# OPTIMAL REPLACEMENT POLICIES FOR MULTI-STATE DETERIORATING SYSTEMS

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# Optimal Replacement Policies for Multi-State Deteriorating Systems

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**Abstract**

In this article we consider state-age-dependent replacement policies for a multi-state deteriorating system. We assume that operating cost rates and replacement costs are both functions of the underlying states. Replacement times and sojourn times in different states are all state-dependent random variables. The optimization criterion is to minimize the expected long run cost rate. A policy improvement algorithm to derive the optimal policy is presented. We show that under reasonable assumptions, the optimal replacement policies have monotonic properties. In particular, when the failure rate functions are nonincreasing or when all the replacement costs and the expected replacement times are independent of state, we show that the optimal policies are only state-dependent. Examples are given to illustrate the structure of the optimal policies in the special case when the sojourn time distributions are Weibull.

# 1. INTRODUCTION

This article studies the optimal replacement policies for a multi-state deteriorating system. We assume that the system is subject to both deterioration and random shocks. The operating condition of the system is characterized by the degree of deterioration and can be classified into a finite number of states. In each state, the system either gradually deteriorates to the next higher state or suddenly fails due to random shocks. Without maintenance action, the system will eventually fail. It is assumed that the system is monitored continuously such that the current state of the system is always known. When failures occur, the system must be replaced with a new identical system. The system can also be replaced at any point of time. The cost and time required to replace the system depend on the state of the system. In general, the operating cost rate, replacement cost, and replacement time increase as the system deteriorates. Therefore, it is desirable to replace the system before it fails to avoid operating or replacing the system in a highly deteriorating condition.

Maintenance policies of stochastically failing systems have been widely investigated in the literature. The papers by McCall [8], Pierskalla and Voelker [12], and Valdez-Flores and Feldman [14] are excellent reviews of the area. Many papers in the literature concentrate on modeling the deteriorating process of a stochastically failure system by a Markov process because of the tractability of the resulting mathematical problems. For example, Luss [7], Mine and Kawai [9, 10], and Ohnishi et al. [11] investigated optimal inspection policies and Anderson [1], Kolesar [6] and Wood [15] studied optimal replacement policies for Markovian deteriorating systems. Under the Markovian formulation, the deterioration of the system is indicated only by the changes of states. It is assumed that the performance of the system within each state does not age, i.e., the probability of moving to a less desirable state does not increase in time or equivalently the failure rate function of the sojourn time distribution in each state is independent of time. Unfortunately, in practice this assumption may be invalid.

The use of a semi-Markov process to model a deteriorating system has been studied before. Feldman [2] and Gottlieb [4] both used a semi-Markov process to model the cumulative damage process of a system that is subject to random shocks. Feldman [2] only considered control limit replacement policies and Gottlieb [4] investigated conditions under which a control limit replacement policy is optimal when the failure rate function need not be increasing and replacement can be made at any time. Furthermore, So [13] used a parametric analysis to establish sufficient conditions for the optimality of control limit replacement policies for semi-Markovian deteriorating systems. The above papers focused on state-dependent policies, that is, an optimal replacement policy is restricted to the class of policies that a replacement action is taken only at transition times. In this paper, we allow a replacement policy to be state-age-dependent. In other words, the system can be replaced at any point of time.

The problem we consider in this paper is closely related to the paper by Kao [5] and Feldman and Joo [3]. Kao studied both state-dependent and state-age-dependent replacement policies for a deteriorating system that is modeled by a semi-Markov process whose sojourn times in different states follow discrete distributions. He used policy iteration methods to derive optimal policies. He also showed that under sufficient conditions, the optimal state-dependent replacement policies are of control limit type. However, he did not investigate the structural properties of the optimal state-age-dependent replacement policies.

Feldman and Joo [3] considered state-age-dependent replacement policies for a shock process. They developed an algorithm for finding the optimal replacement policy under some regularity conditions. They numerically showed that the algorithm is more efficient than Kao's or Gottlieb's algorithms. However, they assumed replacement is instantaneous and there are only two types of replacement costs, after or before failures. Furthermore, it is assumed that the times between shocks form a sequence of independent and identically distributed random variables.

In this paper, we assume that replacement times are random. Replacement cost, operating cost, replacement time distributions and sojourn time distributions in different states are all state-dependent. The goal of this study is to investigate optimal state-age-dependent replacement policies that minimize the expected long run cost rate. We show that under reasonable assumptions on operation cost rates, replacement costs, expected replacement times and failure rate functions, the optimal state-age-dependent replacement policies have monotonic properties. Furthermore, sufficient conditions for the existence of a state-dependent policy that is optimal over the set of all state-age-dependent policies are given. In particular, we show that if the failure rate function in each state is nonincreasing or if the replacement costs and the expected replacement times are independent of state, then the optimal policies are only state-dependent.

The remainder of this paper is organized as follows. The model and the mathematical formulation of the problem are given in Section 2. An iterative algorithm is given in Section 3 to derive the optimal state-age-dependent replacement policy. In Section 4, we study the structures of the optimal policies under various assumptions. In Section 5, we investigate the special case when all the sojourn time distributions are Weibull.

