ERRATA

Page	Line	Reads	Should Read
iii	8	Resoponse	Response
8	1	$\frac{\partial A}{\partial t}$	$\frac{\partial \underline{\underline{A}}}{\partial t}$
8	Eq. (4)	r and r^0	$\underline{r}$ and $\underline{r}^0$
10	13	$j_\lambda^{op}(\underline{r},t)$,	$j_\lambda^{op.}(t)$
10	15	$\underline{J}^{op.}(\underline{r},t)\ \underline{X}_\lambda(\underline{r})$	$\underline{J}^{op.}(\underline{r},t)\cdot\underline{X}_\lambda(\underline{r})$
10	17	$\sum_\lambda$	$\sum_{\lambda'}$
11	6	delete be	
14	4	ttT	t+T
23	19	Eqs. (25)	Eqs. (28)
28	21	U(t-t'),	$U^S(t-t')$
29	4	D(t)	D
44	6	chapter	section
44	12	$i(\cdots)^\tau$	$i(\cdots)_\tau$
44	16	numerator should read	$2\Gamma^+_{M_2M_1}$
47	2	don	do
50	16	posses	possess
51	9	k_λ	$\underline{k}_\lambda$
51	13	k_λ	$\underline{k}_\lambda$
52	11	subscrip	subscript
53	4	and is	and k is

ERRATA (Concluded)

Page	Line	Reads	Should Read
53	9	see Eq. (63b)	see Eq. (65b)
54	5	Eq. (79)	Eq. (79a)
54	7	$\pi\chi_\lambda^2 N$	$2\pi\chi_\lambda^2 N$
55	4	$2\pi N_2\chi_\lambda^2 d_\lambda^2$	$4\pi N_2\chi_\lambda^2 d_\lambda^2$
55	19	" "	" "
58	4	" "	" "
70	7	$\eta_\lambda 2N_2\omega_\lambda^2\chi_\lambda^2 d_\lambda^2\Gamma_o^+$	$\eta_\lambda 4N_2\omega_\lambda^2\chi_\lambda^2 d_\lambda^2\Gamma_o^+$
71	2	apposed	opposed
73	1	$\frac{\eta_\lambda N_2\chi_\lambda^2 d_\lambda^2}{2\Gamma_o^+}$	$\frac{\eta_\lambda N_2\chi_\lambda^2 d_\lambda^2}{\Gamma_o^+}$
73	16	" "	" "
74	7	" "	" "
87	7	Schalow	Schawlow
95	15	complets	completes
112	19	Schalow	Schawlow

THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
Department of Nuclear Engineering

Technical Report

QUANTUM THEORY OF MULTIMODE CAVITIES AND APPLICATION TO THE STEADY-STATE SPECTRUM OF GAS OPTICAL MASERS

P. Lambropoulos
R. K. Osborn

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ABSTRACT

A quantum theoretical treatment of the interaction between the electromagnetic field in a multimode cavity with loss, and an assembly of particles, is presented. The theory is developed in the linear approximation, and the linear susceptibility, which is shown to account for absorption and stimulated emission, is defined. The field is shown to be driven by spontaneous emission from both the active maser material and the loss mechanism, in the absence of any other driving force.

The steady-state power spectrum of an optical maser oscillator is studied. The spectrum is expressed in terms of the spectrum of spontaneous emission from the upper level of the active maser material, and in terms of the susceptibility. Homogeneous broadening, such as natural and collision, and inhomogeneous broadening, such as Doppler and statistical, are taken into consideration. Their effect on the spectrum of spontaneous emission and on the susceptibility is studied. Explicit forms of the susceptibility, valid under various conditions, are obtained. Consideration is also given to a model of a gas optical maser operating in a single mode. Spectrum narrowing is discussed, and a frequency pulling equation, which is found to contain a new term, is obtained.

Comparison to other treatments of the problem is made. It is found that our results are more general in several aspects, and that previous results are obtained as special cases of ours. Possible extensions of the work are also discussed.

CHAPTER I

INTRODUCTION

The extensive literature on the subject eloquently attests to the interest stimulated by the discovery of the maser. As a source of electromagnetic radiation, the maser possesses an unprecedented spectral purity, directionality and intensity, to mention only some of its properties. At the same time, a fairly complex theoretical problem is posed; that is, the description of the electromagnetic field inside a cavity, and its interaction with material systems therein. For certain applications, the properties of the device are much better than needed and the accurate mathematical description appears to be rather unnecessary. It is in part for this reason that severely simplified models seem adequate for the description of the phenomenon. There are certain questions, on the other hand, which make the necessity for a refined and consistent analysis imperative. For example, the coherence time[1,2] of a maser beam is intimately related to the spectral width which requires a fairly accurate mathematical description. In addition, the phenomenon as such has an intrinsic interest independent of the applications.

The present treatment is an attempt to present a deductive derivation of the steady-state spectrum of the power output of a maser oscillator. It is generally agreed that the maser action is a quantum mechanical phenomenon. The mere fact that one deals with emission of radiation by excited atomic systems ought to be fairly convincing in this respect. It is only natural, therefore, that we employ the quantum theoretical formalism. The basic axiom is the choice of the hamiltonian of the system. We take the non-relativistic hamiltonian of an assembly of particles interacting with the radiation field, and with each other. Spin is ignored.

The radiation field is described by the quantum mechanical Maxwell's equations,[3,4] in which the fields and sources are the expectation values of appropriate operators. This enables one to fully account for the quantum effects of the material system to any desired approximation. In Maxwell's theory however, the energy of the field is expressed in terms of the field vectors which are already averages of the respective operators. Quantum mechanically, the energy is given by the expectation values of the squares of those operators.[5] An approximation inherent in Maxwell's theory therefore, is the replacement of averages of products by products of averages. It is presumed that this approximation is justified because of the high photon densities involved in the output of maser oscillators.

The theory has been developed in the linear approximation which leads to the susceptibility function. Mode coupling has been neglected. Among our main concerns has been to avoid introducing phenomenological parameters, especially in studying the interaction of the active maser material with the radiation field. In this effort, damping theory has been used. Although most of the discussion has been devoted to the maser oscillator, the method is applicable to many problems involving the interaction of a material system with the radiation field. In particular, the method illuminates several aspects in the study of the electric susceptibility; namely, the effect of broadening mechanisms.[6]

Chapter II is devoted to the formulation of the problem. The output spectrum is defined Chapter III, in terms of Fourier transforms of truncated functions. Equations for the field operators are developed, and their Fourier transforms are expressed in terms of the Fourier transform of the current operator, thereby reducing the problem to the calculation of the Fourier transform of the current operator $G_\lambda(t)$. The time evolution of $G_\lambda(t)$ is taken up in Chapter IV section 1. An integral equation for the current operator is developed and iterated. In IV-2 the linear approximation is introduced. This linearization generates the response function, for the calculation of which damping theory is used. The result is specialized to a two-level system, in IV-3. The susceptibility, representing the effect of the material system on the field, is defined. In IV-4, the spectrum of spontaneous emission from the upper to the lower level of the two-level system is derived. It is shown that the spontaneous emission drives the field, in the absence of any other driving force. The output spectrum is then expressed in terms of the spectrum of spontaneous emission and the susceptibility. The approximation made in replacing the average of a function by the function of the averages is discussed in IV-5. This approximation is used in Chapter V which is devoted to the application of the results obtained thus far to a model for the gas optical maser. As compared to an actual commercial device, the model is rather idealized. However, devices satisfying several of the conditions imposed by the model can be and have been constructed.[7] The spectrum of spontaneous emission and various forms of the susceptibility are calculated. The effect of various broadening mechanisms is discussed. Moreover, the output spectrum for operation in a single mode is calculated and a frequency pulling equation is derived. In section 6 of this chapter we elaborate somewhat on a quantum mechanical description of the loss along lines similar to those of Ref. 8. Chapter VI is devoted to the comparison of the present theory to other treatments. Finally, the main conclusions are summarized in Chapter VII where possible extensions of the work are also discussed in brief.

The use of damping theory brings this work to close relationship with the work of A. Z. Akcasu[9,10] on the applications of damping theory to the study of line shape. We have developed the formalism in parallel to that of the above references, especially Ref. 9. Expressions for shifts and widths, not dwelt upon here, can be obtained with little or no change therefrom.

The present treatment, being a linearized theory of a non-linear system, cannot answer questions concerning the behavior of the oscillator beyond threshold. For example, it would predict that, when the gain exceeds the losses, the power increases without limit. It is known however, that existing non-linearities stabilize the system. The general formulation nonetheless contains this non-linear behavior.

A question of terminology arises in connection with the frequency pulling equation in section 5 of Chapter V. According to the accepted electrical engineering terminology, the frequency shift due to the detuning between cavity resonance and active material resonance should be called frequency pulling; the shift which depends on the population inversion, and hence on external action (pump action), should be called frequency pushing.

Throughout this study, the term maser will be used in a broad sense covering all maser oscillators irrespective of frequency. Whenever we refer to optical frequencies, we shall use the term "optical maser."

CHAPTER II

FORMULATION OF THE PROBLEM

The system we propose to study consists of an assembly of material particles placed inside a cavity with highly reflecting walls. The cavity may be a more or less closed structure, of the type encountered in microwave applications,[11] or an open structure of the Fabry-Perot type.[12,13,14] In either case, the electromagnetic field inside the cavity is describable in terms of normal modes. A normal mode is a certain spatial distribution of the field vectors which is determined by the geometry of the cavity and the nature of the walls, through the eigenfunctions of a boundary value problem. In the absence of any dissipative interaction, the modes exhibit a harmonic time dependence with characteristic frequencies determined by the eigenvalues of the above boundary value problem. Dissipation (or loss) may result from imperfect reflection at the walls and/or from the geometry of the cavity. The diffraction loss around the edges of the plates of a Fabry-Perot resonator, for example, is an inherently geometric loss. Even in a totally enclosed microwave cavity with perfect walls an opening is necessary for providing coupling with the external world. The effect of this opening is to introduce loss. Thus, in any realistic cavity some loss will be present. As a consequence of the loss, the time dependence of the modes is not purely harmonic but a more complicated function of time. For sufficiently small loss however the Fourier spectra of the modes are peaked functions of frequency, centered at the characteristic frequencies of the cavity.[15]

If $\underline{\mathcal{E}}(\underline{r},t)$ and $\underline{\mathcal{H}}(\underline{r},t)$ are the electric and magnetic fields, respectively, inside the cavity, their time evolution is governed by Maxwell's equations which in Gaussian units read[5]:

$$\underline{\nabla} \times \underline{\mathcal{H}} - \frac{1}{c}\frac{\partial \underline{\mathcal{E}}}{\partial t} = \frac{4\pi}{c} \underline{J}\,(\underline{r},t), \tag{1a}$$

$$\underline{\nabla} \times \underline{\mathcal{E}} + \frac{1}{c}\frac{\partial \underline{\mathcal{H}}}{\partial t} = 0, \tag{1b}$$

$$\underline{\nabla} \cdot \underline{\mathcal{E}} = 4\pi\rho, \tag{1c}$$

$$\underline{\nabla} \cdot \underline{\mathcal{H}} = 0 \;. \tag{1d}$$

where $\underline{J}$ and ρ are the macroscopic current and charge densities, respectively, which shall be referred to as the source terms, and c is the speed of light.

If the macroscopic charge density is zero inside the cavity, one can dispense with the longitudinal part of the electric field and the current. Since this will be the case in all problems to be considered here, $\underline{\mathcal{E}}(\underline{r},t)$ and $\underline{J}\,(\underline{r},t)$ shall indicate the transverse parts of the electric field and the current, respectively, in the remainder of this treatment. Then, the fields can be derived from the vector potential $\underline{A}\,(\underline{r},t)$ through the equations

$$\underline{\mathcal{H}} = \underline{\nabla} \times \underline{A} \;, \tag{2a}$$

$$\underline{\mathcal{E}} = -\frac{1}{c}\frac{\partial A}{\partial t} \tag{2b}$$

It follows from Maxwell's equations that $\underline{A}$ is governed by the equation

$$\nabla^2 \underline{A} - \frac{1}{c^2}\frac{\partial^2 \underline{A}}{\partial t^2} = -\frac{4\pi}{c}\underline{J} . \tag{2c}$$

The field vectors, $\underline{A}$ $(\underline{r},t)$, and $\underline{J}$ $(\underline{r},t)$ are interpreted as the expectation values of appropriate quantum mechanical operators that is,

$$\underline{\mathcal{E}}\,(\underline{r},t) = \mathrm{Tr}\, D\, \underline{\mathcal{E}}^{op.}\,(t), \tag{3a}$$

$$\underline{\mathcal{H}}\,(\underline{r},t) = \mathrm{Tr}\, D\, \underline{\mathcal{H}}^{op.}\,(t), \tag{3b}$$

$$\underline{J}\,(\underline{r},t) = \mathrm{Tr}\, D\, \underline{J}^{op.}\,(t), \tag{3c}$$

$$\underline{A}\,(\underline{r},t) = \mathrm{Tr}\, D\, \underline{A}^{op.}\,(t), \tag{3d}$$

where D is the density operator of the system,[16] and the superscript op. indicates that the quantity it qualifies is an operator. The current operator is given by[18]

$$\underline{J}^{op.}\,(\underline{r},t) = \sum_{\sigma} \frac{e_\sigma}{2m_\sigma}\left\{\left(\underline{p}_\sigma - \frac{e_\sigma}{c}\underline{A}^{op.}\,(r^\sigma)\right)\delta\,(r-r^\sigma) + \right.$$

$$\left. + \delta\,(r-r^\sigma)\left(\underline{p}_\sigma - \frac{e_\sigma}{c}\underline{A}^{op.}\,(r^\sigma)\right)\right\}, \tag{4}$$

where e_σ and m_σ are the charge and mass, respectively, of the σth particle. The summation extends over all particles present inside the cavity and the walls. $\underline{r}^\sigma$ and $\underline{p}^\sigma$ are the position and momentum operators, respectively, of the σth particle.

The time evolution of the operators is governed by Heisenberg's equation of motion. For $\underline{\mathcal{E}}^{op.}$ (t), for example, we have

$$\frac{\partial \underline{\xi}^{op.}(t)}{\partial t} = \frac{i}{h} \left[H, \underline{\xi}^{op.}(t) \right] , \tag{5}$$

where H is the total hamiltonian of the system. The non-relativistic hamiltonian of the system considered here is[2]

$$H = H^R + \sum_{o} \frac{1}{2m_o} \left[\underline{p}_o - \frac{e_o}{c} \underline{A}^{op.}(\underline{r}^o) \right]^2 + H^I , \tag{6}$$

where H^R is the hamiltonian of the free radiation field, and H^I is a term containing particle-particle interactions, interaction between particles and external fields, as well as the hamiltonian of the external world. External fields are to be distinguished from the cavity field. For example, pumping fields are to be considered as external fields as long as they do not have frequency components close to the cavity frequencies of interest, which we assume to be the case.

Let $\{ \underline{X}_\lambda (\underline{r}) \}$, $\lambda = 1,2,3, \ldots$ be a set of eigenvectors satisfying the equations

$$\nabla^2 \underline{X}_\lambda(\underline{r}) + k_\lambda^2 \underline{X}_\lambda(\underline{r}) = 0, \tag{7a}$$

$$\underline{\nabla} \cdot \underline{X}_\lambda(\underline{r}) = 0 , \tag{7b}$$

and boundary conditions appropriate to the cavity under consideration. The domain of definition of these eigenvectors is the interior of the cavity. The k_λ's are the eigenvalues. For a certain class of boundary conditions of interest to us here, the set $\{\underline{X}_\lambda\}$ is orthogonal, real and complete.[15] It will be assumed normalized to unity. Moreover, the set $\{ \frac{1}{k_\lambda} \underline{\nabla} \times \underline{X}_\lambda (\underline{r})\}$ is orthonormal if $\{\underline{X}_\lambda\}$ is orthonormal. Expanding the vector potential, the current, and the field operators in terms of $\{\underline{X}_\lambda\}$ we have

$$\underline{A}^{op.}(\underline{r},t) = \sum_\lambda \frac{\sqrt{4\pi}}{k_\lambda} P_\lambda (t) \underline{X}_\lambda (\underline{r}), \tag{8a}$$

$$\underline{J}^{op.}(\underline{r},t) = \sum_\lambda j_\lambda^{op.}(t) \underline{X}_\lambda (\underline{r}), \tag{8b}$$

$$\underline{\mathcal{E}}^{op.}(\underline{r},t) = \sum_\lambda \sqrt{4\pi}\, \omega_\lambda Q_\lambda (t) \underline{X}_\lambda (\underline{r}), \tag{8c}$$

$$\underline{\mathcal{H}}^{op.}(\underline{r},t) = \sum_\lambda \frac{\sqrt{4\pi}}{k_\lambda} P_\lambda (t) \left(\underline{\nabla} \times \underline{X}_\lambda (\underline{r})\right), \tag{8d}$$

where

$$\omega_\lambda \equiv c\, k_\lambda , \tag{8e}$$

and P_λ, Q_λ are hermitian operators which obey the commutation relations

$$[Q_\lambda, P_{\lambda'}] = i \hbar\, \delta_{\lambda\lambda'}, \tag{9a}$$

$$[Q_\lambda, Q_{\lambda'}] = [P_\lambda, P_{\lambda'}] = 0 . \tag{9b}$$

The operator $j_\lambda^{op.}(\underline{r},t)$ results from the modal decomposition of $\underline{J}^{op.}(\underline{r},t)$, and is given by

$$j_\lambda^{op.}(t) = \int_V \underline{J}^{op.}(\underline{r},t)\, \underline{X}_\lambda (\underline{r})\, d^3 \underline{r} =$$

$$= \sum_\sigma \frac{e_\sigma}{m_\sigma} \underline{X}_\lambda (\underline{r}^\sigma) \cdot \underline{p}_\sigma -$$

$$- \sum_{\lambda'} P_{\lambda'} \frac{\sqrt{4\pi}}{k_{\lambda'}} \sum_\sigma \frac{e^2_\sigma}{m_\sigma c} \underline{X}_\lambda (\underline{r}^\sigma) \cdot \underline{X}_{\lambda'} (r^\sigma) . \tag{10a}$$

The second term in the right side of the above equation gives rise to mode coupling since it contains a summation over all modes. Since we wish to confine the present treatment to the case in which mode coupling can be neglected, we shall neglect all terms for which $\lambda' \neq \lambda$. The remaining term (for $\lambda' = \lambda$) represents a small correction to be the frequency ω_λ of the passive cavity, and we shall assume that ω_λ is so redefined as to incorporate this correction. Thus, we shall take

$$j_\lambda^{op.} = \frac{\omega_\lambda}{\sqrt{4\pi}} G_\lambda , \tag{10b}$$

where, in order to compress notation, we have introduced the operator

$$G_\lambda \equiv \frac{\sqrt{4\pi}}{\omega_\lambda} \sum_o \frac{e_o}{m_o} \underline{X}_\lambda (r^o) \cdot \underline{p}_o \quad . \tag{10c}$$

Combining now Eqs. (2), which are also true in operator form, and Eqs. (8) we obtain

$$\frac{\partial^2}{\partial t^2} P_\lambda (t) + \omega_\lambda^2 P_\lambda (t) = \omega_\lambda^2 G_\lambda(t), \tag{11a}$$

and

$$\frac{\partial}{\partial t} P_\lambda (t) + \omega_\lambda^2 Q_\lambda (t) = 0 . \tag{11b}$$

We shall adopt the Heisenberg picture[18] and hence all operators will be time dependent, their time dependence being determined by the time evolution operator

$$U(t,t_o) \equiv e^{-\frac{i}{\hbar}(t-t_o) H} . \tag{12a}$$

Then, for example,

$$G_\lambda (t) \;=\; U^\dagger(t,t_o)\, G_\lambda (t_o)\; U(t,t_o)\;. \qquad (12b)$$

In the calculations we shall choose $t_o = 0$ and shall omit the time argument whenever it is zero.

CHAPTER III

THE OUTPUT SPECTRUM

In interpreting energy transfer experiments, one would like to calculate the quantity

$$\frac{1}{8\pi} \text{Tr } D \left(\underline{\mathcal{E}}^{\text{op}^2} + \underline{\mathcal{H}}^{\text{op}^2} \right). \tag{13a}$$

This, in general, requires the solution of transport equations whose very formulation is not an easy task.[17] For relatively high field densities however, (13a) can be approximated by

$$\frac{1}{8\pi} \left(\underline{\mathcal{E}}^2 + \underline{\mathcal{H}}^2 \right) \quad , \tag{13b}$$

and this we assume to be the case in this study. Despite this approximation, the Maxwell's equations do provide a quantum mechanical description of the system if the source terms are interpreted correctly, that is as the expectation values of the appropriate operators.

