STATIONARY DISTRIBUTIONS FOR THE SUPERPOSITION OF MARKOV RENEWAL PROCESSES

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This paper studies the steady state behavior of the superposition of p independent Markov renewal processes. The number of Markov renewal processes, p, is fixed, and we are interested in deriving the steady state behavior of the superposed process at the times of occurrence of events. In particular, we define the discrete superposition remaining and current life processes. The stationary distributions and related steady state properties of these processes are given. These results are applied to study both queueing and reliability problems.

1. Introduction and Summary. Let $(X_i, T_i) = \{X_{i,n}, T_{i,n}; n \in \{0, 1, 2, \ldots\}\}$, $i = 1, 2, \ldots, p$, be $p \geq 2$ independent stochastic processes. For each $n \in \mathcal{N} = \{0, 1, 2, \ldots\}$ and $i \in \mathcal{S} = \{1, 2, \ldots, p\}$, $X_{i,n}$ is a random variable taking values in a countable set $\mathcal{E}_i$, and $T_{i,n}$ is a random variable taking value in $\Re_+ = [0, \infty)$ such that $0 = T_{i,0} \leq T_{i,1} \leq T_{i,2} \leq \ldots$. For each $i \in \mathcal{S}$, the stochastic process $(X_i, T_i) = \{X_{i,n}, T_{i,n}; n \in \mathcal{N}\}$ is said to be a Markov renewal process with state space $\mathcal{E}_i$ provided that for all $n \in \mathcal{N}$, $\nu_i, \zeta_i \in \mathcal{E}_i$ and $t \in \Re_+$,

$$\mathcal{P}(X_{i,n+1} = \zeta_i, T_{i,n+1} - T_{i,n} \leq t \mid X_{i,0}, \ldots, X_{i,n} = \nu_i; T_{i,0}, \ldots, T_{i,n})$$

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$$= \mathcal{P}(X_{i,n+1} = \zeta_i, T_{i,n+1} - T_{i,n} \le t \mid X_{i,n} = \nu_i) = Q_i(\nu_i, \zeta_i, t) \quad (1.1)$$

independent of n. The family of probabilities $Q_i = \{Q_i(\nu_i, \zeta_i, t); \nu_i, \zeta_i \in \mathcal{E}_i, t \in \Re_+\}$ is called a semi-Markov kernel of the ith process.

Suppose that there are $p \ge 2$ independent Markov renewal processes in operations simultaneously. Consider the sequence of events formed by superposing the individual processes. Assume that concurrent events cannot occur with positive probability in the superposed process. Hence, at the time when an event occurs in the superposition process, only one of the process, say process i, moves from one state to another state and the process is semi-regenerate. This means that process i probabilistically starts over with initial conditions depending only on the state of the process at that time. In addition, the other processes have age $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_p$ and are in states $E_1, \ldots, E_p$ respectively. D_j s and E_j s are all random variables. Let $T_n > 0$ be the time of occurrence of the nth event in the superposed process. Let $\mathcal{E} = \mathcal{E}_1 \times \ldots \times \mathcal{E}_p$, $\mathcal{S}_i = \{1, 2, \ldots, i-1, i+1, \ldots, p\}$, $\Psi_i = \{\mathbf{s} : \mathbf{s} \in \Re_+^{i-1} \times \{0\} \times \Re_+^{p-i}\}$ and $\Psi = \cup_{i=1}^{p} \Psi_i$.

Define a stochastic process $\{\mathbf{X_n}; n \in \mathcal{N}\}$ such that

$$\mathbf{X_n} = (\mathbf{S_n}, \mathbf{R_n})$$

where

$$\mathbf{S_n} = (S_{1,n}, \ldots, S_{p,n}),$$

and

$$\mathbf{R_n} = (R_{1,n}, \ldots, R_{p,n}) = (R_{1,n}, \ldots, R_{i-1,n}, 0, R_{i+1,n}, \ldots, R_{p,n})$$

if the nth event of the superposed process comes from the ith component process. For $j \in \mathcal{S}_i$, $R_{j,n}$ is the remaining life of the jth component process at time T_n and this component process is in state $S_{j,n} \in \mathcal{E}_j$. Also, the ith component process moves to state $S_{i,n}$ at time T_n and define $R_{i,n} = 0$. Let us call $\{\mathbf{X_n}; n \in \mathcal{N}\}$ the discrete superposition remaining life process (DSRLP). The state space of this process is given by $\Upsilon = \mathcal{E} \times \Psi$ and denote the associated σ-algebra by Ξ.

Similarly, we can define a stochastic process $\{\mathbf{Y_n}; n \in \mathcal{N}\}$ such that

$$\mathbf{Y_n} = (\mathbf{S_n}, \mathbf{C_n}),$$

where

$$\mathbf{S_n} = (S_{1,n}, \ldots, S_{p,n}),$$

and

$$\mathbf{C_n} = (C_{1,n}, \ldots, C_{p,n}) = (C_{1,n}, \ldots, C_{i-1,n}, 0, C_{i+1,n}, \ldots, C_{p,n})$$

if the nth event of the superposed process comes from the ith component process and this ith process moves to state $S_{i,n}$. For $j \in \mathcal{S}$, $C_{j,n}$ is the current life of the jth component process at time T_n and this component process is in state $S_{j,n} \in \mathcal{E}_j$. Note that $C_{i,n}$ is the current life of the ith process at time T_n and it is equal to zero. Let us call $\{\mathbf{Y_n}; n \in \mathcal{N}\}$ the discrete superposition current life process (DSCLP). Again, the state space of the process is given by $\Upsilon = \mathcal{E} \times \Psi$.

In the case when $|\mathcal{E}_1| = |\mathcal{E}_2| = \ldots = |\mathcal{E}_p| = 1$, DSRLP and DSCLP reduce respectively to the remaining life process $\{\mathbf{R_n}; n \in \mathcal{N}\}$ and the current life process $\{\mathbf{C_n}; n \in \mathcal{N}\}$ of the superposition of p independent renewal processes studied in Lam (1990a). When $p = 2$, DSCLP is closely related to the stochastic process $\{{}^1Z_n, {}^2Z_n, I_n, V_n, U_n; n \in \mathcal{N}\}$ presented in Cherry (1972), Cherry and Disney (1983).

In this paper, we show that both DSRLP and DSCLP are Markov chains. Under certain regularity conditions on the individual component Markov renewal processes, DSRLP and DSCLP have a common unique stationary distribution Π. DSRLP and DSCLP are useful in characterizing the steady state behavior of the superposition process at the times of occurrence of events. The continuous time properties of the superposed process are presented in Lam (1990b).

This paper is organized as follows : Preliminaries and notations are given in Section 2. In Section 3, we derive the stationary distributions and related steady state properties of DSRLP and DSCLP. These results are applied to study individual blocking probability of queueing systems with superposition semi-Markovian arrivals, and system availability of a device with components connected both in parallel and in series.

2. Preliminaries and Notations. In this section, we present some definitions and results of Markov renewal theory from the literature which are useful in establishing the characteristics of DSRLP and DSCLP. In addition, we describe the assumptions which are made throughout the rest of the paper. The following definitions are adopted from Cherry and Disney (1983).