## 2. MATHEMATICAL FORMULATION

Consider that the deterioration of a system at any point of time can be classified into one of a finite number of states $0, 1, \cdots, n, n+1$. State 0 represents the process before any deterioration takes place, i.e., it is the initial new state of the system. The intermediate states $1, 2, \cdots, n$ are ordered to reflect their relative degree of deterioration (in ascending order). State $n+1$ represents the terminal state (failure) of the deteriorating process. It is assumed that the time for the system to stay in state $i$ follows a general absolutely continuous distribution $F_i$ with finite mean $\mu_i$ and probability density function $f_i$. Also, assume that $F_i(0) = 0$ and $F_i(t) < 1$ for all $t \in [0, \infty)$. Let $h_i = f_i/\bar{F}_i$, i.e., $h_i$ is the failure rate function of the sojourn time in state $i$.

The deteriorating process of the system is characterized by a semi-Markov process with state space $S = \{0, 1, \cdots, n, n+1\}$. From state $i$, the deteriorating process will make a direct transition to state $i+1$ with probability $p_i$ ($p_n = 0$) due to gradual deterioration, or to state $n+1$ with probability $1 - p_i$ due to a random shock. It is clear that we only need to consider the case when $p_i > 0$ for all $i \in \{0, 1, \cdots, n-1\}$, otherwise we can reduce the number of states. A flow diagram of such a system is illustrated in Figure 1.

We assume that the replacement costs and times depend on the current state of the system. If the system is replaced in state $i \in S$, the replacement cost is $c_i$ and the replacement time follows a distribution function $R_i$ with a finite mean value $r_i$. After the completion of each replacement, the system is renewed (back to state 0). During a replacement, the system is neither operating nor deteriorating and there is a loss of $m$ per unit time. Let $S_n$ denote the set of all the operating states, i.e., $S_n = \{0, 1, \cdots, n\}$. When the system operates in state $i \in S_n$, the operating cost is $a_i$ per unit time.

Here we assume the deterioration system is monitored continuously which means that the current state of the system is always known. A state-age-dependent replacement policy is described as follows. If $E_i$ represents the time instant when the system enters state $i$, then at $E_i$ we make a decision to replace the system $t_i$ units of time later if it remains in state $i$. If the system moves to the state $i+1$ before $t_i$ units of time, a new decision is made at $E_{i+1}$. If the system fails (enters state $n+1$) before $t_i$ units of time, it is replaced immediately after it fails. A state-age-dependent replacement policy, $\delta$, is a sequence of decisions selected at $E_i$, $i \in S$. In other words, $\delta = (\delta(0), \delta(1), \cdots, \delta(n), \delta(n+1)) = (t_0, t_1, \cdots, t_n, t_{n+1})$ which specifies the maximum time allowed for the system staying in each state. In particular, if $t_i = 0$ for some $i \in S$, then the system is replaced as soon as it enters state $i$. On the other hand, if $t_i = \infty$, the system will not be replaced as long as it remains in state $i$. Obviously, a state-dependent replacement policy is a special case of a state-age-dependent replacement policy when $t_i$ is either 0 or $\infty$ for all $i \in S$. Furthermore, since a failed system must be replaced, it is sufficient to consider the class of all replacement policies $\Delta$ with $\delta(n+1) = t_{n+1} = 0$.

Our objective is to find an optimal state-age-dependent replacement policy $\delta^*$ that minimizes the expected long run cost rate. For an infinite time span, it is equivalent to minimizing the expected cost rate of a maintenance cycle which is defined as the time interval from the completion of one replacement to the next. Let $C_\delta(i)$ and $T_\delta(i)$ be respectively the expected cost and time from $E_i$ to the completion of next replacement under the policy $\delta \in \Delta$. They can be calculated recursively using the following equations with $T_\delta(n+1) = r_{n+1}$ and $C_\delta(n+1) = c_{n+1} + mr_{n+1}$.

$$\begin{cases} T_\delta(i) = \int_0^{t_i} \bar{F}_i(u)\,du + \bar{F}_i(t_i)r_i + F_i(t_i)\left[p_i T_\delta(i+1) + (1-p_i)T_\delta(n+1)\right], \\ C_\delta(i) = a_i \int_0^{t_i} \bar{F}_i(u)\,du + \bar{F}_i(t_i)(c_i + mr_i) + F_i(t_i)\left[p_i C_\delta(i+1) + (1-p_i)C_\delta(n+1)\right]. \end{cases} \tag{1}$$

Since $\mu_i$ is finite, it is obvious that both $T_\delta(i)$ and $C_\delta(i)$ are finite for all $t_i \in [0,\infty)$ and $i \in S$.

The objective here is to find $\delta^* = (\delta^*(0), \cdots, \delta^*(n+1)) = (t_0^*, \cdots, t_n^*, 0) \in \Delta$ such that

$$g^* \equiv \inf_{\delta \in \Delta} \frac{C_\delta(0)}{T_\delta(0)} = \frac{C_{\delta^*}(0)}{T_{\delta^*}(0)}. \tag{2}$$

Let $W(g) = \inf_{\delta \in \Delta} [C_\delta(0) - gT_\delta(0)]$. $W(g)$ is clearly a continuous and nonincreasing function in $g \in (0,\infty)$. From Ohnishi et al. [11], solving Equation (2) is equivalent to finding a $g^* \in [0,\infty)$ and a policy $\delta^* \in \Delta$ such that

$$W(g^*) = \inf_{\delta \in \Delta} [C_\delta(0) - g^* T_\delta(0)] = C_{\delta^*}(0) - g^* T_{\delta^*}(0) = 0.$$

Note that when $m = 1$, $a_i = 0$ for $i \in S_n$, and $c_i = 0$ for $i \in S$, $T_\delta(0)$ is the expected cycle time and $C_\delta(0)$ becomes the expected replacement (idle) time of a maintenance cycle. In this case, the expected long run cost rate becomes the expected long run fraction of time that the system is idle which is known as the unavailability of the system. Minimizing unavailability of the system is therefore a special case of the problem considered here.

