SUPPLEMENTARY RESULTS IN THE COMPARISON OF VARIOUS MAINTENANCE STRATEGIES FOR DETERIORATING SYSTEMS

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Supplementary Results in the Comparison of Various Maintenance Strategies for Deteriorating Systems

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The purpose of this report is to provide detailed proofs for some of the results in the comparison of various maintenance strategies for deteriorating systems. The readers should also consult Technical Reports 92-22 and 92-31.

1. EQUIVALENCY OF OBJECTIVE FUNCTIONS

Lemma 1.1. When $X_\delta^\theta(0) > 0$ for all $\delta \in \Delta_\theta$, finding a policy $\delta_\theta^* \in \Delta_\theta$ such that

$$g_\theta^* \equiv \inf_{\delta \in \Delta_\theta} \frac{Y_\delta^\theta(0)}{X_\delta^\theta(0)} = \frac{Y_{\delta_\theta^*}^\theta(0)}{X_{\delta_\theta^*}^\theta(0)} \tag{1}$$

is equivalent to finding a $g_\theta^* \in I\!R^+$ and a policy $\delta_\theta^* \in \Delta_\theta$ such that

$$W_\theta(g_\theta^*) \equiv \inf_{\delta \in \Delta_\theta} \left[Y_\delta^\theta(0) - g_\theta^* X_\delta^\theta(0) \right] = Y_{\delta_\theta^*}^\theta(0) - g_\theta^* X_{\delta_\theta^*}^\theta(0) = 0. \tag{2}$$

Proof. Suppose that Equation (1) holds, then $\dfrac{Y_\delta^\theta(0)}{X_\delta^\theta(0)} \geq \inf_{\delta \in \Delta_\theta} \dfrac{Y_\delta^\theta(0)}{X_\delta^\theta(0)} = \dfrac{Y_{\delta_\theta^*}^\theta(0)}{X_{\delta_\theta^*}^\theta(0)} = g_\theta^*$ for all $\delta \in \Delta_\theta$. When $X_\delta^\theta(0) > 0$, we have $Y_\delta^\theta(0) - g_\theta^* X_\delta^\theta(0) \geq Y_{\delta_\theta^*}^\theta(0) - g_\theta^* X_{\delta_\theta^*}^\theta(0) = 0$ for all $\delta \in \Delta_\theta$, and the equality holds if $\delta = \delta_\theta^*$. Therefore, Equation (2) holds.

Conversely, if Equation (2) holds, then

$$Y_\delta^\theta(0) - g_\theta^* X_\delta^\theta(0) \geq \inf_{\delta \in \Delta_\theta} \left[Y_\delta^\theta(0) - g_\theta^* X_\delta^\theta(0) \right] = Y_{\delta_\theta^*}^\theta(0) - g_\theta^* X_{\delta_\theta^*}^\theta(0) = 0$$

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for all $\delta \in \Delta_\theta$. When $X_\delta^\theta(0) > 0$, it follows that $g_\theta^* = \dfrac{Y_{\delta_\theta^*}(0)}{X_{\delta_\theta^*}(0)} \leq \dfrac{Y_\delta^\theta(0)}{X_\delta^\theta(0)}$ for all $\delta \in \Delta_\theta$. Hence, Equation (1) holds. □

Lemma 1.2. $W_\theta(g)$ is a continuous and nonincreasing function of g.

Proof. Let $g_1 < g_2$ and $\delta_1, \delta_2 \in \Delta_\theta$ such that

$$\begin{aligned} W_\theta(g_1) &= \inf_{\delta\in\Delta_\theta}\left[Y_\delta^\theta(0) - g_1 X_\delta^\theta(0)\right] = Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0) \leq Y_{\delta_2}^\theta(0) - g_1 X_{\delta_2}^\theta(0) \\ W_\theta(g_2) &= \inf_{\delta\in\Delta_\theta}\left[Y_\delta^\theta(0) - g_2 X_\delta^\theta(0)\right] = Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0) \leq Y_{\delta_1}^\theta(0) - g_2 X_{\delta_1}^\theta(0) \end{aligned}$$