Definition 2.1. *For $i \in \mathcal{S}$, the Markov renewal process (X_i, T_i) is said to be conservative if*

$$\mathcal{P}\left(\sup_{n\in\mathcal{N}} T_{i,n} < \infty\right) = 0. \tag{2.1}$$

Definition 2.2. *For $i \in \mathcal{S}$, define the random process $M_i(t)$ associated with the Markov renewal process (X_i, T_i) by $M_i(0) = 0$ and $M_i(t) = n$ for $T_{i,n} \leq t < T_{i,n+1}$. The process (X_i, T_i) is said to be regular if*

$$\mathcal{E}(M_i(t)) < \infty \qquad \textit{for all } t < \infty. \tag{2.2}$$

Definition 2.3. *For $i \in \mathcal{S}$, let $M_{i,e_i}(t)$ be the random process which counts the number of entrances of the Markov renewal process (X_i, T_i) to state $e_i \in \mathcal{E}_i$ during the interval $[0, t)$ including, if applicable, the initial state at time $T_{i,0} = 0$. The Markov renewal process (X_i, T_i) is said to be normal if, regardless of initial state,*

$$\mathcal{E}(M_{i,e_i}(t)) < \infty \qquad \textit{for all } t \in [0, \infty). \tag{2.3}$$

The implications of the definitions above are discussed in Cherry and Disney (1983). In this paper, we exclude conditions which could cause non-regularities and, in particular, we assume that instantaneous transitions are impossible, that is, $Q_i(\nu_i, \zeta_i, 0) = 0$ for all $\nu_i, \zeta_i \in \mathcal{E}_i$ and $i \in \mathcal{S}$. For all component processes, with probability one a non-zero time is spent between transitions. Furthermore, we assume that all the Markov renewal processes to be studied here are conservative, normal and regular.

Let $Q_i(\nu_i, \zeta_i) = Q_i(\nu_i, \zeta_i, \infty)$ and $Q_i = \{Q_i(\nu_i, \zeta_i); \nu_i, \zeta_i \in \mathcal{E}_i\}$. Consider the Markov chain X_i induced by Q_i. By the usual decomposition theorems, we can partition the Markov chain X_i into disjoint classes of recurrent and transient cases. To study the steady state behavior of the superposed process, it is sufficient to consider component processes whose imbedded Markov chains are irreducible recurrent. Throughout the remainder of this paper, we shall assume that X_i is irreducible recurrent non-null and aperiodic for all $i \in \mathcal{S}$. Let λ_i be the stationary distribution of X_i, i.e., for all $\zeta_i \in \mathcal{E}_i$,

$$\sum_{\nu_i \in \mathcal{E}_i} \lambda_i(\nu_i) Q_i(\nu_i, \zeta_i) = \lambda_i(\zeta_i) \qquad \text{and} \qquad \sum_{\nu_i \in \mathcal{E}_i} \lambda_i(\nu_i) = 1. \tag{2.4}$$

For each $i \in \mathcal{S}$ and $\nu_i \in \mathcal{E}_i$, let $F_i(\nu_i,t) = \sum_{\zeta_i \in \mathcal{E}_i} Q_i(\nu_i,\zeta_i,t)$. $F_i(\nu_i,t)$ is the probability distribution of the sojourn time of the ith process in state $\nu_i \in \mathcal{E}_i$. Assume the mean sojourn time in ν_i is $m_i(\nu_i) = \int_0^\infty (1-F_i(\nu_i,t))\,dt < \infty$. Let $\mu_i = \sum_{\nu_i \in \mathcal{E}_i} \lambda_i(\nu_i) m_i(\nu_i)$. We assume that all sojourn time probability distributions are absolutely continuous with respect to the Lebesgue measure and the probability density function is given by $f_i(\nu_i,\cdot)$. Furthermore, for each $i \in \mathcal{S}$, $\nu_i, \zeta_i \in \mathcal{E}_i$, the semi-Markov kernel $Q_i(\nu_i,\zeta_i,t)$ is also an absolutely continuous function in $t \in \Re_+$, i.e., the function $t \to Q_i(\nu_i,\zeta_i,t)$ has a derivative equal to $q_i(\nu_i,\zeta_i,t)$ almost everywhere. The absolute continuity of the functions $t \to Q_i(\nu_i,\zeta_i,t)$ ensure that simultaneous occurrence of events in the superposed process has probability zero.

For each $i \in \mathcal{S}$, let $\{X_i(t); t \in \Re_+\}$ be the semi-Markov process associated with the Markov renewal process (X_i, T_i). From Definition (2.2), $M_i(t)$ is the number of transitions in the ith process during the time interval $(0,t]$. Write $V_i^+(t) = T_{M_i(t)+1} - t$ and $V_i^-(t) = t - T_{M_i(t)}$. V_i^+ is therefore the time until the next jump and V_i^- is the time since the last jump. Both are defined with respect to epoch t, and they are well defined for all $t \in \Re_+$ because all the component processes are conservative. $\{X_i(t), V_i^+(t); t \in \Re_+\}$ and $\{X_i(t), V_i^-(t); t \in \Re_+\}$ are the continuous remaining and current life processes associated with the ith process. From Cinlar (1969), provided that the Markov chain X_i is irreducible recurrent, then

$$\begin{aligned}\lim_{t\to\infty} \mathcal{P}(X_i(t) = \beta_i, V_i^+(t) > z \mid X_i(0) = e_i) &= \lim_{t\to\infty} \mathcal{P}(X_i(t) = \beta_i, V_i^-(t) > z \mid X_i(0) = e_i) \\ &= \frac{\lambda_i(\beta_i)}{\mu_i} \int_z^\infty (1 - F_i(\beta_i, y))\,dy. \qquad (2.5)\end{aligned}$$

For a detailed study of the stationary measures of the continuous remaining and current life processes for Markov renewal processes, see Pyke and Schaufele (1966).

3. Stationary Distributions of DSRLP and DSCLP. In this section, we will show that both DSRLP and DSCLP are Markov chains. Furthermore, the stationary distributions of these processes both exist and are unique. We first define some notations. Let $\mathbf{e} = (e_1, \ldots, e_p)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$ and $I(B)$ be an indicator function such that $I(B) = 1$ whenever event B occurs and zero otherwise. Define the reduced state space $\tilde{\Upsilon} = \mathcal{E} \times \tilde{\Psi}$ where $\tilde{\Psi} = \cup_{i=1}^p \tilde{\Psi}_i$ and $\tilde{\Psi}_i \subseteq \Psi_i$ such that if $\mathbf{s} = (s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, p) \in \tilde{\Psi}_i$, then

1. $s_j \neq s_k \ \forall\ j,k \in \mathcal{S}_i$ and $j \neq k$,

2. $s_j \neq 0 \ \forall\ j \in \mathcal{S}_i$.

