New Better Than Used in Expectation Processes

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Abstract

In this paper, we study the new better than used in expectation and new worse than used in expectation properties of Markov renewal processes. We show that a Markov renewal process belongs to a more general class of stochastic processes encountered in reliability or maintenance applications. We present sufficient conditions such that the first passage times of these processes are new better than used in expectation. The results are applied to the study of shock and repair models, random repair time processes, inventory, and queueing models.

NBUE AND NWUE PROCESSES; INCREASING MARKOV RENEWAL PROCESSES; FIRST PASSAGE TIMES; QUEUEING; RANDOM REPAIR TIME PROCESSES; SHOCK AND REPAIR MODELS; INVENTORY

1 Introduction

First passage times of appropriate stochastic processes have often be used to represent times to failure of devices or systems which are subject to shocks and wear, random repair time, and random interruptions during their operation. The life distribution properties of these processes have therefore been widely investigated in the reliability and maintenance literature. Conditions under which the first passage times have increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU) and new better than used in expectation (NBUE) properties have been studied. It is well known that IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE for life distributions. IFR characterization of birth and death processes, nonnegative diffusion processes, pure jump processes, Levy wear processes and Markov chains were presented in Keilson (1979), Derman, Ross and Schechner (1983), Abdel-Hameed (1984a, b), and Brown and Chaganty (1983) respectively. IFRA properties of Markov jump processes, Poisson shock models and Markov chains were considered by Shaked and Shanthikumar (1987), Ohi and Nishida (1983), Brown and Chaganty (1983) and references there. Marshall and Shaked (1979), and Block and Savits (1981) studied multidimensional IFRA processes. Marshall and Shaked (1983, 1986), and Shanthikumar (1984) considered processes with NBU first passage times. Shock models with NBUE survival were discussed in Block and Savits (1978). Karasu and Özekici (1989) studied new better than used in expectation (NBUE) and new worse than used in expectation (NWUE) properties of increasing Markov processes and Markov chains.

In this paper, we consider the NBUE and NWUE properties of Markov renewal processes. This extends the results in Karasu and Özekici (1989). We introduce a general class of stochastic processes and show that Markov renewal processes are special cases. The class of stochastic processes considered here is closely related to those discussed by Marshall and Shaked (1983), and Shanthikumar (1984). In their papers, they presented conditions such that the first passage times of their processes are NBU. However, as illustrated by an example in Section 2 of this paper, not all NBUE stochastic processes which are used to represent times to failures of certain devices are also NBU. This motivates us to look at a weaker characterization of stochastic processes encountered in reliability or maintenance applications.

In Section 3 of this paper, we consider sufficient conditions such that an increasing Markov renewal process is NBUE or NWUE. The results will then be generalized to characterize the NBUE and NWUE properties of a general Markov renewal process. In Section 4, we introduce the general class of stochastic processes of interest to us, and show that under some conditions, these stochastic processes are NBUE. The results in Sections 3 and 4 are applied to study shock and repair models, random repair processes, inventory and queueing models.

2 An Motivating Example

In this example, we show that not all NBUE processes are also NBU. The set of NBU processes are therefore strictly included in the set of NBUE processes.

Consider a stochastic process $Z = \{Z(t); t \geq 0\}$ with state space $\mathcal{S} \subset \Re^+$. $\Re^+$ here represents the set of all nonnegative real numbers. $Z(0) = 0$ with probability 1. Let $T_x(Z)$ be its first passage time to level $x \in \mathcal{S}$, i.e.,

$$T_x(Z) = \inf\{t \geq 0 : Z(t) > x\}. \tag{2.1}$$

$T_x(Z)$ is said to have NBU (respectively, new worse than used (NWU)) distribution if

$$\mathcal{P}(T_x(Z) > t + s \mid T_x(Z) > t) \leq (\text{respectively}, \geq)\mathcal{P}(T_x(Z) > s) \tag{2.2}$$

for all $t, s \geq 0$. $T_x(Z)$ is said to have NBUE (respectively, NWUE) distribution if

$$\mathcal{E}(T_x(Z) - t \mid T_x(Z) > t) \leq (\text{respectively}, \geq)\mathcal{E}(T_x(Z)) \tag{2.3}$$

or

$$\int_{[0,\infty)} \mathcal{P}(T_x(Z) > t + s)\, ds \leq (\text{respectively}, \geq)\mathcal{P}(T_x(Z) > t) \int_{[0,\infty)} \mathcal{P}(T_x(Z) > s)\, ds \tag{2.4}$$

for all $t \geq 0$. The process Z is said to be NBU (respectively, NWU, NBUE, NWUE) if the first passage times $T_x(Z)$ have NBU (respectively, NWU, NBUE, NWUE) distributions for all $x \in \mathcal{S}$.

Assume that $\{Z(t); t \geq 0\}$ is a Markov process with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and $Z(0) = 0$ with probability 1. Suppose the process can jump from state 0 to state 3 with probability p or to state 1 with probability $1 - p$. Furthermore, the process jumps from state i to state $i + 1$ with probability 1, $i = 1, 2$. Assume that the sojourn times of the process in each state are exponential with mean 1 and let T be the first passage time of the process to a state greater than 2. Obviously,

$$\mathcal{P}(T > t) = pe^{-t} + (1-p)e^{-t}(1 + t + \frac{t^2}{2}) = e^{-t}[1 + (1-p)t + (1-p)\frac{t^2}{2}], \tag{2.5}$$

$$\mathcal{E}(T) = \int_{[0,\infty)} \mathcal{P}(T > t)\, dt = p + 3(1-p) = 3 - 2p, \tag{2.6}$$

and

$$\int_{[0,\infty)} \mathcal{P}(T > t + s)\, dt = e^{-s}[3 - 2p + 2s(1-p) + \frac{s^2}{2}(1-p)]. \tag{2.7}$$

It is easily shown that if $p = 0.5$, then for all $s \geq 0$, $\mathcal{E}(T) - \mathcal{E}(T - s \mid T > s) \geq 0$. However, $\mathcal{P}(T > 0.5) = 0.7961$ and $\mathcal{P}(T > 1.5 \mid T > 1) = 0.8015$. Hence Z is NBUE but not NBU.

