

A connection-theoretic approach to reduction of second-order dynamical systems with symmetry

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We deal with reduction of Lagrangian systems that are invariant under the action of the symmetry group. Unlike the bulk of the literature we do not rely on methods coming from the calculus of variations. Our method is based on the geometrical analysis of regular Lagrangian systems, where solutions of the Euler-Lagrange equations are interpreted as integral curves of the associated second-order differential equation field. In particular, we explain so-called Lagrange-Poincaré reduction [1] and Routh reduction [3] from the viewpoint of that vector field.

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1 The Euler-Lagrange equations in an adapted frame

This contribution is based on references [2] and [4]. Let (x^α) be coordinates on a manifold M and (x^α, u^α) coordinates on its tangent manifold TM . We will assume that the Lagrangian $L(x, u)$ is regular, i.e. that the matrix of functions $(\partial^2 L / \partial u^\alpha \partial u^\beta)$ is everywhere non-singular. Then, the Euler-Lagrange equations may be written explicitly in the form of a set of second-order differential equations $\ddot{x}^\alpha = f^\alpha(x, \dot{x})$ and its solutions can be interpreted as integral curves of the second-order differential equation field $\Gamma = u^\alpha \partial / \partial x^\alpha + f^\alpha(x, u) \partial / \partial u^\alpha$ on TM . The Euler-Lagrange equations may then be expressed in the form $\Gamma(\partial L / \partial u^\alpha) - \partial L / \partial x^\alpha = 0$. These equations, together with the assumption that it is a second-order differential equation field, determine the vector field Γ .

There are two canonical lifts of a vector field $Z = Z^\alpha \partial / \partial x^\alpha$ on M to a vector field on TM . The flow of the so-called complete or tangent lift $Z^c = Z^\alpha \partial / \partial x^\alpha + u^\beta \partial Z^\alpha / \partial x^\beta \partial / \partial u^\alpha$ consists of the tangent maps of the flow of Z . The vertical lift $Z^v = Z^\alpha \partial / \partial u^\alpha$ is tangent to the fibres of $\tau : TM \rightarrow M$ and on $T_m M$ it coincides with $Z(m)$. We use these two concepts to cast the Euler-Lagrange field in terms of a non-coordinate basis. If $\{Z_\alpha\}$ is a basis of vector fields on M , then it can easily be verified that an equivalent expression for the Euler-Lagrange equations is $\Gamma(Z_\alpha^v(L)) - Z_\alpha^c(L) = 0$.

We will assume throughout that the system is invariant under a proper, free (left) action $\psi^M : G \times M \rightarrow M$. Let X_i be the G -invariant horizontal lifts of a coordinate basis of vector fields on M/G (horizontal w.r.t. a principal connection on $\pi^M : M \rightarrow M/G$). We will also need two sets $\{\tilde{E}_a\}$ or $\{\hat{E}_a\}$ of vector fields on M , associated to a basis $\{E_a\}$ of the Lie algebra \mathcal{G} . The vector fields \tilde{E}_a of the ‘moving’ basis are the fundamental vector fields corresponding to the action. If we set locally $\pi^M : U \times G \rightarrow U$ (and $\psi_g^M(x, h) = (x, gh)$), the ‘body-fixed’ basis consists of the vector fields defined by $\hat{E}_a : (x, g) \mapsto T\psi_g^M(\tilde{E}_a(x, e))$. Note that the basis $\{X_i, \hat{E}_a\}$ is invariant, but the basis $\{X_i, \tilde{E}_a\}$ is not. We can now express the Euler-Lagrange equations in either one of these two adapted frames. By doing so, we derive in the next sections both the Lagrange-Poincaré equations [1] and the Lagrange-Routh equations [3] in a relatively straightforward fashion.

2 The mechanical connection and Lagrange-Poincaré reduction

The action ψ^M induces an action $\psi_g^{TM} = T\psi_g^M$ on TM for which $\pi^{TM} : TM \rightarrow TM/G$ is a principal fibre bundle. We assume from now on that the Lagrangian L is invariant under this induced action. Then one can easily show that Γ is an invariant vector field on TM .