## 3. OPTIMAL POLICIES

In this section, we provide an algorithm to find the optimal state-age-dependent replacement policy. Clearly, there are two trivial policies: $\delta_r = (0, 0, \cdots, 0)$ (replacement all the time) and $\delta_f = (\infty, \infty, \cdots, \infty, 0)$ (replacement only at failures). Let $g_r$ and $g_f$ represent the corresponding expected long run cost rates of these two policies. Then, we have

$$g_r = m + \frac{c_0}{r_0}$$

and

$$g_f = \frac{\sum_{i=0}^{n} \left( \prod_{j=0}^{i-1} p_j \right) a_i \mu_i + c_{n+1} + m r_{n+1}}{\sum_{i=0}^{n} \left( \prod_{j=0}^{i-1} p_j \right) \mu_i + r_{n+1}}$$

with $\prod_{j=0}^{-1} p_j = 1$. The optimal expected long run cost rate $g^*$ is therefore bounded above by the minimum of $g_r$ and $g_f$.

For any $i \in S$ and $g \in [0, \infty)$, define $K_i(g) = c_i + (m - g) r_i$ and $V_\delta(i, g) = C_\delta(i) - g T_\delta(i)$ for any $\delta \in \Delta$. Here, $K_i(g)$ represents the expected relative replacement cost (to $g$) in state $i$ and $V_\delta(i, g)$ represents the expected relative cost from state $i$ to the completion of next replacement under the policy $\delta$. Using $V_\delta(n+1, g) = K_{n+1}(g)$ and Equation (1), we have

$$V_\delta(i, g) = (a_i - g) \int_0^{t_i} \bar{F}_i(u)\, du + \bar{F}_i(t_i) K_i(g) + F_i(t_i) [p_i V_\delta(i+1, g) + (1 - p_i) K_{n+1}(g)]$$

for all $i \in S_n$. Given a policy $\delta \in \Delta$, a fixed $g \in [0, \infty)$ and an $i \in S_n$, define a function $t \mapsto v_\delta(i, g, t)$ such that

$$v_\delta(i,g,t) = (a_i - g)\int_0^t \bar{F}_i(u)\,du + \bar{F}_i(t)K_i(g) + F_i(t)\,[p_i V_\delta(i+1,g) + (1-p_i)K_{n+1}(g)]$$

for all $t \in [0, \infty)$. Note that $v_\delta(i, g, t)$ is a continuous function of $t \in (0, \infty)$ with

$$\lim_{t\to 0} v_\delta(i,g,t) \;=\; K_i(g)$$

and

$$\lim_{t\to\infty} v_\delta(i,g,t) \;=\; (a_i - g)\mu_i + [p_i V_\delta(i+1,g) + (1-p_i)K_{n+1}(g)]\,.$$

Since $V_\delta(n+1, g) = K_{n+1}(g)$ for all $\delta \in \Delta$, by backward induction $\lim_{t\to\infty} v_\delta(i, g, t)$ is finite for all $i \in S_n$. Hence, $\inf_{t\in[0,\infty)} v_\delta(i, g, t)$ exists and its value is finite. Since the system can only deteriorate, a stationary policy $\delta_g$ satisfying $W(g) = \inf_{\delta\in\Delta} V_\delta(0, g) = V_{\delta_g}(0, g)$ can be constructed by the following Backward Procedure (BP).

**Step 1:** Set $\delta_g(n+1) = t_{n+1} = 0$ and $V_{\delta_g}(n+1, g) = K_{n+1}(g)$.

**Step 2:** For $i = n, n-1, \cdots, 1, 0$,

Find the smallest nonnegative real number $t_i$ such that

$v_{\delta_g}(i, g, t_i) = \inf_{t\in[0,\infty)} v_{\delta_g}(i, g, t)$. Set $\delta_g(i) = t_i$ and $V_{\delta_g}(i, g) = v_{\delta_g}(i, g, t_i)$.

Observe that $W(0) = \inf_{\delta\in\Delta} C_\delta(0)$ and $W(g) \le 0$ whenever $g \in \{g_r, g_f\}$. Since $W(g)$ is continuous and nonincreasing in $g \in (0, \infty)$, there always exists a $g \in [0, \min(g_r, g_f)]$ and a corresponding $\delta_g$ such that $W(g) = 0$. Furthermore, $\delta^* = \delta_g$ and $g^* = g$ and they can be obtained using the following Policy Improvement Algorithm (PIA).

**Step I:** [Initial Criteria]

Select a tolerance limit $\epsilon > 0$ and an initial policy $\delta^0$, e.g., $\delta_r$.

Set $k = 0$ and $g^k = g_r$ if $\delta^k = \delta_r$.

**Step II:** [Policy Improvement Routine]

Use $g^k$ to construct a policy $\delta_{k+1}$ following BP above.

**Step III:** [Stopping Criterion]

If $\mid W(g^k) \mid = \mid V_{\delta^{k+1}}(0, g^k) \mid < \epsilon$, then STOP. Set $\delta^* = \delta^{k+1}$ and $g^* = g^k$.

**Step IV:** [Value Determination Routine]

Set $g^{k+1} = C_{\delta^{k+1}}(0)/T_{\delta^{k+1}}(0)$, $k = k+1$ and GOTO step II.