The Markov process considered here can be used to model the performance of certain electronic devices. The states 0, 1, 2 and 3 can be used to represent perfect, minor damage, major damage and failure conditions of the unit. The device either fails soon after it is used (fails upon the initial surge of current) or deteriorate gradually until it fails.

A similar example to the one given here was also studied by Marshall and Shaked (1983) to show that Markov process with positive skip (sample paths with positive jumps greater than 1) need not be NBU.

$Z(t)$ here usually represent the state of a device, and the device fails if the state exceeds a certain level x. The first passage time to level x is therefore the lifetime of a device. However, $Z(t)$ here can also represent the virtual waiting time or work-backlog of a queueing system, inventory level in a warehouse, or the content in a dam.

3 NBUE and NWUE properties of Markov Renewal Processes

In this section, we first consider the NBUE and NWUE properties of increasing Markov renewal processes. The results will then be extended to more general Markov renewal processes. It will be shown that a general Markov renewal process can be used to model the accumulated damage to a device that is subjected to both shocks and repairs.

Consider a Markov renewal process $(X, S) = \{X_n, S_n; n \in \{0, 1, 2, \ldots\}\}$ with state space $\mathcal{S} \subset \{0, 1, 2, \ldots\}$ and $S_0 = 0$. Let $Z = \{Z(t), t \geq 0\}$ be the semi-Markov process associated with (X, S). This means that $S_1, S_2, \ldots$ are the successive jump times of Z and $X_1, X_2, \ldots$ are the successive states visited by Z. Assume that $\mathcal{P}(\sup_n S_n < \infty) = 0$. Let the family of probabilities $Q = \{Q(i,j,t) : i,j \in \mathcal{S}, t \geq 0\}$ be the semi-Markov kernel over $\mathcal{S}$. Assume that $\sum_{j \in \mathcal{S}} Q(i,j,\infty) = 1$ for each $i \in \mathcal{S}$. Here, $Q(i,j,t)$ is the probability that knowing the current state is i, the next transition state is to state j and the sojourn time in the current state is less than or equal to t. (X, S) is an increasing Markov renewal process (or Z is an increasing semi-Markov process) if $Q(i,j,t) = 0$ for all $i > j$ and $t \geq 0$. The embedded process X is then an increasing Markov chain with transition probability $Q(i,j,\infty) = Q(i,j)$. Let $F(i,t) = \sum_{j \in \mathcal{S}} Q(i,j,t)$, i.e., it is the distribution of the occupation time during each visit to state i. If $Q(i,j) = 0$ for some pair (i,j), then $Q(i,j,t) = 0$ for all t and define $Q(i,j,t)/Q(i,j) = 1$. With this convention, we define for all $i,j \in \mathcal{S}$ and $t \geq 0$,

$$G(i,j,t) = \frac{Q(i,j,t)}{Q(i,j)}. \tag{3.1}$$

For each pair (i,j), the function $t \to G(i,j,t)$ is the distribution function of the occupation time in state i given the next state will be in state j. Let $T_{ax}(Z)$ be the first passage time of the process Z starting from state a at time $t = 0$ to state $x \in \mathcal{S}$. For $i,j \in \mathcal{S}$, let

$$R(i,j,t) = \sum_{n=0}^{\infty} Q^n(i,j,t) \tag{3.2}$$

be the Markov renewal function. Hence, $R(i,j,t)$ is the expected number of visits to state j in the finite interval $[0,t]$. Write

$$P_t(i,j) = \mathcal{P}(Z(t) = j \mid Z(0) = X_0 = i) \tag{3.3}$$

and

$$U(i,j,t) = \mathcal{E}\left[\int_{[0,t]} I_j(Z(s))\, ds \mid X_0 = i \right] = \int_{[0,t]} P_s(i,j)\, ds \tag{3.4}$$

where $I_j(k) = 1$ or 0 according as $k = j$ or $k \neq j$. Note that $U(i,j,t)$ is called the potential function of Z and it represents the expected time spent in state j during $[0,t]$ by the process starting at i. Furthermore,

$$U(i,j) = U(i,j,\infty) = \mathcal{E}\left[\int_{[0,\infty)} I_j(Z(s))\, ds \mid X_0 = i \right] = \int_{[0,\infty)} P_s(i,j)\, ds. \tag{3.5}$$

Starting at state i, $U(i,j)$ is the total expected time spent in j. From Cinlar (1975a),

$$U(i,j) = R(i,j)m(j) = R(i,j,\infty)m(j) \tag{3.6}$$

where $m(j)$ is the mean sojourn time in state j. For notational convenience, we define the cumulative potential matrix, $\{\bar{U}(i,j); i,j \in \mathcal{S}\}$, by

$$\bar{U}(i,j) = \sum_{k \leq j} U(i,k). \tag{3.7}$$

$\bar{P}_t(i,j)$ and $\bar{R}(i,j)$ are defined similarly.