We first recall a well-known method for reducing and reconstructing integral curves of an invariant vector field. Let $\pi : N \rightarrow N/G$ be a principal fibre bundle and $\psi^N : G \times N \rightarrow N$ the corresponding action. Denote the fundamental vector field of $\xi \in \mathcal{G}$ by $\tilde{\xi}$. Assuming that G is connected, the infinitesimal condition for a vector field Γ on N to be invariant is $[\tilde{\xi}, \Gamma] = 0, \forall \xi \in \mathcal{G}$. An invariant Γ defines a π -related reduced vector field $\check{\Gamma}$ on N/G : the relation $T\pi(\Gamma(v)) = \check{\Gamma}(\pi(v))$ is independent of the choice of $v \in N$ within $\pi(v) \in N/G$. With the aid of a principal connection Ω on $\pi : N \rightarrow N/G$, we can reconstruct the integral curve $v(t)$ of Γ through v_0 from the reduced data as follows: First find the integral curve $\check{v}(t) \in N/G$ of $\check{\Gamma}$ through $\pi(v_0)$. Then, look for its horizontal lift $\check{v}^H(t)$ (this is the curve in N s.t. $\pi \circ \check{v}^H = \check{v}$, $\check{v}^H(0) = v_0$ and $\Omega(\dot{\check{v}}^H) = 0$) and find the solution $g(t) \in G$ of the reconstruction equation $\theta(\dot{g}) = \Omega(\Gamma \circ \check{v}_h)$ with $g(0) = e$ (θ is the Maurer-Cartan form). Then $v(t) = \psi_{g(t)}^N \check{v}^H(t)$ is a reconstructed integral curve of Γ .

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The invariant Lagrangian L reduces to a function l on TM/G with $L = l \circ \pi^{TM}$, and the associated Euler-Lagrange field Γ reduces to a vector field $\tilde{\Gamma}$ on TM/G . The fundamental vector field $\tilde{\xi}$ of the action ψ^M on M is the infinitesimal generator of the 1-parameter group of transformations $\psi_{\exp(t\xi)}^M$. The fundamental vector field of the induced action $T\psi_g^M$ on TM is the infinitesimal generator of $T\psi_{\exp(t\xi)}^M$, and is thus the complete lift $\tilde{\xi}^C$. To conclude, for an invariant Lagrangian, we have $\tilde{E}_a^C(L) = 0$ and $[\tilde{E}_a^C, \Gamma] = 0$.

In view of the reconstruction method, we show first that a principal connection Ω^m on $TM \rightarrow TM/G$ is readily available. The coefficients $\partial^2 L / \partial u^\alpha \partial u^\beta$ of the Hessian of L are functions on TM and they form a so-called tensor field g along the tangent bundle projection $\tau : TM \rightarrow M$. (If $W = X^\alpha(x, u)\partial/\partial x^\alpha + Y^\alpha(x, u)\partial/\partial u^\alpha$ is a vector field on TM , then $\tau_*W = X^\alpha(x, u)\partial/\partial x^\alpha$ is a vector field along τ . The Hessian g acts on 2 vector fields along τ .) We will assume that g is non-singular when restricted to two fundamental vector fields. A vector field W on TM is horizontal for the ‘mechanical’ connection if $g(\tau_*W, \tilde{\eta}) = 0$, for all $\eta \in \mathcal{G}$. In case of an invariant Lagrangian, also the Hessian g is invariant (under the appropriate action) and one can verify that the mechanical connection is principal.

To obtain a coordinate expression of the reduced vector field $\tilde{\Gamma}$, we use the expression of the Euler-Lagrange equations in the invariant basis: $\Gamma(X_i^Y(L)) - X_i^C(L) = 0$ and $\Gamma(\tilde{E}_b^Y(L)) - \tilde{E}_b^C(L) = 0$. The basis $\{X_i, \tilde{E}_a\}$ defines so-called quasi-velocities (v^i, w^a) , which are such that $v_m = v^i X_i(m) + w^a \tilde{E}_a(m) \in T_m M$. Since Γ is a second-order differential equation field, it is of the form $\Gamma = w^a \tilde{E}_a^C + v^i X_i^C + \Gamma^a \tilde{E}_a^V + \Gamma^i X_i^V$. By expressing $[\tilde{E}_a^C, \Gamma] = 0$, one finds that $\tilde{E}_a^C(\Gamma^i) = 0$ and $\tilde{E}_a^C(\Gamma^b) = 0$ which means that Γ^i and Γ^b are invariant functions on TM .

