

PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING  
AND THEIR PERFORMANCE IN SELECTED TEACHING PRACTICES:  
EXPLORING A COMPLEX RELATIONSHIP

by

Charalambos Y. Charalambous

A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
Doctor of Philosophy  
(Education)  
in The University of Michigan  
2008

Doctoral Committee:

Professor Edward A. Silver, Chair  
Professor Deborah L. Ball  
Professor Hyman Bass  
Associate Professor Heather C. Hill, Harvard University

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(VOLUME I)

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In loving memory of my grandfather whose thirst of learning was unparalleled.

## Acknowledgements

As you set out for Ithaka  
hope your voyage is a long one,  
full of adventure, full of discovery. [...]

May there be many [...] summer morning[s] when [...] you come into harbors seen for the first time [...] and may you visit many Egyptian cities to gather stores of knowledge from their scholars.

Keep Ithaka always in your mind. [...]

But don't hurry the journey at all.  
Better if it lasts for years,  
so you are old by the time you reach the island,  
wealthy with all you have gained on the way,  
not expecting Ithaka to make you rich.

Ithaka gave you the marvelous journey.  
Without her you would not have set out.  
She has nothing left to give you now.

And if you find her poor, Ithaka won't have fooled you.  
Wise as you will have become, so full of experience,  
you'll have understood by then what these Ithakas mean.

(Cavafi, "Ithaka," cited in Leontis, Talalay, & Taylor, 2002, p. 80)<sup>1</sup>

As this journey is coming to an end, as I am reaching my Ithaka, probably not as old as Cavafis alludes to in this poem, but still with some grey hair coloring my head, I am looking back to all those people who have accompanied me in this journey and have contributed enormously to my learning and growth over the past years.

I have been privileged to have Dr. Edward Silver as my advisor and dissertation chair. His guidance and support have been unparalleled: he has been generous with his time, insightful in his comments, gentle in challenging and pushing my thinking, as well

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<sup>1</sup> This allegorical poem builds on the fable of Ulysses returning to Ithaka, his homeland, after wandering for ten years.

as motivating and encouraging, especially when needed the most. The other members of my committee, Drs. Deborah Ball, Hyman Bass, and Heather Hill also offered thoughtful and critical comments and suggestions which significantly contributed to the development and refinement of my ideas. The intellectual work and expertise of all my committee members have been a source of inspiration for me and have contributed enormously to advancing my thinking during my studies at the University of Michigan. All of them have ensured that my intellectual journey was indeed “full of adventure, full of discovery.” I owe them all a great debt of gratitude.

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I kept last the person to whom I dedicate this dissertation, my grandfather, whose name I proudly carry. A child of a poor agrarian family, he had the chance to finish only elementary school; yet his thirst for learning was insatiable. From him, I learned to not only treasure learning but also to respect those who help us learn, be them our teachers, parents, or peers. “Wise as [he] became, so full of experiences” throughout his life, he infused within me the love for journeys to Ithakas. I dedicate this dissertation to him as a small token of my deepest appreciation for the many things he taught me deliberately or unwittingly.

What is included in this dissertation could be considered the “final” *product* of my journey – if a text and one’s thinking can ever take a “final” form. Obviously, the whole *process*, the whole journey to Ithaka, cannot be captured in the pages that follow. Nevertheless, I hope that the readers of this dissertation will not “find [it] poor,” to use the poet’s words. For me, this journey has unquestionably not been poor and my reaching to Ithaka marks an end but also a new beginning.



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## **List of Abbreviations**

CCK	Common Content Knowledge
ELMAC	<i>E</i> lementary <i>M</i> asters of <i>A</i> rt with <i>C</i> ertification
GRE	Graduate Record Exam
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
LMT	Learning Mathematics for Teaching (project)
MKT	Mathematical Knowledge for Teaching
MTF	Mathematical Tasks Framework
PE.I	Pre-intervention Interview
PE.S	Pre-intervention Survey
PE.T.	Pre-intervention LMT Test
PO.I	Post-intervention Interview
PO.S.	Post-intervention Survey
PO.T.	Post-intervention LMT Test
PST	Preservice Teacher
QUASAR	Quantitative Understanding: Amplifying Student Achievement and Reasoning (project)
SCK	Specialized Content Knowledge

## **Abstract**

This study examined the relation between teacher knowledge and teaching performance. In particular, it explored the association between preservice teachers' (PSTs) Mathematical Knowledge for Teaching (MKT) and their performance in five teaching practices considered conducive to establishing mathematically rich and intellectually challenging learning environments. Specifically, the study: (1) determined if there is an association between PSTs' MKT and their teaching performance, (2) explored the extent to which this association is mediated by factors such as PSTs' beliefs and background characteristics, and (3) uncovered ways that MKT manifests itself in PSTs' teaching performance. The association of interest was examined for 20 PSTs with respect to one time point (i.e., static perspective) – the beginning of their teacher education program – and with respect to changes between two time points (i.e., dynamic perspective) – before and after an intervention consisting of a two-term mathematics content/methods sequence.

A two-phase, mixed-methods approach was utilized to analyze the data collected through a multiple-choice test (to measure PSTs' MKT), an interview structured around a teaching simulation (to tap PSTs' teaching performance), and a survey (to gauge PSTs' beliefs and background characteristics). In the first phase, non parametric statistics were employed to explore the strength, direction, and robustness of the association of interest. In the second phase, seven PSTs' cases were scrutinized to develop more nuanced characterizations of the association.

The quantitative analysis showed a strong association between the PSTs' MKT and their performance in the teaching practices under investigation. From a static perspective this association was mediated by the PSTs' GRE-quantitative scores, but this factor did not mediate the gains in the PSTs' MKT and teaching performance. The qualitative analysis pointed to two factors – the PSTs' beliefs and their images of teaching – that appeared to inform the PSTs' teaching performance, apart from their knowledge. The cross-case analysis also suggested that the PSTs' knowledge can scaffold their activity in structuring the learning environments considered in the study by helping them maintain an emphasis on the meaning underlying the mathematical procedures at hand. The theoretical, methodological, and practical implications of the study findings are discussed.



## CHAPTER 1

### THE RESEARCH PROBLEM

#### Setting the Context of the Study

The current state: “U.S. mathematics teaching ... [is] characterized by frequent reviews of relatively unchallenging, procedurally oriented mathematics during lessons that are unnecessarily fragmented” (Hiebert, Stigler, Jacobs, Givvin, Garnier, et al., 2005, p. 116)

The vision: “Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. ... The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. ... Students confidently engage in complex mathematical tasks chosen carefully by teachers. ... Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove these conjectures.” (NCTM, 2000, p. 3)

An accumulating body of evidence suggests that selecting cognitively demanding tasks and enacting them as such during mathematics instruction impacts not only how much students learn but also the quality of their learning (Boaler, 2002; Hiebert & Wearne, 1993; Stein & Lane, 1996). However, despite ongoing attempts to reform the mathematics instruction in the United States, this kind of teaching rarely transpires in American classes, as documented by several international and national studies (Hiebert et al., 2005; Hiebert, Gallimore, Garnier, Givvin, Hollingsworth, et al., 2003; Stigler & Hiebert, 1999; Weiss & Parsley, 2004). The TIMSS (Third International Mathematics and Science Study) 1995 Video study, for instance, showed that U.S. students encountered less challenging mathematics compared to their German and Japanese counterparts. The TIMSS 1999 Video study corroborated and extended the findings of its predecessor. Most mathematics problems used in U.S. lessons were stated and solved with an emphasis on executing procedures and getting the correct result, even those

whose mathematical statement asked students to build connections among different mathematical ideas. Less than 1% of the U.S. lessons engaged students in problems that asked them to draw connections – a feature of cognitively challenging tasks – compared with a much higher percentage of such lessons experienced by students in the top performing countries<sup>2</sup> (Jacobs, Hiebert, Givvin, Hollingsworth, Garnier, et al., 2006; Hiebert et al., 2003, p. 104). In addition to the unchallenging content, a constellation of other teaching features appeared to contribute to the lower cognitive level at which mathematics was experienced in these lessons. These include an overemphasis on reviewing and applying familiar procedures without attending to their conceptual underpinnings and a remarkable paucity of opportunities for students to engage in reasoning and conjecturing or in sharing and comparing alternative solutions to problems, features that are considered critical for student intellectual engagement with the content (Hiebert et al., 2005; Jacobs et al., 2006).

One could attribute these differences to the quality of curricula used in the countries under consideration.<sup>3</sup> In fact, for years scholars have blamed the school curricula for the unflattering picture of the teaching of mathematics in the United States, often regarding them as unfocused, repetitive, and undemanding, “a mile wide and an inch deep” (Schmidt, McKnight, & Raizen, 1997). However, affording teachers curricula that include cognitively demanding tasks – such as the *Standards*-based curricula

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<sup>2</sup> For instance, in 37% of the Japanese lessons, 22% of the Dutch lessons, and 12% of the lessons in Hong Kong students were presented with problems that asked them to make connections and explore mathematical relationships. Even more noteworthy, this focus on making connections and exploring relationships was retained during students’ engagement with the problems in about 40% of these lessons (48%, 37%, and 46% of the lessons in Japan, the Netherlands, and Hong Kong, correspondingly).

<sup>3</sup> The interested reader could find detailed discussions of such differences in Schmidt, McKnight, Valverde, Houang, and Wiley (1997) and Valverde, Bianchi, Wolfe, Schmidt, and Houang (2002).

developed during the 1990s<sup>4</sup> – cannot, on its own, elevate the cognitive level at which the content is taught and experienced in mathematics classes, as suggested by a growing body of research.

Weiss and Pasley (2004), for example, found that despite the richness of the curricula made available to teachers, the quality of instruction of several of these teachers did not meet expectations. In particular, of the 364 math and science lessons they examined in their study, only 15% were designed and implemented in a manner that engaged students in important mathematics and science ideas; most of the lessons (59%) were rated of low quality in terms of enhancing students understanding of the targeted concepts. A more recent study that analyzed over 1,600 math and science lessons (Banilower, Pasley, & Weiss, 2006) also showed that teachers tended to reduce the “investigative nature” of the challenging tasks they incorporated in their lessons, often by putting considerable emphasis on procedures and getting right answers. Other smaller-scale studies yielded similar results (Arbaugh, Lannin, Jones, & Park-Rogers, 2006; Castro, 2006; Collopy, 2003; Huberman & Middlebrooks, 2000; Remillard, 2000; Tarr, Chàvez, Reys, & Reys, 2006).<sup>5</sup> Collectively, these studies suggest that although an important leverage for change, curriculum materials cannot on their own ensure that students are engaged in rich and intellectually challenging learning environments. Nor

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<sup>4</sup> A comprehensive review of the *Standards*-based curricula developed during the 1990s appears in Senk and Thompson (2003).

<sup>5</sup> For instance, examining the implementation of a high-school *Standards*-based curriculum, Arbaugh and colleagues (Arbaugh et al., 2006) found that only seven of the 26 lessons maintained the cognitive demand of the curriculum tasks during their enactment. In 11 of these lessons, teachers employed cognitively demanding tasks but failed to maintain the challenge during implementation. Similarly, analyzing a series of lessons taught by four experienced teachers, all of whom were using a middle-grade *Standards*-based curriculum, Castro (2006) found that, when faced with student struggles, these teachers often reduced the cognitive complexity of the assigned tasks.

can the articulation of ambitious visions – such as the one that opens this chapter – or the elaboration of guiding principles and standards for teaching the subject.

Engaging students in mathematically rich and intellectually demanding learning environments constitutes an arduous endeavor, a venture that encompasses several challenges and imposes significant demands on teachers.<sup>6</sup> For example, how can teachers balance maintaining the cognitive complexity of tasks and pique and sustain students' interest? How much information and of what type should they make available to students and when? How can they support students in their exploration of rich and intellectually demanding mathematical tasks without diminishing the cognitive challenge? And foremost, do teachers themselves understand the content thoroughly enough so that they can traverse it with their students? To use the mathematician's Henry Pollak words, can teachers' own knowledge support them in walking not only in paths "carefully laid out through the woods," but also in tracks that "come up against cliffs or thickets" (cited in Lampert, 1990, p. 42), which may very well be the case when seeking to engage students in intellectually challenging tasks?

Starting from the mid 1980s, increased attention has been paid to teacher knowledge of mathematics as a key resource for the work of teaching mathematics. Many studies have focused on "deficit" examples – examples of teachers who failed to capitalize on the rich potential of the curriculum materials made available to them. For instance, several scholars (e.g., Cohen, 1990; Putnam, Heaton, Prawat, & Remillard, 1992; Wilson, 1990) argued that teachers' impoverished understanding of the content impinges on their capacity to implement rich mathematics curricula effectively. Take, for

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<sup>6</sup> Lampert (2001) vividly sketches the challenges that inhere in using rich tasks in instruction. Similar discussions of these challenges appear in Clarke (1997) and Doyle (1992).

instance, the case of Mrs. Oublier, a second-grade teacher who was asked to implement a rich mathematics curriculum that sought to teach mathematics for understanding (Cohen, 1990). As Cohen suggests, because of her weak grasp of the content, Mrs. Oublier failed to capitalize on the multitude of opportunities afforded by the curriculum to engage her students in rich explorations of the content:

Lacking deep knowledge, Mrs. O[ublier] was simply unaware of much mathematical content and many ramifications of the material she taught. Many paths to understanding were not taken in her lessons ...but she seemed entirely unaware of them. Many misunderstandings or inventive ideas that her students might have had would have made no sense to Mrs. O, because her grip on mathematics was so modest. In these ways and many others, her relatively superficial knowledge of this subject insulated her from even a glimpse of many things she might have done to deepen students' understanding (p. 322)

Or consider, the case of Mark, a fifth-grade teacher who, as Wilson (1990) notes, built his lessons around procedural tasks that targeted the mastery of a set of algorithms, mainly because “he could only teach what he understood” (p. 297). Or even think about Valerie and Sandra, two other elementary school teachers who, as Putnam and colleagues (Putnam et al., 1992) maintain, chose intellectually demanding tasks for their lessons but proceduralized these tasks during implementation. Even worse, they distorted the math, leading these scholars to argue that “what teachers know ...guides how they construct lessons, interpret textbooks, and interact with students” (ibid, p. 213). This accumulated experience of the past is compactly articulated in the Teaching Principle of the NCTM *Principles and Standards for School Mathematics* (2000), which emphasizes that the ambitious vision articulated in the opening of this chapter can only be realized if teachers possess the knowledge necessary for establishing classroom environments that challenge and nurture students' intellect:

To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks. (p.17)  
Teachers' actions are what encourage students to think, question, solve problems, and discuss their ideas, strategies, and solutions. The teacher is responsible for creating an intellectual environment

where serious mathematical thinking is the norm. ... If students are to learn to make conjectures, experiment with various approaches to solving problems, construct mathematical arguments and respond to others' arguments, then creating an environment that fosters these kinds of activities is essential (p. 18).

Despite the arguments advanced about the role of teacher knowledge of mathematics in teaching and especially in creating mathematically rigorous and intellectually challenging learning environments, this role is still unclear. Writing in 1992, Fennema and Franke admitted that although no one questions that teacher knowledge is one of the most influential contributors to what transpires in the classroom and ultimately to what students learn, "there is no consensus on what critical knowledge is necessary to ensure that students learn mathematics" (p. 147). About a decade later, in her review of studies of teacher knowledge, Mewborn (2003) echoed Fennema and Franke, acknowledging that "a complicated relationship exists among teachers' knowledge, their teaching practices, and student learning that research has yet to untangle" (p. 47).

Although during the last decade significant steps have been made to disentangle this relationship – as the review of the studies in Chapter 2 suggests – several questions remain unaddressed or partly addressed. For instance, how do teachers mobilize their knowledge in establishing mathematically rich and intellectually demanding learning environments? Is teacher knowledge of mathematics and its teaching sufficient to support them in this role? If not, what other factors might inform their decisions and actions in building such environments? If a constellation of factors informs such decisions and actions, and teacher knowledge is one of them, how might it interact with all these other potential factors in informing germane decisions and actions, and what is its relative weight in this process?

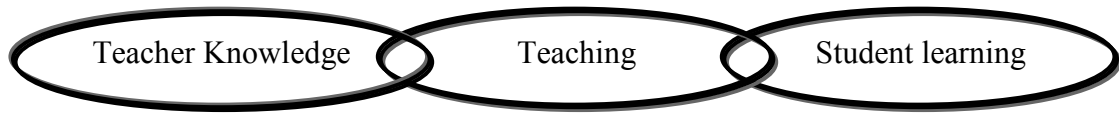
This list of questions is by no means exhaustive. Rather it suggests that despite the significant work to conceptualize and measure teacher knowledge undertaken since the mid 1980s, we are just at the beginning of understanding the extent to which and the means whereby teachers' knowledge of mathematics relates to their decisions and actions conducive to structuring mathematically rich and cognitively challenging learning environments. It is toward this direction of inquiry the present study seeks to contribute.

Exploring the association between teachers' knowledge and their decisions and actions that support the structuring of the aforementioned environments constitutes a broad research territory. To narrow the scope of inquiry, several *design decisions* were made in this study. In the sections that follow, I specify these design decisions, outline the purpose of the study, and present the research questions the study seeks to address. Next, I justify the importance of the inquiry taken up in the study and point to the theoretical and practical contributions of its findings as well as some of its limitations. This chapter concludes with an overview of the dissertation structure.

### Design Decisions

One important goal of exploring the effects of teacher knowledge is to understand how it affects student learning. However, in the chain that links teacher knowledge to student learning (see Figure 1.1), one key link – that of teaching – is often ignored, leading to speculations about the effects of teacher knowledge on student learning. Students learn not merely by being taught by knowledgeable teachers, but because of what these teachers *do* while working with their students on certain topics; in other words, because of the learning environments that knowledgeable teachers can craft by capitalizing on their personal and other contextual resources. Hence, *attempts to*

*understand the effect of teacher knowledge on student learning should pay close attention to teaching.*



*Figure 1.1.* The chain connecting teacher knowledge to student learning.

But if teacher knowledge has been difficult to conceptualize and study, so has teaching. Teaching, as Leinhardt (1993) maintains, “is a complex, dynamic, and ill-structured process” (p. 1).<sup>7</sup> The endemic complexities in studying both teacher knowledge and teaching require that researchers make several design decisions, which unavoidably *frame* any conclusions yielded from a study.<sup>8</sup> For instance, what type of teacher knowledge should one investigate? How should one explore teaching, and particularly, the building of mathematically rich and intellectually challenging learning environments? What components of the work of teaching should the researcher study to capture teachers’ potential to create such environments? And what teacher population should one consider? Novices? Experienced teachers? Could anything be learned from studying prospective teachers? Given that teaching does not happen in a vacuum, what contextual factors should the researcher consider? What might be the drawbacks of ignoring certain contributing factors to the inquiry under examination? How could the researcher minimize complexity in order to study it, but yet, not lose sight of reality?

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<sup>7</sup> Lampert (2001) also discusses this complexity by considering teaching an incessant process of managing dilemmas. She vividly portrays this complexity when she suggests that “working at teaching turns out to be something like navigating in a multidimensional terrain, getting safely across the street while also crossing the city, sighting one island from another while catching the wind that makes it possible to get around the continents and across the ocean” (p. 50). For a similar discussion about the complexities of teaching, see Ball (1993), as well as Ball and Wilson (1996).

<sup>8</sup> I use the word *frame* deliberately to denote that the design decisions shape the conclusions of this study, but also to imply that these decisions set the boundaries within which the conclusions of the study can be generalized. I elaborate this issue in Chapter 3.



With all these considerations in mind, in the present study I made several design decisions, which I justify in the next chapter. Central among these decisions, are those related to the type of teacher knowledge examined in the study and the framework employed to identify and explore practices that contribute to the establishment of rich and cognitively challenging learning environments. In particular, the study builds on the work of Deborah Ball and colleagues on *Mathematical Knowledge for Teaching* (MKT), a type of knowledge distinct to the work of teaching mathematics (Ball & Bass, 2000; 2003a; Ball, Hill, & Bass, 2005; Ball, Thames, Phelps, in press). It also draws on the evolution of the cognitive demands of mathematical tasks as captured by the *Mathematical Tasks Framework* (MTF) proposed by the QUASAR (*Quantitative Understanding: Amplifying Student Achievement and Reasoning*) scholars (Stein, Grover, & Henningsen, 1996; Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2000). In the second chapter, I elaborate the work in these two research areas and explain why I drew on them to frame and explore the inquiry the study undertakes.

To understand how teacher knowledge contributes to establishing mathematically rich and intellectually demanding learning environments I focus on certain *practices of teaching*, the selection of which constitutes another design decision. In line with scholars who have discussed the importance of studying instructional practices to understand the work entailed in teaching (e.g., Shavelson, 1983) or who have actually engaged in this work (e.g., Ball & Bass, 2000, 2003a; Lampert, 2001; Lampert & Ball, 1998; Grossman et al., in press), I use the word practices deliberately to convey a twofold meaning. First, I focus on certain teacher performances, things that teachers *do*, that have the potential to affect the structuring of the learning environments under consideration. Second, the word

practices suggests that these performances are not haphazard; rather, they are patterns of action typically informed by teachers' own knowledge.<sup>9</sup> But in addition to the meaning that the word practices carries, I focus on teachers' practices for another reason which relates to the scaffolding that teachers need to create rich and challenging environments for their students. As already suggested, asking teachers to establish environments that challenge and nurture student intellect puts a heavy burden on teachers' shoulders. Unless researchers and educators alike explore the practices that are conducive to the creation and maintenance of such environments and what it takes to develop such practices, there is little hope that teachers will be able to structure rich learning environments, as the accumulating experience of the past suggests.

