# ON ESTIMATION OF GEOMETRIC PARAMETERS FOR CIRCULAR MEASUREMENTS

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**Abstract**

This paper studies a statistical model of measurements collected on the circumference of circular features. This model takes into consideration the center location variability of different machine parts. Maximum likelihood estimators are derived for both the within and between parts variations as well as the geometric parameters. Statistical procedures are developed to assess the performance of the manufacturing process in producing parts that conform to design specifications, and to keep track of the trend of the manufacturing process over time. An example for a transmission gear carrier is presented to highlight the use of the results.

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# Introduction

A circular feature in a mechanical object (drilled hole, punched hole, cross section of a shaft, etc.) is one of the most basic geometric primitives. Its specification can be described easily by a center and a radius. A circular feature has several functional advantages: it has uniform strength in any direction symmetric to the axis, it can be manufactured by a rotary tool, and its symmetry offers simplicity in assembly. However, due to imperfections introduced in manufacturing, machined parts will not be truly circular. The center and radius will in general be different from their nominal values specified by design engineers.

For example, consider the automatic transmission gear carrier displayed in Figure 1. The primary function of this gear carrier is to hold a number of small pinion gears in a specific location and angular orientation with respect to the centerline of the transmission. The holes on the upper and lower tiers form coaxial hole pairs which determine the eventual location and angular orientation of the shafts for the four pinion gears. Since the entire power of the automobile flows through this gear system, the gear carrier must be held to very tight tolerances for location and angular orientation of the pinion gears. All of these characteristics are obviously controlled by the radius and the center location of the holes drilled in the upper and lower tiers of the carrier. To assess how well the manufacturing process produces circular holes relative to design specifications and the trend of the manufacturing process over time, we must identify suitable methods to model the process and estimate its geometric parameters; its center and radius.

A manufacturing process for circular features will have uncertainty in the positioning of the tool which will cause variability in the center location. Tool wear and vibration can affect the radius and the circularity of the produced features or machined parts. To estimate the geometric parameters, discrete sets of measurements are taken from machined parts using a computer controlled Coordinate Measuring Machine (CMM). The number of measurements and the position on the circumference to take these measurements from each machined part, and the number of machined parts to sample are decision variables. Obviously, if the manufacturing process is fairly consistent (the part to part variation is small relative to variation within a machine part), it may be sufficient to sample a small number of machined parts. On the other hand, if the part variation is large, it may be necessary to obtain a large sample of machined parts in order to implement statistical process control or to assess the performance of the process in producing circular features that conform to the design specifications.

The majority of the literature on circular CMM measurements focuses on defining circularity or verifying a pattern of circular features for a single machined part [10, 14]. Different algorithms have been developed to define the location of the center of an individual circular feature [3, 17, 22]. A pair of concentric circles circumscribing and inscribing all the measurements from the circumference of this single machined part are computed with respect to the reference center. If the radial difference between the two concentric circles is larger than some specified tolerance, the machined part is considered to be nonconforming [10]. Similarly, efficient algorithms are widely studied for composite position tolerance verification [14] to automatically inspect the functionality of machined parts using CMM. While it may be important to inspect the finished machined parts to ensure that they are functional, it is even more important to use the information to assess trends in the manufacturing process and to correct problems before parts drift out of tolerance.

It is the intention of this paper to identify a suitable statistical model for circular measurements. As we will see in the next section, the proposed model captures the variation in center location of different machined parts. The radii of machined parts are assumed to be different and will be estimated from measurements. Statistical models for circular measurements have been studied before in the literature, see for example, Anderson [1], Berman [4, 5], Berman and Culpin [6], Berman and Somlo [7] and Chan [8]. Consistent methods to estimate the center have been developed [1, 6, 8]. These models have been applied to study measurement problems in microwave engineering [4, 7] and possible astronomical significance of various alignments of megalithic stone structures [1, 4]. However, all the statistical models considered in the papers cited above assume that the center of the true circle is fixed but unknown and hence do not describe the between-machine-parts variation of a circular feature manufacturing process.

Based on our model, maximum likelihood estimates are derived for both the *within* and *between*-machined-parts variations, as well as for the geometric characteristics. The geometric parameter estimates are compared statistically with the nominal values. A two-sided confidence interval for the between-machined-parts variability and a tolerance region which captures the population of the center of machined parts are also provided. A simple sampling scheme is obtained which minimizes the variance of the center estimate and takes into consideration the sampling cost of adding an extra machined part to the sample relative to that of taking extra measurements from machined parts. Based on this sampling scheme, statistical process control procedures can be developed to monitor the performance of the manufacturing process over time. An example on the transmission gear carrier is also given to illustrate the use of the results derived in this paper.

# The Model

Let us consider discrete sets of measurements taken from CMM using the following two-stage sampling scheme. For each sampled machined part, first stage measurements are taken to determine the estimated center location of the circular feature of interest. As mentioned in the previous section, this is routinely done to check the circularity or composite position tolerance of finished machined parts. Algorithms are available to find this reference center [2, 10]. We assume that the same method or algorithm is used to estimate the center locations for all machined parts. Next we take second stage measurements on the circumference of the machined parts. The angular differences between measurements can be pre-specified with respect to the center location estimated from the first stage. The same angular differences are used for all sampled machined parts.

We consider the following representation of circular measurements $(X_{ij}, Y_{ij})$ taken from the second stage using the scheme described above (Figure 2). The location of the center of part $i$ is given by $(\xi + A_i, \eta + B_i)$ where $A_1$, $A_2$, $\cdots$, $A_m$, $B_1$, $B_2$, $\cdots$, $B_m$ are assumed to be independent and identically distributed (iid) $N(0, \sigma_A^2)$. Ideally, we would like $\sigma_A^2$, the variance of the center variation of machined parts to be very small or equal to zero and $(\xi, \eta)$

to be equal to the nominal center location. The radius of part $i$, $\rho_i$, is assumed to be fixed but unknown. In the perfect situation, $\rho_1 = \rho_2 = \cdots = \rho_m = \rho$ where $\rho$ is the nominal value of the radius. Let $\tau_{i(j)}$ be the angle of the $j$th measurement of part $i$ and $(\epsilon_{i1}, \delta_{i1}), \cdots, (\epsilon_{in}, \delta_{in})$ be a sequence of independent random vectors of disturbances. This leads to the model

$$\begin{aligned} X_{ij} &= \xi + A_i + \rho_i \cos \tau_{i(j)} + \epsilon_{ij} \\ Y_{ij} &= \eta + B_i + \rho_i \sin \tau_{i(j)} + \delta_{ij} \end{aligned} \tag{1}$$

for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, where $n$ represents the number of measurements taken from the circumference of each machined part and $m$ is the number of machined parts.