Since $X_\delta^\theta(0) \geq 0$ for all $\delta \in \Delta_\theta$, the following inequalities hold.

$$\begin{aligned} W_\theta(g_1) - W_\theta(g_2) &= \left[Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0)\right] - \left[Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0)\right] \\ &\geq \left[Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0)\right] - \left[Y_{\delta_1}^\theta(0) - g_2 X_{\delta_1}^\theta(0)\right] = (g_2 - g_1) X_{\delta_1}^\theta(0) \geq 0. \\ W_\theta(g_1) - W_\theta(g_2) &= \left[Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0)\right] - \left[Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0)\right] \\ &\leq \left[Y_{\delta_2}^\theta(0) - g_1 X_{\delta_2}^\theta(0)\right] - \left[Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0)\right] = (g_2 - g_1) X_{\delta_2}^\theta(0). \end{aligned}$$

Letting $| g_2 - g_1 | \to 0$, result follows. □

2. PROPERTIES OF DETERIORATING SYSTEMS

Since the transitions of the deteriorating system follow a continuous time Markov process, the transition probabilities $P_{ij}(t)$, $i, j \in S$ and $t \in [0, \infty)$ satisfies the following *Kolmogorov's* equations.

Kolmogorov's forward equations:

$$\begin{aligned} \frac{d}{dt}P_{ii}(t) &= -\lambda_i P_{ii}(t) = -\lambda_i e^{-\lambda_i t} && \text{for } 0 \leq i \leq n, \\ \frac{d}{dt}P_{ij}(t) &= -\lambda_j P_{ij}(t) + \beta_{j-1} P_{i,j-1}(t) && \text{for } 0 \leq i < j \leq n, \\ \frac{d}{dt}F_i(t) &= \sum_{j=i}^{n} \alpha_j P_{ij}(t) && \text{for } 0 \leq i \leq n. \end{aligned}$$

Kolmogorov's backward equations:

$$\begin{aligned} \frac{d}{dt}P_{ij}(t) &= -\lambda_i P_{ij}(t) + \beta_i P_{i+1,j}(t) && \text{for } 0 \leq i < j \leq n, \\ \frac{d}{dt}F_i(t) &= \lambda_i \bar{F}_i(t) - \beta_i \bar{F}_{i+1}(t) \\ &= -\lambda_i F_i(t) + \beta_i F_{i+1}(t) + \alpha_i && \text{for } 0 \leq i \leq n. \end{aligned}$$

By solving *Kolmogorov's* equations, explicit formulas for the transition probabilities $P_{ij}(t)$ can be derived easily when $\lambda_0, \lambda_1, \cdots$, and λ_n are all distinct.

$$P_{ij}(t) = \begin{cases} 0 & \text{for } j < i \\ e^{-\lambda_i t} & \text{for } j = i \\ \left(\prod_{u=i}^{j-1} \beta_u\right) \left[\sum_{l=i}^{j} e^{-\lambda_l t} \left(\prod_{k=i,k\neq l}^{j} \frac{1}{\lambda_k - \lambda_l}\right)\right] & \text{for } i < j < n+1 \\ 1 - \sum_{k=i}^{n} P_{ik}(t) & \text{for } j = n+1 \end{cases}.$$

By integrating $P_{ij}(t)$ over $[0, t]$, explicit formulas for $Q_{ij}(t)$ can also be obtained.

$$Q_{ij}(t) = \begin{cases} 0 & \text{for } j < i \\ \dfrac{1 - e^{-\lambda_i t}}{\lambda_i} & \text{for } j = i \\ \left(\prod_{u=i}^{j-1} \beta_u\right) \left[\sum_{l=i}^{j} \left(\frac{1 - e^{-\lambda_l t}}{\lambda_l}\right) \left(\prod_{k=i,k\neq l}^{j} \frac{1}{\lambda_k - \lambda_l}\right)\right] & \text{for } i < j < n+1 \\ t - \sum_{k=i}^{n} Q_{ik}(t) & \text{for } j = n+1 \end{cases}.$$

The expected time to failure μ_i can be easily derived using the following recursive equation with $\mu_{n+1} = 0$.

$$\mu_i = \frac{1}{\lambda_i} + \frac{\beta_i}{\lambda_i}\mu_{i+1} = \frac{1}{\lambda_i} + \sum_{k=i+1}^{n} \frac{1}{\lambda_k} \left(\prod_{l=i}^{k-1} \frac{\beta_l}{\lambda_l}\right).$$

Furthermore, $P_{ij}(t)$ and $Q_{ij}(t)$ have the following properties.

