^*(x_l^*, x_f^*)$, at stage 2. Define $\overline{\Pi}_l(x_l) = \Pi_l(x_l, x_f^*(x_l),$ $p_l(x_l, x_f^*(x_l)),\ p_f(x_l, x_f^*(x_l)))$. Since the consumer distribution is uniform, and hence symmetric about the market center at 0.5, we can without loss of generality restrict $\mathcal{L}$ to locating in the sub-interval $[0, 0.5]$. Then, $\mathcal{L}$'s optimal location satisfies:

$$x_l^* = \arg\max_{x_l \leq 0.5} \overline{\Pi}_l(x_l), \tag{10}$$

or alternatively, satisfies the first-order condition $\frac{\partial \overline{\Pi}_l(x_l)}{\partial x_l} = 0$.

Since we do not have a closed-form expression for equilibrium prices at stage 2, we cannot analytically solve for equilibrium locations in either the simultaneous or the sequential move game. Instead, we numerically solve for equilibria. The results are presented in Section 4. First, to obtain some insight into the tradeoffs faced by the firms, we consider a few special cases of the pricing game at stage 2.

3.1 Analytic results

We assume throughout that, for any pair of locations (x_l, x_f) chosen at stage 1, there exists a pure-strategy pricing equilibrium at stage 2. This assumption is borne out in our numeric calculations. Through much of this section, we also assume that the solution to the first-order conditions (7) does indeed constitute a pair of optimal prices for $\mathcal{F}$ and $\mathcal{L}$. In Theorem 6, we prove this when firms have symmetric locations. More broadly, we confirm in our numeric solution that prices are optimal for each set of parameter values and firm locations.

We begin by showing that, at stage 2, the firm with the higher price has the higher expected demand and higher profit. Recall that the overall expected demand for firm i is $d_i = \int_0^1 q_i(y)dy$, which is also the market share of firm i in our model.

Theorem 2 *In any equilibrium of the pricing subgame at stage 2, for any $i, j = l, f$, if $p_i^* > p_j^*$, then $d_i^* > d_j^*$ and $\Pi_i^* > \Pi_j^*$. That is, the firm with the higher price has a higher market share and a higher profit.*

Proof. Consider the best response conditions for $\mathcal{L}$ and $\mathcal{F}$, equations (7). The best response condition for firm j can be written as

$$\int_0^1 q_j(y)dy \quad = \quad \frac{p_j}{\mu} \int_0^1 q_l(y) q_f(y)dy. \tag{11}$$

It follows immediately that $p_i^* > p_j^* \implies d_i^* = \int_0^1 q_i^*(y)dy > d_j^* = \int_0^1 q_j^*(y)dy$. Since firm i has a higher price and a higher market share than firm j, its profit is also higher. □

Next, we show that location is critical to relative outcomes in the game. The firm that locates closest to the center will, in equilibrium, have a higher price, market share, and profit than the other firm.

Theorem 3 *For each $j = l, f$, if $|x_i - 0.5| < |x_j - 0.5|$, then $p_i^* > p_j^*$, $d_i^* > d_j^*$, and $\Pi_i^* > \Pi_j^*$.*

Proof. Suppose $|x_i - 0.5| < |x_j - 0.5|$. If $x_j < x_i < 0.5$, the result is immediate. Hence, assume that $x_i < 0.5 < x_j$. Then, we have $0.5 - x_i < x_j - 0.5$, or $x_i + x_j > 1$.

Let (p_i^*, p_j^*) denote the price equilibrium for these locations. Then, for each firm l, f, equation (11) must be satisfied. Suppose that $p_i^* \le p_j^*$. Then, it follows from (11) that $\int_0^1 q_i^*(y)dy \le \int_0^1 q_j^*(y)dy$. We will show instead that $\int_0^1 q_i^*(y)dy > \int_0^1 q_j^*(y)dy$, which provides a contradiction.

Let $y_0 = x_i + x_j - 1$, and $y_1 = \frac{x_i + x_j}{2} = \frac{y_0 + 1}{2}$. That is, the point y_1 is the midpoint of the interval $[x_i, x_j]$, and also the midpoint of the interval $[y_0, 1]$. Since $x_i + x_j > 1$, it follows that $y_0 > 0$.

Let z be any point in the interval $[y_0, 1]$. The symmetric image of z about the point y_1 is given by $\overline{z} = 1 + y_0 - z$. Therefore, we have $x_i - z = \overline{z} - x_j$, for any $z \in [y_0, 1]$. Since $p_i^* \le p_j^*$ by assumption, it must be that $q_i(z) \ge q_j(\overline{z})$ for any $z \in [y_0, 1]$. Therefore, the expected demands of firm i over the interval $[y_0, 1]$ must be greater. That is,

$$\int_{y_0}^1 q_i^*(y)dy \ge \int_{y_0}^1 q_j^*(y)dy. \tag{12}$$

Now, consider the interval $[0, y_0]$. By construction, for any point $z \in [0, y_0]$, we have $|x_i - z| < |x_j - z|$. Further, we have assumed $p_i^* \le p_j^*$. Therefore, it follows that $q_i^*(z) > q_j^*(z)$, so that

$$\int_0^{y_0} q_i^*(y)dy > \int_0^{y_0} q_j^*(y)dy. \tag{13}$$

Now, for each firm k, the overall expected demand is $\int_0^1 q_k^*(y)dy = \int_0^{y_0} q_k^*(y)dy + \int_{y_0}^1 q_k^*(y)dy$. Thus, it follows that $p_i^* \le p_j^* \implies \int_0^1 q_i^*(y)dy > \int_0^1 q_j^*(y)dy$. However, by inspection of the best-response condition (11), it must be that $\int_0^1 q_i^*(y)dy > \int_0^1 q_j^*(y)dy \implies p_i^* > p_j^*$, which is a contradiction.

Therefore, it cannot be that $p_i^* \leq p_j^*$, so that we have $p_i^* > p_j^*$. It now follows from Theorem 2 that firm i also has a higher market share and a higher profit. $\square$

Next, consider the special case in which firms have located symmetrically around the market center (0.5), i.e., when $x_l + x_f = 1$. In our numerical results, we show that, for many parameter values, both the sequential and simultaneous games have such symmetric location equilibria (indeed, in the simultaneous move game, we conjecture that the only equilibria that exist satisfy location symmetry). The following theorem shows that if locations are symmetric, then the only pure-strategy price equilibrium has equal prices.

Theorem 4 *Suppose firms' locations are symmetric about the market center (i.e., $x_i + x_j = 1$). Then, $p_i^* = p_j^*$, $d_i^* = d_j^*$, and $\Pi_i^* = \Pi_j^*$. That is, the equilibrium prices, market shares, and profits of the firms are equal.*

Proof.