But what practices should one examine? As Lampert (2001) suggests, teachers engage in a multitude of practices even while teaching a single lesson, all of which affect what transpires in the classroom, and eventually what students learn. This gamut of practices calls for a refinement of the third design decision of the study. For reasons explicated in the next chapter, from the rich panoply of teaching practices, I focus on five considered conducive to establishing mathematically rich and intellectually challenging environments: selecting and using tasks, responding to students' requests for help, using representations, giving explanations, and analyzing student work and contributions. The first two derive from work associated with the unfolding of tasks during instruction, as captured by the MTF; thus, I call them MTF-related practices. The latter three practices are associated with work on MKT; hence, I call them MKT-related practices. In the next

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<sup>9</sup> In Encarta ® World Dictionary (2004, p. 1475) *practice* is defined as "a usual pattern of action, an established way of doing something, especially one that has developed through experience and knowledge." Similarly, Scribner and Cole (1981) define *practice* as "a recurrent, goal-directed sequence of activities using a particular technology and *particular systems of knowledge*" (cited in Brodie, 2004, p. 73, emphasis added).

chapter, I explicate how these practices can help structure the aforesaid rigorous and challenging learning environments.

A fourth design decision pertains to the teacher population the study considers. For reasons explained in the second chapter, this study focuses on *preservice teachers* (PSTs). At first sight, this decision might seem counterintuitive since one could argue that prospective teachers have not yet had the opportunity to develop particular teaching practices. However, as several researchers have pointed out (e.g., Ball, 1988; Ball, Lubienski, & Mewborn, 2001), PSTs enter the teacher preparation programs with already formed ideas about what teaching is or should be and which teaching approaches are more or less suitable. These ideas are often nurtured from their own experiences as students of mathematics.<sup>10</sup> In this study, I explore PSTs' decisions and actions regarding the five practices outlined above via a teaching simulation, the development and use of which I detail in Chapter 3. I investigate PSTs' performance in this simulation by focusing on a particular mathematical topic (fifth design decision) and by decomposing performance in three constituent components-skills for each teaching practice (sixth design decision). These latter design decisions are further elaborated in Chapter 2.

#### Purpose and Research Questions

With these design decisions as boundary conditions, the present study seeks to *understand the association between PSTs' mathematical knowledge for teaching (MKT) and their performance in five teaching practices considered conducive to establishing mathematically rich and intellectually demanding learning environments: selecting and using tasks, responding to students' direct or indirect requests for help, using*

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<sup>10</sup> Citing Lortie (1975), Ball and colleagues argue that by the time student teachers enter preservice training programs, "they have already clocked over 2,000 hours of specialized 'apprenticeship of observation,'" which shapes their ideas about what mathematics is and how it should be taught (p. 437).

*representations, providing explanations, and analyzing student work and contributions.* I approach this overarching goal through both a static and a dynamic perspective.

The *static* perspective pertains to understanding how PSTs' MKT at a *fixed point in time* relates to their performance in the five aforesaid practices. This perspective aims at capturing a snapshot, an "image of the reality," to use Charmaz's (2000) words.<sup>11</sup> To this effect, I ask the following research question (RQ):

RQ1: Does PSTs' MKT relate to their performance in five teaching practices considered conducive to the establishment of rich and intellectually challenging learning environments? If so, how?

Cognizant of several other factors that might be associated with candidate teachers' germane performance – several of which I discuss in the second chapter – I chose this certain time point to be teachers' entrance to a preservice preparation program. In choosing this time point, I sought to narrow the range of the mediating factors that could, in addition to, or in conjunction with MKT, impinge on candidate teachers' relevant performance. However, on entering preservice education programs, PSTs hold several beliefs about mathematics, its teaching and learning, the goals that teaching this subject serves, and their own ability to do and teach mathematics. All these factors could potentially mediate PSTs' performance in the five teaching practices of interest. Performance in these practices might also be associated with PSTs' background characteristics, such as the math content and the math methods courses they had taken and their general knowledge/aptitude. Hence, I approach the first research query by asking the following three subordinate research questions (SRQ):

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<sup>11</sup> Charmaz (2000) clarifies that the researcher cannot capture reality, but an image of it at a certain point in time.

SRQ1.1: To what extent does candidate teachers' MKT at their entrance to a teacher preparation program relate to their performance in the five teaching practices under investigation?

SRQ1.2: Provided that an association between PSTs' MKT and performance in the five practices exists, to what extent is this association mediated by factors such as PSTs' beliefs regarding teaching and learning mathematics, their overarching goals for teaching this subject, their perceived competence, and certain background characteristics (courses taken and general knowledge/aptitude)?

SRQ1.3: How does MKT manifest itself in PSTs' performance in the five teaching practices under consideration?

The second research question considers teachers' knowledge and their performance in the five teaching practices from a *dynamic* perspective. That is, the association between MKT and the five teaching practices is considered in the context of an intervention designed to *leverage changes* in participants' MKT. In particular, I ask:

RQ2: Do changes in the MKT of PSTs who participated in a particular teacher education intervention relate to changes in their performance in five teaching practices considered supportive for structuring rich and cognitively challenging learning environments? If so, how?

Besides leveraging changes in PSTs' MKT, the intervention under exploration was also designed to help prospective teachers hone their ability to provide instructionally appropriate explanations, use representations, and analyze student work and contributions (i.e., the three MKT-related practices the study considers). Hence, in addressing the second research question, I also consider two separate groups of teaching practices: the

three MKT-related practices and the two MTF-related practices. In particular, I ask three subordinate research questions:

SRQ2.1: (a) To what extent are changes in PSTs' MKT associated with changes in their performance as a whole, in the three MKT-related practices and the two MTF-related practices? (b) To what extent are changes in participants' performance in the MKT-related practices associated with changes in their performance in the MTF-related practices?

SRQ2.2: To what extent are the aforementioned associations mediated by other factors, such as PSTs' background characteristics and changes in their beliefs about teaching and learning mathematics, in their overarching goals for teaching this subject, and in their perceived competence?

SRQ2.3: How do changes in MKT appear to play out in changes in PSTs' performance in the MKT-related practices and the MTF-related practices?

The first subordinate question approaches the second query of the study holistically and explores whether any potential changes in PSTs' MKT are related first to changes in their teaching performance as a whole, and then to changes in their performance in the two sets of practices; it also explores whether the changes in performance in the one set of practices covary with the changes in performance in the other set of practices. The second subordinate question explores the extent to which the aforementioned associations are mediated by other factors, such as those considered for the first research question. Finally, in the third subordinate question, I explore in more depth whether changes in PSTs' MKT are associated with any qualitative changes in their MKT- and the MTF-practices. The reader is cautioned that neither the second research

question nor its subordinate questions focus on examining the effectiveness of the intervention per se. This is because the intervention was merely considered a vehicle for triggering changes in PSTs' MKT, which would, in turn, allow for a further exploration of the association between this type of knowledge and performance in the five practices.

This twofold perspective in approaching the overarching goal of the study – by studying the association between knowledge and performance in certain practices at *a fixed point in time* and by further probing this association through studying *changes* in knowledge and teaching practices – combines two complementary approaches that have been pursued in research on teacher knowledge.<sup>12</sup> As such, this bifocal perspective is considered suitable for exploring complex relationships such that between knowledge and performance in teaching practices investigated in this study. Acknowledging the complexity of the relationship under examination, to address the aforementioned research questions, I collect both quantitative and qualitative data and analyze them using a two-phase sequential mixed-method design.

In the first phase, each research question is addressed quantitatively using non-parametric statistics to explore the direction, strength, and robustness of the association between (changes in) MKT and (changes in) PSTs' performance in the five teaching practices. In the second phase, qualitative analysis is pursued to build and elaborate explanations regarding certain ways in which (changes in) MKT may relate to (changes in) PSTs' performance in the five practices. This second phase supports better understanding of the association or lack thereof between MKT and performance in the

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<sup>12</sup> Some studies explored teachers' knowledge at a certain point in time (e.g., Ball, 1988; 1990; Kennedy, 2005; Sherin, 2002; Simon, 1993); other studies explored changes in teachers' knowledge and practices during the course of an intervention designed to leverage such changes (e.g., Sowder, Philipp, Armstrong, & Schappelle, 1998; Swafford, Jones, & Thornton, 1997).

five practices. Although the design of the study does not support establishing causal linkages between the knowledge and the performance in the teaching practices under consideration, its findings help make a *first step* in exploring the direction, strength, and nature of the complex relationship between knowledge and teaching performance.

### Significance

Writing in 1986, Shulman posed a list of questions regarding the manner in which teachers capitalize on their knowledge to improve their practice. Among other queries, he asked: “How does the novice teacher (or even the seasoned veteran) draw on expertise in the subject matter in the process of teaching?” (p. 8). Although a growing number of studies have been conducted since then (cf. Ball et al., in press), and notwithstanding the significant insights garnered from them, the manner in which teachers’ knowledge appears to relate to their teaching performance remains an open question. More recently, Ball and Bass (2003b) extended Shulman’s request for more studies that will help illuminate the association between teacher knowledge and teaching performance, acknowledging that “[s]tudies of mathematical knowledge for teaching [MKT] have barely addressed how teachers mobilize their knowledge in the work of teaching mathematics” (p. 13). This request was further elaborated in a more recent study (Hill, Rowan, & Ball, 2005), in which Ball and colleagues explored the association between teacher knowledge and student learning gains. Given that their study showed that teacher knowledge does affect student learning gains, these scholars urged researchers to explore the particular ways in which teachers’ knowledge influence teaching, and consequently, student learning:

What knowledgeable teachers *do* in classrooms – or how knowing mathematics affects instruction – has yet to be studied and analyzed. Does teacher knowledge of mathematics affect the decisions they make? Their planning? How they work with students, or use their textbooks? How they



manage student confusion or insights, or how they explain concepts?" (p. 401, emphasis in the original)

The present study aims at taking a step toward addressing these questions.

Specifically, by exploring the relationship between teacher knowledge and their *selection and use of tasks*, the study seeks to provide insights into the ways in which teacher knowledge might inform planning decisions and use of available textbooks. By probing the associations between teacher knowledge and the *explanations* they provide as well as the *representations* they use in their teaching, the study aims at exploring how knowledge can be mobilized during instruction to present the content in a manner that supports high-quality learning. Finally, by focusing on *analyzing student work and contributions* and on *how teachers respond to students' requests for help*, the study intends to offer insights into how teachers can capitalize on their knowledge to support student learning not only in ways that are responsive to student needs, but also in a fashion that maintains the integrity and the cognitive challenge of the content. In sum, in line with research efforts currently undertaken (e.g., Hill et al., 2008), the present study seeks to add to the evolving knowledge base on the ways in which teacher knowledge appears to matter for teaching mathematics; more critically, though, this study seeks to unveil how this knowledge can support teachers in making decisions and engaging in actions that are supportive for structuring mathematically rich and cognitively challenging learning environments. It is herein that lies another contribution of the study.

For the past decade, significant advancements have been made in understanding how the tasks teachers select for their lessons and the ways in which these tasks are enacted during instruction can determine the cognitive level at which the content is experienced in mathematics classes, and eventually what students learn (Henningsen &

Stein, 1997; Stein & Smith, 1998; Stein et al., 2007). At the same time, the ongoing work on MKT has contributed to the expanding knowledge base regarding the types of knowledge that appear to matter for teaching the subject effectively, and ultimately for enhancing student learning. The work in both research arenas – the one related to the tasks, and particularly the work on the MTF, and the other to the MKT– has yielded useful insights for teaching mathematics; however, research in these two realms has largely moved onto parallel rather than intersecting paths. The present study aims at the intersection of the two research areas, on the premise that significant insights can be gained by synthesizing the ideas generated in the two fields.<sup>13</sup> In doing so, the study follows recent calls for building on, integrating, and synthesizing existing frameworks, concepts, and methods of inquiry to better understand complex phenomena, such as those investigated in education (National Research Council, 2002, p. 49; Schoenfeld, 1999).

In addition to the theoretical contributions pertaining to examining the association between knowledge and teaching practices and to integrating work in the two aforementioned research arenas, the study findings have certain practical implications, particularly for the preparation of PSTs; I elaborate these implications in the last chapter of the dissertation. The study findings also have implications regarding issues of equity. Previous studies have documented that it is both feasible and beneficial to teach students in disadvantaged school settings in intellectually challenging ways (cf. Silver & Stein, 1996). Yet, two recent studies (Hill, 2007; Hill et al., 2005) have shown that students

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<sup>13</sup> Interestingly enough, about two decades ago, Brophy (1991) pointed to the need for integrating work around teachers' knowledge and their selection and use of instructional tasks. In his concluding comments and synthesis of the chapters that were presented in the second volume of the *Advances in Research and Teaching* – which was devoted to the role of teachers' knowledge of subject matter in their teaching – he urged researchers to explore what he regarded as a link between teacher knowledge and the level at which the content is experienced, claiming that: “[A]t least part of the reason for why poor subject-matter teaching occurs in many classrooms is that the teacher is weak in pedagogical content knowledge or lacks a clear orientation toward teaching the subject” (p. 358).

from lower socioeconomic backgrounds, African American, and Hispanic students are more likely to be taught by low-MKT teachers. If the present study provides evidence about the association between MKT and teaching performance, the question that then arises is whether the unequal distribution of MKT between affluent and disadvantaged school settings documented in the aforesaid studies allows for establishing rich and intellectually challenging learning environments in less affluent areas. I take up this issue in the last chapter of the dissertation.

### Limitations

To facilitate the exploration of the association between PSTs' MKT and their performance in the five teaching practices at hand, in this study I made six major design decisions: focusing on MKT, building on the work around tasks and the MTF; examining five particular teaching practices; exploring the performance of PSTs; exploring this performance via a simulation of teaching; focusing on a specific mathematical topic, and investigating PSTs' performance as a composite of constituent components. A few additional minor design decisions concerning the approaches pursued to collect and analyze the study data were also made; these decisions are discussed in Chapter 3. Unavoidably, all these design decisions impose constraints on the study, the most central being that its findings may have limited transferability. To better understand the findings of this study and their locus of applicability, in the next two chapters, I discuss these design decisions in detail. Specific study limitations are also discussed in the last section of Chapter 3, after I elaborate the design of the study and the data sources.

## Dissertation Overview

This dissertation consists of six chapters. In the first chapter, I have outlined the overarching problem that the study considers and narrowed its scope by outlining six central design decisions. These decisions set the boundaries for the investigation that the study undertakes and consequently limit the applicability of the study findings. Based on these design decisions, I have then presented the two main research questions of the study, explicated its significance, and discussed its implications and limitations.

In Chapter 2, which consists of seven sections, I review related literature to justify the design decisions made in the study. In the first section, I provide an overview of studies on teacher knowledge to justify the focus of the study on MKT (i.e., first design decision). After establishing the association between the environments structured to help students learn mathematics and the quality of student learning, in the next section, I explain the reasons that informed the selection of MTF, as a framework for exploring the association between teacher knowledge and their practices (i.e., second design decision). The third section reviews studies that investigated how teacher knowledge appears to affect their practices. This section helps assemble different teaching practices that have been associated with teacher knowledge. From this gamut of practices, in the fourth section, I focus on five teaching practices and justify the reasons that informed this third design decision. I also explicate how these practices can contribute to the building of rich and intellectually challenging environments. In the fifth section, I detail the inherent challenges of exploring the association between teacher knowledge and their teaching practices and justify the fourth design decision to focus on PSTs and explore their performance via a teaching simulation. In the last two sections of Chapter 2, I explain the

last two design decisions: why I decided to focus on a challenging mathematical topic – the division of fractions – to explore PSTs’ teaching performance and the reasons that informed my decision to conceptualize the teaching performance under consideration as a composite of three subcomponents per teaching practice. Based on this decomposition of performance, in this chapter I also present the analytic framework that guided the exploration of the research questions of the study.

Chapter 3 details the methodology of the study and consists of five sections. The first section, presents the context of the study and provides information about the study participants and the intervention under consideration. The following three sections outline the data collection instruments developed and used in the study as well as the data collection and processing procedures. In sections four and five, I present the measures used in the study and detail the approaches pursued to analyze the collected quantitative and qualitative data. This chapter concludes by considering issues of validity, reliability, and generalizability.

The next two chapters present the findings of the study and are organized by the two research questions. Addressing the first research question (RQ1), Chapter 4 presents the findings pertaining to the exploration of the association between PSTs’ MKT and their performance in the five teaching practices from a static perspective. This chapter is organized in three sections. The first section uses the quantitative data to explore the direction, strength, and robustness of the aforesaid association, thus addressing the first two subordinate questions of RQ1. The second section explores this association from a qualitative perspective, by scrutinizing a set of cases. Looking across these cases, the third section presents the results of a cross-case analysis, which sought to understand how

PSTs' knowledge manifests itself in their teaching performance, thus addressing the third subordinate question of RQ1. This cross-case analysis also aimed at identifying other factors, besides PSTs' knowledge, that appeared to inform their performance in the teaching simulation, thus contributing to answering the second subordinate question of RQ1 from a qualitative perspective.

Chapter 5, which shares the same structure with the previous chapter, addresses the second research question (RQ2); namely it explores the association between changes in PSTs' MKT and changes in their performance in the teaching simulation. The first section of this chapter addresses the two subordinate questions of RQ2, by using quantitative data to explore the direction, strength, and robustness of the aforementioned association. The second section presents the results of a within-case analysis in which the cases considered in the previous chapter are revisited and changes in their knowledge and performance are examined. Building on this within-case analysis, the next section looks across these cases, seeking to understand how changes in PSTs' knowledge appeared to play out in changes in their teaching performance and what other factors appeared to inform PSTs' performance in addition to their knowledge. Taken together, the second and third sections of this chapter contribute to answering the second and third subordinate questions of RQ2.

The dissertation concludes with Chapter 6, in which I summarize the main findings of the study, discuss these findings in light of other pertinent studies, highlight the main theoretical, methodological, and practical contributions and implications, and provide suggestions for future studies.

## CHAPTER 2

### THEORETICAL FOUNDATIONS

#### Overview

The purpose of this study is to clarify the association between teacher knowledge and their performance in a set of teaching practices considered supportive for the structuring of mathematically rigorous and intellectually demanding environments. As explained in Chapter 1, because this research inquiry is broad, six design decisions were made to narrow its scope and facilitate its exploration. This chapter serves two main purposes.

First, it aims at explaining the six design decisions presented in the previous chapter: to focus on *mathematical knowledge for teaching* (MKT); to draw on the *Mathematical Tasks Framework* (MTF) to investigate the association between teacher knowledge and their practices; to explore *five specific teaching practices* considered conducive to establishing mathematically rich and challenging learning environments; to focus on *PSTs' performance in a teaching simulation* as a means to study their performance in the five practices at hand; to explore PSTs' knowledge and performance with respect to the mathematical topic of the division of fractions; and to examine PSTs' performance by decomposing each practice into three subcomponents. Second, the chapter seeks to situate this study within the larger spectrum of research efforts that seek to understand the relationship between teacher knowledge and student learning, and explain its inquiry. To this effect, the literature review presented in this chapter is

organized around the framework depicted in Figure 2.1. This framework elaborates the chain between teacher knowledge and student learning illustrated in Figure 1.1 by decomposing the intermediate link of teaching into two constituent components: teaching practices and the establishment of rich and intellectually challenging environments.

This chapter consists of seven sections. In the first section, I consider the relation between teacher knowledge and student learning. In particular, I draw on previous studies to explore the association between the two outer links of the chain depicted in Figure 2.1. In doing so, I seek to *theoretically justify* the importance of studies that focus on teacher knowledge, one of which is the study undertaken herein. In discussing this association, I ask: Does teacher knowledge really matter for student learning? If so, what *type* of knowledge seems to influence student learning, and how should or could this knowledge be measured? This discussion leads to a justification of the first design decision of the study, namely the selection of MKT as the type of knowledge it explores.

Having theoretically established the association between teacher knowledge and student learning, in the second section I consider how the structuring of mathematically rich and cognitively demanding environments affects student learning. This discussion is organized around two questions: whether the establishment of these environments affects student learning, and if so, what is teachers' role in establishing these environments. This discussion further explains the inquiry of the study, because it suggests that the establishment of such environments matters both for what students learn and for the quality of that learning (i.e., the third and fourth links of the chain). Additionally, it points to the critical role that teachers have in establishing such environments, and thus justifies the connection between the former and the latter two links of the chain of Figure 2.1. But



how could one closely explore the different manifestations of teachers' role in establishing such learning settings? To address this question, I draw on the work associated with the *Mathematical Tasks Framework* (MTF). In particular, I review this work to explicate and elaborate the second design decision of the study.

The second section establishes the critical role that teachers have in engendering rich and challenging environments for student learning. The question that then arises concerns the specific contribution of teacher knowledge to the construction of such environments. Does teacher knowledge matter for structuring rich and challenging environments? And if so, in what particular ways? Both questions, which collectively capture the association between the first and the third link of the chain of Figure 2.1, occupy the third section. Specifically, in this section I review studies that examined the effects of teacher knowledge on teachers' instruction. Although, by and large, these studies *did not directly* explore the association between teacher knowledge and the establishment of rich and challenging environments, they point to certain *teaching practices* through which teacher knowledge appears to influence the structuring of rich and challenging learning settings. Hence, after reviewing these studies, I distill more than a dozen teaching practices and use the MTF to organize them into three categories, corresponding to the phases of planning, task presentation, and task enactment.

In the fourth section of this chapter, I explain the reasons that informed the selection of five teaching practices from those distilled in the previous section. Drawing on relevant literature, I also explicate why I consider these practices conducive to the establishment of rich and cognitively challenging environments. Thus, in conjunction, the third and the fourth sections justify the third design decision of the study: to focus on five

teaching practices and, particularly, explore the association between performance in these practices and a particular type of teacher knowledge, namely MKT. These last two sections also show how the overarching question of exploring teacher knowledge and teachers' performance was gradually narrowed down to the research question investigated in this study. In Figure 2.1, this funneling of the bigger question to the question that this study considers is shown by the trapezoid and the rectangle beneath it.