The angular differences between measurements are given by $\tau_{i(j+1)} - \tau_{i(j)}$, $j = 1, 2, \cdots, n-1$. They are known and are the same for all machined parts. These measurements are all taken with respect to some fixed but usually unknown direction which may represent the positioning of the workpiece relative to the tool during manufacturing. The above is equivalent to saying that

$$\tau_{i(j)} = \theta_{i0} + \theta_j, \qquad j = 1, 2, \cdots, n,$$

where $\theta_j$ is known (with typically $\theta_1 = 0$) and $\theta_{i0}$ is fixed but unknown and can be different from one machined part to another. For $i = 1, 2, \cdots, m$, define

$$\alpha_i = \rho_i \cos \theta_{i0} \qquad \text{and} \qquad \beta_i = \rho_i \sin \theta_{i0}.$$

The model (1) can now be rewritten as

$$\begin{aligned} X_{ij} &= \xi + A_i + \alpha_i \cos \theta_j - \beta_i \sin \theta_j + \epsilon_{ij} \\ Y_{ij} &= \eta + B_i + \alpha_i \sin \theta_j + \beta_i \cos \theta_j + \delta_{ij} \end{aligned} \tag{2}$$

which is linear in the parameters $\xi$, $\eta$, $\alpha_i$ and $\beta_i$. The disturbances $\epsilon_{ij}$ and $\delta_{ij}$ are assumed to be iid $N(0, \sigma^2)$, and they are also assumed to be independent of the $A_i$ and and the $B_i$. These disturbances may represent the measurement errors of the CMM and/or the errors in estimating the center locations from the first stage samples.

Under the above assumptions, it is clear that for each $i = 1, 2, \cdots, m$, $\boldsymbol{X}_i = (X_{i1}, \cdots, X_{in})'$ is a multivariate normal random variable with mean vector

$$\boldsymbol{\mu}_i^x = (\xi + \alpha_i \cos \theta_1 - \beta_i \sin \theta_1, \cdots, \xi + \alpha_i \cos \theta_n - \beta_i \sin \theta_n)'$$

and variance-covariance matrix $\boldsymbol{V} = \sigma^2 \boldsymbol{I}_n + \sigma_A^2 \boldsymbol{J}_n$, where $\boldsymbol{I}_n$ is an $n \times n$ identity matrix and $\boldsymbol{J}_n$ is an $n \times n$ matrix with all entries equal to one. Similarly, we can define $\boldsymbol{Y}_i$ and $\boldsymbol{\mu}_i^y$ so that $\boldsymbol{Y}_i$ is $N(\boldsymbol{\mu}_i^y, \boldsymbol{V})$. Note that $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_m, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_m$ are independent random vectors. The likelihood function is therefore given by

$$\begin{aligned} L &= g(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_m, \boldsymbol{Y}_1, \boldsymbol{Y}_2, \cdots, \boldsymbol{Y}_m) \\ &= \frac{1}{(2\pi)^{mn} \mid \boldsymbol{V} \mid^m} \exp\left\{ -\frac{1}{2} \sum_{i=1}^m \left[ (\boldsymbol{X}_i - \boldsymbol{\mu}_i^x)' \boldsymbol{V}^{-1} (\boldsymbol{X}_i - \boldsymbol{\mu}_i^x) + (\boldsymbol{Y}_i - \boldsymbol{\mu}_i^y)' \boldsymbol{V}^{-1} (\boldsymbol{Y}_i - \boldsymbol{\mu}_i^y) \right] \right\}. \end{aligned}$$

From Graybill [11], pages 171-172,

$$\mid \boldsymbol{V} \mid = \sigma^{2(n-1)} (\sigma^2 + n\sigma_A^2),$$

and

$$\boldsymbol{V}^{-1} = \frac{1}{\sigma^2} \boldsymbol{I}_n - \frac{\sigma_A^2}{\sigma^2(\sigma^2 + n\sigma_A^2)} \boldsymbol{J}_n.$$

Define $\lambda_1 = \sigma^2 + n\sigma_A^2$ and $\lambda_2 = \sigma^2$. After some algebra, the log-likelihood function can be shown to be equal to

$$\begin{aligned} \ln L = & -mn \ln(2\pi) - m(n-1) \ln \lambda_2 - m \ln \lambda_1 \\ & - \frac{1}{2} \left\{ \frac{1}{\lambda_2} \left( \sum_{i=1}^m \sum_{j=1}^n \left[ (X_{ij} - \bar{X}_{i\cdot}) - \alpha_i(\cos\theta_j - \bar{c}) + \beta_i(\sin\theta_j - \bar{s}) \right]^2 \right. \right. \\ & \qquad \left. + \sum_{i=1}^m \sum_{j=1}^n \left[ (Y_{ij} - \bar{Y}_{i\cdot}) - \alpha_i(\sin\theta_j - \bar{s}) - \beta_i(\cos\theta_j - \bar{c}) \right]^2 \right) \\ & \quad + \frac{1}{\lambda_1} \left( \sum_{i=1}^m \sum_{j=1}^n \left[ (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}) - \bar{c}(\alpha_i - \bar{\alpha}) + \bar{s}(\beta_i - \bar{\beta}) \right]^2 \right. \\ & \qquad \left. + \sum_{i=1}^m \sum_{j=1}^n \left[ (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}) - \bar{s}(\alpha_i - \bar{\alpha}) - \bar{c}(\beta_i - \bar{\beta}) \right]^2 \right) \\ & \quad \left. + \frac{mn}{\lambda_1} \left[ (\bar{X}_{\cdot\cdot} - \xi - \bar{c}\bar{\alpha} + \bar{s}\bar{\beta})^2 + (\bar{Y}_{\cdot\cdot} - \eta - \bar{s}\bar{\alpha} - \bar{c}\bar{\beta})^2 \right] \right\} \end{aligned}$$

where

$$\bar{X}_{i\cdot} = \frac{1}{n} \sum_{j=1}^n X_{ij}, \ \bar{X}_{\cdot\cdot} = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n X_{ij}, \ \bar{s} = \frac{1}{n} \sum_{j=1}^n \sin\theta_j,$$

$$\bar{c} = \frac{1}{n} \sum_{j=1}^n \cos\theta_j, \ \bar{\alpha} = \frac{1}{m} \sum_{i=1}^m \alpha_i, \text{ and } \bar{\beta} = \frac{1}{m} \sum_{i=1}^m \beta_i.$$

To derive the maximum likelihood estimators, we differentiate the log-likelihood function with respect to the different parameters and equate the resulting system of equations to zero. It can be verified readily that

$$\begin{aligned}\hat{\xi} &= \bar{X}_{..} - \bar{c}\hat{\alpha} + \bar{s}\hat{\beta},\\ \hat{\eta} &= \bar{Y}_{..} - \bar{s}\hat{\alpha} - \bar{c}\hat{\beta},\end{aligned}$$

where

$$\hat{\alpha} = \frac{1}{m}\sum_{i=1}^{m}\hat{\alpha}_i = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\cos\theta_j - \bar{c}) + \sum_{i=1}^{m}\sum_{j=1}^{n} Y_{ij}(\sin\theta_j - \bar{s})}{m\left[\sum_{j=1}^{n}(\sin\theta_j - \bar{s})^2 + \sum_{j=1}^{n}(\cos\theta_j - \bar{c})^2\right]},$$

and

$$\hat{\beta} = \frac{1}{m}\sum_{i=1}^{m}\hat{\beta}_i = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} Y_{ij}(\cos\theta_j - \bar{c}) - \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}(\sin\theta_j - \bar{s})}{m\left[\sum_{j=1}^{n}(\sin\theta_j - \bar{s})^2 + \sum_{j=1}^{n}(\cos\theta_j - \bar{c})^2\right]}.$$