Lemma 2.1. For each $t \in [0, \infty)$, $P_{ij}(t)$ is totally positive of order 2 (TP_2) in i and $j \in S \setminus \{n+1\}$, that is,

$$\begin{vmatrix} P_{ik}(t) & P_{il}(t) \\ P_{jk}(t) & P_{jl}(t) \end{vmatrix} \geq 0 \text{ for } 0 \leq i < j \leq n \text{ and } 0 \leq k < l \leq n.$$

Proof. Let $\phi_{ij}(t) = P_{ik}(t)P_{jl}(t) - P_{jk}(t)P_{il}(t)$. It is obvious that $\phi_{ii}(t) = 0$, $\phi_{ij}(0) \geq 0$, and $\phi_{ij}(t) = P_{ik}(t)P_{jl}(t) \geq 0$ for all $j > k$. Using *Kolmogorov's* backward equations, we can show that

$$\frac{d}{dt}\phi_{ij}(t) = -(\lambda_i + \lambda_j)\phi_{ij}(t) + \beta_i\phi_{i+1,j}(t) + \beta_j\phi_{i,j+1}(t). \tag{3}$$

Solving Equation (3), we have

$$\phi_{ij}(t) = e^{-(\lambda_i+\lambda_j)t}\left\{\int_0^t e^{(\lambda_i+\lambda_j)u}\left[\beta_i\phi_{i+1,j}(u) + \beta_j\phi_{i,j+1}(u)\right] du + \phi_{ij}(0)\right\}.$$

for all $\delta \in \Delta_\theta$. When $X_\delta^\theta(0) > 0$, it follows that $g_\theta^* = \frac{Y_{\delta_\theta^*}(0)}{X_{\delta_\theta^*}(0)} \leq \frac{Y_\delta^\theta(0)}{X_\delta^\theta(0)}$ for all $\delta \in \Delta_\theta$. Hence, Equation (1) holds. □

Lemma 1.2. $W_\theta(g)$ is a continuous and nonincreasing function of g.

Proof. Let $g_1 < g_2$ and $\delta_1, \delta_2 \in \Delta_\theta$ such that

$$\begin{aligned} W_\theta(g_1) &= \inf_{\delta \in \Delta_\theta} \left[Y_\delta^\theta(0) - g_1 X_\delta^\theta(0) \right] = Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0) \leq Y_{\delta_2}^\theta(0) - g_1 X_{\delta_2}^\theta(0) \\ W_\theta(g_2) &= \inf_{\delta \in \Delta_\theta} \left[Y_\delta^\theta(0) - g_2 X_\delta^\theta(0) \right] = Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0) \leq Y_{\delta_1}^\theta(0) - g_2 X_{\delta_1}^\theta(0) \end{aligned}$$

Since $X_\delta^\theta(0) \geq 0$ for all $\delta \in \Delta_\theta$, the following inequalities hold.

$$\begin{aligned} W_\theta(g_1) - W_\theta(g_2) &= \left[Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0) \right] - \left[Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0) \right] \\ &\geq \left[Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0) \right] - \left[Y_{\delta_1}^\theta(0) - g_2 X_{\delta_1}^\theta(0) \right] = (g_2 - g_1) X_{\delta_1}^\theta(0) \geq 0. \\ W_\theta(g_1) - W_\theta(g_2) &= \left[Y_{\delta_1}^\theta(0) - g_1 X_{\delta_1}^\theta(0) \right] - \left[Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0) \right] \\ &\leq \left[Y_{\delta_2}^\theta(0) - g_1 X_{\delta_2}^\theta(0) \right] - \left[Y_{\delta_2}^\theta(0) - g_2 X_{\delta_2}^\theta(0) \right] = (g_2 - g_1) X_{\delta_2}^\theta(0). \end{aligned}$$

