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APPROXIMATIONS TO THE NONCENTRAL CHI-SQUARE DISTRIBUTIONS WITH APPLICATIONS TO SIGNAL DETECTION MODELS

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PREFACE

The objective of this report is to study power- or energy-measuring devices operating with a steady input of band-limited white Gaussian noise, and possibly a signal added to the noise input. The intent of this report is to evaluate the performance of such devices in detecting signals or in discriminating between signals with slightly different energies.

Common examples of energy-measuring devices are (1) radar receivers with square-law second detectors that average over a number of pulses from the same range, (2) broadband superheterodyne and crystal-video receivers with square-law detectors, used for intercepting radar pulses or for receiving ordinary AM modulated communications signals.

The relevant statistical distributions, the central chi-square and the noncentral chi-square, have been calculated and approximated by a number of authors for various purposes. The interest in this report is in approximations that will allow a comparison of the central versus the noncentral distributions to be used in the detection of the signal-and-noise versus noise-alone problem, and in comparing two noncentral distributions for the increase-in-signal-power problem.

The authors' original interest in this problem stemmed from observing the "suppression effect" in square-law detectors in communication and radar receivers. This is an effective loss of detection efficiency as the signal-to-noise ratio into the detector decreases. Although this effect has been extensively studied for special narrowband and broadband cases, the authors feel that the present treatment of the loss of efficiency in energy-measuring devices forms a broad general basis for understanding the suppression effect.

ABSTRACT

Closed form and tabular approximations for the central and noncentral chi-square distribution are reviewed and compared, and an approximation suitable for application to signal-detection problems chosen. This approximation is used to evaluate the efficiency of energy-detecting devices masked by white Gaussian noise to detect signals, and to discriminate between signals with slightly different energies.

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1. INTRODUCTION

The central χ^2 distribution has been widely investigated and because of its use in statistical applications has been tabulated in more or less detail in a variety of places; see, for example, Pearson and Hartley (Ref. 1). Such a table can give complete coverage of the non-linear part of the function since it depends on a single parameter. The noncentral χ^2 distribution, which has general utility in many applications, depends on two parameters and, for this reason, would require much more space for tabulation. Such tables are not generally available. The usual procedure is either to reduce the problem to one that requires the central chi-square, or to compute the required percentage point or probability level from other tabulated functions.

An approximation to the noncentral χ^2 distribution proposed by Patnaik (Ref.2) is recommended by Pearson and Hartley (Ref. 1). However, the error in the approximation is not stated.

The purpose of this report is to examine various approximations to the noncentral χ^2 and to arrive at some conclusion as to their utility.

2. OBSERVATIONS FROM POPULATIONS HAVING EQUAL VARIANCES

If x_i is a randomly selected variate from a normally distributed population with zero mean and unit variance [$x_i \equiv N(0, 1)$], the probability distribution of x_i is given by

$$P\{x_i \leq X\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-\frac{x_i^2}{2}} \, dx_i \quad . \tag{1}$$

The sum of the squares of n randomly selected variates from the population follows the χ^2 distribution with n degrees of freedom:

$$\begin{aligned} P\{\sum_{i=1}^{n} x_i^2 \leq \chi_o^2\} &= \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_0^{\chi_0^2} t^{\frac{n-2}{2}} e^{-\frac{t}{2}} \, dt \\ &= \int_0^{\frac{\chi_0^2}{2}} \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{\chi^2}{2}\right)^{\frac{n-2}{2}} e^{-\frac{\chi^2}{2}} d\left(\frac{\chi^2}{2}\right) . \end{aligned} \tag{2}$$

Suppose now that x_i is drawn from a normal population with unit variance but with an arbitrary mean: $x \equiv N\,(a_i,\, 1)$. The distribution of x_i is given by

$$P\{x_i \leq X\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-\frac{(x_i - a_i)^2}{2}} \, dx_i \; . \tag{3}$$