Before equations for $\hat{\alpha}_i$, $\hat{\beta}_i$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are given, let us first examine the distributional properties of the center estimator. It can be shown that $(\hat{\xi}, \hat{\eta})'$ is distributed as bivariate normal with mean $(\xi, \eta)'$ and variance-covariance matrix given by

$$\left[\frac{\sigma_A^2}{m} + \frac{\sigma^2}{mn(1 - \bar{c} - \bar{s})}\right] \boldsymbol{I}_2.$$

It follows that, given any $m$ and $n$, the variances of $\hat{\xi}$ and $\hat{\eta}$ are minimized when $\bar{c} = \bar{s} = 0$. This can be achieved, for example, by sampling $n$ measurements equally spaced around the circumference of the circular feature. Since this can be easily implemented and is a sound approach in practice, we use these two conditions to derive the maximum likelihood estimators for $\hat{\alpha}_i$, $\hat{\beta}_i$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ (corrected for bias). In particular,

$$\begin{aligned}
\hat{\xi} &= \bar{X}_{..}, \\
\hat{\eta} &= \bar{Y}_{..}, \\
\hat{\alpha}_i &= \frac{1}{n}\sum_{j=1}^{n}(X_{ij}\cos\theta_j + Y_{ij}\sin\theta_j), \\
\hat{\beta}_i &= \frac{1}{n}\sum_{j=1}^{n}(Y_{ij}\cos\theta_j - X_{ij}\sin\theta_j), \\
\hat{\lambda}_1 &= \frac{1}{2(m-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(\bar{X}_{i\cdot} - \bar{X}_{..})^2 + (\bar{Y}_{i\cdot} - \bar{Y}_{..})^2\right], \\
\hat{\lambda}_2 &= \frac{1}{2m(n-2)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(X_{ij} - \bar{X}_{i\cdot})^2 + (Y_{ij} - \bar{Y}_{i\cdot})^2 - (\hat{\alpha}_i^2 + \hat{\beta}_i^2)\right].
\end{aligned} \tag{3}$$

We can therefore estimate $\sigma_A^2$ by $(\hat{\lambda}_1 - \hat{\lambda}_2)/n$ and $\sigma^2$ by $\hat{\lambda}_2$, respectively. Here we assume $\hat{\lambda}_1 \geq \hat{\lambda}_2$. In the case when $\hat{\lambda}_1 < \hat{\lambda}_2$, an alternative model is proposed in [18]. Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)'$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)'$, $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_n)'$ and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \cdots, \hat{\beta}_n)'$. The following theorem gives the distributional properties of the estimators:

**Theorem 1**

1. $$\hat{\boldsymbol{\varphi}} \equiv \begin{bmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\beta}} \end{bmatrix} \sim N\left(\begin{bmatrix} \xi \\ \eta \\ \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}, \begin{bmatrix} \frac{\sigma^2 + n\sigma_A^2}{mn} & 0 & 0 & 0 \\ 0 & \frac{\sigma^2 + n\sigma_A^2}{mn} & 0 & 0 \\ 0 & 0 & \frac{\sigma^2}{n}\boldsymbol{I}_n & 0 \\ 0 & 0 & 0 & \frac{\sigma^2}{n}\boldsymbol{I}_n \end{bmatrix}\right).$$

2. $\hat{\boldsymbol{\varphi}}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are independent.

3. $\dfrac{2(m-1)\hat{\lambda}_1}{\lambda_1}$ is chi-squared distributed with $2(m-1)$ degrees of freedom.

4. $\dfrac{2m(n-2)\hat{\lambda}_2}{\lambda_2}$ is chi-squared distributed with $2m(n-2)$ degrees of freedom.

The proof of the theorem is given in the appendix.

## Statistical Inferences About Geometric Parameters

Once we have derived the maximum likelihood estimates of $\xi, \eta, \alpha_i, \beta_i, \lambda_1$, and $\lambda_2$, it is possible to make statistical inferences about the geometric parameters. It is easily seen that

$$\frac{(\hat{\xi}-\xi)^2+(\hat{\eta}-\eta)^2}{2\hat{\lambda}_1/mn}$$

has an $F$ distribution with 2 and $2(m-1)$ degrees of freedom. Thus the interior of the circle

$$(\hat{\xi}-\xi)^2+(\hat{\eta}-\eta)^2=\frac{2\hat{\lambda}_1}{mn}F_{1-\gamma:2,2(m-1)}$$

determines a $100(1-\gamma)\%$ confidence region for the center $(\xi, \eta)$, where $F_{\gamma:p,q}$ is the $\gamma$ percentile of the $F$ distribution with $p$ and $q$ degrees of freedom. This confidence region, if obtained when the manufacturing process is assumed to be in statistical control, can be used to monitor the performance of the process. A simple graphical display, similar to the Shewhart control charts, showing an example of 20 possible future $(\hat{\xi}, \hat{\eta})$ (based on the same sample sizes $m$ and $n$), is given in Figure 3. The circles, which replace the upper and lower control limits, are the confidence region. As long as $(\hat{\xi}, \hat{\eta})$ fall inside the circles, we consider that the process is under control. An alternative graphical display could use only one confidence region and plot $(\hat{\xi}, \hat{\eta})$ with distinct symbols.

To study the variability of the center locations among machined parts, we need to look at $\sigma_A^2$. The test statistic

$$F=\hat{\lambda}_1/\hat{\lambda}_2$$

which is distributed as an $F$ with $2(m-1)$ and $2m(n-2)$ degrees of freedom when $\sigma_A^2=0$, can be used to test the hypothesis that there is no significant differences in center location among machined parts. The decision rule for a significance level $\gamma$ test is to reject the hypothesis when $F > F_{1-\gamma:2(m-1),2m(n-2)}$.