We now discuss the conditions under which the above algorithm converges to the optimal policy in a finite number of iterations. For simplicity of notation, define $v(i,g,t) = v_{\delta_g}(i,g,t)$ where $\delta_g$ is the policy constructed using BP for the fixed $g \in [0,\infty)$. By differentiating $v(i,g,t)$ with respect to $t$, we have

$$\frac{d}{dt}v(i,g,t) = (a_i - g)\bar{F}_i(t) + f_i(t)\Gamma_i(g) = \bar{F}_i(t)[(a_i - g) + h_i(t)\Gamma_i(g)] \qquad (3)$$

where $\Gamma_i(g) = p_i V_{\delta_g}(i+1,g) + (1-p_i)K_{n+1}(g) - K_i(g)$. It is obvious that the characteristics of the failure rate function $h_i$ plays an important role in deriving the optimal time for replacement in state $i$. If $h_i$ is monotonic increasing or decreasing, then $\frac{d}{dt}v(i,g,t)$ changes sign at most once in $t \in [0,\infty)$. Hence $v(i,g,t)$ has at most one minimum value in $t$ and $t$ may be 0 or $\infty$. This means that provided the failure rate functions are all monotonic, we would always be able to compute numerically the policy $\delta_g$ for any fixed $g \in [0,\infty)$ to the desired accuracy in a finite number of iterations.

From Steps II and IV, we have $W(g^k) = C_{\delta^{k+1}}(0) - g^k T_{\delta^{k+1}}(0) \leq C_{\delta^k}(0) - g^k T_{\delta^k}(0) = 0 = C_{\delta^{k+1}}(0) - g^{k+1}T_{\delta^{k+1}}(0)$. Hence, $g^k \geq g^{k+1} \geq 0$. Furthermore, $g^k = g^{k+1}$ if and only if $W(g^k) = 0$. These mean that PIA above strictly reduces the cost rate monotonically until the stopping criterion is satisfied. Since the sequence $g_0$, $g_1$, $g_2$, $\cdots$ is monotonic decreasing and bounded below by 0, it converges to a certain limiting value $g'$. We know that the function $W(g)$ is continuous and nonincreasing in $g$, it follows that $W(g_0)$, $W(g_1)$, $\cdots$ is a monotonic increasing sequence and it converges to $W(g')$. It remains to show that $W(g') = 0$. This is clear since

$$| W(g^k) | = | C_{\delta^{k+1}}(0) - g^k T_{\delta^{k+1}}(0) - [C_{\delta^{k+1}}(0) - g^{k+1}T_{\delta^{k+1}}(0)] | \leq (g^k - g^{k+1}) \sup_{\delta \in \Delta} T_\delta(0) \to 0$$

as $k \to \infty$. Hence, $g' = g^*$ and PIA converges to the optimal policy. Given any tolerance limit $\epsilon > 0$, there always a finite $k$ such that $| W(g^k) | < \epsilon$. Therefore, the algorithm converges in a finite number of iterations.

## 4. PROPERTIES OF OPTIMAL POLICIES

In this section, we discuss properties of the optimal state-age-dependent policy $\delta^* = (t_0^*, \cdots, t_n^*, 0)$ which is obtained using PIA given in Section 3. Theorems 1 to 2 investigate conditions under which

$\delta^*$ is only state-dependent. Theorem 3 presents reasonable assumptions such that the optimal time for replacement becomes shorter and shorter as the system deteriorates. These properties facilitate the search of the optimal state-age-dependent policy.

THEOREM 1: If $c_i = c$ and $r_i = r$ for all $i \in S$ and $a_i$ is nondecreasing in $i \in S_n$, then there exists a unique critical state $k^* \in S$ such that $\delta^*(i) = \infty$ if $0 \leq i < k^*$ and $\delta^*(i) = 0$ if $k^* \leq i \leq n+1$. In particular, $k^* = j$ if and only if $a_{j-1} < g^* \leq a_j$ where $j \in S$ with $a_{-1} = -\infty$ and $a_{n+1} = \infty$.

PROOF: Let $K = c + (m - g^*)r$. From Equation (3), we have

$$\frac{d}{dt}v(i, g^*, t) = \bar{F}_i(t)\left\{(a_i - g^*) + h_i(t)p_i\left[V_{\delta^*}(i+1, g^*) - K\right]\right\}. \tag{4}$$

We can now consider the following two cases.

**Case I:** $g^* > a_j$ for some $j \in S_n$
Since $V_{\delta^*}(i, g^*) - K \leq 0$ for all $i \in S$ and $a_i - g^* < 0$ for any $i \leq j$, we have $\frac{d}{dt}v(i, g^*, t) < 0$ for all $t \in [0, \infty)$ and $i \leq j$. This implies that $\delta^*(i) = \infty$ for all $i \leq j$.

**Case II:** $g^* \leq a_j$ for some $j \in S_n$
In this case, we have $\frac{d}{dt}v(n, g^*, t) = (a_n - g^*)\bar{F}_n(t) \geq 0$ for all $t \in [0, \infty)$ which implies that $V_{\delta^*}(n, g^*) = v(n, g^*, 0) = K$ and $\delta^*(n) = 0$. Repeat this argument recursively for $n-1, n-2, \cdots, j+1, j$, we have $\delta^*(i) = 0$ for all $i \geq j$.

