Application of Perturbation Methods to Pricing Credit and Equity Derivatives

by

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To my parents, my wife and my little girl Emma

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Introduction

It is well observed that one-dimensional models, although tractable, are not able to reproduce market data. This is the case for Black-Scholes framework in pricing equity derivatives and simple affine framework in pricing interest rate or credit derivatives. This has spurred a lot of work on developing multi-factor models. But, this extension usually leads to multi-dimensional pricing equations that are in terms of the inverse Laplace transform or that have to be solved by finite difference methods or simulations. Hence, consistent calibration to market data, which involves solving non-trivial inverse problems, becomes difficult.

Perturbation methods were first introduced by Fouque, Papanicolaou, and Sircar to circumvent this issue in pricing equity derivatives with stochastic volatility extending the Black-Scholes framework. [18] gives a collective introduction to this approach based on a series of papers written by them; see also related work [11], [27] and [26]. Since then, perturbation methods continue to find applications in pricing and modeling of other financial derivatives. In this methodology, one starts from a base model and sets some parameter that has been shown empirically important to be stochastic, $\theta_t = f(Y_t, Z_t)$, in which Y_t is a fast evolving factor and Z_t is a slow evolving factor, which follow the dynamics:

$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^1,$$

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta}g(Z_t)dW_2^2,$$

with the small parameter ϵ corresponding to the short time scale of the process Y and the large parameter $1/\delta$ corresponding to the long time scale of the process Z. In fact, Y can be any ergodic fast evolving process. The approximation for prices is given by a leading term plus the correction terms, and the explicit form of these are obtained by solving a sequence of PDEs resulted from an asymptotic expansion. When the payoff function (terminal condition) is smooth, the accuracy of the approximation can be proved by maximum principle arguments. On the other hand, when the payoff function is not smooth, the accuracy of the approximation can be proved by mollifying the payoff function first. As we learn from the work of Fouque, Papanicolaou, and Sircar, the most notable advantage of the perturbation approach over the traditional approaches are: (i) The resulting approximate prices are independent of details of the model specification: independent of specification of function f; (ii) It reduces the number of parameters to be calibrated by grouping them into so-called "market parameters" V^ϵs and V^δs; (iii) Most importantly, it leads to approximate, but *explicit*, closed-form solutions for prices, which greatly facilitates the calibration to market data.

In this thesis, we further explore the application of perturbation methods in modeling and pricing equity and credit derivatives. We are interested in developing hybrid models that can be used in jointly pricing and hedging credit and equity derivatives. Our goal is to bridge previous separate works on pricing credit and equity. Tractable models that are able to capture market behavior are developed by applying the techniques of multi-scale perturbations to the reduced form defaultable stock framework with stochastic volatility, stochastic interest rates. We are also interested in developing efficient models for multi-name credit pricing. The thesis is organized as follows. In Chapter II, we propose a unified framework for pricing credit and equity derivatives that incorporates stochastic volatility, default intensity, and interest rates. We demonstrate the model can be jointly calibrated to the bond and equity options of a same company. It is observed that the implied CDS spread matches the market CDS spread. In Chapter III, we study the pricing of convertible bonds and barrier and lookback options in the framework of Chapter II. Applying perturbation methods, we are able to reduce the dimension of the free-boundary problem for pricing convertible bond and to solve the corresponding Dirichlet and mixed (Dirichlet and Neumann) boundary value problems to approximate prices of barrier and lookback options. In Chapter IV, we extend Linetsky's negative-power intensity model [29] by introducing a fast evolving factor. We show that the resulting approximation for derivatives prices are Linetsky's prices with a *"Greek"* correction term, and we derive the approximations for double barrier options prices. In Chapter V, we study the stochastic parameter effect on a top-down model proposed in [14] for multi-name credit, where the default process is a time-changed birth process. We analyze the effect of stochastic volatility and stochastic mean reversion on loss distributions. We also perform a calibration exercise which shows that the introduction of stochastic parameter bring in more flexibility and improve the fitting to the market data.

CHAPTER II

A Unified Framework for Pricing Credit and Equity Derivatives [1]

2.1 Introduction

Our purpose is to build an intensity-based modeling framework that can be used in trading and calibrating across the credit and equity markets. The same company has stocks, stock options, bonds, credit default swaps on these bonds, and several other derivatives. When this company defaults, the payoffs of all of these instruments are affected; and therefore, their prices all contain information about the default risk of the company.

To build such a model, we specify the default intensity of the company. We also want to match the Treasury yield curve; and hence, we allow for stochastic interest rates. Further, we take into account the fact that the stocks can default along with the bonds. We also account for the stochastic volatility in the modeling of the stocks since even the index options (when there is no risk of default) possess implied volatility skew. We also want to jointly calibrate our model to the implied volatility surface and to the term structure of the corporate bond. Therefore, it is desirable to determine the equity option prices and bond prices explicitly.

To develop a feasible framework that establishes the items listed above, we use the

[1] This chapter is based on [6].

multi-scale modeling approach of [20] (which considers the multi-scale framework in the context of option pricing for stochastic volatility models). The default intensity of the company is driven by two processes that evolve on a slow and fast scale. The volatility of the stock on the other hand evolves only on a fast scale. We use the Vasicek model for the interest rate dynamics. Even though the interest rate is stochastic in our model, we are able to obtain explicit asymptotic pricing formulas. We calibrate the parameters in our pricing formulas to the stock-option implied volatility surface and the yield curve of the defaultable bond observed on a given day. Our model also takes input from the Treasury yield curve, historical stock prices, and historical spot rate data to estimate some of its parameters (see Section 2.4).

After calibrating, we test the effectiveness our model. The model-implied CDS spread time series matches the observed CDS spread time series of Ford Motor Company for over a long period of time; see Figures 2.1 and 2.2. This is a striking observation since we did not make use of the CDS spread data in our calibration. We developed the CDS spread formula under the assumption that if the bond the CDS is written on defaults prior to maturity, it recovers a constant fraction (recovery rate) of its predefault value. Therefore, one of the parameters that is required to determine the model-implied CDS spread is the recovery rate. This parameter is estimated from the option and bond data.

On the equity side, our model is able to produce implied volatility surfaces that match the data closely. We compare the implied volatility surfaces that our model produces to those of [20]. We see that even for longer maturities our model has a prominent skew; compare Figures 2.5 and 2.6. Even when we ignore the stochastic volatility effects, our model fits the implied volatility of the Ford Motor Company well and surpasses performance of the model of [20]; see Figure 2.4. This points to the importance of accounting for the default risk for companies with low ratings. On the other hand, by using index options (when there is no risk of default), we measure the effect of incorporating stochastic interest rates into the prices of options by comparing our results to [18] and [20].

Our modeling framework can be thought of as a hybrid of the models of [18], which only considers pricing options in a stochastic volatility model with constant interest rate, and [30], which only considers a framework for pricing derivatives on bonds. Neither of these models has the means to transfer information from the equity markets to bond market or vice versa, and our framework fills this gap. On the other hand, there is recent literature on pricing options on defaultable stocks; see e.g., [5], [8], [29] and [9]. Since these models take the interest rate to be deterministic, they do not produce reasonable yield spread curves. Therefore, these models transfer the information from the credit market to the stock option market but not vice versa. Also, our model specification differs from those of [8] and [29] since we do not take the default intensity as a function of the stock price. As opposed to the approaches of [8], [29] and [9], instead of simplifying the modeling assumptions (by taking the volatility and intensity to be a function of the stock price) or using the inverse Fourier transform, we use asymptotic expansions to provide explicit pricing formulas for stock options and bonds in a stochastic interest rate framework. On the other hand, our calibration exercise differs from that of [9] since they perform a time series analysis to obtain the parameters of the underlying factors (from the the stock option prices and credit default swap spread time series), whereas we calibrate our pricing parameters to the daily implied volatility surface and bond term structure data. The effort in [9] is spent on jointly estimating the default intensity and the volatility. Our effort is concentrated on daily prediction of the CDS spread only using the data from the bond term structure and implied volatility surface of the options.

The rest of the chapter is organized as follows: In Section 2.2, we introduce our modeling framework and describe the credit and equity derivatives we will consider and obtain an expression for the CDS spread under the assumption that the recovery of a bond that defaults is a constant of its predefault value. In Section 2.3, we introduce the asymptotic expansion method. We obtain explicit (asymptotic) prices for bonds and equity options in Section 2.3.3. In Section 2.4, we describe the calibration of our parameters and discuss our empirical results. Figures are located at the end of the chapter.

2.2 A Framework for Pricing Equity and Credit derivatives

2.2.1 The Model

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a complete probability space supporting (i) correlated standard Brownian motions $\vec{W}_t = (W_t^0, W_t^1, W_t^2, W_t^3, W_t^4)$, $t \geq 0$, with

$$\mathbb{E}[W_t^0, W_t^i] = \rho_i t, \quad \mathbb{E}[W_t^i, W_t^j] = \rho_{ij} t, \quad i,j \in \{1,2,3,4\},\ t \geq 0, \tag{2.1}$$

for some constants $\rho_i, \rho_{i,j} \in (-1,1)$, and (ii) a Poisson process N independent of $\vec{W}$. Let us introduce the Cox process (time-changed Poisson process) $\tilde{N}_t \triangleq N(\int_0^t \lambda_s ds)$, $t \geq 0$, where

$$\begin{aligned} \lambda_t &= f(Y_t, Z_t), \\ dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^2, \quad Y_0 = y, \\ dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_t^3, \quad Z_0 = z, \end{aligned} \tag{2.2}$$

in which ϵ, δ are (small) positive constants and f is a strictly positive, bounded, smooth function. We also assume that the functions c and g satisfy Lipschitz continuity and growth conditions so that the diffusion process for Z_t has a unique strong solution. We model the time of default as

$$\tau = \inf\{t \geq 0 : \tilde{N}_t = 1\}. \tag{2.3}$$

We also take interest rate to be stochastic and model it as an Ornstein-Uhlenbeck process

$$dr_t = (\alpha - \beta r_t)dt + \eta dW_t^1, \quad r_0 = r, \tag{2.4}$$

for positive constants α, β, and η.

We model the stock price as the solution of the stochastic differential equation

$$d\bar{X}_t = \bar{X}_t \left(r_t dt + \sigma_t dW_t^0 - d\left(\tilde{N}_t - \int_0^{t\wedge\tau} \lambda_u du \right) \right), \quad \bar{X}_0 = x, \tag{2.5}$$

where the volatility is stochastic and is defined through

$$\sigma_t = \sigma(\tilde{Y}_t); \quad d\tilde{Y}_t = \left(\frac{1}{\epsilon}(\tilde{m} - \tilde{Y}_t) - \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}} \Lambda(\tilde{Y}_t) \right) dt + \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}} dW_t^4, \quad \tilde{Y}_0 = \tilde{y}. \tag{2.6}$$

Here, Λ is a smooth, bounded function of one variable, which represents the market price of volatility risk. The function σ is also a bounded, smooth function. Note that the discounted stock price is a martingale under the measure $\mathbb{P}$, and at the time of default, the stock price jumps down to zero. The pre-banktruptcy stock price coincides with the solution of

$$dX_t = (r_t + \lambda_t)X_t dt + \sigma_t X_t dW_t^0, \quad X_0 = x. \tag{2.7}$$

It will be useful to keep track of different flows of information. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration of $\vec{W}$. Denote the default indicator process by $I_t = 1_{\{\tau \leq t\}}$, $t \geq 0$, and let $\mathbb{I} = \{\mathcal{I}_t, t \geq 0\}$ be the filtration generated by I. Finally, let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be an enlargement of $\mathbb{F}$ such that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{I}_t$, $t \geq 0$.

Since we will take ϵ and δ to be small positive constants, the processes Y and $\tilde{Y}$ are fast mean reverting, and Z evolves on a slower time scale. See [20] for an exposition and motivation of multi-scale modeling in the context of stochastic volatility models.

We note that our specification of the intensity of default coincides with that of [30], who considered only a framework for pricing credit derivatives. Our stock price specification is similar to that of [29] and [8] who considered a framework for only pricing equity options on defaultable stocks. Our volatility specification, on the other hand, is in the spirit of [18].

[5] considered a similar modeling framework to the one considered here, but the interest rate was taken to be deterministic. In this paper, by extending this modeling framework to incorporate stochastic interest rates we are able to consistently price credit and equity derivatives and produce more realistic yield curve and implied volatility surfaces. We are also be able to take the equity option surface and the yield curve data as given and predict the credit default swap spread on a given day. Testing our model prediction against real data demonstrates the power of our pricing framework.

2.2.2 Equity and Credit Derivatives

In our framework we will price European options, bonds, and credit default swaps of the same company in a consistent way.

1. The price of a European call option with maturity T and strike price K is given by

$$\begin{aligned} C(t;T,K) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(\bar{X}_T - K)^+ 1_{\{\tau>T\}}\middle|\mathcal{G}_t\right] \\ &= 1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)(X_T-K)^+\middle|\mathcal{F}_t\right], \end{aligned} \tag{2.8}$$

in which the equality follows from Lemma 5.1.2 of [7]. (This lemma, which lets us write a conditional expectation with respect to $\mathcal{G}_t$ in terms of conditional expectations with respect to $\mathcal{F}_t$, will be used in developing several identities below). Also, see [29] and [8] for a similar computation.

On the other hand, the price of a put option with the same maturity and strike price is

$$
\begin{aligned}
\mathrm{Put}(t;T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(K-X_T)^+ 1_{\{\tau>T\}}\Big|\mathcal{G}_t\right] + \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) K 1_{\{\tau\le T\}}\big|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}\Bigg(\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)(K-X_T)^+\Big|\mathcal{F}_t\right] \\
&+ K\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right] - K\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)\Big|\mathcal{F}_t\right]\Bigg).
\end{aligned}
\tag{2.9}
$$

2. Consider a defaultable bond with maturity T and par value of 1 dollar. We assume that if the issuer company defaults prior to maturity, the holder of the bond recovers a constant fraction $1-l$ of the pre-default value, with $l \in [0,1]$. The price of such a bond is

$$
\begin{aligned}
B^c(t;T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)1_{\{\tau>T\}} + \exp\left(-\int_t^\tau r_s ds\right)1_{\{\tau\le T\}}(1-l)B^c(\tau-;T)\Big|\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\,\lambda_s)ds\right)\Big|\mathcal{F}_t\right],
\end{aligned}
\tag{2.10}
$$

on $\{\tau > t\}$, see [15] and [32].

3. Consider a credit default swap (CDS) written on B^c, which is a insurance against losses incurred upon default from holding a corporate bond. The protection buyer pays a fixed premium, the so-called CDS spread, to the protection seller. The premium is paid on fixed dates $\mathcal{T} = (T_1, \cdots, T_M)$, with T_M being the maturity of the CDS contract. We denote the CDS spread at time t by $c^{ds}(t;\mathcal{T})$. Our purpose is to determine a fair value for the CDS spread so that what the protection buyer is expected to pay, the value of the premium leg of the contract, is equal to what the protection seller is expected to pay, the value of the protection leg of the contract. For a more detailed description of the CDS contract, see [7] or [31].

The present value of the premium leg of the contract is

$$
\begin{aligned}
\text{Premium}(t;\mathcal{T}) &= c^{ds}(t;\mathcal{T})\,\mathbb{E}\left[\sum_{m=1}^{M}\exp\left(-\int_t^{T_m} r_s ds\right)1_{\{\tau>T_m\}}\Big|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}c^{ds}(t;\mathcal{T})\sum_{m=1}^{M}\mathbb{E}\left[\exp\left(-\int_t^{T_m}(r_s+\lambda_s)ds\right)\Big|\mathcal{F}_t\right].
\end{aligned}
\tag{2.11}
$$

The present value of the protection leg of the contract under our assumption of *recovery of market value* is

$$
\text{Protection}(t;\mathcal{T}) = 1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^{\tau} r_s ds\right)1_{\{\tau\le T_M\}}l\,B^c(\tau-;T_M)\Big|\mathcal{G}_t\right]
\tag{2.12}
$$

Adding (2.10) and (2.12) we obtain

$$
\begin{aligned}
\text{Protection}(t;\mathcal{T}) + B^c(t;T_M) &= \mathbb{E}\left[\exp\left(-\int_t^{T} r_s ds\right)1_{\{\tau>T\}} + \exp\left(-\int_t^{\tau} r_s ds\right)1_{\{\tau\le T\}}B^c(\tau-;T)\Big|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^{T_M} r_s ds\right)\Big|\mathcal{F}_t\right],
\end{aligned}
\tag{2.13}
$$

where the last equality is obtained by setting $l = 0$ in (2.10).

Now, the CDS spread can be determined, by setting $\text{Protection}(t;\mathcal{T}) = \text{Premium}(t;\mathcal{T})$ and using equations (2.11) and (2.13), as

$$
c^{ds}(t;\mathcal{T}) = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[\exp\left(-\int_t^{T_M} r_s ds\right)\Big|\mathcal{F}_t\right] - \mathbb{E}\left[\exp\left(-\int_t^{T_M}(r_s+l\,\lambda_s)ds\right)\Big|\mathcal{F}_t\right]}{\sum_{m=1}^{M}\mathbb{E}\left[\exp\left(-\int_t^{T_m}(r_s+\lambda_s)ds\right)\Big|\mathcal{F}_t\right]}.
\tag{2.14}
$$

2.3 Explicit Pricing Formulas for Credit and Equity Derivatives

2.3.1 Pricing Equation

Let $P^{\epsilon,\delta}$ denote

$$
P^{\epsilon,\delta}(t, X_t, r_t, Y_t, \tilde{Y}_t, Z_t) = \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\lambda_s)ds\right)h(X_T)\Big|\mathcal{F}_t\right].
\tag{2.15}
$$

When $l = 1$ and $h(X_T) = (X_T - K)^+$, $P^{\epsilon,\delta}$ is the price of a call option (on a defaultable stock). On the other hand, when $h(X_T) = 1$, $P^{\epsilon,\delta}$ becomes the price of a defaultable bond.

Using the Feynman-Kac formula, we can characterize $P^{\epsilon,\delta}$ as the solution of

$$\begin{aligned}
&\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}(t,x,r,y,\tilde{y},z) = 0, \\
&P^{\epsilon,\delta}(T,x,r,y,\tilde{y},z) = h(x),
\end{aligned} \tag{2.16}$$

where the partial differential operator $\mathcal{L}^{\epsilon,\delta}$ is defined as

$$\mathcal{L}^{\epsilon,\delta} \triangleq \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3, \tag{2.17}$$

in which

$$\begin{aligned}
&\mathcal{L}_0 \triangleq \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} + \tilde{\nu}^2\frac{\partial^2}{\partial \tilde{y}^2} + (\tilde{m}-\tilde{y})\frac{\partial}{\partial \tilde{y}} + 2\rho_{24}v\tilde{v}\frac{\partial^2}{\partial y\partial \tilde{y}}, \\
&\mathcal{L}_1 \triangleq \rho_2\sigma(\tilde{y})\nu\sqrt{2}x\frac{\partial^2}{\partial x\partial y} + \rho_{12}\eta\nu\sqrt{2}\frac{\partial^2}{\partial r\partial y} + \rho_4\sigma(\tilde{y})\tilde{\nu}\sqrt{2}x\frac{\partial^2}{\partial x\partial \tilde{y}} + \rho_{14}\eta\tilde{\nu}\sqrt{2}\frac{\partial^2}{\partial r\partial \tilde{y}} - \Lambda(\tilde{y})\tilde{\nu}\sqrt{2}\frac{\partial}{\partial \tilde{y}}, \\
&\mathcal{L}_2 \triangleq \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(\tilde{y})x^2\frac{\partial^2}{\partial x^2} + (r+f(y,z))x\frac{\partial}{\partial x} + (\alpha-\beta r)\frac{\partial}{\partial r} + \sigma(\tilde{y})\eta\rho_1 x\frac{\partial^2}{\partial x\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2} - (r+l\,f(y,z))\cdot, \\
&\mathcal{M}_1 \triangleq \sigma(\tilde{y})\rho_3 g(z)x\frac{\partial^2}{\partial x\partial z} + \eta\rho_{13}g(z)\frac{\partial^2}{\partial r\partial z}, \quad \mathcal{M}_2 \triangleq c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2}, \\
&\mathcal{M}_3 \triangleq \rho_{23}\nu\sqrt{2}g(z)\frac{\partial^2}{\partial y\partial z} + \rho_{34}\tilde{\nu}\sqrt{2}g(z)\frac{\partial^2}{\partial \tilde{y}\partial z}.
\end{aligned}$$

2.3.2 Asymptotic Expansion

We construct an asymptotic expansion for $P^{\epsilon,\delta}$ as $\epsilon, \delta \to 0$. First, we consider an expansion of $P^{\epsilon,\delta}$ in powers of $\sqrt{\delta}$

$$P^{\epsilon,\delta} = P_0^{\epsilon} + \sqrt{\delta}P_1^{\epsilon} + \delta P_2^{\epsilon} + \cdots \tag{2.18}$$

By inserting (2.18) into (2.16) and comparing the δ^0 and δ terms, we obtain that P_0^{ϵ} satisfies

$$\begin{aligned}
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_0^{\epsilon} = 0, \\
&P_0^{\epsilon}(T,x,r,y,\tilde{y},z) = h(x),
\end{aligned} \tag{2.19}$$

and that P_1^{ϵ} satisfies

$$\begin{aligned}
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_1^{\epsilon} = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right)P_0^{\epsilon}, \\
&P_1^{\epsilon}(T,x,y,\tilde{y},z,r) = 0.
\end{aligned} \tag{2.20}$$

Next, we expand the solutions of (2.19) and (2.20) in powers of $\sqrt{\epsilon}$

$$P_0^\epsilon = P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0} + \cdots \tag{2.21}$$

$$P_1^\epsilon = P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1} + \epsilon^{3/2}P_{3,1} + \cdots \tag{2.22}$$

Inserting the expansion for P_0^ϵ into (2.19) and matching the $1/\epsilon$ terms gives $\mathcal{L}_0 P_0 = 0$. We choose P_0 not to depend on y and $\tilde{y}$ because the other solutions have exponential growth at infinity (see e.g. [20]). Similarly, by matching the $1/\sqrt{\epsilon}$ terms in (2.19) we obtain that $\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0$. Since $\mathcal{L}_1$ takes derivatives only with respect to y and $\tilde{y}$, we observe that $\mathcal{L}_0 P_{1,0} = 0$. We choose $P_{1,0}$ not to depend on y and $\tilde{y}$.

Now equating the order-one terms in the expansion of (2.19) and using the fact that $\mathcal{L}_1 P_{1,0} = 0$, we get that

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0, \tag{2.23}$$

which is a Poisson equation for $P_{2,0}$ (see e.g. [18]). The solvability condition for this equation requires that

$$\langle \mathcal{L}_2 \rangle P_0 = 0, \tag{2.24}$$

where $\langle\cdot\rangle$ denotes the averaging with respect to the invariant distribution of $(Y_t, \tilde{Y}_t)$, whose density is given by

(2.25)
$$\Psi(y,\tilde{y}) = \frac{1}{2\pi\nu\tilde{\nu}} \exp\left\{-\frac{1}{2(1-\rho_{24}^2)}\left[\left(\frac{y-m}{\nu}\right)^2 + \left(\frac{\tilde{y}-\tilde{m}}{\tilde{\nu}}\right)^2 - 2\rho_{24}\frac{(y-m)(\tilde{y}-\tilde{m})}{\nu\tilde{\nu}}\right]\right\}.$$

Let us denote

$$\bar{\sigma}_1 \triangleq \langle \sigma(\tilde{y})\rangle, \quad \bar{\sigma}_2^2 \triangleq \langle \sigma^2(\tilde{y})\rangle, \quad \bar{\lambda}(z) = \langle f(y,z)\rangle. \tag{2.26}$$

To demonstrate the effect of averaging on $\mathcal{L}_2$, let us write

$$(2.27)\quad \langle\mathcal{L}_2\rangle := \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}_2^2 x^2 \frac{\partial^2}{\partial x^2} + (r+\bar{\lambda}(z))x\frac{\partial}{\partial x} + (\alpha-\beta r)\frac{\partial}{\partial r} + \bar{\sigma}_1\eta\rho_1 x\frac{\partial^2}{\partial x\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2} - (r+l\,\bar{\lambda}(z))\cdot$$

Together with the terminal condition

$$(2.28)\qquad P_0(T,x,r,z) = h(x),$$

equation (2.24) defines the leading order term P_0. On the other hand from (2.23), we can also deduce that

$$(2.29)\qquad P_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)P_0.$$

Matching the $\sqrt{\epsilon}$ order terms in the expansion of (2.19) yields

$$(2.30)\qquad \mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0,$$

which is a Poisson equation for $P_{3,0}$. The solvability condition for this equation requires that

$$(2.31)\qquad \langle\mathcal{L}_2 P_{1,0}\rangle = -\langle\mathcal{L}_1 P_{2,0}\rangle = \langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle P_0,$$

which along with the terminal condition

$$(2.32)\qquad P_{1,0}(T,x,r,z) = 0,$$

completely identifies the function $P_{1,0}$. To obtain the second equality in (2.31) we used (2.29).