Suppose $x_i + x_j = 1$. Then, the point $z = \frac{1}{2}$ lies midway between x_i and x_j. Suppose that $p_i^* < p_j^*$. Now, for every $z \in [0, \frac{1}{2}]$, consider the point $\overline{z} = 1 - z \in [\frac{1}{2}, 1]$. Since $p_i^* < p_j^*$, it follows that $q_i^*(z) > q_j^*(\overline{z})$. Hence, $q_j^*(z) = 1 - q_i^*(z) < q_i^*(\overline{z}) = 1 - q_j^*(\overline{z})$. Therefore, it follows that

$$\int_0^1 q_i^*(y)dy > \int_0^1 q_j^*(y)dy. \tag{14}$$

But this directly contradicts Theorem 2, since $p_i^* < p_j^*$ yet $d_i^* > d_j^*$. Therefore, it cannot be that $p_i^* < p_j^*$.

By the same argument, we can rule out the case that $p_i^* > p_j^*$. Hence, it must be that $p_i^* = p_j^*$. Now, since locations are symmetric about the market center, and the prices are equal, it must be that $d_i^* = d_j^*$, and hence $\Pi_i^* = \Pi_j^*$. $\square$

It is reasonable to conjecture that, as the distance between the firms increases in the location stage, the equilibrium prices will increase in the pricing subgame. The absence of an outside option then implies that overall profits will also go up. While this is hard to prove for the general case (i.e., keeping one player's location fixed while moving the other player further away), we are able to prove this for the symmetric locations case, using the fact that from the theorem above, symmetric locations imply equal prices.

Theorem 5 *Suppose firms' locations are symmetric about the market center (i.e., $x_i + x_j = 1$). Then, equilibrium prices increase as firms' locations become more distant from the center.*

Proof.

Because symmetric locations imply equal prices from Theorem 4, the market share of each firm must be exactly 1/2. Using this, the best response condition for each firm j (11) reduces to:

$$\frac{p_j}{\mu} \int_0^1 q_l(y) q_f(y) dy \quad = \quad \frac{1}{2}. \tag{15}$$

Now consider two sets of locations, with (x_l, x_f) given by $(0.5 - h_1, 0.5 + h_1)$ and $(0.5 - h_2, 0.5 + h_2)$ respectively, where $h_2 > h_1 \geq 0$. Since the firms' locations are symmetric, the equilibrium prices must be equal. Let p_1 denote the equilibrium price of each firm when the firms locate at $(0.5 - h_1, 0.5 + h_1)$. Then, p_1 satisfies equation (15).

Consider what happens when firms locate at $(0.5 - h_2, 0.5 + h_2)$. Both firms must still price equally; let p_2 denote this price. Fix a consumer $y < 1/2$, and let $\delta(y, x_i^j)$ be defined as $|y - x_i^j|$, where $i \in \{l, f\}$, and $j \in \{1, 2\}$ represents the first and second set of locations respectively. Symmetric locations, equal prices, and $h_2 > h_1$ imply that for any $y < 1/2$, we have $\delta(y, x_l^2) - \delta(y, x_l^1) \leq \delta(y, x_f^2) - \delta(y, x_f^1)$. That is, any consumer at $y < 1/2$ sees their nominal distance $\delta(y, x)$ from $\mathcal{L}$ increase by an amount that is no more than the increase in distance from $\mathcal{F}$. Now recall that $\alpha \geq 1$, which ensures that the deterministic consumer utility $-|y-x|^\alpha - p$ is convex in distance. Resultantly, we have $q_l(y, 0.5 - h_2, 0.5 + h_2, p_2, p_2) > q_l(y, 0.5 - h_1, 0.5 + h_1, p_1, p_1)$. The product $q_l(y)q_f(y)$ is therefore smaller for every $y \neq 1/2$ under the second set of locations compared to the first. For (15) to be satisfied, we therefore must have $p_2 > p_1$. □

The theorem above confirms the intuitive expectation that as firms get more distant from each other, equilibrium prices rise. Since there is no outside option, this also implies that equilibrium profits rise.

Thus for any fixed μ, the firms would like to increase the distance from each other in order to sustain higher prices and consequently higher profits. This impulse to move away from each other provides a *repulsive* force. On the other hand, for any pair of firm locations, if one firm moves closer, its expected demand increases if prices stay unchanged. This provides an *attractive* force. In general, of course, the prices will change, and the interplay between the two forces will determine the outcomes.

The equilibrium locations are then the points where the attractive and repulsive forces are in balance. Our numerical results suggest that, in the simultaneous move game, this balance is only attained when locations are symmetric around the market center, while in the sequential move game, locations need not be symmetric.

Finally, we show that, when locations are symmetric about the market center, the solution to the best response equations (7) indeed determines optimal prices; that is, a second-order condition for profit maximization is satisfied.

Theorem 6 *Given symmetric locations x_l and $x_f = 1 - x_l$, the solution to equations (7) maximizes the profit of each firm j, keeping fixed the price of the other firm.*

Proof. Consider the left-hand side of equation (7), which represents $\frac{\partial \Pi_j}{\partial p_j}$ for firm j. For convenience, let $j = l$ in what follows. Differentiating once again with respect to p_l and collecting terms, we have

$$\frac{\partial^2 \Pi_l}{\partial p_l^2} \quad = \quad -\frac{1}{\mu}\left[2\int_0^1 q_l(y)(1-q_l(y))dy - \frac{p_l}{\mu}\int_0^1 q_l(y)(1-q_l(y))dy + \frac{2p_l}{\mu}q_l^2(y)(1-q_l(y))dy\right].$$

Now, using the fact that $q_f(y) = 1 - q_l(y)$, so that $q_l(y) + q_f(y) = 1$, we can write the last two terms in the parentheses as

$$\begin{aligned} & -\frac{p_l}{\mu}\int_0^1 q_l(y)q_f(y)[q_l(y) + q_f(y)]dy + 2\frac{p_l}{\mu}\int_0^1 q_l^2(y)q_f(y)dy \\ = & \quad \frac{p_l}{\mu}\int_0^1 q_l(y)q_f(y)(q_l(y) - q_f(y))dy. \end{aligned}$$

Define $\hat{q}(y) = q_l(y)q_f(y)$. Now, from Theorem 4, symmetry implies that the solution to the best response conditions (7) satisfies $p_l^* = p_f^*$. Since locations are symmetric about the market center, and prices are equal, at any points $y \in [0,1]$ and $z = 1-y$, it follows that $\hat{q}(y)q_l(y) = \hat{q}(z)q_f(z)$. Therefore, $\int_0^1 \hat{q}(y)q_l(y) = \int_0^1 \hat{q}(y)q_f(y)$, so that

$$\frac{p_l}{\mu}\int_0^1 q_l(y)q_f(y)(q_l(y)-q_f(y))dy \quad = \quad 0.$$

Therefore, when locations are symmetric, evaluating the second partial derivative of profit with respect to price at the prices that satisfy the first-order conditions (7), we have

$$\frac{\partial^2 \Pi_l}{\partial p_l^2} \quad = \quad -\frac{1}{\mu}\left[2\int_0^1 q_l(y)(1-q_l(y))dy\right] \quad < \quad 0.$$

Hence, the second-order condition for profit maximization is satisfied for $\mathcal{L}$. A similar analysis shows that the condition is satisfied for $\mathcal{F}$. □

When firm locations are not symmetric, we verify numerically that prices found by solving equations (7) do indeed maximize profits.

