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# On the Pythagorean hull of Q

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The purpose of this paper is to prove that the Pythagorean hull of **Q** is the splitting field in **R** of a special set of rational polynomials.

A real number r will be called a *real radical element* if it lies in some repeated radical extension E of **Q** with $E \subseteq \mathbf{R}$. In other words, r is constructible from rationals by some combination of multiplications, additions and extractions of real n-th roots. A polynomial $f \in \mathbf{Q}[X]$ will be called *solvable by real radicals* if all of its (complex) roots are real radical elements.

I. M. Isaacs [4] proved the following

**Theorem.** Suppose $f \in \mathbf{Q}[X]$ is an irreducible polynomial which splits over **R**. If f has any root which is a real radical element, then the degree of f is a power of 2 and the Galois group of f over **Q** is a 2-group (so all the roots of f are real radical elements that lie in a repeated square root extension of **Q**).

A formally real field K is called *Pythagorean* if every sum of squares in K is a square in K, i.e. if $K^2+K^2=K^2$. To every formally real field K there exists a smallest Pythagorean field which is an extension of K, called the *Pythagorean hull* of K (see [1],[3]) and denoted by Pyth(K). It can be obtained as $\mathrm{Pyth}(K) = \bigcup_{i=0}^{\infty} K_i$, where $K_0=K$ and $K_{i+1}= K_i(\sqrt{1+K^2})$.

Let F be the splitting field in **R** of the set of all polynomials which are solvable by real radicals. The purpose of the present note is to prove the following

**Theorem.** F = Pyth(**Q**).

The proof requires the following preliminary results:

**Lemma 1.**([8],p.173) Pyth(Q) is a Galois extension of **Q**.

A field extension L of K will be called a *2-extension* if L is the compositum of a family $\{K_i\}_{i\in I}$ of finite extensions of K with $Gal(K_i/K)$ a 2-group for all $i \in I$.

**Lemma 2.**([1], p. 50) Pyth(K) is the largest 2-extension of K on which every ordering of K can be extended.

**Proof of the theorem.** Let x be in Pyth(**Q**). Then x is a real radical element and since (by lemma 1) Pyth(**Q**) is a Galois extension of **Q**, the minimal polynomial of x completely splits over Pyth(**Q**), therefore $x \in F$, i.e. Pyth(**Q**)$\subseteq$F. Since F is - according to Isaac's theorem - a 2-extension of **Q**, we have , by lemma 2, F$\subseteq$Pyth(**Q**), hence F=Pyth(**Q**).

**Final comments.** Pythagorean fields play an important role in the axiomatics of Euclidean geometry, since the coordinate field of 'constructive' plane geometry (a geometry in which we may perform only the following two operations: (i) given four points A,B,C,D, to construct the point of intersection of the lines AB and CD, provided they are neither parallel nor identical, and (ii) to lay off the segment AB at C on ray CD) is a Pythagorean ordered field. Axiomatizations of this kind of geometry can be found in [5]-[7], [9]-[11]. Since Pyth(**Q**) is the *prime* model (see [2], p.96) of the theory of Pythagorean ordered fields, the Cartesian plane with Pyth(**Q**) as coordinate field is the prime model of 'constructive' plane geometry. Pyth(**Q**) can be 'geometrically' constructed as follows: let S be the smallest point-set (in the Cartesian plane over **R**) that contains the points (0,0), (1,0), (0,1) and is closed under the operations (i) and (ii); then, according to [5] (or [10]), S = Pyth(**Q**)*x* Pyth(**Q**).

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