Next, we will express the right-hand side of (2.31) more explicitly. To this end, let ψ, κ, and ϕ be the solutions of the Poisson equations

(2.33)

$$\mathcal{L}_0\psi(\tilde{y}) = \sigma(\tilde{y}) - \bar{\sigma}_1 \quad \mathcal{L}_0\kappa(\tilde{y}) = \sigma^2(\tilde{y}) - \bar{\sigma}_2^2, \quad \text{and} \quad \mathcal{L}_0\phi(y,z) = (f(y,z) - \bar{\lambda}(z)),$$

respectively. First observe that

$$(2.34)\quad (\mathcal{L}_2 - \langle\mathcal{L}_2\ \rangle)P_0 = \frac{1}{2}(\sigma^2(\tilde{y}) - \bar{\sigma}_2^2)x^2\frac{\partial^2 P_0}{\partial x^2} + (\sigma(\tilde{y}) - \bar{\sigma}_1)\eta\rho_1 x\frac{\partial^2 P_0}{\partial x\partial r} + l\,(f(y,z) - \bar{\lambda}(z))\left(x\frac{\partial P_0}{\partial x} - P_0\right).$$

Now, along with (2.33), we can write

$$\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)P_0 = \frac{1}{2}\kappa(y,\tilde{y})x^2\frac{\partial^2 P_0}{\partial x^2} + \psi(y,\tilde{y})\eta\rho_1 x\frac{\partial^2 P_0}{\partial x\partial r} + l\,\phi(y,\tilde{y},z)\left(x\frac{\partial P_0}{\partial x} - P_0\right). \tag{2.35}$$

Applying the differential operator $\mathcal{L}_1$ to the last expression yields

(2.36)

$$\begin{aligned}
\langle\mathcal{L}_1\mathcal{L}_0^{-1}(L_2 - \langle\mathcal{L}_2\rangle)\rangle P_0 &= l\,\rho_2\nu\sqrt{2}\langle\sigma\phi_y\rangle x^2\frac{\partial P_0}{\partial x^2} + l\,\rho_{12}\eta\nu\sqrt{2}\langle\phi_y\rangle\frac{\partial P_0}{\partial r}\left(x\frac{\partial P_0}{\partial x} - P_0\right)\\
&+ \rho_4\tilde{\nu}\sqrt{2}\left(\frac{1}{2}\langle\sigma\kappa_{\tilde{y}}\rangle x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right) + \langle\sigma\psi_{\tilde{y}}\rangle\eta\rho_1 x\frac{\partial P_0}{\partial x}x\frac{\partial^2 P_0}{\partial x\partial r}\right)\\
&+ \rho_{14}\eta\tilde{\nu}\sqrt{2}\left(\frac{1}{2}\langle\kappa_{\tilde{y}}\rangle\frac{\partial P_0}{\partial r}x^2\frac{\partial^2 P_0}{\partial x^2} + \langle\psi_{\tilde{y}}\rangle\eta\rho_1 x\frac{\partial^3 P_0}{\partial x\partial r^2}\right)\\
&- \tilde{\nu}\sqrt{2}\left(\frac{1}{2}\langle\Lambda\kappa_{\tilde{y}}\rangle x^2\frac{\partial P_0}{\partial x^2} + \langle\Lambda\psi_{\tilde{y}}\rangle\eta\rho_1 x\frac{\partial^2 P_0}{\partial x\partial r}\right).
\end{aligned}$$

Finally, we insert the expression for P_1^ϵ in (2.22) into (2.20) and collect the terms with the same powers of ϵ. Arguing as before, we obtain that $P_{0,1}$ is independent of y and $\tilde{y}$ and satisfies:

$$\langle\mathcal{L}_2\rangle P_{0,1} = -\langle\mathcal{M}_1\rangle P_0, \quad P_{0,1}(T,x) = 0. \tag{2.37}$$

2.3.3 Explicit Pricing Formula

We approximate $P^{\epsilon,\delta}$ defined in (2.15) by

$$\widetilde{P}^{\epsilon,\delta} = P_0 + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1}. \tag{2.38}$$

It follows from arguments similar to [20] and [30] that for a fixed $(t,x,r,y,\tilde{y},z)$, there exists a constant C such that $|P^{\epsilon,\delta} - \widetilde{P}^{\epsilon,\delta}| \le C\cdot(\epsilon+\delta)$ when h is smooth, and $|P^{\epsilon,\delta} - \widetilde{P}^{\epsilon,\delta}| \le C\cdot(\epsilon\ln(\epsilon)+\delta+\sqrt{\epsilon\delta})$ when h is a put or a call pay-off. In what follows, we will obtain P_0, $P_{1,0}$ and $P_{0,1}$ explicitly.

Our first objective is to develop a closed-form expression for P_0, the solution of (2.24) and (2.28).

Proposition II.1. *The leading order term P_0 in (2.38) is given by:*

(2.39)

$$P_0(t,x,z,r) = B_0^c(t,r;z,T,l)\int_{-\infty}^{\infty} h(\exp(u))\frac{1}{\sqrt{2\pi v(t,T)}}\exp\Big(-\frac{(u-m(t,T))^2}{2v(t,T)}\Big)du,$$

where

$$B_0^c(t,r;z,T,l) \triangleq \exp\big(-l\bar{\lambda}(z)(T-t)+a(T-t)-b(T-t)r\big), \tag{2.40}$$

in which the functions $a(s)$ and $b(s)$ are defined as:

$$a(s) = \left(\frac{\eta^2}{2\beta^2}-\frac{\alpha}{\beta}\right)s+\left(\frac{\eta^2}{\beta^3}-\frac{\alpha}{\beta^2}\right)(\exp(-\beta s)-1)-\frac{\eta^2}{4\beta^3}(\exp(-2\beta s)-1) \tag{2.41}$$

and $b(s) = (1-\exp(-\beta s))/\beta$. On the other hand,

$$\begin{aligned} v_{t,T} = &\left(\bar{\sigma}_2^2+\frac{2\eta\rho_1\bar{\sigma}_1}{\beta}+\frac{\eta^2}{\beta^2}\right)(T-t)+\left(\frac{2\eta\rho_1\bar{\sigma}_1}{\beta^2}+\frac{2\eta^2}{\beta^3}\right)\exp(-\beta(T-t)) \\ &-\frac{\eta^2}{2\beta^3}\exp(-2\beta(T-t))-\left(\frac{2\eta\rho_1\bar{\sigma}_1}{\beta^2}+\frac{3\eta^2}{2\beta^3}\right), \end{aligned} \tag{2.42}$$

and

$$m_{t,T} = \ln(x)+\bar{\lambda}\cdot(T-t)-a(T-t)+b(T-t)r-\frac{1}{2}v(t,T). \tag{2.43}$$

Proof. By applying Feynman-Kac theorem to (2.24) and (2.28), we have that

$$P_0(t,x,z,r) = \mathbb{E}\left[\exp\left(-\int_t^T(r_s+l\bar{\lambda}(z))ds\right)h(S_T)\Big|S_t=x, r_t=r\right], \tag{2.44}$$

where the dynamics of S is given by

$$dS_t = (r_t+\bar{\lambda}(z))S_t dt+\bar{\sigma}_2 S_t d\widetilde{W}_t^0, \tag{2.45}$$

in which $\widetilde{W}^0$ is a Wiener process whose correlation with W^1 is $\bar{\rho}_1 = \frac{\bar{\sigma}_1}{\bar{\sigma}_2}\rho_1$.

Let us define

$$\tilde{P}_0(t,x,z,r) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)h(\widetilde{S}_T)\Big|\widetilde{S}_t=x, r_t=r\right], \tag{2.46}$$

in which

$$d\widetilde{S}_t = r_t \widetilde{S}_t dt + \bar{\sigma}_2 \widetilde{S}_t d\widetilde{W}_t^0. \tag{2.47}$$

Then,

$$P_0(t, x, z, r) = e^{-l\bar{\lambda}(T-t)} \tilde{P}_0(t, x \exp(\bar{\lambda}(z)(T-t)), z, r). \tag{2.48}$$

Now, by following [22] we change the probability measure $\mathbb{P}$ to the forward measure $\mathbb{P}^T$ through the Radon-Nikodym derivative

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\exp\left(-\int_0^T r_s ds\right)}{B(0,T)}, \tag{2.49}$$

where

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t\right]. \tag{2.50}$$

We can obtain the following representation of $\tilde{P}_0$ using the T-forward measure

$$\tilde{P}_0(t, \widetilde{S}_t, z, r_t) = B(t,T)\mathbb{E}^T\left[h(\widetilde{S}_T)|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^T\left[h(F_T)|\mathcal{F}_t\right], \tag{2.51}$$

in which

$$F_t \triangleq \frac{\widetilde{S}_t}{B(t,T)}, \tag{2.52}$$

which is a $\mathbb{P}^T$ martingale. Note that an explicit expression for $B(t,T)$ is available since r_t is a Vasicek model, and it is given in terms of the functions a and b:

$$B(t,T) = \exp(a(T-t) - b(T-t)r_t). \tag{2.53}$$

By applying Itô's formula to (2.52), we observe that the dynamics of F are

$$dF_t = F_t(\bar{\sigma}_1 d\widetilde{W}_t^0 + b(T-t)\eta dW_t^1). \tag{2.54}$$

Given X_t and $B(t,T)$, the random variable $\ln F_T$ is normally distributed with variance

$$
\begin{aligned}
v_{t,T} &= \bar{\sigma}_2^2(T-t) + \eta^2 \int_t^T b^2(T-s)ds + 2\eta\bar{\rho}_1\bar{\sigma}_2 \int_t^T b(T-s)ds \\
&= \left(\bar{\sigma}_2^2 + \frac{2\eta\bar{\rho}_1\bar{\sigma}_2}{\beta} + \frac{\eta^2}{\beta^2}\right)(T-t) + \left(\frac{2\eta\bar{\rho}_1\bar{\sigma}_2}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)\exp(-\beta(T-t)) \\
&- \frac{\eta^2}{2\beta^3}\exp(-2\beta(T-t)) - \left(\frac{2\eta\bar{\rho}_1\bar{\sigma}_2}{\beta^2} + \frac{3\eta^2}{2\beta^3}\right),
\end{aligned} \tag{2.55}
$$

and mean

(2.56)

$$
m(t,T) = \ln F_t - \frac{1}{2}\int_t^T (\bar{\sigma}_2^2 + b^2(T-s)\eta^2 + \bar{\rho}_1\bar{\sigma}_2 b(T-s)\eta)ds = \ln\left(\frac{\widetilde{S}_t}{B(t,T)}\right) - \frac{1}{2}v_{t,T}.
$$

Now the result immediately follows. □

An immediate corollary of the last proposition is the following:

Corollary II.2. *i) When $l=1$, $h(x)=(x-K)^+$, then (2.39) becomes*

$$
C_0(t,x,z,r) = xN(d_1) - KB_0^c(t,r;z,T,1)N(d_2), \tag{2.57}
$$

in which N is the standard normal cumulative distribution function and

$$
d_{1,2} = \frac{\ln \frac{x}{KB_0^c(t,r;z,T,1)} \pm \frac{1}{2}v(t,T)}{\sqrt{v(t,T)}}. \tag{2.58}
$$

ii) When $l=1$, and $h(x)=(K-x)^+$, then (2.39) becomes

$$
Put_0(t,x,z,r) = -x + xN(d_1) - KB_0^c(t,r;z,T,1)N(d_2) + KB_0^c(t,r;z,T,0). \tag{2.59}
$$

iii) When $h(x)=1$, then (2.39) coincides with (3.30) in [30].

Proposition II.3. *The correction term $\sqrt{\epsilon}P_{1,0}$ is given by*

$$
\begin{aligned}
\sqrt{\epsilon}P_{1,0} &= -(T-t)\left(V_1^\epsilon x^2\frac{\partial^2 P_0}{\partial x^2} + V_2^\epsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right)\right) \\
&+ l\,V_3^\epsilon\left(-x\frac{\partial^2 P_0}{\partial x\partial\alpha} - \frac{\partial P_0}{\partial\alpha}\right) + V_4 x^2\frac{\partial^3 P_0}{\partial x^2\partial\alpha} + V_5^\epsilon x\frac{\partial^2 P_0}{\partial\eta\partial x} + V_6^\epsilon x\frac{\partial^2 P_0}{\partial x\partial\alpha},
\end{aligned} \tag{2.60}
$$

in which

$$
\begin{aligned}
&V_1^\epsilon = \sqrt{\epsilon}(l\,\rho_2\nu\sqrt{2}\langle\sigma\phi_y\rangle - \tilde{\nu}\sqrt{2}\frac{1}{2}\langle\Lambda\kappa_{\tilde{y}}\rangle), \quad V_2^\epsilon = \sqrt{\epsilon}(\frac{1}{2}\rho_4\tilde{\nu}\sqrt{2}\langle\sigma\kappa_{\tilde{y}}\rangle),\\
&V_3^\epsilon = \sqrt{\epsilon}(\rho_{12}\eta\nu\sqrt{2}\langle\phi_y\rangle), \quad V_4^\epsilon = -\sqrt{\epsilon}(\frac{1}{2}\rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\kappa_{\tilde{y}}\rangle - \rho_4\tilde{\nu}\sqrt{2}\langle\sigma\psi_{\tilde{y}}\rangle\eta\rho_1 + \rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\psi_{\tilde{y}}\rangle\bar{\sigma}_1\rho_1^2),\\
&V_5^\epsilon = -\sqrt{\epsilon}(\rho_{14}\eta\tilde{\nu}\sqrt{2}\langle(\psi)_{\tilde{y}}\rangle\rho_1), \quad V_6^\epsilon = \sqrt{\epsilon}(-\rho_4\tilde{\nu}\sqrt{2}\langle\sigma\psi_{\tilde{y}}\rangle\eta\rho_1 + \rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\psi_{\tilde{y}}\rangle\bar{\sigma}_1\rho_1^2 - \tilde{\nu}\sqrt{2}\langle\Lambda\psi_{\tilde{y}}\rangle\eta\rho_1).
\end{aligned} \tag{2.61}
$$

Proof. Recall that $P_{1,0}$ is the solution of (2.31) and (2.32) and that the right-hand-side of (2.31) is given by (2.36). The result is a simple algebraic exercise given the following four observations:

1) $x^n \frac{\partial^n}{\partial x^n}$ commutes with $\langle\mathcal{L}_2\rangle$.

2) $-(T-t)(x^n \frac{\partial^n}{\partial x^n})P_0$ solves:

$$\langle\mathcal{L}_2\rangle u = \left(x^n \frac{\partial^n}{\partial x^n}\right) P_0, \quad u(T,x,r;z) = 0. \tag{2.62}$$

3) By differentiating (2.28) with respect to α, we see that $-\frac{\partial P_0}{\partial \alpha}$ also solves

$$\langle\mathcal{L}_2\rangle u = \frac{\partial P_0}{\partial r}, \quad u(T,x,r;z) = 0. \tag{2.63}$$

4) Using 1) and 2) above and the equation we obtain by differentiating (2.28) with respect to η, we can show that $1/\eta \cdot (\bar{\sigma}_1\rho_1 x\frac{\partial^2 P_0}{\partial x \partial \alpha} - \frac{\partial P_0}{\partial \eta})$ solves

$$\langle\mathcal{L}_2\rangle u = \frac{\partial^2 P_0}{\partial r^2}, \quad u(T,x,r;z) = 0. \tag{2.64}$$

$\square$

Remark II.4. By differentiating (2.24) with respect to r, we obtain

$$\langle\mathcal{L}_2\rangle \frac{\partial P_0}{\partial r} = -x\frac{\partial}{\partial x}P_0 + \beta\frac{\partial P_0}{\partial r} + P_0. \tag{2.65}$$

Using observation 2 in the proof of Proposition II.3, we see that

$\frac{1}{\beta}\left(-(T-t)(x\frac{\partial P_0}{\partial x} - P_0) + \frac{\partial P_0}{\partial r}\right)$ solves

$$\langle\mathcal{L}_2\rangle u = \frac{\partial P_0}{\partial r}, \quad u(T,x) = 0. \tag{2.66}$$

Now, it follows from observation 3 in the proof of Proposition II.3 that

$$-\frac{\partial P_0}{\partial \alpha} = \frac{1}{\beta}\left(-(T-t)\left(x\frac{\partial P_0}{\partial x} - P_0\right) + \frac{\partial P_0}{\partial r}\right). \tag{2.67}$$

Using this identity, we can express (2.60) only in terms of the "Greeks".

Next, we obtain an explicit expression for $P_{0,1}$, the solution of (2.37). We need some preparation first. By differentiating (2.24) with respect to z, we see that $\frac{\partial P_0}{\partial z}$ solves

$$\langle \mathcal{L}_2 \rangle u = -\bar{\lambda}'(z)x\frac{\partial P_0}{\partial x} + l\,\bar{\lambda}'(z)P_0, \quad u(T,x,r;z) = 0. \tag{2.68}$$

As a result (see Observation 2 in the proof of Propostion II.3)

$$\frac{\partial P_0}{\partial z} = (T-t)\bar{\lambda}'(z)\left(x\frac{\partial P_0}{\partial x} - l\,P_0\right), \tag{2.69}$$

from which it follows that $-\langle M_1 \rangle P_0$ can be represented as

$$-\langle M_1 \rangle P_0 = -(T-t)\bar{\lambda}'(z)\left(\bar{\sigma}_1\rho_3 g(z)\left(x^2\frac{\partial^2 P_0}{\partial x^2} + (1-l)x\frac{\partial P_0}{\partial x}\right) + \eta\rho_{13}g(z)\left(x\frac{\partial^2 P_0}{\partial x \partial r} - l\frac{\partial P_0}{\partial r}\right)\right). \tag{2.70}$$

Proposition II.5. *The correction term* $\sqrt{\delta}P_{0,1}$ *is given by*

(2.71)

$$\sqrt{\delta}P_{0,1} = V_1^\delta \frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} + (1-l)x\frac{\partial P_0}{\partial x}\right) + V_2^\delta \frac{1}{\beta}\left[x\frac{\partial^2 P_0}{\partial \alpha \partial x} - l\,\frac{\partial P_0}{\partial \alpha} + \frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} - l\,x\frac{\partial P_0}{\partial x} + l\,P_0\right) - (T-t)\left(x\frac{\partial^2 P_0}{\partial r \partial x} - l\,\frac{\partial P_0}{\partial r}\right)\right],$$

in which

$$V_1^\delta = \sqrt{\delta}\bar{\lambda}'(z)\bar{\sigma}_1\rho_3 g(z), \quad V_2^\delta = \sqrt{\delta}\bar{\lambda}'(z)\eta\rho_{13}g(z). \tag{2.72}$$

Proof. We construct the solution from the following observations and superposition since $\langle \mathcal{L}_2 \rangle$ is linear:

1) We first observe that $\frac{(T-t)^2}{2}(x^n\frac{\partial^n}{\partial x^n})P_0$ solves

$$\langle \mathcal{L}_2 \rangle u = -(T-t)\left(x^n\frac{\partial^n}{\partial x^n}\right)P_0, \quad u(T,x,r;z) = 0. \tag{2.73}$$

2) Next, we apply $\langle \mathcal{L}_2 \rangle$ on $(T-t)\frac{\partial P_0}{\partial r}$ and obtain

$$\langle \mathcal{L}_2 \rangle \left((T-t) \frac{\partial P_0}{\partial r} \right) = -\frac{\partial P_0}{\partial r} + (T-t) \left(-x \frac{\partial P_0}{\partial x} + \beta \frac{\partial P_0}{\partial r} + P_0 \right), \tag{2.74}$$

as a result of which we see that

$$\frac{1}{\beta}[-\frac{\partial P_0}{\partial \alpha} - \frac{(T-t)^2}{2}(x\frac{\partial P_0}{\partial x} - P_0) + (T-t)\frac{\partial P_0}{\partial r}] \tag{2.75}$$

solves

$$\langle \mathcal{L}_2 \rangle u = (T-t)\frac{\partial P_0}{\partial r}, \quad u(T,x,r;z) = 0. \tag{2.76}$$

□

2.4 Calibration of the Model

In this section, we will calibrate the loss rate l and the parameters

$$\{\bar{\lambda}, V_1^\epsilon, V_2^\epsilon, V_3^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta, V_2^\delta\}$$

that appear in the expressions (2.39), (2.60), and (2.71) on a daily basis (see, e.g., [20] and [30] for similar calibration exercises carried out only for the option data or only for the bond data). We demonstrate this calibration on Ford Motor Company. Note that there are some common parameters between equity options and corporate bonds. Therefore, our model will be calibrated simultaneously to both of these data sets. We will also calibrate the parameters of the interest rate and stock models to the yield curve data, historical spot rate data and historical stock price data. Next, we test our model by using the estimated parameters to construct an out-of-sample CDS spread time series (3 year and 5 year), which matches real quoted CDS spread data over the time period (1/6/2006 – 6/8/2007) quite well.

We also look at how our model implied volatility matches the real option implied volatility. We compare our results against those of [20]. We see that even when we make the unrealistic assumption of constant volatility, our model is able to produce a very good fit.

Finally, in the context of index options (when $\lambda = 0$), using SPX 500 index options data, we show the importance of accounting for stochastic interest rates by comparing our model to that of [18, 20].

2.4.1 Data Description

- The daily stock price data is obtained from finance.yahoo.com.
- The stock option data is from OptionMetrics under WRDS database, which is the same database used in [9]. For index options, SPX 500 in our case, we use the data from their Volatility Surface file. The file contains information on standardized options, both calls and puts, with expirations of 30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 calender days. Implied volatilities there are interpolated data using a methodolny based on kernel smoothing algorithm. The interpolated implied volatilities are very close to real data because there are a great number of options each day for SPX 500 with different maturities and strikes. But, this is not the case for individual company options, and we find that the results given by using interpolated implied volatilities in this file and data implied volatilities differ. This may due to the fact that there are a limited number of option prices available for individual companies; i.e., there may not be enough data points for the implied volatilities to be accurately interpolated. Therefore, we use the Option Price file, which contains the historical option price information, of the OptionMetrics database when we consider Ford Motor Company's options. We excluded the observations with zero trading volume or with maturity less than 9 days.

- We use the U.S government Treasury yield data from:

  www.treasury.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml

- Corporate bond and CDS data is obtained from Bloomberg.

2.4.2 The Parameter Estimation

The following parameters can be directly estimated from the spot-rate and stock price historical data:

1. The parameters of the interest rate model $\{\alpha, \beta, \eta\}$ are obtained by a least-square fitting to the Treasury yield curve as in [30].

2. $\bar{\rho}_1 = \frac{\bar{\sigma}_1}{\bar{\sigma}_2}\rho_1$, the "effective" correlation between risk-free interest rate r and stock price in (2.45) is estimated from historical risk-free spot rate and stock price data.

3. $\bar{\sigma}_2$, the "effective" stock price volatility in (2.45) is estimated from the historical stock price data.

Now, we detail the calibration method for l, $\bar{\lambda}(z)$ and $\{V_1^\epsilon, V_2^\epsilon, V_3^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta, V_2^\delta\}$, which is carried out in two steps.

