JOINT SIMULATION OF BACKWARD AND FORWARD RECURRENCE TIMES IN A SUPERPOSITION OF INDEPENDENT RENEWAL PROCESSES

C.Y. Teresa Lam
Department of Industrial & Operations Engineering
University of Michigan
Ann Arbor, MI 48109-2117

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C. Y. Teresa Lam
Department of Industrial and Operations Engineering
The University of Michigan
Ann Arbor, MI 48109

Abstract

It is shown that, in a superposition of finitely many independent renewal processes, an observation from the limiting (when $t \to \infty$) joint distribution of backward and forward recurrence times at t can be simulated by simulating an observation of the pair $(UW, (1-U)W)$, where U and W are independent random variables with U ~uniform(0,1) and W distributed according to the limiting total life distribution of the superposition process.

SUPERPOSITION OF RENEWAL PROCESSES; LIMITING TOTAL LIFE DISTRIBUTION; SIMULATION

1 Introduction and Summary

Winter (1989) showed that in a renewal process with interarrival distribution F, an observation from the limiting (when $t \to \infty$) joint distribution of backward and forward recurrence times at t can be simulated by simulating an observation of the pair $(UW, (1-U)W)$, where U and W are independent random variables with U ~uniform(0,1) and W distributed according to the length-biased version of F or the limiting total life distribution. It is shown in this note that the analogous result holds for the superposition of independent renewal processes.

Suppose that there are p independent ordinary renewal processes in operation simultaneously. Let F_i, $i = 1, 2, \ldots, p$, be the probability distribution functions for the successive interarrival times of the ith process with $F_i(0) = 0$ and positive finite mean μ_i. Furthermore, let us assume that F_i is absolutely continuous for all $i = 1, 2, \ldots$. The above conditions ensure that for all component processes, with probability one a non-zero time is spent between transitions. Also, simultaneous occurrence of events in the superposed process has probability zero. Consider the sequence of events formed by pooling the individual processes. In general, the superposed process is not a renewal process. At the time when an event occurs in the superposition process, one of the processes, say process i, probabilistically starts over. In addition, the others have age $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_p$ respectively, where the D_j s are random variables. The age here refers to the time since the last event occurs in a particular component process. Suppose that at time 0, the component processes all have age zero. Let γ_t and β_t be the forward recurrence time and the total life at time t of the superposed process respectively. From Lam and Lehoczky (1991), for $i = 1, \ldots, p$, if F_i satisfies all the conditions stated above and furthermore, it is non-arithmetic, then the limiting distributions of the forward recurrence time and the total life for the superposed process exist and are given by

$$\lim_{t\to\infty} \mathcal{P}(\gamma_t > z) = \begin{cases} \prod_{i=1}^{p}\left[\frac{1}{\mu_i}\int_{(z,\infty)} (1 - F_i(x))\, dx\right] & \text{if } z \geq 0 \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

and

$$\lim_{t\to\infty} \mathcal{P}(\beta_t \leq z) = K(z) = \begin{cases} \frac{1}{\mu}\int_{(0,z]} x\, dG(x) & z \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where

$$G(x) = \begin{cases} 1 - \sum_{i=1}^{p}\left[\left(\frac{1}{\mu_i}\right)\Big/\left(\frac{1}{\mu}\right)\right](1 - F_i(x)) \prod_{j=1, j\neq i}^{p}\left[\int_{(x,\infty)} \frac{(1 - F_j(y))}{\mu_j}\, dy\right] & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and $\mu = \left[\sum_{i=1}^{p} \frac{1}{\mu_i}\right]^{-1}$. From Lam and Lehoczky (1991), Results (1) and (2) above also hold in the superposition of p independent delayed renewal processes.

Furthermore, if δ_t is the backward recurrence time at time t of the superposed process, then for $y, z \geq 0$,

$$\lim_{t\to\infty} \mathcal{P}(\delta_t \geq y \text{ and } \gamma_t \geq z) = \lim_{t\to\infty} \mathcal{P}(\gamma_{t-y} \geq y+z) = \prod_{i=1}^{p}(\frac{1}{\mu_i}\int_{(y+z,\infty)}(1-F_i(x))\,dx). \tag{4}$$

We are now ready to state and prove the following result which is the analogue of the result given in Winter (1989).