Since $g^* \in [0, \min\{g_r, g_f\}]$ and $a_i$ is nondecreasing in $i \in S_n$, it is clear from Case I and Case II that there exists a unique critical state $k^* \in S$ such that $\delta^*(i) = \infty$ for all $0 \leq i < k^*$ and $\delta^*(i) = 0$ for all $k^* \leq i \leq n+1$. Furthermore, if $a_{j-1} < g^* \leq a_j$, then Case I and Case II together imply that $k^* = j$. Conversely, if $k^* = j$, then we know that $\delta^*(i) = \infty$ for all $0 \leq i < j$ and $\delta^*(i) = 0$ for all $j \leq i \leq n+1$. This means that $V_{\delta^*}(i, g^*) = K$ for all $j \leq i \leq n+1$, $\frac{d}{dt}v(i, g^*, t) < 0$ for all $t \in [0, \infty)$ and $0 \leq i < j$, and $\frac{d}{dt}v(i, g^*, t) \geq 0$ for all $t \in [0, \infty)$ and $j \leq i \leq n$. Recall that $0 < \bar{F}_i(t) \leq 1$ for all $i \in S_n$. It follows from Equation (4) that $a_{j-1} < g^*$ and $a_j \geq g^*$. This completes the proof of the theorem. □

Theorem 1 above tells us that in the special case when the replacement costs and the expected replacement times are all identical and independent of state, the optimal time for replacement at any state $i \in S$ is either 0 or $\infty$. This means that in this special case, the optimal state-age-dependent policy is in fact only state-dependent. Furthermore, the optimal state-dependent is of control limit type with critical state $k^*$. For any $k \in S$, let $\delta_k$ be the state-dependent replacement policy such that $\delta_k(i) = \infty$ for $i < k$ and $\delta_k(i) = 0$ for $i \geq k$ and $g(k)$ be the corresponding expected long run cost rate. Then, $g(k)$ can be easily calculated using the following equation.

$$g(k) = \frac{\sum_{i=0}^{k-1} \left( \prod_{j=0}^{i-1} p_j \right) a_i \mu_i + c + mr}{\sum_{i=0}^{k-1} \left( \prod_{j=0}^{i-1} p_j \right) \mu_i + r}$$

with $\prod_{j=0}^{-1} p_j = 1$. Theorem 1 tells us that the critical state $k^*$ is the state $k \in S$ such that $a_{k-1} < g(k) \leq a_k$ and $g^* = g(k^*)$. Note that only the first moment $\mu_i$ of the sojourn time distributions in different states are involved in deriving the optimal state-age-dependent replacement policy in this special case.

THEOREM 2:

(A) If $h_i(t) = \lambda_i$ for all $t \in [0, \infty)$ and $i \in S_n$, then $\delta^*(i) \in \{0, \infty\}$ for all $i \in S$.

(B) If (i) $h_i(t)$ is a nonincreasing function in $t \in [0, \infty)$ for all $i \in S_n$, (ii) $0 \leq a_0 \leq a_1 \leq \ldots \leq a_n$, (iii) $0 < r_0 \leq r_1 \leq \ldots \leq r_n \leq r_{n+1}$, and (iv) $0 = c_0 \leq c_1 \leq \ldots \leq c_n \leq c_{n+1}$, then $\delta^*(i) \in \{0, \infty\}$ for all $i \in S$.

In Theorem 2 above, $h_i(t) = \lambda_i$ for all $t \in [0, \infty)$ is the same as saying that the sojourn time distribution in state $i$, $F_i$, is exponentially distributed with parameter $\lambda_i$. Assumption (i) tells us that $F_i$ is a decreasing failure rate distribution (DFR). In assumptions (ii) to (iv), we assume that the operating cost rate increases as the system deteriorates, the expected replacement times are all strictly positive and it is more expensive and time consuming to replace the system with deterioration. $c_0$ is assumed to be equal to 0 since state 0 is the initial new state of the system before any deterioration takes place. These assumptions are reasonable for deteriorating systems.

For simplicity of notation in the proof of Theorem 2 below, we let $K_i = K_i(g^*)$ for $i \in S$. Also, let $\Gamma_i = \Gamma_i(g^*)$ and $b_i = p_i K_{i+1} + (1 - p_i) K_{n+1} - K_i$ for all $i \in S_n$. Since we assume $c_0 = 0$, it follows that $m = g_r \geq g^*$. Under assumptions (iii) and (iv), it is clear that $K_i$ is nondecreasing in $i \in S$ and $b_i$ is nonnegative for all $i \in S_n$.

PROOF OF THEOREM 2: (A) From Equation (3), we have

$$\frac{d}{dt}v(i,g^*,t) = (a_i - g^* + \lambda_i\Gamma_i)\exp(-\lambda_i t).$$

Note that $a_i - g^* + \lambda_i\Gamma_i$ is independent of $t$. Hence, $v(i,g^*,t)$ is either nonincreasing or nondecreasing for $t \in [0,\infty)$. This implies that $\delta^*(i) \in \{0,\infty\}$ for all $i \in S$.

(B) We consider the following three cases.

**Case I:** If $a_i \geq g^*$, then assumption (ii) implies that $a_j \geq g^*$ for all $j \geq i$. By definition, $\Gamma_n = b_n = K_{n+1} - K_n$ which is always nonnegative. From Equation (3), we have $\frac{d}{dt}v(n,g^*,t) \geq 0$ for all $t \in [0,\infty)$, i.e., $t_n^* = 0$ and $V_{\delta^*}(n,g^*) = K_n$. Hence, $\Gamma_{n-1} = b_{n-1}$ which is nonnegative. Repeat the above argument for $j = n-1,\cdots,i$, we can show that $\frac{d}{dt}v(j,g^*,t) \geq 0$ for all $t \in [0,\infty)$ and this means that $t_j^* = 0$ for all $j \geq i$.