1. Estimation of $l\bar{\lambda}$ and $\{lV_3^\epsilon, lV_2^\delta\}$ from the Corporate Bond Price Data.

The approximate price formula in (2.38) for a defaultable bond is

$$\widetilde{B}^c = B_0^c + \sqrt{\epsilon}B_{1,0}^c + \sqrt{\delta}B_{0,1}^c, \tag{2.77}$$

in which B_0^c is given by (2.40) and

$$\begin{aligned} \sqrt{\epsilon}B_{1,0}^c &= lV_3^\epsilon \frac{\partial B_0^c}{\partial \alpha}, \\ \sqrt{\delta}B_{0,1}^c &= lV_2^\delta \frac{1}{\beta}\left[-\frac{\partial B_0^c}{\partial \alpha} + \frac{(T-t)^2}{2}B_0^c + (T-t)\frac{\partial B_0^c}{\partial r}\right]. \end{aligned} \tag{2.78}$$

We obtain $\{l\bar{\lambda}(z), lV_3^\epsilon, lV_2^\delta\}$ from least-squares fitting, i.e. by minimizing

$$(2.79) \qquad \sum_{i=1}^{n}(B^c_{\text{obs}}(t,T_i) - B^c_{\text{model}}(t,T_i; l\bar{\lambda}, lV_3^\epsilon, lV_2^\delta))^2,$$

where $B^c_{\text{obs}}(t,T_i)$ is the observed market price of a bond that matures at time T_i and $B^c_{\text{model}}(t,T_i; l\bar{\lambda}, lV_3^\epsilon, lV_2^\delta)$ is the corresponding model price obtained from (2.77). For a fixed value of $l\bar{\lambda}(z)$ it follows from (2.77) that $\{lV_3^\epsilon, lV_2^\delta\}$ can be determined as the least squares solution of

$$\left(\frac{\partial B_0^c}{\partial \alpha}, \quad \frac{1}{\beta}\left[-\frac{\partial B_0^c}{\partial \alpha} + \frac{(T_i-t)^2}{2}B_0^c + (T_i - t)\frac{\partial B_0^c}{\partial r}\right]\right)_{1\le i\le n} \begin{pmatrix} lV_3^\epsilon \\ lV_2^\delta \end{pmatrix} = \left(B^c_{\text{obs}}(t,T_i) - B_0^c(t,T_i; l\bar{\lambda})\right)_{1\le i\le n}.$$

Now, we vary $l\bar{\lambda}(z) \in [0, M_1]$ and choose the point $\{l\bar{\lambda}, lV_3^\epsilon, lV_2^\delta\}$ that minimizes (2.79). Here, we take $M_1 = 1$ guided by the results of [30].

2. Estimation of $\{l, V_1^\epsilon, V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta\}$ from the Equity Option Data: These parameters are calibrated from the stock options data by a least-squares fit to the observed implied volatility. We choose the parameters to minimize

(2.80)

$$\sum_{i=1}^{n}(I_{\text{obs}}(t,T_i,K_i) - I_{\text{model}}(t,T_i,K_i; \text{model parameters}))^2$$
$$\approx \sum_{i=1}^{n} \frac{(P_{\text{obs}}(t,T_i,K_i) - P_{\text{model}}(t,T_i,K_i; \text{model parameters}))^2}{\text{vega}^2(T_i,K_i)}$$

in which $I_{\text{obs}}(t,T_i,K_i)$ and $I_{\text{model}}(t,T_i,K_i; \text{model parameters})$ are observed Black-Scholes implied volatility and model Black-Scholes implied volatility, respectively. The right hand side of (2.80) is from [12], page 439. Here, $P_{\text{obs}}(t,T_i,K_i)$ is the market price of a European option (a put or a call) that matures at time T_i and with strike price K_i and $P_{\text{model}}(t,T_i,K_i; \text{model parameters})$ is the corresponding model price which is obtained from (2.38). As in [12] vega(T_i,K_i) is the market implied Black-Scholes vega.

Let $P_0(t, T_i, K_i; \bar{\lambda}(z))$ be either of (2.57) and (2.59) with $K = K_i$ and $T = T_i$. Let us introduce the Greeks,

(2.81)
$$
\begin{aligned}
&g_1 = -(T-t)x^2\frac{\partial^2 P_0}{\partial x^2}, \quad g_2 = -(T-t)x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right), \quad g_3 = \frac{\partial}{\partial \alpha}(x\frac{\partial P_0}{\partial x} - P_0),\\
&g_4 = \frac{\partial}{\partial \alpha}(x^2\frac{\partial^2 P_0}{\partial x^2}), \quad g_5 = x\frac{\partial}{\partial x}(\frac{\partial P_0}{\partial \eta}), \quad g_6 = x\frac{\partial}{\partial x}(\frac{\partial P_0}{\partial \alpha}), \quad g_7 = \frac{(T-t)^2}{2}x^2\frac{\partial^2 P_0}{\partial x^2},\\
&g_8 = \frac{1}{\beta}[x\frac{\partial}{\partial x}(\frac{\partial P_0}{\partial \alpha}) - \frac{\partial P_0}{\partial \alpha} + \frac{(T-t)^2}{2}(x^2\frac{\partial^2 P_0}{\partial x^2} - x\frac{\partial P_0}{\partial x} + P_0) - (T-t)(x\frac{\partial}{\partial x}(\frac{\partial P_0}{\partial r}) - \frac{\partial P_0}{\partial r})],
\end{aligned}
$$

in which each term can be explicitly evaluated by direct differentiation.

Now from (2.38) and the results of Section 2.3.3 (with $l = 1$), we can write

(2.82)
$$
\begin{aligned}
P_{\text{model}}(t, T_i, K_i; \text{model parameters}) &= P_0(t, T_i, K_i; \bar{\lambda}) + V_1^\epsilon g_1(T_i, K_i; \bar{\lambda}) + V_2^\epsilon g_2(T_i, K_i; \bar{\lambda}) + V_3^\epsilon g_3(T_i, K_i; \bar{\lambda})\\
&+ V_4^\epsilon g_4(T_i, K_i; \bar{\lambda}) + V_5^\epsilon g_5(T_i, K_i; \bar{\lambda}) + V_6^\epsilon g_6(T_i, K_i; \bar{\lambda}) + V_1^\delta g_7(T_i, K_i; \bar{\lambda}) + V_2^\delta g_8(T_i, K_i; \bar{\lambda}).
\end{aligned}
$$

First, let us fix the value of l. Then, from step 1, we can infer the values of $\{\bar{\lambda}, V_3^\epsilon, V_2^\delta\}$. Now, the fitting problem in (2.80) is a linear least squares problem for $\{V_1^\epsilon, V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta\}$. Next, we vary $l \in [0, 1]$ and choose $\{l, V_1^\epsilon, V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta\}$ so that (2.80) is minimized.

2.4.3 Model Implied CDS Spead Matches the Observed CDS Spread

Let $\widetilde{B}^c(t, T; l)$ denote the approximation for the price at time t of a defaultable bond that matures at time T, and has loss rate l (see (2.77)). Let $B(t, T)$ be the price of a risk-free bond. Then, the model implied CDS spead with maturity T_M is

(2.83)
$$
c^{ds}_{\text{model}}(t, T_M) = \frac{B(t, T_M) - \widetilde{B}^c(t, T_M; l)}{\sum_{m=1}^{M} \widetilde{B}^c(t, T_m; 1)}.
$$

Recall that we have already estimated all of the model parameters in Section 2.4.2. Therefore, using (2.83) we can plot the model implied CDS spread over time and compare it with the CDS spread data available in the market. This is precisely what we do in Figures 2.1 and 2.2. We look at the time series $c^{ds}_{\text{model}}(t, 3)$ and $c^{ds}_{\text{model}}(t, 5)$ and compare them to the CDS spread time series of the Ford Motor Company. The match seems to be extremely good, which attests to the power of our modeling framework.

By varying T_M in (2.83) we can obtain the model implied term structure of the CDS spread. Figure 2.3 shows the range of shapes we can produce.

2.4.4 Fitting Ford's Implied Volatility

We will compare how well our model fits the implied volatility against the model of [20], which does not account for the default risk and for the randomness of the interest rates. Although, we only calibrate seven parameters (hence we refer to our model as the 7-parameter model) to the option prices (see the second step of the estimation in Section 2.4.2), we have many more parameters than the model of [20], which only has four parameters (we refer to this model as the 4-parameter model). Therefore, for a fair comparison, we also consider a model in which the volatility is a constant. In this case, as we shall see below, there are only three parameters to calibrate to the option prices, therefore we call it the 3-parameter model.

Constant Volatility Model In this case, we take $\bar{\sigma}_1 = \bar{\sigma}_2 = \sigma$ in the expression for P_0 in Corollary II.2 . The expression for $\sqrt{\delta}P_{0,1}$ remains the same as before. However, $\sqrt{\epsilon}P_{1,0}$ simplifies to

$$\sqrt{\epsilon}P_{1,0} = -(T-t)V_1^{\epsilon}x^2\frac{\partial^2 P_0}{\partial x^2} + V_3^{\epsilon}\left(-x\frac{\partial^2 P_0}{\partial\alpha\partial x} + \frac{\partial P_0}{\partial\alpha}\right). \tag{2.84}$$

This model has only three parameters, $l, V_1^{\epsilon}, V_1^{\delta}$ that need to be calibrated to the options prices, as opposed to the 4-parameter model of [20].

As it can be seen from Figure 2.4 as expected our 7-parameter model outperforms the 4-parameter model of [20] as expected and fits the implied volatility data well. But, what is surprising is that the 3-parameter model, which does not account for the volatility but accounts for the default risk and stochastic interest rates, has almost the same performance as the 7-parameter model.

The 7-parameter] model has a very rich implied volatility surface structure, the surface has more curvature than that of the 4-parameter model of [20], whose volatility surface is more flat; see Figures 2.5 and 2.6. (The parameters to draw these figures are obtained by calibrating the models to the data implied volatility surface on June 8 2007.) The 7-parameter model has a recognizable skew even for longer maturities and has a much sharper skew for shorter maturities.

2.4.5 Fitting the Implied Volatility of the Index Options

The purpose of this section is to show the importance of accounting for stochastic interest rates in fitting the implied volatility surface. When we price index options, we set $\bar{\lambda} = 0$ and our approximation in (2.38) simplifies to

$$P^{\epsilon,\delta} \approx P_0 + \sqrt{\epsilon} P_{1,0}, \tag{2.85}$$

in which P_0 is given by Corollary II.2 after settiing $\bar{\lambda}(z) = 0$, and

$$\sqrt{\epsilon} P_{1,0} = -(T-t)\left(V_1^\epsilon x^2 \frac{\partial^2 P_0}{\partial x^2} + V_2^\epsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) + V_4^\epsilon x^2 \frac{\partial^3 P_0}{\partial x^2 \partial \alpha} + V_5^\epsilon x \frac{\partial^2 P_0}{\partial \eta \partial x} + V_6^\epsilon x \frac{\partial^2 P_0}{\partial \alpha \partial x} \tag{2.86}$$

Note that the difference of (2.85) with the model of [20] is that the latter allows for a slow evolving volatility factor to better match the implied volatility at the longer maturities. This was an improvement on the model of [18], which only has a fast scale component in the volatility model. We, on the other hand, by accounting for stochastic interest rates, capture the same performance by using only a fast scale volatility model.

From Figure 2.85, we see that both (2.85) and [20] outperform the model of [18], especially at the longer maturities ($T = 9$months, 1 year, 1.5 years and 2 years), and their performance is very similar. This observation emphasizes the importance of accounting for stochastic interest rates for long maturity contracts.

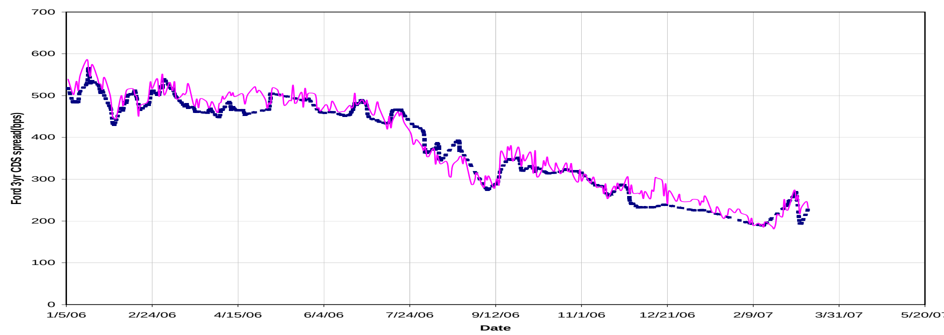

Figure 2.1: Ford 3 year CDS annual spread time series from 1/6/2006-6/8/2007. Spread implied by model is pink solid line, real quoted spread is blue broken line.

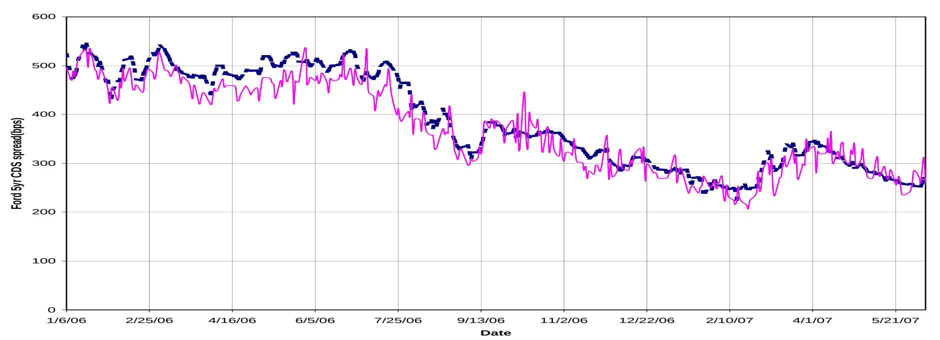

Figure 2.2: Ford 5 year CDS annual spread time series from 1/6/2006-6/8/2007. Spread implied by model is pink solid line, real quoted spread is blue broken line.

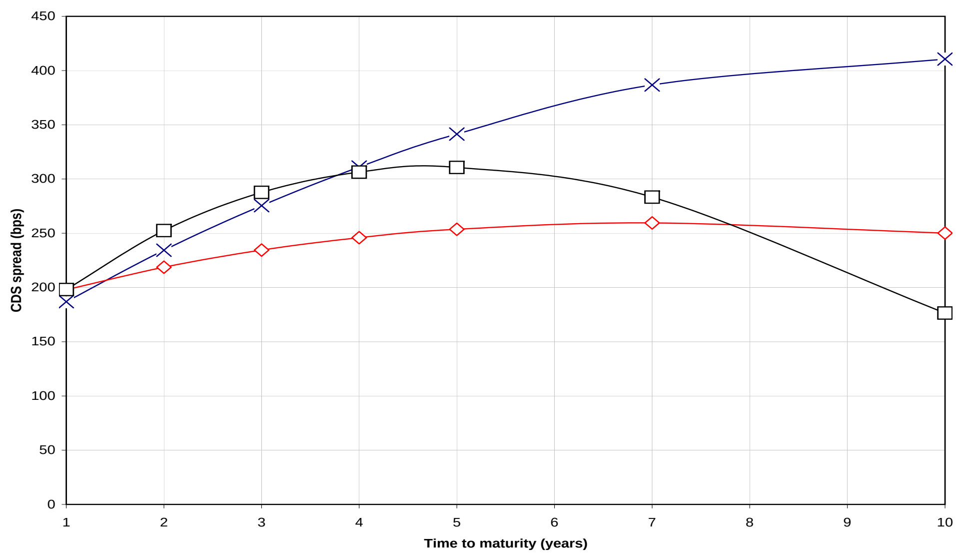

Figure 2.3: CDS Term Structures (2.83) can produce:
Legend
-x-, blue (The parameters are obtained from calibration to 11.13.2006): α=0.0037, β=0.0872 $\eta = 0.0001$, $r = 0.0516$, l(loss rate) $= 0.283$, $\bar{\lambda}(z) = 0.0459$, $[V_3^\epsilon, V_2^\delta] = [0.0425, 0.0036]$.
-squares-, black (The parameters correspond to 6.18.2006): $\alpha = 0.0045$, $\beta = 0.0983$, $\eta = 0.0002$, $r = 0.0516$, $l = 1$, $\bar{\lambda} = 0.012$, $[V_3^\epsilon, V_2^\delta] = [0.0185, 0.0025]$,
-diamonds-, red (The parameters correspond to 9.22.2006): $\alpha = 0.0039$, $\beta = 0.0817$, $\eta = 0.0012$, $r = 0.0496$, l=1, $\bar{\lambda}(z) = 0.017$, $[V_3^\epsilon, V_2^\delta] = [0.0067, 0.0005]$

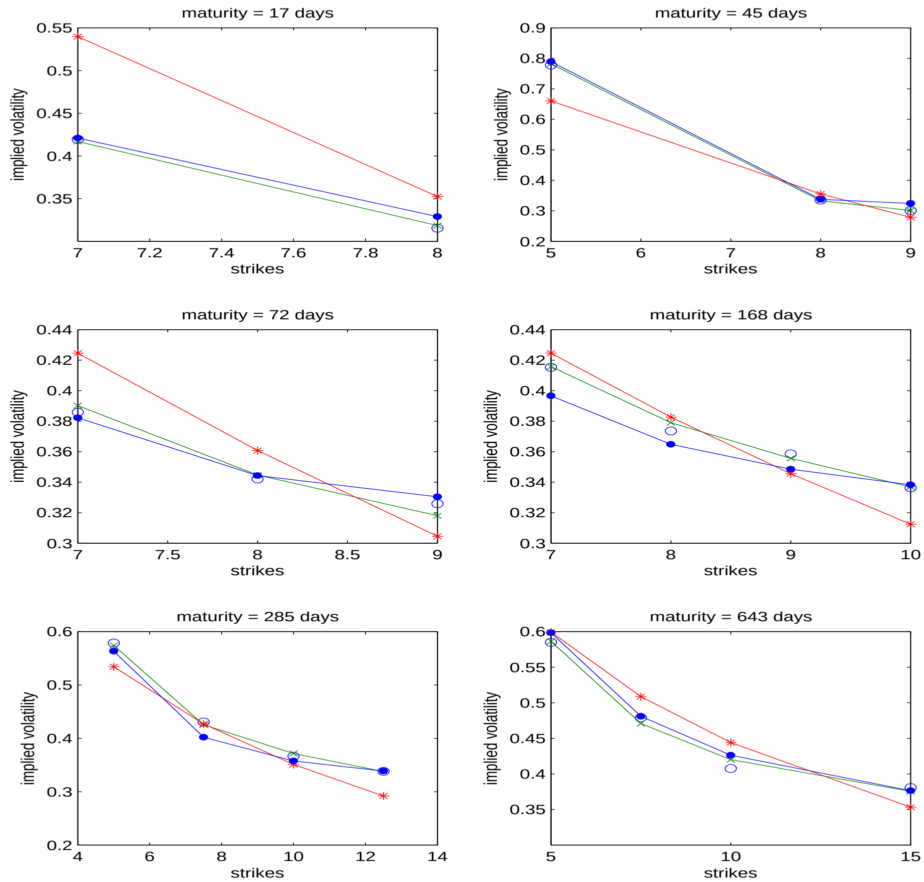

Figure 2.4: Implied volatility fit to the Ford call option data with maturities of [17,45,72,168,285,643] calender days on April 4, 2007. Model is calibrated aross all maturities but we plotted the implied volatilities for each maturity, separately. Here, stock price $(x) = 8.04$, historical volatility $(\bar{\sigma}_2) = 0.3827$, one month treasury rate $(r) = 0.0516$, estimated correlation between risk-free spot rate(one month treasury) and stock price $(\bar{\rho}_1) = -0.0327$. Also $\alpha = 0.0037$, $\beta = 0.0872$, $\eta = 0.0001$ which are obtained with a least-square fitting to the Treasury yield curve on the 4th of April.
Legend:
'o', empty circles = observed data;
'x', green = stochastic vol+stochastic hazard rate+stochastic interest rate = the 7-parameter model;
small full circle, blue = constant vol+stochastic hazard rate+ stochastic interest rate = the 3-parameter model
'*', red = The model of [20] which has constant interest rate+stochastic vol (slow and fast scales) = the 4 parameter model.

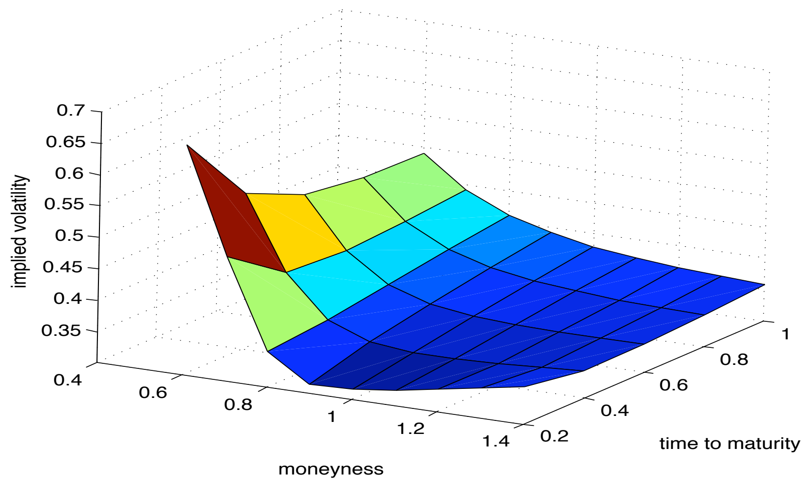

Figure 2.5: Implied volatility surface corresponding to (2.82), the 7-parameter model. Here, $\alpha = 0.0063$, $\beta = 0.1034$, $\eta = 0.012$, $r = 0.0476$ $\bar{\sigma}_2 = 0.2576$, $\bar{\lambda}(z) = 0.027$, $(V_1^\epsilon, V_2^\epsilon, V_3^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta, V_2^\delta) = (0.9960, -0.0014, 0.0009, 0.0104, -0.6514, 0.3340, -0.1837, -0.0001)$.

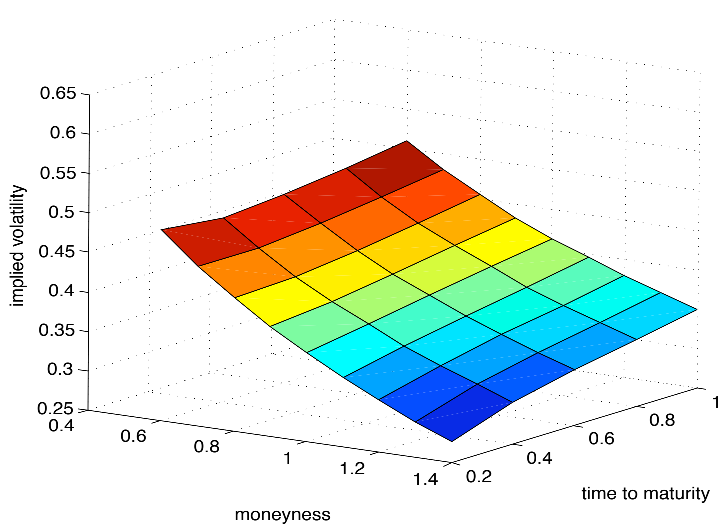

Figure 2.6: Implied Volatility Surface corresponding to the 4-parameter model of [20] when we choose $r = 0.046$, average volatility=0.2546, and the parameters in (4.3) of [20] are choosen to be $(V_2^\epsilon, V_3^\epsilon, V_0^\delta, V_1^\delta) = (-0.0164, -0.1718, 0.0006, 0.0630)$. Note that the parameters here and Figure 2.5 are both obtained by calibrating the models to the data implied volatility surface of Ford Motor Company on June 8, 2007.

