ON ESTIMATION OF PROPORTION OF CONFORMANCE

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Abstract

This paper presents results on point and interval estimation problems for proportion of conformance. Proportion of conformance is defined as the proportion of products with quality characteristic inside the specification limits. Five point estimators are presented and compared with respect to their root mean squared error. Two approximate methods for constructing lower confidence limits are proposed and the performance of each is assessed by simulation. In the case when the nominal value of the quality characteristic is not centered in the middle of the specification range, a modified proportion of conformance is introduced. Numerical examples are also given to illustrate the procedures.

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Introduction

The quality of any manufacturing product is ultimately determined by customer satisfaction. The purpose of any quality improvement process should aim to improve customer satisfaction by reducing and eliminating defects, and to continuously improve processes throughout the organization, thereby reducing sources of variation and improving quality and productivity. The quality of a product can usually be quantified by observable characteristics of the product or the manufacturing process which produces the product. The performance of a quality characteristic can often be specified by a nominal (target) value, and a tolerance (acceptable) region. These specifications are set by engineering requirements or by customers. Ideally, the quality characteristic value should be at the nominal value with no variation. In reality, variation in a manufacturing process is unavoidable and hence there is a statistical distribution associated with the quality characteristic.

Let X be the quality characteristic of interest with a nominal value T, a lower specification limit L and an upper specification limit U. Also, let $p_L = P[X < L]$ and $p_U = P[X > U]$ be the proportion of products with its quality characteristic X below L and above U, respectively. If p_C represents the *proportion of conformance*, that is, the proportion of products with its quality characteristic X within the specification limits L and U, then it is given by

$$p_C = P[L < X < U] = 1 - p_L - p_U. \tag{1}$$

Proportion of conformance is a measure of how well the output of a process meets the specification limits. In recent years, process capability indices have been widely used for assessing the performance of manufacturing processes. However, as pointed out by Pearn, Kotz, and Johnson (1992), the underlying motives for the introduction of process capability indices are clearly related to monitoring the proportion of conforming products. Proportion of conformance is, thus, a more direct and easily understood measure. Furthermore, under proper distributional assumptions, proportion of conformance can usually be estimated not only for univariate but also multivariate quality characteristics over fairly general shapes of tolerance regions. Since there is no consensus on how to extend the capability indices developed for univariate processes to multivariate processes, proportion of conformance is

perhaps a preferable measure of process performance.

The minimum variance unbiased estimator (MVUE) of p_L (and p_U) has been derived by several authors. In particular, Folks, Pierce, and Steward (1965) derived the MVUE of p_L under various distributional assumptions of X. When X is normally distributed, Wheeler (1970) obtained the variance of the MVUE, and Owen and Hua (1977) derived confidence limits on p_L and p_U. Chou and Owen (1984) constructed one-sided simultaneous confidence limits for p_L and p_U when X has a normal distribution. However, there is no known general procedure for constructing confidence intervals for p_C.

In this paper, we consider point and interval estimation of p_C when X is normally distributed with unknown mean μ and variance σ^2. Specifically, we compare five different estimators of p_C with respect to root mean squared error of the estimate. We also present two methods for constructing approximate *lower* confidence limits for p_C. Only lower confidence limits are considered here since, in practice, we would like p_C to be greater than some threshold in order for us to conclude that a process is capable. Simulations are used to compare the five different estimators and to assess the performance of the proposed confidence limits.

When the nominal value T is centered between L and U, i.e., $T = M = (L + U)/2$, the tolerance region for the quality characteristic is symmetric. If T is off-center, the tolerance region is not symmetric. Also, if $T = M$, p_C is maximized when $\mu = T$ (for fixed σ). However, in many assembly-fit processes, the nominal value of the quality characteristic is off-center, indicating that deviation in one direction is less acceptable than deviation in the other. In such cases, we would like to maximize p_C with the additional constraint that the majority of products be produced near the nominal value. We introduce a modified p_C which is maximized when the process mean μ is located at the nominal value T for both symmetric and nonsymmetric tolerance regions. A lower confidence limit is also given for the modified proportion of conformance.

Point Estimation

In this section, we consider five different estimators for p_C defined in equation (1). In the discussion below, let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and variance σ^2. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ be the sample mean and sample variance respectively. Define $\kappa_1 = (\mu - L)/\sigma$ and $\kappa_2 = (U - \mu)/\sigma$. Then, p_C can be written equivalently as

$$p_C = \Phi(\kappa_2) - \Phi(-\kappa_1)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. Let K_1 and K_2 be the estimators of κ_1 and κ_2, respectively, obtained by substituting $\bar{X}$ for μ and S for σ. The five estimators of p_C we consider are:

1. *Uniformly minimum variance unbiased estimator* (UMVUE) :
   From Wheeler (1970), the UMVUE of $\bar{p}_U = 1 - p_U$ is given by

$$\hat{\bar{p}}_U = \begin{cases} 0 & \text{if } K_2 \leq -(n-1)/\sqrt{n} \\ 1 & \text{if } K_2 \geq (n-1)/\sqrt{n} \\ \mathcal{T}_{n-2}(w_U) & \text{otherwise} \end{cases} \tag{2}$$

   where $\mathcal{T}_{n-2}(\cdot)$ is the cumulative distribution function of a Student's t distribution with $n-2$ degrees of freedom and $w_U = [n(n-2)]^{1/2} K_2 [(n-1)^2 - nK_2^2]^{-1/2}$. Similarly, the UMVUE $\hat{p}_L$ for p_L is given by equation (2) with K_2 replaced by K_1. Since

$$I(X_1) = \begin{cases} 1 & \text{if } L < X_1 < U \\ 0 & \text{otherwise} \end{cases}$$

   is an unbiased estimator for p_C, it follows from the Blackwell-Rao Theorem (e.g., see Mood, Graybill and Boes (1974) page 321) that the UMVUE of p_C is

$$\hat{p}_{C(UMVUE)} = E[I(X_1) \mid \bar{X}, S].$$

   Using the arguments given by Johnson and Kotz (1970, page 73) and Wheeler (1970), we find that

$$\hat{p}_{C(UMVUE)} = \hat{\bar{p}}_U - \hat{p}_L.$$

2. *Modified maximum likelihood estimator 1* (MMLE1) :

   Since $\bar{X}$ and $S\sqrt{(n-1)/n}$ are maximum likelihood estimators for μ and σ respectively, it follows that

$$\hat{p}_{C(MMLE1)} = \Phi\left(\sqrt{\frac{n}{n-1}}K_2\right) - \Phi\left(-\sqrt{\frac{n}{n-1}}K_1\right)$$

   is the maximum likelihood estimator for p_C.

3. *Modified maximum likelihood estimator 2* (MMLE2) :

   If we estimate μ by $\bar{X}$ and σ by the biased estimator S, then p_C can be estimated by

$$\hat{p}_{C(MMLE2)} = \Phi(K_2) - \Phi(-K_1).$$

4. *Modified maximum likelihood estimator 3* (MMLE3) :

   Alternatively, we can estimate σ by its unbiased estimator S/c_4 and p_C by

$$\hat{p}_{C(MMLE3)} = \Phi(c_4K_2) - \Phi(-c_4K_1)$$

   where

$$c_4 = \frac{\sqrt{2}\Gamma(n/2)}{\sqrt{n-1}\Gamma((n-1)/2)}$$

   and $\Gamma(\cdot)$ is the gamma function.