4 Numerical analysis of location equilibria

We numerically determine equilibrium locations and prices for a range of (α, μ) parameters. Determining the optimal location essentially requires determining the equilibrium prices for each pair of feasible locations. However, as mentioned above, the equilibrium prices, which satisfy equations (7) do not admit a closed-form solution in general. To determine the equilibria of the overall game numerically, we trace the following steps:

1. Fix a finite location grid on $[0,1]$ for consumers. Let $\left[0, \frac{1}{m}, \frac{2}{m}, \cdots, 1\right]$ represent this grid. The mass at each point on this grid is $\frac{1}{m+1}$.

2. Fix finite location grids on $[0,1]$ for $\mathcal{L}$ and $\mathcal{F}$. Let $\left[0, \frac{1}{2n}, \frac{2}{2n}, \cdots, \frac{1}{2}\right]$ represent the location grid for $\mathcal{L}$, and let $\left[\frac{1}{2}, \frac{n+1}{2n}, \frac{n+2}{2n}, \cdots, 1\right]$ represent the grid for $\mathcal{F}$.

   Notice that we restrict $\mathcal{L}$ to locating in $[0, 0.5]$. As mentioned earlier, given the symmetry of the consumer distribution about the market center 0.5, this is without loss of generality. We also restrict $\mathcal{F}$ to locating in $[0.5, 1]$. While it is intuitive that $\mathcal{F}$ will want to locate on the other side of the market center as $\mathcal{L}$, we first verified numerically that this property holds. That is, for each pair (α, μ) of parameter values, we allowed $\mathcal{F}$ to locate on a grid that spanned $[0,1]$, and found that in each case $\mathcal{F}$ preferred to locate in the sub-interval $[0.5, 1]$.

3. For each point on the location grid of $\mathcal{L}$, determine $\mathcal{F}$'s best response. This is done as follows:

   (a) For each point on $\mathcal{F}$'s location grid, given $\mathcal{L}$'s location, determine equilibrium prices by finding a solution to equations (7). Verify that the second-order condition for optimal prices holds.

   (b) Determine $\mathcal{F}$'s profit at that point.

   (c) $\mathcal{F}$'s best response to $\mathcal{L}$'s location is the point at which $\mathcal{F}$ attains maximum profit.[4]

[4]In the numerical computations, the best response of $\mathcal{F}$ was unique for each set of parameters and fixed $\mathcal{L}$ location.

4. Determine equilibrium in the simultaneous move game as follows. We search for all symmetric equilibria of this game (since the simultaneous-move game is symmetric by construction).

   In any symmetric equilibrium, $\mathcal{L}$ is as far from the market center (at 0.5) as $\mathcal{F}$. Thus, $0.5 - x_l^* = x_f^* - 0.5$, or $x_l^* + x_f^* = 1$. Any point on $\mathcal{F}$'s best response function at which $x_f = 1 - x_l$ thus provides equilibrium locations in the simultaneous-move game. Notice that our numerical procedure finds only pure strategy equilibria.

5. Determine equilibrium in the sequential move game as follows. For each point in $\mathcal{L}$'s location grid, determine the best response of $\mathcal{F}$. Compute $\mathcal{L}$'s profit at this location. $\mathcal{L}$'s optimal action is given by the location that maximizes its profit. Again, we find pure strategy equilibria of the game.

In our numerical analysis, we set $m = 400$. That is, consumers are located at a set of 401 equidistant points in the interval $[0, 1]$, with a mass equal to 1/401 at each point. Further, we set $n = 200$, so that the feasible location grid for $\mathcal{L}$ consists of 201 equidistant points in $[0, 0.5]$, and that for $\mathcal{F}$ consists of 201 equidistant points in $[0.5, 1]$.

Finally, we consider values of μ in the range 0.2 to 2.0, with steps of 0.1, and values of α in the range 0.125 to 3.0, with steps of 0.125. We explored larger values of μ and α, and found qualitatively similar equilibrium patterns. Our results extrapolate to (α, μ) values not explicitly displayed.

4.1 Linear distances model: Follower's best response

Before we turn to equilibria, we first consider the best response of $\mathcal{F}$ for each location $\mathcal{L}$ may choose. In the simultaneous-move game, $\mathcal{L}$'s best response is exactly symmetric. In the sequential move game, $\mathcal{L}$ takes $\mathcal{F}$'s best response into account in choosing its optimal location.

In Figure 2, we exhibit $\mathcal{F}$'s best response for each location of $\mathcal{L}$, in the linear distance model (i.e., fixing $\alpha = 1$). The best responses are shown for three different levels of μ.

Recall that the consumer distribution is symmetric around the market center at 0.5. Thus, in the absence of price competition, firms would wish to be at the center, to minimize their distance to the average consumer. However, price competition is at its most intense when the firms co-locate, and provides a motive for a firm to locate far from its competitor. Since demand at each consumer location is nonlinear in prices and in the idiosyncratic preferences (i.e., μ), the tradeoff between these two effects is also nonlinear.

For low values of μ (in the figure, $\mu = 0.2$), $\mathcal{F}$'s best response is mostly upward-sloping as $\mathcal{L}$ approaches the center, starting with a location of 0. That is, the closer $\mathcal{L}$ is to the market center, the further away from the center $\mathcal{F}$ wishes to be. This is because low values of μ correspond to relatively unimportant idiosyncratic preferences, so the price competition effect dominates. Thus, firms have a natural desire to separate out in location.

As μ increases from 0.2, idiosyncratic preferences become more important, mitigating the effects of price competition. Though not shown in the figure, the nonlinear tradeoff between the distance and price competition effects leads to $\mathcal{F}$ being further away from the center if $\mathcal{L}$ locates at 0 (the left extreme of its location space), and closer to $\mathcal{L}$ if $\mathcal{L}$ locates at 0.5, the market center.