Let $N(t)$ be the number of transitions of the process in the interval $(0,t]$. Then for all $i,j,k \in \mathcal{S}$ and $t \geq 0$, we have

$$\begin{aligned}
& \mathcal{P}(X_{N(t)+1} = k, X_{N(t)} = j \mid X_0 = i) \\
= & \sum_{n=0}^{\infty} \mathcal{P}(X_{n+1} = k, X_n = j, N(t) = n \mid X_0 = i) \\
= & \sum_{n=0}^{\infty} \mathcal{P}(X_{n+1} = k, X_n = j, S_n \leq t < S_{n+1} \mid X_0 = i) \\
= & \sum_{n=0}^{\infty} \int_{[0,t]} Q^n(i,j,ds)\mathcal{P}(X_{n+1} = k, S_{n+1} - S_n > t - s \mid X_n = j) \\
= & \sum_{n=0}^{\infty} \int_{[0,t]} Q^n(i,j,ds)[Q(j,k) - Q(j,k,t-s)] \qquad (3.8) \\
= & \int_{[0,t]} R(i,j,ds)[Q(j,k) - Q(j,k,t-s)]. \qquad (3.9)
\end{aligned}$$

Expression (3.9) follows from (3.8) by the monotone convergence theorem. Furthermore,

$$\mathcal{P}(X_{N(t)} = j \mid X_0 = i) = \int_{[0,t]} R(i,j,ds)[1 - F(j,t-s)]. \qquad (3.10)$$

The interchange of the summation and integral signs is again justified by the monotone convergence theorem. It follows that

$$q_i(j,k,t) = \mathcal{P}(X_{N(t)+1} = k \mid X_{N(t)} = j, X_0 = i) = \frac{\int_{[0,t]} R(i,j,ds)[Q(j,k) - Q(j,k,t-s)]}{\int_{[0,t]} R(i,j,ds)[1 - F(j,t-s)]}. \qquad (3.11)$$

Let M_j be the sojourn time of the process in state j, $j \in \mathcal{S}$,

$$W_i(j,k,t) = \mathcal{E}(M_j)I_k(j) + \sum_{b \in \mathcal{S}} U(b,k)q_i(j,b,t) \qquad (3.12)$$

and

$$\bar{W}_i(j,x,t) = \sum_{k \leq x} W_i(j,k,t). \qquad (3.13)$$

$\bar{W}_i(j,x,t) - \sum_{k \leq x} \mathcal{E}(M_j)I_k(j)$ is the expected time of the process (X,S) spent in states below or equal to x during the time interval $[S_{N(t)+1}, \infty)$ given that $X_0 = i$ and $X_{N(t)} = j$. In the case when $G(i,j,t) = Q(i,j,t)/Q(i,j) = G(i,t)$ independent of j for all $i,j \in \mathcal{S}$ and $t \geq 0$, we have $F(i,t) = G(i,t)$, $q_i(j,k,t) = Q(j,k)$, $W_i(j,k,t) = U(j,k)$ and $\bar{W}_i(j,x,t) = \bar{U}(j,x)$ for all $i,j,k,x \in \mathcal{S}$ and $t \geq 0$.

Increasing Markov chains and Markov processes are both special cases of increasing Markov renewal processes. A review of the literature in the characterization (PF_r densities, IFR, IFRA, NBU) of first passage times of increasing Markov chains and Markov processes are given in Karasu and Özekici (1989). Furthermore, in their paper, they showed that an increasing Markov process and its the embedded Markov chain are NBUE (NWUE) if $\bar{U}(i,j)$ and $\bar{R}(i,j)$ are decreasing (increasing) in i on $i \leq j$ for all $j \in \mathcal{S}$ respectively. In the theorems and corollaries below, we study conditions under which an increasing Markov renewal process is NBUE or NWUE.

Theorem 3.1 *Suppose*

1. (X, S) *is an increasing Markov renewal process,*
2. *the sojourn time in state* i, M_i, *is NBUE for all* $i \in \mathcal{S}$,
3. $\bar{U}(i,j) \geq \bar{W}_i(k,j,t)$ *for all* $i \leq k \leq j$, $i,j \in \mathcal{S}$ *and* $t \geq 0$.

Then (X, S) *is an NBUE Markov renewal process.*

Proof: By condition (1) of the Theorem, for all $t \geq 0$, $a, x \in \mathcal{S}$ and $a \leq x$,

$$\mathcal{P}(T_{ax}(Z) > t \mid X_0 = a) = \mathcal{P}(Z(t) \leq x \mid X_0 = a) = \sum_{a \leq k \leq x} \mathcal{P}(Z(t) = k \mid X_0 = a) = \bar{P}_t(a,x). \tag{3.14}$$

Z or (X, S) is NBUE if

$$p = \bar{P}_t(a,x) \int_{[0,\infty)} \bar{P}_s(a,x)\, ds - \int_{[0,\infty)} \bar{P}_{t+s}(a,x)\, ds \geq 0. \tag{3.15}$$

Note that

$$\int_{[0,\infty)} \bar{P}_{t+s}(a,x)\, ds = \bar{U}(a,x) - \bar{U}(a,x,t) \tag{3.16}$$

is the expected amount of time the process stayed in the set $\{j \in \mathcal{S} : a \le j \le x\}$ in the time interval $[t,\infty)$ given that the process starts from state a at the time origin. Furthermore,

$$\begin{aligned}
\int_{[0,\infty)} \bar{P}_{t+s}(a,x)\,ds &= \sum_{a\le k\le x} \int_{[0,\infty)} \mathcal{P}(Z(t+s)=k \mid X_0=a)\,ds \qquad (3.17)\\
&= \sum_{a\le k\le x} \int_{[0,\infty)} \sum_{u\in\mathcal{S}} \mathcal{P}(Z(t+s)=k \mid Z(t)=u, X_0=a) P_t(a,u)\,ds\\
&= \sum_{a\le u\le x} P_t(a,u) \sum_{u\le k\le x} \int_{[0,\infty)} \mathcal{P}(Z(t+s)=k \mid Z(t)=u, X_0=a)\,ds
\end{aligned}$$

and

$$\bar{P}_t(a,x)\int_{[0,\infty)} \bar{P}_s(a,x)\,ds = \sum_{a\le u\le x} P_t(a,u)\bar{U}(a,x). \qquad (3.18)$$

For $a \le u \le k \le x$,

$$\begin{aligned}
&\int_{[0,\infty)} \mathcal{P}(Z(t+s)=k \mid Z(t)=u, X_0=a)\,ds\\
&= \mathcal{E}\Big[\int_{[0,\infty)} I_k(Z(s+t))\,ds \mid X_{N(t)}=u, X_0=a\Big]. \qquad (3.19)
\end{aligned}$$