5. *Modified maximum likelihood estimator 4* (MMLE4) :

   It is readily verified that $\sqrt{n}K_2$ follows a noncentral t distribution with $n-1$ degrees of freedom and noncentrality parameter $\sqrt{n}\kappa_2$. From Johnson and Kotz (1970, page 203), we know that cK_2 is an unbiased estimator for κ_2 where

$$c = \frac{\sqrt{2}\Gamma((n-1)/2)}{\sqrt{n-1}\Gamma((n-2)/2)}.$$

   Similarly, cK_1 is an unbiased estimator for κ_1. This suggests that we can estimate p_C by

$$\hat{p}_{C(MMLE4)} = \Phi(cK_2) - \Phi(-cK_1).$$

We compare the five estimators listed above using the root mean squared error criterion. The root mean squared error of an estimator $\hat{p}_C$ of p_C is defined as the square root of

$$E_{\kappa_1,\kappa_2}[(\hat{p}_C - p_C)^2]$$

which is a function of κ_1 and κ_2. We want to choose an estimator with the smallest root mean squared error for all practical values of κ_1 and κ_2.

The root mean squared errors of the five estimators can be computed either by numerical integrations or by simulations. Simulations are conducted in this paper to evaluate the root mean squared errors. Given n, a standard normal random deviate Z and an independent chi-squared random deviate Y with $n-1$ degrees of freedom were generated using IMSL (1987) routines RANNOR and RNCHI. For each pair of values κ_1 and κ_2, the resulting values

$$K_1 = \frac{\kappa_1 + Z/\sqrt{n}}{\sqrt{Y/(n-1)}} \quad \text{and} \quad K_2 = \frac{\kappa_2 - Z/\sqrt{n}}{\sqrt{Y/(n-1)}} \tag{3}$$

were used to calculate each of the five estimates of p_C. Monte Carlo estimates of the root mean squared error for each $\hat{p}_C$ were obtained by repeating the procedure above 10000 times. Simulations were carried out for sample size $n = 5(1)15(5)30$ and a wide range of values of κ_1 and κ_2.

Figure 1 shows the root mean squared errors of the five estimates when $n = 5$ and $\kappa_1 = 3.0$. The root mean squared errors of all estimates attain their maximum at $\kappa_2 = 0.0$, corresponding to the situation where the process mean μ is located at the upper specification limit U. This is not surprising since a normal density function has the most mass concentrated around mean μ and with $\mu = U$, a small discrepancy in sample value of K_2 would result in a large difference in $\hat{p}_C$. As κ_2 deviates from zero, the root mean squared errors decrease rapidly for all five estimators. Similar conclusions apply to other values of n, κ_1 and κ_2.

In practice, we expect both κ_1 and κ_2 to be positive and greater than 3.0, meaning that the process mean lies within the specification limits and the process standard deviation consumes no more than one-sixth of the specification range (see for examples, Kane (1986) and McFadden (1993)). In Figure 1, UMVUE has the lowest root mean squared error whereas MMLE4 has the highest for all $\kappa_2 \geq 3$. Keeping $\kappa_1 = 3.0$, the maximum differences of the

root mean squared errors among the five estimates over the range of $\kappa_2 \geq 2.0$ are computed for different sample sizes and are plotted in Figure 2. As the sample size increases, this maximum difference decreases rapidly. When $n \geq 15$, the root mean squared errors of the five different estimation methods become essentially indistinguishable from each other. The performance of UMVUE and MMLE1 are very close for all sample size and all practical value of κ_1 and κ_2. Both can be recommended in practice.

Example

As an illustration, consider $n = 30$, $K_1 = 2.4$ and $K_2 = 3.0$. In this case, $\hat{p}_{C(UMVUE)} = 0.99351$, $\hat{p}_{C(MMLE1)} = 0.99154$, $\hat{p}_{C(MMLE2)} = 0.99045$, $\hat{p}_{C(MMLE3)} = 0.98986$ and $\hat{p}_{C(MMLE4)} = 0.98855$. Note that $\hat{p}_{C(MMLE1)} > \hat{p}_{C(MMLE2)} > \hat{p}_{C(MMLE3)} > \hat{p}_{C(MMLE4)}$. This is as expected since $\sqrt{n/(n-1)} > 1 > c_4 > c$.

Confidence Limits for Proportion of Conformance

We are interested in constructing lower confidence limits on p_C; i.e., with a pre-specified confidence coefficient $1 - \alpha$, we seek $p(\alpha, K_1, K_2)$ such that

$$P_{K_1,K_2}\left[p_C \geq p(\alpha, K_1, K_2)\right] \approx 1 - \alpha.$$

Since

$$p_C = 1 - p_L - p_U$$

we first obtain confidence limits for p_L and p_U and then attempt to find a confidence limit for p_C based on these limits. A $1 - \alpha$ upper confidence limit, $p_1 = p_1(\alpha, K_1)$, for p_L satisfies

$$P[p_L \leq p_1] = 1 - \alpha.$$

Owen and Hua (1977) showed that p_1 can be obtained by solving the equation

$$P_{T_{n-1}}\left[T_{n-1}\left(-\sqrt{n}\,\Phi^{-1}(p_1)\right) \leq \sqrt{n}K_1\right] = 1 - \alpha \tag{4}$$

where $T_\nu(\delta)$ is distributed as a noncentral t distribution with ν degrees of freedom and noncentrality parameter δ, and $\Phi^{-1}(\cdot)$ is the inverse distribution function of the standard

normal. A $1-\alpha$ upper confidence limit, $p_2 = p_2(\alpha, K_2)$, for p_U is similarly obtained by solving the equation

$$P_{T_{n-1}}\left[T_{n-1}\left(-\sqrt{n}\,\Phi^{-1}(p_2)\right) \leq \sqrt{n}K_2\right] = 1-\alpha. \tag{5}$$

Now,

$$\begin{aligned} P[p_L + p_U \leq p_1 + p_2] &\geq P[p_L \leq p_1,\ p_U \leq p_2] \\ &\geq 1 - P[p_L > p_1] - P[p_U > p_2] \\ &= 1 - 2\alpha. \end{aligned}$$

The last inequality is justified by the Bonferroni inequality (e.g. see Graybill (1976), page 360). Thus, a *conservative* lower-limit for a $100(1-2\alpha)\%$ one-sided confidence interval for p_C is given by

$$p_C \geq 1 - p_1 - p_2. \tag{6}$$

Values of p_1 and p_2 can be found in Odeh and Owen (1980, Table 7). They provide tables of p_1 and p_2 for $1-\alpha = 0.5,\ 0.75,\ 0.9,\ 0.95,\ 0.975,\ 0.99,\ 0.995$ and for $n = 2(1)18(3)30,\ 40(20)120,\ 240,\ 600,\ 1000,\ 1200$.