Letting $\mid g_2 - g_1 \mid \rightarrow 0$, result follows. □

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Since the transitions of the deteriorating system follow a continuous time Markov process, the transition probabilities $P_{ij}(t)$, $i, j \in S$ and $t \in [0, \infty)$ satisfies the following *Kolmogorov's* equations.

Kolmogorov's forward equations:

$$\begin{aligned} \frac{d}{dt} P_{ii}(t) &= -\lambda_i P_{ii}(t) = -\lambda_i e^{-\lambda_i t} && \text{for } 0 \leq i \leq n, \\ \frac{d}{dt} P_{ij}(t) &= -\lambda_j P_{ij}(t) + \beta_{j-1} P_{i,j-1}(t) && \text{for } 0 \leq i < j \leq n, \\ \frac{d}{dt} F_i(t) &= \sum_{j=i}^{n} \alpha_j P_{ij}(t) && \text{for } 0 \leq i \leq n. \end{aligned}$$

Kolmogorov's backward equations:

$$\begin{aligned} \frac{d}{dt} P_{ij}(t) &= -\lambda_i P_{ij}(t) + \beta_i P_{i+1,j}(t) && \text{for } 0 \leq i < j \leq n, \\ \frac{d}{dt} F_i(t) &= \lambda_i \bar{F}_i(t) - \beta_i \bar{F}_{i+1}(t) \\ &= -\lambda_i F_i(t) + \beta_i F_{i+1}(t) + \alpha_i && \text{for } 0 \leq i \leq n. \end{aligned}$$

By induction, it is easy to show that $\phi_{ij}(t) \geq 0$ for $0 \leq i < j \leq n$ and $0 \leq k < l \leq n$. □

Lemma 2.2. For each $i \in S$, $P_{ij}(t)$ is totally positive of order 2 (TP_2) in $j \in S \setminus \{n+1\}$ and $t \in [0, \infty)$, that is,

$$\begin{vmatrix} P_{ij}(u) & P_{ij}(v) \\ P_{ik}(u) & P_{ik}(v) \end{vmatrix} \geq 0 \text{ for } 0 \leq j < k \leq n \text{ and } 0 \leq u < v.$$

Proof. Using *Kolmogorov's* backward equations and Lemma 2.1, it is easy to show that for all $t \in (0, \infty)$,

$$\frac{d}{dt}\frac{P_{ik}(t)}{P_{ij}(t)} = \frac{\beta_i\left[P_{i+1,k}(t)P_{ij}(t) - P_{ik}(t)P_{i+1,j}(t)\right]}{[P_{ij}(t)]^2} \geq 0.$$

Hence, $\frac{P_{ik}(t)}{P_{ij}(t)}$ is nondecreasing in t for all $j < k$. Result follows. □

Lemma 2.3. For each $t \in [0, \infty)$, $Q_{ij}(t)$ is totally positive of order 2 (TP_2) in i and $j \in S \setminus \{n+1\}$, that is,

$$\begin{vmatrix} Q_{ik}(t) & Q_{il}(t) \\ Q_{jk}(t) & Q_{jl}(t) \end{vmatrix} \geq 0 \text{ for } 0 \leq i < j \leq n \text{ and } 0 \leq k < l \leq n.$$

Proof. Let $\varphi_{ij}(t) = Q_{ik}(t)Q_{jl}(t) - Q_{jk}(t)Q_{il}(t)$. Obviously, $\varphi_{ii}(t) = 0$ and $\varphi_{ij}(0) = 0$. Now, we prove $\varphi_{ij}(t) \geq 0$ for three different cases.