For n randomly-selected variates from the same or from different populations, the distribution function of the sum of squares is

$$\begin{aligned} P_n &= P\{\sum x_i^2 \leq \chi_o'^2\} \\ &= \int_0^{\frac{\chi_0'^2}{2}} \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{\chi'^2}{2}\right)^{\frac{n-2}{2}} e^{-\frac{\chi'^2}{2}} \, e^{-c^2} \left\{1 + \frac{1}{n} \frac{\chi'^2 c^2}{2} + \right. \end{aligned} \tag{4}$$

$$+ \frac{1}{n(n+2)2!}\left(\frac{\chi'^2 c^2}{2}\right)^2 + \cdots\Big\} \, d\left(\frac{\chi'^2}{2}\right)$$

$$= \int_0^{\frac{\chi_0'^2}{2}} \left(\frac{\chi'}{c}\right)^{\frac{n-2}{2}} e^{-\frac{\chi'^2}{2}-\frac{c^2}{2}} \, I_{\frac{n-2}{2}}(\chi' c) \, d\left(\frac{\chi'^2}{2}\right) \qquad (4)$$

where

$$C^2 = \sum_{i=1}^{n} a_i^2$$

and I is the Bessel function of the first kind with imaginary argument. The distribution function of χ'^2 is called the noncentral Chi-square distribution with n degrees of freedom and parameter c^2.

The noncentral χ^2 distribution function cannot be evaluated directly, nor are tables available which are adequate for most applications.

Fisher (Ref. 3) has given expressions for the exact computation. The complexity of the computations increases rapidly with increasing degrees of freedom. For general use, the computational work is excessively long.

An alternative is to approximate the function by expanding it in an Edgeworth series[1], using enough terms to reduce the maximum error to some specified size. Coefficients of the series depend on the

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1 The Edgeworth series is an expansion in terms of the normal distribution and its derivatives. See Cramer (Ref. 17), p. 227-231.

cumulants[1] of the distribution, which can be determined from the characteristic function of Eq. (4). Marcum (Ref. 4) and Patnaik (Ref. 2) use this method. For small values of n and c^2 the convergence is slow, but it becomes more rapid as either n or c^2 increases.

Patnaik has made use of the Edgeworth expansion to produce a rapidly converging series approximation. He obtains the Edgeworth series expansion of the best fitting central X^2 distribution and subtracts it from the Edgeworth series expansion of the noncentral distribution. The first approximation for the noncentral case is a central X^2, and the second and following terms are correction terms. This method produces high accuracy but requires the use of tables of derivatives of the normal distribution function or tables of the Hermite polynomials. In general, interpolation in these tables is required. The computations are still laborious, but convergence to three significant figures is usually obtained with seven terms, the greatest errors occurring with small n and c^2.

To investigate the possibility of finding simple approximations we consider the limiting distributions of X^2.

As c^2 approaches zero it is clear that the distribution of X'^2 approaches that of X^2. For constant values of c^2 and increasing values of n, the distribution of X'^2 approaches normality, a consequence of the central limit theorem. It will appear later that as the effective number of degrees of freedom of the best fitting central X^2 distribution increases with increasing c^2, the distribution also approaches normality.

To find the best fitting X^2 distribution, we consider the

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1 Cumulants, often called semi-invariates, are coefficients in the series expansion of the log of the characteristic functions, hence the nth cumulant of the random variable which is the sum of several independent variables is the sum of the nth cumulants of these several variables. See Cramer (Ref. 17), p. 187-192.

characteristic function of Eq. (4),

$$\phi(t) = \frac{e^{\frac{\lambda it}{1-2it}}}{(1-2it)^{\frac{n}{2}}} \qquad \left(i = \sqrt{-1}\right) \quad . \tag{5}$$

The formal power series expansion of the logarithm of this characteristic function is

$$\log \phi(t) = \frac{\lambda it}{1-2it} - \frac{n}{2}\log(1-2it)$$

$$= \sum_{r=1}^{\infty} \lambda 2^{r-1}(it)^r - \frac{n}{2}\sum_{r=1}^{\infty}\frac{2^r(it)^r}{r}$$

$$= \sum_{r=1}^{\infty} 2^{r-1}\left(\lambda + \frac{n}{r}\right)(it)^r \quad .$$