Having narrowed the study inquiry and shown how it relates to the bigger research question, in the fifth section I consider inherent challenges in investigating the association between teacher knowledge and performance in the five teaching practices. Central among these challenges is the influence that other factors might have on teachers' performance in the practices examined in the study. This discussion helps justify the fourth design decision of the study, namely to focus on prospective teachers and study their respective performance in the five practices via a teaching simulation.

A question that then arises concerns the mathematical topic(s) around which PSTs' *knowledge* and *performance* in the teaching simulation could be examined. This is the issue taken up in the sixth section of this chapter, which explicates the fifth design decision of the study: to focus on a single mathematical topic, and particularly, on the division of fractions.

## CONTEXT

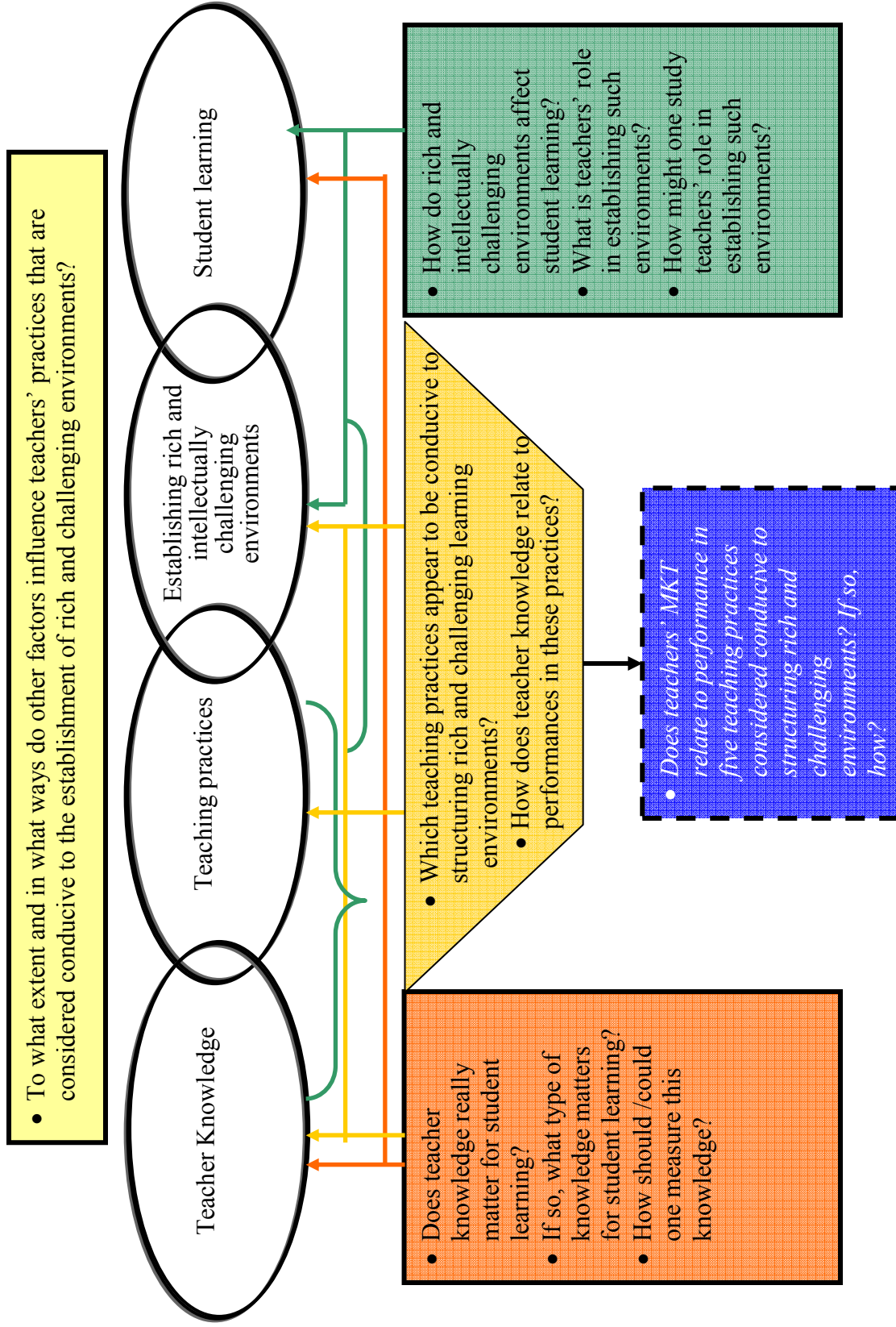


Figure 2.1. An elaboration of the chain connecting teacher knowledge to student learning.

In conjunction, the first six sections explain how the overarching association that this study considers was gradually pinned down in order to be better explored. From this perspective, these sections can be considered to represent the *focusing* function of a microscope. Representing the *magnifying* function of a microscope, the last section details the sixth design decision which concerns how PSTs' performance was theoretically decomposed into subcomponents to be better scrutinized. This section also outlines the analytic framework that guided the exploration of PSTs' performance in the five practices under consideration

### Teacher Knowledge and Student Learning: Is There a Relationship?

This section provides an overview of the evolution of research on teacher knowledge. In this overview, I consider three critical questions that faced researchers trying to understand the effects of teacher knowledge on student learning: whether teacher knowledge matters for student learning; which type of teacher knowledge appears to influence student learning; and how to measure this knowledge. The discussion of these questions seeks to establish the connection between the first and the last link in the chain depicted in Figure 2.1 and to justify the first design decision of the study, namely the selection of MKT as the type of teacher knowledge the study examines.

#### *Research on Teacher Knowledge: Tracing the Path to the Current State of the Art*

Researchers' interest in examining the effect of teacher knowledge on student learning dates back to the late 1960s, when two seminal reports, the *Equality of Educational Opportunity Survey* (Coleman et al., 1966) and the *Inequality* project (Jencks et al., 1972), concluded that teachers and schooling only marginally influence student learning relative to other student background characteristics, such as their socioeconomic

status. Since then, several studies sought to prove that teachers and schooling *do* make a difference in student learning.<sup>14</sup> Given the common-sense assumption that the more teachers know the more effectively they teach, several studies – commonly known as the *educational production function studies* – explored correlations between teacher knowledge and student achievement or performance gains. More often than not, these studies used proxies of teacher knowledge, such as undergraduate and graduate courses taken, degrees earned, and performance on certification exams.

One of the earliest works in this realm was Begle's (1979) meta-analysis of studies conducted between 1960 and 1976 that examined the role of teacher knowledge on student performance. Begle considered three knowledge proxies: the number of math content courses that teachers had taken at the level of *Calculus* or beyond; the number of math methods courses they had taken, and the extent to which they had received an undergraduate major or minor in mathematics during their undergraduate studies. The findings of his meta-analysis were ambiguous: the number of content courses taken was positively associated with student achievement only in 10% of the examined studies and negatively associated with student performance in 8% of the analyzed studies. Similarly, taking math methods courses produced positive effects in 24% of the cases – which was the highest percent of positive effects reported for all the variables examined in the meta-analysis – and negative effects in 6% of the cases. Finally, having a major or minor in mathematics was positively related to student achievement in 9% of the studies and negatively in 4% of the cases examined (pp. 41-43). Interestingly enough, the significant main effects that Begle found – either positive or negative – mostly concerned students' computational proficiency. Associations with more demanding cognitive skills such as

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<sup>14</sup> See Floden (2001) and Koehler and Grouws (1992) for relevant reviews.

comprehension, application, or analysis were observed less frequently, a finding that appears to suggest that teacher knowledge matters more for student performances pertaining to lower rather than higher levels of understanding.

Discouraged by his findings, Begle acknowledged that “the effects of a teacher’s subject matter knowledge . . . on student learning seem to be far less powerful than most of us had realized” (p. 54). Challenging the widely held belief that equated teacher knowledge with teacher effectiveness, Begle maintained that “this belief needs drastic modification” (p. 51). Moving even a step farther, he urged researchers to search for other plausible contributors to student learning instead of maintaining focus on teacher characteristics:

We are no nearer any answers to questions about teacher effectiveness than our predecessors were some generations ago. What is worse, no promising lines of further research have been opened up. Evidently our attempts to improve mathematics education would not profit from further studies of teachers and their characteristics. Our efforts should be pointed in other directions. (pp. 54-55)

Despite Begle’s call to shift research focus, interest in teacher knowledge persisted. The studies that succeeded Begle’s meta-analysis, however, also failed to provide strong conclusive evidence about the effects of teachers’ knowledge on student learning. For instance, using data from the Longitudinal Survey of American Youth to examine the relationship of secondary mathematics teachers’ subject matter preparation and student achievement, Monk (1994) found that the number of mathematics courses that teachers had taken during their undergraduate and graduate studies made a difference, but only up to a certain point. Specifically, every additional content course teachers had taken explained an increase of 1.2% in their students’ test score; however, this trend maintained only for the first five content courses. Each additional content

course past the fifth course predicted only an additional 0.2% of students' score.<sup>15</sup> His analysis also showed that the effect of the number of content courses taken by a teacher was only significant for students in advanced, rather than remedial, courses.<sup>16</sup>

Noteworthy is also that the undergraduate math methods courses that teachers had taken were found to contribute more to student performance gains compared to the math content courses these teachers had taken. In light of this latter finding, Monk suggested that "a good grasp of one's subject area is a necessary but not a sufficient condition for effective teaching" (1994, p. 142).

Rice (2003) also reviewed a wide range of empirical studies that examined the impact of teacher characteristics on their student performance. Her review also yielded mixed findings about the effect of degree holding, teacher coursework, and teachers' test scores on their effectiveness. Whereas several studies showed that high school mathematics teachers who held advanced degrees (mainly master's degrees) were more effective than those who did not hold such degrees, this pattern was not consistent across elementary school teachers. In several of the studies she reviewed, Rice found that teacher coursework in both mathematics and pedagogy was associated with increases in student performance; however, this finding was not consistent across school grades. In particular, the contribution of content coursework was more pronounced at the high school level. On the other hand, pedagogical courses – especially those that also interwove a consideration of the subject matter – contributed to teacher effectiveness in

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<sup>15</sup> The negative associations between the number of courses taken and student achievement reported in this study should be interpreted with caution, since they were not statistically significant at level  $p < .05$ ; additionally the standard errors of their corresponding effects were quite large.

<sup>16</sup> In a similar analysis of the same data set, Monk and King (1994) also found that teachers' subject matter preparation measured by the number of courses that teachers had taken during their undergraduate and graduate studies had small positive effects on student learning but only for the high-achieving students.

all grade levels. Finally, studies on the impact of the National Teacher Examination and other state mandated tests of basic skills and teacher abilities on teacher effectiveness elicited mixed results, ranging from no relationship at all to weak or moderate positive relationships.

Overall, the empirical evidence yielded from the educational production function studies did not meet expectations: some studies reported small positive effects, others showed negligible effects, whereas some others documented negative effects (cf. Ball et al., 2001; Fennema & Franke, 1992; Hill et al., 2005; Hill, Sleep, Lewis, & Ball, 2007). What might account for these unexpected, discouraging, and largely inconsistent findings?

A critical analysis of the educational production function studies suggests that they were deficient in at least two closely related ways. First, they assumed that teacher knowledge could sufficiently be measured via proxy variables. However, these variables cannot accurately capture teacher knowledge, especially the type of knowledge necessary for *teaching* the subject. Earning a master's degree in applied mathematics might deepen a teacher's knowledge of differential equations and numerical analysis, to use an example, but it does not necessarily render the teacher more effective in teaching mathematics. Second, these studies failed to open the black box of instruction. By not examining how teachers capitalize on their personal resources – the most critical among them being their knowledge – these studies could only advance speculations about the manner in which teacher knowledge affects student performance. As Hill and colleagues (Hill et al., 2005) eloquently put it:

Effectiveness in teaching resides not simply in the knowledge a teacher has accrued but how this knowledge is used in the classroom. Teachers highly proficient in mathematics ... will help others learn mathematics ... only *if they are able to use their own knowledge to perform the tasks they*



*must enact as teachers.* (pp. 375-376, emphasis added)

In other words, in the chain that connects teacher knowledge to student learning (see Figure 2.1), the intermediate links that correspond to teaching were taken for granted. In doing so, these studies not only erroneously equated knowing with teaching, but they also “foreclose[d] opportunities to examine closely the mathematical territory and demands of the work [of teaching]” (Ball et al., 2001, p. 450). The need to attend more to the practice of teaching was even suggested by a recurrent, yet overlooked, pattern these studies identified. Despite their equivocal results, more often than not, these studies showed that the number of pedagogical courses that teachers had taken was positively related to their effectiveness, more so than the content courses did.

Attempting to illustrate that teachers do make a difference for student learning, another group of researchers focused on examining the associations between certain *teaching behaviors* (i.e., teaching processes) and student performance (i.e., the final product). In contrast to the studies of the previous paradigm, the studies in this paradigm – largely known as *process-product* studies – *did* open the black box of classroom practice and explored teaching behaviors. Many of the studies of this paradigm examined issues such as the instructional time spent on task, classroom management and organization, the balance between lecturing and other modes of instruction, curriculum pacing, question posing, and the opportunities provided to students for practice and application (Brophy, 1986; Brophy & Good, 1986; Good, 1996; Good, Grouws, & Ebmeier 1983; Reynolds & Muijs, 1999). What they did not examine, though, was how *teacher knowledge* informs these instructional decisions and actions. Additionally, the researchers working in this realm were mostly concerned with identifying *generic*

teaching behaviors applicable to all subject matters.<sup>17</sup> Although the findings of these studies informed the development of programs that sought to enhance student learning in mathematics,<sup>18</sup> scholars in this area seem to have underestimated the disciplinary demands that specific subject matters impose.

Despite their contributions in the field, the studies of both aforementioned research paradigms were limited in substantial ways. Reacting to the limitations of these studies – and especially to those of the second paradigm – in his presidential address delivered to the American Educational Research Association membership, Shulman (1986) called for a significant shift of the research foci:

In reading the literature of research on teaching, it is clear that central questions are left unasked. The emphasis is on how teachers manage their classrooms, organize activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels of their questions, plan lessons, and judge general student understanding. What we miss are questions about the *content* of the lessons taught, *the questions asked*, and *the explanations offered*. ... *Where do teachers explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems or misunderstanding?* ... How does the novice teacher (or even the seasoned veteran) draw on expertise in the subject matter in the process of teaching? What pedagogical prices are paid when the teachers' subject matter competence is itself compromised by deficiencies of prior education or ability?" (1986, p. 8, emphases added)

Moving even further, he advanced a conceptualization of teacher knowledge consisting of three distinct types of knowledge central to the work of teaching (Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). His first category, *subject matter knowledge*, represents the average content knowledge that an educated adult should possess, coupled with an ability to explain why particular propositions hold, why they are true, and how they relate to other propositions. The second category, *pedagogical content*

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<sup>17</sup> A critique of the conceptual and methodological limitations of studies of this paradigm appears in Confrey (1986).

<sup>18</sup> A case in point is the *Active Mathematics Teaching* program developed by Good and his colleagues (Good et al., 1983). Synthesizing several of the teaching behaviors proven to be effective, these scholars proposed a lesson structure for the teaching of mathematics that detailed the time allotted to several parts of the lesson (e.g., warm-up, individual work, etc) and several teacher behaviors throughout the lesson. The comparison of the performance of students who were in classes in which this program was applied to that of students in a control group provided evidence suggesting that this program was effective.

*knowledge*, comprises “a special amalgam of content and pedagogy” and is considered “the province of teachers, their own special form of professional understanding” (Shulman, 1987, p. 8). Components of this knowledge include different forms of representations, analogies, illustrations, examples, explanations, and demonstrations that a teacher can employ, as well as an understanding of what makes learning of specific ideas easy or difficult and how certain strategies or approaches scaffold or impede student learning (Shulman, 1986). The third category, *curricular knowledge*, pertains to teachers’ knowledge of how the curriculum topics are arranged within and across grades and teachers’ awareness of available curriculum materials conducive to learning.

Shulman’s ideas catalyzed the beginning of what could be characterized a new era in research on teacher knowledge, an era marked by two critical features. First, researchers started paying more attention *to the connections between teacher knowledge and teaching* largely absent from the educational production function studies. Second, scholars attended more closely to the *disciplinary demands* of certain subjects; such a serious consideration of the subject matter taught was largely absent from the process-product studies, a gap Shulman identified as “a missing paradigm” in research. Catalytic to attending both to the teaching itself and the content being taught was the notion of *pedagogical content knowledge*, which turned out to be the most influential of Shulman’s ideas. As Ball et al. (in press) observe, researchers in a wide variety of subject areas structured their research upon this notion.<sup>19</sup>

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<sup>19</sup> For instance, Grossmann (1990) explored the content knowledge and the pedagogical content knowledge of six beginning high-school English teachers in the domain of *literature*. Analyzing data collected from interviews and observations of lessons taught by these six beginning teachers, she concluded that subject knowledge, although important, does not suffice to teach literature effectively. Instead, teachers need a type of knowledge unique to teaching – what Shulman called pedagogical content knowledge – which according to Grossman, cannot be developed just by taking content courses in a certain domain or by simply building on one’s teaching experience. In *science*, Magnusson, Krajcik, and Borko (1999)

In mathematics, scholars explored teachers' knowledge with respect to a wide array of specific topics that span different content strands. Among the topics examined were place value and operations with whole or fractional numbers (Ball, 1988, 1990a, 1991; Borko, Eisenhart, Brown, Underhill, Jones, et al., 1992; Ma, 1999), ratios and proportional reasoning (Sowder, Philipp, Armstrong, & Schappelle, 1998; Thompson & Thompson, 1994, 1996); slopes and graphing (Ball, 1988; Ma, 1999); the relationship between the perimeter and area of rectangles (Ball, 1988); and the van-Hiele levels of building understanding of geometric concepts (Swafford, Jones, & Thorton, 1997).<sup>20</sup>

In line with Shulman's arguments, all these studies provided empirical evidence suggesting that the knowledge of the content itself, although important, does not suffice to teach mathematics effectively. Instead, these studies suggested that teachers of mathematics need to understand the content profoundly, in ways that allow them to see connections among and between ideas and to present it in a manner that supports student learning. Collectively, these studies also helped illuminate several other issues such as the knowledge with which prospective teachers enter preservice education programs (e.g., Ball, 1988, 1990), the differences in the knowledge held by in-service teachers in countries with different student performances in international comparative studies (e.g., An, Kulm, & Wu, 2004; Ma, 1999), or the effects that interventions have on enhancing teachers' knowledge (e.g., Sowder et al., 1998; Swafford, et al., 1997). In doing so, these studies not only attended seriously to the content, as recommended by Shulman, but also raised and addressed some of the questions he thought unasked.

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advanced a model for studying the pedagogical content knowledge needed for teaching science; they also argued for the practical implications such a model could have for the development of preservice and in-service teacher education programs in this subject matter.

<sup>20</sup> For a more comprehensive review of the studies that examined teachers' knowledge of several mathematical topics see Ball et al. (2001, pp. 444-447).

Their contribution notwithstanding, these studies still left unanswered questions regarding *the links between teachers' knowledge and their teaching*. For instance, how does teacher knowledge support their efforts to provide explanations? Plan their lessons? Respond to students' questions or misunderstandings? Does knowledge matter for a teacher's selection and use of representations? If so, how does it inform relative decisions and actions? Addressing such questions required refocusing the research inquiry from "one about teachers and what teachers know to one about teaching and what it takes to teach" (Ball et al., 2001, p. 452), a refocusing that would allow a closer examination of *the practice of teaching mathematics* without losing sight of the knowledge necessary to perform the multitude of tasks embedded in teaching the subject.

Starting from the late 1990s and responding to this call, a growing body of scholars has set out to explore the demands of teaching mathematics and the knowledge that can support teachers in their work. Although these researchers have conceptualized the knowledge that teachers need to teach mathematics quite differently and pursued various approaches to explore how teacher knowledge informs instruction, their work shared a common denominator: by and large, they all have studied *knowledge necessary for the work of teaching*. Examples of the work in this realm include the *Learning Mathematics for Teaching* (LMT) work on *Mathematical Knowledge for Teaching* (MKT); Philipp and Sowder's project *Integrating Mathematics and Pedagogy*; Ferrini-Mundi and colleagues' work on understanding the knowledge necessary for teaching algebra; and Sherin's work on analyzing videotaped lessons as a means to foster teacher knowledge.<sup>21, 22</sup> The availability of different concepts and approaches in conceptualizing

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<sup>21</sup> All these projects have been funded by the National Science Foundation (NSF). *Integrating Mathematics and Pedagogy* (IMAP) seeks to explore how PSTs' knowledge and beliefs of mathematics influence

and studying teachers' knowledge of mathematics begs the question: Why focusing on the work associated with MKT to explore the inquiry of the present study? In what follows, I justify the selection of this type of knowledge for the purposes of this study; I also elaborate the work associated with this type of knowledge.

*Focusing on Mathematical Knowledge for Teaching: Justifying the First Design Decision*

Three reasons informed the selection of MKT as the type of teacher knowledge the study considers. First, MKT is *rooted in practice*, thus addressing one of the main limitations of the educational production function studies considered above. This aspect of MKT is pivotal, because the present study aims at examining how teacher knowledge informs their performance in certain *practices*. Second, the work on MKT elaborates Shulman's earlier work, and more critically, closely attends to the specific *disciplinary demands* of teaching mathematics. In so doing, this work addresses one of the main limitations of the process-product studies, which largely ignored the disciplinary demands of teaching certain subjects. This attention to the disciplinary demands of teaching mathematics is key for the purposes of the present study, because, as I argue in the third section, *establishing challenging learning environments in mathematics cannot rely simply on generic teaching skills*. Third, during the last decade a conglomerate of

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their instruction. This project was also designed to explore the effect of early field experiences and mathematics content courses on PSTs' knowledge and beliefs (for more information, see <http://www.sci.sdsu.edu/CRMSE/IMAP/overview.html>). The *Knowledge of Algebra for Teaching project* aims at developing instruments to measure the knowledge that is needed for teaching algebra. The researchers in this project analyze instructional materials, interviews with teachers, and videos of teaching, attempting to map the terrain of the knowledge that is needed to teach algebra (see Ferrini-Mundi, Floden, McCrory, Burrill, & Sandow, 2007). Sherin's project aims at understanding the role of video in teacher learning. She videotapes teachers while teaching math lesson and then organizes "video club meetings" during which these teachers watch and discuss their own teaching or that of colleagues. This approach offers, according to Sherin (2002), a venue for exploring the ways in which teacher knowledge plays out in the work of teaching.