Under the assumption that the above approximation is satisfactory, the energy output per unit time, that is the power output, is given by

$$\sum_{\lambda} \eta_\lambda \, p_\lambda^2 \, (t), \tag{14a}$$

where

$$p_\lambda \, (t) \; \equiv \; \text{Tr } D \, P_\lambda \, (t), \tag{14b}$$

and η_λ is a constant having the dimensions of inverse time, and relating the power output of the λth mode to the energy in that mode stored inside the cavity. In general, the power output will be a function of time. The system shall be said to be in steady state if W_T, defined by

$$W_T \equiv \frac{1}{T}\int_t^{t+T} \sum_\lambda \eta_\lambda \, p_\lambda^2 (t') \, dt', \tag{15a}$$

is independent of t for T large enough. The lower limit of T is determined by the characteristic times of the physical processes taking place inside the cavity. Ultimately we shall approximate the power output by

$$W \equiv \lim_{T\to\infty} W_T. \tag{15b}$$

It should be emphasized that W_T has a physical content, while W is a mathematical quantity by which we approximate W_T. Confining the present treatment to the steady-state we take t = 0 in Eq. (15a). Thus W becomes

$$W = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_\lambda \eta_\lambda \, p_\lambda^2 (t') \, dt'. \tag{15c}$$

We now define

$$p_{\lambda T} (\omega) \equiv \int_0^T p_\lambda (t) \, e^{-i\omega t} \, dt, \tag{16a}$$

which is the Fourier transform of a truncated function[1] equal to $p_\lambda (t)$ for $0 < t < T$ and zero otherwise. In terms of Fourier transforms W_T reads:

$$W_T = \sum_\lambda \frac{\eta_\lambda}{2\pi} \frac{1}{T} \int_{-\infty}^{+\infty} | \, p_{\lambda T} (\omega) |^2 \, d\omega. \tag{16b}$$

Since $p_\lambda(t)$ is assumed to be a real valued function, we shall have

$$p_{\lambda T}(-\omega) = p^*_{\lambda T}(\omega), \tag{16c}$$

where * denotes the complex conjugate. Thus, Eq. (16b) becomes

$$W_T = \sum_\lambda \frac{\eta_\lambda}{\pi T} \int_0^\infty |p_{\lambda T}(\omega)|^2 \, d\omega. \tag{16d}$$

Introducing

$$R_{\lambda T}(\omega) \equiv \frac{1}{\pi T} |p_{\lambda T}(\omega)|^2, \tag{17a}$$

and

$$R_\lambda(\omega) \equiv \lim_{T\to\infty} R_{\lambda T}(\omega), \tag{17b}$$

we obtain

$$W_T = \sum_\lambda \eta_\lambda \int_0^\infty R_{\lambda T}(\omega) \, d\omega, \tag{17c}$$

and

$$W = \sum_\lambda \eta_\lambda \int_0^\infty R_\lambda(\omega) \, d\omega. \tag{17d}$$

$R_\lambda(\omega)$ is identified with the steady-state power spectral density of the λth mode. The steady-state power spectral density of the output is

$$R(\omega) = \sum_\lambda \eta_\lambda R_\lambda(\omega) . \tag{18}$$

Taking the Fourier transform of Eq. (14b) we obtain

$$p_{\lambda T}(\omega) = \mathrm{Tr}\, D \int_0^T P_\lambda(t)\, e^{-i\omega t} \, dt = \mathrm{Tr}\, D\, P_{\lambda T}(\omega), \tag{19}$$

where we have interchanged trace with integration, and we have introduced the operator $P_{\lambda T}(\omega)$ defined as the Fourier transform of $P_\lambda(t)$. Combining Eqs. (16c) and (19) we have

$$|p_{\lambda T}(\omega)|^2 = \left(\mathrm{Tr}\ D\ P_{\lambda T}(\omega)\right)\left(\mathrm{Tr}\ D\ P_{\lambda T}(-\omega)\right).$$

Assuming that we are dealing with field densities high enough to justify the replacement of products of averages by averages of products, and vice versa, we take

$$|p_{\lambda T}(\omega)|^2 = \mathrm{Tr}\ D\ P_{\lambda T}(\omega)\ P_{\lambda T}(-\omega)\ . \tag{20}$$

Then, from Eqs. (17b) and (18) we obtain

$$R(\omega) = \sum_\lambda \eta_\lambda \lim_{T\to\infty} \frac{1}{\pi T}\ \mathrm{Tr}\ D\ P_{\lambda T}(\omega)\ P_{\lambda T}(-\omega), \tag{21}$$

thereby reducing the problem to the calculation of $P_{\lambda T}(\omega)$.

The motion of $P_\lambda(t)$ is governed by Eq.(11a). The operator $G_\lambda(t)$ appearing in the right hand side of this equation contains the coupling between the cavity field and the particle system. Note that G_λ, as defined by Eq. (10c), involves a summation over all particles present. It is convenient at this point to separate G_λ in two parts as follows:

$$G_\lambda \equiv G_\lambda^L + G_\lambda^A, \tag{22}$$

where G_λ^L is an operator involving a summation over the particles of the wall or any other passive material that may exist inside the cavity, while G_λ^A involves a summation over the particles of what we shall call the active material. The latter is the material whose presence gives rise to the maser action and whose quantum effects on the field we wish to study.

Both $G_\lambda^L(t)$ and $G_\lambda^A(t)$ can be expressed in the form of a perturbation expansion in ascending powers of the field operators. If one retains the first two terms of the expansion and then take the Fourier transform of Eq. (11a), after some mathematical manipulations, the following equation is obtained:

$$(-\omega^2 + \omega_\lambda^2)\, P_{\lambda T}(\omega) = \omega_\lambda^2\, G_{\lambda T}^{LS}(\omega) + \omega_\lambda^2\, G_{\lambda T}^{AS}(\omega) -$$

$$-i\omega\gamma_\lambda\, P_{\lambda T}(\omega) + \omega_\lambda^2\, Y_\lambda(\omega)\, P_{\lambda T}(\omega). \qquad (23)$$

The first two terms on the right hand side are independent of the field and account for spontaneous emission from the passive and active material, respectively. The third term, involving the constant γ_λ, accounts for the dissipative effect of the passive material (usually referred to as the loss mechanism) on the field. The constant γ_λ depends on parameters such as the conductivity of the passive material. The fourth term accounts for the effect of the active material on the field. The function $Y_\lambda(\omega)$ is the linear susceptibility and contains the effect of induced emission and absorption. Once the functions on the right side of Eq. (23) are known, one can solve for $P_{\lambda T}(\omega)$, substitute into Eq. (21) and obtain an expression for the output spectrum. Thus, the main task in the remainder of this study will be the determination of the foregoing functions in terms of the dynamical parameters of the system.

Since the main objective of this treatment is the optical maser, it is presumed that spontaneous emission from a material in thermodynamic equilibrium at room termperature will be rather inconsequential at optical frequencies. For this reason, and in order to avoid mathematical complexity we shall neglect G_{λ}^{LS}. The term involving γ_{λ} shall be kept however, since it has an important effect on the output spectrum. Both G_{λ}^{LS} and γ_{λ} have been discussed in Ref. 8, although not in Fourier domain and in a somewhat less general context. A brief discussion is presented in Chapter V, section 6. It should be mentioned that two assumptions inherent in describing the loss in terms of γ_{λ} are: the loss is small, that is

$$\frac{\gamma_{\lambda}}{\omega_{\lambda}} \ll 1 \quad ; \tag{24}$$

and mode coupling due to the loss mechanism can be neglected. It is also assumed that there is no interaction between loss mechanism and active material. At optical frequencies, and with gaseous materials in Fabry-Perot cavities, the foregoing conditions are satisfied. For solid state materials, the loss mechanism and the active material might be correlated. This case is not taken up here. For lower frequencies, spontaneous emission from the passive material (usually referred to as thermal noise) may also be of importance. Then, one will have to retain the term $G_{\lambda T}^{LS}(\omega)$ as well (see also Chapter V-6).

The problem is now effectively reduced to studying the motion of $G_{\lambda}^{A}(t)$, and determining $G_{\lambda T}^{AS}(\omega)$ and $Y_{\lambda}(\omega)$. Since G_{λ}^{L} has been dispensed with, we shall omit the superscript A. Thus, the equation we now have is

$$(-\omega^2 + i\omega\gamma_{\lambda} + \omega_{\lambda}^2)\ P_{\lambda T}(\omega) \ = \ \omega_{\lambda}^2 \int_0^T G_{\lambda}(t)\ e^{-i\omega t}\, dt, \qquad (25)$$

where $G_{\lambda}(t)$ corresponds to the active material only.

In conventional classical electrodynamics, one usually assumes that $Y_{\lambda}(\omega)$ has a real and an imaginary part. The dependence on ω is then taken to be of the form (see for example Ref. 6)

$$\frac{1}{\omega-\omega_0-i\Gamma} \; .$$

The constant Γ is introduced in order to account for losses associated with the susceptibility, and ω_0 is a transition frequency characteristic of the material in question. Of course, this form is valid for values of ω near ω_0. The question arises however, as to when the above form is valid, and how one can determine Γ in terms of the dynamical parameters of the system. The question presents itself also in connection with the spectrum of spontaneous emission. It is precisely these questions that we attempt to answer herein.

CHAPTER IV

METHOD OF SOLUTION

1. TIME EVOLUTION OF G_λ

The time evolution of $G_\lambda(t)$ is governed by the equation

$$\frac{\partial}{\partial t} G_\lambda(t) = \frac{i}{\hbar} [H, G_\lambda(t)], \tag{26}$$

where H is as given by Eq. (6). In order to proceed further a more detailed specification of the system, and hence the hamiltonian is necessary. Thus, assume that the cavity contains two kinds of materials: The active material whose atoms and/or molecules are capable of making radiative transitions, and a second material which shall be termed "the perturber." Both are assumed to be in the gaseous state. The perturber does not interact with the cavity field but it does interact with the active material through collisions. Let H^P be the hamiltonian of the perturber, and V^{PA} the energy of interaction between the active material and the perturber. Note that if collisions with the walls are of importance in perturbing the active material, the hamiltonian of the wall should be included in H^P.

Using the expansion of the vector potential operator in terms of $\{\underline{X}_\lambda\}$, and neglecting the term $\sum_\sigma \frac{e_\sigma^2}{2m_\sigma c^2} \underline{A}^2(r^\sigma)$ as representing effects of higher order, we can write the hamiltonian in the form

$$H = H^E + H^R + H^A + H^P + V^{RA} + V^{PA} + V^{EA}, \tag{27}$$

where H^E is that part of the hamiltonian of the external world which is coupled to the active material, that is the hamiltonian of the pumping mechanism; H^R is the hamiltonian of the free cavity field and is given by

$$H^R = \frac{1}{2}\sum_\lambda (P_\lambda^2 + \omega_\lambda^2 Q_\lambda^2); \tag{28a}$$

H^A is the hamiltonian of the active material; V^{RA} is the energy of interaction between H^R and H^A, and is given by

$$V^{RA} = -\sum_\lambda P_\lambda G_\lambda ; \tag{28b}$$

V^{EA} represents the coupling between pumping mechanism and active material. Recall that G_λ is expressed as a sum over all particles of the active material (see Eq. (10c)), that is over electrons as well as nuclei. Since the materials actually used in cavities consist of atoms and/or molecules, the sum will have to be regrouped into partial sums each of which will represent the particles of one atom or molecule. This however, does not have to be done until a later stage (see also Appendix B).

Combining now Eqs. (26) and (27) we obtain

$$\frac{\partial}{\partial t} G_\lambda(t) = \frac{i}{\hbar}[H^E(t) + V^{EA}(t), G_\lambda(t)] + \tag{29}$$
$$+ \frac{i}{\hbar}[H^\pi(t), G_\lambda(t)] + \frac{i}{\hbar}[H^R(t) + H^P(t), G_\lambda(t)],$$

where we have introduced

$$H^{\pi} \equiv H^{A} + V^{RA} + V^{PA} . \tag{30}$$

The first commutator represents the effect of the pumping mechanism. Since we are interested only in the steady-state, we shall ignore this commutator and account for its effect by assuming that the populations of the states of the active material are kept at certain constant values by means of sufficient pumping. The approximation involved is the neglect of the details of the pumping mechanism, and is useful as long as one is interested in the steady-state spectrum only.

At this point, motivated by Senitzky's work,[8] we consider the equation of motion of $H^{\pi}(t)$, namely

$$\frac{\partial}{\partial t} H^{\pi}(t) = \frac{i}{\hbar} [H, H^{\pi}(t)] =$$

$$= \frac{i}{\hbar} [H^{E}(t) + H^{P}(t) + V^{EA}(t), H^{\pi}(t)] +$$

$$+ \frac{i}{\hbar} [H^{R}(t), H^{\pi}(t)] . \tag{31}$$

Neglecting the first commutator and calculating the second using Eqs. (28) and (30), we obtain

$$\frac{\partial}{\partial t} H^{\pi}(t) = \frac{i}{\hbar} [H^{R}(t), V^{RA}(t)] . \tag{32}$$

Integrating formally we have

$$H^{\pi}(t) = H^{\pi}(o) + \frac{i}{\hbar} \int_{o}^{t} dt_1 [H^{R}(t_1), V^{RA}(t_1)] . \tag{33}$$

Finally, in Eq. (29), we replace $H^R(t)$ and $H^P(t)$ in the third commutator by $H^R(o)$ and $H^P(o)$. Then, using also Eq. (33), we obtain

$$\frac{\partial}{\partial t} G_\lambda(t) = \frac{i}{\hbar} [H^\pi(o) + H^R(o) + H^P(o), G_\lambda(t)] - $$

$$- \frac{1}{\hbar^2} \int_0^t dt_1 \left[[H^R(t_1), V^{RA}(t_1)], G_\lambda(t) \right] . \tag{34a}$$

We now introduce the operator H^S defined by

$$H^S \equiv H^\pi + H^R + H^P =$$

$$= H^A + H^R + H^P + V^{RA} + V^{PA} . \tag{34b}$$

For future use we also define

$$H^O \equiv H^A + H^R + H^P , \tag{34c}$$

and

$$V \equiv V^{RA} + V^{PA} . \tag{34d}$$

Then, H^S reads

$$H^S = H^O + V . \tag{34e}$$

An integral equation equivalent to the integrodifferential equation (34a) is

$$G_\lambda(t) = e^{\frac{i}{\hbar} H^S t} G_\lambda e^{-\frac{i}{\hbar} H^S t} -$$

$$- \frac{1}{\hbar^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \, e^{\frac{i}{\hbar} (t-t_1) H^S} \left[[H^R(t_2), V^{RA}(t_2)], G_\lambda(t_1) \right] \cdot$$

$$\cdot \, e^{-\frac{i}{\hbar} (t-t_1) H^S} , \tag{35a}$$

where operators without time argument are to be understood at t = 0. From Eqs. (25), and the commutation relations for P_λ and Q_λ we have

$$[H^R(t_2), V^{RA}(t_2)] = -\sum_{\lambda'} \omega_{\lambda'}^2 \, i\hbar \, Q_{\lambda'}(t_2) \, G_{\lambda'}(t_2) \, . \quad (35b)$$

Combining Eqs. (35) we obtain

$$G_\lambda(t) = G_\lambda^S(t) + \frac{i}{\hbar}\sum_{\lambda'} \omega_{\lambda'}^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, U^{S\dagger}(t-t_1) \cdot$$

$$\cdot \, [Q_{\lambda'}(t_2) \, G_{\lambda'}(t_2), G_\lambda(t_1)] \, U^S(t-t_1) \, , \quad (36)$$

where we have introduced

$$U^S(t) \equiv e^{-\frac{i}{\hbar} H^S t} \, , \quad (37a)$$

and

$$G_\lambda^S(t) \equiv U^{S\dagger}(t) \, G_\lambda U^S(t) \, . \quad (37b)$$

Equation (36) represents a set of, in principle, infinitely many coupled integral equations. Since this expression is to be substituted into Eq. (25), it is obvious that we shall have infinitely many, coupled equations for $P_{\lambda T}(\omega)$ ($\lambda = 1,2,3...$). In any actual maser oscillator only a finite number of modes oscillate simultaneously.[20,21] This reduces the set of equations to a finite set which still are coupled. Under certain conditions furthermore, the modes may oscillate independently of each other. Limiting this treatment to the case in which the coupling terms can be neglected, we drop from the sum in Eq. (36) all except the λth terms. Then, we have

$$G_\lambda(t) = G_\lambda^S(t) + \frac{i}{\hbar}\omega_\lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, U^{S\dagger}(t-t_1) \, \cdot \quad (38)$$

$$\cdot [Q_\lambda(t_2)\ G_\lambda(t_2),\ G_\lambda(t_1)]\ \overset{S}{U}(t-t_1). \tag{38}$$

Relatively little can be done with this integral equation without further approximations. Thus we resort to an iteration procedure, iterating the equation once and retaining terms up to and including the second order in $G_\lambda^S(t)$. Then, we obtain

$$G_\lambda(t) = G_\lambda^S(t) + \frac{i}{\hbar}\,\omega_\lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2\ U^{S\dagger}(t-t_1)\cdot$$

$$\cdot [Q_\lambda(t_2)\ G_\lambda^S(t_2),\ G_\lambda^S(t_1)]\ U^S(t-t_1)\ . \tag{39}$$

Substituting into Eq. (25) we have

$$(-\omega^2 + i\omega\gamma_\lambda + \omega_\lambda^2)\ P_{\lambda T}(\omega) = \omega_\lambda^2\ G_{\lambda T}^S(\omega) +$$

$$+ \frac{i}{\hbar}\,\omega_\lambda^4 \int_0^T dt\ e^{-i\omega t} \int_0^t dt_1 \int_0^{t_1} dt_2\ U^{S\dagger}(t-t_1)\cdot$$

$$\cdot [Q_\lambda(t_2)\ G_\lambda^S(t_2),\ G_\lambda^S(t_1)]\ U^S(t-t_1), \tag{40a}$$

where we have introduced

$$G_{\lambda T}^S(\omega) \equiv \int_0^T dt\ G_\lambda^S(t)\ e^{-i\omega t}. \tag{40b}$$

The crux of the problem is the handling of the right hand side of Eq. (40a). As will be shown subsequently, the first term corresponds to spontaneous transitions from the excited levels of the active material. The second term represents the response of the material system to the field, in the linear approximation. That is after the operator $Q_\lambda(t_2)$ is taken out of the commutator, as discussed in the following section.

It is perhaps in place to note the difference between the time dependence of operators bearing the superscript S and those which do not. The former's time dependence is determined by $U^S(t) = e^{-\frac{i}{\hbar} H^S t}$, while the latter's is by the time evolution operator for the whole system, namely $U(t) = e^{-\frac{i}{\hbar} Ht}$.

2. THE RESPONSE FUNCTION

Since we wish to confine this treatment to the linear approximation, we write the integrand of the right hand side of Eq. (40a) as follows:

$$Q_\lambda(t_2)\; U^{S\dagger}(t-t_1)\; [G_\lambda^S(t_2),\, G_\lambda^S(t_1)]\; U^S(t-t_1)\ . \qquad (41)$$

In doing so we have neglected the commutator of $Q_\lambda(t_2)$ and $U^S(t-t_1)$. This commutator can be neglected in the zeroeth order approximation, i.e. in the absence of any interaction between field and material system. In the next order approximation, the commutator yields terms linear in G_λ and consequently the corresponding term in (41) will be of third order in G_λ. The approximation involved in (41) therefore, is to neglect terms of order higher than the second in G_λ, consistently with our previous assumptions.

In view of the fact that ultimately we shall take the trace with the density operator, we introduce the function $\Lambda(t,t_1,t_2)$ defined by

$$\Lambda\,(t,t_1,t_2)\ \equiv\ \mathrm{Tr}\, D\, U^{S\dagger}(t-t_1)\; [G_\lambda^S(t_2),\, G_\lambda^S(t_1)]\; U^S(t-t_1), \qquad (42)$$

and we replace (41) by

$$Q_\lambda(t_2) \quad \Lambda(t,t_1,t_2). \tag{43}$$

Formally, the approximation involved in replacing (41) by (43), is replacing the average of a product by the product of the averages. This is done in a way such that the field operator is separated from the particle operators. The resulting function $\Lambda(t,t_1,t_2)$ is a response function representing the effect of the material system on the field. Again quantum effects of order higher than the second in G_λ^S are neglected. The equations for the field operators thus obtained are linear and as will be seen subsequently they account for spontaneous emission, induced emission, and absorption. This is essentially the dielectric approximation.[22] As long as more than one photon processes do not play an important role, the formalism is expected to be adequate for the study of the power spectrum. If photon scattering, for example, becomes important, which may be the case with some solid state materials, a major revision will be necessary.