The definition of the cumulants k_r in terms of this series is

$$\log \phi(t) = \sum_{r=1}^{\infty} \frac{k_r}{r!}(it)^r \quad .$$

From this we obtain the cumulants

$$\begin{aligned} k_1 &= n + \lambda \\ k_2 &= 2(n + 2\lambda) \\ &\circ \quad \circ \quad \circ \quad \circ \\ k_r &= 2^{r-1}(r-1)!(n + r\lambda) \quad . \end{aligned} \tag{6}$$

If only the first two cumulants are used to determine an approximating distribution, and we restrict ourselves to Pearson type III distributions[1],

1 Cramer (Ref. 17), pp. 248-249; type III distributions are a generalization of the χ^2 distribution.

we obtain the density function

$$f(y) = \frac{e^{-\frac{y}{2}} \, y^{\frac{\upsilon-2}{2}}}{2^{\frac{\upsilon}{2}} \, \Gamma(\frac{\upsilon}{2})} \tag{7}$$

where

$$y = \frac{\chi'^2}{\rho} \qquad , \qquad \rho = \frac{n+2c^2}{n+c^2}$$

$$\upsilon = \frac{(n+c^2)^2}{n+2c^2} \quad . \tag{8}$$

Thus we are approximating the distribution of $\frac{\chi'^2}{\rho}$ by the central χ^2 distribution with υ degrees of freedom, υ being in general a fraction.

The normal approximation can be developed independently, but the same formulas are obtained by taking the limiting normal distribution to the χ^2 approximation. It is clear that in the limit the χ'^2 distribution and the χ^2 distribution tend to the same normal distribution, since they have the same first two moments, and the limiting normal distribution is completely determined by the first two moments. As Patnaik has shown, the distribution of χ' approaches normality faster than that of χ'^2, an analogous property to that enjoyed by the χ^2 distribution. We may expect, therefore, that

$$\sqrt{2y} - \sqrt{2\upsilon - 1} \tag{9}$$

or

$$\sqrt{\frac{2\chi'^2(n+c^2)}{n+2c^2}} - \sqrt{\frac{2(n+c^2)^2}{n+2c^2} - 1} \tag{10}$$

is approximately normally distributed with zero mean and unit variance for sufficiently large values of n and c^2. This is based on Fisher's well-known approximation for the χ^2 distribution that $\sqrt{2\chi^2}$ tends to $N(\sqrt{2n-1}, 1)$ as n increases.

A faster converging normal approximation due to Wilson and Hilferty (see Ref. 5) is that

$$\left[\left(\frac{\chi^2}{n}\right)^{\frac{1}{3}} + \frac{2}{9n} - 1\right]\left(\frac{9n}{2}\right)^{\frac{1}{2}} \tag{11}$$

tends to $N(0, 1)$ with increasing n. Accordingly, we may take

$$\left[\left(\frac{y}{\upsilon}\right)^{\frac{1}{3}} + \frac{2}{9\upsilon} - 1\right]\left(\frac{9\upsilon}{2}\right)^{\frac{1}{2}} \tag{12}$$

to be $N(0, 1)$ for υ sufficiently large.

To get some idea as to the accuracy of simple approximations, the probability of exceeding χ^2 was calculated by various methods. In each case the values of y and υ were computed from the parameters n and c^2 and the observed χ^2, using formulas (8). The first two approximations are based on the central χ^2 distribution, the first being obtained by linear interpolation in central χ^2 tables and the second by exact interpolation. The remaining two approximations are based on the normal approximation, using Fisher's approximation in one case and Wilson and Hilferty's in the other. These are shown in Table I. The exact value shown was taken from Patnaik (Ref. 2). The values of $\chi_o'^2$ are shown to the number of significant figures used in the computation. These values were taken mostly from Patnaik's paper, although some which Patnaik had taken from Fisher (Ref. 3) were obtained from Fisher to one more decimal place. Some accuracy in the exact value of the probability was lost by Patnaik in the rounding off of $\chi_o'^2$.