Equation (3.19) gives the expected time spent in state k during the $[t,\infty)$ given that the process starts from state a at time 0 and is in state u at time t. Conditional on $X_{N(t)+1} = b \in \mathcal{S}$, Equation (3.19) becomes

$$\begin{aligned}
&\mathcal{E}\Big[\int_{[t,S_{N(t)+1})} I_k(Z(s))\,ds \mid X_{N(t)}=u, X_0=a\Big]\\
&\quad+\mathcal{E}\Big[\int_{[S_{N(t)+1},\infty)} I_k(Z(s))\,ds \mid X_{N(t)}=u, X_0=a\Big]\\
&= \mathcal{E}\Big[\int_{[t,S_{N(t)+1})} I_k(Z(s))\,ds \mid X_{N(t)}=u, X_0=a\Big]\\
&\quad+\sum_{b\in\mathcal{S}}\mathcal{E}\Big[\int_{[S_{N(t)+1},\infty)} I_k(Z(s))\,ds \mid X_{N(t)+1}=b, X_{N(t)}=u, X_0=a\Big] q_a(u,b,t). \qquad (3.20)
\end{aligned}$$

Since $S_{N(t)+1}$ is a stopping time for the semi-Markov process Z and it is semi-regenerative, this means that

$$\begin{aligned}
&\mathcal{E}\Big[\int_{[S_{N(t)+1},\infty)} I_k(Z(s))\,ds \mid X_{N(t)+1}=b, X_{N(t)}=u, X_0=a\Big]\\
&= \mathcal{E}\Big[\int_{[0,\infty)} I_k(Z(s))\,ds \mid X_0=b\Big] = U(b,k). \qquad (3.21)
\end{aligned}$$

Furthermore, for $a \le u \le k$,

$$\begin{aligned}
& \mathcal{E}\Big[\int_{[t,S_{N(t)+1})} I_k(Z(s))\,ds \mid X_{N(t)} = u, X_0 = a, N(t) = n, S_{N(t)} = s\Big] \\
= \;& I_k(u)\mathcal{E}[M_u - (t-s) \mid M_u > (t-s)] && (3.22) \\
\le \;& I_k(u)\mathcal{E}[M_u]. && (3.23)
\end{aligned}$$

Equation (3.23) follows from Equation (3.22) because M_u is NBUE. Now partially unconditioning both sides of (3.23) with respect to $N(t) = n$ and $S_{N(t)} = s$, we have

$$\mathcal{E}\Big[\int_{[t,S_{N(t)+1})} I_k(Z(s))\,ds \mid X_{N(t)} = u, X_0 = a\Big] \le I_k(u)\mathcal{E}[M_u]. \tag{3.24}$$

Hence, for $a, x \in \mathcal{S}$,

$$p \ge \sum_{a \le u \le x} P_t(a,u)[\bar{U}(a,x) - \bar{W}_a(u,x,t)] \ge 0 \tag{3.25}$$

by condition (3) of the Theorem. (X, S) is therefore NBUE. □

In the case when $G(i,j,t) = G(i,t)$ independent of j for all $i, j \in \mathcal{S}$ and $t \ge 0$, Condition (3) of the theorem can be replaced by $\bar{U}(i,j) \ge \bar{U}(k,j)$ for all $i \le k \le j$, $i, j \in \mathcal{S}$.

Corollary 3.2 *If (X, S) satisfies conditions (1) and (2) of Theorem 3.1 and in addition, $G(i,j,t)$ $= G(i,t)$ is independent of j for all $i, j \in \mathcal{S}$ and $t \ge 0$, $m(i)$ is decreasing in i and $\bar{R}(i,j)$ is decreasing in $i \in \mathcal{S}$ for $i \le j$ and $j \in \mathcal{S}$, then (X, S) is an NBUE Markov renewal process.*

Proof: The proof is the same as the proof of Corollary 4.7 for increasing Markov process in Karasu and Özekici (1989). □

From Theorem 3.4 of Karasu and Özekici (1989), if $\bar{R}(i,j)$ is decreasing in $i \in \mathcal{S}$ for $i \le j$ and $j \in \mathcal{S}$, then the increasing Markov chain X is NBUE. Hence, provided all conditions in Corollary 3.2 hold, the NBUE property of embedded Markov chain X is inherited by the Markov Renewal process (X, S)

Obviously, the NWUE characterization of an increasing Markov renewal process can be carried out in a similar fashion. In particular, we have the following theorem.

Theorem 3.3 *Suppose*

1. (X, S) *is an increasing Markov renewal process,*

2. *the sojourn time in state* i, M_i, *is NWUE for all* $i \in \mathcal{S}$,

3. $\bar{U}(i,j) \leq \bar{W}_i(k,j,t)$ *for all* $i \leq k \leq j$, $i,j \in \mathcal{S}$ *and* $t \geq 0$.

Then (X, S) *is an NWUE Markov renewal process.*

Corollary 3.4 *If* (X,S) *satisfies conditions (1) and (2) of Theorem 3.3 and in addition,* $G(i,j,t)$ $= G(i,t)$ *independent of* j *for all* $i,j \in \mathcal{S}$ *and* $t \geq 0$, $m(i)$ *is increasing in* i *and* $\bar{R}(i,j) - \bar{R}(i,l)$ *is increasing in* $i \in \mathcal{S}$ *for all* $i \leq j$ *and* $l,j \in \mathcal{S}$ *with* $l \leq j$, *then* (X,S) *is an NWUE Markov renewal process.*

Proof: The proof is the same as the proof of Corollary 4.9 for increasing Markov process in Karasu and Özekici (1989). □

Again, provided that the conditions in Corollary 3.4 hold, the NWUE property of the increasing Markov chain X is inherited by (X, S).