The approximation is somewhat similar to the irreversibility approximation[16] frequently used in the description of systems in contact with a thermostat, where the interaction is assumed to be small enough (or the thermostat large enough) for the effect of the system of interest on the thermostat to be negligible. Here, the situation is different in several aspects. The active material, which corresponds to the thermostat, is neither in thermal equilibrium nor is it a large system. In fact, for maser action to take place, it is necessary that the level populations be inverted. However, if we consider as thermostat the active material plus the pumping mechanism, then we do have a large system. Moreover, we may assume that the level populations of the active material are kept at a desired value through sufficient pumping. Thus, the approximation underlying (43) and the subsequent calculations essentially involves the deletion of information concerning the pumping mechanism and the build up of the oscillations, thereby restricting the present treatment to the steady-state. Incidentally, it is important to note that an additional difference from the thermodynamic problem is that here we have a steady but not an equilibrium state.

Substituting (43) into Eq. (40a) we obtain

$$(-\omega^2 + i\omega\gamma_\lambda + \omega_\lambda^2)\, P_{\lambda T}(\omega) = \omega_\lambda^2\, \overset{S}{G}_{\lambda T}(\omega) + $$
$$+ \frac{i}{h}\, \omega_\lambda^4 \int_0^T dt\, e^{-i\omega t} \int_0^t dt_1 \int_0^{t_1} dt_2\, Q_\lambda(t_2)\, \Lambda\,(t,t_1,t_2), \qquad (44)$$

from which one recognizes that $\Lambda(t,t_1,t_2)$ is a functional relating the effect of the material system on the field at all times previous to t, to the field at time t.

Turning now to the calculation of $\Lambda(t,t_1,t_2)$, and using Eqs. (37) and the identity

$$U(t-t') = U^S(t)\, U^{S\dagger}(t'),$$

we obtain

$$\Lambda(t,t_1,t_2) = \Lambda(t,t_1-t_2) =$$

$$\mathrm{Tr}\ U^S(t)\ D\ U^{S\dagger}(t)\ U^S(t_1-t_2)\ G_\lambda\ U^{S\dagger}(t_1-t_2)\ G_\lambda -$$

$$- \mathrm{Tr}\ U^S(t)\ D(t)\ U^{S\dagger}(t)\ G_\lambda\ U^S(t_1-t_2)\ G_\lambda\ U^{S\dagger}(t_1-t_2), \qquad (45)$$

where we have used the identity

$$\mathrm{Tr}\ A\ B = \mathrm{Tr}\ B\ A,$$

which is valid for any two operators A, B. Noting that the second term in the right hand side of Eq. (45) is the complex conjugate of the first, because the corresponding operators are the hermitian adjoints of each other, and setting

$$U^S(t)\ D\ U^{S\dagger}(t) = D^S(t), \qquad (46a)$$

we obtain

$$\Lambda(t,t_1-t_2) = 2i\ \mathrm{Im}\ \mathrm{Tr}\ D^S(t)\ U^S(t_1-t_2)\ G_\lambda\ U^{S\dagger}(t_1-t_2)\ G_\lambda, \qquad (46b)$$

where Im indicates the imaginary part. As indicated above, Λ depends on the difference t_1-t_2 and not on the specific values of t_1 and t_2.

Let now $|\alpha>$ be a representation diagonalizing H^A, $|p>$ a representation diagonalizing H^P, and E_α, E_p the corresponding energy eigenvalues. Then

$$H^A|\alpha> = E_\alpha|\alpha>, \qquad (47a)$$

and

$$H^P|p> \;=\; E_p|p>. \tag{47b}$$

The spectrum of H^P is assumed to be continuous. In addition, let $|\eta>$ be the representation diagonalizing H^R, and let $|o>$ be the vacuum photon field state.[5] We introduce the representation

$$|M> \;\equiv\; |\alpha>|p>|o>, \tag{48a}$$

and we proceed to calculate the trace in Eq. (46b) using this representation. Thus, we have

$$\Lambda(t,t_1-t_2) \;=\; 2i\ \mathrm{Im} \sum_{MM_1M_2M_3} D^S_{MM}(t)\ U^S_{MM_1}(t_1-t_2)$$

$$G_{\lambda,M_1M_2}\ U^{S\dagger}_{M_2M_3}(t_1-t_2)\ G_{\lambda,M_3M}, \tag{48b}$$

where we have neglected the off-diagonal matrix elements of D, and have introduced the simpler notation G_{λ,MM_1} instead of $< M|G_\lambda|M_1 >$.

It is assumed that the eigenvalue problems (47a) and (47b) can be solved and that the corresponding eigenfunctions are known to us. Then, the matrix elements of G_λ can be calculated. We shall have the occasion to elaborate on this point in considerable detail at a later stage (see also Appendix B). The remaining problem is the calculation of the matrix elements of $U^S(t)$. By definition $U^S(t) = -\frac{i}{\hbar} H^S t$, where $H^S = H^o + V$, $H^o = H^A + H^P + H^R$ and $V = V^{RA} + V^{PA}$. As is readily verified, H^o is diagonal in the representation $|M>$.

The calculation of the matrix elements of $U^S(t)$ in this representation is precisely the problem solved by damping theory. As shown in Appendix A, the matrix elements of $U^S(t)$, for $t \geq 0$, are given by

$$U^S_{MM}(t) = e^{-\frac{i}{\hbar}(E_M + s_M - i\gamma_M)\,t}, \qquad (49a)$$

and

$$U^S_{MM'}(t) = V_{MM'}\int_0^t U^S_{MM}(t-\tau)\, U^S_{M'M'}(\tau)\, d\tau, \qquad (49b)$$

where

$$E_M = E_\alpha + E_p, \qquad (49c)$$

$$s_M(\chi) = V_{MM} + PP\sum_{M'\neq M} \frac{|V_{M'M}|^2}{\chi - E_{M'}}, \qquad (49d)$$

$$\gamma_M(\chi) = \frac{2\pi}{\hbar}\sum_{M'\neq M} |V_{M'M}|^2\, \delta(\chi - E_{M'}), \qquad (49e)$$

$$s_M = \hbar\, s_M(E_M), \qquad (49f)$$

$$\gamma_M = \hbar\, \gamma_M(E_M). \qquad (49g)$$

In Eq. (49d), PP indicates that the Cauchy principal part is to be taken whenever an integration over χ is performed. It is important to note that γ_M is non-negative. The quantities s_M and γ_M represent the shift and width of the energy of the state $|M>$ caused by the interaction with the perturber and the vacuum field. A comprehensive discussion of these quantities has been presented by Akcasu.[9,10]

As is seen from Eq. (49b), the off-diagonal matrix elements of $U^S(t)$ are linear in V. If these matrix elements were substituted into Eq. (48b), they would yield terms of at least second order in V^{PA}. However, Eq. (48b) is already of second order in G_λ. Assuming that both of these coupling constants are small quantities, we shall neglect the off-diagonal matrix elements of $U^S(t)$. Then, Eq. (48b) becomes

$$\Lambda(t,t_1-t_2) = 2i\ \mathrm{Im}\sum_{MM_1} D^S_{MM}(t)\ U^S_{MM}(t_1-t_2)$$

$$G_{\lambda,MM_1}\ U^{S\dagger}_{M_1M_1}(t_1-t_2)\ G_{\lambda,M_1M} =$$

$$= 2i\mathrm{Im}\sum_{MM_1} D^S_{MM}(t)\ U^S_{MM}(t_1-t_2)\ U^{S\dagger}_{M_1M_1}(t_1-t_2)\ |G_{\lambda,MM_1}|^2. \qquad (50)$$

Introducing the symbols

$$\omega_M \equiv \frac{E_M}{\hbar},\ \Gamma_M \equiv \frac{\gamma_M}{\hbar},\ S_M \equiv \frac{\Delta_M}{\hbar}, \qquad (51)$$

and using Eq. (49a) we obtain

$$\Lambda(t,t_1-t_2) = 2i\ \mathrm{Im}\sum_{MM_1} D_{MM}\,|G_{\lambda,MM_1}|^2 \qquad (52)$$

$$e^{-i(\omega_M + S_M - i\Gamma_M)(t_1-t_2)}\ e^{i(\omega_{M_1} + S_{M_1} + i\Gamma_{M_1})(t_1-t_2)}\ .$$

Note that for Eq. (49a) to apply we must have $t \geq 0$. The difference (t_1-t_2) appearing in the above equation must, therefore, be non-negative. That this is indeed the case can be readily verified if it is recalled that in Eq. (44) $\Lambda(t,t_1-t_2)$ appears in the integrand of a double integral whose limits of integration are such that $t_1 \geq t_2$.

It is perhaps desirable at this point to iterate some of the physical ideas underlying the calculations presented in this chapter. In calculating the trace, we have neglected the excited states $|\eta >$ of the photon field and have retained only the vacuum state. The necessity for considering the vacuum field stems from the fact that its coupling to a particle system does give rise to a shift and width of the energy levels of the system. The excited states, on the other hand, have been neglected from Eq. (48b) since we wish to confine this treatment to the linear approximation. Indeed, the excited states would yield terms proportional to $< \alpha p\eta|D|\alpha p\eta >$ that is, proportional to the number of photons present. These terms give rise to terms non-linear in Q_λ when substituted into Eq. (44). Thus, what we essentially do is to consider the vacuum and excited fields as two separate dynamical systems, up to a certain point. The excited field is described in terms of $p_\lambda(t) =$ Tr D $P_\lambda(t)$ and $q_\lambda(t) =$ Tr D $Q_\lambda(t)$ which are expectation values of operators The vacuum field cannot be described in terms of expectation values of the field operators. It is taken into consideration in so far as it affects the material system. As will be seen subsequently, its effect appears as a shift and width in the spectrum of spontaneous emission and in the susceptibility. Analogous effects are caused by the perturber. In subsequent chapters, we study these effects in considerable detail for a material system with two internal energy states, usually referred to as a two-level system.

3. THE SUSCEPTIBILITY FUNCTION OF A TWO-LEVEL SYSTEM

The results of the preceding section are now applied to the case of a two-level system which in fact is the central objective of this study. Physically, a two-level system corresponds to an atomic or molecular system, whose transition frequency between two particular levels is close to the frequency of interest, the other transition frequencies being much different. By frequency of interest, we mean the frequency of that mode of the relevant cavity which has the lowest loss. Then, we may disregard the other states of the system and treat it as a two-level system. It is important to note however, that both levels are excited levels, in general, and have finite lifetimes. Ideally, we would desire a four-level system with energies $E_0 < E_1 < E_2 < E_3$. E_0 would be the ground state energy. The pumping would take place from the ground state to $|3\rangle$. The transition $3 \rightarrow 2$ should be very fast compared to $2 \rightarrow 1$ (typically by one order of magnitude), and presumably non-radiative. This scheme would reduce the possibility of saturation of the pumping mechanism, as well as of interference between pumping and cavity fields. The maser action would take place between $|2\rangle$ and $|1\rangle$.

Here, we consider the simpler case of a two-level system, these two levels referring to internal degrees of freedom of the atom. The atom as a whole is subject to thermal motion. To account for the effect of this motion we separate the hamiltonian H^A into two parts as follows:

$$H^A \equiv H^{AI} + H^{Ae} \quad . \tag{53}$$

H^{AI} refers to the internal and H^{Ae} to the external (center of mass) degrees of freedom. Let $|m>$ and $|K>$ be defined by

$$H^{AI}\ |m> \ \equiv \ E_m\ |m>, \tag{54a}$$

and

$$H^{Ae}\ |K> \ \equiv \ E_K\ |K>. \tag{54b}$$

The eigenvector $|\alpha>$ is now written

$$|\alpha> \ = \ |m>|K>, \tag{55a}$$

where

$$E_\alpha \ = \ E_m + E_K\ . \tag{55b}$$

We now assume that H^{AI} possesses only two eigenstates represented by $|1>$ and $|2>$. Their energies will be denoted by E_1 and E_2, where $E_2 > E_1$. The eigenvector $|K>$ is left unspecified for the moment. For the sake of mathematical simplicity we shall assume that the energy eigenstates $|1>$ and $|2>$ are non-degenerate. The presence of degeneracy does not affect the qualitative conclusions and can be handled without difficulty as discussed in Ref. 9, for example. The vector $|M>$ is now written

$$|M> \ = \ |m>|K>|p>|0>, \tag{56}$$

where $m = 1,2$.

Invoking the steady-state assumption we replace $D^S(t)$ by D and assume that the latter can be written as follows:

$$D = \sum_{mKp} |0>|m>|K>|p> D_{mm}\, D_{KK}\, D_{pp} < p| \\ <K|<m|<0|\ . \tag{57a}$$

That is, we assume that the populations of the particle states are kept constant through external means. The effect of the pumping mechanism is accounted for by assuming certain values for D_{mm} in steady-state. The above assumption about D will be used in the calculation of $\Lambda(t,t_1-t_2)$ and the calculation of the spectrum of spontaneous emission, because in both cases only the vacuum field is considered. When we write Tr D $Q_\lambda(t)$, we shall mean the complete density operator of the system. However, its knowledge is not necessary for our purposes since we have equations for the quantities Tr D $Q_\lambda(t)$ and Tr D $P_\lambda(t)$ themselves.

To simplify notation, we denote D_{22} and D_{11} by D_2 and D_1 respectively. These quantities represent the expected values of the populations of the respective levels in steady-state. Using now Eqs. (56) and (57a) we obtain

$$D_{MM} = D_{mm}\, D_{KK}\, D_{pp}. \tag{57b}$$

Noting that G_λ is diagonal in $|p>$ since it does not contain any perturber operators, and that $E_M = E_m + E_K + E_p$, Eq. (52) becomes

$$\Lambda(t,t_1-t_2) = 2i\ \mathrm{Im} \sum_{\substack{mm_1 \\ KK_1p}} D_{mm}\, D_{KK}\, D_{pp} |G_{\lambda;mKm_1K_1}|^2$$

$$e^{-i(\omega_m + \omega_K + \omega_p + S_M - i\Gamma_M)(t_1-t_2)}$$

$$e^{i(\omega_{m_1} + \omega_{K_1} + \omega_p + S_{M_1} + i\Gamma_{M_1})(t_1-t_2)}, \tag{58}$$

where M and M_1 are abbreviations for mKp0 and m_1K_1p0 respectively. Note that p is the same in both. The symbols ω_m, ω_K and ω_p are defined by

$$\omega_m = \frac{E_m}{\hbar}, \; \omega_K = \frac{E_K}{\hbar}, \; \omega_p = \frac{E_p}{\hbar}, \tag{59}$$

consistently with Eq. (51).

As shown in Appendix B, G_λ can be written as follows:

$$G_\lambda = \frac{\sqrt{4\pi}}{\omega_\lambda} \underline{d} \cdot \sum_j \underline{X}_\lambda(\underline{R}j), \tag{60}$$

where $\underline{d}$ is the dynamic electric dipole moment operator and operates only on the internal degrees of freedom; $\underline{R}j$ is the position operator of the center of mass of the jth atom, and the summation extends over all atoms of the active material. In deriving Eq. (60) it has been assumed that $\underline{X}_\lambda(\underline{r})$ does not vary appreciably over the dimensions of the atom, which is essentially the dipole approximation. If we denote the polarization vector of the λth mode by $\underline{\epsilon}_\lambda$, then

$$\underline{X}_\lambda(\underline{R}j) = \underline{\epsilon}_\lambda X_\lambda(\underline{R}j), \tag{61a}$$

and Eq. (60) becomes

$$G_\lambda = \frac{\sqrt{4\pi}}{\omega_\lambda} (\underline{d} \cdot \underline{\epsilon}_\lambda) \sum_j X_\lambda(\underline{R}j). \tag{61b}$$

Assuming that the atoms of the active material are uncorrelated, one can show (see Appendix B) that

$$|G_{\lambda;mKm_1K_1}|^2 = \frac{4\pi}{\omega_\lambda^2} |\langle m|\underline{d}\cdot\underline{\epsilon}_\lambda|m_1\rangle|^2 \sum_j |\langle K|X_\lambda(\underline{R}j)|K_1\rangle|^2. \tag{62}$$

To compress writing we introduce

$$d^2_{\lambda,mm_1} \equiv |\langle m|\underline{d}\cdot\underline{\epsilon}_\lambda|m_1\rangle|^2, \tag{63a}$$

and

$$X^2_{\lambda j,KK_1} \equiv \frac{4\pi}{\omega^2_\lambda} |\langle K|X(\underline{R}j)|K_1\rangle|^2. \tag{63b}$$

Using Eqs. (62) and (63), and noting that ω_p cancels in Eq. (58), the latter becomes

$$\Lambda(t,t_1-t_2) = 2i\ \mathrm{Im}\sum_{mm_1=1}^{2} D_{mm}\, d^2_{\lambda,mm_1}$$

$$\sum_{jKK_1p} D_{KK}\, D_{pp}\, X^2_{\lambda jKK_1}\, e^{-i(\omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} - i\Gamma^+_{MM_1})\, t_1}$$

$$e^{i(\omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} - i\Gamma^+_{MM_1})\, t_2}\ , \tag{64}$$

where we have defined

$$\omega_{mm_1} \equiv \omega_m - \omega_{m_1}, \tag{65a}$$

$$\omega_{KK_1} \equiv \omega_K - \omega_{K_1}, \tag{65b}$$

$$S^-_{MM_1} \equiv S_M - S_{M_1}, \tag{65c}$$

$$\Gamma^+_{MM_1} \equiv \Gamma_M + \Gamma_{M_1}\ . \tag{65d}$$

Substituting Eq. (64) into the second term in the right hand side of Eq. (44), and after a lengthy calculation we obtain

$$\frac{i\omega^4_\lambda}{\hbar}\int_0^T dt\ e^{-i\omega t}\int_0^t dt_1\int_0^{t_1} dt_2\ Q_\lambda(t_2)\,\Lambda(t,t_1-t_2) =$$

$$= \frac{i\omega^4_\lambda}{\hbar}\, Q_{\lambda T}(\omega)\sum_{mm_1=1}^{2} D_{mm}\, d^2_{\lambda,mm_1}\sum_{jKK_1p} D_{KK}\, D_{pp}\, X^2_{\lambda j,KK_1}$$

$$\{[(\omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} - i\Gamma^+_{MM_1})\ (\omega + \omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} - i\Gamma^+_{MM_1})]^{-1} -$$

$$-[(\omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} + i\Gamma^+_{MM_1})(-\omega + \omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} + i\Gamma^+_{MM_1})]^{-1}\}. \tag{66}$$

From Eq. (11b), taking Fourier transforms, we have

$$Q_{\lambda T}\ (\omega) = -\frac{i\omega}{\omega_\lambda^2}\ P_{\lambda T}\ (\omega). \tag{67}$$

Using this equation, and defining

$$Y_\lambda(\omega) = \frac{\omega}{\hbar}\sum_{mm_1=1}^{2} D_{mm}\ d^2_{\lambda,mm_1} \sum_{jKK_1p} D_{KK}\ D_{pp}\ X^2_{\lambda j,KK_1}$$

$$\{[(\omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} - i\Gamma^+_{MM_1})\ (\omega + \omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} - i\Gamma^+_{MM_1})]^{-1} -$$

$$-[(\omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} + i\Gamma^+_{MM_1})\ (-\omega + \omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1} + i\Gamma^+_{MM_1})]^{-1}\}, \tag{68a}$$

Eq. (66) becomes

$$\frac{i\omega_\lambda^4}{\hbar}\int_0^T dt\ e^{-i\omega t}\int_0^t dt_1\int_0^{t_2} Q_\lambda(t_2)\ \Lambda(t,t_1,t_2) =$$

$$= \omega_\lambda^2 Y_\lambda(\omega)P_{\lambda T}(\omega). \tag{68b}$$

$Y_\lambda(\omega)$ is the susceptibility function corresponding to the λth mode. Its value at ω represents the effect of the material system on the ωth Fourier component of the field. The basic steps of the calculation of the triple integral, leading to Eq. (66), are presented in Appendix C. Two approximations have been made: Terms containing the damping factor $e^{-\Gamma^+_{MM_1}T}$ have been neglected, in view of the fact that ultimately we shall take the limit for $T \to \infty$. Moreover, we have neglected terms of the order of $\omega_{mm_1}^{-1}$ as compared to terms of the order of $(\omega-\omega_{mm_1})^{-1}$. This is justified by the fact that we are dealing with frequencies of the order of 10^{10}-10^{15} cps, and narrow spectra (ω_{KK_1} and $S^-_{MM_1}$ are small shifts).

Although the summation over m,m_1 in Eq. (68a) extends from 1 to 2, the equation can be applied to the case of more that two internal states as well. The spacing of the levels however, would have to be small compared to ω_{mm_1}, because otherwise the second of the above assumptions would not be justified.