Confidence intervals are often more informative than the hypothesis tests. Since $\sigma_A^2 = (\lambda_1-\lambda_2)/n$, is a difference of two variances, there is no known method for constructing an exact-size confidence interval on $\sigma_A^2$. Many approximate procedures are available. Wang [20] provides a detailed comparison of the important ones. A $100(1-\gamma)\%$ two-sided confidence interval on $\sigma_A^2$, based on Howe's [15] approximation, is given by

$$\frac{1}{n}\left[\hat{\lambda}_1 - \hat{\lambda}_2 - \sqrt{k_1\hat{\lambda}_1^2 + k_2\hat{\lambda}_2^2}\right] \le \sigma_A^2 \le \frac{1}{n}\left[\hat{\lambda}_1 - \hat{\lambda}_2 + \sqrt{k_3\hat{\lambda}_1^2 + k_4\hat{\lambda}_2^2}\right]$$

where

$$\begin{aligned} k_1 &= \left(1 - 1/F_{1-\gamma/2:2(m-1),\infty}\right)^2, \\ k_2 &= \left(F_{1-\gamma/2:2(m-1),2m(n-2)} - 1\right)^2 - k_1 F^2_{1-\gamma/2:2(m-1),2m(n-2)}, \\ k_3 &= \left(1/F_{\gamma/2:2(m-1),\infty} - 1\right)^2, \end{aligned}$$

and

$$k_4 = \left(1 - F_{\gamma/2:2(m-1),2m(n-2)}\right)^2 - k_3 F^2_{\gamma/2:2(m-1),2m(n-2)}.$$

The "true" center for the $i$th machined part is $(\xi + A_i, \eta + B_i)$, which is a bivariate normal random variable with mean $(\xi, \eta)'$ and variance-covariance matrix $\sigma_A^2 \boldsymbol{I}_2$. What is of interest here is the distribution of the true center of the entire population of machined parts. Statistical tolerance regions are often used in this situation. One computes a region $R$, based on sample data, and makes a statement such as "with 90% confidence we can state that at least 99% of machined parts in the population will have their centers lie inside $R$". (This refers to a 99%-content, 90%-confidence tolerance region.)

There is no known general procedure for constructing tolerance regions for this problem. The difficulty arises from the fact that $\hat{\sigma}_A^2$ has a non-negligible probability of being zero. The "regular" method, such as the one outlined in [9] would produce a tolerance region with the actual confidence much lower that the prespecified value. Wang and Iyer [21] derive an approximate tolerance *interval* for the univariate distribution $N(\xi, \sigma_A^2)$. Following their approach, an approximate tolerance region with confidence $1-\gamma$ is defined to be a circle with center $(\hat{\xi}, \hat{\eta})$ and radius

$$k\sqrt{2\max\left(0, \hat{\lambda}_1 - \hat{\lambda}_2 F_{\gamma/6:2(m-1),2m(n-2)}\right)/n}$$

where $k$ is determined from the data. This tolerance region is conservative. A brief derivation is given in the appendix.

The tolerance zone for the center of a circular feature is usually specified as a circular region with a center located at the nominal value and a radius equal to the allowable deviation of the center from the nominal. Using the parameter estimates of $\xi$, $\eta$ and $\sigma_A^2$, we can also provide an estimate of the proportion of machined parts in the population with centers that lie within the tolerance zone. This can be done by computing the probability integral

$$\int\int_{R'} f(x,y)\,dx\,dy$$

where $f(x,y)$ is the density function of the bivariate normal with mean $(\hat{\xi},\hat{\eta})'$ and variance-covariance matrix $\hat{\sigma}_A^2 \boldsymbol{I}_2$, and $R'$ is the tolerance zone. An efficient algorithm developed by Groenewoud, Hoaglin and Bitalis [13] is available to compute this probability numerically. When $\hat{\lambda}_1 \leq \hat{\lambda}_2$, the distribution of $(\xi + A_i, \eta + B_i)'$ is a degenerate one. In this case, the confidence region on $(\xi, \eta)$ should be used to assess the appropriateness of the center location of the machined part relative to the design specification.

It is also of interest to know whether there is enough evidence in the data to support the hypothesis that the machined parts have a common radius. The null hypothesis is $H_0 : \rho_1 = \rho_2 = \cdots = \rho_m = \rho$, which is equivalent to the hypothesis $H_0 : \alpha_1^2 + \beta_1^2 = \cdots = \alpha_m^2 + \beta_m^2 = \rho^2$. An exact-size test does not appear to be available. An approximate test based on the generalized likelihood-ratio principle [16] is derived as follows.

The parameter space $\Theta$ for the model is

$$\Theta = \{(\xi,\eta,\alpha_i,\beta_i,\lambda_1,\lambda_2) : -\infty < \xi,\eta,\alpha_i,\beta_i < \infty,\ \lambda_1 \geq \lambda_2 > 0\}.$$

We have already derived the values of $\xi,\eta,\alpha_i,\beta_i,\lambda_1$, and $\lambda_2$ in $\Theta$ that maximize the likelihood function $L$, or

$$\sup_{\Theta} L = \frac{e^{-mn}}{(2\pi)^{mn}\hat{\lambda}_1^m \hat{\lambda}_2^{m(n-1)}}.$$

The parameter space $\Theta_0$ for the model under $H_0$ is

$$\Theta_0 = \{(\xi,\eta,\alpha_i,\rho,\lambda_1,\lambda_2) : -\infty < \xi,\eta,\alpha_i < \infty,\ \rho > 0, \lambda_1 \geq \lambda_2 > 0\}.$$

If we replace $\beta_i$ with $\sqrt{\rho^2 - \alpha_i^2}$ in the likelihood function and try to maximize $L$ with respect to $\xi,\eta,\alpha_i,\rho,\lambda_1$, and $\lambda_2$, it will be found that $\hat{\xi},\hat{\eta}$, and $\hat{\lambda}_1$ remain unchanged, while

$$\tilde{\alpha}_i^2 \;=\; \hat{\alpha}_i^2\tilde{\rho}^2/(\hat{\alpha}_i^2 + \hat{\beta}_i^2),$$

$$\tilde{\rho} = \frac{1}{m}\sum_{i=1}^{m}\sqrt{\hat{\alpha}_i^2 + \hat{\beta}_i^2},$$

and

$$\tilde{\lambda}_2 = \frac{1}{2m(n-2)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(X_{ij} - \bar{X}_{i\cdot})^2 + (Y_{ij} - \bar{Y}_{i\cdot})^2 - \tilde{\rho}^2\right],$$

where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are given in (3). This gives

$$\sup_{\Theta_0} L = \frac{e^{-mn}}{(2\pi)^{mn}\hat{\lambda}_1^m\tilde{\lambda}_2^{m(n-1)}}.$$

The generalized likelihood-ratio is then

$$\Lambda = \frac{\sup_{\Theta_0} L}{\sup_{\Theta} L} = \left\{\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(X_{ij} - \bar{X}_{i\cdot})^2 + (Y_{ij} - \bar{Y}_{i\cdot})^2 - (\hat{\alpha}_i^2 + \hat{\beta}_i^2)\right]}{\sum_{i=1}^{m}\sum_{j=1}^{n}\left[(X_{ij} - \bar{X}_{i\cdot})^2 + (Y_{ij} - \bar{Y}_{i\cdot})^2 - \tilde{\rho}^2\right]}\right\}^{m(n-1)}.$$