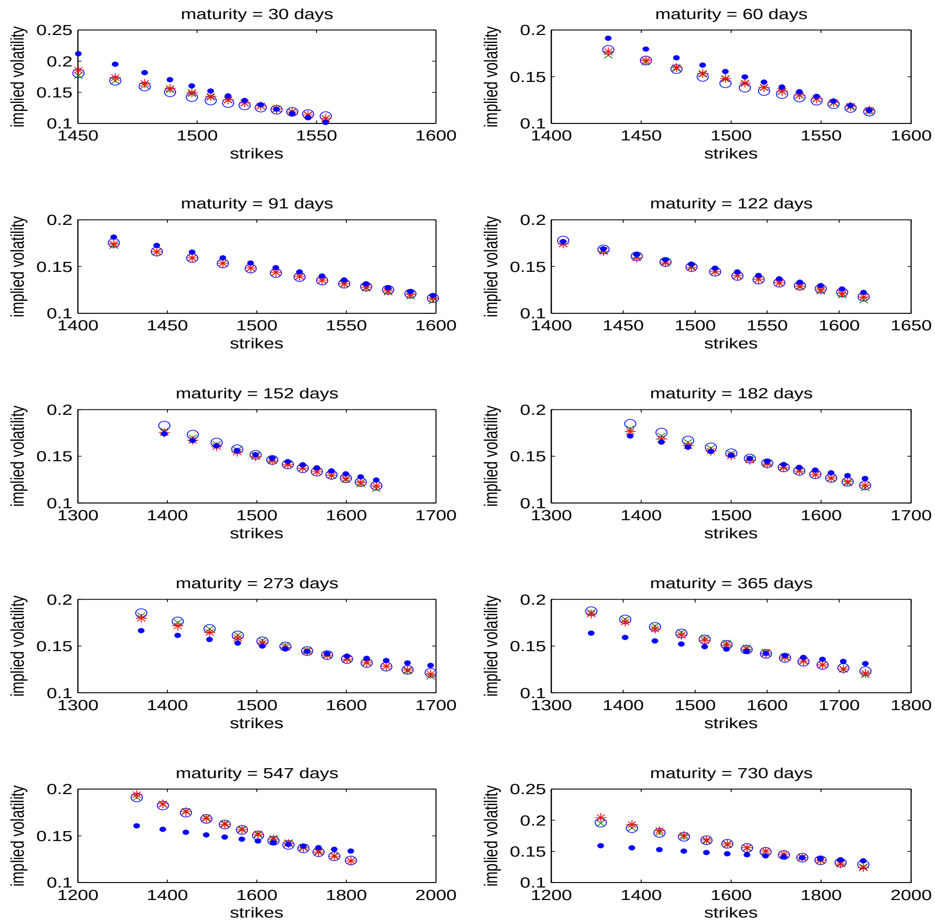

Figure 2.7: The fit to the Implied Volatility Surface of SPX om June 8, 2007 with maturities [30,60,91,122,152,182,273,365,547,730] calender days. Model is calibrated aross all maturities, but we plot the implied volatility fits separately. The parameters are: stock price $(x) = 1507.67$, divident rate $= 0.0190422$, historical volatility $(\bar{\sigma}_2) = 0.1124$, one month treasury rate $(r) = 0.0476$, estimated correlation between risk-free spot rate(one month treasury) and stock price $(\bar{\rho}_1) = 0.020454$. Also, $\alpha = 0.0078$, $\beta = 0.1173$, $\eta = 0.0241$, which are obtained from a least-square fitting to the Treasury yield curve of the same day.

Legend

'o', empty cirles = observed data,
'x'', green = Implied volatility of (2.85),
'*', red = Implied volatility of [20],
small full circle, blue = Implied volatility of [18].

CHAPTER III

Pricing Exotics in the Framework of Chapter II

3.1 Introduction

In this chapter, we demonstrate how the calibrated parameters from Chapter II can be used to price exotic equity and credit derivatives. In particular, we study the pricing of convertible bonds in Section 3.2, as an example of an American-style derivative, we study barrier options in Section 3.3 and lookback options in Section 3.4, as examples of important path-dependent equity options.

A convertible bond is a hybrid equity and credit derivative; hence, we need a unified framework to correctly price it. Although there is vast literature on modeling and pricing of convertible bonds, our framework is advantageous in that we incorporate stochastic interest rates, stochastic default intensity, and stochastic stock price volatility at the same time. More importantly, our framework calibrates to both equity and credit markets very well and we are able infer the recovery rate from the joint calibration,as we have demonstrated in Chapter II. By applying perturbation methods, we are able to reduce the five-dimensional free-boundary pricing problem to a two-dimensional free-boundary problem.

Much of the work on path-dependent options assumes that the underlying asset price follows a one-dimensional diffusion process. Here, in our multi-factor framework, by following the methodology developed in [24], we obtain semi-closed form solutions for the leading term and correction terms of the price approximations by solving the corresponding boundary value PDE problems.

3.2 Convertible Bonds

3.2.1 Pricing Equations

A convertible bond is an ordinary bond that has the option that the holder of the bond can choose, at any time before the expiration time T of the contract, to exchange the bond for a fixed number κ (conversion rate) shares of stock. Usually, convertible bonds also have a call feature, which gives the company the right to purchase the bond back at a fixed price M_c any time before the expiration. The pricing of a convertible bond can be formulated as a linear complimentary problem, see, e.g., [3]. Let $P^{\epsilon,\delta}$ be the price at time $t < T$. We have

$$\text{(I)} \begin{pmatrix} \mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta} = 0 \\ P^{\epsilon,\delta} \geq \kappa x \\ P^{\epsilon,\delta} \leq M_c \end{pmatrix} \text{ or (II)} \begin{pmatrix} \mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta} \geq 0 \\ P^{\epsilon,\delta} = \kappa x \\ P^{\epsilon,\delta} \leq M_c \end{pmatrix} \text{ or (III)} \begin{pmatrix} \mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta} \leq 0 \\ P^{\epsilon,\delta} \geq \kappa x \\ P^{\epsilon,\delta} = M_c \end{pmatrix},$$

with terminal condition

$$P^{\epsilon,\delta}(T, x, y, \tilde{y}, z, r) = \max(1, \kappa x).$$

We can also interpret this as a free boundary problem, with holding region (I), conversion region (II) and calling region (III). We let $x^{\epsilon,\delta}_{fb_1}(t, y, \tilde{y}, z, r)$ denote the free boundary that separates the holding region and conversion region and $x^{\epsilon,\delta}_{fb_2}(t, y, \tilde{y}, z, r)$ denote the free boundary that separates the holding region and the calling region.

Then, in the holding region, we have

$$
\begin{cases}
\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta} = 0, \\
P^{\epsilon,\delta}(t, x^{\epsilon,\delta}_{fb_1}(t,y,\tilde{y},z,r), y, \tilde{y}, z, r) = \kappa x, \\
P^{\epsilon,\delta}(t, x^{\epsilon,\delta}_{fb_2}(t,y,\tilde{y},z,r), y, \tilde{y}, z, r) = M_c, \\
P^{\epsilon,\delta}(T, x, y, \tilde{y}, z, r) = \max(1, \kappa x).
\end{cases}
\tag{3.1}
$$

We follow the methodolny developed in [19] for pricing American options. The idea is to expand (3.1) and the free boundaries in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$ and to solve a free-boundary problem for the leading term and fixed-boundary problems for the correction terms. As in [19], this is assumed to introduce only $O(\sqrt{\epsilon} + \sqrt{\delta})$ error.

3.2.2 Convertible Bond Asymptotics

We look for an asymptotic solution of the form

$$
\begin{aligned}
P^{\epsilon,\delta}(t,x,y,\tilde{y},z,r) &= P_0 + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1} + \cdots , \\
x^{\epsilon,\delta}_{fb_1}(t,y,\tilde{y},z,t,r) &= x_0 + \sqrt{\epsilon}x_{1,0} + \sqrt{\delta}x_{0,1} + \cdots , \\
x^{\epsilon,\delta}_{fb_2}(t,y,\tilde{y},z,t,r) &= \tilde{x}_0 + \sqrt{\epsilon}\tilde{x}_{1,0} + \sqrt{\delta}\tilde{x}_{0,1} + \cdots .
\end{aligned}
$$

Following the asymptotic expansion developed in Chapter II, P_0 doesn't depend on y and $\tilde{y}$. Now, P_0 is independent of y and $\tilde{y}$ on each side of both boundaries x_0 and $\tilde{x}_0$. It follows that x_0 and $\tilde{x}_0$ are independent of y and $\tilde{y}$, also. Hence in the holding region, we have that

$$
\begin{cases}
\langle \mathcal{L}_2 \rangle P_0 = 0, \\
P_0(T, x, z, r) = \max(1, \kappa x), \\
P_0(t, x_0(t,z,r), z, r) = \kappa x, \\
P_0(t, \tilde{x}_0(t,z,r), z, r) = M_c.
\end{cases}
\tag{3.2}
$$

Note that the PDE does not involve any derivatives with respect to z. We just need to compute for one fixed z, in other words, compute with the z-dependent parameters $\bar{\lambda}(z)$, $V_1^\epsilon(z), V_2^\epsilon(z), \cdots, V_6^\epsilon(z), V_1^\delta(z), V_2^\delta(z)$ fixed at the calibrated values. The price of a convertible bond in the framework of constant default intensity ($=\bar{\lambda}(z)$), constant stock volatility ($=\bar{\sigma}$) and Vasicek stochastic interest rates solves the free boundary problem (3.2). The problem of pricing convertible bonds in such a framework has been solved in many places, for example, in [2]. Similar to the American option case in [19], $P_{1,0}$ and $P_{0,1}$ satisfy the following fixed boundary value problems, respectively,

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_{1,0} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0, \\ P_{1,0}(T,x,z,r) = 0, \\ P_{1,0}(t, x_0(t,z,r), z, r) = 0, \\ P_{1,0}(t, \tilde{x}_0(t,z,r), z, r) = 0; \end{cases}$$

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_{0,1}^i = -\langle \mathcal{M}_1 \rangle P_0, \\ P_{0,1}(T,x,z,r) = 0, \\ P_{0,1}(t, x_0(t,z,r), z, r) = 0, \\ P_{0,1}(t, \tilde{x}_0(t,z,r), z, r) = 0. \end{cases}$$

These equations can be solved by using a standard finite-difference scheme, with the differential operators $\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle$ and $-\langle \mathcal{M}_1 \rangle$ given explicitly by equation (2.36) and equation (2.70).

3.3 Barrier Options

In this section, we price the *down-and-out option*, which is a call option with an additional feature that if the underlying's price falls below a barrier B at some time before the expiration time T, the contract becomes worthless. Other kinds of barrier options can be handled analnously. The payoff at expiration time T for a *down-and-out call option* can be expressed as:

$$h(X_T) = (X_T - K)^+ 1_{\{\min\limits_{0\le t\le T} X_T > B\}}.$$

It is worth noting that another easier but interesting case

$$h(X_T) = 1_{\{\min\limits_{0\le t\le T} X_T > B\}}. \tag{3.3}$$

Finding the option value with payoff specified by (3.3) is equivalent to calculating the bond price in a structural credit model with stochastic interest rates and stochastic volatility.

The price of the *down-and-out barrier call option* $P^{\epsilon,\delta}$ satisfies:

$$\begin{cases} \mathcal{L}^{\epsilon,\delta} P^{\epsilon,\delta} = 0 \quad \text{in } x > B, \text{ and } T > 0 \\ P^{\epsilon,\delta}(0, x, r, y, \tilde{y}, z) = (x - K)^+ \\ P^{\epsilon,\delta}(T, B, r, y, \tilde{y}, z) = 0. \end{cases}$$

For convenience, we let T be the time to expiration. Hence, $\mathcal{L}_2$ in Chapter II now takes the form:

$$\mathcal{L}_2 = -\frac{\partial}{\partial T} + \frac{1}{2}\sigma^2(\tilde{y})x^2\frac{\partial^2}{\partial x^2} + (r + f(y,z))x\frac{\partial}{\partial x} + (\alpha - \beta r)\frac{\partial}{\partial r} + \sigma(\tilde{y})\eta\rho_1 x\frac{\partial^2}{\partial x\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2} - (r + f(y,z))\cdot.$$

And $\langle\mathcal{L}_2\rangle$ now takes the form:

$$\langle\mathcal{L}_2\rangle = -\frac{\partial}{\partial T} + \frac{1}{2}\bar{\sigma}_2^2 x^2\frac{\partial^2}{\partial x^2} + (r + \bar{\lambda})x\frac{\partial}{\partial x} + (\alpha - \beta r)\frac{\partial}{\partial r} + \bar{\sigma}_1\eta\rho_1 x\frac{\partial^2}{\partial x\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2} - (r + \bar{\lambda})\cdot.$$

We proceed by approximating $P^{\epsilon,\delta}$ by

$$\widetilde{P^{\epsilon,\delta}} = P_0 + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1},$$

and solve the resulting boundary-value PDE problems from an asymptotic expansion for P_0, $\sqrt{\epsilon}P_{1,0}$ and $\sqrt{\delta}P_{0,1}$.

3.3.1 The Leading Term P_0

P_0 is the solution of the boundary-value problem:

$$
\begin{cases}
\langle \mathcal{L}_2 \rangle P_0 = 0 \quad \text{in } x > B \text{ and } T > 0, \\
P_0(0,x,r) = (x-K)^+, \\
P_0(T,B,r) = 0.
\end{cases}
\tag{3.4}
$$

We can represent P_0 as

$$P_0(0,x,r) = \mathbb{E}\left[\exp\left(-\int_0^T (r_s+\bar{\lambda})ds\right)(S_T-K)^+ 1_{\{\tau>T\}} | S_0 = x, r_0 = r\right].$$

S_t and r_t follow the dynamics of

$$dS_t = (r_t+\bar{\lambda})S_t dt + \bar{\sigma}_2 S_t dW_t^0,$$

$$dr_t = (\alpha - \beta r_t)dt + \eta dW_t^1,$$

$$\mathbb{E}[dW_t^0 dW_t^1] = \bar{\rho}_1 dt,$$

where $\bar{\rho}_1 = \frac{\bar{\sigma_1}}{\bar{\sigma_2}}\rho_1$. τ is the first-passage time to B for S_t conditional on $S_0 = x$ and $r_0 = r$. We change to the forward measure $\mathbb{P}^T$ through the Radon-Nikodym derivative

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\exp(-\int_0^T r_s ds)}{B(T,r)},$$

where

$$B(T,r) = \exp(a(T) - b(T)r),$$

as defined in the proof of Proposition II.1, and we obtain

$$
\begin{aligned}
P_0(0,x,r) &= B(T,r)\exp(-\bar{\lambda}T)\mathbb{E}^T[(S_T-K)^+ 1_{\{\tau>T\}} | S_0 = x, r_0 = r] \\
&= C_0(T,x,r) - B(T,r)\exp(-\bar{\lambda}T)\mathbb{E}^T[(S_T-K)^+ 1_{\{\tau\le T\}} | S_0 = x, r_0 = r],
\end{aligned}
$$

in which $C_0(T,x,r)$ is the leading term for a vanilla European call option price computed in Chapter II. Now, we compute

$\mathbb{E}^T[(S_T-K)^+1_{\{\tau\le T\}}|S_0=x,r_0=r]$ as follows:

$$\begin{aligned}&\mathbb{E}^T[(S_T-K)^+1_{\{\tau\le T\}}|S_0=x,r_0=r]\\&=\int_0^T\int_{-\infty}^{+\infty}\mathbb{E}^T[(S_T-K)^+|r_\tau=\tilde{r},\tau=t]\mathbb{P}^T(r_\tau\in d\tilde{r},\tau\in dt|S_0=x,r_0=r)\end{aligned}$$

We discretize the integrals by dividing $[0,T]$ equally into n_T subintervals, and $[r_{\min},r_{\max}]$ equally into n_r subintervals. We let $l_t=\ln B-\ln S_t$, $l_0=\ln B-\ln x$, $r_0=r$ and $l_b=0$. Under the T-forward measure

$$\begin{aligned}&dl_t=(r_t+\bar{\lambda}-\frac{\sigma^2}{2}-\rho\sigma\eta b(T-t;\beta))dt+\sigma d\tilde{W}_t^0,\\&dr_t=(\alpha-\beta r_t-\eta^2 b(T-t;\beta)dt+\eta d\tilde{W}_t^1,\\&\mathbb{E}^T[d\tilde{W}_t^0 d\tilde{W}_t^1]=\rho dt.\end{aligned}$$

Define $g[l_s=l_b,r_s,s|l_0,r_0,0]$, with $l_0<l_b$ to be the probability density that the first passage time through a constant boundary l_b is at time s, and that the random process r takes on the value r_s at that time. Let $q_{r_i,t_j}(x,r;B)=\Delta t\Delta r g[l_{t_i}=l_b,r_i,t_j|l_0,r_0,0]$. In terms of $q_{r_i,t_j}(x,r;B)$, we can approximate the expectation by

$$\begin{aligned}&\mathbb{E}^T[(S_T-K)^+1_{\{\tau\le T\}}|S_0=x,r_0=r]\\&\approx\sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\mathbb{E}^T[(S_T-K)^+|r_\tau=r_i,\tau=t_j]q_{r_i,t_j}(x,r;B)\\&=\sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\frac{C_0(T-t_j,B,r_i)}{B(T-t_j,r_i)\exp(-\bar{\lambda}(T-t_j))}q_{r_i,t_j}(x,r;B).\end{aligned}$$

We can determine $q_{r_i,t_j}(x,r;B)$ in closed-form iteratively as follows; see [10].

$$
\begin{aligned}
q_{r_i,t_1} &= \Delta r \Psi_{r_i,t_j} \quad \forall i \in (1,2,\cdots,n_r) \\
q_{r_i,t_j} &= \Delta r (\Psi_{r_i,t_j} - \sum_{v=1}^{j-1} \sum_{u=1}^{n_r} q_{r_u,t_v} \psi(r_i,t_j|r_u,t_v)) \\
&\qquad \forall i \in (1,2,\cdots,n_r), \quad \forall j \in (2,\cdots,n_T),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_{r,t} &= \pi(r_t,t|r_0,0) N\left(\frac{\mu(r_t,t|l_0,r_0,0)}{\Sigma(r_t,t|l_0,r_0,0)}\right) \\
\psi(r_t,t|r_s,s) &= \pi(r_t,t|r_s,s) N\left(\frac{\mu(r_t,t|l_s=l_b,r_s,s)}{\Sigma(r_t,t|l_s=l_b,r_s,s)}\right),
\end{aligned}
$$

in which

$$
\begin{aligned}
\mu(r_t,t|l_s=l_b,r_s,s) &= \mathbb{E}_s^T[l_t|r_t] = \mathbb{E}_s^T[l_t] + \frac{\mathrm{Cov}_s^T[l_t,r_t]}{\mathrm{Var}_s^T[r_t]}(r_t - \mathbb{E}_s^T[r_t]) \\
\Sigma^2(r_t,t|l_s=l_b,r_s,s) &= \mathrm{Var}_s^T[l_t|r_t] = \mathrm{Var}_s^T[l_t] - \frac{\mathrm{Cov}_s^T[l_t,r_t]^2}{\mathrm{Var}_s^T[r_t]},
\end{aligned}
$$

$N(\cdot)$ is the standard normal cumulative function and

$$
\pi(r_t,t|r_s,s) = \frac{1}{2\pi \mathrm{Var}_s^T[r_t]} \exp\left(- \frac{(r_t - \mathbb{E}_s^T[r_t])^2}{2\mathrm{Var}_s^T[r_t]}\right).
$$

is the transition density for r_t, which is a Gaussian process. The moments for l_t and r_t are

$$
\begin{aligned}
\mathbb{E}_s^T[l_t] =& l_s - \left(\frac{\alpha}{\beta} - \frac{\eta^2}{\beta^2} + \bar{\lambda} - \frac{\sigma^2}{2} - \frac{\rho\sigma\eta}{\beta}\right)(t-s) \\
& - (r_s - \frac{\alpha}{\beta} + \frac{\eta^2}{\beta^2} + \frac{\rho\sigma\eta}{\beta} e^{-\beta(T-t)} b(t-s;\beta) - \frac{\eta^2}{2\beta} e^{-\beta(T-t)} (b(t-s;\beta)^2,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_s^T[r_t] = r_s e^{-\beta(t-s)} + (\alpha - \frac{\eta^2}{\beta}) b(t-s;\beta) + \frac{\eta^2}{\beta} e^{-\beta(T-t)} b(t-s;2\beta), \\
&\mathrm{Var}_u^T[l_t] = \left(\sigma^2 + 2\frac{\sigma\rho\eta}{\beta} + \frac{\eta^2}{\beta^2}\right)(t-s) - 2(\frac{\sigma\rho\eta}{\beta} + \frac{\eta^2}{\beta^2}) b\left(t-s;\beta\right) + \frac{\eta^2}{\beta^2} b(t-s;2\beta), \\
&\mathrm{Var}_s^T[r_t] = \eta^2 b(t-s;2\beta), \\
&\mathrm{Cov}_s^T[l_t,r_t] = \frac{\eta^2}{\beta} b(t-s;2\beta) - \left(\frac{\eta^2}{\beta} + \rho\eta\sigma\right) b(t-s,\beta).
\end{aligned}
$$

Proposition III.1. *The leading term P_0 for a down-and-out barrier call option can be approximated by*

$$P_0(0,x,r) \approx C_0(T,x,r) - B(T,r)\exp(-\bar{\lambda}T)\sum_{j=1}^{n_T}\sum_{i=0}^{n_r}\frac{C_0(T-t_j,B,r_i)}{B(T-t_j,r_i)\exp(-\bar{\lambda}(T-t_j))}q_{r_i,t_j}(x,r;B).$$

3.3.2 The Correction Terms $\sqrt{\epsilon}P_{1,0}$ and $\sqrt{\delta}P_{0,1}$

$P_{1,0}$ is independent of y, $\tilde{y}$, and it solves

$$\begin{cases} \langle\mathcal{L}_2\rangle P_{1,0} = \langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)\rangle P_0 \quad \text{in } x > B \text{ and } T > 0, \\ P_{1,0}(0,x,r) = 0, \\ P_{1,0}(T,B,r) = 0. \end{cases} \tag{3.5}$$

We define $\hat{P}_{1,0}$ through

$$\begin{aligned} P_{1,0} = \hat{P}_{1,0} + \frac{1}{\sqrt{\epsilon}}\Bigg[&-\frac{1}{\bar{\sigma}_2}\left(V_1^\epsilon\frac{\partial P_0}{\partial\bar{\sigma}_2} + V_2^\epsilon x\frac{\partial}{\partial x}\left(\frac{\partial^2 P_0}{\partial x\partial\bar{\sigma}_2}\right)\right) \\ &+ V_3^\epsilon\left(-x\frac{\partial^2 P_0}{\partial x\partial\alpha} - \frac{\partial P_0}{\partial\alpha}\right) + V_4 x^2\frac{\partial^3 P_0}{\partial x^2\partial\alpha} + V_5^\epsilon x\frac{\partial^2 P_0}{\partial\eta\partial x} + V_6^\epsilon x\frac{\partial^2 P_0}{\partial x\partial\alpha}\Bigg], \end{aligned}$$

By Proposition II.3, and by using the fact that $(T-t)x^2\frac{\partial^2 P_0}{\partial x^2} = \frac{1}{\bar{\sigma}_2}\frac{\partial P_0}{\partial\bar{\sigma}_2}$. $\hat{P}_{1,0}$ solves

$$\begin{cases} \langle\mathcal{L}_2\rangle\hat{P}_{1,0} = 0 \quad \text{in } x > B \text{ and } T > 0, \\ \hat{P}_{1,0}(0,x,r) = 0, \\ \hat{P}_{1,0}(T,B,r) = \frac{1}{\sqrt{\epsilon}}\Big[-\frac{V_2^\epsilon}{\bar{\sigma}_2}x\frac{\partial}{\partial x}\left(\frac{\partial^2 P_0}{\partial x\partial\bar{\sigma}_2}\right) \\ -V_3^\epsilon x\frac{\partial^2 P_0}{\partial x\partial\alpha} + V_4 x^2\frac{\partial^3 P_0}{\partial x^2\partial\alpha} + V_5^\epsilon x\frac{\partial^2 P_0}{\partial\eta\partial x} + V_6^\epsilon x\frac{\partial^2 P_0}{\partial x\partial\alpha}\Big]\Big|_{x=B} = \frac{1}{\sqrt{\epsilon}}g_1(T,r), \end{cases}$$

where we have used the fact that $\frac{\partial P_0}{\partial\bar{\sigma}_2}\big|_{x=B} = 0$ and $\frac{\partial P_0}{\partial\bar{\alpha}}\big|_{x=B} = 0$. To obtain a probability representation for $\hat{P}_{1,0}$ under the T-forward measure, we let $\hat{Q}_{1,0} =$