TABLE I

Approximations to the Probability of Exceeding an Observed Sum of Squares for Various Non-Central Parameters and Degrees of Freedom Using Patnaik's Transformation

n	c^2	υ	χ'^2_o	Exact	I[1]	II[2]	III[3]	IV[4]
2	1	2.25	.17	.05	.0492		.0860	.0500
2	4	3.6	.646	.05	.0329	.0233	.0537	.0313
4	1	4.1667	.91	.05	.0502	.0465	.0700	.0500
4	4	5.3333	1.765	.05	.0427	.0387	.0576	.0414
7	1	7.1111	2.49	.05	.0500	.0489	.0628	.0499
7	4	8.0667	3.664	.05	.0462	.0454	.0580	.0462
2	16	9.5294	6.322	.05	.0389	.0369	.0483	.0379
4	16	11.1111	7.884	.05	.0406	.0400	.0498	.0405
7	16	13.5641	10.257	.05	.0439	.0426	.0512	.0431
2	25	14.0192	12.08	.05	.0406	.0404	.0487	.0408
4	25	15.5741	13.73	.05	.0427	.0417	.0494	.0421
7	25	17.9649	16.23	.05	.0437	.0434	.0504	.0436
16	32	28.8	30.000	.0609	.0594	.0590	.0639	.0590
24	24	32	36.000	.1567	.1556	.1556	.1565	.1553
7	1	7.1111	4.000	.1628	.1635	.1621	.1661	.1610
12	18	18.75	24.000	.2901	.2926	.2920	.2863	.2913
4	10	8.1667	10.000	.3148	.3190	.3179	.3085	.3163
16	8	18	20.000	.3369	.3380	.3380	.3304	.3374
24	24	32	48.000	.5296	.5333	.5332	.5290	.5332
7	16	13.5641	24.000	.5898	.5943	.5949	.5827	.5947
4	4	5.3333	10.000	.7118	.7180	.7197	.7062	.7199
16	8	18	30.000	.7880	.7887	.7902	.7858	.7880
12	6	13.5	24.000	.8174	.8178	.8188	.8162	.8193
16	32	28.8	60.000	.8316	.8326	.8329	.8320	.8332
2	1	2.25	8.64	.95	.9480		.9581	.9515
2	4	3.6	14.64	.95	.9470	.9488	.9555	.9497
4	1	4.1667	11.71	.95	.9490	.9500	.9564	.9506
4	4	5.3333	17.309	.95	.9478	.9491	.9550	.9496
7	1	7.1111	16.004	.95	.9288	.9298	.9341	.9302
7	4	8.0667	21.23	.95	.9491	.9494	.9545	.9497
2	16	9.5294	33.06	.95	.9467	.9474	.9522	.9478
4	16	11.1111	35.43	.95	.9474	.9479	.9523	.9480
7	16	13.5641	38.970	.95	.9476	.9482	.9523	.9483
2	25	14.0192	45.31	.95	.9469	.9478	.9515	.9476
4	25	15.5741	47.61	.95	.9467	.9478	.9515	.9478
7	25	17.9649	51.06	.95	.9476	.9481	.9517	.9481
16	8	18	40.000	.9632	.9626	.9626	.9664	.9626
4	4	5.3333	24.000	.9925	.9909	.9912	.9946	.9911

1 Linear Interpolation in Table 7 of Pearson and Hartley.

2 Exact Interpolation Using Pearson and Hartley's Formulas

3 Normal Approximation Using Fisher's Normal Approximation to the Chi-square Distribution

4 Normal Approximation Using Wilson and Hilferty's Approximation

The conclusion to be drawn from the table is that no one approximation is superior to the others over the whole table. In particular, the more exact approximations (Methods II and IV) are not significantly better than their more easily computed counterparts (Methods I and III). For moderately large υ (say, $\upsilon > 5$) the approximation based on Fisher's normal approximation gives sufficient accuracy for a large number of practical applications. For small values of υ, the Wilson-Hilferty approximation is better at lower probabilities, since the Wilson-Hilferty approximation is more symmetrical.