For any $n \geq 1$, we can now derive

$$\mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, R_{i,n} = 0, 0 < R_{j,n} \leq r_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, R_{k,n-1} = 0, R_{u,n-1} = s_u; u \in \mathcal{S}_k, \mathbf{X_{n-2}}, \ldots, \mathbf{X_0})$$

where $\mathbf{e}, \boldsymbol{\beta} \in \mathcal{E}$, $r_j \in \Re_+$, $j \in \mathcal{S}_i$, $\mathbf{s_k} = (s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \tilde{\Psi}_k$ and $i, k \in \mathcal{S}$. Note that it is sufficient to consider $\mathbf{s_k} \in \tilde{\Psi}_k$, this is because we have assumed that simultaneous occurrence of events in the superposed process has probability zero. By conditioning on

1. if $i = k$, $T_n - T_{n-1} = x$, or equivalently, $R_{j,n} = s_j - x$ for all $j \in \mathcal{S}_k$,

2. if $i \neq k$, $R_{k,n} = x$,

we have

$$\begin{aligned}
&\mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, R_{i,n} = 0, 0 < R_{j,n} \leq r_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, R_{k,n-1} = 0, R_{u,n-1} = s_u; u \in \mathcal{S}_k, \mathbf{X_{n-2}}, \ldots, \mathbf{X_0}) \\
&= \begin{cases} \displaystyle\int_0^{\min_{u \in \mathcal{S}_k} s_u} q_k(e_k, \beta_k, x) \prod_{u \in \mathcal{S}_k} [I(0 < s_u - x \leq r_u) I(\beta_u = e_u)]\, dx & \text{if } i = k \\ \displaystyle\int_0^{r_k} f_k(e_k, s_i + x) I(\beta_k = e_k) I(s_i = \min_{u \in \mathcal{S}_k} s_u) \left[\frac{Q_i(e_i, \beta_i) - Q_i(e_i, \beta_i, s_i)}{1 - F_i(e_i, s_i)}\right] & \\ \displaystyle\qquad \times \prod_{j \in \mathcal{S}_i \cap \mathcal{S}_k} [I(0 < s_j - s_i \leq r_j) I(\beta_j = e_j)]\, dx & \text{if } i \neq k \end{cases} \\
&= \mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, R_{n,i} = 0, 0 < R_{n,j} \leq r_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, R_{n-1,k} = 0, R_{n-1,u} = s_u; u \in \mathcal{S}_k) \\
&= \mathcal{P}_{\mathbf{X}}(\{\boldsymbol{\beta}\} \times A_i \mid (\mathbf{e}, \mathbf{s_k})) \qquad (3.1)
\end{aligned}$$

where $A_i = \{\mathbf{x} = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p) \in \Psi_i \mid x_j \leq r_j, r_j \in \Re_+, j \in \mathcal{S}_i\}$ and $\mathbf{s_k} = (s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \tilde{\Psi}_k$. For $n \geq 1$, $\mathbf{e}, \boldsymbol{\beta} \in \mathcal{E}$, $r_j \in \Re_+$, $j \in \mathcal{S}_i$, and $i \in \mathcal{S}$, define

$$\mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, R_{i,n} = 0, 0 < R_{j,n} \leq r_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, R_{k,n-1} = 0, R_{u,n-1} = s_u; u \in \mathcal{S}_k, \mathbf{X_{n-2}}, \ldots, \mathbf{X_0}) = 0 \qquad (3.2)$$

whenever $\mathbf{s_k} = (s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \Psi_k \backslash \tilde{\Psi}_k$ and $k \in \mathcal{S}$. DSRLP is therefore a Markov chain with state space Υ and stationary transition probabilities given in (3.1) and (3.2). From Breiman (1968), the nonempty set $A \in \Xi$ is called closed for the Markov chain $\{\mathbf{X_n}; n \in \mathcal{N}\}$ if for all $(\mathbf{e}, \mathbf{s}) \in A$, $\mathcal{P}(\mathbf{X_{n+1}} \in A \mid \mathbf{X_n} = (\mathbf{e}, \mathbf{s})) = 1$, i.e., it is almost surely not possible to leave the nonempty measurable set A. Furthermore, the chain is called indecomposable if there are no two disjoint closed sets $A^1, A^2 \in \Xi$. For each $i \in \mathcal{S}$, the Markov chain X_i is assumed to be irreducible recurrent aperiodic in Section 2, this together with the regularity conditions on the sojourn time distributions and semi-Markov kernels ensure that the Markov chain $\{\mathbf{X_n}; n \in \mathcal{N}\}$ with transition probabilities given by (3.1) and 3.2) is indecomposable.

Theorem 3.1. *DSRLP has a unique stationary distribution given by*

$$\mathcal{P}(\mathrm{S} = \boldsymbol{\beta}, R_i = 0, 0 < R_j \leq r_j; j \in \mathcal{S}_i) = \left(\frac{1}{\mu_i} \Big/ \frac{1}{\mu}\right) \lambda_i(\beta_i) \prod_{j=1, j \neq i}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{r_j} (1 - F_j(\beta_j, y))\, dy\right], \tag{3.3}$$

where $\mathrm{S} = (S_1, \ldots, S_p)$, $1/\mu = \sum_{j=1}^{p} 1/\mu_j$, $r_j \in \Re_+$, $j \in \mathcal{S}_i$, $i \in \mathcal{S}$ *and* $\boldsymbol{\beta} \in \mathcal{E}$.

Proof. It is easily checked that

$$\begin{aligned} &\sum_{i=1}^{p} \sum_{\boldsymbol{\beta} \in \mathcal{E}} \mathcal{P}(\mathrm{S} = \boldsymbol{\beta}, R_i = 0, 0 < R_j < \infty; j \in \mathcal{S}_i) \\ = &\sum_{i=1}^{p} \sum_{\boldsymbol{\beta} \in \mathcal{E}} \left(\frac{1}{\mu_i} \Big/ \frac{1}{\mu}\right) \lambda_i(\beta_i) \prod_{j=1, j \neq i}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{\infty} (1 - F_j(\beta_j, y))\, dy\right] = 1. \end{aligned} \tag{3.4}$$

Equation (3.3) is therefore a probability measure on Ξ. By definition a distribution Π is stationary for the stochastic process $\{\mathbf{X_n}; n \in \mathcal{N}\}$ if

$$\Pi(A) = \sum_{\mathbf{e} \in \mathcal{E}} \int_{\Psi} \mathcal{P}(\mathbf{X_2} \in A \mid \mathbf{X_1} = (\mathbf{e}, \mathbf{s}))\, \Pi((\mathbf{e}, \mathbf{ds})) \tag{3.5}$$

for all $A \in \Xi$. Since the Markov chain is indecomposable, whenever the stationary distribution exists, it is unique (see Breiman (1968)). For reason of symmetry, it is sufficient to consider the event

$$A = \{\boldsymbol{\beta}\} \times A_1 \quad \text{and} \quad A_1 = \{\mathrm{s} = (0, s_2, \ldots, s_p) \in \Psi_1 \mid s_j \leq r_j, r_j \in \Re_+, j \in \mathcal{S}_1\}. \tag{3.6}$$

Then the theorem is equivalent to

$$\Pi(\{\beta\} \times A_1) = \left(\frac{1}{\mu_1} \Big/ \frac{1}{\mu}\right) \lambda_1(\beta_1) \prod_{j=2}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{r_j} (1 - F_j(\beta_j, y))\, dy\right]. \tag{3.7}$$

To show that (3.7) satisfies Equation (3.5), we substitute (3.7) into the right hand side of (3.5). Results follow after some tedious but straightforward calculations. The details are given in the Appendix. □

Before the stationary distribution of DSCLP is given, we first define some notations. Let $c_i \in \Re_+$ for all $i \in \mathcal{S}$.