$B(T,r)\exp(-\bar{\lambda}T)\hat{P}_{1,0}$; then, differentiation shows that $\hat{Q}_{1,0}$ solves

$$\begin{cases} (-\frac{\partial}{\partial T}+\mathcal{A})\hat{Q}_{1,0}=0 \quad \text{in } x>B \text{ and } T>0, \\ \hat{Q}_{1,0}(0,x,r)=0, \\ \hat{Q}_{1,0}(T,B,r)=\frac{g_1(T,r)}{\sqrt{\epsilon}B(T,r)\exp(-\bar{\lambda}T)}, \end{cases}$$

in which $\mathcal{A}$ is defined as

$$\mathcal{A}=-\frac{1}{2}\bar{\sigma}_2x^2\frac{\partial^2}{\partial x^2}+(r+\bar{\lambda}-\bar{\rho}_1\bar{\sigma}_2\eta b(T))x\frac{\partial}{\partial x}+(\alpha-\beta r-\eta^2 b(T))\frac{\partial}{\partial r}+\bar{\sigma}_2\eta\bar{\rho}_1 x\frac{\partial^2}{\partial x\partial r}+\frac{1}{2}\frac{\partial^2}{\partial r^2}.$$

$\mathcal{A}$ is the infinitesimal generator of (S_t, r_t) under the T-forward measure. Therefore, we obtain

$$\begin{aligned} \hat{Q}_{1,0} &= \mathbb{E}^T\left[\frac{g_1(\tau,r_\tau)1_{\{\tau\le T\}}}{\sqrt{\epsilon}B(\tau,r_\tau)\exp(-\bar{\lambda}\tau)}|S_0=x,r_0=r)\right] \\ &= \int_0^T\int_{-\infty}^{+\infty}\frac{g_1(\tau,r_\tau)}{\sqrt{\epsilon}B(\tau,r_\tau)\exp(-\bar{\lambda}\tau)}\bigg|_{r_\tau=\tilde{r},\tau=t}\mathbb{P}(r_\tau\in d\tilde{r},\tau\in dt|S_0=x,r_0=r) \\ &\approx \sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\frac{g_1(t_j,r_i)}{\sqrt{\epsilon}B(t_j,r_i)\exp(-\bar{\lambda}t_j)}q_{r_i,t_j}(x,r;B). \end{aligned}$$

Proposition III.2. *The correction term $\sqrt{\epsilon}P_{1,0}$ for a down-and-out barrier call option can be approximated by*

$$\begin{aligned} \sqrt{\epsilon}P_{1,0} \approx\ & B(T,r)\exp(-\bar{\lambda}T)\sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\frac{g_1(t_j,r_i)}{B(t_j,r_i)\exp(-\bar{\lambda}t_j)}q_{r_i,t_j}(x,r;B) \\ & -\frac{1}{\bar{\sigma}_2}\left(V_1^\epsilon\frac{\partial P_0}{\partial\bar{\sigma}_2}+V_2^\epsilon x\frac{\partial}{\partial x}\left(\frac{\partial^2 P_0}{\partial x\partial\bar{\sigma}_2}\right)\right) \\ & +V_3^\epsilon\left(-x\frac{\partial^2 P_0}{\partial x\partial\alpha}-\frac{\partial P_0}{\partial\alpha}\right)+V_4x^2\frac{\partial^3 P_0}{\partial x^2\partial\alpha}+V_5^\epsilon x\frac{\partial^2 P_0}{\partial\eta\partial x}+V_6^\epsilon x\frac{\partial^2 P_0}{\partial x\partial\alpha}. \end{aligned}$$

Similarly, using Proposition II.5 and the additional fact that $\frac{\partial P_0}{\partial r}|_{x=B}=0$, we are able to obtain an approximation for $\sqrt{\delta}P_{0,1}$.

Proposition III.3. *The correction term $\sqrt{\delta}P_{0,1}$ for a down-and-out call option can* *be approximated by*

$$\sqrt{\delta}P_{0,1} \approx B(T,r)\exp(-\bar{\lambda}T)\sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\frac{g_2(t_j,r_i)}{B(t_j,r_i)\exp(-\bar{\lambda}t_j)}q_{r_i,t_j}(x,r;B)$$
$$+V_1^\delta\frac{(T-t)^2}{2}x^2\frac{\partial^2 P_0}{\partial x^2}+V_2^\delta\frac{1}{\beta}\left[x\frac{\partial^2 P_0}{\partial\alpha\partial x}-\frac{\partial P_0}{\partial\alpha}\right.$$
$$\left.+\frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2}-x\frac{\partial P_0}{\partial x}+P_0\right)-(T-t)\left(x\frac{\partial^2 P_0}{\partial r\partial x}-\frac{\partial P_0}{\partial r}\right)\right],$$

where

$$g_2(T,r)=V_2^\delta\frac{1}{\beta}\left[x\frac{\partial^2 P_0}{\partial\alpha\partial x}+\frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2}-x\frac{\partial P_0}{\partial x}\right)-(T-t)x\frac{\partial^2 P_0}{\partial r\partial x}\right]\Bigg|_{x=B}.$$

3.4 Lookback Options

In this section, we price a *lookback put option*, which pays the difference of the realized maximum of the underlying asset price through the option's life time and the asset price at the expiration time T. The payoff can be expressed as

$$h(X_T)=J_T-X_T,$$

where J_t is defined as the running maximum

$$J_t=\max_{0\le s\le t}X_s.$$

In our pricing framework, the price $P^{\epsilon,\delta}$ for such an option solves

$$\begin{cases}\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}=0 \quad \text{in } x<J \text{ and } T>0,\\ P^{\epsilon,\delta}(0,x,J,r,y,\tilde{y},z)=J-x,\\ P_J(T,J,J,r,y,\tilde{y},z)=0.\end{cases}$$

As in the Black-Scholes setting, we use a similarity reduction. Let

$$\xi=\frac{x}{J} \quad \text{and} \quad P^{\epsilon,\delta}(T,x,J,r,y,\tilde{y},z)=JQ^{\epsilon,\delta}(T,x/J,r,y,\tilde{y},z).$$

Then, $Q^{\epsilon,\delta}$ solves

$$\begin{cases} \mathcal{L}^{\epsilon,\delta} Q^{\epsilon,\delta} = 0 \quad \text{in } \xi < 1 \text{ and } T > 0, \\ Q^{\epsilon,\delta}(0,\xi,r,y,\tilde{y},z) = 1-\xi, \\ (Q^{\epsilon,\delta}_{\xi} - Q^{\epsilon,\delta})(T,1,r,y,\tilde{y},z) = 0. \end{cases}$$

Here, $\mathcal{L}^{\epsilon,\delta}$ is defined the same as before, except that x is now replaced by ξ. We approximate $Q^{\epsilon,\delta}$ by

$$\widetilde{Q^{\epsilon,\delta}} = Q_0 + \sqrt{\epsilon} Q_{1,0} + \sqrt{\delta} Q_{0,1}, \tag{3.6}$$

and solve the resulting boundary-value PDE problems from an asymptotic expansion for Q_0, $\sqrt{\epsilon} Q_{1,0}$, and $\sqrt{\delta} Q_{0,1}$. We, then, approximate $P^{\epsilon,\delta}$ by

$$\widetilde{P^{\epsilon,\delta}} = P_0 + \sqrt{\epsilon} P_{1,0} + \sqrt{\delta} P_{0,1},$$

where $P_0(T,x,J,r) = JQ_0(T,x/J,r)$, $P_{1,0}(T,x,J,r) = JQ_{1,0}(T,x/J,r)$ and $P_{0,1}(T,x,J,r) = JQ_{0,1}(T,x/J,r)$.

3.4.1 The Leading Term P_0

We express Q_0 as a solution of

$$\begin{cases} \langle \mathcal{L}_2 \rangle Q_0 = 0 \quad \text{in } \xi < 1 \text{ and } T > 0, \\ Q_0(0,\xi,r) = 1-\xi, \\ (\frac{\partial Q_0}{\partial \xi} - Q_0)(T,1,r) = 0. \end{cases}$$

We now transform the PDE above into a constant-coefficient Dirichlet boundary-value problem. Let

$$\zeta = \ln \xi \quad \text{and} \quad u_0(T,\zeta,r) = Q_0(T,\xi,r).$$

We find that $u_0(T,\zeta,r)$ satisfies

$$\begin{cases} \mathcal{A}u_0 = 0 \quad \text{in } \zeta < 0, \text{ and } T > 0, \\ u_0(0,\zeta,r) = 1 - e^{\zeta}, \\ (\frac{\partial u_0}{\partial \zeta} - u_0)(T,0,r) = 0, \end{cases} \tag{3.7}$$

where

$$\mathcal{A} = -\frac{\partial}{\partial T} + \frac{1}{2}\bar{\sigma}_2^2 \frac{\partial^2}{\partial \zeta^2} + (r + \bar{\lambda} - \frac{1}{2}\bar{\sigma}_2^2)\frac{\partial}{\partial \zeta} + (\alpha - \beta r)\frac{\partial}{\partial r} + \bar{\sigma}_2 \eta \bar{\rho}_1 \frac{\partial^2}{\partial \zeta \partial r} + \frac{1}{2}\eta^2 \frac{\partial^2}{\partial r^2} - (r + \bar{\lambda})\cdot.$$

It follows that $w_0(T,\zeta,r) = \frac{\partial u_0}{\partial \zeta}(T,\zeta,r) - u_0(T,\zeta,r)$ solves the Dirichlet boundary-value problem

$$\begin{cases} \mathcal{A}w_0 = 0 \quad \text{in } \zeta < 0 \text{ and } T > 0, \\ w_0(0,\zeta,r) = -1, \\ w_0(T,0,r) = 0. \end{cases}$$

The function w_0 can be expressed as

$$\begin{aligned} w_0 &= -\mathbb{E}\left[\exp(-\int_0^T (r_t + \bar{\lambda})dt) 1_{\{\tau > T\}} | \ln S_0 = \zeta, r_0 = r\right], \\ &= B(T,r)\exp(-\bar{\lambda}T)\mathbb{E}^T[1_{\{\tau \le T\}} | \ln S_0 = \zeta, r_0 = r] - B(T,r)\exp(-\bar{\lambda}T), \end{aligned}$$

where S_t, and r_t are the same stochastic processes as in Section 3.3.1, and here τ is the first-passage time of $\ln S_t$ to 0, i.e., S_t to 1. Similar to Section 3.3, we can discretize the integrals and approximate w_0 by

$$\tilde{w}_0 = B(T,r)\exp(-\bar{\lambda}T) \sum_{j=1}^{n_T} \sum_{i=1}^{n_r} q_{r_i,t_j}(e^{\zeta}, r; 1) - B(T,r)\exp(-\bar{\lambda}T).$$

We now solve the ODE

$$w_0(T,\zeta,r) = \frac{\partial u_0}{\partial \zeta}(T,\zeta,r) - u_0(T,\zeta,r)$$

to recover $u_0(T,\zeta,r)$. We have

$$u_0(T,\zeta,r) = e^{\zeta}\int_0^{\zeta} e^{-\kappa} w_0(T,\kappa,r)d\kappa + e^{\zeta}u_0(T,0,r). \tag{3.8}$$

To find $u_0(T,0,r)$, we substitute $w_0(T,\zeta,r)$ in (3.8) to (3.7) and set $\zeta = 0$. We find

$$\begin{cases} -\frac{\partial u_0}{\partial T}(T,0,r) + ((\alpha+\bar{\sigma}_2\eta\bar{\rho}_1) - \beta r)\frac{\partial u_0}{\partial r}(T,0,r) + \frac{1}{2}\eta^2\frac{\partial^2 u_0}{\partial r^2}(T,0,r) = -\frac{1}{2}\bar{\sigma}_2^2\frac{\partial w_0}{\partial \zeta}(T,0,r), \\ u_0(0,0,r) = 0. \end{cases}$$

We recognize that

$$((\alpha+\bar{\sigma}_2\eta\bar{\rho}_1) - \beta r)\frac{\partial}{\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2}$$

is the infinitesimal generator of a process r_t following the dynamics of

$$dr_t = (\alpha+\bar{\sigma}_2\eta\bar{\rho}_1 - \beta r_t)dt + \eta dW_t.$$

Hence, we can express $u_0(T,0,r)$ as

$$u_0(T,0,r) = \frac{1}{2}\bar{\sigma}_2^2\mathbb{E}\left[\int_0^T \frac{\partial w_0}{\partial \zeta}(t,0,r_t)dt | r_0 = r\right].$$

Since r_t can be easily solved to be

$$r_t = e^{-\beta t}r_0 + \frac{\alpha+\bar{\sigma}_2\eta\bar{\rho}_1}{\beta}(1-e^{-\beta t}) + \eta e^{-\beta t}\int_0^t e^{\beta s}dW_s,$$

and, hence, normally distributed with mean

$$\tilde{m}_r = e^{-\beta t}r_0 + \frac{\alpha+\bar{\sigma}_2\eta\bar{\rho}_1}{\beta}(1-e^{-\beta t}),$$

and variance $\tilde{v}_r = \frac{\eta^2}{2\beta}(1-e^{-2\beta t})$, we can write out $u_0(T,0,r)$ as

$$u_0(T,0,r) = \frac{1}{2}\bar{\sigma}_2^2\int_0^T\int_{-\infty}^{+\infty}\frac{\partial w_0}{\partial \zeta}(t,0,\kappa)\frac{1}{\sqrt{2\pi\tilde{v}_r}}\exp(-\frac{(\kappa-\tilde{m}_r)^2}{2\tilde{v}_r}d\kappa.$$

The above calculations, thus, lead to a semi-closed form solution for P_0.

Proposition III.4. *The leading term P_0 for a lookback put option can be approximated by*

$$x\left(\int_0^{\zeta} e^{-\kappa}\tilde{w}_0(T,\kappa,r)d\kappa + \frac{1}{2}\bar{\sigma}_2^2\int_0^T\int_{-\infty}^{+\infty}\frac{\partial\tilde{w}_0}{\partial\zeta}(t,0,\kappa)\frac{1}{\sqrt{2\pi\tilde{v}_r}}\exp(-\frac{(\kappa-\tilde{m}_r)^2}{2\tilde{v}_r}d\kappa\right)\Big|_{\zeta=\ln(x/J)},$$

where

$$\tilde{w}_0 = B(T,r)\exp(-\bar{\lambda}T)\sum_{j=1}^{n_T}\sum_{i=1}^{n_r} q_{r_i,t_j}(e^{\zeta},r;1) - B(T,r)\exp(-\bar{\lambda}T),$$
$$\tilde{m}_r = e^{-\beta t}r_0 + \frac{\alpha+\bar{\sigma}_2\eta\bar{\rho}_1}{\beta}(1-e^{-\beta t}),$$
$$\tilde{v}_r = \frac{\eta^2}{2\beta}(1-e^{-2\beta t}).$$

3.4.2 The Correction Terms $\sqrt{\epsilon}P_{1,0}$ and $\sqrt{\delta}P_{0,1}$

$Q_{1,0}$ defined in (3.6) satisfies

$$\begin{cases} \langle\mathcal{L}_2\rangle Q_{1,0} = \langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2-\langle\mathcal{L}_2\rangle)\rangle Q_0 \text{ in } \xi<1, \text{ and } T>0, \\ Q_{1,0}(0,\xi,r)=0, \\ (\frac{\partial Q_{1,0}}{\partial\xi}-Q_{1,0})(T,1,r)=0. \end{cases}$$

Let

$$Q_{1,0} = \hat{Q}_{1,0} + \frac{1}{\sqrt{\epsilon}}\Bigg[-\frac{1}{\bar{\sigma}_2}\left(V_1^{\epsilon}\frac{\partial Q_0}{\partial\bar{\sigma}_2} + V_2^{\epsilon}\xi\frac{\partial}{\partial\xi}\left(\frac{\partial^2 Q_0}{\partial\xi\partial\bar{\sigma}_2}\right)\right)$$
$$+ V_3^{\epsilon}\left(-\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha} - \frac{\partial Q_0}{\partial\alpha}\right) + V_4\xi^2\frac{\partial^3 Q_0}{\partial\xi^2\partial\alpha} + V_5^{\epsilon}\xi\frac{\partial^2 Q_0}{\partial\eta\partial\xi} + V_6^{\epsilon}\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha}\Bigg].$$

By applying Proposition II.3, and the boundary conditions for Q_0 and $Q_{1,0}$, we find $\hat{Q}_{1,0}$ to satisfy

$$\begin{cases} \langle\mathcal{L}_2\rangle\hat{Q}_{1,0} = 0 \quad \text{in } \xi<1 \text{ and } T>0, \\ \hat{Q}_{1,0}(0,\xi,r)=0, \\ (\frac{\partial\hat{Q}_{1,0}}{\partial xi}-\hat{Q}_{1,0})(T,1,r) = g_1(T,r), \end{cases}$$

where

$$g_1(T,r) = \Bigg[\frac{\partial}{\partial \xi}\Bigg(- \frac{V_2^\epsilon}{\bar\sigma_2}\xi\frac{\partial}{\partial \xi}\left(\frac{\partial^2 Q_0}{\partial\xi\partial\bar\sigma_2}\right) - V_3^\epsilon\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha} + V_4\xi^2\frac{\partial^3 Q_0}{\partial\xi^2\partial\alpha} + V_5^\epsilon\xi\frac{\partial^2 Q_0}{\partial\eta\partial\xi} + V_6^\epsilon\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha}\Bigg) - \Bigg(- \frac{V_2^\epsilon}{\bar\sigma_2}\xi\frac{\partial}{\partial \xi}\left(\frac{\partial^2 Q_0}{\partial\xi\partial\bar\sigma_2}\right) - V_3^\epsilon\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha} + V_4\xi^2\frac{\partial^3 Q_0}{\partial\xi^2\partial\alpha} + V_5^\epsilon\xi\frac{\partial^2 Q_0}{\partial\eta\partial\xi} + V_6^\epsilon\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha}\Bigg)\Bigg]\Bigg|_{\xi=1}.$$

Let $\zeta = \ln\xi$, $\hat u_{1,0}(T,\zeta,r) = \hat Q_{1,0}(T,\xi,r)$, and

$$\hat w_{1,0} = \frac{\partial \hat u_{1,0}}{\partial\zeta} - \hat u_{1,0}(T,\zeta,r)$$

We find

$$\begin{cases} \mathcal{A}\hat w_{1,0} = 0 \quad \text{in } \zeta < 0 \text{ and } T > 0, \\ \hat w_{1,0}(0,\zeta,r) = 0, \\ \hat w_{1,0}(T,0,r) = g_1(T,r), \end{cases}$$

in which

$$\mathcal{A} = -\frac{\partial}{\partial T} + \frac{1}{2}\bar\sigma_2^2\frac{\partial^2}{\partial\zeta^2} + (r+\bar\lambda-\frac{1}{2}\bar\sigma_2^2)\frac{\partial}{\partial\zeta} + (\alpha-\beta r)\frac{\partial}{\partial r} + \bar\sigma_2\eta\bar\rho_1\frac{\partial^2}{\partial\zeta\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2} - (r+\bar\lambda)\cdot.$$

Similar to how we determined w_0 in Section 3.4.1, $\hat w_{1,0}$ can approximated by

$$\tilde w_{1,0} = B(T,r)\exp(-\bar\lambda T)\sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\frac{g_1(t_j,r_i)}{B(t_j,r_i)\exp(-\bar\lambda t_j)}q_{r_i,t_j}(e^\zeta,r;1).$$

We have

$$\hat u_{1,0}(T,\zeta,r) = e^\zeta\int_0^\zeta e^{-\kappa}\hat w_{1,0}(T,\kappa,r)d\kappa + e^\zeta\hat u_{1,0}(T,0,r),$$

and

$$\hat u_{1,0}(T,0,r) = \int_0^T\int_{-\infty}^{+\infty}\left(\frac{1}{2}\bar\sigma_2^2\frac{\partial\hat w_{1,0}}{\partial\zeta}(t,0,\kappa) + (r+\bar\lambda-\frac{1}{2}\bar\sigma_2^2)g_1(T,\kappa) + \bar\sigma_2\eta\bar\rho_1\frac{\partial\hat w_{1,0}}{\partial r}(T,0,\kappa)\right)\frac{1}{\sqrt{2\pi\tilde v_r}}\exp\left(-\frac{(\kappa-\tilde m_r)^2}{2\tilde v_r}\right)d\kappa.$$

In summary, we have the following proposition:

Proposition III.5. *The correction term $\sqrt{\epsilon}P_{1,0}$ can be approximated by*

$$\Bigg[x\Bigg(\int_0^\zeta e^{-\kappa}\tilde w_{1,0}(T,\kappa,r)d\kappa + \int_0^T\int_{-\infty}^{+\infty}\tilde g_1(T,\kappa)\frac{1}{\sqrt{2\pi\tilde v_r}}\exp\left(-\frac{(\kappa-\tilde m_r)^2}{2\tilde v_r}\right)d\kappa\Bigg) + J\Bigg(-\frac{1}{\bar\sigma_2}\left(V_1^\epsilon\frac{\partial Q_0}{\partial\bar\sigma_2} + V_2^\epsilon\xi\frac{\partial}{\partial\xi}\left(\frac{\partial^2 Q_0}{\partial\xi\partial\bar\sigma_2}\right)\right) + V_3^\epsilon\left(-\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha} - \frac{\partial Q_0}{\partial\alpha}\right) + V_4\xi^2\frac{\partial^3 Q_0}{\partial\xi^2\partial\alpha} + V_5^\epsilon\xi\frac{\partial^2 Q_0}{\partial\eta\partial\xi} + V_6^\epsilon\xi\frac{\partial^2 Q_0}{\partial\xi\partial\alpha}\Bigg)\Bigg]\Bigg|_{\zeta=\ln(x/J),\xi=x/J},$$

where

$$\tilde{g}_1(T,\kappa) = \frac{1}{2}\bar{\sigma}_2^2 \frac{\partial \tilde{w}_{1,0}}{\partial \zeta}(t,0,\kappa) + (r+\bar{\lambda} - \frac{1}{2}\bar{\sigma}_2^2) g_1(T,\kappa) + \bar{\sigma}_2 \eta \bar{\rho}_1 \frac{\partial \tilde{w}_{1,0}}{\partial r}(T,0,\kappa).$$

By applying similar arguments, we can deduce the approximation for $\sqrt{\delta}P_{0,1}$.