<sup>22</sup> For a more extensive list of the projects that focused on teacher knowledge with respect to teaching, see Hill et al. (2007).

researchers working on MKT have developed, used, and validated instruments to explore the different facets of this knowledge. These instruments allow researchers to obtain *valid measures* of teacher knowledge. Given that one of the greatest challenges faced by researchers working in the area of teacher knowledge relates to developing and validating instruments to measure this construct,<sup>23</sup> the availability of such instruments is critical; it is more so for this study which seeks to explore the association between teacher knowledge and their performance in certain practices.

The fact that this type of knowledge and its associated work incorporates these three traits – rooted in practice, disciplinary-based, and validly measurable – distinguishes it from other aforementioned notions or approaches on teacher knowledge that share some, but not necessarily all, of these traits. I will now elaborate each of these three MKT-features in turn.

#### *Mathematical Knowledge for Teaching: Teacher Knowledge Rooted in Practice*

In the late 1990s, strongly convinced that teaching mathematics effectively requires more than just content knowledge or pedagogical knowledge, Ball and Bass set out to understand the mathematical knowledge necessary to teach elementary school mathematics well (Ball & Bass, 2000, 2003a, 2003b). Their approach lied exactly at the opposite end of that followed by the researchers in the educational production function studies, in which it was not only assumed that the proxies used reflected teachers' knowledge, but also that this presumed knowledge could support teachers in the multitude of tasks related to teaching the subject. Ball and Bass scrutinized *actual*

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<sup>23</sup> This difficulty is vividly captured in Monk and King's (1994) acknowledgment that, although proxies are not ideal for measuring what teachers know, it is often hard to obtain relative information through alternative measures: "We would have much preferred data measuring more directly what teachers know about the subject(s) being taught, but this kind of information is difficult to obtain" (p. 56).

*records of practice* – videotapes of teaching mathematics in elementary school grades, copies of student work, teacher plans, notes, and reflections<sup>24</sup> – seeking to untangle and understand the several demands embedded in teaching mathematics as well as the configurations of knowledge necessary to efficiently respond to these demands.

A key product of this work was the notion of *MKT*, a type of professional knowledge rooted in the practice of teaching that synthesizes the multitude of requirements entailed in the work of teaching mathematics. According to Ball and colleagues' conceptualization (Ball et al., 2005, in press), teachers of mathematics need to be able to explain why certain mathematical algorithms are applicable and make sense;<sup>25</sup> they should be capable of analyzing, evaluating, and modifying textbook tasks so that they can meet their lesson goals but also respond to their students' needs; they need to diagnose the difficulties students might have when assigned a task; they should be able to listen to and analyze their students' thinking and decipher the source of students' errors; and they need to make judicious selections of representations and use these representations wisely during instruction to help students develop meaning and see connections among and between different representational systems. Teaching mathematics also requires being able to make meaning of unconventional approaches

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<sup>24</sup> See Lampert and Ball (1998) for examples of such records of practice.

<sup>25</sup> The following two examples illuminate this claim. First, consider the algorithm of the division of fractions. While for most professions it might be sufficient to know that when dividing fractions one needs to “invert the second fraction and multiply,” such knowledge is not sufficient for the work of teaching mathematics. The teacher of mathematics needs to know not only the algorithm of the division of fractions but also why the algorithm works. Such an understanding will allow the teacher to present the content in ways that help students make meaning of the mathematical procedure at hand, but also build connections among and between different mathematical ideas. Second, take the example of long division. Whereas for professionals in other fields long division might only evoke distant school memories, the teacher of mathematics has a lot of questions to contemplate before teaching this topic. Why, while in all other operations with whole numbers (addition, subtraction, and multiplication) one usually starts with “manipulating” the digits in the units place and then moves to the digits on the left (e.g., tens, hundreds, etc), does division require one to start the other way around, namely from left to right? And what does it mean to “carry down” a number? What are the mathematical warrants for such an action in division and what makes a similar approach inapplicable to operations with other numbers?



students might propose when solving problems, as well as carefully selecting and sequencing mathematical examples and using mathematical language and symbols with care to support students' gradual construction of mathematical ideas. Finally, it encompasses activities such as writing and grading assessments, explaining class work to parents, and attending to concerns for equity.

Obviously, to efficiently perform all these tasks, teachers of mathematics need not only know a different corpus of mathematics than that construed by well-educated adults in other mathematically intensive professions, like engineering, architecture, or physics; they should know and understand mathematics differently than other professionals, in ways that can substantially support them in the multitude of activities entailed in teaching mathematics.

*Mathematical Knowledge for Teaching: A Disciplinary-Based Type of Teacher Knowledge*

Shulman's conceptualization of teacher knowledge, and particularly his idea of pedagogical knowledge for teaching, initially fermented the development of MKT. This ferment was enriched with Ball's earlier work (1988, 1990) which closely attended to the *disciplinary demands* of teaching mathematics. In analyzing teachers' subject-matter knowledge, Ball focused, in particular, on two types of knowledge: teachers' knowledge *of* mathematics and their knowledge *about* mathematics. The former type of knowledge captures teachers' substantive knowledge of mathematics concepts and procedures; the latter type captures teachers' syntactic knowledge of the field (cf. Schwab, 1964, 1978), and addresses questions such as where knowledge in mathematics comes from, how it is

constructed, how it changes, how truth is established in mathematics, and what it means to know and do mathematics.

The fermentation of Shulman's and Ball's ideas led to the development of some initial hypotheses about the knowledge that teachers need in the work of teaching mathematics, which were then empirically tested by closely observing and analyzing the practice of teaching, as explained above. This process, though, was bidirectional: the initial theoretical framework offered the lens through which to analyze practice; the analysis of the practice of teaching helped refine the initial hypotheses. This bidirectional process led to the advancement of a theoretical framework that conceptualizes MKT as comprising *four distinct domains*: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching.<sup>26</sup> I briefly elaborate each of these components below.

*Common Content Knowledge* (CCK) is the “mathematics knowledge and skill used in settings other than teaching ... [and thus are] not special to the work of teaching” (Ball et al., in press). For instance, the teacher of mathematics, like every well-educated adult, needs to be able to compute  $35 \times 25$  accurately, to identify the power of 10 that is equal to 1, to name a number that lies between 1.1 and 1.11, to recognize that the square is a special type of rectangle or that the diagonals of a parallelogram are not always perpendicular (Ball et al., in press; Hill & Ball, 2004). Despite being a necessary ingredient of teachers' knowledge, common content knowledge is not sufficient for the work of teaching. It is here that the remaining three components of MKT come into play.

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<sup>26</sup> Recently, two additional components of this knowledge have been proposed (i.e., *Horizon Knowledge* and *Knowledge of Content and Curriculum*, see Ball et al., in press); however, these components are yet underdeveloped.

*Specialized Content Knowledge* (SCK) is the mathematical knowledge and skill needed by individuals engaged in teaching mathematics. Such knowledge allows teachers to appraise students' methods of solving problems, assess novel approaches that students propose, and determine whether these approaches are generalizable to other problems. It also supports teachers in identifying patterns of student error, explicating why certain algorithms work or make sense, explaining a mathematical idea by selecting appropriate examples and representations, linking representations and models to their underlying meaning, evaluating students' explanations and justifications, and choosing and developing workable definitions (Ball et al., in press; Hill & Ball, 2004). For instance, although a well-educated adult is expected to be able to divide two fractions, he or she is not expected to know if alternative algorithms are suitable for dividing fractions. However, a competent mathematics teacher could benefit from possessing such knowledge.

*Knowledge of Content and Students* (KCS) intertwines knowledge of mathematical notions with knowledge of how students think or come to understand these ideas. This knowledge renders teachers capable of anticipating plausible student thinking trajectories, predicting student difficulties when engaged with specific mathematical ideas or processes, and hearing and interpreting students' thinking (ibid). Evidently, to efficiently engage in these activities, teachers need not only understand the content, but also be familiar with students' mathematical thinking and common student misconceptions and errors. For example, knowing that students often consider 5.8 smaller than 5.67, because in the "world of whole numbers" 67 is larger than 8, helps the teacher

easily recognize such errors in student work, understand their source, and design suitable interventions.

*Knowledge of Content and Teaching* (KCT) refers to the type of knowledge that combines knowing about teaching and knowing about mathematics. This knowledge aids teachers in selecting and sequencing examples to gradually lead students to develop certain mathematical ideas or in considering the relative strengths and limitations of available representations and models. For instance, being aware of the different models of subtraction (e.g., take-away and comparison) and division (i.e., partitive and measurement), and more so knowing the limitations and the affordances of each, equips teachers with the roadmap necessary for structuring teaching in ways that support learning.

These four components of MKT are apparently not just an expansion of Shulman's ideas. Collectively, they elaborate Shulman's conceptualization of teacher knowledge *by seriously attending to the disciplinary demands of teaching mathematics*. Consider, for example, the practice of postulating and testing conjectures which, according to Lakatos (1976), is fundamental for the generation and validation of mathematical knowledge.<sup>27</sup> Or think about making educated choices of representations and examples in solving problems, a distinct characteristic of efficient problem solvers, as Polya's (1957) seminal work on problem solving suggests. Consider, finally, the work that teachers do with their students while searching for patterns or making generalizations. This work lies squarely at the heart of doing mathematics, which is itself regarded as the science of generating and exploring patterns (cf. Devlin, 2000; Thurston, 1994).

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<sup>27</sup> For examples of the application of this practice in teaching see Lampert (1990, 2001).

*Mathematical Knowledge for Teaching: A Validly-Measurable Type of Knowledge*

Besides advancing the notion of MKT and mapping the terrain of this knowledge by seriously attending to the disciplinary demands of the subject and its teaching (the two strengths of MKT reviewed above), starting from 2001, Ball and Bass have worked closely with psychometricians, practicing teachers, mathematics educators, and other scholars in the context of the *Learning Mathematics for Teaching* (LMT) project to develop instruments to measure this knowledge.<sup>28</sup> The development of these instruments comprises an additional strength of the work pursued in this arena, because it allows for the obtainment of valid measures of teachers' knowledge with respect to teaching the subject. So far, the analysis of data collected via these instruments has provided converging evidence that validates the theoretical assumptions underpinning the construct of MKT: that it is multidimensional; that it relates to student learning; and that it integrates mathematical reasoning and pedagogical thinking.

Hill, Schilling, and Ball (2004), for example, found that items used to measure Common Content Knowledge loaded to a different factor than those related to measuring specialized content knowledge, a finding that supports the claim that MKT is a multidimensional rather than a unidimensional construct. Teachers' MKT was also found to be related to gains in students' performance, after controlling for key student- and teacher-covariates, such as students' socioeconomic status and teachers' credentials and experience (Hill et al., 2005). Specifically, students taught by teachers in the top

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<sup>28</sup> So far, the LMT research group has developed hundreds of items to examine elementary teachers' Common Content Knowledge and Specialized Content Knowledge in three areas: numbers and operations; patterns, functions, and algebra, and geometry. Recently, the group has extended the scope of their work to cover other mathematical strands, such as data analysis and probability and the other two types of knowledge (i.e., Knowledge of Content and Students and Knowledge of Content and Teaching); the group has also developed items for middle-school teachers.

knowledge quartile were found to experience gains in their scores equivalent to about two weeks of instruction, compared to their counterparts taught by average teachers. Additionally, the effect of teacher knowledge was comparable to that of students' socioeconomic status, leading these scholars to conclude that "while teachers' mathematical knowledge would not by itself overcome the existing achievement gap, it could prevent the gap from growing" (p. 44). In a different study, Hill and Ball (2004) also found gains in teachers' MKT as a consequence of their participation in professional development seminars; these gains were larger for those teachers who attended programs that focused on proof, analysis, and use of representations, all of which integrate mathematical reasoning and pedagogical thinking.

These findings collectively suggest that the developed MKT instruments tap a multidimensional type of knowledge necessary for the work of teaching, and consequently for student learning. Therefore, these findings not only provide evidence that supports the association between the first and fourth link depicted in the chain in Figure 2.1,<sup>29</sup> but also address – at least to some extent – the three critical questions that faced scholars in the area of teacher knowledge: whether teacher knowledge matters for student learning, what knowledge matters for this learning, and how to measure it.

Having explored the association between the first and the last link of the chain in Figure 2.1 and having justified the first design decision of the study, I now consider the type of student learning forged in rich and intellectually challenging environments.

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<sup>29</sup>Additional evidence about this association comes from a recent study conducted at the other side of the Atlantic, by the COACTIV (*Cognitive Activation*) researchers in Germany. In this study, Baumert and colleagues (cited in Krauss et al., 2008) explored the Pedagogical Content Knowledge (PCK) of a representative sample of secondary mathematics teachers, whose students participated in the PISA (Program for International Student Assessment) 2003-2004 study. These scholars found that students in classes taught by teachers with higher PCK scores exhibited significantly higher gains in mathematics.

## Building Rich and Cognitively Challenging Learning Environments: What Is Taught and How Affects What Gets Learned

This section considers student learning in mathematically rich and cognitively challenging environments. Like its predecessor, this section has a twofold goal. First, by discussing how the structuring of rich and intellectually challenging environments influences student learning, it seeks to theoretically establish the association between the third and fourth link of the chain depicted in Figure 2.1. Since this discussion also highlights the critical role of teachers in establishing such environments, this section also suggests an association between the former and the latter two links of the chain. Second, this section aims at justifying the second design decision of the study, namely to frame the research inquiry around the work associated with the MTF.

### *Student Learning in Rich and Intellectually Challenging Environments: Focusing on Instructional Tasks and Their Enactment*

The process-product studies previously discussed should be credited for making a significant first step toward uncovering what transpires during instruction. However, these studies were deficient in two ways. First, as discussed above, they approached teaching from a *generic* perspective, and thus, failed to consider the disciplinary demands of teaching different subject matters. Second, they largely ignored the cognitive complexity and richness of the content presented and worked on during math lessons.

Responding to the prevailing lack of attention to the cognitive demands of the content in which students are engaged, in the early 1980s Doyle proposed that scholars seriously attend to the type of learning environments that teachers build for student learning. Central to his proposal was the notion of instructional or academic tasks, which

he considered “the basic treatment unit in classrooms” (Doyle, 1983, p. 162), and consequently, the “proximal causes of [student] learning from teaching” (Doyle, 1988, p. 168). For Doyle, focusing on the tasks in which students engage during instruction offered a promising path to explore the associations between teaching and learning, on the following two premises:

First, the concept of task connects student information processing with environmental conditions. Knowing the task that students are working on gives access to the kinds of *cognitive processes that are likely to be necessary to accomplish the task*. Second, a task is more than just content. It also includes the situation in which content is embedded. It is possible, therefore, to *incorporate teaching variables into the conceptualization of classroom conditions that affect achievement*. (Doyle, 1983, p. 162, emphases added)

The first fundamental premise of Doyle’s approach was that instructional tasks largely determine the work that students *do* in mathematical classrooms and hence the meaning they come to construct of the ideas explored. By advancing this argument, Doyle was suggesting a critical shift from a quantitative approach to studying instruction – one manifested in the work of the process-product researchers – to a more qualitative perspective. For example, although it might be true that the time students spend on academic work matters for student learning (what the process-product researchers called “time-on-task”), this instructional aspect tells very little about the quality of their learning. What are students doing while they are on-task? Are they reproducing previously learned facts? Are they practicing well-established formulas and procedures? Or are they applying their prior knowledge to solve new tasks, to identify patterns, to make generalizations and test predictions? Addressing these questions was, according to Doyle, critical for understanding student thinking and learning. Hence, he suggested that scholars explore the *cognitive demands* of tasks – namely, the cognitive processes that students are required to use in accomplishing these tasks. For instance, some tasks cannot



promote higher level thinking, if they require only reproducing previously learned facts and rules or applying formulae and algorithms. Other tasks have more potential to promote understanding, since they expect students to engage in higher cognitive processes, such as interpreting problems, developing plans, making judicious choices of operations, drawing inferences, and explaining and justifying answers.

The cognitive demands of a task, however, cannot on their own determine the cognitive level at which students engage with it. Rather attention should be given to how the task is worked on during instruction, because, although the task itself creates the space for particular learning opportunities, it is the work that is done on and around the task that largely determines whether these learning opportunities are seized.<sup>30</sup> Take, for example, Mrs. Oublier, as discussed in the introductory section of the first chapter. Although Mrs. Oublier was furnished with rich mathematical tasks that could potentially engage her second graders in high-level thinking, as Cohen (1990) argues, she mainly proceduralized these tasks and had students approach them algorithmically, thus eliminating the cognitive challenge.

There is, however, the other side of the coin. Consider, for example, the following task that Lampert assigned to her fifth graders during the first day of school: “Find additions with answers between the answers to  $3 + 4$  and  $7 + 9$ ” (Lampert, 2001, p. 67). At first sight, one could argue that this task could afford nothing but just opportunities to retrieve learned number facts: three plus four is seven, and seven plus nine is sixteen; thus, any addition yielding a sum between seven and sixteen could work. Although this is

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<sup>30</sup> To emphasize that tasks offer only probabilistic opportunities for student learning, Christiansen and Walther (1986, cited in Watson & Mason, 2007) distinguished between mathematical *tasks* and mathematical *activity*, the latter representing the activity which arises as the tasks get enacted by the teacher and the students in the classroom.

a viable approach to solving this task, Lampert asked her students to work on this task quite differently. She asked them to find solutions to the problem without figuring out the sums of the two given additions, an approach that called for pattern identification and reasoning. After discussing different solutions to the problem, Lampert's class concluded that any addition that included any number between three and seven (i.e., the first addends of the given additions) and any number between four and nine (i.e., the second addends of the given additions) as its addends would work, under certain conditions.<sup>31</sup>

These two examples underline the key role that teachers play in establishing mathematically rich and intellectually demanding environments. Teachers' critical role in enacting these tasks with their students captures Doyle's second fundamental premise, according to which focusing on the work done on tasks allows teaching variables to be incorporated into the exploration of the factors that affect student learning. This idea is elaborated in the *Principles and Standards for School Mathematics* (NCTM, 2000), where it is suggested that

[w]orthwhile tasks alone are not sufficient for effective teaching. Teachers must also decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them, and thus, eliminating the challenge. (p. 19)

It then follows that research efforts to investigate student learning should not be limited to an exploration of tasks as they appear in the curriculum, but should rather focus on teachers' and students' work on these tasks.

These two fundamental premises of Doyle's work on tasks not only point to an association between the third and the fourth links of the chain presented in Figure 2.1, but also suggest associations with the other two links of this chain (i.e., teacher knowledge

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<sup>31</sup> For more on how "ordinary tasks" can be transformed into high-level tasks, see Silver, Kilpatrick, and Schlesinger (1990).

and teaching practices).<sup>32</sup> Several studies which followed different methodological approaches to explore the enactment of tasks during instruction provide converging evidence corroborating these associations.

Hiebert and Wearne (1993), for example, examined the teaching of place value and multi-digit addition and subtraction in six second-grade classes. In four of these classes, teachers followed the conventional textbook approach: they demonstrated or reviewed how to find the answer to a given problem and then had students work individually to solve several similar problems. In the remaining two classes an alternative approach was used, which emphasized the construction of relationships between place value and computation strategies. The examination of student learning gains revealed that the students in these latter two classes outperformed their counterparts in place-value understanding, as well as in routine and novel computation. Although Hiebert and Wearne were careful enough to not argue that task selection and enactment *determines* student learning, they did claim that *instructional tasks and their enactment appear to mediate the relationship between teaching and learning*.

Additional evidence regarding the effect of instructional tasks and their enactment on student learning comes from the two *TIMSS* video-studies (Stigler & Hiebert, 1999; Hiebert et al., 2003) that examined the instructional practices in countries with different student achievement. Although mathematics instruction in these countries varied significantly, certain features prevailed in the teaching of mathematics in all high-achieving countries. Specifically, even though the teachers across the high-achieving

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<sup>32</sup> Several scholars appear to have ignored or underestimated this fundamental premise. For instance, many of the studies that explored the effect of *Standards*-based curricula have largely attributed student learning gains to the rich mathematical tasks embedded in these curricula without actually examining how these curricula were enacted. For a more detailed discussion of this issue, see Stein et al. (2007).

countries employed intellectually demanding problems with different frequency (e.g., such problems were more frequently employed in Japan and in the Netherlands than in Hong-Kong or the Czech Republic), in most cases, the teachers in these countries enacted the problems in a manner that maintained the challenge. In contrast, with almost no exception, when such problems were employed in the United States, the teachers succumbed to students' persistent requests for help and lessened the demand. Furthermore, even though certain features appeared to traverse mathematics instruction in the participating countries, a closer inspection of the teaching of mathematics in the top-ranking countries revealed that these instructional features were geared toward constructing meaning (Hiebert et al., 2005; Jacobs et al., 2006; Stigler & Hiebert, 2004). For instance, in Czech lessons, review was part of a broader system that included more challenging content and drew students' attention to conceptual aspects of mathematics. While reviewing old material, Czech teachers asked their students to explain why certain procedures are suitable for solving a given task. Reviewing old material was a prevalent characteristic of the American lessons as well; yet, more often than not, American students were asked to practice familiar procedures and only rarely were they required to justify their answers, identify patterns, or make generalizations. Likewise, teachers in Hong Kong placed more emphasis on procedures compared to their American counterparts; yet, the former teachers also drew their students' attention to the conceptual underpinnings of the procedures considered during their math lessons. Such an instructional feature was rarely observed in American lessons.