The Bonferroni inequality has been used by other authors in many similar problems, especially, in interval estimation for variance components, e.g., see Williams (1962), Wang (1991, 1992). In variance components problems, it was shown, analytically or numerically, that the confidence coefficient $1-2\alpha$ can be replaced by $1-\alpha$. This also seems to be the case for the interval in (6) as indicated by the simulation studies described below.

To evaluate the true confidence coefficient of the interval (6), simulation studies described in the previous section were again carried out. For each value of κ_1 and κ_2, K_1 and K_2 were computed as given in equation (3). Also, p_1 and p_2 were obtained by solving equations (4) and (5), respectively. The procedure was repeated 10000 times and the percentage of times that

$$1 - p_1 - p_2 \leq p_C = \Phi(\kappa_2) - \Phi(-\kappa_1)$$

was recorded. Table 1 reports the results for $1-\alpha = 0.95$, $n = 10, 30, 50$, and for some selected values of κ_1 and κ_2. Since the interval in (6) is symmetric about κ_1 and κ_2, only cases with $\kappa_2 \geq \kappa_1$ are shown in the table.

The results indicate that the proposed interval is successful in maintaining the stated confidence level. The sample size n seems not to play an important role on the converge of the interval except when either κ_1 or κ_2 are negative, a rare situation in practice. In Table 1, the proposed confidence interval is conservative when κ_1 is near κ_2, and as the difference between κ_1 and κ_2 increases (or p_U becomes smaller) the confidence coefficient approaches to $1-\alpha$ (0.95 in this example). The latter can be verified by observing that

$$\begin{aligned} P[p_C \geq 1-p_1-p_2] &= P[p_L + p_U \leq p_1 + p_2] \\ &= P[p_L < p_1] + P[p_L + p_U \leq p_1 + p_2,\ p_L > p_1] - \\ &\quad P[p_L + p_U \geq p_1 + p_2,\ p_L < p_1 < p_L + p_U] \end{aligned}$$

and the second and third terms vanish as p_U approaches to zero. A similar argument can also be used to verify the findings reported by Kushler and Hurley (1992) regarding the coverage of several confidence intervals for the process capability index $C_{pk} = \min(\kappa_1/3, \kappa_2/3)$. That is, the intervals are conservative when μ is near T and give nominal coverage when μ moves away from T.

In the case when κ_1 is near κ_2, an alternative lower confidence limit $p^* = p^*(\alpha, K_1, K_2)$ is derived in the appendix and is given by

$$p^* = \Phi\left(\frac{1}{\sqrt{n}} + \max(K_1, K_2)\sqrt{\frac{\chi^2_{\alpha:n-1}}{n-1}}\right) - \Phi\left(\frac{1}{\sqrt{n}} - \min(K_1, K_2)\sqrt{\frac{\chi^2_{\alpha:n-1}}{n-1}}\right).$$

The simulated confidence coefficient corresponding to confidence limit p^* is given in Table 2. Again, the interval is successful in maintaining the nominal confidence level and the sample sizes considered have little effect on the performance of the interval. For the purpose of comparing with confidence limit $1-p_1-p_2$, Table 2 also includes the simulated value of $E[p^*]/E[1-p_1-p_2]$.

From Tables 1 and 2, we can conclude that confidence limit $1-p_1-p_2$ is less conservative and produces a tighter bound for most cases. On the other hand, the confidence limit p^* is easier to compute and performs better when both κ_1 and κ_2 are small and are close to each other. However, p^* is derived based on the assumption that the probabilities of $K_1 < 0$ or $K_2 < 0$ are negligible, meaning that μ lies between L and U. Although in practice, we

expect the process mean to lie within the specification limits, nevertheless, it is a limitation for p^*.

Examples

For illustration, again consider $n = 30$. The 95% lower confidence limits $1 - p_1 - p_2$ and p^* for p_C are tabulated in Table 3 for a few combinations of K_1 and K_2. For other values of α, n, K_1 and K_2, a computer program to compute these lower confidence limits is described in Lam and Wang (1993). Table 3 confirms that the confidence limit $1 - p_1 - p_2$ provides a tighter bound for p_C.

Modified Proportion of Conformance

It is common that the nominal value of a quality characteristic is not centered between the upper and lower specifications. This generally occurs in an assembly-fit process where deviation of the quality characteristic in one direction is less acceptable than deviation in the other. The primary goal for design engineers in this case is not to simply maximize the proportion of conformance. Instead, the engineers want to maximize proportion of conformance under the additional constraint that the majority of products be produced near to the nominal value.

Consider the two processes A and B in Figure 3, both processes have the same standard deviation. However, the mean of process A is located at the center of the specification range M and the mean of process B is located at the nominal value T. Obviously, the proportion of conformance p_C of process A will be greater than that of process B even though the mean of process B is at the nominal value. However, process B is preferable to process A if the intent of the design engineers is to penalize deviation toward the lower specification less than deviation toward the upper specification. This important fact is ignored in computing the proportion of conformance of processes A and B.

This problem can be overcome by defining a modified proportion of conformance as discussed in Littig and Lam (1993). This modified proportion of conformance is maximized

when the process mean is located at the nominal value. Since deviation in one direction is penalized more than deviation in the other, it is reasonable to use different distance scales in measuring deviation from the nominal value. In particular, we choose constants d_1 and d_2 such that

$$\frac{T-L}{d_1} = \frac{U-T}{d_2}.$$

The modified proportion of conformance, p_C^m, is then given by

$$p_C^m = \begin{cases} \Phi\left(\frac{U-T}{d_2\sigma} + \frac{T-\mu}{d_1\sigma}\right) - \Phi\left(-\frac{\mu-L}{d_1\sigma}\right) & \text{if } T \geq \mu \\ \Phi\left(\frac{U-\mu}{d_2\sigma}\right) - \Phi\left(-\frac{T-L}{d_1\sigma} - \frac{\mu-T}{d_2\sigma}\right) & \text{if } T \leq \mu \end{cases} \quad (7)$$

In this paper, we choose $d_2 = 1$ whenever $T - L \geq U - T$ and $d_1 = 1$ whenever $T - L \leq U - T$. This choice of d_1 and d_2 ensures that processes C and D in Figure 4 has the same modified proportion of conformance. This is a desirable property since processes C and D and their corresponding design specifications are mirror reflection of each other and should be considered as performing equally well in meeting design specifications. Furthermore, it is readily verified that p_C^m defined in equation (7) is indeed maximized when the process mean μ is located at the nominal value T. Also, if the mean of process E is $T - r(T - L)$ and the mean of process F is $T + r(U - T)$ where $0 \leq r \leq 1$ (Figure 5), then both processes have the same modified proportion of conformance. This is reasonable since both processes have the identical proportion of allowable process mean deviation below and above the nominal value.