Proposition III.6. *The correction term* $\sqrt{\epsilon}P_{1,0}$ *can be approximated by*

$$\begin{aligned}
&\Bigg[x\Bigg(\int_0^{\zeta} e^{-\kappa}\tilde{w}_{0,1}(T,\kappa,r)d\kappa + \int_0^T \int_{-\infty}^{+\infty} \tilde{g}_2(T,\kappa)\frac{1}{\sqrt{2\pi\tilde{v}_r}}\exp\Bigg(-\frac{(\kappa-\tilde{m}_r)^2}{2\tilde{v}_r}\Bigg)d\kappa\Bigg)\\
&+ J\Bigg(V_1^{\delta}\frac{(T-t)^2}{2}\xi^2\frac{\partial^2 P_0}{\partial \xi^2} + V_2^{\delta}\frac{1}{\beta}\Bigg(\xi\frac{\partial^2 P_0}{\partial\alpha\partial\xi} - \frac{\partial P_0}{\partial \alpha}\\
&+ \frac{(T-t)^2}{2}\left(\xi^2\frac{\partial^2 P_0}{\partial\xi^2} - \xi\frac{\partial P_0}{\partial \xi} + P_0\right) - (T-t)\left(\xi\frac{\partial^2 P_0}{\partial r\partial\xi} - \frac{\partial P_0}{\partial r}\right)\Bigg)\Bigg]\Bigg|_{\zeta=\ln(x/J),\xi=x/J},
\end{aligned}$$

where

$$\tilde{g}_2(T,\kappa) = \frac{1}{2}\bar{\sigma}_2^2 \frac{\partial \tilde{w}_{0,1}}{\partial \zeta}(T,0,\kappa) + (r+\bar{\lambda} - \frac{1}{2}\bar{\sigma}_2^2) g_2(T,\kappa) + \bar{\sigma}_2 \eta \bar{\rho}_1 \frac{\partial \tilde{w}_{0,1}}{\partial r}(T,0,\kappa),$$

in which

$$\begin{aligned}
g_2(T,r) = V_2^{\delta}\frac{1}{\beta}\Bigg[&\frac{\partial}{\partial\xi}\Bigg(\xi\frac{\partial^2 P_0}{\partial\alpha\partial\xi} + \frac{(T-t)^2}{2}\left(\xi^2\frac{\partial^2 P_0}{\partial\xi^2} - \xi\frac{\partial P_0}{\partial\xi}\right) - (T-t)\xi\frac{\partial^2 P_0}{\partial r\partial\xi}\Bigg)\\
&- \Bigg(\xi\frac{\partial^2 P_0}{\partial\alpha\partial\xi} + \frac{(T-t)^2}{2}\left(\xi^2\frac{\partial^2 P_0}{\partial\xi^2} - \xi\frac{\partial P_0}{\partial\xi}\right) - (T-t)\xi\frac{\partial^2 P_0}{\partial r\partial\xi}\Bigg)\Bigg]\Bigg|_{\xi=1},
\end{aligned}$$

and

$$\tilde{w}_{0,1} = B(T,r)\exp(-\bar{\lambda}T)\sum_{j=1}^{n_T}\sum_{i=1}^{n_r}\frac{g_2(t_j,r_i)}{B(t_j,r_i)\exp(-\bar{\lambda}t_j)}q_{r_i,t_j}(e^{\zeta},r;1).$$

CHAPTER IV

Extension of Linetsky's Negative-Power Intensity Model

4.1 Introduction

In this chapter we study a parsimonious extension of Linetsky's [29] one-factor reduced-form framework for pricing equity and credit derivatives subject to default. We introduce one additional factor into the framework, so that one company's default intensity is no longer only dependent on its stock price. Applying perturbation methods, we show in Section 4.2 that the correction term can be easily solved, and can be nicely expressed as a *Greek* letter of the leading term. In this new framework, we study the pricing of a double barrier option, as an important path-dependent option. The Laplace transform of the price is obtained in closed form in Section 4.3.1 and is then inverted analytically using eigenfunction expansions in Section 4.3.2.

4.2 Stochastic Default Intensity

In [29], the default intensity of bankruptcy is a negative power of the stock price:

$$h(S) = \alpha S^{-p}, \quad \alpha > 0, \quad p > 0,$$

where α is constant. Hence, the model is one-dimensional, and the default intensity of a specific company is only dependent on its stock price proces. Here, we extend this framework by allowing α to be stochastic. In particular, we assume that α is driven by a fast evolving mean reverting process. Under a pricing measure:

$$\alpha_t = \alpha(Y_t)$$
$$dY_t = [\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t)]dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^1,$$

where α is now a bounded and strictly positive function of the diffusion process Y. Under the risk neutral pricing measure, the pre-default stock price follows the dynamics:

$$dS_t = (r - q - \alpha(Y_t)S^{-p})S_t dt + \sigma S_t dW_t^0, \quad S_0 = S > 0,$$

in which W_t^0 and W_t^1 are independent Brownian motions. This setting is reasonable in that we already have strong correlation between the default intensity and the stock price with the presence of negative power of the stock price in the default intensity. As in Chapter II, the valuations of European call and put options and bond reduce to computing expectation of form:

$$P^\epsilon(T, S, y) = e^{-rT}\mathbb{E}\left[\exp\left(-\int_0^T \alpha(Y_t)S_t^{-p}dt\right)\psi(S_T)\middle| S_0 = S, Y_0 = y\right].$$

When $\psi(S_T) = (S_T - K)^+$, P^ϵ is the price of a call option on a defaultable stock with strike K. When $\psi(S_T) = 1$, P^ϵ is the price of a defaultable bond witth unit face value and zero recovery. An application of the Feynman-Kac Theorem gives us:

$$\begin{cases} \mathcal{L}^\epsilon P^\epsilon = 0, \\ P^\epsilon(0, S, y) = \psi(S), \end{cases}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

and

$$\mathcal{L}_0 := \nu^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y},$$
$$\mathcal{L}_1 := \rho\sigma\nu\sqrt{2}S\frac{\partial^2}{\partial S \partial y} - \Lambda(y)\nu\sqrt{2}\frac{\partial}{\partial y},$$
$$\mathcal{L}_2 := -\frac{\partial}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q + \alpha(y)x^{-p})S\frac{\partial}{\partial S} - (r + \alpha(y)x^{-p})\cdot .$$

Let the approximation for P^ϵ be given by

$$\tilde{P}^\epsilon = P_0 + \sqrt{\epsilon}P_1.$$

Following our usual formal expansion procedures, P_0 and P_1 are independent of y, and if we let $\langle\cdot\rangle$ denote operator of taking average with respect to the invariant distribution of Y_t and let $\langle\alpha(y)\rangle = \bar{\alpha}$, we have P_0 and P_1 satisfy the following initial value problems, respectively,

$$\begin{cases} \langle\mathcal{L}_2\rangle P_0 = 0, \\ P_0(0, S) = \psi(S), \end{cases}$$

and

$$\begin{cases} \langle\mathcal{L}_2\rangle P_1 = \langle\mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)P_0 = \mathcal{A}P_0 \text{in } T > 0, \\ P_0(0, S) = 0, \end{cases}$$

where $\langle\mathcal{L}_2\rangle$ is the operator

$$\langle\mathcal{L}_2\rangle = -\frac{\partial}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - q + \bar{\alpha}x^{-p})S\frac{\partial}{\partial S} - (r + \bar{\alpha}x^{-p})\cdot .$$

If we let $\phi(y)$ be a solution of $\mathcal{L}_0\phi(y) = \alpha(y) - \bar{\alpha}$, $\mathcal{A}P_0$ can be written out as

$$\mathcal{A}P_0 = -V^\epsilon\left[S^{-p+1}\frac{\partial P_0}{\partial S} + S^{-p}P_0\right],$$

where

$$V^\epsilon = -\nu\sqrt{2}\langle\Lambda(y)\phi'(y)\rangle.$$

Observe that

$$\begin{cases} \langle \mathcal{L}_2 \rangle \frac{\partial P_0}{\partial \bar{\alpha}} = -S^{-p+1} \frac{\partial P_0}{\partial S} + S^{-p} P_0, \\ \frac{\partial P_0}{\partial \bar{\alpha}}(0, S) = 0. \end{cases}$$

Hence,

$$P_1 = V^\epsilon \frac{\partial P_0}{\partial \bar{\alpha}},$$

and

$$\tilde{P}^\epsilon = P_0 + V^\epsilon \frac{\partial P_0}{\partial \bar{\alpha}}.$$

4.3 Double Barrier Options

We consider a double barrier option that pays $\psi(S_T)$ at the maturity but becomes worthless when stock price either go below a lower barrier L or go above a upper barrier U. The price $P^\epsilon(T, S)$ for such an option satisfies:

$$\begin{cases} \mathcal{L}^\epsilon P^\epsilon = 0 \text{ in } T > 0 \text{ and } S \in (L, U), \\ P^\epsilon(0, S) = \psi(S), \\ P^\epsilon(T, L) = 0, \\ P^\epsilon(T, U) = 0. \end{cases}$$

We approximate P^ϵ by

$$\tilde{P}^\epsilon = P_0 + \sqrt{\epsilon} P_1.$$

It follows from the asymptotic expansion that P_0 and P_1 satisfy the following PDEs with initial and boundary conditions, respectively,

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_0 = 0 \text{ in } T > 0 \text{ and } S \in (L, U), \\ P_0(0, S) = \psi(S), \\ P_0(T, L) = 0, \\ P_0(T, U) = 0, \end{cases}$$

and

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_1 = \mathcal{A} P_0 \quad \text{in } T > 0 \text{ and } S \in (L, U), \\ P_1(0, S) = 0, \\ P_1(T, L) = 0, \\ P_1(T, U) = 0. \end{cases}$$

We observe that given P_0, the expression $V_1^\epsilon \frac{\partial P_0}{\partial \bar{\alpha}}$ also satisfies the zero boundary conditions as P_0 does. This observation, together with our calculations from the previous section, yields that

$$P_1 = V_1^\epsilon \frac{\partial P_0}{\partial \bar{\alpha}}. \tag{4.1}$$

Therefore, we can focus on effort on finding an expression for the leading term $P_0(T, S)$. By the Feynman-Kac Theorem, we can write $P_0(T, S)$ as

$$P_0(T, S) = \mathbb{E}[e^{-rT} e^{\int_0^T \bar{\alpha} \tilde{S}_t^{-p} dt} 1_{\{\gamma_{L,U} > T\}} \psi(\tilde{S}_T) | \tilde{S}_0 = S],$$

where $\tilde{S}_t$ is the solution of the SDE

$$d\tilde{S}_t = (r - q + \bar{\alpha} \tilde{S}_t^{-p}) \tilde{S}_t dt + \sigma \tilde{S}_t dW_t,$$

and $\gamma_{L,U}$ is the first time that $\tilde{S}_t$ hits L or U. Using equation (2.2) in [29], we obtain

$$P_0(T, S) = S \hat{\mathbb{E}}[e^{-qT} \tilde{S}_T^{-1} 1_{\{\gamma_{L,U} > T\}} \psi(\tilde{S}_T) | \tilde{S}_0 = S], \tag{4.2}$$

where $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = e^{-\sigma B_T - \frac{1}{2}\sigma^2 T}$.

We make the following changes of variable:

$$X_t = \beta S_t^p, \quad \tau = \frac{p^2\sigma^2 T}{4},$$

with $\beta = p\sigma^2/(4\bar{\alpha})$. Then, by Itô's formula, the process X solves the SDE:

$$dX_t = (2(\eta+1)X_t + 1)dt + 2X_t dW_t,$$

where $\eta = \frac{2}{p\sigma^2}(r - q + \sigma^2/2)$. We can express $P_0(T,S)$ as

$$P_0(T,S) = e^{-qT} S\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)],$$

in which $\gamma_{a,b}$ is the first time that X hits $a = \beta L^p$ or $b = \beta U^p$, and

$$\chi_\psi(y) := (y/\beta)^{-1/p}\psi((y/\beta)^{1/p}).$$

4.3.1 Laplace Transform

We calculate the Laplace Transform of the leading term for the price of a double barrier call option, for which the payoff function is

$$\psi(S_T) = (S_T - K)^+.$$

Let $p(t;x,y)$ denote the transition density of X. The resolvent kernel or Green's function $G_s(x,y) := \int_0^\infty e^{-st}p(t;x,y)dt$ with $s > 0$ is the Laplace transform of the transition density. Let $k = \beta K^p$. The Laplace transform of $\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)]$ is given by

(4.3)

$$\begin{aligned}
\mathcal{L}\big[\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)]\big] &= \int_0^\infty e^{-st}\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}(X_\tau/\beta)^{-1/p}((X_\tau/\beta)^{1/p} - K)^+] \\
&= \int_0^\infty e^{-st}\left(\int_k^b (y/\beta)^{-1/p}((y/\beta)^{1/p} - K)p(t;x,y)dy\right)dt \\
&= \int_k^b ((y/\beta)^{-1/p}((y/\beta)^{1/p} - K)G_s(x,y)dy.
\end{aligned}$$

In what follows, we explicitly calculate $G_s(x,y)$. The function $u(x,\tau) = \hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)]$ solves

$$\begin{cases} -\frac{\partial u}{\partial \tau} + 2x^2\frac{\partial^2 u}{\partial x^2} + ((\eta+1)x+1)\frac{\partial u}{\partial x} = 0, \quad l < x < u \quad \text{in } \tau > 0 \text{ and } x \in (a,b), \\ u(x,0) = \chi_\psi(x), \\ u(a,\tau) = 0, \\ u(b,\tau) = 0. \end{cases}$$

The Laplace transform $\mathcal{L}u = U$ is the solution of the boundary value ODE problem

$$\begin{cases} \mathcal{D}U(x) = sU(x) - \chi_\psi(x), \\ U(a) = 0, \quad U(b) = 0, \end{cases}$$

where $\mathcal{D} = 2x^2\frac{d}{dx^2} + (2(\eta+1)+1)\frac{d}{dx}$ is the infinitesimal generator of the diffusion process X. The operator $\mathcal{D}$ can be written as

$$\mathcal{D} = \frac{1}{2}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\left(\frac{1}{s(x)}\frac{d}{dx}\right),$$

in which $m(x) = \frac{1}{2}x^{\eta-1}e^{-1/2x}$ is called the speed density, and $s(x) = x^{-\eta-1}e^{1/2x}$ is called the scale density. We see that $G_s(x,y)$ is the unique continuous solution of

$$\begin{cases} -\mathcal{D}U(x) + sU(x) = \delta(x-y), \quad x \in (a,b), \\ U(a) = 0, \quad U(b) = 0. \end{cases}$$

In other words, $G_s(x,y)$ is the Green's function for the ordinary differential operator $-\mathcal{D} + s\cdot$ with the boundary conditions. The Green's function can be constructed as follows; see [17] p. 354. Let $v_a(x)$ and $v_b(x)$ be the nonzero solutions of the initial value problems

$$\mathcal{D}v_a(x) - sv_a(x) = 0, \quad v_a(a) = 0,$$

$$\mathcal{D}v_b(x) - sv_b(x) = 0, \quad v_b(b) = 0,$$

and let W be the Wronskian:

$$W = v_a v_b' - v_b v_a'$$

Then, the Green's function takes in the form

$$G_s(x,y) = \frac{v_a(x \wedge y) v_b(x \vee y)}{-2y^2 W(y)}.$$

Let $\psi_s(x)$ and $\phi_s(x)$ the unique solutions of the ODE

$$\mathcal{D}U(x) = sU(x),$$

such that ψ_s is increasing and ϕ_s is decreasing. The above equation can be reduced to the Whittaker differential equation [33] by the Liouville substitution [29]

$$U(x) = x^{\frac{1-\kappa}{2}} e^{\frac{1}{4x}} w\left(\frac{1}{2x}\right).$$

Hence $\psi_s(x)$ and $\phi_s(x)$ can be readily solved as given in [29]

$$\psi_s(x) = x^{\frac{1-\eta}{2}} e^{\frac{1}{4x}} W_{\frac{1-\eta}{2},\mu(s)}\left(\frac{1}{2x}\right),$$
$$\phi_s(x) = x^{\frac{1-\eta}{2}} e^{\frac{1}{4x}} \mathcal{M}_{\frac{1-\eta}{2},\mu(s)}\left(\frac{1}{2x}\right),$$

where $\mathcal{M}$ is the regularized first Whittaker function, and W is the second Whittaker function (see [33]) and $\mu(s) = \frac{1}{2}\sqrt{2s+\eta}$. Let

$$\Delta_s(A,B) := \phi_s(A)\psi_s(B) - \psi_s(A)\phi_s(B).$$

Then v_a and v_b take the form

$$v_a(x) = \Delta_s(a,x) \quad v_b(x) = \Delta_s(x,b),$$

and the Green's function can be expressed as

$$G_s(x,y) = \frac{-m(y)}{w_s \Delta_s(a,b)} \Delta_s(a, x \wedge y) \Delta_s(x \vee y, b), \tag{4.4}$$

in which w_s is the Wronskian of ϕ_s and ψ_s with respect to the scale density $s(x)$ and is given by

$$w_s = \frac{1}{2\Gamma(\mu(s) + \frac{\eta}{2})}.$$

We express $G_s(x, y)$ in terms of ϕ_s and ψ_s

$$\begin{aligned} G_s(x,y) = \frac{-m(y)}{w_s \Delta_s(a,b)} [&\phi_s(a)\psi_s(b)\psi_s(x \wedge y)\phi_s(x \vee y) - \phi_s(a)\phi_s(b)\psi_s(x \wedge y)\phi_s(x \vee y) \\ &- \psi_s(a)\psi_s(b)\phi_s(x \wedge y)\phi_s(x \vee y) + \psi_s(a)\phi_s(b)\phi_s(x \wedge y)\phi_s(x \vee y), \end{aligned}$$

and substitute this expression into equation (4.3) to obtain the Laplace transform.

Proposition IV.1. *The Laplace transform of* $\hat{\mathbb{E}}[1_{\{\gamma_{a,b} > \tau\}} \chi_\psi(X_\tau)]$*, when* ψ *is a call payoff function, is given by:*

for $x \leq k$

$$\begin{aligned} \frac{1}{w_s \Delta_s(l,u)} \Big[&\phi_x(a)\psi_s(b)\psi_s(x)J_s(k,b) - \phi_s(a)\phi_s(b)\psi_s(x)I_s(k,b) \\ &- \psi_s(a)\psi_s(b)\phi_s(x)J_s(k,b) + \psi_s(a)\phi_s(b)\phi_s(x)I_s(k,b) \Big]; \end{aligned}$$

for $x > k$

$$\begin{aligned} \frac{1}{w_s \Delta_s(l,u)} \Big[&\phi_s(a)\psi_s(b)\phi_s(x)I_s(k,x) - \phi_s(a)\phi_s(u)\psi_s(x)I_s(k,x) \\ &- \psi_s(a)\psi_s(b)\phi_s(x)J_s(k,x) + \psi_s(a)\phi_s(b)\phi_s(x)J_s \\ &\phi_s(a)\psi_s(b)\psi_s(x)J_s(x,b) - \phi_s(a)\phi_s(b)\psi_s(x)I_s(x,b) \\ &- \psi_s(a)\psi_s(b)\phi_s(x)J_s + \psi_s(a)\phi_s(b)\phi_s(x)I_s \Big], \end{aligned}$$

in which

$$I_s(A,B) = \int_A^B m(y)(y/\beta)^{-1/p}((y/\beta)^{1/p} - K)\psi_s(y)dy,$$
$$J_s(A,B) = \int_A^B m(y)(y/\beta)^{-1/p}((y/\beta)^{1/p} - K)\phi_s(y)dy.$$

By substituting the explicit formulas for $m(y)$, $\phi_s(y)$ and $\psi_s(y)$, the definite integrals I_s and J_s can be found explicitly by using known integrals (Mathematica) in term of confluent hypergeometric and Whittaker functions. Let I_s and J_s be the corresponding indefinite integrals. We have

$$
\begin{aligned}
I_s = &-2y^{(1+\eta)/2}e^{-1/4y}W_{\frac{-(v+1)}{2},\mu(s)}\left(\frac{1}{2y}\right) + \beta^{1/p}K\frac{\pi}{\sin(2\mu(s)\pi)}\Bigg(2^{-3-\frac{3}{2}\eta-\mu(s)}y^{-\eta-\frac{3}{2}-1/p-\mu(s)} \\
&\frac{\Gamma(\eta+\frac{3}{2}+\frac{1}{p}+\mu(s))}{\Gamma(\frac{3}{2}+\eta+\mu(s))}\ {}_2\mathrm{F}_2\left[\eta+\frac{5}{2}+\mu(s),\eta+\frac{3}{2}+\frac{1}{p}+\mu(s);1+\mu(s),\eta+\frac{5}{2}+\frac{1}{p}+\mu(s);-\frac{1}{2y}\right] \\
&2^{-3-\frac{3}{2}\eta+\mu(s)}y^{-\eta-\frac{3}{2}-1/p+\mu(s)}\frac{\Gamma(\eta+\frac{3}{2}+\frac{1}{p}-\mu(s))}{\Gamma(\frac{5}{2}-\eta-\mu(s))} \\
&{}_2\mathrm{F}_2\left[\eta+\frac{5}{2}-\mu(s),\eta+\frac{3}{2}+\frac{1}{p}-\mu(s);1-\mu(s),\eta+\frac{5}{2}+\frac{1}{p}-\mu(s);-\frac{1}{2y}\right]\Bigg), \\
J_s = &-\frac{4}{2\mu(s)-\eta}y^{(1+\eta)/2}e^{-1/4y}\mathcal{M}_{\frac{-(v+1)}{2},\mu(s)}\left(\frac{1}{2y}\right) + \beta^{1/p}K2^{-3-\frac{3}{2}\eta-\mu(s)}y^{-\eta-\frac{3}{2}-1/p-\mu(s)} \\
&\Gamma(\eta+\frac{3}{2}+\frac{1}{p}+\mu(s)){}_2\mathrm{F}_2\left[\eta+\frac{5}{2}+\mu(s),\eta+\frac{3}{2}+\frac{1}{p}+\mu(s);1+\mu(s),\eta+\frac{5}{2}+\frac{1}{p}+\mu(s);-\frac{1}{2y}\right],
\end{aligned}
$$

where ${}_2\mathrm{F}_2$ is the generalized hypergeometric function; see [34] and we have used the identity

$$
W_{\kappa,\mu}(z) = \frac{\pi}{\sin(2\mu\pi)}\left(\frac{\mathcal{M}_{\kappa,-\mu}(z)}{\Gamma(1/2+\mu-\kappa)} - \frac{\mathcal{M}_{\kappa,\mu}(z)}{\Gamma(1/2-\mu-\kappa)}\right).
$$

4.3.2 Eigenfunction Expansion

The inversion of the Laplace transform relies on results from Sturm-Liouville theory that the spectrum (eigenvalues) of the Sturm-Liouville problem:

$$
\mathcal{D}U(x) + \lambda U(x) = 0, \quad U(a) = U(b) = 0, \quad a, b > 0
$$

is simple and discrete. Let $\{\lambda_n\}_{n=1}^{\infty}$ denote the eigenvalues and $\{\theta_n\}_{n=1}^{\infty}$ be the corresponding eigenvectors. The eigenvalues are positive and can be listed as an increasing sequence

$$
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots
$$

with $\lambda_n \to \infty$ as $n \to \infty$. $\{\theta_n\}_{n=1}^{\infty}$ form an orthogonal basis for the Hilbert space $\mathrm{L}^2([a,b],m)$. We let $\{\theta_n\}_{n=1}^{\infty}$ be the normalized eigenvectors. For spectral classification of one-dimensional Sturm-Liouville problems, please refer to [28] and references therein. The Green's function for the ordinary differential operator $-\mathcal{D} - \lambda\cdot$, with zero boundary conditions at a and b can be represented as

$$g(x, y; \lambda) = m(y) \sum_{n=1}^{\infty} \frac{\theta_n(x)\theta_n(y)}{\lambda_n - \lambda}.$$

The convergence is uniform for $x, y \in [a, b]$, see [17] p. 375. We see that $G_s(x, y) = g(x, y; -s)$ is meromorphic in s (analytical except for poles; see [1]) with simple poles at $-\lambda_1, -\lambda_2, \cdots$. The residue at pole $s = -\lambda_n$ is $m(y)\theta_n(x)\theta_n(y)$. We can, then, invert the Laplace transform, and by applying the Cauchy Residue Theorem, we obtain

(4.5)

$$\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{s\tau}\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)]ds$$

$$= \sum_{n=1}^{\infty} e^{-\lambda_n \tau}\theta_n(x) \int_k^u (y/\beta)^{-1/p}((y/\beta)^{1/p} - K)\theta_n(y)m(y)dy, \tag{4.6}$$

in the case of a call option. It remains to determine $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$.