Further evidence that supports the claim that immersing students in cognitively challenging environments shapes student learning comes from a recent study that

examined the effects of teacher qualifications and practices on student achievement. In this study, Croninger, Lareson, and VonSecker (2006) used multilevel models to analyze data collected from roughly 500 mathematics lessons observed in 66 mathematics classes that largely enrolled students from disadvantaged socioeconomic backgrounds. Their analysis showed that student achievement measured on a state-mandated mathematics test was not associated with any of the variables related to teacher preparation, degrees, and years of experience. In contrast, it was significantly related to a variable these scholars used to capture the cognitive level at which the content was experienced: the ratio of the average percentage of time that teachers made higher cognitive demands on students (e.g., asked students to solve higher order tasks or share and discuss multiple solutions to a problem) to the average percentage of time that teachers made lower demands on students (e.g., had students solve routine tasks and elicited only simple answers from them). This effect was more prevalent in the high-poverty classes,<sup>33</sup> which suggests that students in disadvantaged settings might benefit more than the students in less affluent settings from instructional practices centered on solving demanding mathematical tasks and on asking students to provide alternative strategies and methods. However, as these researchers noted, such practices were rarely observed in disadvantaged school settings, leading them to lament that “what appears to influence achievement most occurs so infrequently in mathematics lessons at the upper elementary grades” (p. 18).

Insights about how task selection and enactment impacts the quality of teaching and subsequently, student learning can also be gleaned from Boaler’s ethnographic study

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<sup>33</sup> This effect was not significant for the low-poverty classes. This finding might be an artifact of the sample of the study (which was skewed toward more disadvantaged students) and/or the collinearity effects with the variable “linking content” used in the study, which also captured the cognitive demands imposed on students.

(2002) of the mathematics taught and learned in two schools that pursued different instructional approaches. Mathematics instruction in Amber Hill, the first school considered in her study, placed more emphasis on students' obtaining correct answers. In most of the cases, the teachers in this school delivered the new content in a clear and structured way, usually without using more than one approach. They then had students work on related exercises to practice the introduced mathematical ideas. Students were rarely, if ever, provided with opportunities to draw connections among and between the different ideas or to situate these ideas within the broader territory of mathematics. Attempting to ensure that their students would experience success in mathematics, these teachers reduced the cognitive demands of the tasks, by breaking the problems down into smaller parts, posing leading questions, and essentially walking students to the answer. By doing so, teachers and students were trapped in a vicious circle, the former believing that students would not or could not think, and the latter avoiding thinking, and thus, confirming their teachers' expectations. Boaler recounts,

In the majority of [the] lessons I watched, the teachers would respond to the students' inability to answer questions by offering them a multiple-choice question, with one of two correct answers (e.g., "Well is it 4 or 5?"). The students would select an answer, and if this was right the teachers moved on, "So, is it the length or the width?", and so it proceeded. If students selected the wrong answer, the teachers would repeat it using a disbelieving tone, which was an indication that the students should go for the other answer. (p. 27)

Teaching in Phoenix Park, the second school considered in the study, was quite different. Not only did the teachers in this school employ rich open-ended tasks, but they also helped their students learn how to learn from these tasks. In contrast to the students in Amber Hill, the students in this school were also expected to explain their reasoning and justify their answers. The teachers at Phoenix Park would additionally pose insightful questions that led students to the heart of mathematical issues. "When students were stuck," Boaler recounts, "teachers asked them to explain what they knew so far, they

listened to students carefully, and they selected appropriate questions and interventions that helped students move forward” (p. 83).

Using several measures of student learning and performance, Boaler found that the students in the latter school outperformed those of the former school. More importantly, the students at Phoenix Park appeared to develop a *qualitatively* different kind of mathematical knowledge. In Boaler’s words,

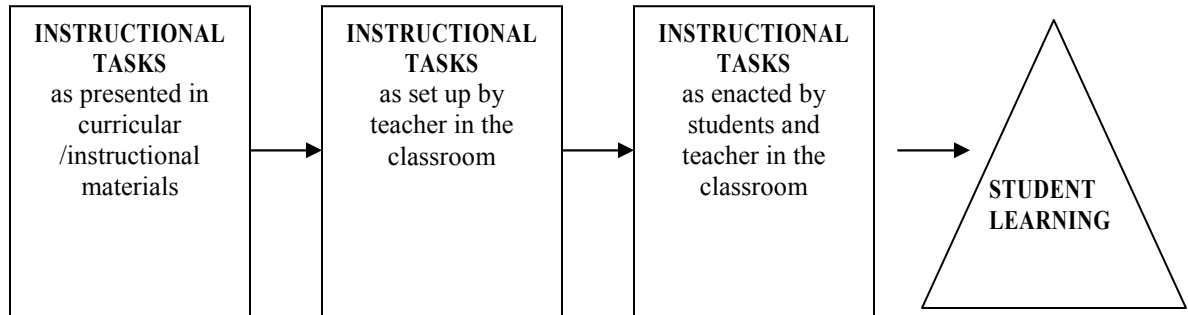
[t]he Phoenix Park students did not have a greater knowledge of mathematical facts, rules, and procedures, but were more able to make use of the knowledge they did have in different situations. [These] students showed that they were flexible and adaptable in their use of mathematics, probably because they understood enough about the methods they were using to utilize them in different situations (p. 104).

In sum, regardless of the differences in their theoretical frameworks and research paradigms, the studies examined above provide converging evidence that engaging students in mathematically rich and intellectually demanding environments affects both how much they learn as well as the quality of their learning. Collectively, these studies also point to the critical role that teachers have in crafting and sustaining rich and intellectually challenging settings. The question that then arises is how to more closely explore and better understand teachers’ work in creating such environments. I address this question by drawing on research on the Mathematical Tasks Framework (MTF).

*The Unfolding of Tasks during Instruction: The Mathematical Tasks Framework and Its Associated Work*

Building on Doyle’s work on tasks (1983, 1988, 1992) and particularly on his caveat that in exploring the effect of instructional tasks on student learning one needs to consider not only the content of tasks, but also teachers’ role in presenting and enacting these tasks with their students, the QUASAR researchers proposed the *Mathematical Tasks Framework* (MTF). The MTF is an analytic framework that traces the cognitive

demands of tasks in different phases, and explores how their unfolding affects student learning (see Figure 2.2).



*Figure 2.2. The Mathematical Tasks Framework: A representation of the unfolding of mathematical tasks during instruction (adapted from Henningsen & Stein, 1997, p. 528).*

According to the MTF, instructional tasks pass through three phases: first, as they are written in curriculum materials, second, as they are presented by the teacher in the classroom, and third, as they are enacted by students and the teacher (Stein & Smith, 1998; Stein et al., 2000). The manner in which tasks are enacted during instruction shapes what students learn. As Stein and Smith (1998) explain,

Tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for student thinking. (p. 269)

To systematically explore the cognitive demands of tasks during their implementation, the QUASAR researchers proposed the *Task Analysis Guide*, a classification scheme that categorizes the cognitive demands of tasks. In developing this scheme, Stein and colleagues (Stein & Smith, 1998; Stein et al., 1996) adopted the first two categories proposed by Doyle that pertain to lower intellectually demanding tasks (e.g., memorization tasks and procedural or routine tasks). Appropriating Doyle's third category (comprehension or understanding tasks), they created two categories of tasks



that impose high cognitive demands on students. These tasks concern either working on procedures that are linked to their underlying meaning (i.e., procedures with connections to concepts, meaning, and/or understanding) or solving complex problems that do not lend themselves to algorithmic and easily predictable solution paths (i.e., doing mathematics). After analyzing a series of mathematics lessons, the QUASAR scholars enriched this scheme, by including another two categories. The first category captures the cases in which students are involved in activities that have no mathematical focus (non-mathematical activities); the second category describes the cases in which students engage in unsystematic or unproductive explorations of tasks that could potentially be enacted at a higher level. Figure 2.3 presents the different categories of tasks according to their cognitive demands.

<b>Nonmathematical demands</b>	<b>Low-level demands</b>	<b>High-level demands</b>
<p><i>No mathematical activity is involved:</i> Students do not engage in any activity that has mathematical focus (either students are off-task, or the task requires no mathematical thinking).</p>	<p><i>Memorization:</i> Students reproduce previously learned facts, rules, or definitions or commit facts, rules, or definitions to memory.</p> <p><i>Procedures without connections to concepts, meaning, and/or understanding:</i> Students employ a procedure to solve the task; however, they do not engage in considering how and why the procedure works or is suitable for the particular task.</p> <p><i>Unsystematic and/or nonproductive exploration:</i> Although these tasks could potentially engage students in higher level thinking, students fail to make systematic and sustained progress in developing mathematical strategies or understanding.</p>	<p><i>Procedures with connections to concepts, meaning, and/or understanding:</i> Students solve the task by using one or more procedures which draw their attention to deeper mathematical ideas.</p> <p><i>Doing mathematics:</i> Students engage in solving complex tasks that require non-algorithmic thinking and for which there is no predictable solution path.</p>

*Figure 2.3. The Tasks Analysis Guide: A scheme for analyzing the cognitive demands of tasks during enactment (based on Stein & Lane, 1996, pp. 58-59).*

Using the MTF and the Task Analysis Guide, Stein and colleagues analyzed instructional episodes from 144 middle-grade mathematics lessons, focusing on the unfolding of tasks during instruction (Henningsen & Stein, 1997; Stein et al., 1996). Their analysis showed that teachers were more successful in selecting high-level tasks and presenting them as such than in preserving the cognitive complexity of the tasks during enactment. Three fourths of the tasks used in these lessons were cognitively demanding and were presented as such. However, more than half of these tasks (58%) declined into procedures without connections or unsystematic exploration during their enactment.

The extent to which the cognitive complexity was preserved during instruction was found to significantly affect student learning. Specifically, Stein and Lane (1996) found that student learning gains were relatively small if students were engaged in algorithmic instructional tasks that could be solved with a single, easily accessible strategy. Learning gains were moderate in classes in which high-level tasks were selected, but their enactment failed to preserve their cognitive demands. Finally, students who were in classes in which challenging mathematical tasks were selected and enacted as such displayed the highest learning gains.

The purpose of the QUASAR project was not only to investigate if and how rich and intellectually challenging environments affect student learning but also to intervene and create such environments for students in disadvantaged urban schools. The effects of this experimentation exceeded expectations; as one of the participating teachers put it, “it was a revolution of the possible” (Silver & Stein, 1996). Specifically, the students in the intervention classes gradually developed an increased capacity for complex mathematical

thinking and reasoning; they performed better than other students in disadvantaged urban districts on the National Assessment Evaluation Progress (NAEP) test and they did fairly well compared to the national NAEP sample. Finally, an increased number of these students were also qualified for placement in ninth-grade algebra courses and did well when taking such courses.

### *Justifying the Second Design Decision*

The work of QUASAR researchers associated with the MTF not only provides tools to closely explore how teachers can affect the structuring of rigorous and intellectually challenging environments for students, but also provides evidence suggesting that this work is critical to student learning. This twofold contribution of the QUASAR work justifies the *second design decision* of the study, namely to draw on the MTF to explore the association between teachers' knowledge and their performance in teaching practices considered conducive to establishing rich and cognitively demanding environments.

In particular, the MTF offers an analytic framework that traces the cognitive demands of instructional tasks at three important phases of their unfolding. More critically, though, it also underscores the pivotal role that teachers can have in each of these phases. Hence, *the MTF helps identify three different areas for exploring the role of teachers' knowledge in structuring environments that support the learning of rich and challenging mathematics:*

- First, the researcher can explore how teachers *use curriculum materials* (i.e., teacher guides and student textbooks) *when planning their lessons*: What are they focusing on?

Which tasks do they select for their lessons and which do they omit? What informs these decisions?

- Second, the analyst can investigate how *teachers present these tasks to their students*: What explanations do they give? What examples do they use, and what representations do they employ in presenting the content? Do they modify the curriculum tasks they select? If so, in what particular ways, and what might inform their decisions? What implications might these modifications have on the cognitive level of the curriculum tasks?
- Third, the researcher can examine how teachers work with students during *the enactment of these tasks*: How do teachers respond to students' requests for support? What clarifications do they provide when students appear to stumble? How do the different decisions that teachers make during the enactment of tasks affect the cognitive level at which the content is experienced?

In addition to the three entrance points that the MTF offers for exploring teachers' work around instructional tasks, the work of the QUASAR researchers provides evidence that the selection, presentation, and enactment of tasks affect the quality of student learning. Thus, although the present study does *not* explore how teachers' decisions and actions influence what students learn (i.e., the last link of the chain presented in Figure 2.1) the findings of the QUASAR studies, along with those of studies also reviewed in this section, suggest that the association investigated in the present study should not be regarded as being unrelated to student learning.<sup>34</sup>

Thus far, I reviewed pertinent studies to illustrate the connection between teacher knowledge and student learning (i.e., first and last links in Figure 2.1); I also discussed

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<sup>34</sup> This argument warrants further investigation, which, though, lies beyond the scope of this study.

how the establishment of rich and challenging learning environments affects the quality of student learning (i.e., third and fourth links of the chain). In the next section, I review studies that point to the association between teacher knowledge and the structuring of rigorous and intellectually challenging environments. The review of these studies also helps identify different teaching practices that, on the one hand seem to be conducive to the establishment of such environments, and on the other hand, appear to be informed by teachers' knowledge.

### Establishing Mathematically Rich and Cognitively Challenging Environments: Making Teacher Knowledge Part of the Equation

The argument that teachers' profound knowledge of mathematics – manifested in their breadth and depth of knowledge – matters for teaching mathematics well is not new. In fact, centuries ago, Aristotle argued that “Teaching is the highest form of understanding,” to emphasize the depth of knowledge necessary for teaching. Similarly, a Chinese proverb holds that if teachers want to give students a cup of water, they need to have a bucket of water of their own (cited in An, 2004, p. 23), an assertion that points to the breadth of knowledge teachers must possess. Although few would refute that teachers' profound understanding of mathematics constitutes a critical resource for teaching mathematics effectively, the association between teacher knowledge and the quality of their instruction, and especially *the association between teacher knowledge and teachers' establishing of mathematically rich and challenging environments*, largely remains unclear. In fact, many studies have explicitly or more tacitly attributed certain deficiencies of teachers' instruction to teachers' impoverished knowledge without though obtaining direct measures of the knowledge itself. Fewer are the studies that explored

how teacher knowledge affects their instruction. Nevertheless, both types of studies – those of conjecture and those that provide more direct evidence – legitimize considering teacher knowledge part of the equation that determines their establishment of rich and cognitively challenging learning environments. These studies also point to how teacher knowledge might inform different teaching practices that support the establishment of such learning settings.

In this section, I review both types of studies to make the case that teacher knowledge – and particularly their MKT – matters for creating the kind of learning settings this study considers. In doing so, this section serves as a *theoretical justification* for the inquiry the study pursues. The reader is cautioned, though, that in reviewing past research, I deliberately single out one factor, namely teacher knowledge. Other factors that might inform relevant decisions and actions are considered in the fifth section of this chapter, in which I discuss the challenges inherent in exploring the association between teacher knowledge and their practices. Additionally, the reader is notified that the review that follows places more emphasis on studies that provide direct evidence about the association between teacher knowledge and their instruction. Hence, the unequal treatment of the aforementioned two categories of studies does not reflect the proportion of studies clustered in each of these two categories.

#### *Associations between Teacher Knowledge and Their Instruction: Studies of Conjecture*

A series of studies that examined teachers' practices and how teachers respond to calls to reform their teaching considered teacher knowledge a significant determinant of their structuring and delivering of mathematics lessons. Central among these studies are those published during the 1990s. In these studies, the researchers identified several

deficiencies in the lessons they observed, which they attributed to teachers' impoverished knowledge of mathematics and its teaching. Such deficiencies include teachers' selective use or modification of textbook tasks in ways that denuded them of their cognitive complexity (e.g., Mark's proceduralization of complex mathematical tasks, as discussed in Wilson, 1990); their failure to establish connections among several mathematical ideas (e.g., Mrs. Oublier's treatment of the notion of estimation as unrelated to other mathematical ideas and processes, as described in Cohen, 1990); and their imprecise explanations and problematic use of manipulatives (e.g., Carol's and Cathy's explanations of the two-digit subtraction that conflicted their use of manipulatives, as outlined in Ball, 1990b and Peterson, 1990, correspondingly). These studies also documented teachers' inappropriate use of representations and analogies that obscured or distorted the content (e.g., Sandra's use of the representation of a car going back and forth to explain inverse functions, as presented in Heaton, 1992, and Jim's analogy of "naked numbers" to support students' work on finding common denominators for adding fractions, as discussed in Remillard, 1992). They also pointed to critical mathematical errors that these teachers made during instruction (e.g., Valerie's inappropriate calculation of averages as described in Putnam, 1992 and Sandra's nonsensical computations of volume, as outlined in Heaton, 1992).

In sum, these studies point to various ways in which teachers' assumed lack of knowledge can compromise their instruction, and particularly their efforts to structure mathematically rich and cognitively demanding environments: through their problematic selection and modification of tasks; their inappropriate or inefficient use of manipulatives, analogies, and representations; their giving of explanations; their inability

to establish connections among several ideas and processes; and their making of serious mathematical errors. Although teachers' lack of profound mathematical knowledge was considered one of the main factors contributing to teachers' problematic instruction, the foregoing studies provide only tentative hypotheses about the role of teachers' knowledge in structuring rich and challenging learning environments largely because they did not obtain direct measures of teachers' knowledge.

More recent studies have also alluded to the pivotal role that teachers' knowledge plays in instruction. Focusing on teachers' use of *Standards*-based curricula – developed with the intent to help teachers create rich and challenging environments – these studies consider teacher knowledge to contribute significantly to teachers' enactment of these curricula. Consider, for example, the case of two middle-grade teachers, Gina and Bonnie, who both used the same *Standards*-based curriculum but differed significantly in its enactment (Manouchehri & Goodman, 2000). In enacting the textbook tasks, Gina used probing questions, synthesized answers, and generally helped her students construct meaning. Bonnie, on the other hand, structured her lessons around practicing algorithms and formulae. Manouchehri and Goodman largely attribute these differences in practice to the teachers' respective mathematical knowledge; in fact, they contend that teachers' knowledge exerts the greatest influence on how these two teachers evaluated and used their mandated textbook, planned their instruction, and interacted with their students.

Likewise, in discussing teachers' difficulties in implementing the practice of having students share and compare multiple solutions when solving mathematical problems, Silver and colleagues (Silver, Ghouseini, Gosen, Charalambous, & Strawhun, 2005) postulated that several of these difficulties might stem from limitations in teachers'



knowledge. Such limitations were assumed to impinge on teachers' realization of this practice in their teaching in several ways. For example, they were associated with teachers' difficulties in planning a lesson that incorporates this practice and with teachers' discomfort in monitoring students' work or orchestrating the sharing of multiple solutions.

Huberman and Middlebrooks (2000) also discuss the role of teacher knowledge in enacting cognitively demanding curricula. These scholars examined how teachers in six schools implemented a reform program (i.e., the *Voyages of the Mimi*) which offered opportunities to explore rich and demanding mathematics. Their analysis of a rich corpus of data suggested that, although many of the teachers in these schools endorsed this program, they could not implement it. Even its most enthusiastic supporters were found to implement a watered-down version that emphasized the application of algorithms and afforded students miniscule opportunities for rich explorations and discussions.

Explaining their gradual dilution of the program, many teachers argued that it was too complicated and demanding to implement, and hence they transformed it in ways they could handle. This finding points to the ways in which teachers' weak knowledge might limit their interpretation and enactment of rich and demanding mathematics curricula.

Similar results were also found in a recent study (Nicol & Crespo, 2006), which examined PSTs' understanding and use of textbooks. All four teachers participating in this study admitted that their own understanding of and familiarity with the content influenced how they interpreted and used the available curriculum materials.

There is, though, the other side of the coin. For example, Rosebery's study (2005) of Mary showcases how a strong understanding of the content and its teaching supports

the teacher in planning and enacting her lessons. An accomplished teacher with about 30 years of experience, Mary was observed while planning and teaching a fifth-grade lesson on distance, time, and speed. While planning her lesson, she drew on her own understanding of the content to appropriate an activity designed to help students understand speed as a function of the distance traveled in a certain time period. While teaching this lesson, Mary also improvised to respond to her students' difficulties without, though, lessening the richness and complexity of the content.

Finally, arguments about the role teachers' knowledge plays in their building of rich and challenging environments can be traced in comparative studies that explored either teachers' knowledge or their instructional practices. By focusing on only one aspect – either teachers' knowledge or their instruction – these studies advanced *hypotheses* about the potential association between knowledge and instructional practices. One such example is Ma's (1999) widely cited study in which she compared Chinese and American teachers' mathematical knowledge. Her study revealed significant differences in the depth, breadth, and thoroughness of teachers' knowledge in the two countries, differences allegedly reflected in their teaching of mathematics. Despite the reasonableness of this association, Ma's study did not provide any direct evidence linking teachers' knowledge to their practice. At the other end of the spectrum, certain studies did scrutinize the instructional practices of teachers in different countries, but did not obtain any measures of teachers' knowledge. Although some researchers explicitly attributed the higher level at which mathematics is experienced in the East Asian countries to the profound knowledge of teachers in these countries (e.g., Leung, 2005), this argument warrants further investigation.

In sum, the studies reviewed above make a strong case that teachers' knowledge or lack thereof appears to inform teaching practices that contribute to the establishment of rich and challenging learning environments. These practices range from using appropriate representations, analogies, and manipulatives, to providing explanations and engaging students in rich mathematical discussions. However, most of these studies mainly drew inferences about teachers' knowledge from exactly the same data source used to delineate teachers' practices (e.g., classroom observations). This limitation suggests that a more valid and thorough approach would require obtaining separate measures of teachers' knowledge and their practices, an approach that was pursued in the studies reviewed in what follows.