**Case II:** If $a_i < g^*$ and $\Gamma_i > 0$, then from Equation (3), $\frac{d}{dt}v(i,g^*,t)$ changes its sign at most once in $t \in [0,\infty)$ from positive to negative since $h_i(t)$ is a nonincreasing function in $t$. This result implies that the infimum of $v(i,g^*,t)$ is either $v(i,g^*,0)$ or $v(i,g^*,\infty)$. In other words, $t_i^*$ is one of the end points, 0 or $\infty$.

**Case III:** If $a_i < g^*$ and $\Gamma_i \leq 0$, then $\frac{d}{dt}v(i,g^*,t) \leq 0$ for all $t \in [0,\infty)$, which implies that the infimum of $v(i,g^*,t)$ is $v(i,g^*,\infty)$ and $t_i^* = \infty$.

These three cases indicate the optimal decision $\delta^*(i)$ is either 0 or $\infty$. □

Given any $\delta \in \Delta$, if $k = \min_{i \in S}\{i : \delta(i) = 0\}$, then the system is replaced as soon as it enters the critical state $k$ or at failure. This together with Theorem 2 tell us under the given conditions, the optimal state-age-dependent policy, $\delta^*$, constructed using PIA in Section 3 is only state-dependent. Hence, to find the optimal state-age-dependent replacement policy, it is sufficient to consider the set of all state-dependent replacement policies $\{\delta_k : k \in S\}$. The expected long run cost rate $g(k)$

can be calculated by the following equation.

$$g(k) = \frac{\sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1} p_j\right) a_i\mu_i + \left(\prod_{j=0}^{k-1} p_j\right)(c_k + mr_k) + \left(1 - \prod_{j=0}^{k-1} p_j\right)(c_{n+1} + mr_{n+1})}{\sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1} p_j\right) \mu_i + \left(\prod_{j=0}^{k-1} p_j\right) r_k + \left(1 - \prod_{j=0}^{k-1} p_j\right) r_{n+1}}. \tag{5}$$

The optimal expected long run cost rate $g^*$ can be obtained by searching a $k^* \in S$ such that $g^* = g(k^*) = \min_{k\in S} g(k)$.

THEOREM 3: In addition to assumptions (ii) to (iv) of Theorem 2, if (I) $h_i(t)$ is a nondecreasing function in both $i \in S_n$ and $t \in [0, \infty)$, (II) $1 \geq p_0 \geq p_1 \geq \cdots \geq p_{n-1} > p_n = 0$, and (III) $p_i c_{i+1} + (1 - p_i)c_{n+1} - c_i$ and $p_i r_{i+1} + (1 - p_i) r_{n+1} - r_i$ are both nondecreasing in $i \in S_n$, then there exist $h^*$ and $k^*$ with $0 \leq h^* \leq k^* \leq n + 1$, such that

$$\delta^*(i) = \begin{cases} \infty & \text{if } i < h^*, \\ t_i^* & \text{if } h^* \leq i < k^*, \\ 0 & \text{if } k^* \leq i \leq n+1. \end{cases}$$

Furthermore, $\infty > t_i^* \geq t_j^* > 0$ for $h^* \leq i \leq j < k^*$.

Assumption (I) tells us that $F_i$ is an increasing failure rate distribution (IFR) for all $i \in S_n$ and the failure rates also increase with deterioration. In assumption (II), we assume that the probability of transition into failure state from intermediate states are nondecreasing as the system deteriorates. Assumption (III) means that the expected marginal replacement costs and times increase as the system deteriorates. Theorem 3 tells us that under reasonable assumptions, the optimal state-age-dependent replacement policy has the monotonic property in which the optimal time for replacement becomes shorter and shorter as the system deteriorates.

PROOF OF THEOREM 3: Under assumption (III), it is clear that $b_i$ is nondecreasing in $i \in S_n$. Define

$$\mathcal{W}_i(t) = \frac{1}{\bar{F}_i(t)} \frac{d}{dt} v(i, g^*, t) = a_i - g^* + h_i(t)\Gamma_i. \tag{6}$$

We will use backward mathematical induction to show that

(a) $\mathcal{W}_i(t)$ in nondecreasing in $i \in S_n$ for all $t \in [0, \infty)$, and

(b) $t_i^*$ is nonincreasing in $i \in S_n$.

For state $n$, it is clear that $\Gamma_n = b_n = K_{n+1} - K_n \geq 0$. Using Equation (6) above, we have the following two cases.

**Case I:** If $a_n \geq g^*$, then $\frac{d}{dt}v(n, g^*, t) \geq 0$ for all $t \in [0, \infty)$ and $t_n^* = 0$ from the Proof of Theorem 2 (B) Case I.

**Case II:** If $a_n < g^*$, then $\frac{d}{dt}v(n, g^*, t)$ changes its sign at most once in $t \in [0, \infty)$ from negative to positive since $h_n(t)$ is a nondecreasing function in $t$. Hence, $\frac{d}{dt}v(n, g^*, t) \leq 0$ for all $t \in [0, t_n^*)$.