Further increases in μ are characterized by further mitigation of price competition, so that at $\mu = 0.95$, $\mathcal{F}$'s best response is downward-sloping, and for yet higher values of μ, $\mathcal{F}$ comes in even

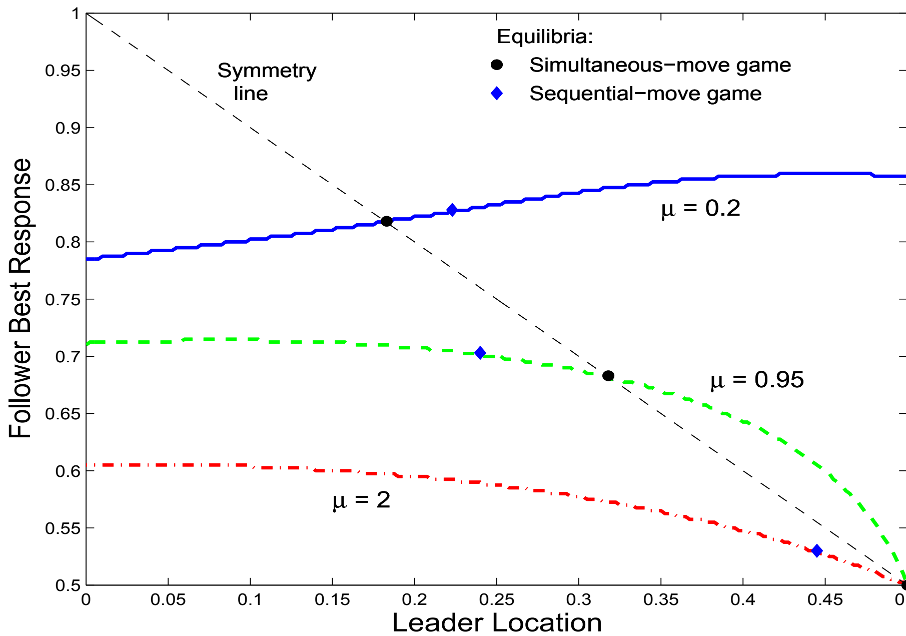

Figure 2: $\mathcal{F}$'s best response given $\mathcal{L}$'s location, for $\alpha = 1$

closer to the market center at each $\mathcal{L}$ location. That is, as μ increases in this range, $\mathcal{F}$'s entire best response curve falls toward 0.5. Notice also that $\mathcal{F}$'s best response is most sensitive to changes in μ when $\mathcal{L}$ locates at the market center, and least sensitive when $\mathcal{L}$ locates at 0.

4.1.1 Equilibria in the simultaneous-move game for $\alpha = 1$

Now, consider equilibria of the simultaneous-move game. As mentioned earlier, in a symmetric equilibrium, $x_f = 1 - x_l$. This line is shown as the dashed "Symmetry line" in Figure 2. All points where the best response functions intersect with this line represent symmetric equilibria in the simultaneous move game (these points are the solid circles in the figure). As may be seen, for low values of μ, the equilibrium is unique (in the class of symmetric pure-strategy equilibria), and is characterized by both firms being in the interior of their respective location spaces.

Initially, as μ increases, firms move further away from the center (not shown in the figure), and then move back in toward the center once idiosyncratic preferences are sufficiently important. At $\mu = 0.95$, there are two pure-strategy equilibria in the simultaneous-move game: one with locations of $\mathcal{L}$ and $\mathcal{F}$ at (0.32,0.68), and the other with both firms at the market center, (0.5, 0.5). When idiosyncratic preferences are even more important (in the figure, at $\mu = 2$), price competition is relatively weak, and the distance effect dominates. As a result, both firms locate at the market center in equilibrium.

4.1.2 Equilibria in the sequential-move game for $\alpha = 1$

Figure 2 also shows the location equilibria of the sequential-move game (the equilibrium points are shown as solid diamonds). Low levels of idiosyncratic preferences (i.e., low values of μ) are characterized by a first-mover advantage. That is, $\mathcal{L}$ is closer to the market center than $\mathcal{F}$ (since the equilibrium point is above the symmetry line when $\mu = 0.2$), and hence by Theorem 3, has a higher profit. Conversely, high levels of idiosyncratic preferences are characterized by a first-mover disadvantage: $\mathcal{F}$ is closer to the center than $\mathcal{L}$.

Compare the sequential-move game equilibrium locations with those in the simultaneous-move game. At low values of μ (e.g., $\mu = 0.2$), $\mathcal{F}$'s best response function is upward-sloping at the simultaneous-move game equilibrium. That is, by moving in closer to the market center than in the simultaneous-move equilibrium, $\mathcal{L}$ can drive $\mathcal{F}$ further away from the center. This results in $\mathcal{L}$ being closer to the market center in the equilibrium of the sequential-move game.

Conversely, at higher values of μ ($\mu = 0.95$ and 2), $\mathcal{F}$'s best response function is downward-sloping at the simultaneous-move game equilibrium locations. In this case, if $\mathcal{L}$ were to come in closer to the center, $\mathcal{F}$ would respond by moving in as well, which would exacerbate price competition between the firms. Thus, $\mathcal{L}$ chooses to move away from the center, conceding a location advantage to $\mathcal{F}$.

Therefore, as compared to the simultaneous-move game equilibrium locations, $\mathcal{L}$'s choice in the sequential-move game appears to be dominated by the price competition effect. When $\mathcal{L}$ can come in towards the market center while still maintaining a distance between the firms (which happens for low values of μ), it does so. When moving towards the center would increase price competition, it moves further away from the center.