The quantity $-2\log\Lambda$ has approximately a chi-squared distribution with $m-1$ degree of freedom. Hence the hypothesis of equal $\rho_i$ is rejected if $-2\log\Lambda > \chi^2_{1-\gamma:m-1}$, for a pre-determined significance level $\gamma$, where $\chi^2_{\gamma:p}$ is the $\gamma$ percentile of the chi-squared distribution with $p$ degrees of freedom. If the hypothesis is not rejected, a pooled estimate of the radius is

$$\hat{\rho} = \frac{1}{m}\sum_{i=1}^{m}\sqrt{\hat{\alpha}_i^2 + \hat{\beta}_i^2}.$$

If Taylor's theorem is used to expand $\sqrt{\hat{\alpha}_i^2 + \hat{\beta}_i^2}$ about $\alpha_i$ and $\beta_i$, it can be shown that

$$\mathrm{E}(\hat{\rho}) = \rho + \frac{\sigma^2}{2n\rho} + \frac{\sigma^4}{8n^2\rho^3} + \frac{3\sigma^6}{16n^3\rho^5} + \cdots$$

That is, $\hat{\rho}$ is biased. However, if $\rho >> \sigma$ (which should be the case in practice), the bias is small. Similarly, we can evaluate the variance of $\hat{\rho}$, which is given by

$$\mathrm{Var}(\hat{\rho}) = \frac{\sigma^2}{mn} - \frac{\sigma^4}{2mn^2\rho^2} - \frac{\sigma^6}{2mn^3\rho^4} - \frac{13\sigma^8}{64mn^4\rho^6} - \cdots \tag{4}$$

Using Equation (4) and estimating $\rho$ and $\sigma^2$ by $\hat{\rho}$ and $\hat{\lambda}_2$ respectively, a 3-*sigma* control chart for $\hat{\rho}$ (based on the same sample sizes $m$ and $n$) can be used to monitor the radius of the future machined parts.

## Sampling Plans

Frequently, one is faced with the following question. One could sample 5 machined parts and obtain 10 measurements from each part, or one could sample 10 machined parts and obtain 5 measurements from each part, or follow some other scheme. The answer depends on the statistical objective and the relative costs associated with sampling and measuring each machined part. A reasonable cost function is given by

$$\text{total cost} = c_1 m + c_2 mn \tag{5}$$

where $c_2$ is the cost of taking a measurement on each part sampled, and $c_1$ is the cost associated with sampling each machined part. If the total cost is fixed and the objective is to obtain a center estimate with the smallest possible variance, the value of $n$ that results is [19]

$$n = \sqrt{\frac{c_1 \sigma^2}{c_2 \sigma_A^2}}.$$

The value of $m$ is found from (5) given the fixed cost by solving the cost function. In practice, $m$ and $n$ must be whole numbers, and the smallest practical values they can have are 2 and 3, respectively.

## An Example

Consider the transmission gear carrier problem discussed earlier in the paper. A gear carrier has four different coaxial hole pairs with top and bottom holes centered nominally at (44.45,0), (0,44.45), $(-44.45, 0)$, and $(0, -44.45)$ millimeters with respect to the center of the part. The tolerance zone for the hole centers is a circle with center at the nominal and radius of 0.1 mm. To control the angular orientation of the shaft for each hole pair, a tolerance zone for the relative location of the center of the bottom hole with respect to those of the top hole is usually specified. In this problem, the angular tolerance is a circular region with a radius of 0.075 mm centered at the realized top hole center. The angular tolerance is tighter than the location tolerance since the angle of the pinion shaft is a more critical element for proper gear function. The nominal value for the radius of each hole is 5.5 mm.

Suppose we are interested in studying the location of hole 2 in the upper tier of a gear carrier (Figure 4). Measurements are routinely taken and the center locations of all eight holes are estimated. A reference frame is chosen with the origin at the part center of the upper tier, and the x-axis going through the estimated center of hole 1 as shown in Figure 4. With respect to this reference frame, second stage measurements are taken using a CMM on the circumference of hole 2 with known angular differences of 60 degrees relative to its estimated center location from previous measurements. The same procedure is repeated for 5 randomly chosen gear carriers.

We can now fit (2) to this dataset. The maximum likelihood estimates are summarized in Table 1. In this case, the estimated centers of hole 2 for all 5 gear carriers have met the specified tolerance. The 95% confidence region for $(\xi, \eta)$ is a circle centered at $(-0.0050, 44.4582)$ with radius 0.0513. The $F$ statistic for testing $H_0 : \sigma_A^2 = 0$ is 422.15. Thus, there are significance differences in the center location of hole 2 among the 5 gear carriers. A 95% confidence interval on $\sigma_A^2$ is given by $(0.0007, 0.0054)$.

Table 1: Maximum Likelihood Estimate of Hole Center Locations and Radii of Hole 2 for the Upper Tier of 5 Gear Carriers

| Gear Carrier $i$ | $\bar{X}_{i.}$ | $\bar{Y}_{i.}$ | $\hat{\alpha}_i$ | $\hat{\beta}_i$ | $\hat{\rho}_i$ |
|---|---|---|---|---|---|
| 1 | −0.0315 | 44.4272 | −4.6131 | 2.9947 | 5.4999 |
| 2 | 0.0134 | 44.5056 | 3.3391 | −4.3732 | 5.5022 |
| 3 | −0.0331 | 44.4699 | −2.4306 | −4.9340 | 5.5001 |
| 4 | 0.0205 | 44.4966 | −5.1926 | 1.8087 | 5.4986 |
| 5 | 0.0056 | 44.3918 | −1.6928 | −5.2357 | 5.5026 |
| $\hat{\xi} = -0.0050$, $\hat{\eta} = 44.4582$, $\hat{\sigma}_A = 0.0384$, $\hat{\sigma} = 0.0046$ | | | | | |

Since $\sigma_A^2$ is significantly different from 0, we can construct a tolerance region for the distribution of the true centers. A conservative 95%-content 90%-confidence tolerance region is found to be a circle centered at $(-0.0050, 44.4582)$ with radius 0.1991.

Let $R'$ be the tolerance zone of the center location of hole 2, i.e., $R'$ is a circular region with center at (0,44.45) and radius 0.1. The estimated proportion of gear carriers with center location of hole 2 that lie within $R'$ is 0.9628. The value of the test statistic for testing the hypothesis of a common $\rho$ is 3.7777 which has a p-value of 0.563. Thus, we can use $\hat{\rho} = 5.5007$ to estimate the radius of hole 2.

Suppose now we are interested in studying the coaxial hole pair 1 of the gear carrier. A reference frame is chosen such that its origin is located at the projection of the estimated center location of hole 1 of the upper tier onto the lower tier, and the x-axis goes through the part center of the lower tier (Figure 5). With respect to this reference frame, second stage measurements are taken on the circumference of hole 1 in the lower tier with known angular differences of 60 degrees relative to the estimated center location of this hole from previous measurements. The same procedure is again repeated for 5 randomly chosen gear carriers. The results are summarized in Table 2.