Case 1: For $j > k$, since $Q_{jk}(t) = 0$, it follows that $\varphi_{ij}(t) = Q_{ik}(t)Q_{jl}(t) \geq 0$

Case 2: For $0 \leq i < j = k < l \leq n$, from *Kolmogorov's* forward and backward equations, we can show that $\frac{d}{dt}Q_{ii}(t) = -\lambda_i Q_{ii}(t)$ and $\frac{d}{dt}Q_{ij}(t) = -\lambda_i Q_{ij}(t) + \beta_i Q_{i+1,j}(t)$ for $0 \leq i < j \leq n$. Taking the first derivative of $\varphi_{ij}(t)$ with respect to t, we have

$$\frac{d}{dt}\varphi_{ij}(t) = -(\lambda_i + \lambda_j)\varphi_{ij}(t) + \beta_i\varphi_{i+1,j}(t) + \beta_j Q_{j+1,l}(t)Q_{ij}(t) - Q_{il}(t). \tag{4}$$

Solving Equation (4), we obtain

$$\varphi_{ij}(t) = \int_0^t e^{-(\lambda_i+\lambda_j)(t-u)}\left[\beta_i\varphi_{i+1,j}(u) + \beta_j Q_{j+1,l}(u)Q_{ij}(u) - Q_{il}(u)\right] du.$$

Let $\eta_i(t) = \beta_j Q_{j+1,l}(t)Q_{ij}(t) - Q_{il}(t)$. Since $\varphi_{jj}(t) = 0$, if we can show that $\eta_i(t) \geq 0$ for all $0 \leq i \leq j-1$, then it will hold by induction that $\varphi_{ij} \geq 0$ for $0 \leq i \leq j$. By differentiating $\eta_i(t)$ with respect to t, we have

$$\frac{d}{dt}\eta_i(t) = -\lambda_i\eta_i(t) + \beta_i\eta_{i+1}(t) + \beta_j P_{j+1,l}(t)Q_{ij}(t). \tag{5}$$

property of TP_2 functions, Equation (7) changes its sign at most once in i and t and the possible change is from negative to positive for any c. This result implies that the failure rate function $h_i(t)$ is nondecreasing in i and t. Since $F_i(t) = 1 - e^{-\int_0^t h_i(u)\,du}$, $F_i(t)$ is therefore nondecreasing in i. □

Lemma 2.6. If α_i is nondecreasing in $i \in S \setminus \{n+1\}$, then μ_i is nonincreasing in i.

Proof. It is obvious that for $i \in S \setminus \{n+1\}$,

$$\mu_i - \mu_{i+1} = \int_0^\infty [\bar{F}_i(t) - \bar{F}_{i+1}(t)]\ dt = \int_0^\infty [F_{i+1}(t) - F_i(t)]\ dt.$$

From Lemma 2.5, it is clear that μ_i is nonincreasing in i. □

3. PROPERTIES OF COST AND TIME STRUCTURES

Given that the system starts in state i, the expected operating cost to failure $A_i(\infty)$ can be easily calculated by the following recursive equation providing that $A_{n+1}(\infty) = 0$. (Note that $A_i(\infty) = \mu_i$ when $a_0 = a_1 = \cdots = a_n = 1$.)

$$A_i(\infty) = \frac{a_i}{\lambda_i} + \frac{\beta_i}{\lambda_i} A_{i+1}(\infty) = \frac{a_i}{\lambda_i} + \sum_{k=i+1}^{n} \frac{a_k}{\lambda_k}\left(\prod_{l=i}^{k-1} \frac{\beta_l}{\lambda_l}\right).$$

If the system satisfies the following assumptions, it can be shown that the optimal policies of various strategies have structural properties.

(A1) $0 < \lambda_0 \le \lambda_1 \le \ldots \le \lambda_n$

(A2) $0 < \alpha_0 \le \alpha_1 \le \ldots \le \alpha_n$

(A3) $0 \le r_0 \le r_1 \le \ldots \le r_n \le r_{n+1} - q$

(A4) $0 < \dfrac{C_0 + M}{r_0 + q} \le \dfrac{C_1 + M}{r_1 + q} \le \ldots \le \dfrac{C_{n+1} + M}{r_{n+1} + q} \le \dfrac{C_{n+1}}{r_{n+1}}$

(A5) $\dfrac{a_0}{\lambda_0} - (C_0 + mr_0) \le \dfrac{a_1}{\lambda_1} - (C_1 + mr_1) \le \ldots \le \dfrac{a_n}{\lambda_n} - (C_n + mr_n)$

Assumptions **(A1)** to **(A5)** above imply the following properties.