The maximum likelihood estimator of p_C^m can be readily obtained by replacing $\bar{X}$ for μ and $S\sqrt{(n-1)/n}$ for σ in equation (7). Judging by the results on p_C in the previous section, this should be an adequate estimator for p_C^m. Let $\rho = (U-T)/(T-L)$ be the relative location of T to the specification limits L and U. In particular, $\rho = 1$ means that the nominal value is centered between the specification limits and $p_C^m = p_C$. It is readily verified that equation (7) is equivalent to

$$p_C^m = 1 - p_L^m - p_U^m$$

where

$$p_L^m = \begin{cases} \Phi\left(-\dfrac{\kappa_1}{\max(1,1/\rho)}\right) & \text{if } \dfrac{\kappa_2-\rho\kappa_1}{1+\rho} \geq 0 \\ \Phi\left(-\dfrac{(\kappa_1+\kappa_2)+(\kappa_1-\kappa_2/\rho)}{(1+1/\rho)\max(1,\rho)}\right) & \text{if } \dfrac{\kappa_2-\rho\kappa_1}{1+\rho} \leq 0 \end{cases}$$

and

$$p_U^m = \begin{cases} \Phi\left(-\dfrac{(\kappa_1+\kappa_2)+(\kappa_2-\rho\kappa_1)}{(1+\rho)\max(1,1/\rho)}\right) & \text{if } \dfrac{\kappa_2-\rho\kappa_1}{1+\rho} \geq 0 \\ \Phi\left(-\dfrac{\kappa_2}{\max(1,\rho)}\right) & \text{if } \dfrac{\kappa_2-\rho\kappa_1}{1+\rho} \leq 0 \end{cases}$$

In the equation above, $(\kappa_2-\rho\kappa_1)/(1+\rho) = (T-\mu)/\sigma$. Hence, we would expect $\mid \kappa_2-\rho\kappa_1 \mid$ to increase as μ deviates from the nominal value T. The same technique used to derive the confidence limit for p_C can be extended to obtain the lower confidence bound for the modified proportion of conformance p_C^m. In particular, this confidence limit can be obtained by solving for $p_1^m = p_1^m(\alpha, K_1, K_2)$ and $p_2^m = p_2^m(\alpha, K_1, K_2)$ from the following equations. If $(K_2-\rho K_1)/(1+\rho) \geq 0$, the equations are

$$\begin{aligned} &P_{T_{n-1}}\left[T_{n-1}\left(-\sqrt{n}\max(1,1/\rho)\,\Phi^{-1}(p_1^m)\right) \leq \sqrt{n}K_1\right] = 1-\alpha, \\ &P_{T_{n-1}}\left[T_{n-1}\left(-\sqrt{n}\max(1,1/\rho)\,\Phi^{-1}(p_2^m)\right) \leq \frac{\sqrt{n}[(K_1+K_2)+(K_2-\rho K_1)]}{1+\rho}\right] = 1-\alpha. \end{aligned} \tag{8}$$

Otherwise, we solve for p_1^m and p_2^m in the following equations.

$$\begin{aligned} &P_{T_{n-1}}\left[T_{n-1}\left(-\sqrt{n}\max(1,\rho)\,\Phi^{-1}(p_1^m)\right) \leq \frac{\sqrt{n}[(K_1+K_2)+(K_1-K_2/\rho)]}{1+1/\rho}\right] = 1-\alpha, \\ &P_{T_{n-1}}\left[T_{n-1}\left(-\sqrt{n}\max(1,\rho)\,\Phi^{-1}(p_2^m)\right) \leq \sqrt{n}K_2\right] = 1-\alpha. \end{aligned} \tag{9}$$

A $1-\alpha$ lower-limit one-sided confidence interval for the modified proportion of conformance is then given by $1-p_1^m-p_2^m$. Again, we carried out simulation studies described earlier to evaluate the true confidence coefficient of this lower confidence limit. Simulations were carried out for $1-\alpha = 0.95$, $n = 10, 30, 50$, $\rho = 0.25, 0.5, 0.75$ and a variety of combinations of κ_1 and κ_2. Note that the lower confidence bound is no longer symmetric about κ_1 and

κ_2 and we have to consider both $\kappa_1 \geq \kappa_2$ and $\kappa_1 < \kappa_2$. Since $p_C^m = p_C$ when $\rho = 1$ and the definition of p_C^m ensures that processes C and D in Figure 4 have the same modified proportion of conformance, it is only necessary to consider $\rho < 1$ in our simulation.

Table 4 reports the results for $\rho = 0.75$. The results for $\rho = 0.25, 0.5$ are very similar and hence omitted here. From Table 4, it is clear that the proposed confidence interval is again conservative. Also, as $\kappa_2 - \rho\kappa_1$ deviates from zero (μ deviates from T) and keeping κ_1 or κ_2 fixed (p_L^m or p_U^m become smaller), the confidence coefficient approaches $1 - \alpha$. This fact can also be verified using a similar argument as given for p_C.

Examples

The 95% lower confidence limit $1 - p_1^m - p_2^m$ for p_C^m is tabulated in Table 5 for $n = 30$, $\rho = 0.75$ and some common combinations of K_1 and K_2. For other values of α, n, ρ, K_1 and K_2, a computer program in Lam and Wang (1993) can be used. Note that the lower confidence limit is not symmetric in K_1 and K_2. For example, when $K_1 = 2.4$ and $K_2 = 3.0$ (with $K_2 - \rho K_1 = 1.2$), the 95% lower confidence limit is 0.8954, while $K_1 = 3.0$ and $K_2 = 2.4$ (with $K_2 - \rho K_1 = 0.15$), the 95% lower confidence limit is 0.9208. This is not surprising since $K_2 - \rho K_1$ is smaller in the later case indicating that even though both cases have the same sample standard deviation, the sample mean of the later case is closer to the nominal value.

Conclusions

We have considered both point and interval estimation for the proportion of conformance and a modified proportion of conformance. Proportion of conformance measures how well the output of a process meets the specification limits. If the objective is to meet design specifications and at the same time to require that the majority of products be produced near the nominal value, then the modified proportion of conformance can be used. The computer program used to obtain point estimates and confidence limits for p_C and p_C^m is described in Lam and Wang (1993).

All statistical procedures developed in this paper are based on the assumption that the quality characteristic of interest can be modeled well by a normal distribution. As discussed in Littig and Lam (1993), many quality characteristics such as flatness and goodness of surface finish are modeled better by a skewed distribution such as a three-parameter gamma distribution. Also, in many applications quality characteristics and tolerance regions are multi-dimensional such as the hole location problem in a gear carrier with a circular tolerance region (see, for example, Littig, Lam and Pollock (1993)). These topics as well as the statistical tolerance interval problems for the distribution of $\hat{p}_C$ will be the subject of future research.