Recall that $\underline{d}$ is the dynamic electric dipole moment operator. Assuming that the atoms of the active material do not exhibit a permanent electric dipole moment, in either of the two states, the diagonal matrix elements of d_λ will vanish. Introducing the simpler notation

$$d_\lambda^2 \equiv d^2_{\lambda,12} = d^2_{\lambda,21} , \tag{69a}$$

and

$$\omega_0 \equiv \omega_{21} = -\omega_{12} , \tag{69b}$$

recalling that M stands for $mKp0$, and M_1 for m_1K_1p0, and performing the summation over m, m_1 in Eq. (68a), we obtain

$$Y_\lambda(\omega) = \frac{\omega}{\hbar} d_\lambda^2 \sum_{jKK_1p} D_{KK} D_{pp} X^2_{\lambda j,KK_1} \cdot$$

$$\cdot\Big[\frac{1}{(\omega_0+\omega_{KK_1}+ S^-_{2Kpo,1K_1po} + i\,\Gamma^+_{2Kpo,1K_1po})} \cdot$$

$$\cdot \frac{D_2}{(\omega-\omega_0-\omega_{KK_1} - S^-_{2Kpo,1K_1po} -i\,\Gamma^+_{2Kpo,1K_1po})} -$$

$$- \frac{1}{(\omega_o - \omega_{KK_1} - S^{-}_{1Kpo,2K_1po} + i\,\Gamma^{+}_{1Kpo,2K_1po})} \cdot$$

$$\cdot \frac{D_1}{(\omega - \omega_o + \omega_{KK_1} + S^{-}_{1Kpo,2K_1po} - i\,\Gamma^{+}_{1Kpo,2K_1po})} \;] \,. \qquad (70)$$

Again, we have neglected terms of the order of $(\omega + \omega_o)^{-1}$ as compared to terms of the order of $(\omega - \omega_o)^{-1}$. The above equation can be simplified somewhat if one notes that the shift S, as well as ω_{KK_1}, are small quantities as compared to ω_o. Thus we may replace $\omega_o + \omega_{KK_1} + S^{-}_{2Kpo,1K_1po} + i\,\Gamma^{+}_{2Kpo,1K_1po}$ by $\omega_o + i\,\Gamma^{+}_{2Kpo,1K_1po}$, and $\omega_o - \omega_{KK_1} - S^{-}_{1Kpo,2K_1po} + i\,\Gamma^{+}_{1Kpo,2K_1po}$ by $\omega_o + i\,\Gamma^{+}_{1Kpo,2K_1po}$ in the denominators of Eq. (70). Note that the same approximation cannot be made in the remaining factors because there, the quantities S and ω_{KK_1} are compared to $(\omega - \omega_o)$ which is of the same order. Upon making the above approximations, and noting that

$$S^{-}_{1Kpo,2K_1po} = - S^{-}_{2K_1po,1Kpo}, \qquad (71a)$$

and

$$\Gamma^{+}_{1Kpo,2K_1po} = \Gamma^{+}_{2K_1po,1Kpo}, \qquad (71b)$$

Eq. (70) simplifies to

$$Y_\lambda(\omega) = \frac{\omega}{\hbar} d_\lambda^2 \sum_{jKK_1p} D_{KK}\, D_{pp}\, X^2_{\lambda j,KK_1} \cdot$$

$$\cdot [\, \frac{D_2}{(\omega_o + i\Gamma^{+}_{2Kpo,1K_1po})\,(\omega - \omega_o - \omega_{KK_1} - S^{-}_{2Kpo,1K_1po} - i\Gamma^{+}_{2Kpo,1K_1po})} -$$

$$- \frac{D_1}{(\omega_o + i\Gamma^{+}_{2K_1po,1Kpo})\,(\omega - \omega_o + \omega_{KK_1} - S^{-}_{2K_1po,1Kpo} - i\Gamma^{+}_{2K_1po,1Kpo})} \,] \qquad (72)$$

The remaining task is to average over the states of the center of mass and the perturber.

4. POWER SPECTRAL DENSITY OF SPONTANEOUS TRANSITIONS

We have found that $P_{\lambda T}(\omega)$ obeys Eq. (44) which in terms of the susceptibility function reads

$$(-\omega^2+i\omega\gamma_\lambda+\omega_\lambda^2-\omega_\lambda^2 Y_\lambda(\omega))\, P_{\lambda T}(\omega) = \omega_\lambda^2\, G^S_{\lambda T}(\omega). \qquad (73)$$

The power output spectrum is

$$R(\omega) = \sum_\lambda \eta_\lambda\, R_\lambda(\omega)$$

where

$$R_\lambda(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} \mathrm{Tr}\, D\, P_{\lambda T}(\omega)\, P_{\lambda T}(-\omega). \qquad (74b)$$

From Eqs. (73) and (74b) we obtain

$$R_\lambda(\omega) = \frac{\omega_\lambda^4 \lim\limits_{T\to\infty} \frac{1}{T} \mathrm{Tr}\, D\, G^S_{\lambda T}(\omega)\, G^S_{\lambda T}(-\omega)}{\pi\left|-\omega^2+i\omega\gamma_\lambda+\omega_\lambda^2 - \omega_\lambda^2 Y_\lambda(\omega)\right|^2}. \qquad (75)$$

The numerator represents the power spectral density of spontaneous transitions, as will be shown in this section. It provides the force that drives the field, since no other driving force has been assumed, which is the case in actual maser oscillators. If an additional driving force, such as an external field, is present, its power spectral density should be added to the numerator.

Using Eq. (40b), we obtain,

$$\mathrm{TrD}G^{S}_{\lambda T}(\omega)G^{S}_{\lambda T}(-\omega) = \mathrm{TrD}\int_0^T\int_0^T dt\,dt'\; e^{-i\omega(t-t')}G^{S}_{\lambda}(t)G^{S}_{\lambda}(t'). \tag{76}$$

To insure that the right hand side remains real after the transformation to follow we write it as follows:

$$\mathrm{TrD}G^{S}_{\lambda T}(\omega)G^{S}_{\lambda}(-\omega) = \frac{1}{2}\,\mathrm{TrD}\int_0^T\int_0^T dt\,dt'[e^{-i\omega(t-t')}G^{S}_{\lambda}(t)G^{S}_{\lambda}(t') + $$
$$+ e^{i\omega(t-t')}G^{S}_{\lambda}(t')G^{S}_{\lambda}(t)].$$

Introducing a new variable τ, defined by $t \equiv t'+\tau$, and after some manipulations, we obtain

$$\mathrm{TrD}G^{S}_{\lambda T}(\omega)G^{S}_{\lambda T}(-\omega) = \mathrm{Re}\int_0^T d\tau(T-\tau)\; e^{-i\omega\tau}\;[\mathrm{TrD}G^{S}_{\lambda}(\tau)G_{\lambda} + $$
$$+ \mathrm{TrD}G_{\lambda}G^{S}_{\lambda}(-\tau)],$$

where we have interchanged trace with integration and Re indicates the real part. Introducing

$$I_{\lambda}(\omega) \equiv \lim_{T\to\infty}\frac{1}{T}\,\mathrm{TrD}G^{S}_{\lambda T}(\omega)G^{S}_{\lambda T}(-\omega), \tag{77a}$$

in order to compress writing, and making use of the steady-state assumption, we obtain

$$I_{\lambda}(\omega) = 2\lim_{T\to\infty}\mathrm{Re}\int_0^T d\tau e^{-i\omega\tau}\,\mathrm{TrD}G^{S}_{\lambda}(\tau)G_{\lambda}. \tag{77b}$$

From the definition of $G_\lambda^S(\tau)$ we have,

$$\mathrm{Tr}\, D\, G_\lambda^S(\tau)\, G_\lambda =$$

$$= \sum_{MM_1M_2M_3} D_{MM}\, U^{S+}_{MM_1}(\tau)\, G_{\lambda,M_1M_2}\, U^S_{M_2M_3}(\tau)\, G_{\lambda,M_3M}, \tag{78a}$$

where we have neglected the off-diagonal matrix elements of the density operator, and $|M\rangle$ is as defined by Eq. (56). The subsequent calculations and approximations are parallel to those of the preceding chapter; namely, we retain only the diagonal matrix elements of $U^S(\tau)$ and use the results of damping theory; we assume that G_λ is off-diagonal in the representation $|m\rangle$, and that D can be written as in Eq. (57a). Then, for a two-level system, we obtain

$$\mathrm{Tr}\, D\, G_\lambda^S(\tau)\, G_\lambda = \sum_{mm_1=1}^{2} D_{mm}\, d^2_{\lambda,mm_1} \sum_{jKK_1p} D_{KK}\, D_{pp}\, X^2_{\lambda j,KK_1}$$

$$e^{i(\omega_{mm_1} + \omega_{KK_1} + S^-_{mKpo,m_1K_1po} + i\Gamma^+_{mKpo,m_1K_1po})\tau} . \tag{78b}$$

Substituting into Eq. (77b) and neglecting the term which is proportional to $(\omega+\omega_o)^{-1}$ we obtain

$$I_\lambda(\omega) = D_2\, d^2_\lambda \sum_{jKK_1p} D_{KK}\, D_{pp}\, X^2_{\lambda j,KK_1}$$

$$\frac{\Gamma^+_{M_2M_1}}{(\omega-\omega_o-\omega_{KK_1}-S^-_{M_2M_1})^2 + (\Gamma^+_{M_2M_1})^2} , \tag{79a}$$

where

$$|M_2\rangle = |2\rangle|K\rangle|p\rangle|o\rangle, \tag{79b}$$

and

$$|M_1\rangle = |1\rangle|K_1\rangle|p\rangle|o\rangle. \tag{79c}$$

$I_\lambda(\omega)$ is the spectrum of spontaneous transitions $|2\rangle \rightarrow |1\rangle$ into the λth mode. As can be seen from Eqs. (73) and (75), the spontaneous emission is the force driving the field. At low values of population inversion, that is of the quantity $(D_2 - D_1)$, the spontaneous emission spectrum has a dominant effect on the spectrum of the output. As the degree of inversion increases, the induced emission takes over. For a quantitative discussion see section 5 of Chapter V.

The spontaneous emission spectrum is represented by Eq. (79a) as a formal average over the states of the center of mass of the active material, and the states of the perturber. A similar average appears in Eq. (72) which represents the susceptibility function of a two-level system. In most practical problems, one has to resort to numerical calculation in order to perform the averages. For a gaseous active material and a Fabry-Perot cavity however, the calculation is simplified considerably. As a consequence, one is able to obtain useful results in a more or less closed form, as shown subsequently.

From Eqs. (75) and (77a) we have

$$R_\lambda(\omega) = \frac{\omega_\lambda^4}{\pi} \frac{I_\lambda(\omega)}{|-\omega^2 + i\omega\gamma_\lambda + \omega_\lambda^2 - \omega_\lambda^2 Y_\lambda(\omega)|^2} . \tag{80}$$

Thus the output spectrum is expressed, through Eqs. (74a) and (80), in terms of the spectrum of spontaneous emission and the susceptibility function. We shall now use these results to study the spectrum of a gas optical maser, in steady-state.

5. STATISTICAL APPROXIMATION

Before embarking on the calculation of the averages in Eqs. (72) and (79a) we discuss briefly the approximation involved in replacing the average of a function by the function of the averages. This approximation shall be referred to as the statistical approximation. Following Akcasu[9] we consider a function Z = f(A,B) where A and B are functions of some set τ of stochastic variables. It is assumed that we have a probability distribution $P(\tau)$ defined on τ. The mean value of Z is then defined by

$$\bar{Z} = \int f(A,B)\, P\,(\tau)\, d\tau. \tag{81}$$

The mean values of A and B, which are denoted by $\bar{A}$ and $\bar{B}$, are defined in a similar fashion. If α and β are the deviations of A and B from their mean values we shall have

$$A = \bar{A} + \alpha, \tag{82a}$$

and

$$B = \bar{B} + \beta. \tag{82b}$$

Expanding Z in a Taylor series we obtain

$$\bar{Z} = f(\bar{A},\bar{B}) + \frac{1}{2}\left[\overline{\alpha^2}\left(\frac{\partial^2 f}{\partial A^2}\right)_{\bar{A},\bar{B}} + \right.$$

$$+ 2\,\overline{\alpha\beta}\left(\frac{\partial^2 f}{\partial A \partial B}\right)_{\bar{A},\bar{B}} + \overline{\beta^2}\left(\frac{\partial^2 f}{\partial B^2}\right)_{\bar{A},\bar{B}}\Bigg] + \ldots \tag{83a}$$

Terms linear in α and β don not appear since the mean values of α and β vanish by definition. Here, we shall assume that the mean value of $\alpha\,\beta$ also vanishes. This is the case for example, when α and β depend on different sets of stochastic variables. Thus we take

$$\bar{Z} = f(\bar{A},\bar{B}) + \frac{1}{2}\left[\overline{\alpha^2}\left(\frac{\partial^2 f}{\partial A^2}\right)_{\bar{A},\bar{B}} + \overline{\beta^2}\left(\frac{\partial^2 f}{\partial B^2}\right)_{\bar{A},\bar{B}}\right] . \tag{83b}$$

We apply now this result to two functions which will be of interest to us in connection with the spontaneous emission spectrum and the susceptibility function. First we consider the function Z_1 defined by

$$Z_1 = \frac{A}{B^2+A^2} . \tag{84}$$

Using Eq. (83b) we obtain

$$\bar{Z}_1 = \frac{\bar{A}}{\bar{B}^2+\bar{A}^2}\left[1 - (\overline{\beta^2}-\overline{\alpha^2})\,\frac{\bar{A}^2-3\bar{B}^2}{(\bar{A}^2+\bar{B}^2)^2}\right] . \tag{85a}$$

Assuming that the second term inside the square brackets is small compared to 1, and using the approximation $(1-x) \simeq (1+x)^{-1}$, the above equation becomes

$$\overline{Z}_1 = \frac{\overline{A}}{\overline{B}^2\left[1 - \frac{3(\overline{\beta^2}-\overline{\alpha^2})}{\overline{A}^2+\overline{B}^2}\right] + \overline{A}^2\left[1 + \frac{\overline{\beta^2}-\overline{\alpha^2}}{\overline{A}^2+\overline{B}^2}\right]} \quad . \tag{85b}$$

If it can be assumed that the correction terms are small we have

$$\overline{Z}_1 = \frac{\overline{A}}{\overline{A}^2+\overline{B}^2} , \tag{86a}$$

which is replacing the average of the function Z_1 by the function of the averages. If however, we replace the average of Z_1 by the ratio of the average, that is if we take $\overline{Z}_1 \simeq \frac{\overline{A}}{\overline{B^2}+\overline{A^2}}$, we obtain

$$\overline{Z}_1 = \frac{\overline{A}}{\overline{B}^2 + \overline{A}^2 + \overline{\beta^2}} , \tag{86b}$$

where we have assumed that $\overline{\alpha^2} \ll \overline{A}^2$. As discussed in Ref. 9, Eq. (86b) is a better approximation than Eq. (86a), and it is the former that we shall use in this treatment. In any event, one can go back and use Eq. (85b) if greater precision is desired.

We now consider the function

$$Z_2 = \frac{1}{B-iA} . \tag{87}$$

Using Eq. (83b) we find

$$\overline{Z}_2 = \frac{1}{\overline{B}-i\overline{A}}\left[1 + \frac{\overline{\beta^2}}{(\overline{B}-i\overline{A})^2} - \frac{\overline{\alpha^2}}{(\overline{B}-i\overline{A})^2}\right] . \tag{88}$$

If the correction terms inside the square brackets can be assumed to be small compared to unity then what we obtain is the function of the averages. If this approximation is not satisfactory, which will be the case in the calculation of the susceptibility where we shall have a function of the form Z_2, but we can nevertheless assume that $\overline{\alpha^2} \ll \bar{A}^2$, then we obtain

$$\bar{Z}_2 = \frac{1}{\bar{B}-i\bar{A}} \left[1 + \frac{\overline{\beta^2}}{(\bar{B}-i\bar{A})^2} \right] . \tag{89}$$

In this study we shall use Eqs. (86b) and (89). For further discussion of the statistical approximation in connection with the function Z_1, Ref. 9 should be consulted.

CHAPTER V

APPLICATION TO A GAS OPTICAL MASER

1. THE SPECTRUM OF SPONTANEOUS EMISSION

The model for the gas optical maser we shall consider consists of a tube of length L (typically 100 cm), containing a gaseous active material.[21] The side walls of the tube are transparent to light, while the end plates are highly reflecting, with reflectivity of the order of 99% or better. This structure forms a Fabry-Perot cavity with a high quality factor. It has been shown[12] that, the modes of this cavity which have the lowest loss are the even symmetric modes whose frequencies are

$$\omega_{\lambda} = \frac{\pi\lambda c}{L}, \tag{90}$$

where c is the velocity of light, and λ a large integer of the order of 10^6. For a typical He-Ne optical maser, the separation of these frequencies is of the order of 160 Mc/sec. There are also modes of next lowest loss which posses odd radial symmetry, and their frequencies differ from the frequencies of the previous modes by, typically, 1 Mc/sec. Here, we shall neglect these modes. Moreover, it has been shown that, the transverse field of the even symmetric modes does not vary appreciably over the diameter of the tube, usually of the order of 2-3 cm. These modes correspond to propagation of light along the axis of the tube, and inside the tube one has a standing wave pattern. In order to simplify the analysis, we shall ignore the variation of the mode vectors over the diameter, and shall take

$$\underline{X}_\lambda(\underline{r}) = \underline{\epsilon}_\lambda \chi_\lambda \operatorname{Sin} k_\lambda z, \tag{91a}$$

where χ_λ is a constant normalization factor, and k_λ is the wave number related to the frequency as follows:

$$\omega_\lambda = c\, k_\lambda \,. \tag{91b}$$

The z-axis is taken along the axis of the tube and the x,y-axes on a plane perpendicular to the axis. If we introduce a vector k_λ defined by

$$\underline{k}_\lambda = (o,o,k_\lambda), \tag{91c}$$

where the numbers inside the parenthesis are the cartesian components of k_λ, we shall have

$$\underline{X}_\lambda(\underline{r}) = \underline{\epsilon}_\lambda \frac{\chi_\lambda}{2i} \left(e^{i\underline{k}_\lambda \cdot \underline{r}} - e^{-i\underline{k}_\lambda \cdot \underline{r}}\right). \tag{91d}$$

For each mode, only one polarization is present the other being eliminated by using windows of the Brewster's[1,21] angle type at the ends of the tube.

The active material inside the cavity is assumed to consist of an assembly of uncorrelated atoms (or molecules) whose center of mass is subject to thermal motion. Thus, the states of the center of mass shall be taken to be free particle states given by[5]

$$|K\rangle = (2\pi)^{-3/2} e^{-i\underline{K}\cdot\underline{R}} , \tag{92}$$

where, $\underline{R}$ is the position operator of the center of mass, and $\underline{K}$ the wave vector. The energy of the state $|K\rangle$ is

$$E_K = \frac{\hbar^2 K^2}{2m} , \tag{93}$$

where m is the mass of the atom. Recalling the definition of $X_{\lambda j,KK_1}$ as given by Eq. (63b), and using Eqs. (92) and (91d) we obtain

$$X_{\lambda j,KK_1} = \frac{\chi_\lambda \sqrt{4\pi}}{2i\omega_\lambda} \Big[\delta\,(\underline{K}-\underline{K}_1+\underline{k}_\lambda) - \delta\,(\underline{K}-\underline{K}_1-\underline{k}_\lambda) \Big] . \tag{94a}$$

Note that the subscrip j in the left hand side, which refers to the jth atom, now becomes redundant and will be deleted. Whenever a summation over j occurs, as in Eqs. (72) and (79), it will be replaced by multiplication by N, the number of atoms of the active material. From Eq. (94a) we now have

$$X^2_{\lambda,KK_1} = \frac{\pi\chi^2_\lambda}{\omega^2_\lambda} \Big[\delta\,(\underline{K}-\underline{K}_1+\underline{k}_\lambda) - \delta\,(\underline{K}-\underline{K}_1-\underline{k}_\lambda) \Big] . \tag{94b}$$

Moreover, we shall assume that D_{KK} is a Maxwellian distribution with temperature T, or mean energy $\frac{3}{2}\Theta$, where

$$\Theta = k\,T\,, \tag{95a}$$

and is the Boltzmann's constant. The summation over K is then replaced by integration, according to

$$\sum_K D_{KK} \rightarrow b^3\ \pi^{-3/2} \int d^3\underline{K} e^{-b^2K^2}, \tag{95b}$$

where

$$b^2 = \frac{\hbar^2}{2m\Theta}\,. \tag{95c}$$

Recalling now the definiton of ω_{KK_1} (see Eq. (63b)), and using Eq. (93) we obtain

$$\omega_{KK_1} = \frac{\hbar}{2m}\,(K^2 - K_1^2). \tag{96a}$$

Because of the presence of the delta functions (or Kronecker deltas, if a discrete spectrum of K's is assumed) in Eq. (94b), the only terms that will survive in a summation over K and K_1 are those for which $\underline{K} = \underline{K}_1 \pm \underline{k}_\lambda$. Using this relationship Eq. (96a) yields

$$\omega_{KK_1} = \frac{\hbar\omega_\lambda^2}{2mc^2} \pm \frac{\hbar}{m}\,(\underline{K}_1\cdot\underline{k}_\lambda).$$

Neglecting $h\omega_\lambda^2/2mc^2$ as small compared to ω_λ, and introducing

$$S_d \equiv \frac{\hbar}{m}\,(\underline{K}\cdot\underline{k}_\lambda), \tag{96b}$$

we have

$$\omega_{KK_1} = \pm S_d \, . \tag{96c}$$

In Eq. (96b) we have changed K_1 to K. The same will be done in the summation $\sum_{KK_1}$ from which K has now disappeared.