*Associations between Teacher Knowledge and Their Instruction: Studies of Direct Evidence*

The studies that examined both teachers' knowledge and practices can be clustered in three main categories. The first category encompasses "constraint" studies that explored how teachers' limited knowledge compromises their teaching practices. The second category includes "affordance" studies that showcased how a strong understanding of the content and its teaching supports teachers in engaging students in worthwhile and challenging mathematics. The third category combines both aforementioned approaches and includes studies that explored how differences in knowledge are reflected in teachers' instructional approaches. I consider each category in turn.

### *“Constraint” Studies*

In one of the earliest works in this realm, Borko and colleagues (Borko et al., 1992; Eisenhart, Borko, Underhill, Brown, Jones, et al., 1993) explored PSTs’ knowledge and practices during their fieldwork placement. From the eight cases discussed in their studies, of special interest is the case of Ms. Daniel. A novice middle-grade teacher, Ms. Daniel had a superficial and mostly procedural knowledge of fractions. Her limited knowledge of the content, identified through three interviews conducted over a school year, substantially constrained her instructional approach. In selecting instructional tasks for her lesson, Ms. Daniel mostly chose tasks that had little potential to engage students in constructing meaning, simply due to the fact that she herself could not understand the more conceptual tasks of her curriculum. In teaching this lesson, Ms. Daniel could not provide adequate and reasonable explanations as to why the algorithm employed in the division of fractions makes sense, nor could she use a suitable representation to explain this procedure. While reflecting on her enactment of the lesson, she regretted having spent time on trying to provide a conceptual explanation for the algorithm of the division of fractions, and preferred that she had given students more time to practice the algorithm. Hence, her limited understanding of fractions seems to have also restricted her opportunities to learn from her own practice. Further insights about the ways in which teacher knowledge appears to constrain teachers’ instructional approaches can be gleaned from three studies that explored in-service teachers’ knowledge and practices.

Stein, Baxter, and Leinhardt (1990) explored the subject-matter knowledge<sup>35</sup> of a fifth-grade teacher, Mr. Gene, and his respective instructional practices on functions. His

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<sup>35</sup> The authors defined subject-matter knowledge more broadly to include not only what Ball (1988) identifies as knowledge *of* mathematics and knowledge *about* mathematics, but also knowledge of how to teach mathematics, which corresponds to Shulman’s pedagogical content knowledge.

narrow understanding of the content – gauged by an interview and a task sorting activity – appears to have limited his instruction in three significant ways. First, because his own understanding of functions lacked key ideas (for instance, that a function can represent a many-to-one correspondence) he presented the content in very limited ways, by using an incomplete definition and a narrow analogy, both of which illuminated only the one-to-one correspondence of functions. Although the definition and the analogy he used could work well for his fifth-grade students, these “tools” could hinder his students’ learning of functions in subsequent grades. Second, during several instances, Mr. Gene overemphasized limited truths. For instance, he talked about the “two-out-of-three” rule, according to which when given any two of the following three pieces of information – input, output, and the function rule – one can always determine the third, an idea that contradicted the student textbook.<sup>36</sup> In addition, he proposed the “checking” procedure, according to which, when one plots a function, all the points ought to be on a straight line, a procedure that applies only to linear graphs. His presentation of functions thus distorted the integrity of the math in consequential ways (cf. Ball, 1993b). Third, because his understanding lacked critical connections among the different representational forms of functions – tables, graphs, algebraic equations, and ordered pairs – he missed important opportunities to forge such connections when presenting the content.

Thompson and Thompson (1994, 1996) also discussed the various ways in which a teacher’s *condensed* knowledge hampered his interaction with a student around the concept of rate. This teacher could not identify the conceptual subtleties embedded in the notion of rate, and thus failed to transform the content in order to make it

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<sup>36</sup> Because some functions represent many-to-one mappings, this rule does not always support finding the input number, when one is given the output number and the function rule. Similarly, more than one function rule can be used to describe the same input and output numbers.

comprehensible. Without having an appreciation of the complexities of the content, he also explained the concept of rate by providing a single example. This limited the likelihood that the student with whom he interacted understood the concept by viewing it through multiple perspectives. The teacher’s packed understanding of the content also prevented him from listening closely to and analyzing the student’s thinking; instead, the teacher largely interpreted the student’s explanations in terms of his own understanding of the content.

A similar case is that of Tom, a middle-grade teacher discussed in Sowder et al. (1998). Like Bill, the teacher considered in Thompson and Thompson, Tom had a strong mathematical background. However, his pedagogical knowledge of the content was not robust. Consequently, in teaching the division of fractions, a topic he personally found particularly challenging, Tom avoided introducing the cryptic “invert-and-multiply” algorithm, but suggested an equally esoteric approach illustrated in Figure 2.4. According to this approach, students “could follow the arrows and multiply.”

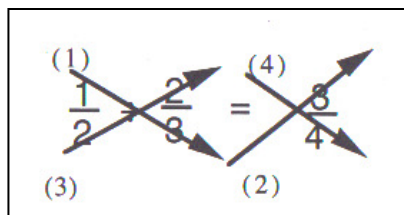


Figure 2.4. Tom’s approach for dividing fractions (cited in Sowder et al., 1998, p. 135).

The cases of Tom and Bill both suggest that teachers’ impoverished or compact understanding of the content cannot support them in “*psychologizing* the content,” (cf. Dewey, 1906, p. 30), that is in transforming it in ways both mathematically legitimate and pedagogically appropriate. Tom transformed the content in a fashion that made it palatable to his students, yet he distorted its integrity. Bill, on the other hand, retained the

integrity of the content, but failed to make it accessible and comprehensible. In addition, both teachers missed the opportunity to engage their students in rich and challenging mathematical explorations of the content. Because of his absurd treatment of the content, Tom failed to even afford his students such opportunities. Bill, on the other hand, furnished the student with whom he was working such opportunities but, because of his sophisticated and compact understanding of the content, he made it difficult for the student to follow him. Bill was just exploring the mathematical terrain on his own, whereas Tom defied the challenge to even enter it for himself.

Finally, in a recent study, Leikin and Levav-Waynberg (2007) made a first step in exploring the association between teacher knowledge and the practice of having students share and discuss multiple solutions to a problem, which supports the establishment of rich learning environments. In their study, these scholars analyzed data from interviews and focus-group meetings with 12 secondary teachers; these teachers were also asked to solve six problems from different topic areas in different ways and to then propose an example of a problem that admitted multiple solutions. Overall, these teachers could not solve the problems in more than one way unless multiple solutions to these problems were explicitly presented in their curricular materials. These teachers also mentioned that they would avoid employing tasks that admitted several solutions. Hence, Leikin and Levav-Waynberg concluded that teachers' knowledge appears to affect teachers' willingness and ability to engage in the practice of sharing multiple solutions and building connections among them.

The "affordance" studies considered next illuminate the association between teachers' knowledge and their teaching approaches from a different perspective.

### *“Affordance” Studies*

Lloyd and Wilson’s study (1998) of a teacher’s instruction of functions represents a counterexample to Stein and colleagues’ study (1990) discussed above. In their study, Lloyd and Wilson documented how a high-school teacher’s comprehensive and well-connected knowledge of functions – gauged through a series of baseline and post-lesson interviews – contributed to his rich enactment of a *Standards*-based unit on functions. In teaching this unit, the teacher, Mr. Allen, defined functions appropriately, emphasized the conceptual nature of functions, supported students in building complex links among several representations, and led productive classroom discussions that probed for student explanations and meaning. Although this study and that of Stein and colleagues outline the practices of two teachers working at different grade levels, both studies highlight – one from a constraint and another from an affordance perspective – how teachers’ knowledge or lack thereof can support them or constrain them in selecting definitions, using and building connections among different representations, and focusing the classroom discussions on making meaning, rather than on applying algorithmic rules of limited applicability.

A series of studies that focus on the practices of exceptional teachers also showcase how teachers’ strong mathematical knowledge can support them in teaching mathematics effectively. I briefly review two examples of exceptional teachers and their practices below.

The first example concerns Magdalene Lampert, who has taught elementary school children for about ten years. Her profound mathematical knowledge was manifested in several instances of her instructional approaches, as captured by her



recounting of her own teaching or as discussed by several other scholars who have analyzed her instruction (Ball, 1991b; Lampert, 1986, 1989, 1990, 2001; Leinhardt & Steele, 2005). To start with, when planning her lessons, Lampert carefully thought about the mathematics she wanted to teach without losing sight of the bigger mathematical picture and how the topics she was planning to address figured in this picture. Instead of treating the curriculum as a list of topics, concepts, or procedures to be covered, she used to focus on big mathematical ideas and deliberately selected instructional tasks and problems to help her students see connections among and between different concepts and procedures (cf. Lampert, 2001, pp. 213-263). During instruction, and responding to her students' struggles, she was often seen skillfully adapting the problems she had originally chosen without diminishing their cognitive complexity. In introducing new concepts or procedures, she often employed carefully chosen representations to scaffold students' understanding and help them see the underlying meaning of certain mathematical processes.<sup>37</sup> In Lampert's classes these representations were not end in themselves but vehicles for exploration and thinking; they were "tools" that supported her students' reasoning (Lampert, 1989, p. 233).<sup>38</sup> When explaining a mathematical concept or procedure, Lampert also used to build her explanations around useful examples that clarified the meaning of the concepts and the procedures under investigation (cf. Leinhardt & Steele, 2005). In monitoring her students, she used to pose probing questions to elicit their thinking, and often pressed them to explain and justify their answers.

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<sup>37</sup> See, for example, Lampert (1986) and Ball (1991) for a discussion of Lampert's representation of jugs and glasses to help students understand the distributive property of multiplication with respect to addition or subtraction. Also, see Lampert (2001, pp. 191-196) for a discussion of her impromptu crafting and use of a linear representation to help students solve a speed-distance problem.

<sup>38</sup> A close dissection of a series of Lampert's lessons on functions revealed that Lampert was using representations very deliberately to first display and connect several aspects of the big idea of functions, and second to offer her students "stepping stones" that would help them build an understanding of intermediate subskills and procedures (Leinhardt & Steele, 2005, p. 137).

Moreover, she followed her students' thinking painstakingly, and almost always, asked for greater clarification, while also steering the discussion toward the big mathematical ideas she wanted her students to explore.

One could legitimately argue that several factors might have contributed to Lampert's exceptional teaching, including her rich repertoire of instructional routines and her beliefs about teaching and learning mathematics. Yet, one cannot refute the pivotal role that her strong mathematical knowledge appears to have played in several aspects of her teaching, ranging from her selection of tasks, representations, and examples to providing explanations or co-constructing such explanations with her students.

A parallel argument holds for Ball's instruction in a third-grade class. Rather than offering an extensive summary of her instructional approaches and how she appeared to mobilize and utilize her knowledge, I briefly discuss four instructional episodes from her teaching, each of which showcases how a teacher's knowledge of mathematics and its teaching can inform her instructional decisions and actions.

The first episode comes from a lesson in which Ball assigned the following problem to her students: "Erasers cost 2 cents, and pencils cost 7 cents. How many different combinations of erasers and pencils can you buy if you want to spend exactly 30 cents?" (Ball & Bass, 2003a, p. 35). At first sight, this problem seems quite simple. However, as Ball explains, she intentionally chose the numbers of this problem to initiate a discussion on even and odd numbers. She was expecting that her students would see that an even number of pencils ought to be bought to amount 30 cents. Indeed, while working on this problem, students generated a list of conjectures about even and odd numbers which provided the basis for a series of lessons on proving conjectures. This

deliberate selection of the numbers embedded in a problem spotlights how a teacher's knowledge of the content can support her while selecting and modifying tasks so that she structures rich learning environments for her students.

The next two episodes concern Ball's work around representations and manipulatives. In a lesson on fractions, Ball asked her students to share a dozen cookies among the members of their family (Ball, 1993a). To figure out how many cookies each of the five family members would get, Cassandra, one of the students, drew a table with five columns and successfully divided up ten of the cookies. To share the remaining two cookies, she drew two circles, which she then partitioned unequally. Identifying the difficulties inherent in partitioning cyclical representations into equal parts, Ball suggested that the student draw a rectangular representation and split it into equal parts. Although at first sight such a shift from one type of representation to another might seem trivial, it is not; in fact, it constitutes an "inherently difficult and uncertain" task that requires considerable knowledge and skill (ibid, p. 161). How convoluted this task can be is also depicted in another episode, during which Ball's third graders were using pattern blocks to figure out the sum of  $\frac{1}{6} + \frac{1}{6}$  (Ball, 1992). Her students arrived at two different answers (i.e.,  $\frac{2}{12}$  and  $\frac{2}{6}$ ); even more worrisome, one student suggested that both answers can be correct: the first being the answer that one gets from manipulating the numbers and the second the answer that one gets from manipulating the pattern blocks. Instead of clarifying the right answer for them, Ball decided that she needed to work more on the notion of unit, a pivotal mathematical idea which, when developed, would allow her students use the manipulatives appropriately. In both episodes outlined above, Ball's knowledge of the content and its teaching appeared to scaffold her in making decisions in

situ regarding how to support her students' use of representations and manipulatives.

Without this knowledge, confronted with her students' struggles and disagreements, Ball might have just stepped in and done the thinking for them.

The last episode considered from Ball's teaching comes from a lesson on subtraction (Ball & Bass, 2000, pp. 91-93). In this lesson, Ball assigned the following problem: "Joshua ate 16 peas on Monday and 32 peas on Tuesday. How many more peas did he eat on Tuesday than he did on Monday?" When asked to share their solutions, students proposed at least three different approaches. Using the number line posted on a wall, Sean started from 16 and counted up until he reached 32. Betsy used beansticks and built up the number of peas that Joshua ate on Monday and Tuesday. Then, she matched as many "Monday peas" she could with "Tuesday peas," and concluded that the leftover peas represented those that Joshua ate more on Tuesday. Mei disagreed with Betsy's method, and argued that the problem should be solved by taking away 16 from 32. To build on these students' different approaches and orchestrate a productive classroom discussion, Ball needed to decipher how each of these students had approached the problem, and the affordances and limitations of each approach. Carrying out this task successfully again hinged on Ball's thorough knowledge of the content and its teaching.

What can we learn from the studies of Mr. Allen, Lampert, and Ball about how a strong mathematical knowledge base can inform decisions and actions that relate to crafting mathematically rich and cognitively demanding environments? Synthesizing the results of these studies, one could argue that teachers' profound mathematical knowledge is pivotal for all three phases of task unfolding the MTF considers. First, teacher knowledge appears to inform planning decisions, including the selection of problems,

activities, examples, and representations, as well as their adaptation in ways that both attend to the targeted mathematical ideas but also to the students' needs and abilities. Second, teachers' thorough knowledge appears to inform their decisions and actions related to the presentation of the content. These include using appropriate definitions, representations, and analogies, giving explanations that elucidate the big mathematical ideas, and further adapting the employed examples and tasks to make them accessible without diluting their cognitive challenge. Finally, these studies also suggest that in working with their students on certain tasks, teachers can also draw on their knowledge to analyze and understand students' diverse thinking approaches.

The arguments made about the ways in which teacher knowledge appears to inform their instructional decisions and actions are also corroborated by the last category of studies that combine a "constraint" and an "affordance" perspective.

#### *"Constraint" and "Affordance" Studies*

Within this category one can distinguish three subcategories: *within-teacher comparative studies*, which compare the same teacher's knowledge and instructional approaches in two different topics; *between-teacher comparative studies*, which contrast different teachers' knowledge and practices with respect to the same topic; and *intervention studies*, which trace changes in teachers' knowledge and instructional practices. I discuss examples of each of the three subcategories in turn.

*Within-teacher comparative studies.* In this subcategory, I consider two examples, the first focusing on an in-service teacher, and the second on a PST.

Based on data collected in the context of the *Cognitively Guided Instruction* project, Fennema and colleagues (Fennema & Franke, 1992; Fennema, Franke,

Carpenter, & Carey, 1993) reported on the case of Ms. Jackson, an elementary school teacher, who was observed to teach lessons in two different content areas: additive operations with whole numbers and fractions. The knowledge measures obtained in this study suggested that Ms. Jackson was much stronger in the former than in the latter area, a difference reflected in her instructional decisions and actions. When teaching addition and subtraction, Ms. Jackson always built her lessons around a variety of problems that differed in their structural characteristics; she also engaged students in comparing and contrasting different situations, seeking to help them develop an understanding of the underlying concepts. On the contrary, when teaching fractions, she mainly employed a single problem structure, avoided asking clarification questions that could help students see connections among and between the ideas examined, and was rarely successful at following and capitalizing on her students' thinking and contributions.

A similar example of how teachers' knowledge can support or constrain their instructional decisions and actions comes from a study of 16 preservice secondary mathematics teachers who were observed while teaching different topics during their field placement (Kahan et al., 2003). Of particular interest in this study is the case of Ms. Lehava. Mathematically strong in the area of circles and their properties, Ms. Lehava designed and enacted a successful mathematics lesson that supported students' consideration of relevant mathematical ideas. However, she was not equally strong in the area of combinatorics. Hence, although she designed a task that asked students to consider the distinct chords that connect eight points on the circumference of a circle, she failed to support students' work on this task; after they named ten such chords, Ms. Lehava shifted to another activity. In her post-lesson reflection, she admitted that she was not

familiar with the idea of combinatorics and thus did not know how to handle students' contributions. As Kahan and colleagues argue, her impoverished knowledge of combinatorics "narrowed [her] scope of what was possible" (p. 247).

*Between-teacher comparative studies.* As already mentioned, although many between-teacher comparative studies have explored either teachers' knowledge or their instructional approaches, there is a relative scarcity of studies that have investigated both knowledge *and* instruction. The three studies reviewed below seem to be the exception to this situation.

An and colleagues explored the pedagogical content knowledge and the instructional practices of Chinese and American teachers (An, 2004; An, et al., 2004). The participants in these scholars' study were 28 fifth- to eighth-grade American teachers and 33 fifth- and sixth-grade Chinese teachers. Data on teachers' pedagogical content knowledge on fractions, ratios, and proportions were collected from all participants; these scholars also conducted classroom observations and post-lesson discussions with five Chinese and five American teachers. The analysis of these data showed that the Chinese teachers possessed richer and more profound knowledge in the study-relevant areas relative to their American counterparts. For instance, whereas the American teachers mainly referred to the part-whole interpretation of fractions, the Chinese teachers also referred to the concept of unit fraction. Like their American counterparts, the Chinese teachers were aware of multiple representations they could employ in their teaching. Chinese teachers, though, were additionally capable of comparing and contrasting different representations and deciding under which circumstances one representation was better than another, an area in which American teachers performed poorly. For example,

whereas most of the American teachers argued that they would only use an area representation to introduce fraction multiplication, the Chinese teachers would use both the area model and a repeated addition interpretation. They also considered the latter representation more suitable for multiplying a fraction with a whole number and the former more applicable when multiplying two fractions.

An and colleagues attributed the differences identified in these teachers' instructional approaches to the disparities they found in teachers' knowledge. In particular, the Chinese teachers employed both more and more diverse representations to help their students overcome their misconceptions. They were also seen to explicitly help their students draw connections among different representations. The Chinese teachers were also adept at posing questions that elicited deep and critical thinking. The American teachers, on the other hand, often failed to help students see connections between the concrete materials they were using and the abstract ideas these materials were supposed to represent. Additionally, while the Chinese teachers encouraged abstract thinking and problem solving, the American teachers mostly taught facts. Although one could reasonably question whether these instructional differences were solely attributable to differences in teachers' knowledge, one cannot deny that their knowledge might have, at least partly, contributed to these results.

Similar arguments about how teacher knowledge appears to inform instruction were presented in two recent studies that explored associations between teachers' MKT and their instructional practices. In the first study, Hill and colleagues (Hill, et al., 2008) explored whether teachers who differed in their level of MKT also differed in the



mathematics quality of their instruction.<sup>39</sup> Analyzing 90 lessons taught by ten teachers with different levels of MKT, these researchers found that teachers with higher levels of MKT committed fewer mathematical errors, responded more appropriately to their students, and chose examples that helped students construct meaning of the targeted concepts and processes. In contrast, teachers with lower MKT scores were not very successful at selecting and sequencing mathematical examples and tasks, restating textbook definitions, and using representations effectively.

Drawing on a subset of the teachers considered in the previous study, Charalambous (2006) analyzed the instruction of a high-MKT and a low-MKT teacher, focusing, in particular, on their selection, presentation, and enactment of instructional tasks. This analysis showed differences in the unfolding of tasks in the two teachers' lessons. Specifically, although both teachers were found to select both high and low demanding tasks to incorporate in their teaching, Karen, the high-MKT teacher, largely maintained the demand of these tasks at their intended level during their presentation. One of the factors assumed to contribute to her ability to maintain the cognitive complexity of the content was her deliberate sequencing of the instructional tasks she was using so that students could gradually develop understanding without consistently resorting to her for clarifications and guidance. In contrast, Lisa, the low-MKT teacher, was not equally successful at maintaining the richness and cognitive complexity of the content: in several cases, she lowered the demand of highly complex tasks either by her

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<sup>39</sup> These scholars defined *mathematics quality of instruction* as a conglomerate of several dimensions that characterize the rigor and richness of the mathematics of the lesson, including “accurate use of mathematical language, the avoidance of mathematical errors or oversights, the provision of mathematical explanations when warranted, the connection of classroom work to important mathematical ideas, and the work of ensuring all students access to mathematics” (Learning Mathematics for Teaching, 2006, p. 3).

presentation of these tasks or by her failure to draw students' attention to the underlying meaning of the mathematical procedures explored in her lessons.

The quality of the mathematics experienced in the two teachers' classes also differed remarkably. Karen's lessons offered a balance between high and low demanding tasks. Even more critically, the enactment of tasks in Karen's lessons supported drawing connections between the targeted procedures and their underlying meaning. In contrast, the mathematics experienced in Lisa's classes was dominated by an emphasis on remembering rules and algorithms. By placing so much emphasis on getting the correct answer and on remembering and applying rules, Lisa often failed to capitalize on students' contributions to elevate the cognitive demands of the tasks. In interacting with her students, Lisa was also observed further reducing the cognitive challenge of the tasks, by rushing to provide hints on their most challenging aspects.