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Appendix

We seek $p^* = p^*(\alpha, K_1, K_2)$ such that

$$\begin{aligned}
1-\alpha &= P\left[P[L < X < U] \geq p^*\right] \\
&= P_{\bar{X},S}\left[P[\bar{X} - K_1 S < X < \bar{X} + K_2 S] \geq p^*\right] \\
&= P_{\bar{X},S}\left[P\left[\frac{\bar{X}-\mu}{\sigma} - \frac{K_1 S}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\bar{X}-\mu}{\sigma} + \frac{K_2 S}{\sigma}\right] \geq p^*\right] \\
&= P_{Z,Y}\left[\Phi\left(\frac{Z}{\sqrt{n}} + K_2\sqrt{\frac{Y}{n-1}}\right) - \Phi\left(\frac{Z}{\sqrt{n}} - K_1\sqrt{\frac{Y}{n-1}}\right) \geq p^*\right]
\end{aligned}$$

where Z is a standard normal random variable, Y is a chi-squared random variable with $n-1$ degrees of freedom and is independent of Z. If the probabilities that $K_1 < 0$ and $K_2 < 0$

are negligible, or if μ lies between L and U, a result described in Wald and Wolfowitz (1946) can be used to approximate the above equation. Specifically, for $u > 0$,

$$E_Z\left[\Phi\left(Z/\sqrt{n}+u\right)-\Phi\left(Z/\sqrt{n}-u\right)\right]$$

is closely approximated by

$$\Phi\left(1/\sqrt{n}+u\right)-\Phi\left(1/\sqrt{n}-u\right)$$

the difference being of the order $1/n^2$. Making use of this approximation, an approximate lower confidence limit p^* can be obtained from the following equation.

$$P_Y\left[\Phi\left(\frac{1}{\sqrt{n}}+K_2\sqrt{\frac{Y}{n-1}}\right)-\Phi\left(\frac{1}{\sqrt{n}}-K_1\sqrt{\frac{Y}{n-1}}\right)\geq p^*\right]=1-\alpha. \qquad (10)$$

Let q be the solution of

$$\Phi\left(\frac{1}{\sqrt{n}}+K_2 q\right)-\Phi\left(\frac{1}{\sqrt{n}}-K_1 q\right)=p^*$$

then by equation (10)

$$P_Y\left[\sqrt{Y/(n-1)}\geq q\right]=1-\alpha$$

or

$$q=\sqrt{\chi^2_{\alpha:n-1}/(n-1)}$$

where $\chi^2_{\alpha:\nu}$ is the α percentile of a chi-squared distribution with ν degrees of freedom. Taking into account of symmetry, the result follows.

References

[1] Chou, Y.; and Owen, D. B. (1984). "One-Sided Confidence Regions on the Upper and Lower Tail Areas of Normal Distribution". *Journal of Quality Technology* 16, pp. 150-158.

[2] Folks, J. L.; Pierce, D. A.; and Stewart, C. (1965). "Estimating the Fraction of Acceptable Product". *Technometrics* 7, pp. 43-50.

[3] Graybill, F. A. (1976). *Theory and Applications of the Linear Model.* Duxbury Press, North Scituate, MA.

[4] IMSL Inc. (1987). *The IMSL Math/Library.* Houston, Texas.

[5] Johnson, N. L.; and Kotz. S. (1970). *Continuous Univariate Distributions-1.* John Wiley & Sons, New York.

[6] Kane, V. E. (1986). "Process Capability Indices". *Journal of Quality Technology* 18, pp. 41-52.

[7] Kushler, R. H.; and Hurley, H. (1992). "Confidence Bounds for Capability Indices". *Journal of Quality Technology* 24, pp. 188-195.

[8] Lam, C. T.; and Wang, C. M. (1993). "Lower Confidence Limits for Proportion of Conformance". submitted to *Journal of Quality Technology.*

[9] Littig, S. J.; and Lam, C. T. (1993). "Case Studies in Process Capability Measurement". *ASQC: 47th Annual Quality Congress* pp. 569-575.

[10] Littig, S. J.; Lam, C. T.; and Pollock, S. M. (1993). "Capability Measurements for Multivariate Processes : Definitions and and Example for a Gear Carrier". Technical report #42, Industrial and Operations Engineering Department, University of Michigan.

[11] Mood, A. M.; Graybill, F. A.; and Boes, D. C. (1974). *Introduction to the Theory of Statistics.* McGraw-Hill, New York.

[12] McFadden, F. R. (1993). "Six-Sigma Quality Programs". *Quality Progress* 26, 6, pp. 37-42.

[13] Odeh, R. E.; and Owen, D. B. (1980). *Tables for Normal Tolerance Limits, Sampling Plans, and Screening.* Marcel Dekker, Inc., New York.

[14] Owen, D. B.; and Hua, T. A. (1977). "Tables of Confidence Limits on the Tail Area of the Normal Distribution". *Communications in Statistics - Simulation and Computation* B6, pp. 285-311.

[15] Pearn, W. L.; Kotz, S.; and Johnson, N. L. (1992). "Distributional and Inferential Properties of Process Capability Indices". *Journal of Quality Technology* 24, pp. 216-231.

[16] Wald, A.; and Wolfowitz, J. (1946). "Tolerance Limits for a Normal Distribution". *Annals of Mathematical Statistics* 14, pp. 45-55.

[17] Wang, C. M. (1991). "Approximate Confidence Intervals on Positive Linear Combinations of Expected Mean Squares". *Communications in Statistics - Simulation and Computation* B20, pp. 81-96.

[18] Wang, C. M. (1992). "Approximate Confidence Intervals on Linear Combinations of Expected Mean Squares". *Journal of Statistical Computation and Simulation* 43, pp. 229-241.

[19] Wheeler, D. J. (1970). "The Variance of an Estimator in Variables Sampling". *Technometrics* 12, pp. 751-755.

[20] Williams, J. S. (1962). "A Confidence Interval for Variance Components". *Biometrika* 49, pp. 278-281.

Key Words: *Capability Indices, Confidence Interval, Modified Proportion of Conformance, Point Estimation.*

Table 1. Simulated confidence coefficients for confidence limit $1 - p_1 - p_2$ of p_C

$1 - \alpha = 0.95$

		n		
κ_1	κ_2	10	30	50
1.0	1.0	0.9889	0.9937	0.9941
1.0	2.0	0.9760	0.9772	0.9783
1.0	3.0	0.9603	0.9570	0.9563
1.0	4.0	0.9496	0.9481	0.9496
1.0	5.0	0.9479	0.9471	0.9490
1.0	6.0	0.9474	0.9471	0.9490
2.0	2.0	0.9728	0.9738	0.9742
2.0	3.0	0.9604	0.9586	0.9583
2.0	4.0	0.9491	0.9486	0.9491
2.0	5.0	0.9470	0.9475	0.9486
2.0	6.0	0.9469	0.9474	0.9486
3.0	3.0	0.9657	0.9661	0.9654
3.0	4.0	0.9542	0.9521	0.9507
3.0	5.0	0.9472	0.9471	0.9475
3.0	6.0	0.9468	0.9470	0.9474
4.0	4.0	0.9621	0.9611	0.9614
4.0	5.0	0.9503	0.9498	0.9493
4.0	6.0	0.9459	0.9478	0.9480
5.0	5.0	0.9597	0.9587	0.9588
5.0	6.0	0.9496	0.9479	0.9493
6.0	6.0	0.9577	0.9574	0.9573
4.0	-1.0	0.9823	0.9596	0.9538
5.0	-2.0	0.9948	0.9558	0.9517
6.0	-3.0	0.9999	0.9561	0.9518
7.0	-1.0	0.9514	0.9491	0.9491
8.0	-2.0	0.9506	0.9500	0.9497
9.0	-3.0	0.9518	0.9517	0.9512