Recall the following fact from the general theory of ordinary differential equations: *If the coefficients of a linear ordinary differential equation depend analytically on a complex parameter λ, then the solution satisfying a fixed set of initial conditions also depends analytically on λ,*; see [17] p. 370. It follows that $\Delta_s(a, x)$ and $\Delta_s(x, b)$ are analytic in s. Therefore the poles

$$-\lambda_1, -\lambda_2, \cdots, -\lambda_n \cdots$$

are precisely the zeros of $w_s\Delta_s(a, b)$. These zero can be found numerically, as Linetsky did in [13] for pricing under a CEV process. The Green's function fails to exist at these poles. This failure happens precisely when $\Delta_s(a, x) = \Delta_{-\lambda_n}(a, x)$ and

$\Delta_s(x,b) = \Delta_{-\lambda_n}(x,b)$ are linearly dependent, i.e. multiple of each other. In that case, they are nonzero solutions of

$$\mathcal{D}U(x) + \lambda_n U(x) = 0$$

that satisfy both boundary conditions

$$U(a) = 0 \text{ and } U(b) = 0.$$

Hence, they are eigenfunctions of the Sturm-Liouville problem with eigenvalue λ_n, although not normalized. Let

$$R_{\lambda_n} = \frac{\Delta_{-\lambda_n}(a,x)}{\Delta_{-\lambda_n}(x,b)} \text{ and } C_{\lambda_n} = \frac{d}{ds}(w_s\Delta_s(a,b))\bigg|_{s=-\lambda_n}.$$

By using (4.4), the residue of $G_s(x,y)$ at $s = -\lambda_n$ can expressed as

$$\frac{-m(y)\Delta_{-\lambda_n}(a,x)\Delta_{-\lambda_n}(a,y)}{R_{\lambda_n}C_{\lambda_n}}.$$

On the other hand, as we have obtained earlier, the residue in term of θ_n is

$$m(y)\theta_n(x)\theta_n(y).$$

Therefore

$$(\theta_n(x))^2 = \frac{(\Delta_{-\lambda_n}(a,x))^2}{-R_{\lambda_n}C_{\lambda_n}}.$$

By substituting this into equation (4.5), we obtain the eigenfunction expansion for $\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)]$ for call option in explicit form.

Proposition IV.2. $\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)]$, *when* ψ *is a call payoff function, can be represented as*

$$\begin{aligned}
&\hat{\mathbb{E}}[1_{\{\gamma_{a,b}>\tau\}}\chi_\psi(X_\tau)] \\
&= \sum_{n=1}^{\infty} e^{-\lambda_n\tau}\frac{\Delta_{-\lambda_n}(a,x)}{-R_{\lambda_n}C_{\lambda_n}}\int_k^u (y/\beta)^{-1/p}((y/\beta)^{1/p} - K)\Delta_{-\lambda_n}(y)m(y)dy \\
&= \sum_{n=1}^{\infty} e^{-\lambda_n\tau}\frac{\Delta_{-\lambda_n}(a,x)}{-R_{\lambda_n}C_{\lambda_n}}\left(\phi_{-\lambda_n}(a)I_{-\lambda_n} - \psi_{-\lambda_n}(a)J_{-\lambda_n}\right).
\end{aligned}$$

Corollary IV.3. *The leading term P_0 for a double barrier option is given by*

$$P_0 = e^{-qT} S \sum_{n=1}^{\infty} e^{-\lambda_n \tau} \frac{\Delta_{-\lambda_n}(a,x)}{-R_{\lambda_n} C_{\lambda_n}} \Big(\phi_{-\lambda_n}(a) I_{-\lambda_n} - \psi_{-\lambda_n}(a) J_{-\lambda_n} \Big),$$

and the correction term P_1 is given by

$$\begin{aligned} P_1 =& e^{-qT} S^{p+1} \left(\frac{p\sigma^2}{4\bar{\alpha}^2} \right) \sum_{n=1}^{\infty} \Bigg[e^{-\lambda_n \tau} \frac{\phi_{-\lambda_n}(a) I_{-\lambda_n} - \psi_{-\lambda_n}(a) J_{-\lambda_n}}{R_{\lambda_n} C_{\lambda_n}} \left(\frac{1}{2x} \right)^{1/2+\mu(-\lambda_n)} x^{\frac{1-\eta}{2}} \\ & (\phi_{-\lambda_n}(a)\, U[\eta/2 + \mu(-\lambda_n), 1 + 2\mu(-\lambda_n), 1/2x] - \psi_{-\lambda_n}(a) {}_1F_1[\eta/2 + \mu(-\lambda_n), 1 + 2\mu(-\lambda_n), 1/2x]) \Bigg]. \end{aligned}$$

where U is the second confluent hypergeometric function (Tricomi function, see [33])

.

Proof. This directly follows from (4.2) and (4.1). $\square$

CHAPTER V

Multi-scale Time-Changed Birth Models for Multi-Name Credit Derivatives

5.1 Introduction

Copulas have been the standard approach in the financial industry for creating correlation structures and pricing multi-name credit derivatives. However, copula models have some well known drawbacks, most notably their static character. Copula models do not take into account the time evolution of joint default risks, therefore cannot be used to price more exotic, multi-period instruments, such as tranche forwards and tranche options. This has motivated recent work in developing alternative approaches to multi-name credit risk modeling. Several recent papers proposed a top-down approach, in which one models the portfolio loss process directly as a jump process, whose default intensity λ_t represents the conditional rate of occurrence of the next default. We are interested in the top-down framework proposed in [14], in which the portfolio loss process is modeled as a time-changed birth process. Under this general setting, [14] analyzed and implemented a particular parametric specification, where the time change activity rate is a CIR process. The advantage of this model over other top-down models is its tractability: tranche prices can be expressed in closed form. On the other hand, there are limitations of this model, as also pointed out in [14]. Between default events, the default intensity volatility and mean reversion level are constant. These parameters have to be dependent on the number of defaults N, and the dependence is simple: they increase as N increases. This undesired feature is a result of using the birth process, whose intensity is always increasing as a process to be time changed. The model turns out to be a special specification of the class of affine point process models introduced in [16], but with gained tractability at the cost of reduced flexibility.

In this chapter, we propose a remedy to those issues by introducing stochastic parameter fluctuations driven by the combination of a fast evolving factor and a slow evolving factor. The motivation is to go beyond the affine family of models and bring in more flexibility. By properly specifying the fast and slow parameterization of the fast and slow processes, we are able to keep the tractability and improve the fit to the market data. Multi-scale stochastic modeling for multi-name credit derivatives is also discussed in [21]. They consider stochastic parameter extension to a bottom up model where individual default intensities are given by correlated Ornstein-Uhlenbeck processes. One problem is that intensities are Guassian and may become negative. Also, although explicit approximations were computed, calibration to the real data with the bottom-up approach is difficult and was not discussed in [21].

The chapter is organized as follows: We describe the top-down approach introduced in [14] in Section 5.2. In Section 5.3, we study a stochastic volatility extension. In Section 5.4, we discuss an extension, in which the stochastic mean reversion level is stochastic. In Section 5.5, we implement the calibration of the models to the market data. Figures and Tables showing the calibration results are placed at the end of the chapter.

5.2 Modeling

5.2.1 Time-Changed Birth Process

We consider the model proposed in [14], in which correlated default arrivals are modeled directly under a risk-neutral pricing measure through a time-changed birth process N. More precisely, suppose that N^0 is a birth process with intensity

$$\lambda_t^0 = \theta_1 + \theta_2 N_t^0.$$

The time-changed birth process is defined by

$$N_t = N_{T_t}^0,$$

where

$$T_t = \int_0^t X_s ds.$$

We assume that the activity rate X is independent of the birth process N^0, and follows the dynamics:

$$dX_t = \kappa(\mu - X_t)dt + \sigma\sqrt{X_t}dW_t^0.$$

By (4) in [14], the intensity λ for process N is given by

$$\lambda_t = X_t(\theta_1 + \theta_2 N_t).$$

Thanks to the above specification of X, N is a counting process whose intensity satisfies

$$d\lambda_t = \kappa(\mu\eta_t - \lambda_t)dt + \sigma\sqrt{\eta_t\lambda_t}dW_t^0 + \frac{\theta_2}{\eta_t}\lambda_t dN_t,$$

where $\eta_t = \theta_1 + \theta_2 N_t$. Now, for a portfolio of credit securities that are issued by n names, we define $N^n = N \wedge n$ to be the default process that counts the number of defaults in the portfolio. Let

$$\lambda_t^n = \lambda_t 1_{\{N_t < n\}},$$

λ^n be the intensity of N^n.

5.2.2 Tranche and Index Swaps

A credit derivative index is a portfolio of defaultable assets. Investors may invest in contracts based on a tranche of the index specified by a lower attachment point $\underline{K} \in [0,1]$ and an upper attachment point $\overline{K} \in [\underline{K}, 1]$. The product of $K = \overline{K} - \underline{K}$ and the index notional(face value) I is the tranche notional. Usually, the firms in the index with n names has the same notional, and we normalize the index notional I to \$1and notional for each firm to $\$1/n$. The loss rate l at default is often assumed to be constant and the cumulative loss process L^n is defined in terms of N^n by

$$L_t^n = \frac{lN_t^n}{n}.$$

The cumulative tranche loss at time t is defined by

$$U_t = (L_t - \underline{K})^+ - (L_t - \overline{K})^+.$$

A tranche swap is an insurance against losses between $\underline{K}$ and $\overline{K}$. An index swap is a tranche swap for which $\underline{K} = 0$ and $\overline{K} = 1$. The protection buyer in the swap pays a fraction of the tranche notional $F \cdot K$ as an upfront fee and a premium to the protection seller on future dates $(t_1, \cdots, t_i, \cdots, t_M)$, with t_M being the maturity of the contract. The amount of the premium paid at each payment date t_i is a fixed fraction *spr* (usually quoted as an annual rate) of $K - U_{t_i}$, the difference between the tranche notional and the cumulative tranche loss to t_i. This fraction is called the *tranche spread.* The protection seller, on the other hand, compensates the protection buyer for the default losses that occur before the maturity of the contract. We assume that the compensation for a loss is paid at the very next premium payment date t_i. Therefore, the present value of the premium leg is

$$\text{premium}_{t,T} = spr \sum_{i=1}^{M} c_i e^{-r(t_i - t)}(K - \mathbb{E}_t[U_{t_i}]),$$

where c_i is the day count fraction for period i. Usually, payments are made quarterly, and $c_i = 0.25$. On the other hand, the present value of the protection leg is

$$\text{protection}_{t,T} = \sum_{i=1}^{M} e^{-r(t_i-t)}(\mathbb{E}_t[U_{t_i}] - \mathbb{E}_t[U_{t_{i-1}}]) - F \cdot K.$$

The tranche swap spread spr is determined, as for CDS spread, by equating the premium leg and the protection leg. We obtain

$$spr = \frac{\sum_{i=1}^{M} e^{-r(t_i-t)}(\mathbb{E}_t[U_{t_i}] - \mathbb{E}_t[U_{t_{i-1}}]) - F \cdot K}{\sum_{i=1}^{M} c_i e^{-r(t_i-t)}(K - \mathbb{E}_t[U_{t_i}])}. \tag{5.1}$$

For the equity tranches (tranches with the lowest attachment points), the market convention is to charge an upfront payment from the protection buyer, while fixing the spread at certain level s^*, say 500bsp. In this case,

$$\begin{aligned} F = \frac{1}{K}\Bigg[&\sum_{i=1}^{M} e^{-r(t_i-t)}(\mathbb{E}_t[U_{t_i}] - \mathbb{E}_t[U_{t_{i-1}}]) \\ &-s^* \sum_{i=1}^{M} c_m e^{-r(t_i-t)}(K - \mathbb{E}_t[U_{t_i}])\Bigg]. \end{aligned} \tag{5.2}$$

is quoted. We will show that spr in (5.1) and F in (5.2) can be calculated explicitly. For this purpose, we compute $\mathbb{E}_t[U_T]$, which can be expressed using the distribution of N_T as follows

$$\mathbb{E}_t[U_T] = \sum_{k=0}^{n-N_t^n} U_T\left(\frac{lk}{n}\right)\mathbb{P}[N_T^n - N_t^n = k|\mathcal{G}_t],$$

in which

$$\mathbb{P}[N_T^n - N_t^n = k|\mathcal{G}_t] = \begin{cases} \mathbb{P}[N_T - N_t = k|\mathcal{G}_t] & \text{if } k < n - N_t^n \\ \mathbb{P}[N_T - N_t \geq k|\mathcal{G}_t] & \text{if } k = n - N_t^n \\ 0 & \text{if } k < n - N_t^n. \end{cases}$$

The probability distribution of $N_T - N_t$ is given by

$$\mathbb{P}[N_T - N_t = k|\mathcal{G}_t] = \frac{\Gamma(C_t + k)}{\Gamma(C_t)k!} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mathcal{T}_{t,T}(\theta_1(m + C_t), 0), \tag{5.3}$$

where $C_t = N_t + \frac{\theta 1}{\theta 2}$ and

$$\mathcal{T}_{t,T}(s, \xi) = \mathbb{E}_t \left[e^{-s \int_t^T X_u du + i\xi X_T} |\mathcal{G}_t \right]; \tag{5.4}$$

see [14].

5.2.3 Tranche and Index Options

The mark-to-market value at t of a swap is

$$M_{t,T}(spr, F) = \text{protection}_{t,T} - \text{premium}_{t,T}(spr, F)$$

Tranche and Index options are derivatives on the mark-to-market value, which allows investors to bet on the future spreads. The value of a option on a swap with payoff function h, maturity T, exercise date $T^* < T$ and strike spread S and upfront rate F at time t is

$$\exp(-r(T^* - t))\mathbb{E}[h(M_{T^*,T}(S, F))|\mathcal{G}_t]$$

From the previous section, $M_{T^*,T}(S, F)$ is function of $N_{T^*}^n$ and X_{T^*}. We write

$$M_{T^*,T}(S, F) = \widetilde{M}_{T^*,T}(N_{T^*}^n, X_{T^*}; S, F)$$

The expectation can be computed using joint of distribution of $N_{T^*}^n - N_t^n$ and X_{T^*}:

$$\begin{aligned} \mathbb{E}[h(M_{T^*,T}(S, F))|\mathcal{G}_t] &= \mathbb{E}[h(\widetilde{M}_{T^*,T}(N_{T^*}^n, X_{T^*}; S, F)|\mathcal{G}_t] \\ &= \int_0^\infty \sum_{k=0}^{n-N_t^n} h(\widetilde{M}_{T^*,T}(k + N_t^n, x; S, F)\mathbb{P}[N_{T^*}^n - N_t^n = k, X_{T^*} \in dx|\mathcal{G}_t], \end{aligned}$$

in which

$$\mathbb{P}[N_{T^*}^n - N_t^n = k, X_{T^*} \in dx|\mathcal{G}_t] = \begin{cases} \mathbb{P}[N_{T^*} - N_t = k, X_{T^*} \in dx|\mathcal{G}_t] & \text{if } k < n - N_t^n \\ \mathbb{P}[N_{T^*} - N_t \geq k, X_{T^*} \in dx\mathcal{G}_t] & \text{if } k = n - N_t^n \\ 0 & \text{if } k < n - N_t^n. \end{cases}$$

where the joint distribution of $N_{T^*} - N_t$ and X_{T^*} is given by

$$\mathbb{P}[N_{T^*} - N_t = k, X_{T^*} \in dx | \mathcal{G}_t] \tag{5.5}$$

$$= \frac{\Gamma(C_t + k)}{\Gamma(C_t) k!} \sum_{m=0}^{k} \frac{(-1)^m}{2\pi} \binom{k}{m} \int_{-\infty}^{\infty} \mathcal{T}_{t,T^*}(\theta_1(m + C_t), \xi) e^{-ix\xi} d\xi dx; \tag{5.6}$$

see [14].

5.3 Stochastic Volatility

In this section, we extend the top-down model described in the previous section to incorporate multi-scale stochastic volatility. The volatility of the time-change activity rate depends on a fast evolving factor Y and a slowly evolving factor Z:

$$\sigma_t = f(Y_t, Z_t),$$

where the function $f(y, z)$ is a strictly positive, bounded, smooth function. Specifically, the activity rate X is modeled as the solution of

$$dX_t = \kappa(\mu - X_t)dt + f(Y_t, Z_t)\sqrt{X_t} dW_t^0.$$

The processes Y and Z are modeled by

$$dY_t = \frac{1}{\epsilon} X_t (m - Y_t) dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \sqrt{X_t} dW_t^1,$$

$$dZ_t = \delta X_t c(Z_t) dt + \sqrt{\delta}\sqrt{X_t} g(Z_t) dW_2^2,$$

$$\mathbb{E}_t[dW_t^0 dW_t^i] = \rho_i dt, \quad i \in \{1, 2\}, \quad \mathbb{E}_t[dW_t^1 dW_t^2] = \rho_{1,2} dt,$$

in which ϵ, δ are small positive constants, and the functions $c(z)$ and $g(z)$ are assumed to be smooth. We assume that $f^2(y, z) \leq 2\kappa\mu$ in order to guarantees the process X never hits zero; see [14]. The intensity λ follows the dynamics

$$d\lambda_t = \kappa(\mu\eta_t - \lambda_t)dt + f(Y_t, Z_t)\sqrt{\eta_t \lambda_t} dW_t^0 + \frac{\theta_2}{\eta_t} \lambda_t dN_t,$$

where $\eta_t = \theta_1 + \theta_2 N_t$. Note that the intensity process inherits the stochastic volatility from the activity rate process X. We see from Section 5.2.2 that in order to obtain prices for multi-name credit derivatives, we need to compute

$$\mathcal{T}_{t,T}(s,\xi) = \mathbb{E}_t\left[e^{-s\int_t^T X_u du + i\xi X_T} \middle| \mathcal{G}_t\right] = u^{\epsilon,\delta}(t,x,y,z;s,\xi).$$

The function $u^{\epsilon,\delta}$ satisfies the PDE

$$\begin{cases} \mathcal{L}^{\epsilon,\delta} u^{\epsilon,\delta} = 0 \text{ in } t < T, \\ u(T,x,y,z;s,\xi) = e^{i\xi x}, \end{cases}$$

where

$$\mathcal{L}^{\epsilon,\delta} := \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3,$$

in which

$$\begin{aligned}
\mathcal{L}_0 &:= x\left(\nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}\right) = x\tilde{\mathcal{L}}_0, \\
\mathcal{L}_1 &:= \sqrt{2}\nu\rho_1 f(y,z) x \frac{\partial^2}{\partial x \partial y}, \\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2} f^2(y,z) x \frac{\partial^2}{\partial x^2} + \kappa(\mu - x)\frac{\partial}{\partial x} - sx\cdot, \\
\mathcal{M}_1 &:= \rho_2 g(z) f(y,z) x \frac{\partial^2}{\partial x \partial z}, \\
\mathcal{M}_2 &:= x\left(c(z)\frac{\partial}{\partial z} + \frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2}\right), \\
\mathcal{M}_3 &:= x\nu\sqrt{2}\rho_{1,2} g(z) \frac{\partial^2}{\partial y \partial z}.
\end{aligned}$$

We approximate $u^{\epsilon,\delta}$ by

$$\tilde{u}^{\epsilon,\delta} = u_0 + \sqrt{\epsilon} u_{1,0} + \sqrt{\delta} u_{0,1}.$$

By matching the powers of ϵ and δ, we obtain the PDEs satisfied by u_0, $u_{1,0}$ and $u_{0,1}$. We will solve these PDEs explicitly. We denote $\bar{\sigma}^2(z) = \langle f^2(y,z)\rangle$ and let $\phi(y,z)$ be the solution of

$$\tilde{\mathcal{L}}_0 \phi(y,z) = f^2(y,z) - \bar{\sigma}^2(z).$$

We find that u_0 is independent of y, and it solves

$$\begin{cases} \langle \mathcal{L}_2 \rangle u_0 = 0 \\ u_0(T, x, z; s, \xi) = e^{i\xi x}. \end{cases}$$

We see that u_0 is the transform $\mathcal{T}_{t,T}(s, \xi)$ defined in (5.4) for X_t with fixed volatility at $\bar{\sigma}(z)$ and, therefore, is given by

$$u_0 = e^{\alpha(T-t)+\beta(T-t)x}, \tag{5.7}$$

where functions α and β are defined as

$$\begin{aligned} \alpha(t) &= \frac{\kappa\mu(ac-d)}{bcd} \log \frac{c + de^{bt}}{c+d} + \frac{\kappa\mu}{c} t, \\ \beta(t) &= \frac{1 + ae^{bt}}{c + de^{bt}}, \end{aligned}$$

in which

$$\begin{aligned} c &= \frac{\kappa + \sqrt{\kappa^2 + 2\bar{\sigma}^2 s}}{-2s}, \\ d &= (1 - ic\xi) \frac{-\kappa + i\bar{\sigma}^2 \xi + \sqrt{\kappa^2 + 2\bar{\sigma}^2 s}}{-i2\kappa\xi - \bar{\sigma}^2\xi^2 - 2s}, \\ a &= i(c+d)\xi - 1, \\ b &= \frac{d(-\kappa - 2cs) + a(-\kappa c + \bar{\sigma}^2)}{ac - d}. \end{aligned}$$

The correction term $u_{1,0}$ is independent of y, and it solves

$$\begin{cases} \langle \mathcal{L}_2 \rangle u_{1,0} = \left\langle \mathcal{L}_1 \tilde{\mathcal{L}}_0^{-1} \frac{1}{x} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \right\rangle u_0 = \frac{1}{\sqrt{2}} \rho_1 \nu \langle f \phi_y \rangle (z) x \frac{\partial^3 u_0}{\partial x^3}, \\ u_{1,0}(T, x, z; s, \xi) = 0. \end{cases}$$

Observe that $\frac{\partial^3 u_0}{\partial x^3} = \beta^3(T-t) u_0$. Letting $V_1^\epsilon = \sqrt{\epsilon} \frac{1}{\sqrt{2}} \rho_1 \nu \langle f \phi_y \rangle (z)$, we have

$$\begin{cases} \langle \mathcal{L}_2 \rangle u_{1,0} = V_1^\epsilon / \sqrt{\epsilon} \beta^3(T-t) u_0, \\ u_{1,0}(T, x, z; s, \xi) = 0. \end{cases}$$

It can be checked that

$$u_{1,0} = \frac{V_1^\epsilon}{\sqrt{\epsilon}}(D_1(T-t)x + D_2(T-t))u_0.$$

in which $D_1(t)$ and $D_2(t)$ are solutions of

$$D_1'(t) + (-\bar{\sigma}^2\beta(t) + \kappa)D_1(t) + \beta^3(t) = 0, \quad D_1(0) = 0,$$

$$D_2'(t) - \kappa\mu D_1(t) = 0, \quad D_2(0) = 0.$$

The correction term $u_{0,1}$ is independent of y, and it solves

$$\begin{cases} \langle\mathcal{L}_2\rangle u_{0,1} = -\langle\mathcal{M}_1\rangle u_0 = -\rho_2 g(z)\langle f\rangle(z)x\frac{\partial^2 u_0}{\partial x\partial z}, \\ u_{0,1}(T,x,z;s,\xi) = 0. \end{cases}$$

Observe that $\frac{\partial u_0}{\partial z}$ solves

$$\begin{cases} \langle\mathcal{L}_2\rangle\frac{\partial u_0}{\partial z} = -\bar{\sigma}(z)\bar{\sigma}'(z)x\frac{\partial^2 u_0}{\partial x^2} = -\bar{\sigma}(z)\bar{\sigma}'(z)\beta^2(T-t)xu_0, \\ \frac{\partial u_0}{\partial z}(T,x,z;s,\xi) = 0. \end{cases}$$

As a result

$$\frac{\partial u_0}{\partial z} = -\bar{\sigma}(z)\bar{\sigma}'(z)(D_3(T-t)x + D_4(T-t))u_0,$$

where $D_3(t)$ and $D_4(t)$ solve

$$D_3'(t) + (-\bar{\sigma}^2\beta(t) + \kappa)D_3(t) + \beta^2(t) = 0, \qquad D_3(0) = 0,$$

$$D_4'(t) - \kappa\mu D_3(t) = 0, \qquad D_4(0) = 0.$$

Letting $V_2^\delta = \sqrt{\delta}\rho_2 g(z)\langle f\rangle(z)\bar{\sigma}(z)\bar{\sigma}'(z)$, we can write the PDE for $u_{0,1}$ as

$$\begin{cases} \langle\mathcal{L}_2\rangle u_{0,1} = V_2^\delta/\sqrt{\delta}\big((D_3(T-t) + \beta(T-t)D_4(T-t))xu_0 + D_3(T-t)\beta(T-t)x^2u_0\big), \\ u_{0,1}(T,x,z;s,\xi) = 0. \end{cases}$$