*Intervention studies.* These studies portray the instructional approaches that teachers pursued before and after an intervention designed to deepen their knowledge in a specific topic. Three such studies are considered.

Among the earlier intervention studies that explored changes in teachers' knowledge and the impact of these changes on teachers' instructional approaches are the studies conducted in the context of the *Cognitively Guided Instruction* (CGI) project, designed to deepen teachers' knowledge of arithmetic word problems (Carpenter, Fennema, & Franke, 1996; Carpenter et al., 1999; Fennema & Franke, 1992). One of these studies focused on a first-grade teacher, Fran, who participated in a four-week CGI summer workshop (Lubinski, 1993). This professional development experience helped Fran deepen and extend her knowledge of the different types of addition and subtraction

word problems and the various ways in which children approach such problems. This deepening and broadening of her knowledge was reflected in the changes she introduced in planning and teaching word problems. After her participation in the workshop, Fran would select a wide variety of problems for her lessons; she would purposely sequence her presentation of these problems, starting from problems accessible to all her students, and then increasingly elevating their cognitive complexity. Fran would also follow her students' thinking, elicit more explanations, and had her students share and compare different approaches. In sum, as Lubinski concludes, this teacher's "developing pedagogical content knowledge ... has influenced her instructional decisions and encouraged a learning environment based on children's thinking" (p. 201).

In a more recent study, Sowder et al. (1998) explored the practices of five middle-school teachers over a period of two years during which these teachers participated in an intervention designed to enhance their knowledge of rational numbers and proportional reasoning. These scholars found that the teachers' practices improved as their content knowledge deepened. In particular, at the end of the intervention, the teachers were less dependent on the prescribed curriculum and more willing to introduce new mathematical ideas in their teaching; they held higher expectations for their students, whom they pressed for further explanations and justifications. After developing a better understanding of the content, these teachers emphasized conceptual understanding more and paid less attention to procedural competence. At the end of the intervention, they reported confidence in teaching complicated concepts and to encourage multiple solutions to the problems they assigned.

Of particular interest is the case of one of these teachers, Shey, who, after participating in the intervention, attempted to engage his students in richer and more challenging mathematical tasks that required multiplicative thinking. Even though his students originally wrestled with the assigned tasks, Shey did not succumb to their explicit requests for help and do the thinking for them. In contrast, he posed different variations of the problems to help his students start thinking in a multiplicative mode and pushed them for more explanations and clarifications. Sowder et al. (1998) argue that these changes in his instructional approach reflected changes in his own understanding of the content. They maintain that the deepening of his understanding of the content “led him to raise his expectations of his students, resulting in [his] assigning more challenging tasks and expecting his students to provide more conceptual explanations for their reasoning” (ibid, p. 93).

Finally, in a similar study, Swafford et al. (1997) examined the effects of an intervention designed to enhance teachers’ content and pedagogical content knowledge in geometry. The comparison of teachers’ pre- and post- test scores showed significant gains in their geometry content knowledge. Follow-up observations of eight of these teachers showed key shifts in their instructional approaches in the respective domain. Specifically, after the intervention, these teachers allotted more instruction time to geometry; furthermore, they selected and enacted more open-ended and cognitively demanding tasks, and were more prone to elicit students’ thinking and explanations by posing more exploratory questions. In addition, they were more confident in their ability to respond to but also provoke higher levels of students’ thinking. Based on these findings, Swafford and colleagues concluded that enhancing teachers’ content knowledge

of geometry and their knowledge of how students' learning in geometry develops can improve teachers' instructional practices in this mathematical strand.

Overall, all the studies explored in this section suggest that teacher knowledge informs several teaching practices which, in turn, can help establish rich and cognitively demanding environments. In what follows, I use the MTF to organize these teaching practices.

*Teacher Knowledge and Teaching Practices Conducive to Establishing Rich and Cognitively Demanding Environments: A Synthesis of Findings*

To synthesize the findings of the studies reviewed in this section, I draw on the MTF and consider three phases in which teacher knowledge appears to inform their relative decisions and actions. Specifically, I consider *planning*, during which teachers review the available curriculum materials to plan their lesson, *presentation*, during which teachers present the content or the new tasks to their students, and *enactment*, during which teachers work with their students on the assigned tasks or the content under investigation. Although these three phases are presented and discussed as distinct, in reality they cannot be clearly distinguished.<sup>40</sup> Thus, in Figure 2.4, these phases are represented as intersecting ellipses.

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<sup>40</sup> Consider, for instance, the phases of planning and presentation. The examples, representations, and explanations that a teacher uses during the presentation of a task are usually selected during lesson planning. On the other hand, reacting to certain classroom factors, the teacher might spontaneously redesign and re-present a task to support student learning. Similarly, the phases of presentation and enactment are often blurred, since a teacher often moves back and forth between presenting the new content and working with his or her students on it. For example, while responding to students' struggle with a task, the teacher might provide an additional explanation or some further clarifications to support the students' work.

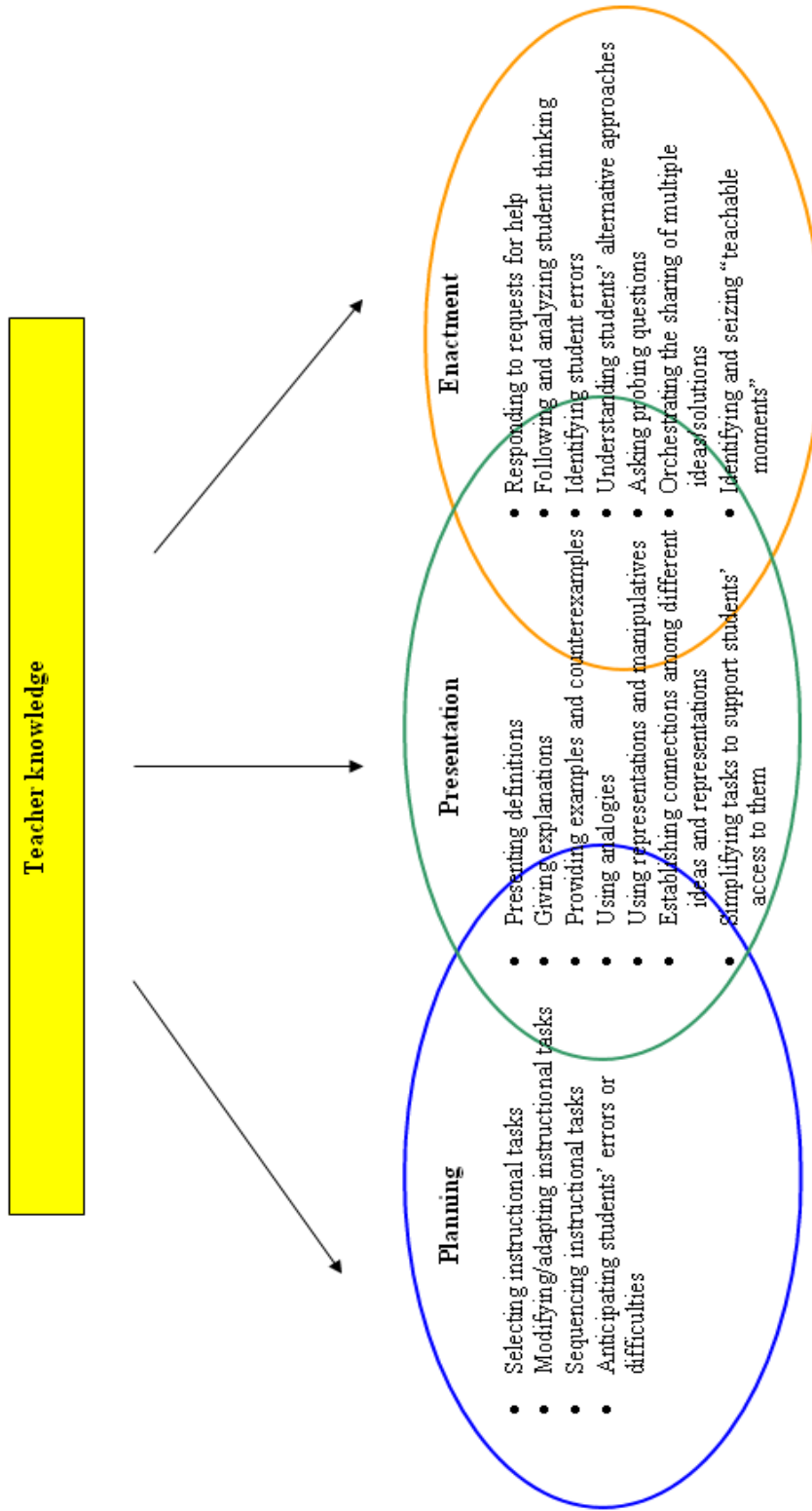


Figure 2.5. The potential role of teacher knowledge in informing decisions and actions during planning, task presentation, and task enactment.

### *The Planning Phase*

The studies reviewed in this section suggest that having a deep and thorough knowledge of the content can help teachers effectively navigate the mathematical terrain. This profound knowledge enables teachers to not only understand the content to be taught and its subtleties, but also see connections among and between the different ideas and procedures under investigation. Recall, for instance, how Lampert's skillful navigation of the mathematical terrain informed her planning and allowed her to situate each lesson within the bigger picture of mathematics. When a teacher lacks such a deep and thorough understanding of the content or when his or her knowledge of the content is compact – as the case of Bill discussed in Thompson and Thompson (1994, 1996) suggest – the teacher is less likely to see these connections or the subtleties of the content that need to be unpacked to scaffold student learning.

This understanding of the content and its teaching is consequential for several teaching practices associated with planning lessons amenable to establishing mathematically rich and intellectually demanding environments. To start with, consider teachers' *selection of instructional tasks*. The case of Ms. Daniel in Borko et al. (1992) suggests that teachers who do not understand the content well are less likely to select cognitively demanding tasks, a hypothesis also supported by the cases of Bonnie and Gina considered in Manouchehri and Goodman (2000) who both used the same textbook but selected different tasks. The case of Shey, as discussed in Sowder et al. (1998), also suggests that as teachers' understanding of the content becomes more thorough, teachers are more inclined to select richer and more challenging tasks for their students.

Analogous arguments can be made for the practice of *modifying instructional tasks*. The more that teachers understand the concepts and procedures they are teaching and the more unpacked their knowledge is with respect to these notions and procedures, the more capable they are of adapting existing tasks to facilitate students' rich explorations of the content. This was the case with Ball in the episode on the erasers-and-pencils problem. In this episode, she intentionally selected certain numbers for the problem she used to create a platform for her students to explore even and odd numbers. In contrast, teachers who find rich and cognitively demanding curriculum tasks challenging for themselves are more likely to water them down to minimize their complexity and, consequently, the risks associated with teaching more challenging content.<sup>41</sup> This was the case with the teachers discussed in Huberman and Middlebrooks (2000), who proceduralized many of the tasks of the *Voyages of the Mimi*, or the case of Mark considered in Wilson (1990).

The studies reviewed above also suggest that a comprehensive understanding of the content and an ability to “psychologize” it to scaffold student thinking and learning is fundamental to the *sequencing of instructional tasks*. Recall for instance how skillfully Lampert sequenced her instructional tasks to help her students see connections among and between different ideas. Or even consider the case of Fran, as discussed in Lubiski (1993), who, after participating in the intervention, was seen selecting a variety of tasks and sequencing them in a manner that would facilitate students' access to them. Consider,

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<sup>41</sup> As Doyle (1983) explains, teachers run several risks when assigning students cognitively challenging tasks. These risks stem from students' lack of familiarity with such tasks, the fact that not all students are able to effectively work on such tasks, and the extra time that these tasks require for completion. Hence, teachers are often inclined to adjust the cognitive demands of these tasks to meet the different ability levels of their students or to cope with time limitations. Alas, as he observes, “the type of tasks which cognitive psychology suggests will have the greatest long-term consequences for improving the quality of academic work are precisely those which are the most difficult to install in classrooms” (p. 186).



also, the case of Karen, as discussed in Charalambous (2006), whose purposeful sequencing of tasks helped her create a rich and intellectually challenging environment for her students but, more critically, sustain this environment during task enactment.

Finally, a strong knowledge of the content and a good grasp of the ways in which students might engage with this content appear to be critical to *anticipating potential student errors or difficulties*. This, in turn, can enable the teacher to plan in advance in order to both maintain the task complexity and support students so that they remain engaged. A case in point here is Mary, the experienced teacher discussed in Rosebery (2005) who appropriated an activity because she anticipated that her students would not be able to make sense of the content, had she presented the activity as proposed to her.

In sum, a profound and flexible understanding of the content and a good grasp of its teaching in ways that scaffold student learning appear to be essential for planning lessons that can potentially engage students in explorations of rich and challenging mathematics. This knowledge enables teachers to develop plans that help them effectively navigate the mathematical terrain with their students. It can also support them in following alternative routes to arrive at the targeted mathematical ideas or processes, but still respect the integrity of the mathematics.

#### *The Task Presentation Phase*

The studies reviewed above also point to several ways in which teachers' knowledge of mathematics and how they hold this knowledge matters for their presentation of the content in ways that support the structuring of rich and challenging learning environments.

First, to help students engage in such environments teachers need to present the new task clearly and in ways that initiate students' initial engagement with it. Such a presentation often requires that teachers use *precise definitions* that are both comprehensible to students of a particular age but at the same time do not distort the mathematics. The cases of Mr. Gene documented in Stein et al. (1990) and Mr. Allen considered in Lloyd and Wilson (1998) suggest that to give appropriate definitions teachers themselves need a thorough understanding of the content.

At several points during a math lesson, teachers also need to *deliver coherent and meaningful explanations* that draw their students' attention to the meaning of the content under consideration. To provide such explanations, teachers ought to have a strong mathematical background and the ability to unpack their own knowledge to make it transparent to students. Several cases of the teachers reviewed in the preceding studies corroborate this claim. Consider, for instance, Tom, the middle-grade teacher discussed in Sowder et al. (1998). Because he did not know why the algorithm of the division of fractions works, he refrained from providing any explanation at all for this rule. Instead, he encouraged his students to follow a meaningless gimmick. But what if students press for more conceptual explanations? The case of Ms. Daniels in Borko et al. (1992) suggests that teachers with impoverished knowledge are less likely to be able to provide such conceptual explanations, since teachers' limited understanding of the content compromises their potential to draw their students' attention to the underlying meaning of a mathematical process.

In providing explanations, teachers often employ (counter)examples or use certain analogies. However, not all examples or analogies are equally appropriate. In fact, the use

of such examples and analogies can occasionally result in focusing students' attention on non-mathematical ideas or ideas tangentially related to the content or even distort the mathematics under consideration. The studies reviewed above suggest that teachers' knowledge is critical to crafting and presenting mathematically rigorous examples, counterexamples, and analogies that are comprehensible to a particular student population. Recall, for instance, the case of Bill discussed in Thompson and Thompson (1994, 1996). Because of his very packed understanding of the concept of rate, Bill was not able to offer transparent examples to the student with whom he was working. Also consider the mathematically thin analogies that the elementary teachers in Heaton (1992) and Remillard (1992) presented to their students. On the other hand, recall the carefully selected examples that Lampert was using in her teaching which, as Leinhardt and Steele (2005) argue, were key to clarifying the meaning of the concepts and procedures discussed in her lessons.

Teachers' knowledge of the content and the way in which they hold this knowledge also appear to be crucial for their *selection and use of representations and manipulatives* and, consequently, to the richness and complexity of the learning environments that teachers structure. Recall, for instance, Carol's and Cathy's use of manipulatives that was not consistent with the algorithm they were trying to explain (in Ball, 1990b and Peterson, 1990, correspondingly). Contrast these cases to Ball's deliberate shift from one representation to another, when she realized the inherent difficulties in equipartitioning a cyclical representation. Or consider her ability to decipher the key mathematical idea of unit that her students had not developed, and the lack of which prevented them from appropriately using the pattern blocks she made

available to them. Consider, finally, the Chinese teachers described in An (2004) who not only possessed a rich repertoire of representations but knew when to use each of them and for which particular instructional purposes.

The establishment of rich and challenging environments can also be significantly facilitated when the teacher is able to help her students *see and build connections* among and between different mathematical ideas and representations. Acknowledging the critical role of building such connections, the authors of the *Principles and Standards for School Mathematics* (2000) identified *Connections* as a distinct process standard that K-12 teachers should have in their agenda, on the premise that when students are guided to build connections, their understanding becomes deeper and more lasting (p. 64). The studies previously reviewed imply that teachers' knowledge can support them in helping their students see and build such connections. Compare for instance, Mr. Gene, the teacher in Stein et al. (1990) to Mr. Allen considered in Lloyd and Wilson (1998). Because Mr. Gene's understanding of the different representations of functions was short of such connections, he could not help his students see such connections, let alone support them in building them. As Stein and colleagues eloquently argue, "[a]lthough circumstances were ripe for the development of connections between functions and graphs, in Mr. Gene's lessons these two representations remained as isolated islands" (p. 658). In contrast, in Mr. Allen's class, graphs, functions, and algebraic equations were connected to each other.

Finally, in presenting the content teachers are often faced with a dilemma. What if students struggle with the tasks and seem to be unable to make significant progress? As several studies have documented (e.g., Stein et al., 2001; Henningsen & Stein, 1997),

confronted with such a situation, teachers are more often than not inclined to diminish the cognitive complexity of the assigned tasks, even during task presentation, to facilitate students' access to them. A characteristic example is the case of Lisa considered in Charalambous (2006). Lisa often diminished the cognitive demand and richness of the tasks she was using, even during their presentation, to make them accessible to her students. The case of Lampert, in contrast, suggests that teachers can elevate the demands of mathematical tasks during presentation. Recall, for instance, how Lampert increased the cognitive complexity of a simple task that included two one-digit additions, as discussed in the second section of this chapter. Such upgrading of the cognitive demand of instructional tasks requires a deep and broad understanding of the mathematical content and its teaching on the part of the teachers.

#### *The Task Enactment Phase*

The phase of task enactment is also critical to maintaining the cognitive complexity and richness of the content. In fact, several studies have suggested or explicitly shown that it is mostly during this phase that teachers water down the intellectual demands of the content to manage complexity and to respond to students' requests for support and guidance (Doyle, 1983, 1992; Henningsen & Stein, 1997; Huberman & Middlebrooks, 2000; Stein et al., 2000). The preceding literature review suggests that teacher knowledge might help them make these decisions and take those actions that support but not replace student thinking, thus maintaining the complexity and richness of the assigned tasks. A case in point here is how Shey supported his students when the latter were not able to solve the assigned multiplicative-structure problems. Instead of offering students the answer to these problems or providing them with hints,

They modified the tasks he was using but retained their richness and complexity.

Similarly, when confronted with her students' conflicting answers to the addition  $\frac{1}{6} + \frac{1}{6}$ , Ball could have just stepped in and clarified what the correct answer was. Instead, Ball realized that students' confusion and struggle stemmed from their weak understanding of the concept of unit, a realization that informed her subsequent lessons.

To support student work without diluting the richness of the content, teachers also need to be able to follow and analyze their student thinking and work, including errors that students might commit or non-conventional approaches they might propose. Recall, for instance, the situation in which Ball had to respond to the three different solutions that Sean, Betsy, and Mei proposed. The three different solutions created a space for a rich exploration of the pertinent mathematical ideas. But such an exploration required that Ball appropriately analyze each student's contribution and understand how she could capitalize on them to help students successfully explore the mathematical terrain at hand. The case of Bill considered in Thompson and Thompson (1994, 1996) also corroborates this argument. Although Bill did structure a rich and challenging learning environment for the student with whom he was working, he largely failed to analyze and understand the student's thinking, thus exploring the mathematical terrain on his own without facilitating the student's access to it.<sup>42</sup>

In addition to effectively analyzing and building on student thinking, other practices appear to also help teachers maintain the richness and the intellectual complexity of the assigned tasks. These include *asking probing questions to elicit*

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<sup>42</sup> The cases of Ball and Bill also suggest that it takes more than just a profound understanding of the content to follow, analyze, and understand student thinking; teachers' propensity to listen carefully to their students is also critical. However, this propensity alone cannot support teachers in the work of analyzing and understanding students' work and contributions.

*students' explanations and justifications* or *orchestrating the sharing of multiple solutions* in such a manner that students are prompted to build connections among different ideas or representations. The studies reviewed above suggest that these practices are more likely to be undertaken by teachers with a more profound knowledge of mathematics and its teaching. Take for example, the case of Ms. Jackson considered in Fennema and Franke (1992). Because of her own deep and thorough knowledge of additive word problems, when working with students on such problems she often asked probing questions and pushed her students to share and compare multiple solutions. In contrast, her thin understanding of fractions could not support her in engaging in similar practices when teaching fractions. Thus, her teaching approach in this latter area was remarkably different: more often than not, she accepted just a single solution and refrained from eliciting students' explanations and justifications. Leikin and Levav-Waynberg (2007) reached a similar conclusion: the teachers they studied were less inclined to ask students to share and compare different solutions to a problem because of their own difficulties in seeing multiple solutions and identifying connections among those solutions.

The studies reviewed above also suggest that a teacher with a weak grasp of the content and its teaching is less equipped to identify "teachable moments," namely instances when students propose ideas or pose questions that create rich opportunities for mathematical explorations; this teacher is also less willing to seize these opportunities. The cases of Ms. Lehava (in Kahan et al., 2003) and Ms. Daniel (in Borko et al., 1992) exemplify how a teacher's narrow understanding of the content and its teaching can impinge on the teacher's ability and inclination to identify and capitalize on such

opportunities. Weak in the area of combinatorics, Ms. Lehava could not see the opportunity that was created by her students' exploration of the chords in a circle. Similarly, when confronted by students' request to teach division more conceptually, Ms. Daniel failed to do so, and suggested that students continue working on a set of algorithmic exercises.