1. For $k \in \mathcal{S}$, let $U_k \subset \Re_+^p$ such that $(s_1, \ldots, s_p) \in U_k$ if and only if
   (a) $(s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \tilde{\Psi}_k$,
   (b) $0 < s_j + s_k - \min_{u \in \mathcal{S}_k} s_u \le c_j$ for all $j \in \mathcal{S}_k$,
   (c) $0 < s_j < c_j$ for all $j \in \mathcal{S}_k$,
   (d) $0 < \min_{u \in \mathcal{S}_k} s_u < s_k \le \min_{j \in \mathcal{S}_k}(s_j + s_k - \min_{u \in \mathcal{S}_k} s_u)$,
2. For $k \in \mathcal{S}$ and $i \in \mathcal{S}_k$, let $W_{k,i} \subset \Re_+^p$ such that $(s_1, \ldots, s_p) \in W_{k,i}$ if and only if
   (a) $(s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \tilde{\Psi}_k$,
   (b) $0 < s_j < c_j$ for all $j \in \mathcal{S}_k$,
   (c) $0 < s_j + s_k \le c_j$ for all $j \in \mathcal{S}_i \cap \mathcal{S}_k$,
   (d) $0 < s_k \le c_k$.

Now, for any $n \ge 1$, $\mathbf{e}, \boldsymbol{\beta} \in \mathcal{E}$, $c_j \in \Re_+$, $j \in \mathcal{S}_i$, $\mathbf{s_k} = (s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \tilde{\Psi}_k$ and $i, k \in \mathcal{S}$, we can derive

$$\mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, C_{i,n} = 0, 0 < C_{j,n} \le c_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, C_{k,n-1} = 0, C_{u,n-1} = s_u; u \in \mathcal{S}_k,$$
$$\mathbf{Y_{n-2}}, \ldots, \mathbf{Y_0})$$

by conditioning on

1. if $i = k$, $T_n - T_{n-1} = s_k - \min_{u \in \mathcal{S}_k} s_u$, or equivalently, $C_{n,v} = s_k$ and $s_v = \min_{u \in \mathcal{S}_k} s_u$,

2. if $i \neq k$, $T_n - T_{n-1} = s_k$, or equivalently, $C_{n,k} = s_k$.

This gives

$$\mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, C_{i,n} = 0, 0 < C_{j,n} \leq c_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, C_{k,n-1} = 0, C_{u,n-1} = s_u; u \in \mathcal{S}_k, \mathbf{Y_{n-2}}, \ldots, \mathbf{Y_0})$$

$$= \begin{cases} \displaystyle\int_0^\infty q_k(e_k, \beta_k, s_k - \min_{u \in \mathcal{S}_k} s_u) \prod_{u \in \mathcal{S}_k} \left[\left(\frac{1 - F_u(e_u, s_u + s_k - \min_{l \in \mathcal{S}_k} s_l)}{1 - F_u(e_u, s_u)} \right) I(\beta_u = e_u) \right] I(U_k)\, ds_k & \text{if } i = k \\ \displaystyle\int_0^\infty \frac{q_i(e_i, \beta_i, s_i + s_k)}{1 - F_i(e_i, s_i)} (1 - F_k(e_k, s_k)) I(\beta_k = e_k) \prod_{u \in \mathcal{S}_i \cap \mathcal{S}_k} \left[\left(\frac{1 - F_u(e_u, s_u + s_k)}{1 - F_u(e_u, s_u)} \right) I(\beta_u = e_u) \right] I(W_{k,i})\, ds_k & \text{if } i \neq k \end{cases}$$

$$\begin{aligned} &= \mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, C_{i,n} = 0, 0 < C_{j,n} \leq c_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, C_{k,n-1} = 0, C_{u,n-1} = s_u; u \in \mathcal{S}_k) \\ &= \mathcal{P}_{\mathbf{Y}}(\{\boldsymbol{\beta}\} \times B_i \mid (\mathbf{e}, \mathbf{s_k})) \end{aligned} \tag{3.8}$$

where $B_i = \{\mathbf{x} = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_p) \in \Psi_i \mid x_j \leq c_j, c_j \in \Re_+, j \in \mathcal{S}_i\}$ and $\mathbf{s_k} = (s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \tilde{\Psi}_k$. Again, for any $n \geq 1$, $\mathbf{e}, \boldsymbol{\beta} \in \mathcal{E}$, $c_j \in \Re_+$, $j \in \mathcal{S}_i$ and $i \in \mathcal{S}$, define

$$\mathcal{P}(\mathbf{S_n} = \boldsymbol{\beta}, C_{i,n} = 0, 0 < C_{j,n} \leq c_j; j \in \mathcal{S}_i \mid \mathbf{S_{n-1}} = \mathbf{e}, C_{k,n-1} = 0, C_{u,n-1} = s_u; u \in \mathcal{S}_k, \mathbf{Y_{n-2}}, \ldots, \mathbf{Y_0}) = 0 \tag{3.9}$$

whenever $\mathbf{s_k} = (s_1, \ldots, s_{k-1}, 0, s_{k+1}, \ldots, s_p) \in \Psi_k \backslash \tilde{\Psi}_k$ and $k \in \mathcal{S}$. DSCLP is therefore a Markov chain with state space Υ and stationary transition probabilities given by (3.8) and (3.9). Just like DSRLP, the Markov chain $\{\mathbf{Y_n}; n \geq 1\}$ is indecomposable. The stationary distribution of DSCLP is given in the theorem below.

Theorem 3.2. *DSCLP has a unique stationary distribution given by*

$$\mathcal{P}(\mathbf{S} = \boldsymbol{\beta}, C_i = 0, 0 < C_j \leq c_j; j \in \mathcal{S}_i) = \left(\frac{1}{\mu_i}\Big/\frac{1}{\mu}\right) \lambda_i(\beta_i) \prod_{j=1, j\neq i}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{c_j} (1 - F_j(\beta_j, y))\, dy\right], \tag{3.10}$$

where $\mathbf{S} = (S_1, \ldots, S_p)$, $1/\mu = \sum_{j=1}^{p} 1/\mu_j$, $c_j \in \Re_+$, $j \in \mathcal{S}_i$, $i \in \mathcal{S}$ *and* $\boldsymbol{\beta} \in \mathcal{E}$.

Proof. The proof is essentially the same as that for Theorem (3.1). Again, we need to verify for all $A = \{\boldsymbol{\beta}\} \times B_1 \in \Xi$ and $B_1 = \{\mathbf{s} = (0, s_2, \ldots, s_p) \in \Psi_1 \mid s_j \leq c_j, c_j \in \Re_+, j \in \mathcal{S}_1\}$,

$$\Pi(A) = \sum_{\mathbf{e}\in\mathcal{E}} \int_{\Psi} \mathcal{P}(\mathbf{Y_2} \in A \mid \mathbf{Y_1} = (\mathbf{e}, \mathbf{s}))\, \Pi((\mathbf{e}, \mathbf{ds})) \tag{3.11}$$

and

$$\Pi(A) = \left(\frac{1}{\mu_1}\Big/\frac{1}{\mu}\right) \lambda_1(\beta_1) \prod_{j=2}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{c_j} (1 - F_j(\beta_j, y))\, dy\right]. \tag{3.12}$$

By substituting Expression (3.12) into the right hand side of (3.11) and simplify, result follows. Again, the details are given in the Appendix. □

It immediate follows from Theorems (3.1) and (3.2) that when the superposed process is in equilibrium, the probability is μ/μ_i for an event to come from the ith component process. Furthermore,

$$\mathcal{P}(\mathbf{S} = \boldsymbol{\beta}) = \sum_{i=1}^{n} \left(\frac{1}{\mu_i}\Big/\frac{1}{\mu}\right) \lambda_i(\beta_i) \prod_{j=1, j\neq i}^{p} \left[\frac{\lambda_j(\beta_j) m_j(\beta_j)}{\mu_j}\right], \tag{3.13}$$

and

$$\mathcal{P}(S_j = \beta_j, 0 < R_j \leq r_j; j \in \mathcal{S}_i \mid R_i = 0) = \prod_{j=1, j\neq i}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{r_j} (1 - F_j(\beta_j, y))\, dy\right]. \tag{3.14}$$

Hence, conditional on knowing an event comes from the ith component process of an equilibrium superposed process, then $\{(S_j, R_j)\}_{j=1, j\neq i}^{p}$ are independent pairs of random variables and (S_j, R_j) follows the limiting continuous remaining or current life distribution of the jth component process given in Equation (2.5). The same result holds for DSCLP.