We seek a solution of form

$$u_{0,1} = \frac{V_2^\delta}{\sqrt{\delta}}(D_5(T-t)x^2 + D_6(T-t)x + D_7(T-t))u_0,$$

and we find $D_5(t)$, $D_6(t)$, and $D_7(t)$ solve the ODEs

$$\begin{aligned}
&D_5'(t) + 2(-\bar{\sigma}^2\beta(t) + \kappa)D_5(t) + D_3(t)\beta(t) = 0, && D_5(0) = 0,\\
&D_6'(t) + (-\bar{\sigma}^2\beta(t) + \kappa)D_6(t) - (\bar{\sigma}^2 + 2\kappa\mu)D_5(t) + (D_3(t) + \beta(t)D_4(t)) = 0, && D_6(0) = 0,\\
&D_7'(t) - \kappa\mu D_6(t) = 0, && D_7(0) = 0.
\end{aligned}$$

Notice that since the terminal condition is smooth, so the arguments in [20] can be adapted to show that for fixed (t, x, y, z), there exists a constant C such that

$$|u^{\epsilon,\delta} - \tilde{u}^{\epsilon,\delta}| < C \cdot (\epsilon + \delta).$$

Now, let $\tilde{u}^{(m)} = u_0^{(m)} + \sqrt{\epsilon}u_{1,0}^{(m)} + \sqrt{\delta}u_{0,1}^{(m)}$ be the approximation for $\mathcal{T}_{t,T}(\theta_1(m + C_t), \xi)$. We have

$$\begin{aligned}
u_0^{(m)} &= e^{\alpha(T-t;m)+\beta(T-t;m)},\\
\sqrt{\epsilon}u_{1,0}^{(m)} &= V_1^\epsilon(D_1^{(m)}(T-t)x + D_2^{(m)}(T-t))u_0^{(m)},\\
\sqrt{\delta}u_{0,1}^{(m)} &= V_2^\delta(D_3^{(m)}(T-t)x^2 + D_5^{(m)}(T-t) + D_7^{(m)}(T-t))u_0^{(m)}.
\end{aligned}$$

where we have let $X_t = x$. We see that the approximation for the loss distribution density and joint distribution density is linear in V_1^ϵ and V_2^δ:

$$\begin{aligned}
&\mathbb{P}(N_T - N_t = k|\mathcal{G}_t) \approx \frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}(-1)^{(m)}\binom{k}{m}u_0^{(m)}\\
&+ V_1^\epsilon\frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}(-1)^{(m)}\binom{k}{m}(D_1^{(m)}(T-t)x + D_2^{(m)}(T-t))u_0^{(m)}\\
&+ V_2^\delta\frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}(-1)^{(m)}\binom{k}{m}(D_3^{(m)}(T-t)x^2 + D_5^{(m)}(T-t) + D_7^{(m)}(T-t))u_0^{(m)}\bigg|_{\xi=0};
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}[N_T - N_t = k, X_T \in dx|\mathcal{G}_t]\\
&\approx \frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}\frac{(-1)^m}{2\pi}\binom{k}{m}\int_{-\infty}^{\infty}u_0^{(m)}e^{-ix\xi}d\xi dx\\
&+ V_1^\epsilon\frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}\frac{(-1)^m}{2\pi}\binom{k}{m}\int_{-\infty}^{\infty}(D_1^{(m)}(T-t)x + D_2^{(m)}(T-t))u_0^{(m)}e^{-ix\xi}d\xi dx\\
&+ V_2^\delta\frac{\Gamma(C_t+k)}{\Gamma(C_t)k!}\sum_{m=0}^{k}\frac{(-1)^m}{2\pi}\binom{k}{m}\int_{-\infty}^{\infty}(D_3^{(m)}(T-t)x^2 + D_5^{(m)}(T-t) + D_7^{(m)}(T-t))u_0^{(m)}e^{-ix\xi}d\xi dx
\end{aligned}$$

5.4 Stochastic Mean Reversion Level

As pointed out earlier, the dependence of the mean reversion level on N through $\mu\eta_t = \mu(\theta_1 + \theta_2 N_t)$ may reduce the flexibility of the model. This is a side-effect of using the birth process, whose intensity is increasing. Indeed, the disadvantage is clear if we consider the case when the volatility $\sigma = 0$. In that case the activity rate process X follows the dynamics

$$dX_t = \kappa(\mu - X_t)dt.$$

The intensity of N_t under this assumption follows

$$d\lambda_t = \kappa(\mu\eta_t - \lambda_t)dt + \frac{\theta_2}{\eta_t}\lambda_t dN_t, \tag{5.8}$$

where $\eta_t = \theta_1 + \theta_2 N_t$. This setup can be compared to the Hawkes model proposed in [16] and implemented in [23] and [4]. In the Hawkes model, the intensity follows the dynamics

$$d\lambda_t = \kappa(\mu - \lambda_t) + \theta dL_t.$$

Recall that L_t denotes the loss process and is related to N_t by a constant factor when the loss at default rate is assumed to be constant. While the implementation of the Hawkes process model requires numerical methods, the model specified by (5.8) can be solved analytically. On the other hand, while the Hawkes process model fits to the market data well, see [23], the model described above fits the market data poorly.

We are looking to counteract the effect of the increasing intensity of the birth process by allowing the mean reversion level of the activity rate μ to be stochastic. More specifically, μ now depends on a fast evolving factor Y and a slow evolving factor Z:

$$\mu_t = \mu(Y_t, Z_t),$$

where the function $\mu(y,z)$ is strictly positive and bounded. Now the activity rate X follows the dynamics

$$dX_t = \kappa(\mu(Y_t, Z_t) - X_t)dt + \sigma\sqrt{X_t}dW_t^0.$$

The fast process is modeled by

$$dY_t = \frac{1}{\epsilon}\left((m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\tilde{\Lambda}(Y_t, Z_t)\right)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^1,$$

and the slow process is modeled by

$$dZ_t = \left(\delta c(Z_t) - \sqrt{\delta}g(Z_t)\tilde{\Gamma}(Y_t, Z_t)\right)dt + \sqrt{\delta}g(Z_t)dW_2^2,$$

where $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are the market prices of the risk of fluctuations in intensity level, which resulted from the measure change from the physical measure to risk-neutral measure. As usual, ϵ and δ are small positive constants, and the functions $c(z)$ and $g(z)$ are assumed to be smooth. Also the Brownian motions are correlated:

$$\mathbb{E}_t[dW_t^0 dW_t^i] = \rho_i dt, \quad i \in \{1, 2\}, \quad \mathbb{E}_t[dW_t^1 dW_t^2] = \rho_{1,2}dt,$$

We need to compute

$$\mathcal{T}_{t,T}(s, \xi) = \mathbb{E}_t\left[e^{-s\int_t^T X_u du + i\xi X_T}\middle|\mathcal{G}_t\right] = u^{\epsilon,\delta}(t, x, y, z; s, \xi).$$

$u^{\epsilon,\delta}$ satisfies the PDE

$$\begin{cases} \mathcal{L}^{\epsilon,\delta}u^{\epsilon,\delta} = 0 \text{ in } t < T, \\ \\ u(T, x, y, z; s, \xi) = e^{i\xi x}, \end{cases}$$

where

$$\mathcal{L}^{\epsilon,\delta} := \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3,$$

and

$$\begin{aligned}
\mathcal{L}_0 &:= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \\
\mathcal{L}_1 &:= -\nu\sqrt{2}\tilde{\Lambda}(y,z)\frac{\partial}{\partial y} + \sqrt{2}\nu\rho_1\sigma\sqrt{x}\frac{\partial^2}{\partial x \partial y}, \\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x \frac{\partial^2}{\partial x^2} + \kappa(\mu(y,z)-x)\frac{\partial}{\partial x} - sx\cdot, \\
\mathcal{M}_1 &:= -g(z)\tilde{\Gamma}(y,z)\frac{\partial}{\partial z} + \rho_2 g(z)\sigma\sqrt{x}\frac{\partial^2}{\partial x \partial z}, \\
\mathcal{M}_2 &:= c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2}, \\
\mathcal{M}_3 &:= \nu\sqrt{2}\rho_{1,2}g(z)\frac{\partial^2}{\partial y \partial z}.
\end{aligned}$$

We formally expand $u^{\epsilon,\delta}$ in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$:

$$u^{\epsilon,\delta} = u_0 + \sqrt{\epsilon}u_{1,0} + \sqrt{\delta}u_{0,1} + \cdots,$$

and further expand $u_{1,0}$ and $u_{0,1}$ in powers of σ:

$$u_{1,0} = u_{1,0,0} + \sigma u_{1,0,1} + \cdots,$$

$$u_{1,0} = u_{0,1,0} + \sigma u_{0,1,1} + \cdots.$$

We approximate $u^{\epsilon,\delta}$ by

$$\tilde{u}^{\epsilon,\delta} = u_0 + \sqrt{\epsilon}u_{1,0,0} + \sqrt{\delta}u_{0,1,0}$$

As for the stochastic volatility case, since the terminal condition is smooth, the arguments in [20] can be adapted to show that for fixed (t,x,y,z), there exists a constant C such that

$$|u^{\epsilon,\delta} - \tilde{u}^{\epsilon,\delta}| < C \cdot (\sigma\sqrt{\epsilon} + \sigma\sqrt{\delta} + \epsilon + \delta).$$

We let $\bar{\mu}(z) = \langle \mu(y,z) \rangle$ and $\phi(y,z)$ be the solution of

$$\mathcal{L}_0\phi(y,z) = \mu(y,z) - \bar{\mu}(z).$$

We find that u_0 is independent of y, and it solves

$$\begin{cases} \langle\tilde{\mathcal{L}}_2\rangle u_0 = 0, \\ u_0(0,x,z;s,\xi) = e^{i\xi x}. \end{cases}$$

where

$$\langle\widetilde{\mathcal{L}}_2\rangle = -\frac{\partial}{\partial\tau} + \frac{1}{2}\sigma^2 x \frac{\partial^2}{\partial x^2} + \kappa(\bar{\mu}(z) - x)\frac{\partial}{\partial x} - sx\cdot$$

and we have let $\tau = T - t$. Therefore u_0 is given by (5.7). To compute the correction terms, we define $u_{0,0}$ to be the solution of

$$\begin{cases} \langle\widehat{\mathcal{L}}_2\rangle u_{0,0} = 0, \\ u_{0,0}(0,x,z;s,\xi) = e^{i\xi x}. \end{cases}$$

where

$$\langle\widehat{\mathcal{L}}_2\rangle = -\frac{\partial}{\partial\tau} + \kappa(\bar{\mu}(z) - x)\frac{\partial}{\partial x} - sx\cdot$$

This quasi-linear PDE in x and τ can be readily solved by method of characteristics; see [25]: The initial curve is parametrized by

$$\tau = 0, \quad x = a, \quad \omega = e^{i\xi a}.$$

The characteristic differential equations are

$$\frac{d\tau}{db} = -1, \quad \frac{dx}{db} = \kappa(\bar{\mu}(z) - x), \quad \frac{d\omega}{db} = sx\omega$$

This leads the parametric representation of the solution

$$\begin{aligned}
&\tau = -b, \\
&x = \bar{\mu}(z) + (a - \bar{\mu}(z))e^{-\kappa b}, \\
&\omega = \exp\left(s\bar{\mu}(z)b + \frac{s(a - \bar{\mu}(z))}{\kappa}(1 - e^{-\kappa b})\right)e^{i\xi a}.
\end{aligned}$$

We solve for a,b in terms of x and τ and substitute into ω. We obtain

$$u_{0,0} = \omega(\tau, x) = \exp\left(-s\bar{\mu}(z)\tau + \frac{s}{\kappa}(x - \bar{\mu}(z))(e^{-\kappa\tau} - 1) + i\xi(xe^{-\kappa\tau} + \bar{\mu}(z)(1 - e^{-\kappa\tau})) \right).$$

By a formal expansion, the correction term $u_{1,0,0}$ is independent of y, and it solves

$$\begin{cases} \langle \widehat{\mathcal{L}}_2 \rangle u_{1,0,0} = \left\langle \widehat{\mathcal{L}}_1 \mathcal{L}_0^{-1}(\widehat{\mathcal{L}}_2 - \langle \widehat{\mathcal{L}}_2 \rangle) \right\rangle u_{0,0} = \frac{V_1^\epsilon}{\sqrt{\epsilon}} \frac{\partial u_{0,0}}{\partial x}, \\ u_{1,0,0}(0, x, z; s, \xi) = 0, \end{cases}$$

where $\widehat{\mathcal{L}}_1 := -\nu\sqrt{2}\tilde{\Lambda}(y, z)\frac{\partial}{\partial y}$, and $V_1^\epsilon = -\sqrt{\epsilon}\kappa\nu\sqrt{2}\langle \tilde{\Lambda}\phi_y \rangle(z)$. Note that

$$\frac{\partial u_{0,0}}{\partial x} = \left(\frac{s}{\kappa}(e^{-\kappa\tau} - 1) + i\xi e^{-\kappa\tau} \right) u_{0,0}.$$

The correction term $u_{0,1,0}$ is independent of y and it solves

$$\begin{cases} \langle \tilde{\mathcal{L}}_2 \rangle u_{0,1,0} = -\langle \widehat{\mathcal{M}}_1 \rangle u_{0,0} = g(z)\langle \tilde{\Gamma} \rangle(z) \frac{\partial u_{0,0}}{\partial z}, \\ u_{0,1,0}(0, x, z; s, \xi) = 0, \end{cases}$$

where $\widehat{\mathcal{M}}_1 := -g(z)\tilde{\Gamma}(y, z)\frac{\partial}{\partial z}$. Observe that

$$\frac{\partial u_{0,0}}{\partial z} = \bar{\mu}'(z)\left(-s\tau + \left(\frac{s}{\kappa} + i\xi \right)(1 - e^{-\kappa\tau}) \right) u_{0,0}.$$

By letting $V_2^\delta = \sqrt{\delta}\bar{\mu}'(z)g(z)\langle \tilde{\Gamma} \rangle(z)$, we can express the initial value problem for $u_{0,1,0}$ as

$$\begin{cases} \langle \tilde{\mathcal{L}}_2 \rangle u_{0,1,0} = \frac{V_2^\delta}{\sqrt{\delta}} \left(-s\tau + \left(\frac{s}{\kappa} + i\xi \right)(1 - e^{-\kappa\tau}) \right) u_{0,0}, \\ u_{0,1,0}(0, x, z; s, \xi) = 0. \end{cases}$$

We recognize that the initial value problems for $u_{1,0,0}$ and $u_{0,1,0}$ are again quasi-linear first-order equations, which can be readily solved by the method of characteristics. We obtain

$$u_{1,0,0} = \frac{V_1^\epsilon}{\sqrt{\epsilon}} D_1(\tau) u_{0,0},$$
$$u_{0,1,0} = \frac{V_2^\delta}{\sqrt{\delta}} D_2(\tau) u_{0,0},$$

where

$$D_1(\tau) = \frac{s}{\kappa^2}(e^{-\kappa\tau} + \tau - 1) + \frac{i\xi}{\kappa}(e^{-\kappa\tau} - 1),$$
$$D_2(\tau) = \frac{s\tau^2}{2} + \left(\frac{s}{\kappa} + i\xi\right)\left(\frac{1}{\kappa}(1 - e^{-\kappa\tau}) - \tau\right).$$

These corrections enter the loss distribution in a similar fashion as the stochastic volatility corrections.

5.5 Calibration to Market Tranche Prices

Model calibration involves determining of the model parameters

$$\Theta = (X_0, \mu, \kappa, \bar{\sigma}, \theta_1, \theta_2, V_1^{\epsilon}, V_2^{\delta})$$

for stochastic volatility and

$$\Theta = (X_0, \bar{\mu}, \kappa, \sigma, \theta_1, \theta_2, V_1^{\epsilon}, V_2^{\delta})$$

for stochastic mean reversion, that yields model prices that best match the market data. We use the CDX.NA.HY.10 (CDX High Yield index portofolio of $n = 100$ North American constituents) index tranche price data obtained from Bloomberg on June 16, 2008. We take the risk-free rate $r = 0.02$ and the loss at default rate $l = 0.6$. The model is fitted to market tranche quotes across maturities of 5yr and 7yr. The goodness-of-fit is measured by the root-mean-squared error (RMSE) defined as:

$$\sqrt{\frac{1}{\#}\sum_k \left(\frac{\text{MarketMid}(k) - \text{Model}(\Theta)}{\text{MarketBid}(k) - \text{MarketAsk}(k)}\right)^2}.$$

where # is the number of data points. We choose the parameters to minimize the RMSE by solving a constrained nonlinear least square problem:

$$\underset{\Theta}{\text{argmin}} \sum_k \left(\frac{\text{MarketMid}(k) - \text{Model}(\Theta)}{\text{MarketBid}(k) - \text{MarketAsk}(k)}\right)^2$$

subject to $2\kappa\mu \geq \bar{\sigma}^2$ for stochastic volatility and $2\kappa\mu \geq \sigma^2$ for stochastic mean reversion.

Directly implementation of (5.3) and (5.5) using double precision would result in great cancellation errors. Here, we implement high-precision arithmetics in our C++ program. We used the APPREC package, available from: http://crd.lbl.gov/ dhbailey/mpdist/. Implementation of (5.3) also involves discretizing the integral and applying a fast Fourier transform(FFT). For this purpose, we used the fftw package available from: http://www.fftw.org.

We compare our calibration results to the model without stochastic parameter corrections. Our calibration exercise shows that the introduction of the correction terms improves the fit to market data. Our models can also be compared to a bottom-up stochastic volatility intensity model with seven parameters proposed in [21]. Our calibration shows that it is not able to fit the market data well. The model of [21] assumes that the dynamics and the starting points of the intensities are the same for all names in the portfolio. Specifically,

$$d\lambda_t^{(i)} = \kappa(\theta - \lambda_t^{(i)})dt + \sigma(Y_t, Z_t)dW_t^{(i)}, \quad \lambda_t^{(i)} = \lambda;$$

$$\mathbb{E}[dW_t^{(i)} dW_t^{(j)}] = \rho dt, \quad i \neq j$$

The fast process Y is modeled by

$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^y,$$

and the slow process Z is modeled by

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_2^z,$$

$$\mathbb{E}[dW_t^{(i)} dW_t^{(y)}] = \rho_y dt, \quad \mathbb{E}[dW_t^{(i)} dW_t^{(z)}] = \rho_z dt, \quad \mathbb{E}[dW_t^{(y)} dW_t^{(z)}] = \rho_y dt.$$

The leading term and correction terms in the approximation for the loss distribution are given explicitly by (13), (32) and (33) in [21], respectively.

We summarize the data and results of the calibration in Table 5.1. We also show the loss distribution implied by the Models in Sections 2 to 4 in Figure 5.1.

Maturity	Contract	MarketBid	MarketAsk	Model1	Model2	Model3	Model4
	0-10%	88.05%	88.55%	89.92%	87.94%	88.51%	90.65%
	10-15%	66.089%	66.589%	64.24%	66.60%	66.18%	74.62%
5Yr	15-25%	1050.29	1059.709	1018.5	1057.3	1061.6	963.4985
	25-35%	523.13	530.38	525.25	528.372	508.4975	85.6576
	35-100%	149.85	154.149	130.0824	148.1399	158.2778	0.0618
	0-10%	90.849%	91.33%	92.65%	91.47%	92.99%	91.31%
	10-15%	74.73%	75.099%	74.89%	74.97%	74.47%	77.72%
7Yr	15-25%	1176.25	1186.25	1118.2	1173.0	1175.7	1271.2
	25-35%	616.69	624.559	690.5859	623.4425	627.8967	301.5964
	35-100%	164.330	168.0	163.9533	167.7249	169.9020	2.0454
RMSE				3.9946	0.5786	1.6869	31.7728

Table 5.1: Calibration results.
The columns "MarketBid" and "MarketAsk" contain the market bid and ask quotes of CDX.NA.HY.10 (CDX High Yield index portofolio of $n = 100$ North American constituents) on June 16, 2008. Data Source: Bloomberg.
The column of "Model1" contains calibrated prices for the model without stochastic fluctuations of the parameters, i.e., the model of [14].
The column of "Model2" contains calibrated prices for the model with stochastic volatility, described in Section 5.3.
The column of "Model3" contains calibrated prices for model with stochastic mean reversion level, described in Section 5.4.
The column of "Model4" contains calibrated prices for a bottom-up model with stochastic volatility and symmetric names, proposed in [21].
For calibrations, we assumed the risk-free rate $r = 0.03$ and the loss at default rate $l = 0.6$.
The calibrated parameters are:
Model1
$X_0 = 1.4508$, $\mu = 1.2117$, $\kappa = 0.1836$, $\sigma = 0.6670$, $\theta_1 = 4.6965$, $\theta_2 = 0.00067895$
Model2
$X_0 = 1.5679$, $\mu = 0.9502$, $\kappa = 0.2042$, $\bar{\sigma} = 0.5054$, $\theta_1 = 4.6301$, $\theta_2 = 0.0008758$
$V_1^{\epsilon} = 0.1662$, $V_2^{\delta} = 0.0744$
Model3
$X_0 = 1.433$, $\bar{\mu} = 1.0297$, $\kappa = 0.8131$, $\sigma = 1.2650$, $\theta_1 = 4.6982$, $\theta_2 = 0.0011$
$V_1^{\epsilon} = -0.2258$, $V_2^{\delta} = 0.1102$.
Model4
$X_0 = 0.091$, $\theta = 0.0732$, $\kappa = 0.4685$, $\sigma = 0.0469$, $\rho_X = 0.7825$
$V_1(z) = -1.9940e - 07$, $V_3(z) = -6.5243e - 8$.

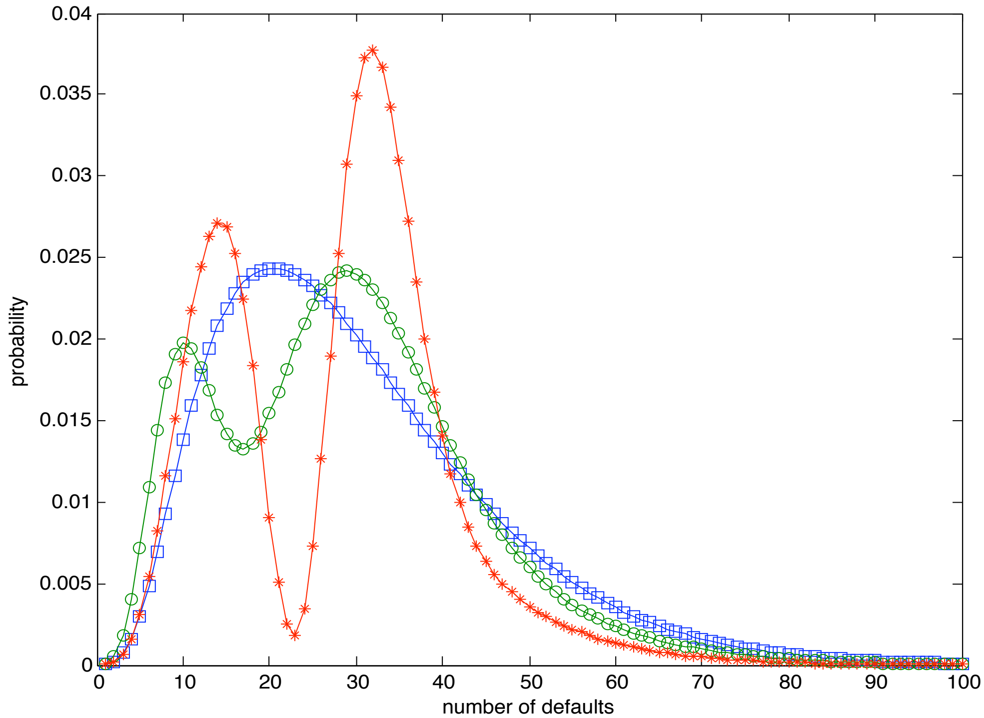

Figure 5.1: 5yr Loss Distribution implied by the models calibrated to 5+7Y CDX.NA.HY.10 June 16, 2008 data:
Legend
-squares-, blue(no correction)
'-o', green(stochastic volatility)
'-*'-, red(stochastic mean reversion level)

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