Table 2. Simulated confidence coefficients for confidence limit p^* of p_C

$1 - \alpha = 0.95$

		n					
κ_1	κ_2	10		30		50	
1.0	1.0	0.9815[a]	1.470[b]	0.9750[a]	1.169[b]	0.9707[a]	1.116[b]
1.0	2.0	0.9786	1.121	0.9737	1.036	0.9747	1.021
1.0	3.0	0.9738	1.020	0.9701	0.999	0.9691	0.997
1.0	4.0	0.9726	0.986	0.9688	0.992	0.9663	0.993
1.0	5.0	0.9719	0.977	0.9682	0.989	0.9658	0.992
1.0	6.0	0.9718	0.974	0.9681	0.991	0.9658	0.992
2.0	2.0	0.9795	1.043	0.9737	1.014	0.9701	1.010
2.0	3.0	0.9742	0.998	0.9718	0.995	0.9717	0.995
2.0	4.0	0.9714	0.975	0.9707	0.988	0.9709	0.993
2.0	5.0	0.9712	0.967	0.9705	0.988	0.9707	0.992
2.0	6.0	0.9712	0.955	0.9705	0.987	0.9707	0.992
3.0	3.0	0.9765	1.000	0.9720	1.000	0.9694	1.000
3.0	4.0	0.9694	0.989	0.9676	0.997	0.9692	0.998
3.0	5.0	0.9668	0.983	0.9676	0.996	0.9686	0.997
3.0	6.0	0.9668	0.981	0.9676	0.996	0.9686	0.998
4.0	4.0	0.9735	0.997	0.9710	1.000	0.9681	1.000
4.0	5.0	0.9665	0.994	0.9651	0.999	0.9668	1.000
4.0	6.0	0.9643	0.992	0.9649	0.999	0.9667	1.000
5.0	5.0	0.9697	0.998	0.9696	1.000	0.9674	1.000
5.0	6.0	0.9643	0.998	0.9629	1.000	0.9643	1.000
6.0	6.0	0.9677	0.999	0.9672	1.000	0.9661	1.000

[a]simulated confidence coefficient

[b]simulated value of $E[p^*]/E[1 - p_1 - p_2]$

Table 3. Lower confidence limits for p_C

$1 - \alpha = 0.95$ and $n = 30$

K_1	K_2	$1 - p_1 - p_2$	p^*
2.4	3.0	0.9519	0.9490
3.0	3.0	0.9771	0.9789
3.0	4.0	0.9875	0.9842
4.0	4.0	0.9979	0.9979
4.0	6.0	0.9989	0.9984

Table 4. Simulated confidence coefficients for confidence limit of p_C^m

$1 - \alpha = 0.95$ and $\rho = 0.75$

		n		
κ_1	$\kappa_2 - \rho\kappa_1$	10	30	50
1.0	0.0	0.9899	0.9957	0.9958
1.0	1.0	0.9767	0.9792	0.9800
1.0	2.0	0.9648	0.9639	0.9633
2.0	0.0	0.9732	0.9777	0.9786
2.0	1.0	0.9625	0.9611	0.9625
2.0	2.0	0.9515	0.9512	0.9513
3.0	0.0	0.9670	0.9684	0.9675
3.0	1.0	0.9559	0.9539	0.9541
3.0	2.0	0.9480	0.9479	0.9481
4.0	0.0	0.9624	0.9626	0.9625
4.0	1.0	0.9518	0.9506	0.9515
4.0	2.0	0.9464	0.9479	0.9482
5.0	0.0	0.9597	0.9591	0.9598
5.0	1.0	0.9506	0.9486	0.9505
5.0	2.0	0.9469	0.9470	0.9486
6.0	0.0	0.9581	0.9579	0.9577
6.0	1.0	0.9494	0.9484	0.9504
6.0	2.0	0.9462	0.9470	0.9491

		n		
κ_2	$\kappa_1 - \kappa_2/\rho$	10	30	50
1.0	0.0	0.9838	0.9893	0.9904
1.0	1.0	0.9750	0.9823	0.9819
1.0	2.0	0.9620	0.9662	0.9648
2.0	0.0	0.9683	0.9704	0.9699
2.0	1.0	0.9615	0.9659	0.9645
2.0	2.0	0.9516	0.9559	0.9538
3.0	0.0	0.9624	0.9626	0.9625
3.0	1.0	0.9559	0.9570	0.9562
3.0	2.0	0.9487	0.9518	0.9517
4.0	0.0	0.9591	0.9586	0.9595
4.0	1.0	0.9523	0.9526	0.9528
4.0	2.0	0.9475	0.9499	0.9508
5.0	0.0	0.9572	0.9570	0.9570
5.0	1.0	0.9508	0.9511	0.9510
5.0	2.0	0.9481	0.9492	0.9503
6.0	0.0	0.9560	0.9560	0.9561
6.0	1.0	0.9503	0.9501	0.9506
6.0	2.0	0.9485	0.9487	0.9501

Table 5. Lower confidence limits for p_C^m

$1-\alpha = 0.95$, $\rho = 0.75$ and $n = 30$

K_1	K_2	$1 - p_1^m - p_2^m$
2.4	3.0	0.8954
2.4	4.0	0.9082
2.4	6.0	0.9104
3.0	2.4	0.9208
3.0	3.0	0.9428
3.0	4.0	0.9542
3.0	6.0	0.9560
4.0	2.4	0.9538
4.0	3.0	0.9788
4.0	4.0	0.9880
4.0	6.0	0.9894
6.0	2.4	0.9633
6.0	3.0	0.9884
6.0	4.0	0.9987
6.0	6.0	0.9998