Let T be the interevent time of the equilibrium superposed process and $X_i(\beta_i)$ be a random variable whose distribution function is given by $F_i(\beta_i, t)$. By Theorem (3.1), the mean interevent time of this equilibrium process is given by

$$
\begin{aligned}
&\int_0^\infty \mathcal{P}(T > t)\,dt \\
= &\sum_{i=1}^{p}\sum_{\beta\in\mathcal{E}}\int_0^\infty \mathcal{P}(\mathbf{S}=\beta, R_i=0, \min\{R_1,\ldots,R_{i-1},X_i(\beta_i),R_{i+1},\ldots,R_p\} > t)\,dt \\
= &\sum_{i=1}^{p}\sum_{\beta\in\mathcal{E}}\int_0^\infty \left(\frac{1}{\mu_i}\Big/\frac{1}{\mu}\right)\lambda_i(\beta_i)\prod_{j=1,j\neq i}^{p}\left[\frac{\lambda_j(\beta_j)}{\mu_j}\int_t^\infty (1-F_j(\beta_j,y))\,dy\right](1-F_i(\beta_i,t))\,dt \\
= &\ \mu\sum_{\beta\in\mathcal{E}}\prod_{j=1}^{p}\left[\frac{\lambda_j(\beta_j)}{\mu_j}\int_0^\infty (1-F_j(\beta_j,y))\,dy\right] \\
= &\ \mu. \qquad (3.15)
\end{aligned}
$$

Note that alternatively, we can define a stochastic process $\{\mathbf{X}'_\mathbf{n}; n \in \mathcal{N}\}$ such that

$$\mathbf{X}'_\mathbf{n} = (\mathbf{S_n}, \mathbf{R}'_\mathbf{n})$$

where

$$\mathbf{S_n} = (S_{1,n},\ldots,S_{p,n}),$$

and

$$\mathbf{R}'_\mathbf{n} = (R_{1,n},\ldots,R_{p,n}) = (R_{1,n},\ldots,R_{i-1,n},X_i(S_{i,n}),R_{i+1,n},\ldots,R_{p,n})$$

if the nth event of the superposed process comes from the ith component process and this ith process moves to state $S_{i,n}$. For $j \in \mathcal{S}$, $R_{j,n}$ is the remaining life of the jth component process at time T_n and this component process is in state $S_{j,n} \in \mathcal{E}_j$. Note that the remaining life of the ith process is given by $R_{i,n} = X_i(S_{i,n})$ where $X_i(S_{i,n})$ has distribution function $F_i(S_{i,n},t)$. Let us call $\{\mathbf{X}'_\mathbf{n}; n \in \mathcal{N}\}$ the alternative discrete superposition remaining life process (ADSRLP). The state space of this process is given by $\Upsilon = \mathcal{E} \times \Re_+^p$. Just like DSRLP and DSCLP, we can derive the transition probabilities for ADSRLP and show that it is a Markov chain. Also, similar arguments used in Theorems (3.1) and (3.2) can be applied to show that under the regularity conditions stated in Section 2 of this paper, ADSRLP has a unique stationary distribution and it is given by

$$
\begin{aligned}
&\mathcal{P}(\mathbf{S}=\beta, 0 < R_i \le r_i; i \in \mathcal{S}) \qquad (3.16)\\
= &\ \mu\left[\frac{\lambda_i(\beta_i)}{\mu_i}F_i(\beta_i,r_i)\right]\prod_{j=1,j\neq i}^{p}\left[\frac{\lambda_j(\beta_j)}{\mu_j}\int_0^{r_j}(1-F_j(\beta_j,y))\,dy\right],
\end{aligned}
$$

where $\mathrm{S} = (S_1, \ldots, S_p)$, $1/\mu = \sum_{j=1}^{p} 1/\mu_j$, $r_j \in \Re_+$, $j \in \mathcal{S}$ and $\beta \in \mathcal{E}$.

4. Applications. In this section, the results obtained in the previous section are applied to the study of the individual blocking probability of queueing systems with steady state superposition of semi-Markovian arrivals, and system availability of a device with components connected both in parallel and in series.

Example 1: Queues with superposition semi-Markovian arrivals

Consider a single server queueing system with no waiting room. The arrival process is the superposition of p point processes. For the ith arrival process, suppose there are N_i types of customers. Assume $N_i < \infty$ for all $i \in \mathcal{S}$. Let $X_{i,n}$ be the type of the nth arriving customer in the ith stream, $n \in \mathcal{N}$, and the instants of arrivals be $T_{i,0}, T_{i,1}, T_{i,2}, \ldots$. Suppose $(X_i, T_i) = \{X_{i,n}, T_{i,n}; n \in \mathcal{N}\}$ can be modeled by a Markov renewal process. We assume that the service times are independent and identically distributed whose common distribution function is $1 - e^{-ax}$, $x \in \Re_+$. There is no waiting room in this system. This means that a customer arrives and finds the server busy leaves the system and never returns. Let X and $X_i(e_i)$ be independent random variables whose distribution function are given by $1 - e^{-ax}$ and $F_i(e_i, x)$ respectively. Suppose the arrival process is in equilibrium, let I be a random variable such that $I = 1$ if a customer arrives and finds the server busy and zero otherwise. Let t_1 denotes the arrival instant of this arbitrary customer, and t_0 be the arrival instant of the immediately preceding customer. The customer arrives at time t_1 and finds the system blocked if and only if the service of a preceding customer is not completed. It does not matter when this service started. The residue service time after the time instant t_0 is exponentially distributed. For $i = 0, 1$, let $(\mathbf{S}(t_i), \mathbf{R}(t_i)) = (S_1(t_i), \ldots, S_p(t_i), R_1(t_i), \ldots, R_p(t_i))$ be the state of the equilibrium superposition remaining life process at time t_i. By Theorem (3.1), we have

$$\begin{aligned}
\mathcal{P}(I = 1) &= \sum_{i=1}^{p} \mathcal{P}(I = 1, R_i(t_0) = 0) \qquad (4.17) \\
&= \sum_{i=1}^{p} \sum_{\mathbf{e} \in \mathcal{E}} \mathcal{P}(\mathrm{S}(t_0) = \mathbf{e}, R_i(t_0) = 0, \\
&\qquad X > \min\{R_1(t_0), \ldots, R_{i-1}(t_0), X_i(e_i), R_{i+1}(t_0), \ldots, R_p(t_0)\})
\end{aligned}$$

$$= 1 - \sum_{i=1}^{p} \sum_{\mathbf{e}\in\mathcal{E}} \mathcal{P}(\mathbf{S}(t_0) = \mathbf{e}, R_i(t_0) = 0,$$

$$X \leq \min\{R_1(t_0), \ldots, R_{i-1}(t_0), X_i(e_i), R_{i+1}(t_0), \ldots, R_p(t_0)\})$$

$$= 1 - \sum_{i=1}^{p} \left(\frac{1}{\mu_i} \Big/ \frac{1}{\mu}\right) \sum_{\mathbf{e}\in\mathcal{E}} \lambda_i(e_i) \int_0^\infty a e^{-ax} (1 - F_i(e_i, x))$$

$$\times \left[\prod_{j=1, j\neq i}^{p} \frac{\lambda_j(e_j)}{\mu_j} \int_x^\infty (1 - F_j(e_j, y))\, dy \right] dx.$$