We know that $r_i + q \le r_{n+1}$ from **(A3)**. Result therefore follows. □

Property 3.6. Recall that $b_i(g) = a_i - \lambda_i K_i(g) + \beta_i K_{i+1}(g) + \alpha_i[K_{n+1}(g) - \tilde{M}]$. For $g \in \left[0, m + \frac{C_0 + M}{r_0 + q}\right]$, $b_i(g)$ is a nondecreasing function in $i \in S \setminus \{n+1\}$.

Proof. Given any $g \in \left[0, m + \frac{C_0 + M}{r_0 + q}\right]$ and $i \in S \setminus \{n, n+1\}$, using Property 3.5, we have

$$\begin{aligned}
& b_{j+1}(g) - b_j(g) \\
= \; & \{[a_{j+1} - \lambda_{j+1} K_{j+1}(g)] - [a_j - \lambda_j K_j(g)]\} + \beta_{j+1} K_{j+2}(g) - \beta_j K_{j+1}(g) \\
& + (\alpha_{j+1} - \alpha_j)\left[K_{n+1}(g) - \tilde{M}\right] \\
\ge \; & \{[a_{j+1} - \lambda_{j+1} K_{j+1}(g)] - [a_j - \lambda_j K_j(g)]\} + (\beta_{j+1} - \beta_j) K_{j+1}(g) + (\alpha_{j+1} - \alpha_j) K_{j+1}(g) \\
= \; & \{[a_{j+1} - \lambda_{j+1} K_{j+1}(g)] - [a_j - \lambda_j K_j(g)]\} + (\lambda_{j+1} - \lambda_j) K_{j+1}(g).
\end{aligned}$$

Furthermore, from assumptions **(A1)**, **(A3)** and **(A5)**, it is clear that $a_j - \lambda_j K_j(g)$ is nondecreasing in $j \in S \setminus \{n+1\}$. Result follows. □

4. PROPERTIES OF OPTIMAL POLICIES

Sequential Inspection Strategy:

Lemma 4.1. For each fixed $g \in [0, g_{\min}]$, if $D_{\delta_g}(i) \ne R$, then

$$G^*(i, g, \infty) - V^*(i, g) \le \frac{\beta_i}{\lambda_i}[G^*(i+1, g, \infty) - V^*(i+1, g)]$$

for all δ_g constructed in **Step 2** of the algorithm and $i \in S \setminus \{n+1\}$.

Proof. If $D_{\delta_g}(i) \ne R$, then $V^*(i,g) = G^*(i,g,t_i^*(g))$ and $\left.\frac{d}{dt}G^*(i,g,t)\right|_{t=t_i^*(g)} = 0$. Using *Kolmogorov's* backward equations, we have

$$\begin{aligned}
\frac{d}{dt}G^*(i,g,t) \;=\; & \frac{1}{1 - P_{ii}(t)}\{a_i - \lambda_i G^*(i,g,t) + \beta_i V^*(i+1,g) + \alpha_i K_{n+1}(g) - g \\
& + \beta_i [1 - P_{i+1,i+1}(t)][G^*(i+1,g,t) - V^*(i+1,g)]\}.
\end{aligned}$$

Since $\left.\frac{d}{dt}G^*(i,g,t)\right|_{t=t_i^*(g)} = 0$ and $V^*(i+1,g) \le G^*(i+1,g,t_i^*(g))$, we have

$$a_i - \lambda_i G^*(i,g,t_i^*(g)) + \beta_i V^*(i+1,g) + \alpha_i K_{n+1}(g) - g \le 0. \tag{8}$$