3. OBSERVATIONS FROM POPULATIONS HAVING UNEQUAL VARIANCES

So far we have assumed that the x_i were selected from populations having variances equal to one. When both the means and the variances vary, we write $x_i \in N(b_i, v_i)$. Under these conditions the distribution function of $\psi^2 = \Sigma x_i^2$ no longer satisfies the conditions of the central limit theorem. However, a sufficient condition for the distribution of ψ to be asymptotically normal is that the set $\{v_i\}$ be bounded. While this condition is always satisfied in physical experiments, the upper bound on the v_i may be so large that convergence to the limiting distributions is extremely slow. The limiting distributions have the same form as before, when fitted by the first two moments. We have

$$f(y) = \frac{e^{-\frac{y}{2}} \, y^{\frac{\upsilon-2}{2}}}{2^{\frac{\upsilon}{2}} \Gamma\left(\frac{\upsilon}{2}\right)} \tag{7}$$

where

$$y = \frac{\Psi^2}{\rho} \quad , \quad \rho = \frac{\Sigma V_i^2 + 2\Sigma b_i^2}{\Sigma V_i + \Sigma b_i V_i^{-1}} \quad ,$$

$$\nu = \frac{\left(\Sigma V_i + \Sigma b_i V_i^{-1}\right)^2}{\Sigma V_i^2 + 2\Sigma b_i^2} \quad . \tag{13}$$

4. CONDITIONAL DISTRIBUTIONS UNDER LINEAR RESTRAINTS

It is well known that for the central χ^2 distribution, if the x_i are subject to s linear restraints, then $\sum_{i=1}^{n} x_i^2$ follows the χ^2 distribution with n-s degree of freedom. The distribution of χ'^2 has a similar property. Bateman (Ref. 6) gives the general proof from which the result given by Patnaik follows as a special case.

Suppose the x_i are subject to s orthogonal linear restraints

$$\sum_{i=1}^{n} C_{\ell i} X_i = \rho_\ell \tag{14}$$

with

$$\sum_i C_{\ell i} C_{mi} = \delta_\ell^m \tag{15}$$

where the $c_{\ell i}$, ρ_ℓ are constants. Let $E(x_i) = a_i$. Then

$$\sum_{i=1}^{n} X_i^2 - \sum_{\ell-1}^{s} \rho_\ell^2 \tag{16}$$

is distributed as χ'^2 with n-s degree of freedom and parameter

$$C^2 = \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{s} \left(\sum_{i=1}^{n} a_i C_{\ell i} \right)^2 \quad . \tag{17}$$

For the conditional distribution of ψ^2 (defined in the preceding section) under s linear restraints, the moments can be determined from the conditional characteristic function, and the best fitting χ^2 distribution can be determined by fitting a Type III curve, using the first two moments.

5. APPLICATION OF THE NONCENTRAL CHI-SQUARE TO DECISION MODELS OF SIGNAL DETECTION AND DIFFERENTIAL DISCRIMINATION

A class of decision theory models which has been applied with some success to both electronic and psychophysical problems (Refs. 7-12, 15, 16) is the following.

A point in a given finite dimensional space is considered to be an "observation." This point is distributed, if viewed repeatedly, according to one of several different possible probability distributions, called "hypotheses," H_0, H_1... . The decision task is to optimize the procedure for deciding which hypothesis holds, on the basis of one "observation." The problems arising in detection always consider two possible alternatives. Whenever all of the parameters of these two distributions are known, the ratio of the two probability density functions is of particular importance. It is called the likelihood ratio, and it has been shown (Ref. 7) that it is the relevant statistic in this decision. That is to say that optimum decisions assume some critical value of likelihood ratio and decide for one hypothesis whenever the likelihood ratio of the observation is greater than this critical value and for the other hypothesis whenever the likelihood ratio of the observation is not greater than the critical value.

Two types of problem lead to a consideration of the noncentral chi-square distribution.