4.1.3 Complete equilibria for linear distances

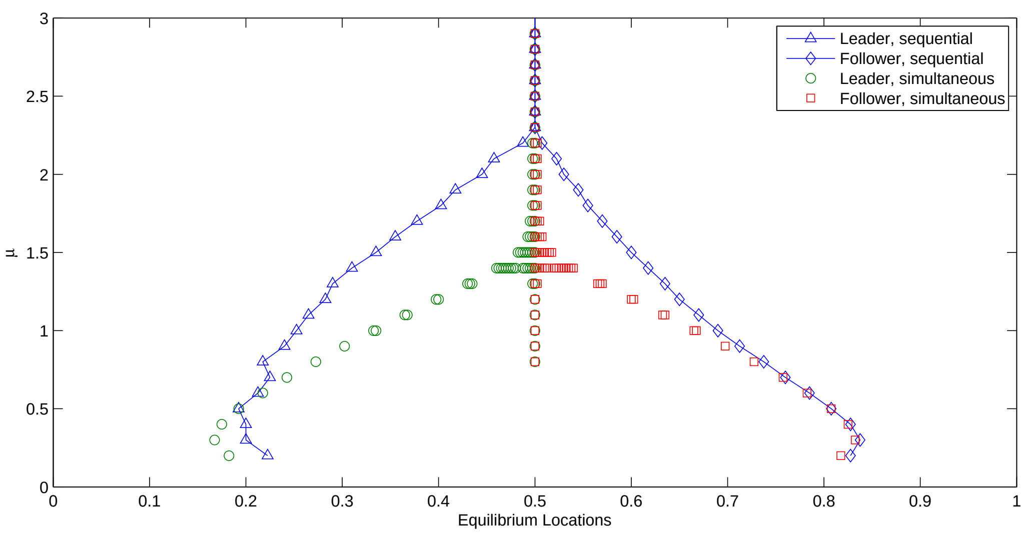

Figure 3: Complete equilibrium locations for $\alpha = 1$.

Figure 3 shows all equilibrium locations for different values of μ for $\alpha = 1$. Consider the simultaneous move game first. As discussed above, for extremely low values of μ, the unique pure-strategy equilibria are characterized by players maintaining a sufficient distance between them. As μ increases, not only do the players' locations get closer to the market center, but both players locating at the center can also be sustained in equilibrium. For some values of μ, the figure shows more than one non-central equilibrium pair. This is due to the imprecision introduced by taking a finite location grid in the numerical analysis. As the follower's best response curve in Figure 2 becomes tangential to the symmetry line, the numerical computation reports multiple equilibria in the region where the two curves are close to coinciding.

As μ increases further, idiosyncratic factors dominate and the advantage of players locating away from each other vanishes. Thus, the only equilibrium sustained in the simultaneous move game is the one where both players locate at the center.

Figure 3 also shows the equilibrium locations in the sequential move game. The overall pattern is the same: initially, both players maintain some amount of distance between each other; the distance initially increases with increasing μ, and then decreases as both players move towards the market center. The reason for this overall pattern is the same as in the simultaneous move game: increasing importance of the idiosyncratic factors (increasing μ) reduces the benefits of maintaining inter-firm distance.

It is also straightforward from the figure to identify regions under which one of the players dominates the other in terms of profit and market share. From Theorem 3, the firm closer to the market center has a higher price, market share, and profit. Therefore, we find that for low values of μ (approximately in the range [0.2,0.8]), the first mover maintains an advantage, while for intermediate values of μ (in the approximate range [0.8, 2.2]), the follower has the advantage. As in the simultaneous move game, once μ is sufficiently high only the central equilibrium is obtained.

An interesting observation is that for μ in the range approximately [1.8, 2.2], the simultaneous move game has both players locating at the center while separation is maintained in the sequential move game. In fact, for a larger range of μ values, the equilibrium distance in the sequential move game is significantly greater than the equilibrium distance in the simultaneous move games. Recall that there is no outside option in this game, and prices increase with increase in distance. Therefore, in this range of μ, the sequential move game results in higher prices faced by consumers, and overall higher profits earned by the two firms.

A brief explanation of this phenomenon is as follows. For these values of μ, the idiosyncratic factors matter somewhat, but there is still room for differentiation between firms. Re-examining Figure 2, it may be noted that, at $\mu = 2$, the best-response curve of Follower is downward-sloping at the point of intersection with the symmetry line. In the simultaneous move game, the strength of the idiosyncratic factors allows the two players to move in closer to each other. However, in the sequential game, the downward-sloping best response curve of $\mathcal{F}$ indicates that there are higher profits to be made if both players move away from each other. Since the slope of the line is less than 45 degrees, the movement by $\mathcal{F}$ away from the market center is slower than the movement of $\mathcal{L}$. That is, in equilibrium $\mathcal{F}$ is closer to the market center than $\mathcal{L}$, and as a result earns a higher profit than $\mathcal{L}$. However, since $\mathcal{L}$ is maximizing her own profit rather than minimizing her relative disadvantage, it is still in $\mathcal{L}$'s best interest to sustain this asymmetric equilibrium.

4.2 Equilibria as α and μ vary

We now consider equilibria in the game for different values of α and μ. Thus, we allow the attribute preferences to vary (by varying α), along with variation in the idiosyncratic preferences. This allows us to study the relative impact of the attribute and idiosyncratic preferences, and their impact on equilibrium locations and firm advantages.

4.2.1 Simultaneous-move game

The equilibria for the simultaneous move game are shown in Figure 4. We identify three classes of equilibria in the simultaneous move game. All the equilibria we found are symmetric in location (so $x_l^* + x_f^* = 1$). Thus, by Theorem 4, the firms having equal prices and profits in all these equilibria. As already seen in the case of $\alpha = 1$, multiple equilibria may exist in the simultaneous move game.

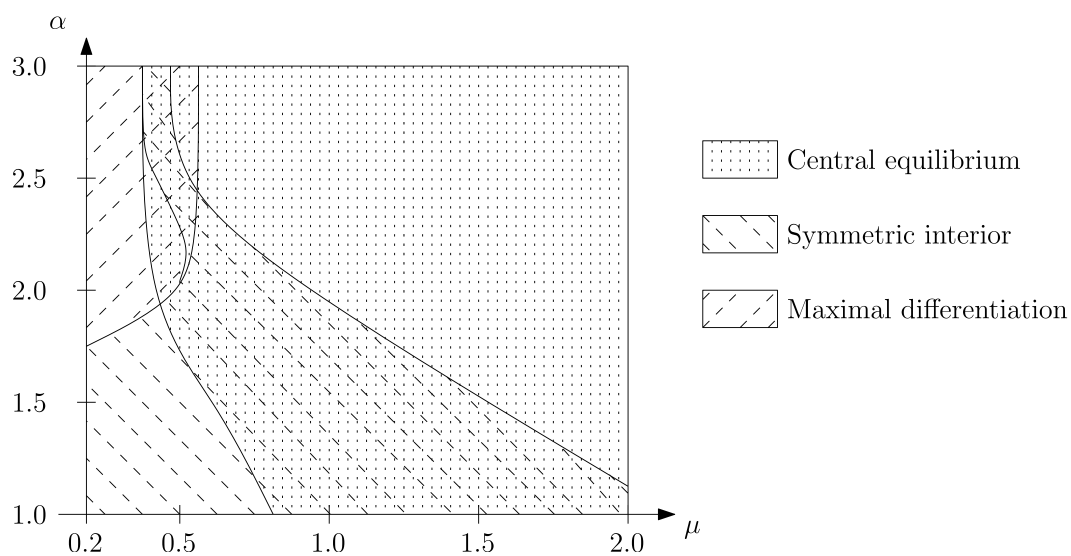

Figure 4: Regions of different equilibria in simultaneous move game

1. Maximal differentiation: The firms are at the extremes of the location space, with $x_l^* = 0$ and $x_f^* = 1$. These equilibria are observed for $\alpha \geq 1.7$, with small values of μ.