Cases I and II above imply that

$$\frac{d}{dt}v(n, g^*, t) \leq 0$$

for all $t \in (0, t_n^*)$ and therefore

$$\int_0^{t_n^*} \frac{d}{du}v(n, g^*, u)\, du \leq 0.$$

Next observe that

$$V_{\delta^*}(i, g^*) = v(i, g^*, t_i^*) \quad = \quad \int_0^{t_i^*} \frac{d}{du}v(i, g^*, u)\, du + K_i$$

and hence,

$$\Gamma_i = p_i \int_0^{t_{i+1}^*} \frac{d}{du}v(i+1, g^*, u)\, du + b_i. \tag{7}$$

From the assumptions given in the theorem, we know that $a_n \geq a_{n-1}$, and $b_n \geq b_{n-1} \geq 0$. If $h_i(t)$ is nondecreasing in $i$, then $h_n(t) \geq h_{n-1}(t)$ for $t \in [0, \infty)$. Hence, from Equations (6) and (7), $\mathcal{W}_n(t) \geq \mathcal{W}_{n-1}(t)$ for $t \in [0, \infty)$. This result in turn implies $t_{n-1}^* \geq t_n^*$.

For some state $k \in S_n \setminus \{0, 1\}$, assume the following two conditions hold for all $i \geq k$.

(C1) $\frac{d}{dt}v(i, g^*, t) \leq 0$ for all $t \in (0, t_i^*)$, and

(C2) $\mathcal{W}_i(t) \geq \mathcal{W}_{i-1}(t)$ for all $t \in [0, \infty)$.

We have shown that conditions (C1) and (C2) both hold for $k = n$. Note that condition (C2) indicates that $t_{i-1}^* \geq t_i^*$ for all $i \geq k$ and

$$\frac{d}{dt}v(i, g^*, t) \geq \frac{\bar{F}_i(t)}{\bar{F}_{i-1}(t)} \frac{d}{dt}v(i-1, g^*, t) \tag{8}$$

for all $t \in [0, \infty)$.

In the following discussion, we show that conditions (C1) and (C2) also hold for $i = k - 1$. Using Equation (6), consider the following three cases.

**Case I:** If $a_{k-1} \geq g^*$, then $\frac{d}{dt}v(k-1, g^*, t) \geq 0$ for all $t \in [0, \infty)$ and $t^*_{k-1} = t^*_k = \cdots = t^*_n = 0$ from the proof of Theorem 2 (B) Case I.

**Case II:** If $a_{k-1} < g^*$ and $\Gamma_{k-1} > 0$, then $\frac{d}{dt}v(k-1, g^*, t)$ changes its sign at most once in $t \in [0, \infty)$ from negative to positive since $h_{k-1}(t)$ is a nondecreasing function in $t$. Hence, $\frac{d}{dt}v(k-1, g^*, t) \leq 0$ for all $t \in [0, t^*_{k-1})$.

**Case III:** If $a_{k-1} < g^*$ and $\Gamma_{k-1} \leq 0$, then $\frac{d}{dt}v(k-1, g^*, t) \leq 0$ for all $t \in [0, \infty)$ and $t^*_{k-1} = \infty$. These three cases indicate that

$$\frac{d}{dt}v(k-1, g^*, t) \leq 0$$

for all $t \in (0, t^*_{k-1})$. Since $\bar{F}_i(t) = \exp\left[-\int_0^t h_i(u)\,du\right]$ and $h_i(t)$ is nondecreasing in $i$, it is clear that $\bar{F}_i(t)$ is nonincreasing in $i$ and $\frac{\bar{F}_i(t)}{\bar{F}_{i-1}(t)} \leq 1$for all $i \in S_n$ and $t \in [0, \infty)$. Using (C1) and Equation (8), we have

$$0 \geq \frac{d}{dt}v(k, g^*, t) \geq \frac{\bar{F}_k(t)}{\bar{F}_{k-1}(t)}\frac{d}{dt}v(k-1, g^*, t) \geq \frac{d}{dt}v(k-1, g^*, t)$$

for all $t \in (0, t^*_k) \subseteq (0, t^*_{k-1})$. Therefore,

$$0 \geq \int_0^{t^*_k} \frac{d}{du}v(k, g^*, u)\,du \geq \int_0^{t^*_k} \frac{d}{du}v(k-1, g^*, u)\,du \geq \int_0^{t^*_{k-1}} \frac{d}{du}v(k-1, g^*, u)\,du.$$

It follows from assumption (II) that

$$0 \geq p_{k-1}\int_0^{t^*_k} \frac{d}{du}v(k, g^*, u)\,du \geq p_{k-2}\int_0^{t^*_{k-1}} \frac{d}{du}v(k-1, g^*, u)\,du$$

and therefore $\mathcal{W}_{k-1}(t) \geq \mathcal{W}_{k-2}(t)$ for all $t \in [0, \infty)$. Again, this result implies $t^*_{k-2} \geq t^*_{k-1}$.

We have shown that if conditions (C1) and (C2) hold for all $i \geq k \in S_n \setminus \{0, 1\}$, then they also hold for all $i \geq k - 1 \in S_n \setminus \{0, n\}$. By the principle of mathematical induction, the proof of the theorem is completed. □

## 5. Special Case

In this section, we investigate the special case when the sojourn time distribution in state $i$, $F_i$, follows a Weibull distribution for all $i \in S_n$. Weibull distribution is chosen here since it includes Exponential distribution as a special case and both Weibull and Exponential distributions are widely used in probabilistic reliability modeling.