Case I: The null hypothesis is that the observations are distributed according to a completely symmetric normal (multivariate) distribution. This is called "white Gaussian noise" in engineering problems. The mean of the distribution is the origin of the space, and in engineering problems the variance per coordinate is $N_o/2$, i.e., half the noise power per cps. The "signal" hypothesis is a composite hypothesis: each simple hypothesis is a simple translation of the null hypothesis, with mean displaced $c\sqrt{N_o/2}$ from the origin, and these means are uniformly distributed over an n-1 dimensional sphere about the origin. It should be obvious that the only relevant coordinates of the observation are those in the n-dimensional space containing this sphere. Because of symmetry, the radius in this n-dimensional subspace is monotone with likelihood ratio. If the space is normalized to have unit variance on each coordinate axis, the sphere will have radius c. The null hypothesis is now the normalized central chi-square distribution, and the signal hypothesis is the noncentral chi-square distribution with parameter c^2.

Case II: The null hypothesis is, as in Case I, simple white Gaussian noise. The signal hypothesis is just a translation of the mean to some point $c\sqrt{N_o/2}$ from the origin. A non-optimum decision can be based on the radius of the observation in some n-dimensional subspace which contains the translation vector. Although this is a non-optimum procedure, it does arise when the actual axis on which the signal mean lies is unknown but can logically be bounded to some subspace. Such is the case when the signals are sine-wave-like signals with uncertain phase and starting times. The distribution of the measured statistic, the radius, is the "chi" distribution, central for the null hypothesis, and noncentral for the signal hypothesis. The parameter c is $\sqrt{\frac{2E}{N_o}}$, where E is the signal energy and N_o the noise power per cps.

In this latter case it is customary to compute the efficiency (Ref. 13) of the decision device relative to the optimum device. This efficiency is the ratio of the energy E_{11} necessary for the optimum decision device to reach the same performance as that achieved by the non-optimum device which used energy E_{12}

$$\eta = \frac{E_{11}}{E_{12}} \quad . \tag{18}$$

In this, Case II, when the signal is specified exactly, E_{11} can be determined as

$$E_{11} = \left[\frac{\Delta\mu(\chi')}{\sigma(\chi')} \right]^2 \frac{N_o}{2} \tag{19}$$

where $\Delta\mu(\chi')$ is the difference in the means of the two "chi" distributions, and $\sigma(\chi')$ is the standard deviation.

So far, the use of the chi-square distribution in two specific detection cases has been discussed. "Detection" usually carries the connotation in white Gaussian noise that one of the hypotheses has mean at the origin. When the two hypotheses that are possible on both "signal" hypotheses with different values of the parameter "c", the label "differential discrimination" is often used. In computing the efficiency for such a situation and for signals specified exactly and differing only in amplitude, the "energy" E_{11} or E_{12} referred to is the energy of the difference signal, which is proportional to the square of the difference of the rms voltages of the signals.

5.1 Application in Detection Model

The problem in detection is the comparison of the observation χ' (or χ'^2) under the two alternative hypotheses H_0, that $c = 0$; and H_c, that is some specific non-zero value. Clarke, Birdsall, and Tanner (Ref. 14) have suggested that in comparison of two normal but unequal variance hypotheses, the average measure $\sqrt{d_e}$ be used, where

$$\sqrt{d_e} = \frac{\mu_c - \mu_0}{\sqrt{\sigma_c^2 + \sigma_0^2}} \quad . \tag{20}$$

The corresponding efficiency η is then

$$\eta = \frac{d_e}{c^2} \quad . \tag{21}$$

The means and variances are obtained from equation (10):

$$H_0: \quad \mu_0 = \sqrt{n - .5} \quad , \quad \sigma_0^2 = .5 \quad ,$$

$$H_c: \quad \mu_c = \frac{2(n+c^2)^2 - (n+2c^2)}{2(n+c^2)} \quad , \quad \sigma_c^2 = \frac{(n+2c^2)}{2(n+c^2)} \quad . \tag{22}$$

These yield an efficiency of

$$\eta = \frac{\left[\sqrt{\dfrac{2(n+c^2)^2 - (n+2c^2)}{2(n+c^2)}} - \sqrt{n+.5}\right]^2}{c^2\left(\dfrac{2n+3c^2}{2n+2c^2}\right)}$$

$$= \frac{2n+2c^2}{2n+3c^2} \cdot \frac{1}{c^2}\left(\sqrt{n+c^2 - \frac{n+2c^2}{2n+2c^2}} - \sqrt{n+.5}\right)^2 \quad . \tag{23}$$