2. Central location: Both firms are at the market center, with $x_l^* = x_f^* = 0.5$. For each value of α, these equilibria are observed when μ is sufficiently high, with the threshold value of μ falling as α increases.

3. Symmetric interior location: Firms are strictly in the interior of the location space, with $x_l^* \in (0, 0.5)$, $x_f^* \in (0.5, 1)$, and $x_f^* = 1 - x_l^*$. These equilibria are observed for all values of α, with a range of μ that declines as α increases.

4.2.2 Elasticities of profit

To explain these equilibria, we consider the sensitivity of profit with respect to price and distance. Since the level of profit changes with α and μ, we use elasticity as a scale-free measure of sensitivity. Formally, the elasticity of the profit of firm i with respect to its own price is defined as $\eta_p^\Pi = \frac{\partial \Pi_i(x_l, x_f, p_l, p_f)/\partial p_i}{\Pi_i(x_l, x_f, p_l, p_f)/p_i}$. Thus, the elasticity varies with the locations and prices of both firms. Further, although α and μ are suppressed as arguments of the profit function, both profit and elasticity vary with α and μ as well.

To standardize our elasticity computation, we fix firm locations at $x_f = 0.25$ and $x_l = 0.75$. We choose these locations because they are the center of the respective location spaces for the two firms, and provide a notion of average elasticity (average taken over all firm locations). Given these locations, we determine equilibrium prices $p_l^*(x_l, x_f)$ and $p_f^*(x_l, x_f)$ for each (α, μ) pair (since the locations are symmetric, the equilibrium prices are equal across the firms). Defining a change in price for $\mathcal{L}$ of $\Delta_p = 0.0025$, we then compute the elasticity of profit of $\mathcal{L}$ with respect to its price for a given (α, μ) pair as follows:

$$\eta_p^\Pi = \frac{(\Pi_l(x_l, x_f, p^*(x_l, x_f) + \Delta_p, p^*(x_l, x_f)) - \Pi_l(x_l, x_f, p^*(x_l, x_f), p^*(x_l, x_f)))/\Delta_p}{\Pi_l(x_l, x_f, p^*(x_l, x_f), p^*(x_l, x_f))/p^*(x_l, x_f)}$$

In computing the elasticity of firm i's profit with respect to its own location, we are careful to ensure that firms choose optimal prices given the new location. As before, we define $x_l = 0.25$ and $x_f = 0.25$. Define a change of location for $\mathcal{L}$ of $\Delta_x = 0.0025$. For a given (α, μ) pair, we compute the elasticity of $\mathcal{L}$'s profit with respect to its own location as follows:

$$\eta_x^\Pi = \frac{(\Pi_l(x_l + \Delta_x, x_f, p_l^*(x_l + \Delta_x, x_f), p_f^*(x_l + \Delta_x, x_f)) - \Pi_l(x_l, x_f, p^*(x_l, x_f), p^*(x_l, x_f)))/\Delta_x}{\Pi_l(x_l, x_f, p^*(x_l, x_f), p^*(x_l, x_f))/x_l}$$

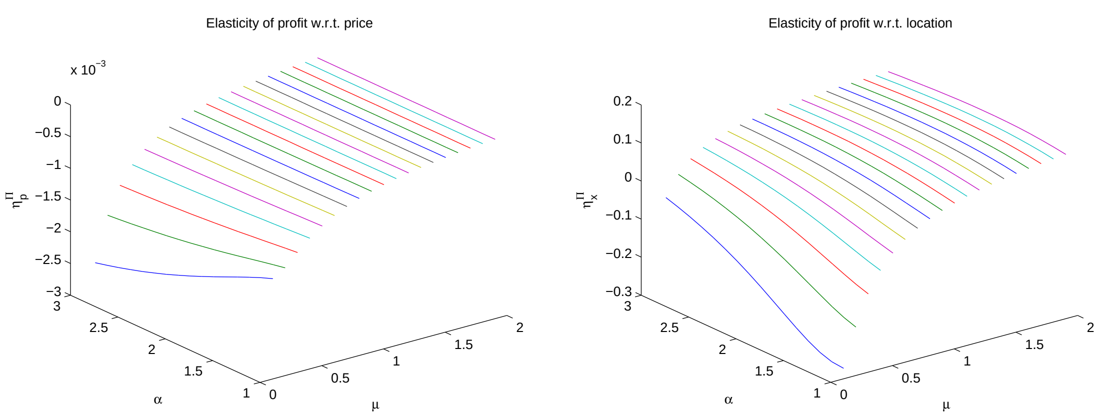

Figure 5: Elasticities of $\mathcal{L}$'s profit with respect to (a) price, and (b) location.

Figure 5 displays the elasticities of $\mathcal{L}$'s profit with respect to location and price (η_x^Π and η_p^Π), as α and μ vary. Consider first the elasticity of profit with respect to price (shown in the left of Figure 5). Note that the elasticity is consistently negative; since the initial price is optimal given the firm locations, any change in price leads to a smaller profit. For small μ, the elasticity increases in magnitude as α increases. As α becomes high, distance cost becomes less relevant (since the distance from any consumer to either firm is less than 1, increasing α reduces the importance of distance in consumer utility). Thus, consumers are more sensitive to price, resulting in an increased profit elasticity. For large μ, the elasticity is more or less flat in α, and relatively small in magnitude: as idiosyncratic preferences become more important, consumers are less sensitive to price.

Next, consider the elasticity with respect to location, η_x^Π (shown on the right in Figure 5). When μ is small, η_x^Π is negative, with a large magnitude for α close to 1 (i.e., when distance cost is more relevant to the consumer). At these values of (α, μ), there is a strong repellent force on the firm. That is, fixing $\mathcal{F}$ at $x_f = 0.75$, firm $\mathcal{L}$ would like to move further away than 0.25, to locations closer to zero. For large μ, the elasticity η_x^Π is positive, but small in magnitude. Idiosyncratic preferences are more important when μ is large, so the impact of location is small and firms are willing to locate closer together. Hence the positive η_x^Π, pointing to an attractive force between firms.

It is important to note that we measure elasticities at the fixed locations of $x_l = 0.25$ and $x_f = 0.75$. These elasticities vary across locations, especially when μ is small. Although equilibrium locations themselves change as α and μ change, nevertheless the elasticities we construct at fixed locations provide insight into the outcomes of the overall game.