Theorem 1 *Let U and W be two independent positive random variables, with U ~uniform(0,1) and $W \sim K$ with K as in Result (2). Then*

$$\mathcal{P}(UW > y, (1-U)W > z) \;=\; \begin{cases} \prod_{i=1}^{p}\frac{1}{\mu_i}\int_{(y+z,\infty)}(1-F_i(x))\,dx & \text{if } y \geq 0,\ z \geq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Proof: Let $X_1, \ldots, X_p$ be independent random variables such that X_i, $i = 1, \ldots, p$, follows that limiting forward recurrence or backward recurrence distribution of the ith component process. Also, let $T = \min\{X_1, \ldots, X_p\}$. As in Winter (1989), for $y, z \geq 0$,

$$\begin{aligned} \mathcal{P}(UW > y, (1-U)W > z) &= \mathcal{P}(W > y+z, \frac{y}{W} < U < 1 - \frac{z}{W}) \\ &= \frac{1}{\mu}\int_{(y+z,\infty)}(1-G(u))\,du \end{aligned} \tag{6}$$

By substituting Equation (3) into (6) and using the definitions of X_i, $i = 1, 2, \ldots, p$ and T, we have for $y, z \geq 0$,

$$\begin{aligned} \mathcal{P}(UW > y, (1-U)W > z) &= \int_{(y+z,\infty)} \sum_{i=1}^{p}\frac{1}{\mu_i}(1-F_i(u)) \prod_{j=1, j\neq i}^{p}\left[\int_{(u,\infty)}\frac{(1-F_j(w))}{\mu_j}\,dw\right]du \\ &= \sum_{i=1}^{p}\mathcal{P}(y+z < X_i < \min_{j\neq i} X_j) \\ &= \mathcal{P}(T > y+z) = \prod_{i=1}^{p}\frac{1}{\mu_i}\int_{(y+z,\infty)}(1-F_i(w))\,dw \end{aligned} \tag{7}$$

This completes the proof of Theorem 1.

2 Concluding Remarks

Theorem 1 and Equation (4) above show how one can simulate a joint observation of the pair (δ_t, γ_t) for very large t. It can be obtained by computing $(uw, (1-u)w)$, where u is a simulated observation of U ~uniform(0,1) and w is a simulated observation of $W \sim K$, with U and W independent. This is, of course, practical only when the F_i s are such that the distribution function K is easily derived. This includes the cases when $F_i = F$ for all $i = 1, 2, \ldots, p$ and F is uniform, shifted exponential, gamma or beta. In particular, the results below are easily verified.

1. When $F_i = F \sim \text{uniform}(0,1)$,

$$G(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1-(1-x)^{2p-1} & \text{if } 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \tag{8}$$

   and

$$K(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1-(1-z)^{2p-1}[1+(2p-1)z] & \text{if } 0 < z < 1 \\ 1 & \text{if } z \geq 1 \end{cases} . \tag{9}$$

2. When $F_i = F$,

$$F(x) = \begin{cases} 1-\exp[-\lambda(x-1)] & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

   and $\lambda > 0$. We have after some tedious but straightforward calculations,

$$G(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1-\left(\frac{1}{1+\lambda}\right)^{p-1}[1+\lambda(1-x)]^{p-1} & \text{if } 0 < x < 1 \\ 1-\left(\frac{1}{1+\lambda}\right)^{p-1}\exp[-\lambda p(x-1)] & \text{if } x \geq 1 \end{cases} \tag{11}$$

$$
K(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ 1-\left(\dfrac{1+\lambda(1-z)}{1+\lambda}\right)^{p-1}\left(\dfrac{1+\lambda pz+\lambda(1-z)}{1+\lambda}\right) & \text{if } 0 < z < 1 \\ 1-\left(\dfrac{1}{1+\lambda}\right)^{p-1}\left(\dfrac{1+\lambda p}{1+\lambda}\right)+\left(\dfrac{1}{1+\lambda}\right)^{p}[1+\lambda p & \\ \quad -(1+\lambda p+\lambda p(z-1))\exp(-\lambda p(z-1))] & \text{if } z \geq 1 \end{cases} \tag{12}
$$

Similarly, when F is gamma or beta, we can derive both distribution functions G and K. The calculations may be tedious but they involve only integrations of polynomials on bounded intervals when F is beta. In the case when F is gamma, we require to evaluate definite integrals of product of exponential functions and polynomials. These integrals can be calculated easily using integration by parts.

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References

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