For large n (n > 10) the second term in each radical is small compared to the first term. If these second terms are ignored, the expression for efficiency becomes dependent only on the ratio of c^2 to n, as follows:

$$\eta = \frac{2n+2c^2}{2n+3c^2} \frac{\left\{\sqrt{n+c^2}-\sqrt{n}\right\}^2}{c^2} \tag{24}$$

or

$$\eta = \frac{2+2\frac{c^2}{n}}{2+3\frac{c^2}{n}} \frac{\left(\sqrt{1+\frac{c^2}{n}}-1\right)^2}{\frac{c^2}{n}} \quad . \tag{25}$$

Equation (25) is plotted in Fig. 1.

5.2 Application in Differential Discrimination Model

The model of differential discrimination is that two hypotheses are compared, H_{c_1} and H_{c_2}, where the parameters c_1 and c_2 are both large but approximately equal. Specifically, the c_i should be large enough so that simple detection is nearly perfect. The following analysis will assume that the difference between c values is considerably less than the smaller c value. The equation for the mean and variance of X' can be rewritten as

$$\sigma^2(\chi') = \frac{n+2c^2}{2n+2c^2} \quad ,$$

$$\mu_c(\chi') = \sqrt{n+c^2+.5\sigma^2} \quad . \tag{26}$$

If a small change of c to c + ϵ is made, the variance remains relatively unchanged, and the mean increases to

$$\mu_{c+\epsilon}(\chi') = \sqrt{n+c^2+.5\sigma^2+2c\epsilon+\epsilon^2}$$

$$= \sqrt{n+c^2+.5\sigma^2}\sqrt{1+\frac{2(c+\epsilon)\epsilon}{n+c^2+.5\sigma^2}} \quad . \tag{27}$$