Introducing the foregoing simplifications into Eq. (79), we obtain

$$I_\lambda(\omega) = D_2 d_\lambda^2 \frac{\pi \chi_\lambda^2 N}{\omega_\lambda^2} \sum_{Kp} D_{KK} D_{pp} \cdot$$

$$\cdot \left[\frac{\Gamma^+_{M_2'M_1'}}{(\omega-\omega_o-S_d-S^-_{M_2'M_1'})^2 + (\Gamma^+_{M_2'M_1'})^2} + \right. \tag{97a}$$

$$+ \left. \frac{\Gamma^+_{M_2'M_1'}}{(\omega-\omega_o+S_d-S^-_{M_2'M_1'})^2 + (\Gamma^+_{M_2'M_1'})^2} \right] ,$$

where

$$|M_2'\rangle = |K\rangle|2\rangle|p\rangle|o\rangle \tag{97b}$$

$$|M_1'\rangle = |K\rangle|1\rangle|p\rangle|o\rangle \tag{97c}$$

An additional approximation has been made in Eq. (97a). It has been assumed that the significant effect of the center of mass motion is contained in S_d, and K_1 has been replaced by K in $\Gamma^+_{M_2M_1}$ and $S^-_{M_2M_1}$.

Since D_{KK} depends only on the magnitude of the vector $\underline{K}$, and since we integrate over all $\underline{K}$-space, both terms inside the square brackets in Eq. (97a) yield the same quantity, when averaged. It suffices therefore to retain one of the two multiplied by two, thus obtaining

$$I_\lambda(\omega) = \frac{2\pi N_2 \chi_\lambda^2 d_\lambda^2}{\omega_\lambda^2} \sum_{Kp} D_{KK}\, D_{pp} \frac{\Gamma^{+}_{M_2'M_1'}}{(\omega-\omega_0-S_d-S^{-}_{M_2'M_1'})^2 + (\Gamma^{+}_{M_2'M_1'})^2}, \tag{97d}$$

where $N_2 = ND_2$ is the expected number of atoms of the active material, in the upper state $|2\rangle$, in steady-state. The above equation gives the spectrum of spontaneous emission as a superposition of Lorentzians. Subsequently, we shall use the statistical approximation to replace the right hand side of (Eq. 97d) by a single Lorentzian.

2. THE SPECTRUM OF SPONTANEOUS EMISSION IN THE STATISTICAL APPROXIMATION

The spectrum $I_\lambda(\omega)$ as represented by Eq. (97d) has the form of z_1 as defined by Eq. (84). Setting

$$A = \Gamma^{+}_{M_2'M_1'}, \tag{98a}$$

and

$$B = \omega-\omega_0-S_d-S^{-}_{M_2'M_1'}, \tag{98b}$$

we have

$$I_\lambda(\omega) = \frac{2\pi N_2 \chi_\lambda^2 d_\lambda^2}{\omega_\lambda^2} \sum_{Kp} D_{KK} D_{pp} \frac{A}{B^2+A^2}. \tag{99}$$

We now introduce

$$\Gamma_2 \equiv \sum_{Kp} D_{KK} D_{pp} \Gamma_{M_2'} = \sum_{Kp} D_{KK} D_{pp} \Gamma_{K2po} , \tag{100a}$$

$$\Gamma_1 \equiv \sum_{Kp} D_{KK} D_{pp} \Gamma_{M_1'} = \sum_{Kp} D_{KK} D_{pp} \Gamma_{K1po} , \tag{100b}$$

and

$$\Gamma_o^+ \equiv \Gamma_2 + \Gamma_1 = \overline{A} .$$

Γ_1 and Γ_2 are the widths of the lower and upper states of the active material, in vacuum, and averaged over the states of the center of mass and the perturber. Similarly, we introduce

$$S_2 \equiv \sum_{Kp} D_{KK} D_{pp} S_{M_2'} = \sum_{Kp} D_{KK} D_{pp} S_{K2po} , \tag{101a}$$

$$S_1 \equiv \sum_{Kp} D_{KK} D_{pp} S_{M_1'} = \sum_{Kp} D_{KK} D_{pp} S_{K1po} , \tag{101b}$$

and

$$S_o \equiv S_2 - S_1 . \tag{101c}$$

The subscript o in Γ_o^+ and S_o should not be confused with the vacuum state appearing in the right hand sides of the above equations. Again, S_1 and S_2 are the shifts of the lower and upper states of the active material, in vacuum, and averaged over the states of the center of mass and the perturber. Akcasu[9] discusses the averaged widths and shifts of the states in considerable detail. Here, we simply note that both shift and width can be separated in two parts: One due to the vacuum field, and another due to the perturber. For further information see Ref. 9.

Observing that $\bar{S}_d$, being the average over all $\underline{K}$-space of an odd function of $\underline{K}$, vanishes, and using Eqs. (101), we obtain

$$\bar{B} = \omega-\omega_o-S_o. \qquad (102)$$

Moreover, we have

$$\overline{\beta^2} = \sum_{Kp} D_{KK}D_{pp}\ (B-\bar{B})^2. \qquad (103a)$$

Combining Eqs. (98b) and (102) we obtain

$$\overline{\beta^2} = \sum_{K} D_{KK}S_d^2 + \left(\sum_{Kp} D_{KK}D_{pp}(S_{M_2'M_1'})^2-S_o^2\right), \qquad (103b)$$

where again use of the facts that $\bar{S}_d = o$, and that S_d does not depend on p has been made.

We now define

$$\Gamma_d^2 \equiv \sum_{K} D_{KK}\ S_d^2 = \sum_{K} D_{KK}\frac{\hbar^2}{m^2}\ (\underline{K}\cdot\underline{k}_\lambda)^2, \qquad (104a)$$

$$\Gamma_s^2 \equiv \sum_{Kp} D_{KK}D_{pp}(S_{M_2'M_1'})^2-S_o^2, \qquad (104b)$$

and

$$\Gamma_e^2 \equiv \Gamma_o^{+2} + \Gamma_d^2 + \Gamma_s^2 \ . \tag{104c}$$

With the foregoing definitions, and combining equations (103b), (102), (100c), (99) and (86b), we obtain

$$I_\lambda(\omega) = \frac{2\pi N_2 \chi_\lambda^2 d_\lambda^2}{\omega_\lambda^2} \frac{\Gamma_o^+}{(\omega-\omega_o-S_o)^2 + \Gamma_e^2} \ . \tag{105}$$

In obtaining this result we have assumed that the statistical fluctuations of the widths can be neglected. This assumption is inherent in the condition $\overline{\alpha^2} \ll \overline{A}^2$ under which Eq. (86b) has been derived.

Thus, the spectrum of spontaneous emission into the λth mode is shown to be a Lorentzian centered at ω_o+S_o and having an effective width Γ_e. This effective width consists of three terms. The first term Γ_o^{+2} is the sum of the widths arising from the interaction with the vacuum field (natural width), and the interaction with the perturber (collision broadening). Also the third term Γ_s^2 is due to the same interaction but it is different in nature. It appears as a width, while actually is due to the statistical fluctuations of the shifts. It is usually referred to as statistical broadening. The second term Γ_d^2 is due to the recoil of the center of mass of the atom when it emits a photon. This is essentially the Doppler broadening. For a Maxwellian distribution of center of mass velocities, Γ_d^2 can be readily calculated. The calculation is carried out in Appendix D with the result

$$\Gamma_d^2 = \omega_\lambda^2 \frac{kT}{mc^2} . \qquad (106)$$

In the limit of zero temperature or infinite mass it vanishes as it should.

Up to this point, ω_o has denoted the frequency of the transition $2 \rightarrow 1$. Then, S_o is the shift due to the interaction with the vacuum field and the perturber, averaged over the states of the perturber and the motion of the center of mass. In interpreting experiments however, it may be preferable to include the vacuum shift in ω_o. Then, S_o should be reinterpreted as due to the interaction with the perturber only and averaged as before.

3. THE SUSCEPTIBILITY FUNCTION IN THE STATISTICAL APPROXIMATION

We now proceed to calculate the susceptibility function for a gas optical maser, in the statistical approximation. The starting point is Eq. (72). Since D_{KK} depends on the magnitude of $\underline{K}$ only, and since $\omega_{KK_1} = \pm S_d$, we may choose one of the signs and then multiply by two because we average over the whole $\underline{K}$-space. Moreover, assuming that the recoil effect is adequately accounted for by S_d, we replace $S^-_{2Kpo,1K_1po}$ and $S^-_{2K_1po,1Kpo}$ by $S^-_{2Kpo,1Kpo}$, and $\Gamma^+_{2Kpo,1K_1po}$ and $\Gamma^+_{2K_1po,1Kpo}$ by $\Gamma^+_{2Kpo,1Kpo}$. Then, using also Eqs. (97b) and (97c), Eq. (72) becomes

$$Y_\lambda(\omega) = \frac{2\pi N\omega\chi_\lambda^2 d_\lambda^2}{\hbar\omega_\lambda^2} \sum_{Kp} \frac{(D_2-D_1)}{(\omega_o+i\ \Gamma^+_{M_2'M_1'})} .$$

$$\frac{D_{KK}\ D_{pp}}{\omega-\omega_o-S_d-S^-_{M_2'M_1'}-i\Gamma^+_{M_2'M_1'}} \tag{107a}$$

where we have used Eq. (94b) and have renamed K_1 to K. Since we have already assumed (see V-2) that the statistical fluctuations of the widths can be neglected, we replace the two factors in the right hand side of Eq. (107a) by their averages. After some mathematical manipulations, and using Eqs. (100) we obtain

$$Y_\lambda(\omega) = \frac{2\pi N\omega\chi_\lambda^2 d_\lambda^2}{\hbar\omega_\lambda^2\ \omega_o^2}\ [\ (D_2-D_1)\,(\omega_o-i\ \Gamma_o^+)]$$

$$\sum_{Kp} \frac{D_{KK}\ D_{pp}}{\omega-\omega_o-S_d-S^-_{M_2'M_1'}\ -i\ \Gamma^+_{M_2'M_1'}}\ , \tag{107b}$$

where we have replaced $\omega_o^2+(\Gamma_o^+)^2$ by ω_o^2 in the denominator, since $\Gamma_o^+ \ll \omega_o$. We shall also introduce Z, defined by

$$Z \equiv (D_2-D_1)\ (\omega_o-i\Gamma_o^+)\ , \tag{107c}$$

because it will appear in several of the following equations.

The problem is now reduced to the calculation of $\mathfrak{z}(\omega)$, defined by

$$\mathfrak{z}(\omega) = \sum_{Kp} \frac{D_{KK}\ D_{pp}}{\omega-\omega_o-S_d-S^-_{M_2'M_1'}-i\ \Gamma^+_{M_2'M_1'}}\ . \tag{108}$$

The statistical fluctuations of $\Gamma^+_{M_2'M_1'}$ will be neglected, as was done in previous instances. On the contrary, we shall retain the statistical fluctuations of the shift and as will be shown subsequently, two different methods of averaging suggest themselves, depending on whether

Γ_s^2 is negligible as compared to Γ_d^2 or not. Consider first the more general case in which both Γ_s^2 and Γ_d^2 are to be retained. Identifying $\omega-\omega_0-S_d-S^-_{M_2'M_1'}$ with B, and $\Gamma^+_{M_2'M_1'}$ with A, the right hand side of Eq. (108) assumes the form of the function z_2 difined by Eq. (84). As shown in IV-5, the average value of z_2 can be approximated as in Eq. (89), provided one assumes that $\overline{\alpha^2} \ll \overline{A^2}$. Thus, combining Eqs. (89), (100), (101), (102), (103) and (104), we obtain

$$\mathfrak{z}(\omega) = \frac{1}{(\omega-\omega_0-S_0)-i\,\Gamma_0^+}\left[1 + \frac{\Gamma_d^2+\Gamma_s^2}{[(\omega-\omega_0-S_0)-i\Gamma_0^+]^2}\right] . \tag{109}$$

If the correction term inside the square brackets can be neglected as compared to unity, one has

$$\mathfrak{z}(\omega) = \overset{\circ}{\mathfrak{z}}(\omega) \equiv \frac{1}{\omega-\omega_0-S_0-i\,\Gamma_0^+} , \tag{110}$$

which defines $\overset{\circ}{\mathfrak{z}}(\omega)$. In phenomenological treatments of the problem,[6] one obtains an expression for the susceptibility function resembling $\overset{\circ}{\mathfrak{z}}(\omega)$. Actually, S_0 is entirely ignored, and Γ_0^+ is replaced by the effective width Γ_e, appearing in the spectrum of spontaneous emission (see Eq. (105)) which is again assumed on phenomenological grounds. However, Eq. (109) shows that the form $\overset{\circ}{\mathfrak{z}}(\omega)$ involves at least two assumptions; namely that both Doppler and statistical broadening are zero (or very small). In addition, even if this is so and even if

S_o is zero (or negligible), the imaginary part of the denominator is not the effective width Γ_e.

In the limit of zero temperature or infinite mass of the emitting atom, Γ_d^2 vanishes as Eq. (106) shows. Then, one still has a correction due to statistical broadening. It appears therefore, that Eq. (109) is useful when Γ_s^2 is either comparable to Γ_d^2 or much larger. In the second case in fact, one may neglect Γ_d^2 entirely. In order to obtain higher order corrections, if necessary, one can consider the Taylor series given in Eq. (83a) and supplement it with additional terms. In conventional line shape experiments and interpretations, the second order corrections seem to be adequate. The spectra of optical masers however, are extremely narrow and one should be prepared to go to higher order corrections when relevant experiments with well stabilized masers become feasible.

There is a third case, not discussed thus far, namely the case in which $\Gamma_d^2 \gg \Gamma_s^2$. Then, Γ_s^2 can be neglected from Eq. (109), and the resulting expression gives the susceptibility function with a correction, due to the motion of the center of mass, of the first order in the temperature. For this case, in which Γ_s^2 is negligible, we shall now proceed to obtain higher order corrections.

4. THE SUSCEPTIBILITY FUNCTION FOR THE CASE OF NEGLIGIBLE STATISTICAL BROADENING

In this section we calculate $\chi(\omega)$ for the case in which the statistical fluctuations of both $S^-_{M_2'M_1'}$ and $\Gamma^+_{M_2'M_1'}$ can be neglected. Then, we replace the above quantities by their average values in Eq. (108), which now becomes

$$\mathcal{J}(\omega) = \sum_K \frac{D_{KK}}{\omega-\omega_o-S_d-S_o-i\,\Gamma_o^+} \quad . \tag{111}$$

The only unaveraged quantity in this expression is S_d which does not depend on p. Note that S_d cannot be replaced by its average since $\overline{S}_d = o$ and one would lose all information about its effect. Using Eqs. (95b) and (96b) we obtain

$$\mathcal{J}(\omega) = b^3\pi^{-3/2}\int \frac{d^3\underline{K}\; e^{-b^2K^2}}{\omega-\omega_o-S_o-i\,\Gamma_o^+ - \frac{\hbar}{m}\,(\underline{K}\cdot\underline{k}_\lambda)} \quad . \tag{112}$$

The direction of $\underline{k}_\lambda$ is fixed and has been chosen as the z-axis. Let ϑ be the angle between $\underline{K}$ and $\underline{k}_\lambda$, ϕ the azimuthal angle, and $\mu \equiv \cos\vartheta$. Then $\mathcal{J}(\omega)$ becomes

$$\mathcal{J}(\omega) = \frac{2b^3m}{\hbar\sqrt{\pi}}\int_{-1}^{+1} d\mu \int_0^\infty dK \frac{e^{-b^2K^2}}{\frac{m}{\hbar}\,(\omega-\omega_o-S_o) - Kk_\lambda\mu - i\,\frac{m}{\hbar}\,\Gamma_o^+} \;, \tag{113}$$

where we have performed the integration over ϕ. Observing now that the denominator has the integral representation

$$\frac{1}{\frac{m}{\hbar}\,(\omega-\omega_o-S_o)-Kk_\lambda\mu-i\,\Gamma_o^+\frac{m}{\hbar}} = i\int_0^\infty dx\; e^{-i[\frac{m}{\hbar}\,(\omega-\omega_o-S_o)-Kk_\lambda\mu-i\Gamma_o^+\frac{m}{\hbar}]x} \;, \tag{114}$$

and substituting into Eq. (113) we obtain

$$\mathcal{Z}(\omega) = \frac{2ib^3 m}{\sqrt{\pi}\,\hbar} \int_0^\infty dx\, e^{-[\Gamma_0^+ + i(\omega-\omega_0-S_0)]\frac{m}{\hbar}x} \,.$$

$$\cdot \int_0^\infty K^2 e^{-b^2K^2} dK \int_{-1}^{+1} e^{iKk_\lambda \mu x} d\mu \,, \qquad (115)$$

where we have interchanged the order of integrations. The calculation of the integral is relatively straightforward, albeit somewhat lengthy, and is presented in Appendix E. The result is

$$\mathcal{Z}(\omega) = i\,\frac{bm\sqrt{\pi}}{\hbar\, k_\lambda}\,\mathcal{E}\left[\frac{bm}{\hbar k_\lambda}\left(\Gamma_0^+ + i(\omega-\omega_0-S_0)\right)\right] , \qquad (116a)$$

where the function $\mathcal{E}(z)$, for any complex number z, is defined by

$$\mathcal{E}(z) = e^{z^2}\,\mathrm{Erfc}(z), \qquad (116b)$$

and the complementary error function is defined[23] as follows:

$$\mathrm{Erfc}(z) = \int_z^\infty e^{-t^2}\,dt \,. \qquad (116c)$$

Combining now Eqs. (116a), (107) and (108), we obtain the following expression for the susceptibility:

$$Y_\lambda(\omega) = \frac{2\pi^{3/2} N\omega \chi_\lambda^2 d_\lambda^2}{\hbar\omega_\lambda^2\,\omega_0^2}\, Z\, i\left(\frac{bm}{\hbar k_\lambda}\right)$$

$$\mathcal{E}\left[\frac{bm}{\hbar k_\lambda}\left(\Gamma_0^+ + i\,(\omega-\omega_0-S_0)\right)\right] . \qquad (117a)$$

Recalling the definition of b (see Eqs. (95)) we have

$$\frac{bm}{\hbar k_\lambda} = \frac{1}{\omega_\lambda}\left(\frac{m\,c^2}{2kT}\right)^{1/2} . \qquad (117b)$$

The argument of the function $\mathcal{E}$ in Eq. (117a) is therefore, inversely proportional to the square root of the temperature, and directly proportional to the square root of the mass, provided the width Γ_o^+ and shift S_o are slowly varying functions of the temperature and the mass.

In order to investigate the behavior of $Y_\lambda(\omega)$ for

$$\left| \frac{bm}{\hbar k_\lambda} \left(\Gamma_o^+ + i(\omega - \omega_o - S_o) \right) \right| \gg 1 , \tag{118}$$

we note that the function $\mathcal{E}(z)$ has the asymptotic expansion[23]

$$\mathcal{E}(z) \sim \frac{1}{2z} \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1.3\ldots(2m+1)}{(2z^2)^m} \right\} , \tag{119}$$

which is valid for $|z| \to \infty$, and $|\arg z| < \frac{3\pi}{4}$. In Eq. (117a) we have $\mathrm{Re}\, z > o$ and consequently the condition for the argz is satisfied under all circumstances.