Result (4.17) is a generalization of the result in Willie (1990) to steady state superposition of semi-Markovian arrivals. Suppose that there are p independent $M/G/1$ queues. For the ith queue, the waiting room is of size $N_i - 1 < \infty$. It is well known that the departure process from a $M/G/1/N_i$ queue can be modeled by a Markov renewal process whose state are queue lengths after departure and $\{T_{i,n+1} - T_{i,n}\}_{n\in\mathcal{N}}$ are times between departure. From Cinlar (1975a), if the imbedded Markov chain of the ith Markov renewal process is irreducible aperiodic with finitely many states and all states are recurrent, then the stationary distribution of this Markov chain, λ_i, exists and is unique. Suppose the merged output of these p independent $M/G/1/N_i$ queues with finite waiting room becomes the input of a single server queue with no waiting room. Call this queue the second stage queue. When the input to the second stage queue is in equilibrium, Equation (4.17) above gives the blocking probability of an arbitrary customer to this second stage queue. Note that the p $M/G/1$ queue here can be bulk service queues or the service times of these queues can be state dependent. For details, see Cinlar (1975b) or Lam (1990b). The result above also applies to the case when $N_i = \infty$ and the traffic intensity of the ith $M/G/1$ queue is less than 1.

Example 2: System Availability

Consider a device with p components connected in parallel. The device is not working when all the components are under repair. The ith component consists of a finite number of subcomponents connected in series. There are $N_i < \infty$ different types of subcomponents for component i, $i \in \mathcal{S}$. If any one subcomponent fails, the whole component fails. Let $X_{i,n}$ be the type of the subcomponent causing the nth failure in the ith component, and $T_{i,n}$ be the times of successive failures. The time $T_{i,n+1} - T_{i,n}$ between two failures is the sum of the repair time of the subcomponent which failed at $T_{i,n}$ and a failure free interval following the repair. We suppose that subcomponent of type $\beta_i \in \mathcal{E}_i = \{1, 2, \ldots, N_i\}$ in the ith component has exponential lifetimes with parameter $a(i, \beta_i)$, and suppose its repair time has distribution $\varphi(i, \beta_i, \cdot)$. Under these assumptions, the failure process of the device can be modeled by the superposition of p independent Markov renewal processes. In particular, for each $i \in \mathcal{S}$, the semi-Markov kernel Q_i is given by

$$Q_i(e_i, \beta_i, t) = \int_0^t a(i, \beta_i) e^{-a(i)u} \varphi(i, e_i, t-u)\, du \tag{4.18}$$

where $e_i, \beta_i \in \mathcal{E}_i$ and $a(i) = \sum_{\beta_i \in \mathcal{E}_i} a(i, \beta_i)$. Also,

$$F_i(e_i, t) = \sum_{\beta_i \in \mathcal{E}_i} Q_i(e_i, \beta_i, t) = \int_0^t a(i) e^{-a(i)u} \varphi(i, e_i, t-u)\, du. \tag{4.19}$$

From Cinlar (1975a), Example (10.6.23), $\lambda_i(\beta_i) = a(i, \beta_i)$ for all $\beta_i \in \mathcal{E}_i$ and $i \in \mathcal{S}$. If the mean repair time of type β_i subcomponent of the ith component is $b(i, \beta_i)$, then $m_i(\beta_i) = b(i, \beta_i) + (1/a(i))$. Let $X_i(\beta_i)$ be a random variable whose distribution function is given by $\varphi(i, \beta_i, \cdot)$ and $a(i)b(i) = \sum_{\beta_i \in \mathcal{E}_i} a(i, \beta_i) b(i, \beta_i)$. Suppose that the failure process of device is in equilibrium. Let I be a random variable such that $I = 1$ if a failure in one of the components results in a failure of the device. This means that at some time t of the equilibrium superposed process, one component fails and all the other components are undergoing repair. Let $\mathbf{c_i} = (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_p) \in \Re_+^{p-1}$ and

$$H(\boldsymbol{\beta}, i, \mathbf{c_i}) = \left(\frac{1}{\mu_i} \Big/ \frac{1}{\mu}\right) \lambda_i(\beta_i) \prod_{j=1, j \neq i}^{p} \left[\frac{\lambda_j(\beta_j)}{\mu_j} \int_0^{c_j} (1 - F_j(\beta_j, y))\, dy \right]. \tag{4.20}$$

We can now derive the steady state probability that the device fails at some time t and for the ith component, subcomponent of type β_i is undergoing repair at that time.

$$\begin{aligned}
&\mathcal{P}(I = 1, \mathrm{S} = \boldsymbol{\beta}) \\
= &\sum_{i=1}^{p} \mathcal{P}(I = 1, \mathrm{S} = \boldsymbol{\beta}, C_i = 0) \\
= &\sum_{i=1}^{p} \mathcal{P}(\mathrm{S} = \boldsymbol{\beta}, C_i = 0, C_j \leq X_j(\beta_j); j \in \mathcal{S}_i)
\end{aligned}$$

$$= \sum_{i=1}^{p} \int_{\Re_+^{p-1}} \prod_{j=1,j\neq i}^{p} \left[\frac{1-\varphi_j(\beta_j,c_j)}{1-F_j(\beta_j,c_j)} dc_j \right] H(\boldsymbol{\beta}, i, \mathbf{dc_i})$$

$$= \left(1 \Big/ \sum_{k=1}^{p} \frac{1}{1+a(k)b(k)} \right) \sum_{i=1}^{p} \frac{a(i,\beta_i)}{1+a(i)b(i)} \prod_{j=1,j\neq i}^{p} \left[\frac{a(j,\beta_j)b(j,\beta_j)}{1+a(j)b(j)} \right]. \tag{4.21}$$

The asymptotic continuous time behavior of this system is studied in Lam (1990b).

Appendix. To show that Expression (3.7) satisfies Equation (3.5)):

First, let us define some notations. For $m \geq 1$, let $\tilde{\Re}_+^m \subseteq \Re_+^m$ such that if $(x_1, \ldots, x_m) \in \tilde{\Re}_+^m$, then

1. $x_i \neq x_j \; \forall i,j \in \{1,2,\ldots,m\}$ and $i \neq j$,
2. $x_i \neq 0 \; \forall i \in \{1,2,\ldots,m\}$.