First, consider the region with (approximately) $\mu \geq 0.4$ and $\alpha \geq 3 - \mu$. For these parameter values, idiosyncratic preferences are important (μ is large) and distance costs relatively unimportant (α is also large). In this region, the location elasticity of profit, η_x^Π, is positive, so $\mathcal{L}$ would like to come in closer toward the center, when $\mathcal{F}$ is fixed at $x_f = 0.75$. Further, the price elasticity of profit, η_p^Π, is small in magnitude, so profit is relatively invariant to price. Thus, there is an attractive force between the firms, driving them to locate towards each other. The most that they can get towards each other is the center, resulting in the central equilibrium.

Next, consider the region with $\mu \leq 0.4$ and $\alpha \geq 1.5$. Here, idiosyncratic preferences are relatively unimportant, as are distance costs. Thus, consumers are highly sensitive to price. As a result, the price elasticity η_p^Π is negative and has large magnitude. The desire to avoid price competition results in the location elasticity, η_x^Π, also being negative. There is a strong repulsive force causing firms to differentiate in order to raise prices. Hence, the equilibrium is characterized by maximal differentiation.

Finally, consider the remaining region: small-to-moderate values of both μ and α. Here, distance costs are important, resulting in a negative location elasticity, η_x^Π. Idiosyncratic preferences are only moderately important, so η_p^Π is small in magnitude. The negative location elasticity points to gains from differentiation, whereas the small price elasticity suggests the gains are limited. As a result, the equilibrium is characterized by symmetric interior locations.

4.2.3 Sequential-move game

We now turn to the sequential game. The equilibria are depicted in Figure 6.

We find that there are five classes of equilibria. The first three of these correspond to similar equilibria in the simultaneous move game, with firms having equal prices and profits. The last two exhibit either a first-mover advantage or a first-mover disadvantage.

1. Maximal differentiation: The firms are at the extremes of the location space, with $x_l^* = 0$ and

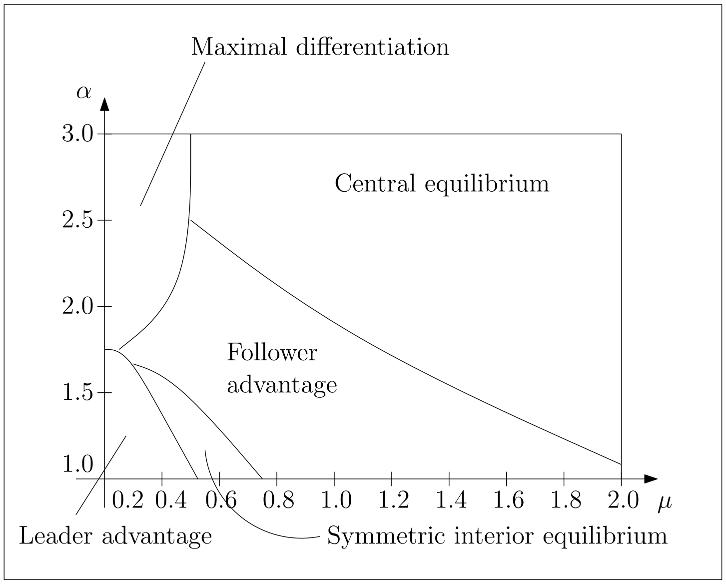

Figure 6: Regions of different equilibria in sequential move game

$x_f^* = 1$. The parameter region for which these equilibria are found roughly corresponds to the maximal differentiation region of the simultaneous move game.

2. Central equilibrium: Both firms are at the market center, with $x_l^* = x_f^* = 0.5$. Again, the parameter region corresponds roughly to the central equilibrium region in the simultaneous move game

3. Symmetric interior equilibrium: Firms are strictly in the interior of the location space, with $x_l^* \in (0, 0.5)$, $x_f^* \in (0.5, 1)$, and $x_f^* = 1 - x_l^*$. Observe that the parameter region in which these equilibria hold is a small subset of the symmetric interior region of the simultaneous move game. In other areas where the simultaneous move game has a symmetric interior equilibrium, we find that one of the two firms has a profit advantage in the sequential game.

4. Leader advantage: Firm $\mathcal{L}$ is closer to the center than the follower, with $0.5 - x_l^* < x_f^* - 0.5$. Theorem 3 implies that $\mathcal{L}$ has a profit advantage in these equilibria, which is confirmed in our numerical results. The parameter region in which these equilibria obtain is roughly the triangle bounded by $(\mu = 0.2, \alpha = 1.8)$ and $(\mu = 0.45, \alpha = 1)$.

   The maximum Leader advantage we observe is when $\alpha = 1.5$ and $\mu = 0.2$. At these values, we find $x_l^* = 0.2525$, $x_f^* = 0.9775$, $p_l^* = 0.9382$ and $p_f^* = 0.8098$. Thus, $\mathcal{L}$ is approximately halfway between one end of the market and the market center, while $\mathcal{F}$ is almost at the other end of the market. This locational advantage of $\mathcal{L}$ is reflected in the higher profit observed: $\Pi_l^* = 0.5036$, with $\Pi_f^* = 0.3751$, so that the leader has a profit advantage of 34.24%.

5. Follower advantage: Firm $\mathcal{F}$ is closer to the center $(0.5 - x_l^* > x_f^* - 0.5)$, so, by Theorem 3, has a profit advantage. As shown in Figure 6, the parameter region in which these equilibria obtain consists of intermediate values of μ, for α ranging from 1 to approximately 2.5.

The maximum follower advantage is when $\alpha = 2.5$ and $\mu = 0.6$. At these values, $\mathcal{L}$ locates at $x_l^* = 0$ and $\mathcal{F}$ locates at $x_f^* = 0.7425$, with prices $p_l^* = 1.2648$ and $p_f^* = 1.3846$. The resultant profits are $\Pi_l^* = 0.6038$ and $\Pi_f^* = 0.7236$ respectively, translating to a 16.56% profit disadvantage for $\mathcal{L}$.

In comparing the equilibria of the simultaneous and sequential move games, a striking feature is the close overlap of the regions with central equilibrium and maximal differentiation. This is no coincidence: we prove below that if the sequential-move game leads to a symmetric equilibrium, then the same symmetric equilibrium must also be an outcome of the simultaneous move game.