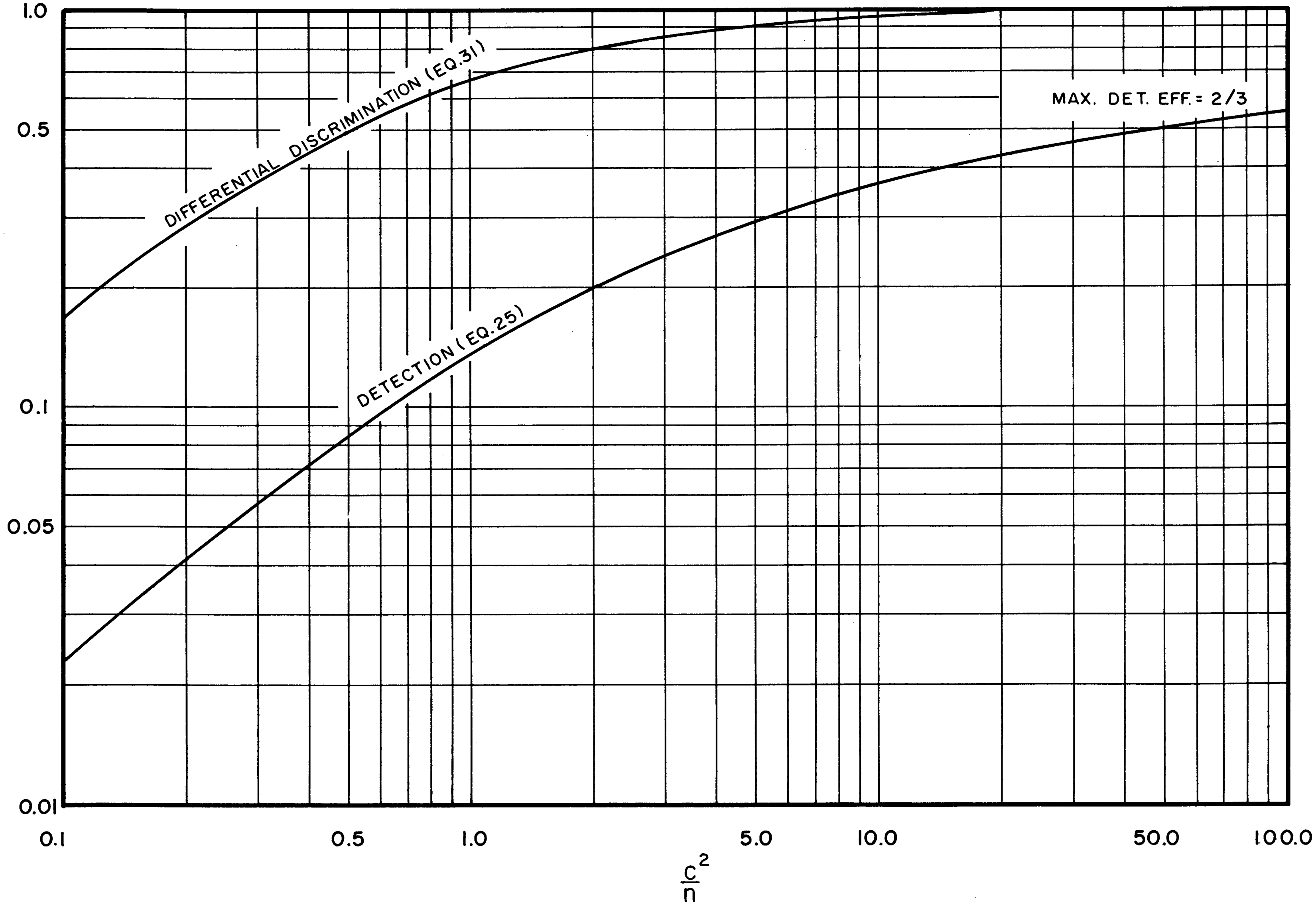

FIG.1 NON-CENTRAL CHI-SQUARE ANALYSIS OF DETECTION AND DIFFERENTIAL DISCRIMINATION EFFICIENCY, n>10.

Under the foregoing assumption that the difference between the c values is much less than the smaller, a good approximation to the second radical is the first two terms of the power series expansion, yielding

$$\mu_{c+\epsilon}(\chi') = \sqrt{n + c^2 + .5\sigma^2}\left(1 + \frac{c\epsilon}{n + c^2 + .5\sigma^2}\right) ,$$

$$= \mu_c(\chi') + \frac{c}{\epsilon\sqrt{n + c^2 + .5\sigma^2}} . \qquad (28)$$

The discriminability of two such hypotheses (d') is measured by the ratio of the difference of the means divided by the standard deviation.

$$d' = \frac{c}{\frac{\epsilon}{\sigma}\sqrt{n + c^2 + .5\sigma^2}} . \qquad (29)$$

The efficiency of differential discrimination is the ratio of d'^2 to ϵ^2

$$\eta_{D.D.} = \frac{c^2}{n + c^2 + .5\sigma^2} \cdot \frac{1}{\sigma^2} . \qquad (30)$$

This can be further simplified with very little change by noting that the range of $.5\sigma^2$ is from .25 to .50, which is very small compared to $c^2 + n$. Dropping $.5\sigma^2$ in the denominator, and expressing σ^2 in terms of c^2 and n, we obtain

$$\eta_{D.D.} = \frac{c^2}{c^2 + .5n} . \qquad (31)$$

This is also plotted in Fig. 1.

6. CONCLUSION

The various approximations to the non-central chi-square distribution (the distribution of χ'^2) available in the literature have been reviewed and compared. It is concluded that for n > 10 the Fisher approximation

$$\sqrt{\chi'^2} , \quad N\left(\sqrt{\frac{2(n + c^2)^2 - (n + 2c^2)}{2(n + c^2)}} \sqrt{\frac{n + 2c^2}{2n + 2c^2}}\right) \qquad (32)$$

is the simplest and quite adequate for use in models of detection and differential discrimination. Based on this approximation the efficiency of a specific decision device has been determined for detection and discrimination in additive white Gaussian noise.

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