Table 2: Maximum Likelihood Estimate of Hole Center Locations and Radii of Hole 1 in the Lower Tier Relative to the Same Hole in the Upper Tier for 5 Gear Carriers

| Gear Carrier $i$ | $\bar{X}_{i.}$ | $\bar{Y}_{i.}$ | $\hat{\alpha}_i$ | $\hat{\beta}_i$ | $\hat{\rho}_i$ |
|---|---|---|---|---|---|
| 1 | 0.0319 | 0.0273 | 3.7042 | 4.0644 | 5.4991 |
| 2 | 0.0415 | −0.0145 | −5.3582 | 1.2447 | 5.5008 |
| 3 | 0.0151 | −0.0233 | −3.7030 | 4.0719 | 5.5038 |
| 4 | 0.0479 | 0.0157 | −5.0828 | −2.0984 | 5.4989 |
| 5 | 0.0077 | −0.0113 | 5.2800 | −1.5468 | 5.5018 |
| $\hat{\xi} = 0.0288$, $\hat{\eta} = -0.0012$, $\hat{\sigma}_A = 0.0194$, $\hat{\sigma} = 0.0049$ | | | | | |

In this coaxial case, the estimated centers of hole 1 in the lower tier for all 5 gear carriers lie inside the circular tolerance zone with radius 0.075 and centered at the projection of the estimated center location of hole 1 in the upper tier onto the lower tier. However, the 95% confidence region for $(\xi, \eta)$ is a circle centered at $(0.02882, -0.0012)$ with radius 0.026, and it lies completely in the positive-x half plane. This suggests that the drilling tool has consistently drifted to the right after machining has begun. Actions would need to be taken to correct this problem so that the drill will maintain its angular orientation perpendicular to the drilling surface throughout the machining process. Other statistics can be similarly obtained.

# Concluding Remarks

We have proposed a model of measurements collected on the circumference of circular features. The model uses a random effect to capture the variation in center locations. We have derived, under the assumptions of normality and constant angular difference, maximum likelihood estimators of the geometric parameters of interest. Statistical methods commonly used in random effects models can then be employed to make inferences about the model.

In this paper, we assume that all the $A_i$ and $B_i$ are independent. In some problems, it may be more reasonable to assume that $(A_1, B_1)$, $(A_2, B_2)$, $\cdots$, $(A_m, B_m)$ are independent bivariate random variables but $A_i$ is correlated with $B_i$. Also, we assume $\rho_i$ are different. In other situations, it may be more appropriate to assume that $\rho_i = \rho$, in (1). The proposed statistical tolerance region for the population of center locations is quite conservative which will be the subject of future research.

# Appendix

First, we define the following. Let $\boldsymbol{Z}_i = (\boldsymbol{X}_i', \boldsymbol{Y}_i')'$, $\boldsymbol{\mu}_i = (\boldsymbol{\mu}_i^{x\prime}, \boldsymbol{\mu}_i^{y\prime})'$, $\hat{\varphi}_i = (\bar{X}_{i\cdot}, \bar{Y}_{i\cdot}, \hat{\alpha}_i, \hat{\beta}_i)'$, $\boldsymbol{c} = (\cos\theta_1, \cdots, \cos\theta_n)'$, $\boldsymbol{s} = (\sin\theta_1, \cdots, \sin\theta_n)'$ and $\boldsymbol{l}_n$ be an $n \times 1$ column vector whose entries are all equal to one. Also, let

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_n - \frac{1}{n}\boldsymbol{l}_n\boldsymbol{l}_n' & 0 \\ 0 & \boldsymbol{I}_n - \frac{1}{n}\boldsymbol{l}_n\boldsymbol{l}_n' \end{bmatrix}, \ \boldsymbol{B}' = \begin{bmatrix} \boldsymbol{l}_n & 0 & \boldsymbol{c} & -\boldsymbol{s} \\ 0 & \boldsymbol{l}_n & \boldsymbol{s} & \boldsymbol{c} \end{bmatrix}, \text{ and } \boldsymbol{C}' = \begin{bmatrix} \boldsymbol{c} & -\boldsymbol{s} \\ \boldsymbol{s} & \boldsymbol{c} \end{bmatrix}.$$

**Proof of Theorem 1:**

The proof of Parts 1 and 3 of Theorem 1 is trivial. To prove Part 2, observe that $\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}$ is independent of $\bar{X}_{\cdot\cdot}$, $X_{ij} - \bar{X}_{i\cdot}$, $\hat{\alpha}_i$ and $\hat{\beta}_i$. Similarly, $\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}$ is independent of $\bar{Y}_{\cdot\cdot}$, $Y_{ij} - \bar{Y}_{i\cdot}$, $\hat{\alpha}_i$ and $\hat{\beta}_i$. Therefore, $\hat{\lambda}_1$ is independent of $\hat{\varphi}$ and $\hat{\lambda}_2$.

The following can be verified to hold for all $i = 1, 2, \cdots, m$.

1. $\boldsymbol{Z}_i \sim N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma})$ where $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{V} & 0 \\ 0 & \boldsymbol{V} \end{bmatrix}$,
2. $\sum_{j=1}^{n} \left[(X_{ij} - \bar{X}_{i\cdot})^2 + (Y_{ij} - \bar{Y}_{i\cdot})^2 - (\hat{\alpha}_i^2 + \hat{\beta}^2)\right] = \boldsymbol{Z}_i'\boldsymbol{D}\boldsymbol{Z}_i$ where $\boldsymbol{D} = \boldsymbol{A} - \frac{1}{n}\boldsymbol{C}'\boldsymbol{C}$,
3. $\hat{\varphi}_i = \frac{1}{n}\boldsymbol{B}\boldsymbol{Z}_i$ and $\boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{D} = 0$,
4. $\frac{\boldsymbol{D}\boldsymbol{\Sigma}}{\lambda_2} = \boldsymbol{D}$ and $\boldsymbol{D}^2 = \boldsymbol{D}$, i.e., $\boldsymbol{D}$ is an idempotent matrix,
5. $\boldsymbol{\mu}_i'\boldsymbol{D}\boldsymbol{\mu}_i = 0$,
6. $\text{trace}(\boldsymbol{D}) = \text{trace}(\boldsymbol{A}) - \frac{1}{n}\text{trace}(\boldsymbol{c}\boldsymbol{c}' + \boldsymbol{s}\boldsymbol{s}' + \boldsymbol{c}\boldsymbol{c}' + \boldsymbol{s}\boldsymbol{s}') = 2(n-2)$.

Using Theorems 4.4.3 and Theorem 4.5.2 in Graybill [12], we complete the proof of the theorem.