Retaining the first two terms in Eq. (119), and after substituting into Eq. (117a) we find

$$Y_\lambda(\omega) \cong \pi^{3/2} \frac{N\omega\chi_\lambda^2 d_\lambda^2}{\hbar\omega_\lambda^2 \omega_{\lambda o}^2} \frac{Z}{(\omega-\omega_o-S_o-i\Gamma_o^+)} \cdot \left[1 + \frac{3\,\Gamma_d^2}{(\omega-\omega_o-S_o-i\Gamma_o^+)^2} \right], \tag{120a}$$

where we have used the fact that

$$\left(\frac{bm}{\hbar k_\lambda} \right)^{-2} = 2\,\Gamma_d^2 , \tag{120b}$$

as can be seen by comparing Eqs. (117b) and (106). For the sake of comparison, we give the expression for $Y_\lambda(\omega)$ resulting from the considerations in V-3. It is obtained by combining Eqs. (107), (108) and (109), and has the form

$$Y_\lambda(\omega) = 2\pi \frac{N\omega\chi_\lambda^2 d_\lambda^2}{\hbar\omega_\lambda^2\omega_0^2} \frac{Z}{(\omega-\omega_0-S_0-i\Gamma_0^+)} \left[1 + \frac{\Gamma_d^2+\Gamma_s^2}{(\omega-\omega_0-S_0-i\Gamma_0^+)^2}\right]. \tag{120c}$$

If Γ_s^2 can be neglected as compared to Γ_d^2, the two expressions assume the same form except for two differences. The coefficient in Eq. (120a) is slightly smaller than that of Eq. (120c), their ratio being approximately 0.9. The correction term in Eq. (120a) on the other hand, is three times larger than the corresponding term in Eq. (120c). In view of the drastic approximations made in calculating $Y_\lambda(\omega)$, the above differences are not too surprising. It is presumed that in the extreme case in which Γ_s^2 is entirely ignorable, Eq. (117a) (from which the asymptotic expansion has been derived) gives a better approximation. In addition, it has the advantage of expressing $Y_\lambda(\omega)$ in effectively closed form. In the case in which Γ_s^2 cannot be ignored, it is Eq. (120c) that must be used.

As mentioned above, Eq. (117a) is in effectively closed form. This enables one to obtain correction terms up to any desired order in the quantity $\frac{bm}{\hbar k_\lambda}\left(\Gamma_0^+ + i(\omega-\omega_0-S_0)\right)$, for large values of this quantity, through the asymptotic expansion of $\mathcal{E}(z)$. A further advantage of Eq. (117a) is that one can obtain approximate expressions for $Y_\lambda(\omega)$ in the case in which we have

$$\left| \frac{bm}{\hbar k_\lambda} \left(\Gamma_0^+ + i(\omega-\omega_0-S_0)\right) \right| \ll 1. \qquad (121a)$$

This we now proceed to discuss.

The function $\mathcal{E}(z)$ has the following series representation[23]:

$$\mathcal{E}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{\Gamma\left(\frac{n}{2}+1\right)} , \qquad (121b)$$

where $\Gamma(x)$ is the gamma function. Retaining the first three terms of the series and substituting into Eq. (117a), we obtain

$$Y_\lambda(\omega) = \frac{2\pi^{3/2} N\omega \chi_\lambda^2 d_\lambda^2}{\hbar \omega_\lambda^2 \omega_0^2} \, Z \, i \left[\frac{1}{\sqrt{2}\,\Gamma_d^2} - \frac{\Gamma_0^+ + i(\omega-\omega_0-S_0)}{\sqrt{\pi}\,\Gamma_d^2} + \frac{[\Gamma_0^+ + i(\omega-\omega_0-S_0)]^2}{2\sqrt{2}\,\Gamma_d^3} \right], \qquad (122)$$

where we have used Eq. (120b). It should be emphasized again that this equation contains the inherent assumption that the statistical broadening is ignorable. Presumably, in the range of validity of this expansion, the above assumption is likely to be satisfied, since inequality (121a) also implies relatively large Doppler broadening.

As an attempt to decide about the form of $Y_\lambda(\omega)$ that should be used in the analysis of an actual system, we consider briefly the first He-Ne gas optical maser developed at the Bell Telephone Laboratories.

According to Bennet's paper,[20] the maser consists of a discharge tube 100 cm long and with an inside diameter of 1.5 cm, filled with He at 1mm Hg pressure and Ne at 0.1mm Hg. The transition used in the maser action is the $2s \rightarrow 2p_4$ (Paschen notation)[24] transition of Ne. The associated frequency is approximately 1.64×10^{15} cps. The Doppler width is estimated to be of the order of 800 Mc/sec, while the natural width of the order of 50 Mc/sec. The lowest loss cavity modes have a width of the order of 0.5 Mc/sec and the modes are separated by 160 Mc/sec. At room temperature and for m = 20amu, Eq. (117b) yields

$$\frac{bm}{\hbar k_\lambda} \cong 3.56 \times 10^{-10} \text{ sec.} \tag{123}$$

Under these circumstances, the whole power output will be practically within at most 10^9 cps about $\omega_\lambda = 1.64 \times 10^{15}$ cps. Interpreting Γ_0^+ as the natural width and noting that in the present case it is much smaller than 10^9 cps, we conclude that, for those values of ω for which we have a substantial amount of power, we shall have

$$|i(\omega-\omega_0-S_0) + \Gamma_0^+| < 10^9 \text{ cps.}$$

Combining this with Eq. (123) we find

$$\left| \frac{bm}{\hbar k_\lambda} \left(\Gamma_0^+ + i(\omega-\omega_0-S_0)\right) \right| < 0.356. \tag{124}$$

In any event therefore, the argument of $\mathcal{E}$ in Eq. (117a) is smaller than unity and it is the series expansion rather than the asymptotic expansion that one should use. Moreover, since the right hand side of Eq. (124) is not much smaller than unity, for a different system the inequality might be reversed.

Attempting to analyse the foregoing optical maser in terms of $Y_\lambda(\omega)$ would have several weak points. The most serious difficulty arises from the fact that the maser exhibited strong mode coupling. Also the frequency stability of the system was not particularly satisfactory becuase of fluctuations of the mechanical construction. Our analysis is aimed particularly, albeit not inevitably, at the single mode operation of a well stabilized maser. According to a recent report,[7] such systems have been constructed, and the hope that one will be able to perform measurements on such systems, in the immediate future, can be hardly considered as optimistic. A third difficulty comes from the fact that most existing treatments, dealing with the interpretation of actual experiments, are phenomenological. And it is not always clear what the parameters quoted really represent.

In the foregoing discussion on the dependence of $Y_\lambda(\omega)$ on the mass and temperature, we have ignored the dependence of S_0 and Γ_0^+ on these quantities, by assuming that their variation is slow. The dependence nevertheless exists and it may be imperative to take it into consideration in actual situations. A fairly extensive study of this problem is presented in Ref. 9 whose formulation we have followed closely. Here, we simply note that one is ultimately faced with the necessity of numerical calculations, if comparison with experimental results is contemplated.

5. THE STEADY-STATE OUTPUT SPECTRUM

The steady-state power spectrum of the λth mode $(R_\lambda(\omega))$ is given by Eq. (80). Combining this equation with Eqs. (22) and (105), we obtain the following expression for the output spectrum $R(\omega)$:

$$R(\omega) = \sum_{\lambda} \frac{\eta_\lambda 2N_2 \omega_\lambda^2 \chi_\lambda^2 d_\lambda^2 \Gamma_0^+}{[(\omega-\omega_0-S_0)^2+\Gamma_e^2]\left|-\omega^2+i\omega\gamma_\lambda+\omega_\lambda^2-\omega_\lambda^2 Y_\lambda(\omega)\right|^2} \quad . \qquad (125)$$

The expression for the spontaneous emission spectrum, appearing above, has been derived under the assumption that the mode involved represents photons travelling along the axis of the tube. Photons travelling at any angle with respect to the axis are lost after a small number of reflections, and consequently no appreciable amount of energy in those modes can build up inside the cavity. Moreover, from the modes with longitudinal propagation, only the ones lying within one or two widths of the spontaneous emission line will oscillate. Thus, although the summation in Eq. (125) was initially understood over all modes which in principle are infinite, for the gas maser this summation is effectively reduced to a small number of terms. A further reduction comes from the assumption, made at an earlier stage, that only the lowest loss modes oscillate. Under these circumstances, the number of modes that need be considered in a typical He-Ne gas optical maser may be as low as three. Incidentally, this is another aspect that greatly simplifies the analysis of a gas maser as apposed to a solid-state maser[25] where more modes must be considered. Eq. (125) gives the steady-spectrum in a general form, in terms of the susceptibility $Y_\lambda(\omega)$ and the shifts and widths of the relevant states. To proceed further one will have to decide about the form of $Y_\lambda(\omega)$ that is appropriate to the system under consideration, and the values of the parameters involved. Here we shall discuss two special cases.

Assume that we have a well stabilized maser operating in a single mode. That is, most of the energy is concentrated in one mode, the other modes having practically no energy at all. Then, the summation in Eq. (125) reduces to one term and although the index λ is now unnecessary, we shall retain it for notational convenience. The transition frequency ω_o and the cavity mode frequency ω_λ are assumed to be of the same order, typically 10^{15} cps. Due to the anticipated narrow spectra, we introduce the following approximations:

$$\frac{\omega}{\omega_o} \approx \frac{\omega}{\omega_\lambda} \approx \frac{\omega_o}{\omega_\lambda} \approx 1 , \qquad (126a)$$

and

$$\omega+\omega_\lambda \approx \omega+\omega_o \approx \omega_\lambda+\omega_o \approx 2\omega_\lambda \approx 2\omega_o . \qquad (126b)$$

In addition, we assume that Γ_d^2 and Γ_s^2 can be neglected. We then have $\Gamma_e^2 \cong \Gamma_o^{+2}$, and Eq. (120c) yields

$$\omega_\lambda^2 Y_\lambda(\omega) \;\tilde{=}\; 2\, g_\lambda^2 \omega_\lambda \frac{Z}{\omega-\omega_o-S_o-i\Gamma_o^+} \tag{127a}$$

where we have introduced

$$g_\lambda^2 = \frac{\pi N \chi_\lambda^2 d_\lambda^2}{\hbar\omega_\lambda^2} \; . \tag{127b}$$

Observing that

$$(\omega-\omega_o-S_o)^2+(\Gamma_o^+)^2 = |\omega-\omega_o-S_o-i\Gamma_o^+|^2 \; ,$$

we have

$$|-\omega^2+i\omega\gamma_\lambda+\omega_\lambda^2-\omega_\lambda^2 Y_\lambda(\omega)|^2 \; [(\omega-\omega_o-S_o)^2+(\Gamma_o^+)^2] =$$

$$= 4\,\omega_\lambda^2 |(\omega_\lambda-\omega)\;(\omega-\Omega_o-i\Gamma_o^+) + i\,\frac{\gamma_\lambda}{2}\;(\omega-\Omega_o-i\Gamma_o^+) -$$

$$- g_\lambda^2\,(D_2-D_1)\,\omega_o+ig_\lambda^2(D_2-D_1)\,\Gamma_o^+|^2 \; , \tag{128a}$$

where we have introduced

$$\Omega_o \equiv \omega_o+S_o \; , \tag{128b}$$

and have used the expression for Z given by Eq. (107d). Note that the approximation $\omega_\lambda \approx \omega_o \approx \Omega_o$ is valid since S_o is much smaller than ω_o. Substituting into Eq. (125) and dividing numerator and denominator by (Γ_o^+) we obtain

$R_\lambda(\omega) =$

$$\frac{\dfrac{\eta_\lambda N_2 \chi_\lambda^2 d_\lambda^2}{2\Gamma_o^+}}{[(\omega-\omega_\lambda)+\frac{\gamma_\lambda}{2\Gamma_o^+}(\omega-\Omega_o)+g_\lambda^2(D_2-D_1)]^2+[\frac{(\omega_\lambda-\omega)(\omega-\Omega_o)}{\Gamma_o^+}+\frac{\gamma_\lambda}{2}-\frac{g_\lambda^2}{\Gamma_o^+}(D_2-D_1)\omega_o]^2} \quad . \tag{129}$$

The width γ_λ of a good Fabry-Perot cavity is of the order of 0.5 Mcps, while, according to Bennet,[20] the natural width of the upper level of the maser transition is of the order of 50 Mcps. Here, Γ_o^+ is the sum of the widths of both levels and contains both natural and collision widths. We may assume therefore, that $\gamma_\lambda \ll \Gamma_o^+$. Moreover, we assume that the maser is stabilized well enough to have $|\Omega_o-\omega_\lambda| < \gamma_\lambda/2$. Although usual gas optical masers are not so stable, stabilities of the order of 10^{10} over periods of several hours have been reported recently.[7] Under the foregoing conditions, most of the power output is expected to be concentrated within a few γ_λ's about ω_λ and Ω_o. That is, the power is essentially contained in a frequency range such that $|\omega-\omega_\lambda| \ll \Gamma_o^+$. This implies that $\frac{|(\omega-\Omega_o)(\omega_\lambda-\omega)|}{\Gamma_o^+} \ll (\omega-\Omega_o)$. Since $(\omega-\Omega_o)$ is of the order of γ_λ we may neglect $\frac{(\omega-\Omega_o)(\omega_\lambda-\omega)}{\Gamma_o^+}$ from the denominator of Eq. (129) which now simplifies to

$$R(\omega) = \frac{\dfrac{\eta_\lambda N_2 \chi_\lambda^2 d_\lambda^2}{2\,\Gamma_o^+}}{[(\omega-\omega_\lambda)+\frac{\gamma_\lambda}{\Gamma_o^+}(\omega-\Omega_o)+g_\lambda^2(D_2-D_1)]^2+[\frac{\gamma_\lambda}{2}-\frac{g_\lambda^2}{\Gamma_o^+}(D_2-D_1)\omega_o]^2} \quad . \tag{130}$$

If ω_m is the value of ω at which $R(\omega)$ attains its maximum value, we shall have

$$\omega_m - \omega_\lambda + (\omega_m - \Omega_o)\,\frac{\gamma_\lambda}{2\Gamma_o^+} + g_\lambda^2 (D_2 - D_1) \;=\; 0 \quad .$$

Solving this equation for ω_m and retaining only terms linear in $\gamma_\lambda/2\Gamma_o^+$ we obtain

$$\omega_m \;=\; \omega_\lambda \;-\; (\omega_\lambda - \Omega_o)\,\frac{\gamma_\lambda}{2\Gamma_o^+} - g_\lambda^2 (D_2 - D_1) \;. \tag{131}$$

Also, if we neglect terms of order higher than the first in $\frac{\gamma_\lambda}{2\Gamma_o^+}$, $R(\omega)$ becomes

$$R(\omega) \;=\; \frac{\dfrac{\eta_\lambda N_2 \chi_\lambda^2 d_\lambda^2}{2\Gamma_o^+}}{(\omega-\omega_m)^2 + \left(\dfrac{\gamma_\lambda}{2} - \dfrac{g_\lambda^2}{\Gamma_o^+}(D_2-D_1)\,\omega_o\right)^2} \quad . \tag{132}$$

To the extent that the conditions under which the above equation has been derived are satisfied, the power output spectrum of a well stabilized optical gas maser operating in a single mode has a Lorentzian shape. The line is centered at ω_m and has a full-width at half-maximum

$$(\delta\omega)_{1/2} \;=\; \gamma_\lambda - \frac{2g_\lambda^2}{\Gamma_o^+}(D_2-D_1)\,\omega_o \quad . \tag{133}$$

This width is larger or smaller than the cavity mode width γ_λ, depending on whether (D_2-D_1) is negative or positive respectively. (Note that (D_2-D_1) varies from +1 to -1). The width decreases as the degree of inversion, that is (D_2-D_1), increases. Results similar to Eq. (132) have been derived (through different arguments) and discussed elsewhere.[25,26] Thus, we shall not dwell on it any further. However, Eq. (131) deserves further attention since it contains a term which does not appear in previous treatments.

This equation is usually referred to as the "linear frequency pulling" equation. If the term $g_\lambda^2(D_2-D_1)$ is neglected, the equation agrees with the well-known result obtained for the first time in Ref. 26. Here, we obtain an additional correction term. This term is presumably a consequence of the more refined model we have used. In Ref. 26, as well as in other treatments, one introduces a phenomenological width which masks the fact that this width is due to many effects which give rise to separate widths and shifts for each level, as shown in earlier chapters. One cannot expect therefore, such models to predict effects associated with this fine structure, so to speak, of the effective width. Since (D_2-D_1), for most masers, will vary between 0.5 and 1, the order of magnitude of $g_\lambda^2(D_2-D_1)$ will be determined by g_λ^2. It is not a priori obvious therefore, that this term is ignorable under all circumstances. It is perhaps illuminating to compare this term to the term $2g_\lambda^2(D_2-D_1)\,\omega_0/\Gamma_0^+$ which accounts for the spectrum narrowing. For an ammonia maser for example,[26] we have $\frac{\omega_0}{\Gamma_0^+} \sim 10^6$. For an optical gas maser, we have[20] $\frac{\omega_0}{\Gamma_0^+} \sim 10^8$. Thus, the frequency shift term is, typically, seven orders of magnitude smaller than the narrowing term. In usual devices, one would not expect this shift term to be of importance. In a well stabilized maser operating in a single mode however, it might represent a significant effect.

As a second special case, let us consider an optical maser operating in a single mode and assume that the spontaneous emission spectrum is very broad compared to γ_λ. The previous example suggests that, for population inversion high enough, the output spectrum will be narrower than γ_λ. One may assume therefore, that the spectrum of spontaneous emission is constant over the frequency range of interest, and replace it by its value at the center of the line, that is Ω_o. Thus in Eq. (125) we replace $(\omega-\omega_o-S_o)^2 + \Gamma_e^2$ by Γ_e^2. Again, this assumption is valid if the maser is stabilized well enough for Ω_o and ω_λ to differ by an amount of the order of γ_λ at most. Let furthermore, $y_{1\lambda}(\omega)$ and $y_{2\lambda}(\omega)$ be the real and imaginary parts, respectively, of $Y_\lambda(\omega)$. That is

$$Y_\lambda(\omega) = y_{1\lambda}(\omega) + i\, y_{2\lambda}(\omega) \; . \tag{134}$$

Then, using also Eqs. (126), we obtain

$$R(\omega) = \frac{\dfrac{\eta_\lambda N_2 \chi_\lambda^2 d_\lambda^2 \Gamma_o^+}{2\,\Gamma_e^2}}{\left(\omega-\omega_\lambda + \frac{\omega_\lambda}{2}\, y_{1\lambda}(\omega)\right)^2 + \left(\frac{\gamma_\lambda}{2} - \frac{\omega_\lambda}{2}\, y_{2\lambda}(\omega)\right)^2} \tag{135}$$

This, in general, is not a Lorentzian since $y_{1\lambda}$ and $y_{2\lambda}$ depend on ω. If these quantities are slowly varying functions of ω, or perhaps constant, then the spectrum does become a Lorentzian centered at

$$\omega_\lambda - \frac{\omega_\lambda}{2}\, y_{1\lambda} \; , \tag{136a}$$

and having a full-width at half-maximum

$$\gamma_\lambda - \omega_\lambda y_{2\lambda} . \tag{136b}$$

It is seen therefore, that the real part of the susceptibility appears as a shift of the frequency of oscillation with respect to the cavity mode frequency, while the imaginary part appears as a width. The detailed structure of the spectrum is contained in Eq. (135). This equation is likely to correspond more closely to the spectrum of an actual gas optical maser than Eq. (129) does, with the additional complication of considering two or three more modes. For example, according to Bennet's[20] estimates, the Bell Telephone Laboratories He-Ne gas maser would have $\Gamma_e > 800$ Mcps and $\gamma_\lambda \approx 0.5$ Mcps. The distance between modes was 160 Mcps. For single mode operation therefore, all conditions under which Eq. (135) was derived are satisfied. Moreover, it is conjectured that even for two-mode operation, in which Ω_0 lies between two cavity modes, Eq. (135) will approximate the actual spectrum adequately, when summed over the modes in question. It must be emphasized that the comparison of the present theory, as well as of other theories, to experimental results is hindered mainly by the inadequate frequency stability of usual devices.

6. ON THE LOSS MECHANISM

As pointed out in Chapter III, the coupling of the cavity field to the loss mechanism gives rise to the damping constant γ_λ and a driving term representing the fluctuations of the loss mechanism. The problem has been discussed by Senitzky[8] and in this section we shall elaborate somewhat on his method.