Theorem 7 *If for some (α, μ), the sequential game has a symmetric equilibrium, the same set of locations and prices constitute an equilibrium of the corresponding simultaneous move game.*

Proof. Since symmetric locations imply equal prices by Theorem 4, let $(x_l^*, 1 - x_l^*, p^*, p^*)$ denote the equilibrium locations and prices in the sequential game. Suppose for contradiction that these locations and prices do not constitute an equilibrium in the simultaneous game. Then, given that firm $\mathcal{L}$ is located at x_l^*, the best response of $\mathcal{F}$ in the simultaneous move game must be given by some $x'_f \neq x_f^*$. That is, $\Pi_f(x_l^*, x'_f, p_l^*(x_l^*, x'_f), p_f^*(x_l^*, x'_f)) > \Pi_f(x_l^*, x_f^*, p^*, p^*)$. But then, in the sequential game as well, when $\mathcal{L}$ locates at x_l^*, firm $\mathcal{F}$ should locate at x'_f rather than at x_l^*, contradicting the assumption that $(x_l^*, 1 - x_l^*, p^*, p^*)$ constitutes an equilibrium outcome of the sequential game. $\square$

Note that the converse is not true; in fact, if it were, there would be no asymmetry in profits in the sequential move game. However, Theorem 7 indicates the following: if (α, μ) are such that we are in the region of Maximal differentiation or Central equilibrium, then entry order does not matter because whether the game is played with sequential entry or simultaneous entry, the same equilibria will arise.

4.2.4 Asymmetric equilibria in sequential move game

Let us now examine the asymmetric equilibria, where one or the other firm has an advantage. If we examine the elasticities of profit in Figure 5, we see that the asymmetric equilibria occur approximately in the regions where η_x^Π is negative. At these values of μ and α, firms prefer to be differentiated: if locations are fixed at 0.25 for $\mathcal{L}$ and 0.75 for $\mathcal{F}$, firm $\mathcal{L}$ would rather be further from the market center.

Consider first the case when η_x^Π is negative, and large in magnitude. This means that once $\mathcal{L}$ has chosen its location, $\mathcal{F}$ has a strong incentive to locate far away from the center (and also far away from $\mathcal{L}$) in order to try and maintain a sufficient distance to keep prices high. An example of this is when $\alpha = 1$ and $\mu = 0.2$. This strong repellent force on $\mathcal{F}$ is reflected in the best-response curve: an examination of Figure 2 shows that $\mathcal{F}$'s best-response is upward-sloping in $\mathcal{L}$'s location for small values of μ, meaning that a move close to the market center by $\mathcal{L}$ results in $\mathcal{F}$ moving away from the market center. Note that there is a limit to this effect as well; the slope of $\mathcal{F}$'s best-response is less than 45 degrees, so that $\mathcal{F}$'s response to a move by $\mathcal{L}$ is relatively small in magnitude. Thus $\mathcal{L}$ does not find it optimal to go to the extreme of locating at the market center. Indeed, equilibrium locations are $(x_l^* = 0.2225,\ x_f^* = 0.8275)$.

In general, if the repellent force is strong enough, $\mathcal{F}$ will move farther from the market center as $\mathcal{L}$ moves a little closer, resulting in a profit advantage for $\mathcal{L}$. This occurs when product attribute preferences are strong (i.e., α is low, so price competition is a potential issue) and idiosyncratic preferences are weak (i.e., μ is too low to mitigate price competition).

In contrast, consider the case when η_x^Π is negative but small in magnitude, such as when $\alpha = 1$ and $\mu = 0.95$. For a fixed location of $\mathcal{L}$, the repellent force on $\mathcal{F}$ is no longer as strong, due to the importance of idiosyncratic preferences in consumer choice. However, the fact that η_x^Π is negative means that there are still gains to be made by differentiation. How is $\mathcal{L}$ to realize this gain? Since price competition is weaker than when μ is low, $\mathcal{L}$ rationally anticipates that $\mathcal{F}$ will not differentiate enough, and thus takes the initiative of differentiation by locating further away from the market center. This phenomenon is reflected in the best-response curve: Figure 2 shows that $\mathcal{F}$'s best response is now downward-sloping with $\mathcal{L}$'s location, but at an angle less than 45 degrees. In fact, this shows that movement away from the market center by $\mathcal{L}$ results in $\mathcal{F}$ also moving away from the center, but not by as much as $\mathcal{L}$. Therefore, even if $\mathcal{F}$ ends up closer to the market center than $\mathcal{L}$ (resulting in a profit advantage for $\mathcal{F}$), $\mathcal{L}$ is still better off compared to the simultaneous move equilibrium.

In effect, the first-mover disadvantage occurs when $\mathcal{L}$ takes action to "expand the pie," so that both firms earn a higher profit than in the simultaneous move equilibrium. That is, even though $\mathcal{L}$ has conceded a profit advantage to $\mathcal{F}$, it does increase its own profit (compared to the equilibrium profit in the simultaneous-move game) in the process.

4.3 Implications on firm entry order

Whenever the sequential-move game results in a symmetric equilibrium, the same equilibrium holds in the simultaneous-move game. In such an event, entry order clearly does not matter.

However, when asymmetric equilibria appear in the sequential-move game, the order of entry matters. If product attribute preferences are strong and idiosyncratic preferences are weak, we are in a region of first-mover advantage. There will therefore be a "rush to market" to realize this advantage. Conversely, with strong product attribute preferences and strong idiosyncratic preferences, we are in a region of first-mover disadvantage. Thus, both firms would prefer that the other firm move first.

In reality, of course, several other factors may also affect the timing of entry timing, including technology, entry barriers, evolving customer demand, and incomplete information. These, in conjunction with the impact of location and pricing, will then determine the optimal time to enter.

5 Conclusion

In this paper, we consider a duopoly location model with an allowance for idiosyncratic preferences that may affect a consumer's choice over the two products. We find that the interplay between product attribute preferences and idiosyncratic preferences is critical in determining whether there is a first-mover advantage or disadvantage in the model. Strong price competition leads to a first-mover advantage. Idiosyncratic preferences mitigate price competition, and, in conjunction with strong product attribute preferences, imply a first-mover disadvantage.

While we explicitly consider a Hotelling setting in this paper, we have confirmed that our insights into first-mover advantage do carry over more broadly to location models. Two other popular location models are those of Lane (1980), and the Defender model of Hauser and Shugan (1983). In Lane's model, the leader again captures the market center and commands a profit which is more than twice that of the follower. We performed a numerical analysis similar to that in Section 4 in the Lane model (after adding the random utility term), and obtained qualitatively similar results about the existence of a first-mover advantage or disadvantage, depending on the strength of product attribute preferences and idiosyncratic preferences. We are not aware of a sequential location analysis of the Defender model.

Our analysis, of course, leaves many open questions. Important extensions include consideration of multiple product attributes and a larger number of firms.

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