**Deviation of the Tolerance Region for the True Centers:**

If $X$ is $N(\xi, \sigma_A^2)$ and $Y$ is $N(\eta, \sigma_A^2)$, then the $100(1-\beta)$%-content, $100(1-\gamma/2)$%-confidence tolerance intervals for $X$ and $Y$ are given by (Wang and Iyer [21])

$$| X - \hat{\xi} | \leq k\sqrt{\max\left(0, \hat{\lambda}_1 - \hat{\lambda}_2 F_{\gamma/6:2(m-1),2m(n-2)}\right)/n},$$

and

$$| Y - \hat{\eta} | \leq k\sqrt{\max\left(0, \hat{\lambda}_1 - \hat{\lambda}_2 F_{\gamma/6:2(m-1),2m(n-2)}\right)/n},$$

respectively. It is easily shown by Bonferroni inequality that

$$(X - \hat{\xi})^2 + (Y - \hat{\eta})^2 \leq 2k^2 \max\left(0, \hat{\lambda}_1 - \hat{\lambda}_2 F_{\gamma/6:2(m-1),2m(n-2)}\right)/n$$

is a $100(1-\beta)\%$-content confidence region with confidence level greater than or equal to $1-\gamma$.

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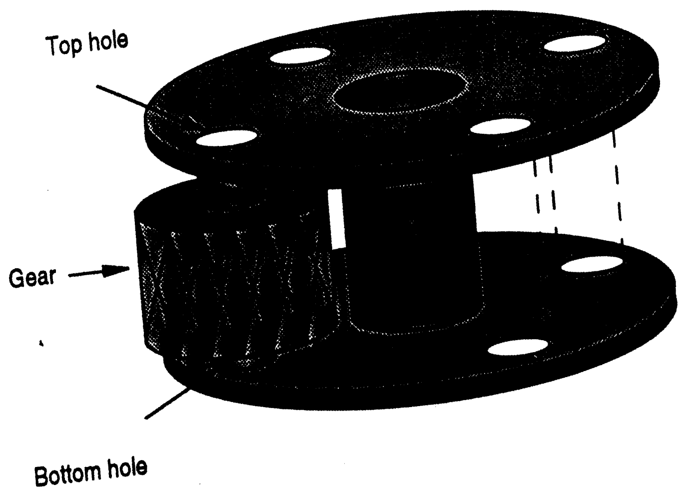


Figure 1: A Four Hole Gear Carrier

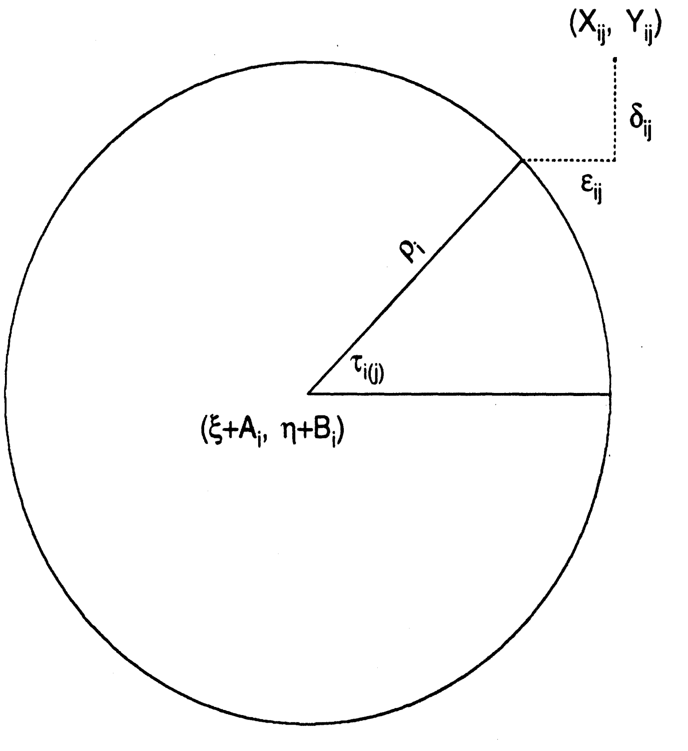


Figure 2: Definition of Circular Measurements for the Second Stage Sampling Scheme

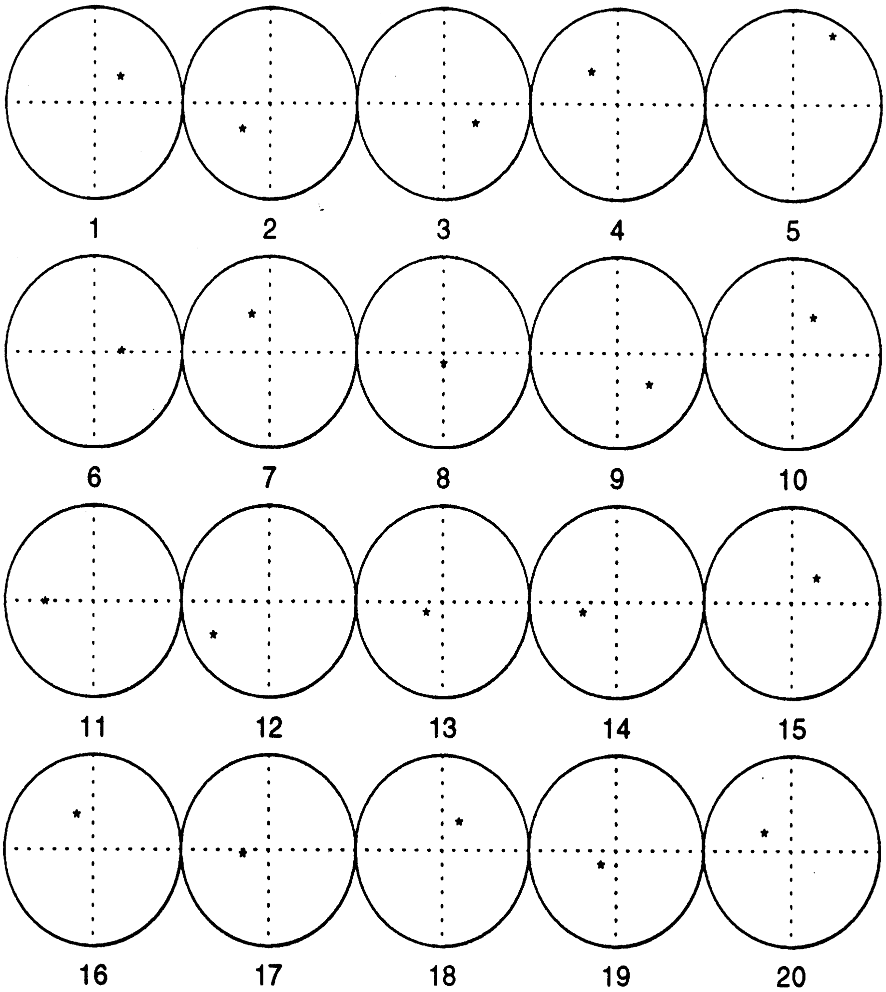


Figure 3: A Simple Graphical Display for Twenty Possible Future $(\hat{\xi}, \hat{\eta})$ (Based on Same Sample Sizes $m$ and $n$)