Note that $\Re_+^m \backslash \tilde{\Re}_+^m$ is a set of Lebesgue measure zero in $\Re_+^m$. From equations (3.1), (3.5) and (3.7),

$$\left[\frac{1}{\mu} \Big/ \left(\prod_{i=1}^{p} \frac{1}{\mu_i} \right) \right] \left(\prod_{j=2}^{p} \frac{1}{\lambda_j(\beta_j)} \right) \sum_{\mathbf{e}\in\mathcal{E}} \int_{\Psi} \mathcal{P}(\mathbf{X_2} \in \{\boldsymbol{\beta}\} \times A_1 \mid \mathbf{X_1} = (\mathbf{e},\mathbf{s})) \Pi((\mathbf{e},\mathbf{ds}))$$

$$= \sum_{k=2}^{p} \sum_{e_1\in\mathcal{E}_1} \int_{\tilde{\Re}_+^{p-1}} \left(\int_0^{r_k} f_k(\beta_k, s_1 + x) I(s_1 = \min_{u\in\mathcal{S}_k} s_u)(Q_1(e_1,\beta_1) - Q_1(e_1,\beta_1,s_1)) \right.$$

$$\left. \times \prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k} I(0 < s_j - s_1 \leq r_j)\, dx \right) \lambda_1(e_1)\, ds_1 \prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k} [(1 - F_j(\beta_j, s_j))\, ds_j]$$

$$+ \sum_{e_1\in\mathcal{E}_1} \int_{\tilde{\Re}_+^{p-1}} \left(\int_0^{\min_{u\in\mathcal{S}_1} s_u} q_1(e_1,\beta_1,x) \prod_{u\in\mathcal{S}_1} I(0 < s_u - x \leq r_u)\, dx \right)$$

$$\times \lambda_1(e_1) \prod_{j\in\mathcal{S}_1} [(1 - F_j(\beta_j, s_j))\, ds_j]$$

$$= \sum_{k=2}^{p} \sum_{e_1\in\mathcal{E}_1} \int_{\Re_+^{p-2}} \left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0, s_j - r_j\}}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u} (1 - F_k(\beta_k, s_1))(Q_1(e_1,\beta_1) - Q_1(e_1,\beta_1,s_1))\, ds_1 \right)$$

$$\times \lambda_1(e_1) \prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k} [(1 - F_j(\beta_j, s_j))\, ds_j]$$

$$-\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-2}}\left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0,s_j-r_j\}}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u}(1-F_k(\beta_k,s_1+r_k))\right.$$

$$\left.\times(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,s_1))\,ds_1\right)\lambda_1(e_1)\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$+\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-1}}\left(\int_{\max_{j\in\mathcal{S}_1}\{0,s_j-r_j\}}^{\min_{u\in\mathcal{S}_1} s_u} q_1(e_1,\beta_1,x)\,dx\right)\lambda_1(e_1)\prod_{j\in\mathcal{S}_1}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$=\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-2}}\left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0,s_j-r_j\}}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u}(1-F_k(\beta_k,s_1))(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,s_1))\,ds_1\right)$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$-\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-2}}\left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0,s_j-r_j\}+r_k}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u+r_k}(1-F_k(\beta_k,v))\right.$$

$$\left.\times(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,v-r_k))\,dv\right)\lambda_1(e_1)\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$-\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-1}}(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,\min_{u\in\mathcal{S}_1} s_u))I(\max_{j\in\mathcal{S}_1}\{0,s_j-r_j\}\le\min_{u\in\mathcal{S}_1} s_u)$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$+\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-1}}(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,\max_{u\in\mathcal{S}_1}\{0,s_u-r_u\}))I(\max_{j\in\mathcal{S}_1}\{0,s_j-r_j\}\le\min_{u\in\mathcal{S}_1} s_u)$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$=\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-2}}\left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0,s_j-r_j\}}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u}(1-F_k(\beta_k,s_1))(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,s_1))\,ds_1\right)$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$-\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-2}}\left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0,s_j-r_j\}+r_k}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u+r_k}(1-F_k(\beta_k,v))\right.$$

$$\left.\times(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,v-r_k))\,dv\right)\lambda_1(e_1)\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$-\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-2}}\left(\int_{\max_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\{0,s_j-r_j\}}^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k} s_u}(1-F_k(\beta_k,s_k))(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,s_k))\,ds_k\right)$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$+\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-1}}I(0=\max_{j\in\mathcal{S}_1}\{0,s_j-r_j\}\le\min_{u\in\mathcal{S}_1}s_u)(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,0)) \tag{A.1}$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$+\sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\int_{\Re_+^{p-1}}(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,s_k-r_k))I(s_k-r_k=\max_{j\in\mathcal{S}_1}\{0,s_j-r_j\}\le\min_{u\in\mathcal{S}_1}s_u)$$

$$\times\lambda_1(e_1)\prod_{j\in\mathcal{S}_1}[(1-F_j(\beta_j,s_j))\,ds_j]$$

$$= \int_{\Re_+^{p-1}}I(s_j\le r_j; j\in\mathcal{S}_1)\lambda_1(\beta_1)\prod_{j\in\mathcal{S}_1}[(1-F_j(\beta_j,s_j))\,ds_j] \tag{A.2}$$

$$= \left[\frac{1}{\mu}\Big/\left(\prod_{i=1}^{p}\frac{1}{\mu_i}\right)\right]\left(\prod_{j=2}^{p}\frac{1}{\lambda_j(\beta_j)}\right)\Pi(\{\beta\}\times A_1) \tag{A.3}$$

Expression (A.2) above follows from (A.1) because λ_1 is the stationary distribution of the Markov chain induced by Q_1 and $Q_1(e_1,\beta_1,0)=0$.

To show that Expression (3.12) satisfies Equation (3.11)):

First, let us define some notations and transformations. Let $c_i\in\Re_+$ for all $i\in\mathcal{S}$. Also, note that for $k\in\mathcal{S}$ and $i\in\mathcal{S}_k$, U_k and $W_{k,i}$ are defined in Section 3.

1. For $k\in\mathcal{S}$ and $v\in\mathcal{S}_k$, let $U_{k,v}\subset U_k$ such that $(s_1,\ldots,s_p)\in U_{k,v}$ if and only if $s_v=\min_{u\in\mathcal{S}_k}s_u$ and $(s_1,\ldots,s_p)\in U_k$. Obviously, $\bigcup_{v\in\mathcal{S}_k}U_{k,v}=U_k$.

2. For $k \in \mathcal{S}$ and $v \in \mathcal{S}_k$, let $U'_{k,v} \subset \Re^p_+$ such that $(y_1, \ldots, y_p) \in U'_{k,v}$ if and only if

   (a) $(y_1, \ldots, y_{v-1}, 0, y_{v+1}, \ldots, y_p) \in \tilde{\Psi}_v$,

   (b) $0 < y_j \leq c_j$ for all $j \in \mathcal{S}_v \cap \mathcal{S}_k$,

   (c) $0 < y_v < y_k \leq \min_{j \in \mathcal{S}_v \cap \mathcal{S}_k} \{y_j, c_v\}$.

   Define a transformation $f : U_{k,v} \to U'_{k,v}$ such that $(y_1, \ldots, y_p) = f(s_1, \ldots, s_p)$ and

   (a) $y_j = s_j + s_k - s_v$ for all $j \in \mathcal{S}_v \cap \mathcal{S}_k$,

   (b) $y_j = s_j$ if $j = v, k$.

   Obviously, the transformation is one to one and onto. Furthermore, the Jacobian determinant is equal to one.