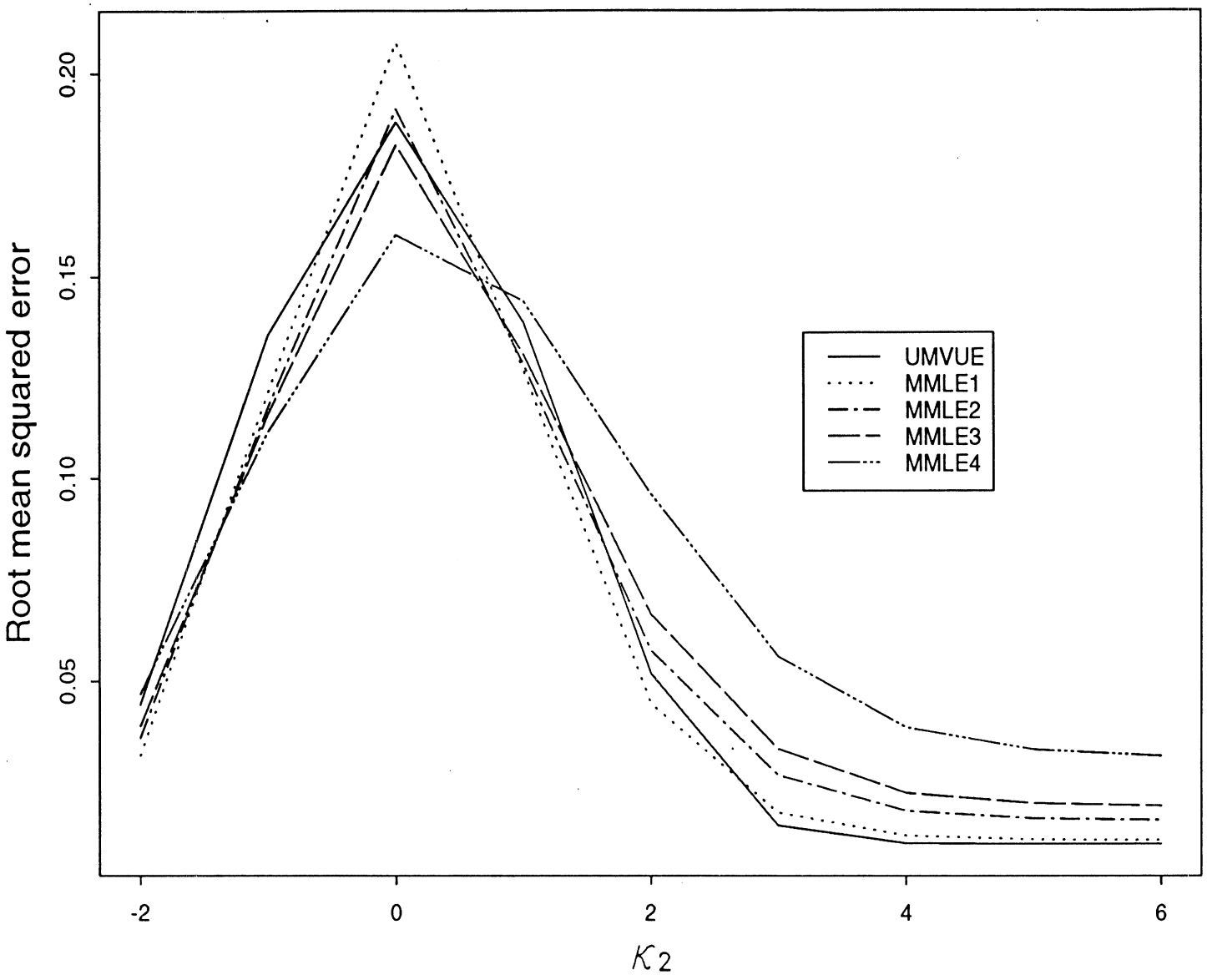

Figure 1: Root mean squared error of five different estimators for p_C when $n = 5$ and $\kappa_1 = 3.0$

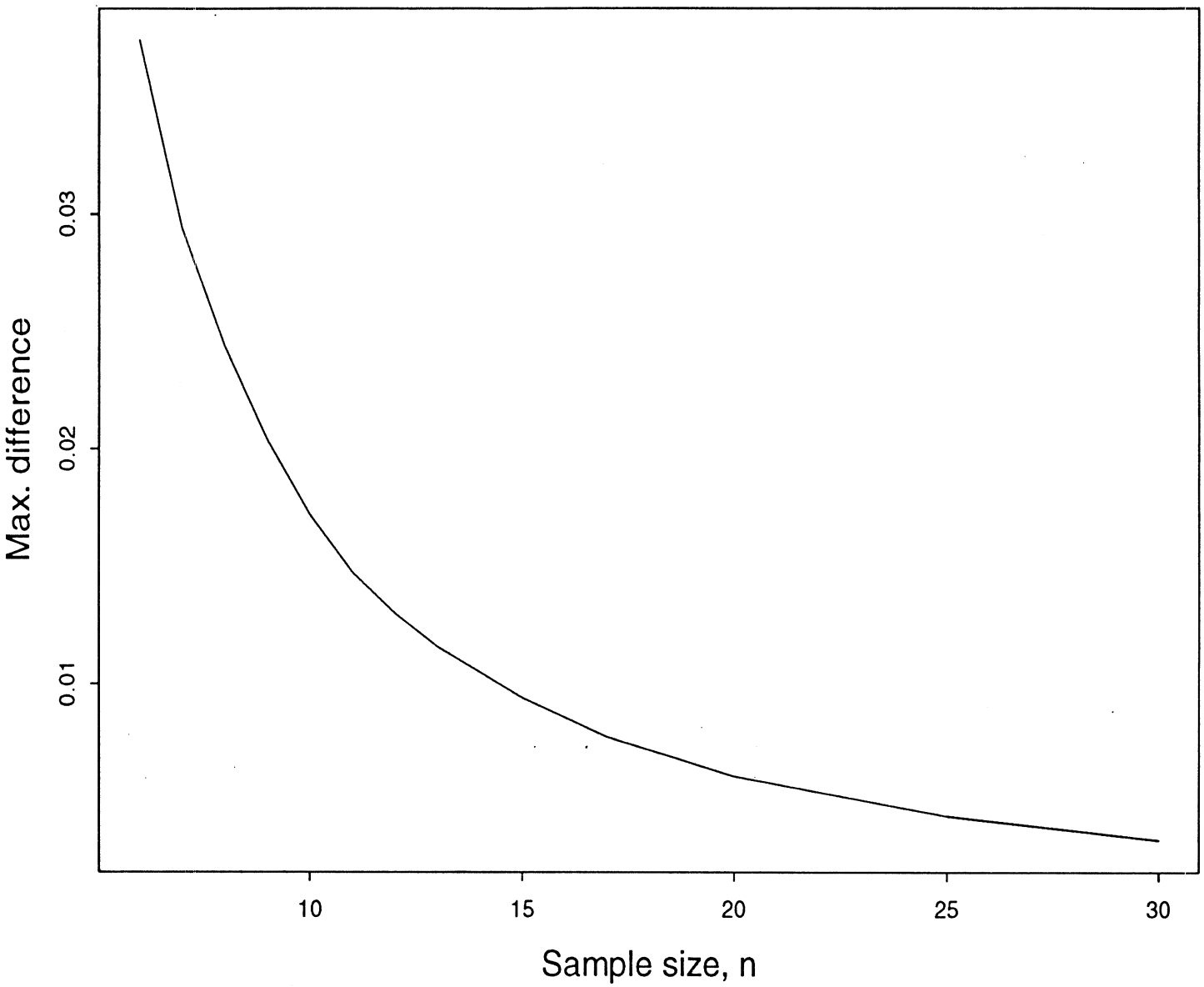

Figure 2: Maximum difference in root mean squared errors among the five estimators over the range of $\kappa_2 \geq 2.0$ and $\kappa_1 = 3.0$