Consider a device that is subjected to shocks and assume that the amount of damage to the device at time t can be modeled by an increasing Markov renewal process, (X, S), with discrete state space $\mathcal{S}$. The device fails when the accumulated damage exceeds x and $Z(0) = a \leq x$. The survival function $\bar{H} = 1 - H$ of the device is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \bar{P}_k \mathcal{P}(N(t) = k) \tag{3.26}$$

where $N(t)$ is the number of shocks that occur in the interval $(0,t]$ and $\bar{P}_k = 1 - P_k$ is the probability of surviving k shocks. Provided that the conditions in Theorem 3.1 hold, the distribution function H is NBUE, i.e.,

$$\bar{H}(t) \int_{[0,\infty)} \bar{H}(s)\, ds \geq \int_{[t,\infty)} \bar{H}(s)\, ds. \tag{3.27}$$

Furthermore,

$$\bar{P}_k = \sum_{j \leq x} Q^k(a,j). \tag{3.28}$$

The same argument used in the proof of Theorem 3.4 in Karasu and Özekici (1989) applies to show that if $\bar{R}(i,j)$ is decreasing in $i \in \mathcal{S}$ for $i \leq j$ and $j \in \mathcal{S}$, then P_k is an NBUE sequence, i.e., for $j = 0, 1, 2, \ldots$,

$$\bar{P}_j \sum_{k=0}^{\infty} \bar{P}_k \geq \sum_{k=j}^{\infty} \bar{P}_k. \tag{3.29}$$

The shock model considered here is a special case of the shock model discussed in Block and Savits (1978).

More generally, let $(\hat{X}, \hat{S})$ be a Markov renewal process (not necessarily increasing) with state space $\mathcal{S}$ and semi-Markovian kernel $\hat{Q}$. Let $\hat{Z}$ be the associated semi-Markov process, and define

$$Z(t) = \sup\{\hat{Z}(s); 0 \leq s \leq t\}, \tag{3.30}$$

in other words, $Z(t)$ is the maximum level the process $\hat{Z}$ has ever attained during $[0,t]$. We adjoin a distinguished state Δ to the state space and define $\hat{X}_\infty = \Delta$, $\hat{S}_\infty = \infty$. For a subset A of $\mathcal{S}$, define $N = \inf\{n \geq 1 : \hat{X}_n \in A\}$. Set $N = \infty$ if no such N exists. Then

$$F_A(i,j,t) = \mathcal{P}(\hat{X}_N = j, \hat{S}_N \leq t \mid \hat{X}_0 = i) \tag{3.31}$$

is the probability that, starting from state i, the process $\hat{Z}$ enters A for the first time at state $j \in A$ and this happens at or before t. For $i \in \mathcal{S}$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then, for all $j \in A_i$ and $t \geq 0$,

$$Q(i,j,t) = F_{A_i}(i,j,t) \tag{3.32}$$

is the probability that the first transition of Z is to state j and this happens before t. From Cinlar (1975b), Z is an increasing semi-Markov process, and Q is the corresponding semi-Markovian kernel. As noted in Cinlar (1975b), F_A satisfies the equation

$$F_A(i,j,t) = \hat{Q}(i,j,t) + \sum_{k \in \mathcal{S}} \int_{[0,t]} Q_A(i,k,ds) F_A(k,j,t-s) \tag{3.33}$$

where $Q_A(i,k,t) = 0$ if $k \in A$ and $Q_A(i,k,t) = \hat{Q}(i,k,t)$ if $k \in A^c$ for all $i,k \in \mathcal{S}$, $t \geq 0$. This is a Markov renewal equation with defective semi-Markov kernel Q_A. From Cinlar (1969), it is not difficult to show that the solution of Equation (3.33) exists and is unique. More specifically, for all $i,j \in \mathcal{S}$ and $t \geq 0$,

$$F_A(i,j,t) = \sum_{k \in \mathcal{S}} \int_{[0,t]} R_A(i,k,ds) \hat{Q}(k,j,t-s) \tag{3.34}$$

where $R_A(i,j,t) = \sum_{n=0}^{\infty} Q_A^n(i,j,t)$. Now for all $a \le x$, if $T_{ax}(\hat{Z})$ is the first passage time of the process $\hat{Z}$ starting from state a at time $t = 0$ to state $x \in \mathcal{S}$, then

$$\mathcal{P}(T_{ax}(\hat{Z}) > t \mid \hat{X}_0 = a) = \mathcal{P}(T_{ax}(Z) > t \mid Z(0) = a). \tag{3.35}$$

It follows immediately that $\hat{Z}$ is NBUE (respectively, NWUE) if and only if the increasing semi-Markov process Z is NBUE (respectively, NWUE).

Consider a device that is subjected to both shocks and repairs. Assume that the amount of damage to the device at time t can be modeled by a Markov renewal process, $(\hat{X}, \hat{S})$, with discrete state space $\mathcal{S}$. If $\hat{Z}(\hat{S}_n) < \hat{Z}(\hat{S}_n^-)$, the device is said to be under repair at time $\hat{S}_n$. On the other hand, if $\hat{Z}(\hat{S}_n) \ge \hat{Z}(\hat{S}_n^-)$, then at epoch $\hat{S}_n$, a shock with magnitude $\hat{Z}(\hat{S}_n) - \hat{Z}(\hat{S}_n^-)$ affects the device and causes damage. The device fails when the total damage exceeds x and $\hat{Z}(0) = a \le x$. The survival function $\bar{H}$ of the device is again given by Equation (3.26) where $N(t)$ is now the number of events that occur during the interval $(0,t]$ in the increasing Markov renewal process Z and $\bar{P}_k = \sum_{j \le x} Q^k(a,j)$. Provided that all the conditions in Theorem (3.1) hold for the increasing semi-Markov process Z, the distribution function H is again NBUE.