Recall that (see Eq. (27)) the total hamiltonian of the system was

$$H = H^E + H^R + H^A + H^P + V^{RA} + V^{PA} + V^{EA} .$$

The loss is the result of the interaction of the radiation field inside the cavity with some other system (e.g. the walls of the cavity, or the host crystal in a solid-state maser). Let H^L be the hamiltonian of this system which we refer to as the loss mechanism. Also, let V^{RL} be the energy of interaction between the radiation field and the loss mechanism. Considering the interaction of each particle of the latter with the radiation field, as we did with the active material, we can write V^{RL} as follows:

$$V^{RL} = - \sum_{\lambda} P_\lambda G_\lambda^L , \qquad (137a)$$

where G_λ^L is the current operator of the loss mechanism whose definition is analogous to the definition of G_λ (see Eq. (10c)). The total hamiltonian now becomes

$$H = H^E + H^L + H^R + H^A + H^P + V^{RA} + V^{PA} + V^{EA} + V^{RL} . \qquad (137b)$$

It is assumed that H^L is coupled only to H^R and not to H^A or H^P. Following Chapters III and IV-1, and treating the loss mechanism as in Ref. 8, we obtain

$$\left(-\omega^2+\omega_\lambda^2-\omega_\lambda^2 Y_\lambda(\omega)\right) \quad P_{\lambda T}(\omega) \quad = \quad \omega_\lambda^2 G^S_{\lambda T}(\omega) + \omega_\lambda^2 \; G^L_{\lambda T}(\omega) \; +$$

$$+ \frac{i}{\hbar}\omega_\lambda^4 \int_0^T dt \; e^{-i\omega t} \int_0^t dt_1 \int_0^{t_1} dt_2 \; e^{\frac{i}{\hbar}(t-t_1)H^L} \, \cdot$$

$$\cdot \; [Q_\lambda(t_2)\; G^L_\lambda(t_2),\; G^L_\lambda(t_1)] \quad e^{-\frac{i}{\hbar}(t-t_1)H^L}, \tag{138a}$$

where

$$G^L_\lambda(t) \quad = \quad e^{\frac{i}{\hbar}H^L} \; G^L_\lambda \; e^{-\frac{i}{\hbar}H^L t} \; . \tag{138b}$$

We now treat the last term in the right hand side of Eq. (138a) according to Chapter IV. That is, we pull $Q_\lambda(t_2)$ out of the commutator and we replace the operator multiplying $Q_\lambda(t_2)$ by its expectation value. Moreover, we assume that H^L has an energy spectrum densly spaced, and that its density matrix is diagonal in the energy representation, its diagonal matrix elements being

$$D_{nn} \quad = \quad \frac{e^{-E_n/kT^L}}{\sum\limits_n e^{-E_n/kT^L}} \quad , \tag{139}$$

where

$$H^L|n\rangle \quad = \quad E_n|n\rangle \; .$$

In calculating the trace with the density matrix we have a summation over the states $|n\rangle$. Assuming that the spectrum is dense enough for this summation to be replaced by integration, and after a rather lengthy calculation which is presented in Ref. 8, one obtains

$$\left(-\omega^2+\omega_\lambda^2-\omega_\lambda^2 Y_\lambda(\omega)\right) P_{\lambda T}(\omega) = \omega_\lambda^2 G^S_{\lambda T}(\omega) + \omega_\lambda^2 G^L_{\lambda T}(\omega) - \tag{140}$$

$$-i\omega\gamma_\lambda P_{\lambda T}(\omega) \quad .$$

The quantity γ_λ is a constant which arose from the expectation value of the operator that multiplied $Q_\lambda(t_2)$. It is the susceptibility of the loss mechanism, and it turns out to be a constant, that is independent of ω, because of the assumptions made about the properties of H^L. Additional assumptions introduced during the calculation are: γ_λ is small compared to ω_λ and no mode coupling exists. In fact the question of mode coupling does not arise at all in Ref. 8 because a single harmonic oscillator is considered there. Thus, one has a model for the quantum mechanical description of loss. Eq. (140) shows therefore that the equations we used in the present treatment are in agreement with the above model, except for one difference. In our equations we did not have the term $\omega_\lambda^2 G^L_{\lambda T}(\omega)$ appearing in Eq. (140). It should be clear from the considerations in IV-4 that this term, when one calculates the output spectrum, will give rise to the quantity

$$I^L_\lambda(\omega) \equiv \lim_{T\to\infty} \frac{1}{T} \operatorname{Tr} D^L G^L_{\lambda T}(\omega)\, G^L_{\lambda T}(-\omega), \tag{141a}$$

where D^L is the density operator of the loss mechanism, and $G^L_{\lambda T}(\omega)$ is defined by

$$G^L_{\lambda T}(\omega) = \int_0^T dt\, e^{-i\omega t}\, e^{\frac{i}{\hbar} H^L t}\, G^L_\lambda\, e^{-\frac{i}{\hbar} H^L t} \quad . \tag{141b}$$

$I_{\lambda}^{L}(\omega)$ represents the spectrum of spontaneous emission from the loss mechanism, and in calculating the output spectrum it will be added to the spectrum of spontaneous emission from the upper level of the active material. When we consider a gas optical maser therefore, $I_{\lambda}^{L}(\omega)$ represents spontaneous emission from the walls of the tube at room temperature, and at frequencies of the order of 10^{15} cps. In principle, this contribution is present. But it is extremely unlikely that its neglect could be of any importance, as far as the output spectrum is concerned. Even if one considers an optical maser amplifier, in which case $I_{\lambda}^{L}(\omega)$ would constitute noise, its effect can presumably be neglected since it will be masked by the much more important term of spontaneous emission from the upper maser level. For masers in the range of microwaves however, spontaneous emission from the cavity walls may not be ignorable in which case $I_{\lambda}^{L}(\omega)$ would have to be taken into consideration.

CHAPTER VI

COMPARISON WITH OTHER THEORIES

The present study was motivated by the work of Wagner and Birnbaum.[25] Although their work is particularly aimed at the solid-state maser, their formulation is rather general. They describe the electromagnetic field classically, in terms of the cavity modes, that is p_λ and q_λ. The active material is treated as an assembly of fluctuating dipoles, with no permanent dipole moment. By "fluctuating" is meant that dipoles which are in the upper state can decay to the lower state spontaneously. If a field is present, dipoles in the upper state can decay and dipoles in the lower state can make transitions to the upper state at a rate which is proportional to: the number of photons present (that is the square of the field), the number of dipoles in the respective levels, and the square of the coupling constant which is the matrix element of the dipole moment. The fluctuation represents spontaneous emission. Taking the spontaneous emission as the driving force, and calculating the induced dipole moment by using second order perturbation theory, they are able to obtain an equation for the spectrum similar to Eq. (125). They assume that the spontaneous emission spectrum has the form

$$\frac{1}{(\omega-\omega_0)^2 + \Gamma^2} \,, \qquad (142a)$$

where Γ is a phenomenological width including all broadening effects. This width is also used in their calculation of the susceptibility function which has the form

$$\frac{g_\lambda^2(D_2-D_1)}{\omega-\omega_o-i\,\Gamma} \quad . \tag{142b}$$

In both (142a) and (142b), we have omitted non-germane multiplicative factors.

Structurally, (142a) resembles our expression for the spectrum of spontaneous emission, except for two differences. First, no shift appears in (142a). Secondly, we have seen that the width Γ_e consists of several parts and we have exhibited explicit formulas for them, indicating their dependence on the dynamical parameters of the system. The issue however, becomes even more important when one considers the susceptibility function. As we saw in Chapter V, $Y_\lambda(\omega)$ takes the form (142b) only if the Doppler and statistical broadening can be neglected. Then, if in addition we neglect the shift S_o or reinterpret ω_o, our results reduce to those of Wagner and Birnbaum. If the Doppler and the statistical broadening cannot be neglected, then (142b) does not coincide with our expression for the susceptibility. Nevertheless, under certain conditions we were able to express $Y_\lambda(\omega)$ in the form of a series whose first term was similar to (142b). However, one difference still remains. That is, the quantity Γ, appearing in the first term of the series for $Y_\lambda(\omega)$, is not the effective width Γ_e, appearing in the spontaneous emission spectrum. While, according to (142a) and (142b) the same Γ appears under all circumstances. Moreover, we have shown that, under certain conditions, $Y_\lambda(\omega)$ may have a form entirely different than (142b) (see Eq. (122)).

In addition to the above differences in the results obtained, the present treatment also differs in the derivation of the equations. In fact, Wagner and Birnbaum do not derive their equations. They rather construct them. Here, we construct the hamiltonian, and then derive the equations through Heisenberg's equations, making suitable approximations. This approach has the advantage of exhibiting the approximations involved, and lends itself to generalizations in order to account for phenomena such as mode coupling, non-linear effects etc. Also, it has the intellectually pleasing feature that one does not have to assume that the spontaneous emission is the driving force, since it inevitably follows from the formulation. Actually, it was shown that the field is, in principle, driven by spontaneous emission from both the active material and the loss mechanism.

The foregoing differences stem mainly from the difference in the degree of refinement of the two models. Lumping all effects into a constant Γ, as in (142a), has the advantage of leading to simpler expressions. At the same time however, one loses considerable information about the relative importance of several aspects that may alter the results even qualitatively.

Part of this treatment is also related to Senitzky's[8] work. In order to study the electromagnetic field inside a cavity, Senitzky has considered the problem of a single harmonic oscillator coupled to material systems. Indeed, each mode of the cavity corresponds to a harmonic oscillator, and it seems reasonable to consider a single harmonic oscillator, if one wishes to study the single mode operation. This would undoubtedly be correct in an entirely enclosed, perfect cavity with only one mode excited. Of course, perfect cavity implies no coupling with the external world, and one would have to redefine the connection between theory and measurement. In any event, the problem treated here is not of this nature. The cavity is quite open and clearly, when an atom placed inside the cavity emits spontaneously a photon of wavelength 10^{-5} cm, it does not know that it is inside the cavity. It emits as if it were in free space. When the emission is induced, the presence of the cavity is felt strongly because the induced emission is proportional to the number of photons present in the final state, and it is the cavity that selects the photons which stay in it for a relatively long time. If one considers a single harmonic oscillator and attempts to calculate the spectrum of spontaneous emission, as we did in IV-4, no natural broadening is found. In fact, if the collision and the Doppler broadening are neglected, the spectrum becomes a delta function. This is to be expected since, the natural broadening is intimately connected with the fact that the atom can decay into a continuous spectrum. We have avoided this difficulty by considering not a single oscillator but the whole field as represented by the vector potential $\underline{A}$. In order to obtain equations for p_λ and q_λ we expand $\underline{A}(\underline{r},t)$ in terms of the cavity modes. However, when we develop and study the time evolution of $G_\lambda(t)$, we retain the coupling of the particle system to the whole radiation field. The coupling term V^{RA} is contained in $U^S(t)$. Thus, when we calculate the matrix elements of $U^S(t)$ we expand the vector potential not in terms of the cavity modes but of plane waves, thereby being able to account for natural broadening (see also Appendix A). Natural broadening is of quantitative importance in some cases while it is not in other cases. Obtaining it or introducing it phenomenologically however, is a matter of consistency of the formulation. This, we regard as an essential difference between the present approach and Ref. 8. In addition, here we have considered not a single mode but a multimode cavity, we have formulated the problem as a many-body problem in terms of the density operator, and we have employed Heitler's damping theory which, to our knowledge, has not been applied to the maser problem thus far.

The method used in the calculation of the spectrum of spontaneous emission is the generalization of a technique developed by Ekstein and Rostoker[27]. These authors have not considered broadening effects and their results are expressed in terms of delta functions. The introduction of broadening requires a different treatment of the autocorrelation function. Their result is recaptured by taking $\Gamma_o^+ = o$.

Papers dealing with problems directly or indirectly connected with our work abound in the research journals. Refs. 28-31 and the references already cited constitute only a small sample. The first paper dealing with the maser is the paper by Gordon, Zeiger and Townes[26]. This paper was later extended to the optical maser by Shallow and Townes[28]. Some of the results of the first paper are special cases of ours. The second contains all the fundamental ideas that led to the construction of the first optical maser but the analysis is rather qualitative. More closely related to our work are the papers by Kemeny[30] and McCumber[22]. In addition to the different techniques that they use, their emphasis is more on the mathematical than the physical aspects of the problem. Lastly, one cannot fail to mention Lamb's[32] work differing from ours in intention and contentconsiderably.

CHAPTER VII

CONCLUSIONS

The present theory is a basically linear theory of a multimode cavity in which mode coupling can be neglected. In so far as the theory is valid, it has been shown that: The electromagnetic field inside the cavity, in the absence of any other driving force, is, in principle, driven by spontaneous emission from material systems existing inside the cavity as well as the loss mechanism. For a gas optical maser, the field is effectively driven by spontaneous emission from the active material only. The effect of the material system on the field is represented by the susceptibility. A model for a gas optical maser has been studied, and explicit expressions for the spectrum of spontaneous emission and the susceptibility have been derived. For operation in a single mode and adequate frequency stability, one finds that the output has a Lorentzian shape whose width decreases as the population inversion increases. Moreover, a new term is found in the equation determining the center frequency of the Lorentzian. Since line shape measurements on lines of the narrowness of the optical maser output are not available, the only test of the theory has been the comparison with other theories. This comparison suggests that we have a more refined model capable of accounting for several phenomena that other models do not account for.

Further work along the same lines could be directed toward calculating the spectrum of the output for more than one mode oscillating simultaneously. However, such a calculation would be more meaningful if comparison to experiment were feasible. Also, one might attempt to extend the theory to include mode coupling. It is quickly recognized though, that the mathematics will become very complex. Perhaps the only thorough treatment of mode coupling existing today is Lamb's[32] work. His equations are extremely cumbersome and the whole work leans heavily on numerical calculations. Thus, it appears that mode coupling inescapably leads to mathematical complexity independently of the underlying model.

APPENDIX A

DAMPING THEORY

In this appendix we present a brief derivation of Eqs. (49) by using damping theory. The discussion follows that of Ref. 9.

Let H be the hamiltonian of a system. It is assumed that H can be written

$$H = H^{o} + V , \tag{A1}$$

and that the eigenvalue problem

$$H^{o}|n\rangle = E_n|n\rangle , \tag{A2}$$

can be solved. Then, the problem we wish to solve is to calculate the matrix elements of U(t), where

$$U(t) = e^{-\frac{i}{\hbar}Ht} , \tag{A3}$$

in the representation $\{|n\rangle\}$.

We introduce the resolvent operator R(z)[18], defined by

$$R(z) \equiv \frac{1}{z-H} \tag{A4}$$

where z is a complex number. The operator U(t) is the inverse Laplace transform of R(z), that is

$$U(t) = \frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} dz\, R(z)\, e^{-itz/\hbar} \tag{A5}$$

where $\epsilon > o$. The problem is now reduced to finding the matrix elements of $R(z)$.

Let N and Q be two new operators defined by

$$R = N + N\,Q\,N, \tag{A6}$$

and the condition that N be diagonal and Q non-diagonal in the representation $\{|n>\}$. Then, NQN will be non-diagonal, and consequently N and NQN will be the diagonal and non-diagonal parts of R respectively in the representation $\{|n>\}$. Introducing the operator R_o defined by

$$R_o \equiv \frac{1}{z-H^o}, \tag{A7}$$

R(z) can be expanded as follows:

$$R(z) \equiv \frac{1}{z-H^o-V} = \frac{1}{1-R_oV} R_o = R_o \frac{1}{1-VR_o} =$$

$$= \sum_{n=o}^{\infty} (R_oV)^n R_o = \sum_{n=o}^{\infty} R_o (VR_o)^n. \tag{A8}$$

To obtain integral equations for N one writes Eq. (A4) in the form

$$(z-H^o-V)\;R = 1, \tag{A9}$$

which by virture of Eq. (A6) becomes

$$(z-H^o-V)\;(N+NQN) = 1. \tag{A10}$$

Equating the diagonal and non-diagonal operators on both sides we obtain

$$Q = V_{nd} + [V_{nd}NQ]_{nd} - [(VNQ)_d\ NQ]_{nd}, \tag{A11}$$

and

$$N = [z-H^o - \frac{\hbar}{2}\Gamma(z)], \tag{A12}$$

where we have defined

$$\frac{\hbar}{2}\Gamma(z) \equiv (V+VNQ)_d . \tag{A13}$$

The subscripts d and nd denote the diagonal and non-diagonal parts, respectively, of an operator.

From Eqs. (A6) and (A12) follows that the diagonal matrix elements of R are given by

$$R_{nn}(z) = \frac{1}{z-E_n-\Gamma_{nn}(z)} . \tag{A14}$$

To find the off-diagonal matrix elements of R, we iterate Eq. (A11) treating N as independent of Q. Thus, we have

$$Q = V_{nd} + V_{nd}NV_{nd} + \dots . \tag{A15}$$

Keeping the first term only and using Eq. (A6) we obtain

$$R_{mn}(z) = N_{mm}\ V_{mn}\ N_{nn} =$$

$$= \frac{V_{mn}}{[z-E_m-\frac{\hbar}{2}\Gamma_{mm}]\ [z-E_n-\frac{\hbar}{2}\Gamma_{nn}]} . \tag{A16}$$

The matrix elements of U(t) therefore are:

$$U_{mn}(t) = \frac{1}{2\pi i} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} dz \frac{V_{mn}\, e^{-itz/\hbar}}{[z-E_m-\frac{\hbar}{2}\Gamma_{mm}]\,[z-E_n-\frac{\hbar}{2}\Gamma_{nn}]}, \quad \text{(A17)}$$

for $m \neq n$. For $m = n$, we have

$$U_{nn}(t) = \frac{1}{2\pi i} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} dz \frac{e^{-itz/\hbar}}{z-E_n-\frac{\hbar}{2}\Gamma_{nn}} . \quad \text{(A18)}$$

From these equations follows that the Laplace transform of $U_{mn}(t)$ is the product of the transforms of $U_{mm}(t)$ and $U_{nn}(t)$. The inversion integral in Eq. (A17) therefore, can be expressed as the convolution of $U_{mm}(t)$ and $U_{nn}(t)$ as follows:

$$U_{mn}(t) = V_{mn} \int_0^t U_{mm}(t-\tau)\, U_{nn}(\tau)\, d\tau =$$

$$= V_{mn} \int_0^t U_{mm}(\tau)\, U_{nn}(t-\tau)\, d\tau . \quad \text{(A19)}$$

Thus, the problem reduces to calculating the inversion integral for $U_{mm}(t)$ only.

One first investigates the analyticity of the integrand in Eq. (A18), and in particular the analyticity of $\Gamma_{nn}(z)$. By substituting Eq. (A15) into Eq. (A13) and replacing N by $(z-H^o)^{-1}$ we obtain

$$\frac{\hbar}{2}\Gamma(z) = V_d + [V \frac{1}{z-H^o} V_{nd}]_d + \ldots \quad . \quad \text{(A20)}$$

Retaining the first two terms only we have

$$\frac{\hbar}{2}\,\Gamma_{nn}(z) = V_{nn} + \sum_{n'\neq n} \frac{|V_{n'n}|^2}{z-E_{n'}} . \tag{A21}$$

This equation shows that the singularities of $\Gamma_{nn}(z)$ lie on the portion $x>E_o$ of the real axis, where E_o is the lowest eigenvalue of H^o. The singularities are simple poles when the spectrum of H^o is discrete. When part of the spectrum of H^o is continuous $\Gamma_{nn}(z)$ has a branch cut along that part of the real axis which corresponds to the continuous spectrum. It can also be shown that $\mathrm{Im}\Gamma_{nn}(z)$ and $\mathrm{Re}z$ have always opposite signs. Hence, the denominator in Eq. (A18) can vanish only on the real axis. Noting that,

$$\lim_{\epsilon\to o} \frac{1}{x\pm i\epsilon} = PP\,\frac{1}{x} \mp i\pi\,\delta(x) ,$$

where PP denotes the Cauchy principal part, Eq. (A21) yields

$$\lim_{\epsilon\to o} \frac{\hbar}{2}\,\Gamma\,(x+i\epsilon) = \mathscr{S}_n(x) - i\,\frac{\hbar}{2}\,\gamma_n(x), \tag{A22}$$

where

$$\gamma_n(x) = \frac{2\pi}{\hbar}\sum_{n'\neq n} |V_{n'n}|^2\,\delta(x-E_{n'}) , \tag{A23}$$

and

$$\mathscr{S}_n(x) = V_{nn} + PP\sum_{n'\neq n} \frac{|V_{n'n}|^2}{x-E_{n'}} . \tag{A24}$$

It follows then that the integrand in Eq. (A18) is analytic in the complex plane cut by $\mathrm{Im}z = o$ and $\mathrm{Re}z>E_o$. Expressing the complex integral as a real integral by shifting the path of integration properly, we obtain

$$U_{nn}(t) = \frac{\hbar}{2\pi} \int_{E_o}^{\infty} \frac{dx e^{-ixt/\hbar} \gamma_n(x)}{[x-E_n-s_n(x)]^2 + \left(\frac{\hbar}{2}\gamma_n(x)\right)^2} \qquad \text{(A25)}$$

For a continuous spectrum and since the quantities $s_n(x)$ and $\gamma_n(x)$ are small quantities the integrand attains its maximum value near $x = E_n$. The main contribution to the integral thus comes from the vicinity of E_n. Assuming that $s_n(x)$ and $\gamma_n(x)$ are slowly varying functions of x, we replace them by their values at E_n. Moreover, for $E_n \gg E_o$ we can extend the integration to $-\infty$. Then, $U_{nn}(t)$ is approximated by

$$U_{nn}(t) = e^{-\frac{i}{\hbar}(E_n + s_n - i\gamma_n)t}, \qquad \text{(A26a)}$$

for $t > o$, where

$$\gamma_n \equiv \hbar\, \gamma_n(E_n) , \qquad \text{(A26b)}$$

and

$$s_n \equiv \hbar\, s_n(E_n). \qquad \text{(A26c)}$$

This complets the derivation of Eqs. (49) which we have used in the present treatment. The presentation has been rather sketchy with several subtle questions passed over. More elaborate discussions of the theory can be found in Refs. 9, 10, 18, and 19.