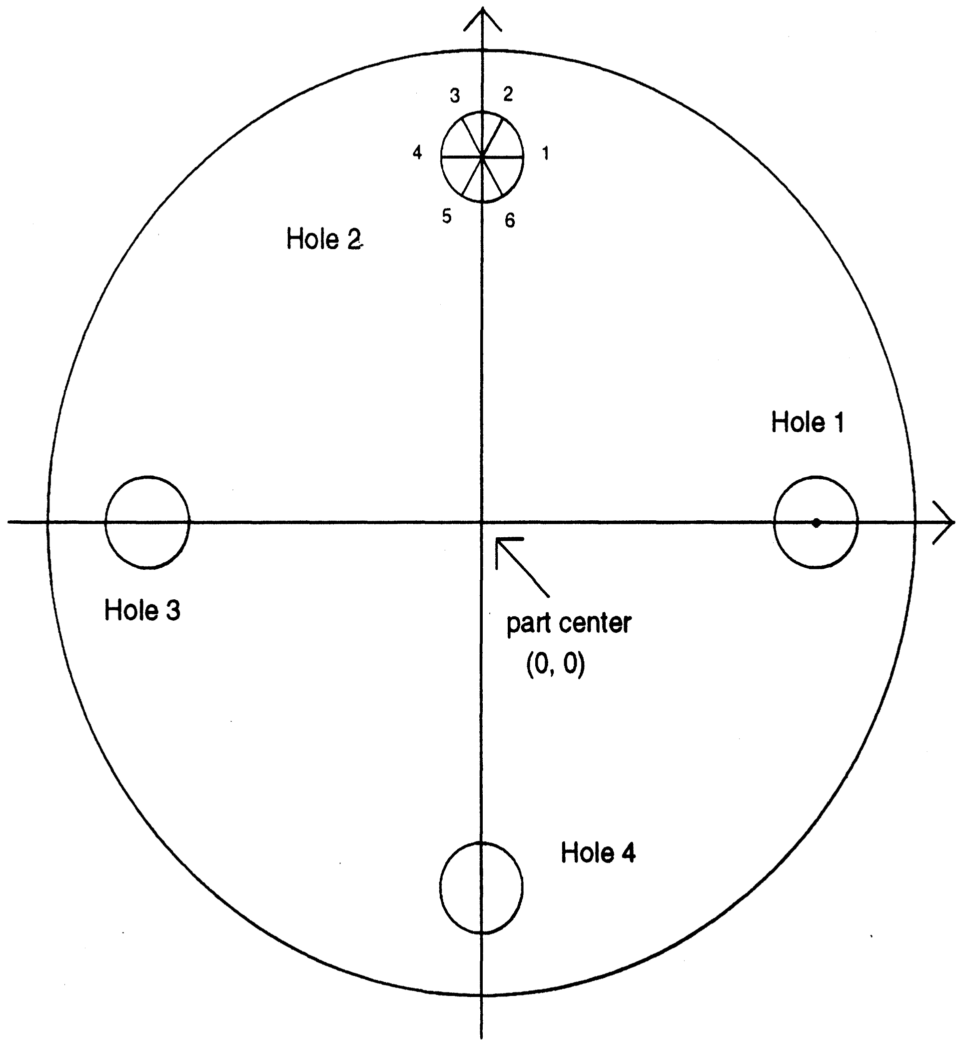


Figure 4: Upper Tier of a Four Hole Gear Carrier with Reference Frame Going Through Part Center and Estimated Center of Hole 1

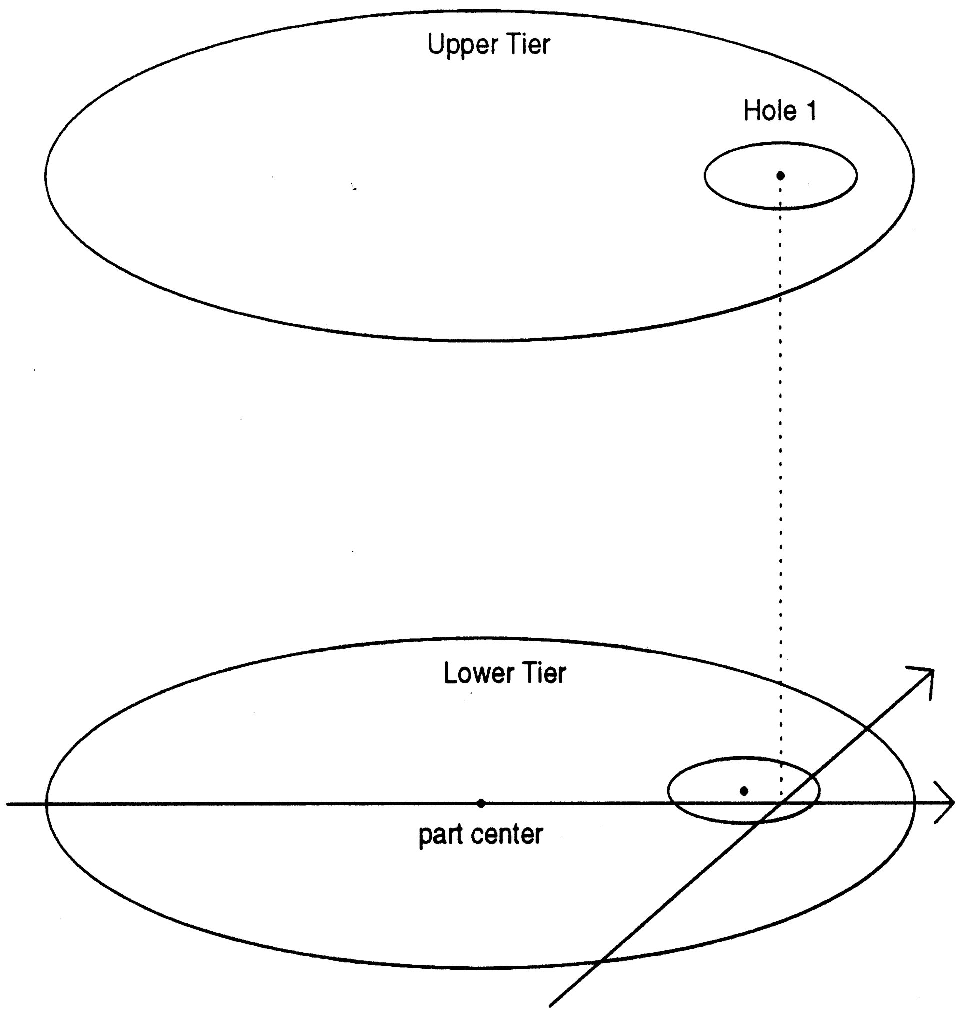


Figure 5: Reference Frame for the Coaxial Hole Problem