From now on, we use coordinates $(x^\alpha) = (x^i, x^a)$ on M s.t. (x^i) are coordinates on U and (x^a) are coordinates on a fibre. The functions x^i, v^i and w^a can be interpreted as invariant functions on TM and induce coordinates on TM/G . Given that the Lie brackets of the basis vector fields are $[\tilde{E}_a, \tilde{E}_b] = C_{ab}^c \tilde{E}_c, [\tilde{E}_a, \tilde{E}_b] = -C_{ab}^c \tilde{E}_c, [\tilde{E}_b, \tilde{E}_a] = 0 = [\tilde{E}_a, X_i], [X_i, X_j] = K_{ij}^a \tilde{E}_a$ (curvature) and $[X_i, \tilde{E}_a] = \gamma_i^b C_{ab}^c \tilde{E}_c$ if $X_i = \partial/\partial x^i - \gamma_i^b(x^i, x^a)\tilde{E}_b$, a small calculation reveals that $T\pi^M \circ \tilde{E}_a^C = (\gamma_i^c C_{ac}^b v^i + C_{ac}^b w^c)\partial/\partial w^b \circ \pi^M, T\pi^M \circ \tilde{E}_a^V = \partial/\partial w^a \circ \pi^M, T\pi^M \circ X_i^C = (\partial/\partial x^i + (-K_{ij}^a v^j + \gamma_i^c C_{cb}^a w^b)\partial/\partial w^a) \circ \pi^M$ and $T\pi^M \circ X_i^V = \partial/\partial v^j \circ \pi^M$. The reduced vector field $\tilde{\Gamma}$ is thus $\tilde{\Gamma} = v^j \partial/\partial x^j + \Gamma^j \partial/\partial v^j + \Gamma^a \partial/\partial w^a$ and the above Euler-Lagrange equations for $L = l \circ \pi^M$ become the so-called Lagrange-Poincaré equations (d/dt stands for $\tilde{\Gamma}$)

$$\frac{d}{dt} \left(\frac{\partial l}{\partial v^i} \right) - \frac{\partial l}{\partial x^i} = (K_{ik}^a v^k + \gamma_i^c C_{bc}^a w^b) \frac{\partial l}{\partial w^b} \quad \text{and} \quad \frac{d}{dt} \left(\frac{\partial l}{\partial w^a} \right) = (\gamma_i^c C_{ac}^b v^i + C_{ac}^b w^c) \frac{\partial l}{\partial w^b}.$$

3 Routh reduction

In this section we use the adapted basis $\{X_i, \tilde{E}_a\}$ and its associated quasi-velocities (v^i, v^a) . The Euler-Lagrange equations become $\Gamma(X_i^Y(L)) - X_i^C(L) = 0$ and $\Gamma(\tilde{E}_b^Y(L)) - \tilde{E}_b^C(L) = 0$. From the last equation it is clear that for an invariant Lagrangian ($\tilde{E}_b^C(L) = 0$) solutions lie on a fixed level set $\tilde{E}_a^V(L) = \mu_a$ of the ‘momentum’ (denoted by N_μ). If we assume that g is non-singular when restricted to two fundamental vector fields, N_μ is a submanifold of TM and we can rewrite $\tilde{E}_a^V(L) = \mu_a$ in the form $v^a = \iota^a$. In that case, there are coefficients A_i^b, B_i^b and C_a^b such that the vector fields $\tilde{X}_i^C = X_i^C + A_i^a \tilde{E}_a^V, \tilde{X}_i^V = X_i^V + B_i^a \tilde{E}_a^V$ and $\tilde{E}_a^C = \tilde{E}_a^C + C_a^b \tilde{E}_b^V$ are tangent to each level set N_μ (they are not complete or vertical lifts). The restriction of the Euler-Lagrange field to N_μ takes therefore the form $\Gamma = \iota^a \tilde{E}_a^C + v^i \tilde{X}_i^C + \Gamma^i \tilde{X}_i^V$. Let $\mathcal{R} = L - v^a \tilde{E}_a^V(L)$ be the Routhian, and \mathcal{R}^μ its restriction to N_μ . The Euler-Lagrange equation in X_i can be cast in the form $\Gamma(\tilde{X}_i^V(\mathcal{R}^\mu)) - \tilde{X}_i^C(\mathcal{R}^\mu) = -\mu_a R_{ij}^a v^j$, where $[X_i, X_j] = R_{ij}^a \tilde{E}_a$ are components of the curvature. If g is non-singular, then so is also $\tilde{X}_i^V \tilde{X}_j^V(\mathcal{R})$ and the above determines the functions Γ^i .

The G -action on TM induces only a G_μ -action on N_μ ($g_\mu \in G_\mu$ if $\text{ad}_{g_\mu}^* \mu = \mu$). When restricted to N_μ, Γ is G_μ -invariant and it reduces to a vector field $\tilde{\Gamma}$ on N_μ/G_μ . Much as before, we can define a principal connection, this time on the principal fibre bundle $N_\mu \rightarrow N_\mu/G_\mu$, and obtain an expression for $\tilde{\Gamma}$ and the reduced ‘Lagrange-Routh’ equations.

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