APPENDIX B

ON THE MATRIX ELEMENTS OF G_λ

The operator G_λ is defined by

$$G_\lambda = \frac{\sqrt{4\pi}}{\omega_\lambda} \sum_{\sigma} \frac{e_\sigma}{m_\sigma} \underline{X}_\lambda(\underline{r}_\sigma)\cdot\underline{p}_\sigma \tag{B1}$$

where the summation is over all particles that is, electrons as well as nuclei. Assuming that the particles are grouped into atoms, we separate the summation in two parts as follows:

$$G_\lambda = \frac{\sqrt{4\pi}}{\omega_\lambda} \sum_{j} \sum_{\sigma_j} \frac{e_{\sigma_j}}{m_{\sigma_j}} \underline{X}_\lambda(\underline{r}_{\sigma_j})\cdot\underline{p}_{\sigma_j} \tag{B2}$$

where j is an index referring to the atoms and σ_j refers to the σth particle of the jth atom. Assuming now that the mode vector $\underline{X}_\lambda(\underline{r})$ does not vary appreciably over the dimensions of the atom, we replace $\underline{X}_\lambda(\underline{r}_{\sigma_j})$ by $\underline{X}_\lambda(\underline{R}j)$, where Rj is the position operator of the center of mass of the jth atom. This is essentially the dipole approximation. Then, introducing the operator $\underline{d}$ defined by

$$\underline{d} = \sum_{\sigma_j} \frac{e_{\sigma_j}}{m_{\sigma_j}} \underline{p}_{\sigma_j} , \tag{B3}$$

Eq. (B2) becomes

$$G_\lambda = \frac{\sqrt{4\pi}}{\omega_\lambda} \underline{d}\cdot\sum_{j} \underline{X}_\lambda(\underline{R}j) . \tag{B4}$$

Note that $\underline{d}$ operates only on the internal degrees of freedom of the atom.

The hamiltonian H^{π} (see Eqs. (30)) can be written

$$H^{\pi} = \sum_j H^{\pi}j. \tag{B5}$$

Assuming that the atoms of the active material are uncorrelated and that they do not interact between each other, we shall have

$$[H^{\pi}j, H^{\pi}j'] = o . \tag{B6}$$

Then, we introduce the operators $H_j^S = H^R + H^P + H_j^{\pi}$ and

$$U_j^S(t) = e^{-\frac{i}{\hbar} H_j^S t} , \tag{B7}$$

which defines $U_j^S(t)$. Introducing furthermore, $G_{\lambda j}$ defined by

$$G_{\lambda j} \equiv \frac{\sqrt{4\pi}}{\omega_\lambda} \underline{d} \cdot \underline{X}_\lambda(\underline{R}j), \tag{B8}$$

we have

$$G_\lambda = \sum_j G_{\lambda j} . \tag{B9}$$

Combining now Eqs. (B5)-(B9) we obtain

$$G_\lambda^S(t) = U^{S\dagger}(t)\, G_\lambda\, U^S(t) =$$

$$= \sum_j U_j^{S\dagger}(t)\, G_\lambda\, U_j^S(t) = \sum_j G_{\lambda j}^S(t), \tag{B10}$$

which defines $G^S_{\lambda j}(t)$. From this equation it is also seen that

$$[G^S_{\lambda j}(t),\ G^S_{\lambda j'}(t\ )] \ = \ o\ , \tag{B11}$$

for all j,j'.

From Eq. (42) we have

$$\Lambda\ (t,t_1,t_2) \ = \ \mathrm{Tr}\ D\ \Lambda^{op.}(t,t_1,t_2), \tag{B12a}$$

where

$$\Lambda^{op.}(t,t_1,t_2) \ = \ U^{S\dagger}(t-t_1)\ [G^S_\lambda(t_2),\ G^S_\lambda(t_1)]\ U^S(t-t_1). \tag{B12b}$$

In the linear approximation and for uncorrelated atoms we have

$$\Lambda^{op.}(t,t_1,t_2) \ = \sum_j \ \Lambda^{op.}_j(t,t_1,t_2) \ =$$

$$= \sum_j \ U^{S\dagger}_j(t-t_1)\ [G^S_{\lambda j}(t_2),\ G^S_{\lambda j}(t_1)]\ U^S_j(t-t_1) \tag{B13}$$

which defines Λ^{op}_j. This equation justifies Eqs. (62) and (64).

Also, in calculating the spectrum of spontaneous emission we have to calculate the quantity Tr D $G^S_\lambda(\tau)G_\lambda$. By virtue of Eq. (B10) we have

$$\mathrm{Tr}\ D\ G^S_\lambda(\tau)\ G_\lambda \ = \ \mathrm{Tr}\ D\sum_j G^S_{\lambda j}(\tau)\ G_{\lambda j} +$$

$$+\ \mathrm{Tr}\ D\sum_{\substack{jj' \\ (j\neq j')}} \ G^S_{\lambda j}(\tau)\ G_{\lambda j'} \quad . \tag{B14}$$

For uncorrelated atoms, and if the off-diagonal matrix elements of D can be neglected, the second term vanishes. The first term gives rise to the right hand side of Eq. (78b).

To summarize: the results of this Appendix show that, in calculating the spectrum of spontaneous emission, as well as the susceptibility, we may take

$$|G_{\lambda,mK,m_1K_1}|^2 = \frac{4\pi}{\omega_\lambda^2} |\langle m|\underline{d}\cdot\underline{\epsilon}_\lambda|m_1\rangle|^2 \cdot$$

$$\cdot \sum_j |\langle K|X_\lambda(Rj)|K_1\rangle|^2 , \tag{B15}$$

where

$$\underline{X}_\lambda(\underline{r}) = \underline{\epsilon}_\lambda X_\lambda(\underline{r}) , \tag{B16}$$

provided the atoms of the active material interact with the electromagnetic field independently of each other.

APPENDIX C

ON THE CALCULATION OF THE SUSCEPTIBILITY

For the purposes of this Appendix, we introduce $\Phi_{\lambda T}(\omega)$ defined by

$$\Phi_{\lambda T}(\omega) = \int_0^T dt e^{-i\omega t} \int_0^t dt_1 \int_0^{t_1} dt_2 \, Q_\lambda(t_2) \, \Lambda(t,t_1-t_2), \qquad \text{(C1)}$$

where

$$\Lambda(t,t_1-t_2) = 2i \, \text{Im} \sum_{mm_1=1}^{2} D_{mm} \, d^2_{\lambda,mm_1} \sum_{jKK_1p} D_{KK} D_{pp} X^2_{\lambda jKK_1}$$

$$e^{-i(\omega_{mm_1}+\omega_{KK_1}+S^-_{MM_1}-i\Gamma^+_{MM_1})t_1} \; e^{i(\omega_{mm_1}+\omega_{KK_1}+S^-_{MM_1}-i\Gamma^+_{MM_1})t_2} . \qquad \text{(C2)}$$

We also introduce the symbols

$$\Omega_{MM_1} \equiv \omega_{mm_1}+\omega_{KK_1}+S^-_{MM_1}, \qquad \text{(C3a)}$$

and

$$B_n \equiv D_{mm} \, d^2_{\lambda,mm_1} \, D_{KK} \, D_{pp} \, X^2_{\lambda j,KK_1} , \qquad \text{(C3b)}$$

where all subscripts are lumped into n and B_n is real. Then, we have

$$\Lambda(t,t_1-t_2) = 2i \, \text{Im} \sum_n B_n \, e^{-i(\Omega_{MM_1}-i\Gamma^+_{MM_1})t_1}$$

$$e^{i(\Omega_{MM_1}-i\Gamma^+_{MM_1})t_2} , \qquad \text{(C4a)}$$

which, if written as the difference of the right side and its complex conjugate, becomes

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$$+ \frac{Q_{\lambda T}(\omega)}{(\Omega_{MM_1}-i\Gamma^+_{MM_1})\ (\omega+\Omega_{MM_1}-i\Gamma^+_{MM_1})} - \frac{Q_{\lambda T}(\omega)}{\omega(\Omega_{MM_1}-i\Gamma^+_{MM_1})} . \qquad (C8)$$

The first term in the right hand side of the above equation shall be neglected as containing the damping factor $e^{-\Gamma^+_{MM_1}T}$, in view of the fact that ultimately we shall let $T \to \infty$. Moreover, recall that $\Omega_{MM_1} = \omega_{mm_1} + \omega_{KK_1} + S^-_{MM_1}$ where ω_{mm_1} shall be a frequency of the order of 10^{10} - 10^{15} cps, and ω_{KK_1}, $S^-_{MM_1}$, $\Gamma^+_{MM_1}$ small quantities as compared to ω_{mm_1}. For the purpose of investigating orders of magnitude therefore, we may neglect these small quantities. If we denote by ω_o the maser transition frequency, the term $(\omega+\Omega_{MM_1})$ will give rise to resonance terms of the form $(\omega-\omega_o)$. In view of the narrow spectral lines involved, we shall be interested in ω's of approximately the same order as ω_o which means that $\omega-\omega_o \ll \omega_o$. Thus, we shall have

$$(\omega\ \omega_o)^{-1} \ll \Big(\omega_o(\omega-\omega_o)\Big)^{-1},$$

and consequently the third term in the right side of Eq. (C8) will be much smaller than the second. Retaining therefore only the second term we have

$$\phi_{\lambda T_1}(\omega) = \frac{Q_{\lambda T}(\omega)}{(\Omega_{MM_1}-i\Gamma^+_{MM_1})\ (\omega+\Omega_{MM_1}-i\Gamma^+_{MM_1})} . \qquad (C9a)$$

Note that this term contains also antiresonance terms of the order of $\Big(\omega_o(\omega+\omega_o)\Big)^{-1}$ which should be neglected if the approximation is to be consistent. This is done at a later stage in Chapter VI.

The calculation of $\phi_{\lambda T_2}(\omega)$ proceeds along the same lines and the same approximations are made. The result is

$$\phi_{\lambda T_2}(\omega) = \frac{Q_{\lambda T}(\omega)}{(\Omega_{MM_1}+i\Gamma^{+}_{MM_1})\ (-\omega+\Omega_{MM_1}+i\Gamma^{+}_{MM_1})} \ . \qquad \text{(C9b)}$$

Combining now Eqs. (C3),(C5) and (C9) we obtain

$$\Phi_{\lambda T}(\omega) = Q_{\lambda T}(\omega) \sum_{mm_1=1}^{2} D_{mm}\, d^{2}_{\lambda,mm_1} \sum_{jKK_1p} D_{KK}\, D_{pp}\, X^{2}_{\lambda jKK_1}$$

$$\{[(\omega_{mm_1}+\omega_{KK_1}+S^{-}_{MM_1}-i\Gamma^{+}_{MM_1})\ (\omega+\omega_{mm_1}+\omega_{KK_1}+S^{-}_{MM_1}-i\Gamma^{+}_{MM_1})]^{-1} -$$

$$\text{(C10)}$$

$$- [(\omega_{mm_1}+\omega_{KK_1}+S^{-}_{MM_1}+i\Gamma^{+}_{MM_1})\ (-\omega+\omega_{mm_1}+\omega_{KK_1}+S^{-}_{MM_1}+i\Gamma^{+}_{MM_1})]^{-1}\}$$

from which Eq. (66) follows.

APPENDIX D

CALCULATION OF THE DOPPLER WIDTH

We have, by definition,

$$\Gamma_d^2 = \sum_K D_{KK} \frac{\hbar^2}{m^2} (\underline{K} \cdot \underline{k}_\lambda)^2 . \tag{D1}$$

Assuming a Maxwellian distribution of velocities for the center of mass motion, $\sum_K D_{KK}$ becomes

$$\sum_K D_{KK} \to b^3 \pi^{-3/2} \int d^3 \underline{K}\, e^{-b^2 K^2} , \tag{D2}$$

where

$$b^2 = \frac{\hbar^2}{2m\Theta} , \tag{D3}$$

and Θ is the mean energy. Since for the integration over $\underline{K}$ the vector $\underline{k}_\lambda$ is fixed, we transform to spherical coordinates taking $\underline{k}_\lambda$ as the z-axis. Calling ϑ the angle between $\underline{k}_\lambda$ and $\underline{K}$ and setting $\nu = \cos\vartheta$, we obtain

$$\Gamma_d^2 = \frac{\hbar^2 b^3}{\pi^{3/2} m^2} k_\lambda^2 \int_0^{2\pi} d\phi \int_{-1}^{+1} \nu^2 d\nu \int_0^\infty K^4 e^{-b^2 K^2} dK. \tag{D4}$$

Using the relation[33]

$$\int_0^\infty x^{2n} e^{-\alpha^2 x^2} dx = \frac{1.3\ldots(2n-1)}{2^{n+1} \alpha^{2n}} \sqrt{\frac{\pi}{\alpha^2}} ,$$

carrying out the integrations, and using (D3) we obtain

$$\Gamma_d^2 = k_\lambda^2 \frac{\Theta}{m} \quad .$$

Noting that $\Theta = kT$ where k is the Boltzmann's constant and T the temperature, and that $k_\lambda^2 = \omega_\lambda^2 / c^2$, we have

$$\Gamma_d^2 = \omega_\lambda^2 \frac{k\,T}{mc^2} \quad . \tag{D4}$$

APPENDIX E

CALCULATION OF $\mathcal{Z}(\omega)$

From Eq. (115) we have

$$\mathcal{Z}(\omega) = \frac{2ib^3m}{\sqrt{\pi}\,\hbar}\int_0^\infty dx\, e^{-(\sigma+i\eta)x}\int_0^\infty K^2 e^{-b^2K^2}\, dK.$$

$$\cdot\int_{-1}^{+1} e^{-iKk_\lambda \mu x}\, d\mu\ , \qquad \text{(E1)}$$

where we have introduced

$$\sigma \equiv \frac{m}{\hbar}\Gamma_o^+\ , \qquad \text{(E2)}$$

and

$$\eta \equiv \frac{m}{\hbar}(\omega-\omega_o-S_o)\ . \qquad \text{(E3)}$$

Integrating with respect to μ we obtain

$$\int_{-1}^{+1} e^{-iKk_\lambda \mu x}\, d\mu = \frac{2}{Kk_\lambda x}\ \text{Sin}\ (Kk_\lambda x). \qquad \text{(E4)}$$

Then, Eq. (E1) becomes

$$\mathcal{Z}(\omega) = \frac{2ib^3m}{\sqrt{\pi}\ h}\int_0^\infty \frac{e^{-(\sigma+i\eta)x}}{k_\lambda x} dx\int_0^\infty e^{-b^2t}\ \text{Sin}(k_\lambda x\sqrt{t})dt, \qquad \text{(E5)}$$

where we have made the transformation

$$K^2 = t. \qquad \text{(E6)}$$

From Ref. 33, Vol. II, p. 57, we have

$$\int_0^{\infty} e^{-st} \operatorname{Sin} \lambda \sqrt{t}\, dt = \frac{\lambda}{2s}\sqrt{\frac{\pi}{S}}\, e^{-\lambda^2/4s}, \tag{E7}$$

which is valid for s > o. Using this formula we obtain

$$\int_0^{\infty} e^{-b^2 t} \operatorname{Sin}(k_\lambda x \sqrt{t})\, dt = \frac{k_\lambda x \sqrt{\pi}}{2b^3}\, e^{-\frac{k_\lambda^2 x^2}{4b^2}}. \tag{E8}$$

Substituting into Eq. (E5) it becomes

$$\mathcal{Z}(\omega) = i\,\frac{m}{\hbar} \int_0^{\infty} dx\, e^{-(\sigma+i\eta)x - \frac{k_\lambda^2}{4b^2} x^2} . \tag{E9}$$

This can also be written

$$\mathcal{Z}(\omega) = i\,\frac{m}{\hbar} \int_0^{\infty} dx\, e^{-\alpha^2 x^2 - \sigma x} \operatorname{Cos}(\eta x)\, dx +$$

$$+ \frac{m}{\hbar} \int_0^{\infty} dx\, e^{-\alpha^2 x^2 - \sigma x} \operatorname{Sin}(\eta x)\, dx, \tag{E10}$$

where we have introduced

$$\alpha^2 \equiv \frac{k_\lambda^2}{4b^2} \tag{E11}$$

The above integrals are Fourier Cosine and Sine transforms and from Ref. 34 pages 15 and 74 we have

$$\int_0^{\infty} e^{-\alpha^2 x^2 - \sigma x} \operatorname{Sin}(\eta x)\, dx =$$

$$= -\frac{1}{4}\, i\, \frac{\sqrt{\pi}}{\alpha} \left\{ e^{\frac{1}{4\alpha^2}(\sigma - i\eta)^2} \operatorname{Erfc}\left[\frac{1}{2\alpha}(\sigma - i\eta)\right] - \right.$$

$$-e^{\frac{1}{4\alpha^2}(\sigma+i\eta)^2}\ \mathrm{Erfc}\left[\frac{1}{2\alpha}(\sigma+i\eta)\right]\} \ , \qquad \text{(E12)}$$

and

$$\int_0^\infty e^{-\alpha^2 x^2-\sigma x}\ \mathrm{Cos}\ (\eta x)\ dx =$$

$$= \frac{1}{4}\ \frac{\sqrt{\pi}}{\alpha}\{\ e^{\frac{1}{4\alpha^2}(\sigma-i\eta)^2}\ \mathrm{Erfc}\left[\frac{1}{2\alpha}(\sigma-i\eta)\right] + \qquad \text{(E13)}$$

$$+\ e^{\frac{1}{4\alpha^2}(\sigma+i\eta)^2}\ \mathrm{Erfc}\left[\frac{1}{2\alpha}(\sigma+i\eta)\right]\ \} \ ,$$

where Erfc(z) is defined by

$$\mathrm{Erfc}(z) = \int_z^\infty e^{-t^2}\ dt \ . \qquad \text{(E14)}$$

Eqs. (E12) and (E13) are valid for $\mathrm{Re}\alpha^2 > o$ and $\eta > o$. The first condition is always satisfied since α^2 is real and positive. But η assumes positive as well as negative values. Consider first the case of positive η. Then $\eta = |\eta|$ and therefore,

$$\mathrm{Cos}\ (\eta x) = \mathrm{Cos}\ (|\eta| x)\ \text{and}\ \mathrm{Sin}\ (\eta x) = \mathrm{Sin}\ (|\eta| x).$$

Using these relations in Eq. (E10) and then Eqs. (E12) and (E13), which are now applicable since $|\eta| > o$, after some straightforward manipulations we obtain

$$\mathcal{J}(\omega) = i\ \frac{m\sqrt{\pi}}{2\alpha\hbar}\ e^{\frac{1}{4\alpha^2}(\sigma+i|\eta|)}\ \mathrm{Erfc}\left(\frac{\sigma+i|\eta|}{2\alpha}\right) \ , \qquad \text{(E15)}$$

which is valid for $\eta > o$. Let now $\eta < o$. Then $\eta = -|\eta|$ and

$$\text{Cos}\,(\eta x) \;=\; \text{Cos}\,(|\eta|x) \text{ and } \text{Sin}\,(\eta x) \;=\; -\text{Sin}\,(|\eta|x).$$

Again, using these relations in Eq. (E10) and proceeding as before we obtain

$$\mathcal{J}(\omega) \;=\; i\,\frac{m\sqrt{\pi}}{2\alpha\hbar}\,e^{\frac{1}{4\alpha^2}\,(\sigma-i|\eta|)^2}\,\text{Erfc}\left(\frac{\sigma-i|\eta|}{2\alpha}\right), \qquad \text{(E16)}$$

which is valid for $\eta < o$. But for $\eta > o$ we have $|\eta| = \eta$, and for $\eta < o$ we have $-|\eta| = \eta$. Consequently Eqs. (E15) and (E16) can be combined to the single equation

$$\mathcal{J}(\omega) \;=\; i\,\frac{m\sqrt{\pi}}{2\alpha\hbar}\,e^{\left(\frac{\sigma+i\eta}{2\alpha}\right)^2}\,\text{Erfc}\,\left(\frac{\sigma+i\eta}{2\alpha}\right) \qquad \text{(E17)}$$

which is valid for all values of η, including zero. That the case $\eta = o$ is included in Eq. (E17) can be readily verified by calculating the integral in Eq. (E9) for $\eta = o$ and comparing the result to what Eq. (17) gives for $\eta = o$.

Introducing now the function $\mathcal{E}(z)$ defined by

$$\mathcal{E}(z) \;=\; e^{z^2}\,\text{Erfc}(z), \qquad \text{(E18)}$$

and using Eqs. (E2), (E3), (E11) and (E17) we obtain

$$\mathcal{J}(\omega) \;=\; i\,\frac{bm\sqrt{\pi}}{\hbar k\lambda}\,\mathcal{E}\left[\,\frac{bm}{\hbar k\lambda}\,\left(\Gamma_o^+ + i(\omega-\omega_o-S_o)\right)\,\right]. \qquad \text{(E19)}$$

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