4 General NBUE processes

The stochastic processes considered in this section are very general. They include Markov renewal processes, shock models considered by Esary, Marshall and Proschan (1973), A-Hameed and Proschan (1975), and Block and Savits (1978), and NBU processes studied by Marshall and Shaked (1983), and Shanthikumar (1984).

Let $(A_n, Y_n)_1^\infty$ be a sequence of (not necessarily independent and identically distributed) random variables. Suppose Y_n are all nonnegative, $S_n = \sum_{i=1}^n Y_i$, $n = 1, 2, \ldots$ and $S_0 = 0$. Let $N(t) = \sup\{n : S_n \le t\}$. We say that $\{N(t); t \ge 0\}$ is a point process associated with the sequence $(A_n, Y_n)_1^\infty$, and at time S_n, an event of magnitude A_n occurs. For fixed $a_1, \ldots, a_j$ (some of the a_i s may be negative) and $y_1, \ldots, y_j$ (all the y_i s are nonnegative), define the sequence $(h_j((a_i, y_i), i = 1, \ldots, j; \cdot))_0^\infty$ and $h_0(\cdot)$ of deterministic nonnegative real valued functions on $\Re^+$.

Consider the stochastic process $\{Z(t); t \geq 0\}$ defined by

$$Z(t) = \begin{cases} h_0(t) & \text{if } N(t) = 0 \\ h_{N(t)}((A_i, Y_i), i = 1, 2, \ldots, N(t); t - S_{N(t)}) & \text{if } N(t) \geq 1 \end{cases} . \tag{4.1}$$

This means that the process moves deterministically between events. The deterministic pattern between events is determined by the location and magnitude of earlier events. In particular, in the time interval $[S_j, S_{j+1})$, the function h_j governs the behavior of the process. Note that the process Z stays nonnegative. However, unlike Marshall and Shaked (1983), and Shanthikumar (1984), we are not restricting the process to jump an amount A_n at time S_n. An increasing Markov renewal process is a special case of the general stochastic process introduced here. In particular, it satisfies the following conditions. $(A_n)_1^\infty$ is a sequence of discrete random variables taking values on $\mathcal{S}$. Assume $A_0 = a$ with probability 1 and for all $n \geq 1$, $\mathcal{P}(A_n = j \mid A_{n-1} = i) = Q(i,j)$. $Y(0) = 0$ and for $n \geq 1$, $\mathcal{P}(Y_n \leq t \mid A_{n-1} = i) = F(i,t)$. For all $n \geq 1$, $\mathcal{P}(A_n = j, Y_n \leq t \mid A_{n-1} = i) = Q(i,j,t)$. $h_0(t) = a$ for all $t \geq 0$. For $j = 1, 2, \ldots$ and $t \geq 0$, $h_j((A_i, Y_i), i = 0, 1, \ldots, j; t) = A_j$. Before we state sufficient conditions such that the process Z is NBUE, let us define some notations.

For each $m \in \{0, 1, 2, \ldots\}$, let $S_n^m = \sum_{i=m+1}^{m+n} Y_i$ and $N_m(t) = \sup\{n : S_n^m \leq t\}$. Define the stochastic processes $\{Z_m(t); t \geq 0\}$ and $\{W_m(t); t \geq 0\}$ by

$$Z_m(t) = Z(S_m + t) = \begin{cases} h_0(t) & \text{if } N_0(t) = 0 \\ h_{m+N_m(t)}((A_i, Y_i), i = 1, 2, \ldots, m + N_m(t); t - S^m_{N_m(t)}) & \text{otherwise} \end{cases} \tag{4.2}$$

and

$$W_m(t) = \begin{cases} h_0(t) & \text{if } N_m(t) = 0 \\ h_{N_m(t)}((A_{m+i}, Y_{m+i}), i = 1, 2, \ldots, N_m(t); t - S^m_{N_m(t)}) & \text{if } N_m(t) \geq 1 \end{cases} . \tag{4.3}$$

Observe that Z_m is obtained from Z by shifting the origin to S_m. W_m is defined in the same way as Z, but with the sequence $(A_{m+i}, Y_{m+i})_{i=1}^\infty$. Also, for $t \geq 0$, let $\hat{Z}(t) = \max_{0 \leq u \leq t} Z(u)$ and $\hat{W}_m(t) = \max_{0 \leq u \leq t} W_m(u)$. $\hat{Z}$ and $\hat{W}_m$ are therefore transformations of Z and W_m which trace their historical maximum. Let $H(Z, t)$ represents the history of the process Z upto time t.

For each $x \in \mathcal{S}$, define $\tau_x = \inf\{t : h_0(t) > x; t \geq 0\}$. Set $\tau_x = \infty$ if no such τ_x exists. For

$n = 1, 2, \ldots$, define

$$Y_n^x = \begin{cases} Y_n & \text{if } Y_n \le \tau_x \\ \tau_x & \text{if } Y_n > \tau_x \end{cases} . \tag{4.4}$$

Furthermore, from Shanthikumar (1984), a stochastic process $\{X(t); t \ge 0\}$ stochastically dominates a process $\{Y(t); t \ge 0\}$ $(X \ge_{st} Y)$ if for every nondecreasing functional f

$$\mathcal{E}[f(X)] \ge \mathcal{E}[f(Y)] \tag{4.5}$$

whenever the expectations exist.

For Theorem 4.1 below and the examples in this section, we will assume $(A_n, Y_n)_1^\infty$ is a sequence of independent random variables.