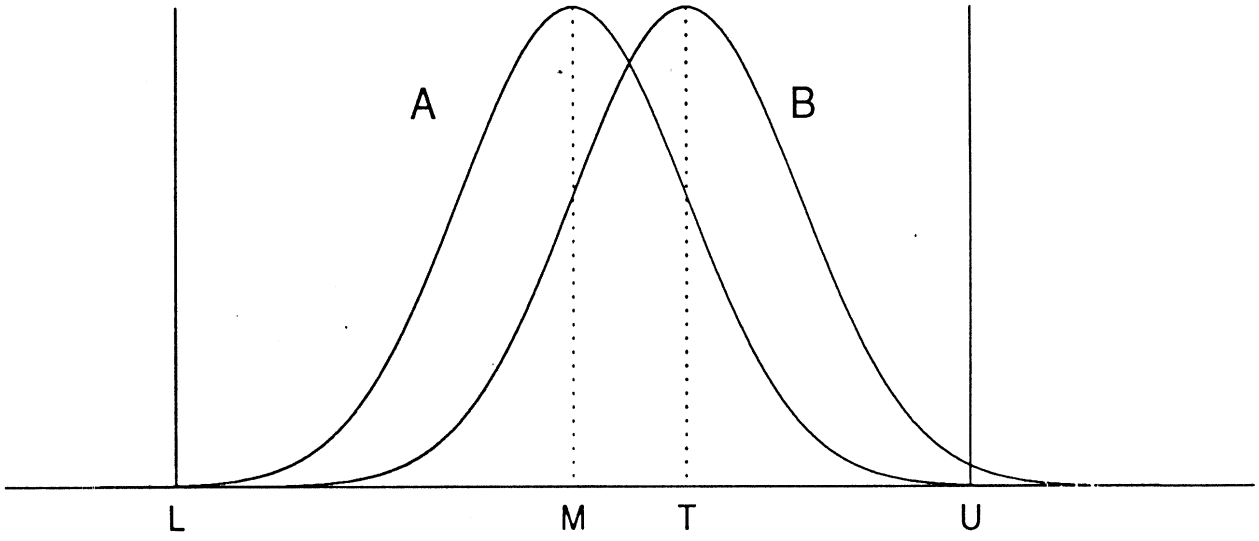

Figure 3: Process B is preferable to process A since it is centered at the nominal value T

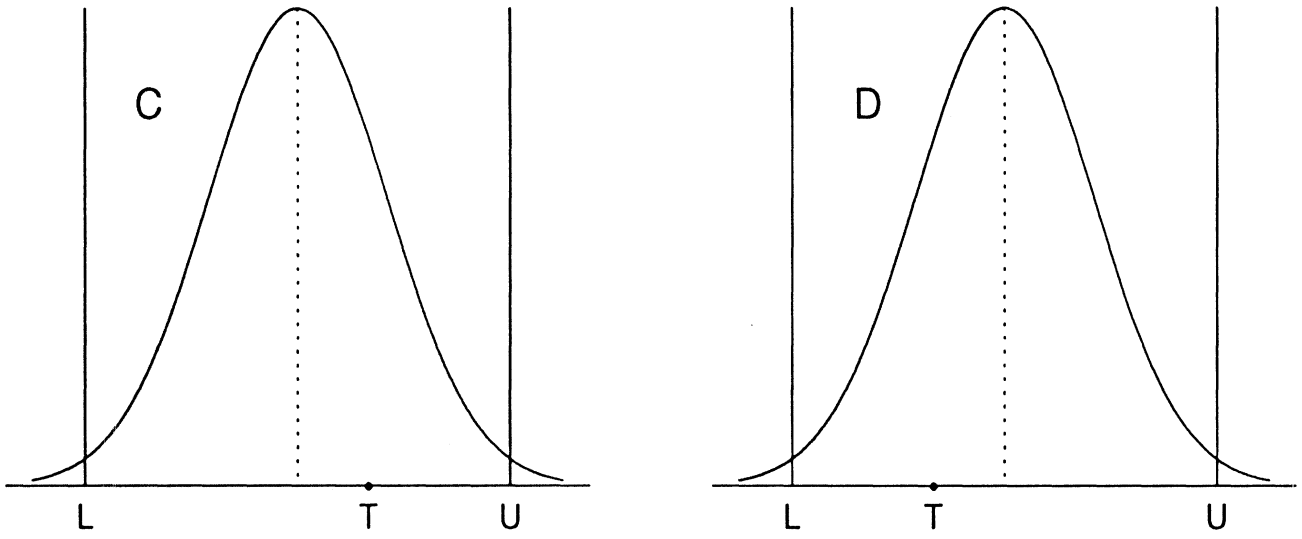

Figure 4: Processes C and D are mirror reflection of each other

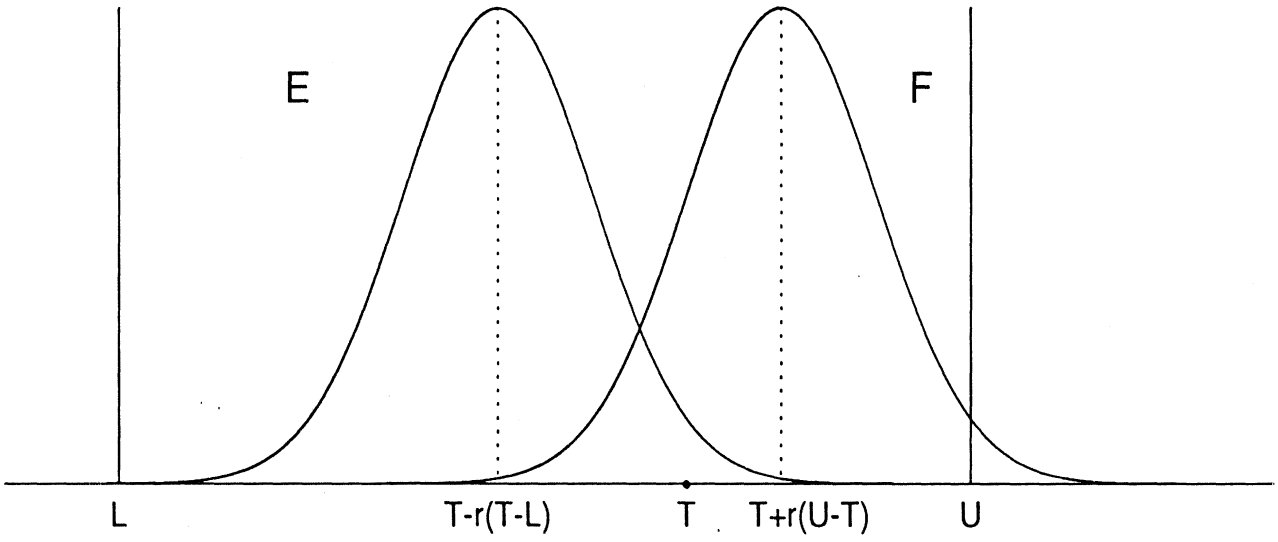

Figure 5: Processes E and F have the same modified proportion of conformance for any $0 < r < 1$