3. For $k \in \mathcal{S}$ and $i \in \mathcal{S}_k$, let $W'_{k,i} \subset \Re^p_+$ such that $(y_1, \ldots, y_p) \in W'_{k,i}$ if and only if

   (a) $(y_1, \ldots, y_i, 0, y_{i+1}, \ldots, y_p) \in \tilde{\Psi}_i$,

   (b) $0 < y_j \leq c_j$ for all $j \in \mathcal{S}_i \cap \mathcal{S}_k$,

   (c) $0 < y_k \leq \min_{j \in \mathcal{S}_i \cap \mathcal{S}_k} \{y_j, c_k\}$.

   Define a transformation $g : W_{k,i} \to W'_{k,i}$ such that $(y_1, \ldots, y_p) = g(s_1, \ldots, s_p)$ and

   (a) $y_j = s_j + s_k$ for all $j \in \mathcal{S}_i \cap \mathcal{S}_k$,

   (b) $y_j = s_j$ if $j = i, k$.

   Again, the transformation is one to one and onto. Furthermore, the Jacobian determinant is equal to one.

Now substituting Expression (3.12) into the right hand side of (3.11) and using the transformations defined above, we have

$$\left[\frac{1}{\mu} \Big/ \left(\prod_{i=1}^{p} \frac{1}{\mu_i}\right)\right] \left(\prod_{j=2}^{p} \frac{1}{\lambda_j(\beta_j)}\right) \sum_{\mathbf{e} \in \mathcal{E}} \int_{\Psi} \mathcal{P}(\mathbf{Y_2} \in \{\beta\} \times B_1 \mid \mathbf{Y_1} = (\mathbf{e}, \mathbf{s}))\, \Pi((\mathbf{e}, \mathbf{ds}))$$

$$= \sum_{k=2}^{p} \sum_{e_1 \in \mathcal{E}_1} \int_{\Re_+^p} \lambda_1(e_1) q_1(e_1, \beta_1, s_1 + s_k)(1 - F_k(\beta_k, s_k)) \left[\prod_{u \in \mathcal{S}_1 \cap \mathcal{S}_k} (1 - F_u(\beta_u, s_u + s_k)) \right] \times I(W_{k,1}) \left[\prod_{u \in \mathcal{S}} ds_u \right]$$
$$+ \sum_{e_1 \in \mathcal{E}_1} \int_{\Re_+^p} \lambda_1(e_1) q_1(e_1, \beta_1, s_1 - \min_{u \in \mathcal{S}_1} s_u) \left[\prod_{u \in \mathcal{S}_1} (1 - F_u(\beta_u, s_u + s_1 - \min_{l \in \mathcal{S}_1} s_l)) \right] \times I(U_1) \left[\prod_{u \in \mathcal{S}} ds_u \right]$$

$$= \sum_{k=2}^{p} \sum_{e_1 \in \mathcal{E}_1} \int_{\Re_+^p} \lambda_1(e_1) q_1(e_1, \beta_1, y_1 + y_k)(1 - F_k(\beta_k, y_k)) \left[\prod_{u \in \mathcal{S}_1 \cap \mathcal{S}_k} (1 - F_u(\beta_u, y_u)) \right] \times I(W'_{k,1}) \left[\prod_{u \in \mathcal{S}} dy_u \right]$$
$$+ \sum_{v=2}^{p} \sum_{e_1 \in \mathcal{E}_1} \int_{\Re_+^p} \lambda_1(e_1) q_1(e_1, \beta_1, s_1 - s_v)(1 - F_v(\beta_v, s_1))$$
$$\times \left[\prod_{u \in \mathcal{S}_1 \cap \mathcal{S}_v} (1 - F_u(\beta_u, s_u + s_1 - s_v)) \right] I(U_{1,v}) \left[\prod_{u \in \mathcal{S}} ds_u \right]$$

$$= \sum_{k=2}^{p} \sum_{e_1 \in \mathcal{E}_1} \int_{\Re_+^p} \lambda_1(e_1) q_1(e_1, \beta_1, y_1 + y_k)(1 - F_k(\beta_k, y_k)) \left[\prod_{u \in \mathcal{S}_1 \cap \mathcal{S}_k} (1 - F_u(\beta_u, y_u)) \right] \times I(W'_{k,1}) \left[\prod_{u \in \mathcal{S}} dy_u \right]$$
$$+ \sum_{v=2}^{p} \sum_{e_1 \in \mathcal{E}_1} \int_{\Re_+^p} \lambda_1(e_1) q_1(e_1, \beta_1, y_1 - y_v)(1 - F_v(\beta_v, y_1)) \left[\prod_{u \in \mathcal{S}_1 \cap \mathcal{S}_v} (1 - F_u(\beta_u, y_u)) \right] \times I(U'_{1,v}) \left[\prod_{u \in \mathcal{S}} dy_u \right]$$

$$= \sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\int_0^{c_j}\right]\left[\int_0^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k}\{y_u,c_k\}}\left(\int_0^{\infty}\lambda_1(e_1)q_1(e_1,\beta_1,y_1+y_k)\,dy_1\right)\right.$$

$$\left.\times(1-F_k(\beta_k,y_k))\,dy_k\right]\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}(1-F_j(\beta_j,y_j))\,dy_j\right]$$

$$+\sum_{v=2}^{p}\sum_{e_1\in\mathcal{E}_1}\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_v}\int_0^{c_j}\right]\left[\int_0^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_v}\{y_u,c_v\}}\left(\int_0^{y_1}\lambda_1(e_1)q_1(e_1,\beta_1,y_1-y_v)\,dy_v\right)\right.$$

$$\left.\times(1-F_v(\beta_v,y_1))\,dy_1\right]\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_v}(1-F_j(\beta_j,y_j))\,dy_j\right]$$

$$= \sum_{k=2}^{p}\sum_{e_1\in\mathcal{E}_1}\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}\int_0^{c_j}\right]\left[\int_0^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_k}\{y_u,c_k\}}\lambda_1(e_1)(Q_1(e_1,\beta_1)-Q_1(e_1,\beta_1,y_k))\right.$$

$$\left.\times(1-F_k(\beta_k,y_k))\,dy_k\right]\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_k}(1-F_j(\beta_j,y_j))\,dy_j\right]$$

$$+\sum_{v=2}^{p}\sum_{e_1\in\mathcal{E}_1}\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_v}\int_0^{c_j}\right]\left[\int_0^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_v}\{y_u,c_v\}}\lambda_1(e_1)(Q_1(e_1,\beta_1,y_1)\right.$$

$$\left.\times(1-F_v(\beta_v,y_1))\,dy_1\right]\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_v}(1-F_j(\beta_j,y_j))\,dy_j\right]$$

$$= \sum_{v=2}^{p}\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_v}\int_0^{c_j}\right]\left[\int_0^{\min_{u\in\mathcal{S}_1\cap\mathcal{S}_v}\{y_u,c_v\}}\lambda_1(\beta_1)(1-F_v(\beta_v,y_1))\,dy_1\right]$$

$$\times\left[\prod_{j\in\mathcal{S}_1\cap\mathcal{S}_v}(1-F_j(y_j))\,dy_j\right]$$

$$= \left[\frac{1}{\mu}\Big/\left(\prod_{i=1}^{p}\frac{1}{\mu_i}\right)\right]\left(\prod_{j=2}^{p}\frac{1}{\lambda_j(\beta_j)}\right)\Pi(A) \tag{A.4}$$

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