Theorem 4.1 *Suppose*

1. $(A_n, Y_n)_1^\infty$ *is a sequence of independent random variables.*
2. $Z(0) = h_0(0) = 0$ *with probability 1.*
3. *For every realization* $(a_i, y_i)_1^\infty$ *of* $(A_i, Y_i)_1^\infty$, *we have for all* $t \ge 0$,
   (a) $h_1((a_1, y_1); t) \ge h_0(t)$,
   (b) $h_{m+j}((a_i, y_i), i = 1, \ldots, m+j; t) \ge h_j((a_{m+i}, y_{m+i}), i = 1, \ldots, j; t)$, $m = 0, 1, 2, \ldots$, $j = 1, 2, \ldots$,
   (c) $\mathcal{P}(h_j((A_{m+1}, Y_{m+1}), (a_{m+i}, y_{m+i}), i = 2, \ldots, j; t) > p \mid Y_{m+1} > s)$
   $$\ge \mathcal{P}(h_j((A_{m+1}, Y_{m+1}), (a_{m+i}, y_{m+i}), i = 2, \ldots, j; t) > p),$$
   $p, s \ge 0$, $m = 0, 1, \ldots$, $j = 1, 2, \ldots$.
4. *For each* $x \in \mathcal{S}$ *and* $n = 1, 2, \ldots$, Y_n^x *is NBUE, i.e.,* $\mathcal{E}(Y_n^x - s \mid Y_n^x > s) \le \mathcal{E}(Y_n^x)$ *whenever* $0 \le s < \tau_x$.
5. $\int_0^\infty \mathcal{P}(\hat{W}_m(t) \le x)\, dt \le \int_0^\infty \mathcal{P}(\hat{Z}(t) \le x)\, dt$ *for all* $x \in \mathcal{S}$ *and* $m = 0, 1, 2, \ldots$.

Then $\{Z(t); t \ge 0\}$ *is an NBUE process.*

Remarks:

1. Conditions (3(a)) and (3(b)) require $Z_m(t) \geq W_m(t)$ with probability 1 for all $t \geq 0$. In most applications, this condition is easily verified. Some examples are given below.

   Furthermore, we have $Z(S_m) \geq W_m(0) = Z(0) = h_0(0)$. Hence, if $Z(t)$ represents the accumulated damage of a device at time t and events mean repair, this means that a used device cannot be repaired to a better condition than a new one.

2. Condition (3(c)) in the Theorem is weaker than Condition (iv) of Theorem 3.2 in Shanthikumar (1984) for NBU processes. In his paper, he required that

   $$\mathcal{P}(h_j((A_{m+1}, Y_{m+1}), (a_{m+i}, y_{m+i}), i = 2, \ldots, j; t) > p \mid Y_{m+1} = s_1)$$
   $$\geq \mathcal{P}(h_j((A_{m+1}, Y_{m+1}), (a_{m+i}, y_{m+i}), i = 2, \ldots, j; t) > p \mid Y_{m+1} = s_2),$$

   whenever $s_1 \geq s_2$, $t, p, s_1, s_2 \geq 0$, $m = 0, 1, \ldots$, $j = 1, 2, \ldots$.

3. Note that if $h_0(t) = 0$ for all $t \geq 0$ and Y_n is NBUE, $n = 1, 2, \ldots$, then Condition (4) of the theorem holds. Also, if Y_n is NBU, $n = 1, 2, \ldots$, then it is trivially true that Y_n^x is also NBU for all $x \in \mathcal{S}$. Hence, Y_n^x is NBUE.

4. If $(A_n, Y_n)_1^\infty$ is a pair of renewal sequences, then Z and W_m are the same process distributionally. Condition (5) of Theorem 4.1 always holds. If $\hat{W}_m \geq_{st} \hat{Z}$, $m = 0, 1, 2, \ldots$, then $\mathcal{P}(\hat{W}_m(t) \leq x) \leq \mathcal{P}(\hat{Z}(t) \leq x)$ for all $x \in \mathcal{S}$ and $t \geq 0$. Condition (5) of Theorem 4.1 follows. Section 4 of Shanthikumar (1984) discussed situations such that $\hat{W}_m \geq_{st} \hat{Z}$ holds for independent non-renewal sequences.

   If $T_x(W_m)$ and $T_x(Z)$ are respectively the first passage times of the processes W_m and Z to level $x \in \mathcal{S}$, then Condition (5) of the theorem implies that $\mathcal{E}(T_x(W_m)) \leq \mathcal{E}(T_x(Z))$ for all $x \in \mathcal{S}$.

Before the proof of Theorem 4.1 is given, let us look at some examples of the stochastic process Z encountered in applications.

Example 1 : A random repair time process

Suppose $(A_n, Y_n)_1^\infty$ is a pair of renewal sequences. Let $Z(t)$ represents the wear and repair process of an equipment at time t. Assume that wear starts at 0 at time 0 and the rate of wear, r, is constant independent of time. At time S_n ($n > 0$), the equipment is repaired and the process Z is set to level $\max\{Z(S_n^-) - A_n, \min\{Z(S_n^-), a\}\}$ and $a > 0$. This means that once the equipment is used, it is not possible to repair it to perfect condition. In this example, we have

1. $h_0(t) = rt$, $r > 0$, $t \geq 0$,

2. $h_j((A_i, Y_i), i = 1, 2, \ldots, j; t) = \max\{Z(S_j^-) - A_j, \min\{Z(S_j^-), a\}\} + rt$, $j = 1, 2, \ldots$, $t \geq 0$.

Now, provided that Y_n is NBU for all $n = 1, 2, \ldots$, it is easily verified that all conditions in Theorem 4.1 are satisfied and Z is therefore NBUE.

In general, let $a(x,t)$ be a nonnegative real valued function defined on $\Re^+ \times \Re^+$. Assume that at time S_n, the equipment is repaired to level $\max\{Z(S_n^-) - A_n, \min\{Z(S_n^-), a(Z(S_n^-), S_n)\}\}$ which depends on both level and age of the equipment. In particular, Z is still NBUE when $a(x,t)$ is monotonically increasing in both x and t. Furthermore, we can also assume the rate of wear to be both level and age dependent (Shanthikumar (1984)), i.e.,

$$\frac{dZ(t)}{dt} = r(Z(t), t), \qquad S_{n-1} < t < S_n, \qquad n = 1, 2, \ldots$$

$r(x,t)$ here is a nonnegative real valued function defined on $\Re^+ \times \Re^+$ and it increases on t.