Let $F_i$ be a Weibull distribution with a scale parameter $\alpha_i > 0$ and a shape parameter $\beta_i > 0$, i.e.,

$$F_i(t) = \begin{cases} 1 - \exp(-\alpha_i t^{\beta_i}) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The failure rate function is given by $h_i(t) = \alpha_i \beta_i t^{\beta_i - 1}$ for all $t \in [0, \infty)$. It is clear that $F_i$ is an IFR distribution if and only if $\beta_i \geq 1$. When $\beta_i = 1$, it reduces to an exponential distribution with a constant failure rate $\alpha_i$. $F_i$ is a DFR if and only if $\beta_i \leq 1$. From Theorem 2, we know that when $\beta_i \leq 1$ for all $i \in S_n$, we only need to consider state-dependent policies. When $\beta_i > 1$ for all $i \in S_n$, PIA can be used to find $g^*$ and $\delta^*$. In particular,

$$\frac{d}{dt} v(i, g, t) = \left(a_i - g + \alpha_i \beta_i t^{\beta_i - 1} \Gamma_i\right) \exp(-\alpha_i t^{\beta_i}) = 0$$

can be solved explicitly with solution equal to 0, $\infty$ or $\left(\dfrac{g - a_i}{\alpha_i \beta_i \Gamma_i}\right)^{1/(\beta_i - 1)}$.

For example, consider a 5-state system ($n = 3$) with Weibull sojourn time distributions. Let $a_0 = 1$, $a_1 = 1.5$, $a_2 = 2$, $a_3 = 2.5$, $\mu_0 = 100$, $\mu_1 = 90$, $\mu_2 = 80$, $\mu_3 = 70$, $c_0 = 0$, $c_1 = 20$, $c_2 = 60$, $c_3 = 120$, $c_4 = 200$, $r_0 = 10$, $r_1 = 11$, $r_2 = 13$, $r_3 = 16$, $r_4 = 20$, $p_0 = p_1 = p_2 = 0.9$, and $m = 15$. Note that the above cost and time parameters satisfy the assumptions given in Theorems 2 and 3. Depending on the shape and scale parameters of the Weibull sojourn time distributions, we have the following results.

1. [Exponential Case] $\alpha_i = 1/\mu_i$ and $\beta_i = 1$ for all $i \in S_n$: It is readily verified that $g(0) = 15$, $g(1) = 2.83$, $g(2) = 2.68$, $g(3) = 2.85$, and $g(4) = 3.09$. Hence, $k^* = 2$, $g^* = 2.68$.

2. [DFR Case] $\alpha_i = \sqrt{2/\mu_i}$ and $\beta_i = 0.5$ for all $i \in S_n$: From Equation (5), $g(k)$ depends only on the first moment $\mu_i$ of the sojourn time distributions in different states, the optimal policy and cost rate are therefore the same as in the exponential case.

3. [IFR Case] $\alpha_i = \pi/(2\mu_i)^2$ and $\beta_i = 2$ for all $i \in S_n$. By applying PIA, we have $\delta^*(0) = 312.03$, $\delta^*(1) = 66.54$, $\delta^*(2) = 20.79$, $\delta^*(3) = 1.50$, $\delta^*(4) = 0$, and $g^* = 2.56$. Note that the optimal policy is of control limit type and the optimal times for replacement decrease as the system deteriorates.

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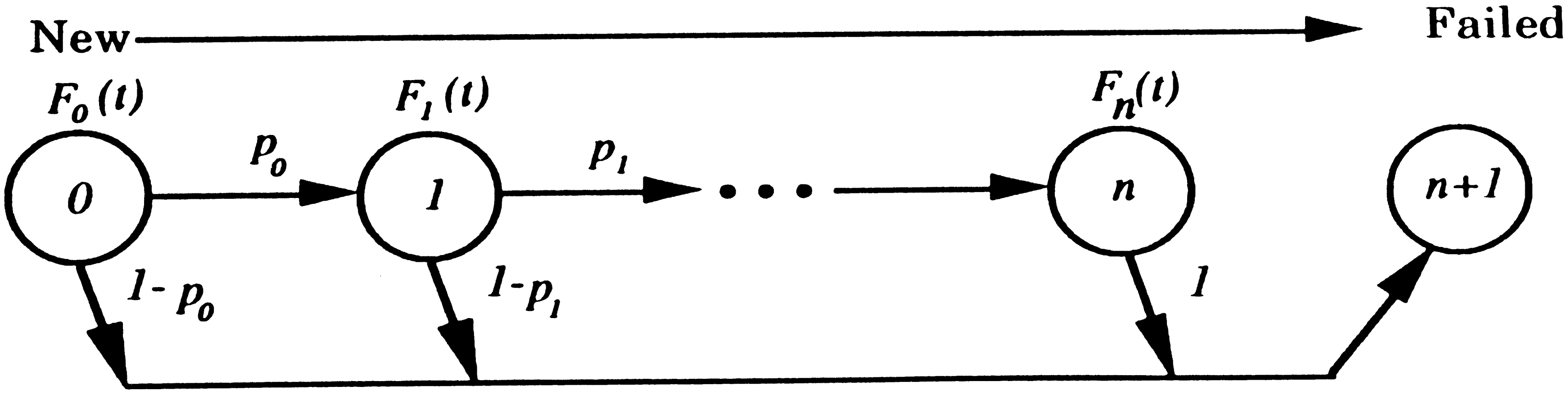


Figure 1: A flow diagram of a deteriorating system