Example 2 : Inventory model

Let $Z(t)$ represents the net inventory level of an item at time t in a warehouse of finite capacity $x > 0$. Assume $Z(0) = 0$. At time t, the items are consumed at rate $r(Z(t)) \geq 0$ depending on the net inventory position at that time. This means that

$$\frac{dZ(t)}{dt} = -r(Z(t)), \qquad S_{n-1} < t < S_n, \qquad n = 1, 2, \ldots.$$

Assume backlogging is permitted. $Z(t)$ is therefore negative if the backorder level at time t is positive. Furthermore, items are replenished at time S_n. The number of items received at time S_n is equal to $\max\{A_n, -Z(S_n^-)\}$, the maximum of the positive random variable A_n and the backorder level at that time. Let $Z^+(t) = \max\{0, Z(t)\}$. $Z^+(t)$ is therefore the on-hand inventory level at time t. Assume that

1. $(A_n, Y_n)_1^\infty$ is a pair of renewal sequence.

2. Y_n is NBUE, $n = 1, 2, \ldots$, i.e., given that there is no shipment of items in the last certain period of time, the expected additional time until the next shipment is less than or equal to the expected time between shipments.

3. $\mathcal{P}(A_n > p \mid Y_n > s) \geq \mathcal{P}(A_n > p)$, $n = 1, 2, \ldots$, $p, s \geq 0$.

Then from Theorem 4.1, the first passage time of on-hand inventory to level $x \in \mathcal{S}$ (the time taken to fill up an empty warehouse) is NBUE. Z^+ is still NBUE even if we assume that the maximum number of backorder permitted is $b \geq 0$ (or $Z(t) \geq -b$ for all $t \geq 0$).

Example 3 : Queueing model

Let $(A_n, Y_n)_1^\infty$ be a non-renewal sequence of independent random variables. Let $Z(t)$ represents the work-backlog or the virtual waiting time in a queueing system at time t. Assume $Z(0) = 0$. At time t, the service rate is $r(Z(t)) \geq 0$ depending on the work-backlog at that time. Suppose the nth customer arrives at time S_n increases the work-backlog by A_n to $Z(S_n) + A_n$. Assume that $A_{n+1} \geq_{st} A_n$, $Y_n \geq_{st} Y_{n+1}$, and Y_n is NBUE, $n = 1, 2, \ldots$. Also, $\mathcal{P}(A_n > p \mid Y_n > s) \geq \mathcal{P}(A_n > p)$, $n = 1, 2, \ldots$, $p, s \geq 0$. Then from Shanthikumar (1984), Conditions (1) and (2) above implies that $\hat{W}_m \geq_{st} \hat{Z}$, $m = 0, 1, 2, \ldots$. It follows from Theorem 4.1 that the first passage time of the work-backlog to level $x \in \mathcal{S}$ is NBUE.

Proof of Theorem 4.1: For $x \in \mathcal{S}$ and $m = 0, 1, 2, \ldots$, let

$$T_x^* = T_x(W_m) - Y_{m+1}^x. \tag{4.6}$$

Consider the conditional expectation

$$\begin{aligned}
& \mathcal{E}(T_x(Z) - t \mid T_x(Z) > t, N(t) = m, H(Z, S_m)) \\
= \;& \mathcal{E}(T_x(Z) - S_m - (t - S_m) \mid T_x(Z) - S_m > t - S_m, N(t) = m, H(Z, S_m)) \\
= \;& \mathcal{E}(T_x(Z_m) - (t - S_m) \mid T_x(Z_m) > t - S_m, N(t) = m, H(Z, S_m)) && (4.7) \\
\leq \;& \mathcal{E}(T_x(W_m) - (t - S_m) \mid T_x(W_m) > t - S_m, Y_{m+1} > t - S_m \geq 0) && (4.8) \\
= \;& \mathcal{E}(T_x^* \mid T_x(W_m) > t - S_m, Y_{m+1} > t - S_m \geq 0)
\end{aligned}$$

$$
\begin{aligned}
& + \mathcal{E}(Y^x_{m+1} - (t - S_m) \mid T_x(W_m) > t - S_m, Y_{m+1} > t - S_m \geq 0) \\
= \;& \mathcal{E}(T^*_x \mid Y^x_{m+1} > t - S_m \geq 0) + \mathcal{E}(Y^x_{m+1} - (t - S_m) \mid Y^x_{m+1} > t - S_m \geq 0) \qquad (4.9) \\
\leq \;& \mathcal{E}(T^*_x) + \mathcal{E}(Y^x_{m+1}) \qquad (4.10) \\
= \;& \mathcal{E}(T_x(W_m)) \\
\leq \;& \mathcal{E}(T_x(Z)). \qquad (4.11)
\end{aligned}
$$

Conditions (1), (3(a)) and (3(b)) of the theorem lead (4.7) into (4.8). Note that

1. $\mathcal{E}(T^*_x \mid Y^x_{m+1} > t - S_m \geq 0) \leq \mathcal{E}(T^*_x)$ by Conditions (1) and (3(c)) of the theorem,
2. $\mathcal{E}(Y^x_{m+1} - (t - S_m) \mid Y^x_{m+1} > t - S_m \geq 0) \leq \mathcal{E}(Y^x_{m+1})$ because Y_{m+1} and S_m are independent and Y^x_{m+1} is NBUE.

Equation (4.10) therefore follows from (4.9). Finally, (4.11) follows from Condition (5) of the theorem. Now partially unconditioning both sides of (4.11) with respect to the history and m (including the case $m = 0$), we have

$$\mathcal{E}(T_x(Z) - t \mid T_x(Z) > t) \leq \mathcal{E}(T_x(Z)). \qquad (4.12)$$

□

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