Overall, the preceding analysis suggests that teachers' ability to create and sustain mathematically rich and challenging learning environments for their students cannot merely hinge on their *generic teaching skills*, such as their efficient time- and classroom-management or their capability to hold students accountable for high-level and quality work. These generic teaching skills, although necessary, are apparently subservient to creating such learning settings. Instead, as the preceding studies collectively suggest, teacher knowledge is more critical to crafting and maintaining such environments. If teachers know the content, and are able to transform it so that they can facilitate students' access to it, and still maintain its richness and complexity, they are more likely to engender and sustain the kind of learning environments considered in this study.

#### *Situating the Present Study within the Bigger Research Picture*

The preceding synthesis of findings suggests that teachers' knowledge informs certain teaching practices that span the phases of planning, task presentation, and task enactment, and which are conducive to the structuring of rich and challenging learning environments. Hence, this synthesis of findings supports Figure 2.1, which presents teaching practices as the link connecting teacher knowledge to their establishment of rich and intellectually demanding environments. It is in this space that the present study is situated. Specifically, the study seeks to extend the research seeking to understand the



association, if any, between teacher knowledge and their performance in certain teaching practices; it does so in five ways.

First, in contrast to most of the studies considered in this section, the present study *makes the exploration of the association between teacher knowledge and their practices its focal inquiry*. Second, whereas many of the aforementioned studies merely focused on one teacher, the present study explores the association between teachers' knowledge and their practices by considering a *cohort* of PSTs. Third, with the exception of the intervention studies, the studies reviewed above investigated teachers' knowledge and its effects only at one point in time. The present study explores the association between teacher knowledge and their practices both at a fixed point in time and between two time points. This latter exploration allows for the investigation of how changes in PSTs' MKT after their participation in an intervention relate to changes in their performance in a set of teaching practices. Fourth, the study provides insights into the ways in which a specific type of teacher knowledge, their MKT, informs their instructional practices (see the results reported in the last sections of Chapters 4 and 5). In doing so, this study extends the work associated with MKT and makes a first step in addressing Hill and colleagues' (Hill et al., 2005) request to understand how teachers' MKT informs their instructional practices. Finally, compared to the foregoing studies, the present study covers a wider terrain by exploring a conglomerate of teaching practices that span the three MTF-phases (i.e., planning, presentation, and enactment); it also focuses on teaching practices that, as the previous analysis suggested, are critical to building and sustaining rich and intellectually challenging settings for student learning.

Establishing Mathematically Rich and Intellectually Challenging Environments:  
Focusing on Five Teaching Practices

In the preceding sections, I considered the association between teacher knowledge and student learning (i.e., section one) and the relationship between establishing rich and the intellectually challenging environments and the quality of student learning (i.e., section two). In the third section, I also reviewed relevant studies suggesting that teacher knowledge informs several teaching practices that appear to contribute to the structuring of the learning settings this study considers. I also used the MTF to organize these practices into three categories that correspond to the phases of planning, task presentation, and task enactment. In this section, I explain the reasons that informed the third design decision of the study: to focus on a subset of the practices identified above. Besides elaborating those reasons, I also explicate how each of the five teaching practices lends itself to creating environments that can potentially immerse students in rich and cognitively challenging mathematics.

*Criteria Informing the Selection of the Five Teaching Practices: The Third Design Decision*

The decision to focus on the five teaching practices this study considers was based on three main reasons: first, that these practices collectively span the three phases of task unfolding; second, that they could be explored with some integrity in an in-vitro situation, which relates to the fourth design decision considered in the next section of this chapter; and third, that they recognize the central role that teachers have in structuring rich and intellectually challenging learning environments. I elaborate upon each of these reasons in turn.

As explained in the second and third sections of this study, teachers can affect the structuring of the aforementioned environments by their decisions and actions at three different phases of task unfolding: while interpreting the textbook materials and selecting tasks for their lessons; while presenting these tasks during the lesson; and while interacting with their students around these tasks. The research explored in the second section clearly showed that each phase is *critical* to establishing the rich and challenging environments this study considers. Hence, attempts to understand how teacher knowledge informs relevant teaching practices should not be limited only to a single phase. Instead, as the studies reviewed in the third section suggest, considering teaching practices that span all three phases can provide more insights into the association between teacher knowledge and their instruction. Even more, by considering practices from all three phases, a first step can be made toward comparing the relative weight that teacher knowledge appears to have in informing decisions and actions with respect to each of these phases. Taking into consideration these reasons, in this study, I focus on the practice of *selecting and using tasks* from the phase of lesson planning; the practices of *providing explanations* and *using representations* from the phase of task presentation; and the practices of *analyzing student work and contributions* and *responding to students' direct or indirect requests for help* from the phase of task enactment.

The practices considered in this study are *composites* of the practices presented in Figure 2.5. In particular, the practice of selecting and using tasks includes selecting, adapting/modifying, and sequencing tasks. Similarly, the practice of providing explanations also encompasses the use of examples and analogies, since it is possible that teachers use these “tools” to structure their explanations. Similarly, the practice of using

representations also includes building connections among different representations or representational systems. Finally, by referring to the practice of analyzing student work and contributions, I also consider the subordinate practice of making sense of students' errors.

Apparently, these five composite practices are not the only ones that could contribute to establishing the environments the study considers. Take, for instance, a teacher's skill in posing questions to elicit students' explanations or justifications or the practice of identifying and seizing teachable moments. "If these practices are also important," the reader might ask, "what might justify focusing on the five aforementioned practices?" In addition to considering these five practices critical to crafting rich and cognitively challenging environments, an argument that I elaborate in the next part of this section, a more pragmatic reason informed the selection of these five practices: that they be explored with some integrity in an in-vitro situation. Consider, for instance, the practice of posing questions to elicit students' explanations. This practice usually unfolds as a sequence of interactions between the teacher and the students: the teacher asks a probing question, a student answers this question, other students then build on the first student's contribution, and the teacher might then follow up with yet another question, which generates another cycle of teacher-student and student-student interactions. This complex and iterative pattern of interactions was considered difficult to capture and study with integrity in a teaching simulation. Equally difficult to be captured and studied is the practice of seizing teachable moments since the decisions made to capitalize on such moments are typically informed by several contextual factors. From this perspective, the

five practices this study considers could be regarded as a convenience sample of practices.

Finally, the five teaching practices recognize the central role of teachers in structuring the environments under consideration. This role starts from the phase of planning a lesson and extends to how the teacher first presents the content and then works with his or her students on the content. Responding to this last justification of the third design decision of the study, the critical reader might legitimately counterargue that teachers are only one of the constituent parts of what transpires during instruction; even more critically, what students learn, this reader might maintain, is not so much determined by the environments structured by teachers, but by the manner in which students themselves work within, utilize, and capitalize on these environments. Accordingly, this reader could also claim that attention should rather be placed on how students engage the tasks presented to them and not so much on what the teacher does to structure such environments.

In this study, I do not dismiss the fact that how students engage with the tasks is what eventually determines the quality of their learning. Nevertheless, I do challenge the tendency to pit the so-called *student-* or *learner-centered* modes of instruction against what has been called *teacher-centered* instruction, a tendency that might be implicit in arguments such those just considered. Largely nurtured by misinterpreting the NCTM documents and by the polarized positions adopted during the recent tumultuous period of the Math Wars (cf. Schoenfeld, 2004; Wilson, 2003; Wilson, Cooney, & Stinson, 2005, pp. 108-109), this tendency seems to pose a series of artificial dilemmas: Who should play the active role during instruction, the teacher *or* students? Which type of teaching is

more effective: teacher-directed *or* student-centered? Should the teacher be just a facilitator *or* actively lead the classroom discussion? Should the teacher lecture *or* create a space for students to discover and construct knowledge?

The position that I adopt in this study is that these dichotomies are unnecessary and that *both* the teacher-directed and the student-centered modes of instruction can forge the type of environments that nurture students' intellect.<sup>43</sup> Consider for instance, a teacher whose direct teaching can help students build connections between the procedures explored during a lesson and their underlying meaning. Should this type of instruction be rejected, simply because it is more teacher-centered and hence more "traditional"? And is this type of instruction less favorable than immersing students into unsystematic and non-productive explorations of the content, which can very well result from a teacher's taking a less active, more "facilitative" role? As the *Task Analysis Guide* presented in section two (see Figure 2.3) suggests, whereas the first type of instruction has the potential to

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<sup>43</sup> Several scholars endorse this position. For instance, Lampert (1990) clearly considers different modes of instruction – be they teacher-centered or student-oriented – suitable for student learning. Explaining how she herself has tried to scaffold her students' understanding, she mentions: "Sometimes I straightforwardly told students what kinds of activities were and were not appropriate. At other times, I modeled the roles that I wanted them to be able to take in relation to themselves and to one another. And at other times, I did mathematics with them, just as the dance instructor dances with the learner so that the learner will know what it feels like to be interacting with someone who knows how to do what he or she is trying to learn how to do. Just as the dance instructor knows traditional forms of dance and demonstrates them, I demonstrated the conventions of mathematics discourse to my students, but we also reinvented them as we did mathematics together" (p. 42). Along similar lines, Chazan and Ball (1998) argue that teachers should have an active role, even in student-centered environments. Drawing on their own instruction, these scholars maintain that teachers should not only facilitate classroom discussions, but also intervene and reinvigorate these discussions, by interjecting substantive mathematical comments. Whereas Chazan and Ball do not advocate that teachers do the thinking for their students, they do critique the stance which holds that "the teacher should never tell." Such exhortations, they argue, prevent us from better understanding the work of teaching. Hence, they recommend "less embracing or rejecting of particular lessons and more effort aimed at developing understanding of and reasoning about practice" (p. 9). Leinhardt (2001) is equally concerned with the exhortation that teachers' role be minimal. She maintains: "As researchers try to emphasize the need for teachers to listen and learn from the mental constructions and social goals of their students, the unfortunate suggestion or assumption has been made that any action on the part of the teacher ... is deleterious rather than supportive and facilitative. This assumption leaves us in the untenable position of suggesting that the best teaching is to do nothing and that, therefore, we should not strive to work on and improve the most directly improvable portion of the educational system" (p. 337).

engage students in high-level thinking, the second is less likely to do so. Hence, when talking about teaching practices that are conducive to establishing rich and intellectually challenging learning environments, I deliberately avoid associating these practices with traditional or reform teaching or, alternatively, with teacher- or student-centered instructional approaches. Instead, I focus on whether the decisions teachers make and their actions can *potentially* create learning settings that furnish students the opportunity to engage with rich and intellectually challenging mathematics.

Such environments can be fostered by teachers' selection of cognitively challenging tasks or by engaging students in activities that can forge high-level thinking (e.g., identifying and considering patterns and making generalizations). They can also be promoted by the richness of the content considered in a lesson and the extent to which instruction actually helps students focus on the underlying meaning of the explored procedures, regardless of whether such meaning is directly presented by the teacher (e.g., through his or her explanations or use of representations) or is co-constructed via the teacher-student interactions.<sup>44</sup> Whether students will actually construct this high-level understanding these environments are designed to promote cannot be guaranteed. However, the results of the QUASAR studies considered in the second section of this chapter clearly show that this type of learning is less likely to be forged when teachers

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<sup>44</sup> Support for this argument comes from Leung's (2005) analysis of the mathematics instruction in the two East Asian countries that participated in the TIMSS 1999 video-study (i.e., Japan and Hong Kong). His analysis revealed that more often than not the instructional approaches pursued in these countries were teacher-directed. Even so, the students in these countries were exposed to more and more complex and advanced content than the content experienced by their counterparts in the United States. Hence, instead of only focusing on the teaching method – teacher-centered or student-centered – pursued in teaching mathematics, Leung suggests that attention also be paid to the richness of the content considered during math lessons. As he points out, “without quality content, quality learning will not take place – no matter how ingenious the teaching method” (p. 210).

employ lower level tasks and maintain this low demand during task presentation;<sup>45</sup> neither can it be promoted by limiting teachers' role to just facilitating students' interactions and work on the content.

Having outlined the three criteria that informed the selection of the teaching practices the study considers and having clarified that these practices should not be associated with any particular type of instruction, I now justify my argument that each of these practices is conducive to the structuring of the environments discussed above, thus theoretically establishing the link between the second and third ellipses of the chain depicted in Figure 2.1.

*The Five Teaching Practices and their Potential to Establish Rich and Cognitively Challenging Learning Environments*

The studies presented in the second and third sections of this chapter provided some warrants about the potential of the selected five practices to create rich and intellectually demanding learning environments. The literature reviewed below further corroborates the argument that these practices are conducive to engendering and maintaining such environments.

*Selecting and Using Tasks*

Evidence regarding the potential of selecting and using tasks to engender rich and intellectually demanding environments for student learning comes from the studies considered in the second section of this chapter. Collectively, these studies, along with the MTF, suggest that teachers' selection, modification, and sequencing of tasks

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<sup>45</sup> Two QUASAR studies (Henningesen & Stein, 1997; Stein et al., 1996) showed that in almost all cases, the cognitive challenge of the tasks teachers employed in their instruction declined rather than inclined during task enactment. Only two of the 144 tasks examined in these studies were found to be implemented at a higher level than that at which they were originally set up.



constitute the first step in determining whether students will indeed be afforded the opportunities to consider rich and challenging mathematics. Additional evidence that corroborates this claim comes from studies that explored teachers' interpretation and use of curriculum materials. From the gamut of these studies, I focus on three in which scholars examined different teachers' use of the *same* curriculum materials. These cross-case studies allow for the exploration of the mediating role that teachers have in using the curriculum materials to structure learning opportunities for their students.

Collopy (2003) investigated two upper elementary teachers' use of a curriculum that emphasized problem solving and mathematical explorations. While using this curriculum, one of these teachers, Ms. Clark, deliberately omitted most of the curriculum tasks that could potentially afford opportunities for rich investigations of the content and largely selected tasks that emphasized carrying out procedures. While using the same curriculum, Ms. Ross, the other teacher, selected tasks that mostly pertained to rich explorations and making meaning of mathematical procedures. This disparity in the selection and use of curriculum materials was consequential for the type of mathematics experienced in each of the two classes. Ms. Clark's students mainly had opportunities to practice procedures without attending to the underlying meaning of these procedures. In contrast, the students in Ms. Ross's class approached the content from a more conceptual standpoint, which honed both their conceptual understanding of the content and their procedural fluency of the explored mathematical procedures.

In a similar but larger scale study, Remillard and Bryans (2004) explored how eight elementary grade 1-4 teachers working in an urban district school used the same curriculum to structure their mathematics instruction. These scholars followed the eight

teachers over the course of two years and conducted anywhere from 4 to 27 classroom observations of the different teachers. The analysis of these data showed remarkable differences in the teachers' selection, use, adaptation, and sequencing of the curriculum tasks. Whereas some of these teachers taught lessons that emphasized meaning-making, strategy development, and reasoning, other teachers used the curriculum rather selectively and placed more emphasis on memorization, and drill-and-practice tasks. These latter teachers more often than not ignored tasks or introduced and structured the tasks quite differently from the manner suggested in the curriculum. As Remillard and Bryans observe, these differences in teachers' selection, adaptation, and sequencing of tasks led to different opportunities for student learning.

Similar results were found in a more recent study that focused on middle-grade teachers' use of teacher guides. In this study, Castro (2006) explored four experienced teachers' use of a mathematics curriculum that emphasizes problem solving and building of connections. Although this curriculum largely involves cognitively demanding tasks, these teachers interpreted and used the curriculum tasks differently. For instance, whereas Alicia, one of these teachers, focused mostly on the procedural aspects of the curriculum tasks, Susan, another teacher, attended more to the conceptual and investigative nature of the tasks. Hence, there were notable differences in the mathematics experienced in these two teachers' classes.

Collectively, all three studies point to teachers' mediating role between the written and the enacted curricula, largely by their selection, use, and modification of curriculum tasks. This mediation, according to these studies, shapes the learning opportunities made available to students, thus corroborating the association between the

practice of task selection and use and the structuring of rich and intellectually challenging environments. However, these studies largely attributed the differences in teachers' selection and use of curriculum tasks to teachers' beliefs about teaching and learning and their orientations toward the curriculum materials. None of these studies explored the role of teacher knowledge in informing teachers' respective decisions and actions, which is the focus of the present study.

### *Providing Explanations*

The explanations considered in this study are those described by Leinhardt (2001) as *instructional explanations*, which she defines as explanations “designed to explicitly teach – to specifically communicate some portion of the subject matter to ... learners” (p. 340). These explanations, Leinhardt suggests, can be presented by a textbook, a computer, the teacher or a student, or can be jointly constructed by the entire class, while working on a task. In this study, I focus on the explanations that teachers provide.

Designing and delivering coherent and meaningful explanations is regarded a central teaching practice in which teachers engage at several instances: when presenting ideas to their students, when responding to students' questions, or when scaffolding students struggling with an assigned task. Hence, this practice is considered fundamental to the learning process and one that significantly shapes the learning opportunities made available to students (Leinhardt, 1988; Leinhardt, Putnam, Stein, & Baxter, 1991). Even more, teacher explanations not only clarify subject matter but can potentially influence the knowledge that students develop *about* mathematics, since these explanations model both the types of questions that can be asked in a domain and the manner in which such questions are answered. As Leinhardt (2001) explicates, teacher explanations not only

“demonstrate, convince, structure, and convey, [but they also] ... suggest the appropriate metacognitive behavior for working in a given discipline” (p. 340).

The ways in which teachers’ explanations shape learning opportunities was clearly presented in a sequence of studies in which Leinhardt compared and contrasted the explanations given by novice teachers and those presented by teachers who were characterized as effective based on measures of their students’ learning gains (Leinhardt, 1987, 1988, 1989). These studies showed that the latter compared to the former more frequently built explanations on representations, examples and analogies known to their students, capitalized on students’ prior skills and knowledge, gave more complete and well-connected explanations for the concepts or procedures at hand, and more often than not presented explanations free of mathematical errors. The differences in the richness and intellectual complexity of the content experienced in these classes due to the explanations of expert and novices teachers is well exemplified by the following comparison of the explanations that two such teachers provided with respect to the procedure of finding equivalent fractions (Leinhardt, 1988).

In discussing this procedure, the novice teacher talked about “reducing fractions,” a term which actually distorts the content, since the process under consideration, as its name suggests, results in finding *equivalent* fractions. When at the end of the lesson her students neither conceptually understood this process nor they were fluent in applying it, this teacher decided to re-teach the lesson the next day, but did so in exactly the same manner. When reconsidering this process in a subsequent lesson, she explained that it pertained to “doing the same thing to the numerator and the denominator of a given fraction,” an explanation that actually *describes* the process of finding equivalent

fractions – and rather vaguely – but does not explain *why* this process works. Compare this approach to that of the expert teacher. In introducing the procedure of finding equivalent fractions, the expert teacher used manipulatives (i.e., paper folding) to first help her students understand that two fractions that appear to be different can actually represent the same numerical quantity. She then spent a considerable amount of time working on the notion of equivalence and the meaning of the equal sign. Next, she helped her students understand why this procedure makes sense, by pointing out that by multiplying the numerator and the denominator by the same number essentially one multiplies the given fraction by one (i.e., the neutral element of multiplication). She then worked with her students on applying this procedure to a set of exercises deliberately selected and sequenced: they first considered known unit fractions (e.g.,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ) and gradually moved to less familiar fractions or non unit-fractions (e.g.,  $\frac{3}{4}$ ).

Overall, as Leinhardt (1988) concludes, whereas both these teachers were teaching the same procedure of finding equivalent fractions, the expert’s lesson “built upon a fairly strong linkage to basic mathematical principles, and it proceeded by justifying each procedural move in terms of these principles” (p. 63). In contrast, the novice teacher’s lesson mainly consisted of outlining an arbitrary set of steps. Also, although both teachers pursued what can be called a teacher-centered instruction, there were significant differences in the mathematics experienced, and probably in the mathematics learned, in these two classes.

Leinhardt’s research also legitimizes exploring the association between providing explanations and teacher knowledge, in general, and MKT in particular. As she and colleagues maintain (Leinhardt et al., 1991), “[t]he ways in which teachers design

explanations – the examples they select, the demonstrations they perform – reflect knowledge of subject matter and knowledge of how to teach subject matter” (p. 89). These scholars also suggest that to design and deliver coherent and meaningful explanations teachers should not only know the content, but also possess a gamut of relevant examples, counterexamples, analogies, and representations. Even more critically, they should know when these tools can be used to exemplify and illuminate a particular concept. These two latter aspects represent components of the *Specialized Content Knowledge*, one of the constituent domains of MKT considered in this study.

### *Using Representations*

The practice of using representations during mathematics lessons can be considered a subcomponent of the preceding teaching practice. However, for analytic purposes, in this study I explore this practice separately from the practice of giving explanations. Before explicating the association between this practice and its potential to structure rich and high-level learning environments, some clarifications are in order.

I use the term *representations* to denote what several scholars (Goldin, 1998; Kaput, 1987; von Glaserfeld, 1987) have identified as *external* representations, a term that encompasses a conglomerate of tools, such as pictures, diagrams, graphs, mathematical symbols, and concrete manipulative materials. However, because these tools are not self-enacting, I do not focus on the tools per se, but rather on how teachers mobilize them in their teaching. In fact, I consider the practice of using representations tantamount to Ball’s (1992, 1993a) term of building and working within a *representational context*. This term implies that what determines student learning is *the work that is done with and around* representations and not if representations are used in

instruction. Given the purpose of this study, I focus on teachers' use of these representations rather than on examining the joint use of them by teachers and students.

The importance of using representations to scaffold student learning is well exemplified by the fact that in the most recent NCTM document (NCTM, 2000) *representations* constitute one of the five endorsed process principles.<sup>46</sup> In this document, representations are discussed as powerful tools for thinking that make mathematical ideas more concrete and transparent and can thus scaffold student learning. The document also suggests that

[r]epresentations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. (p. 67)

But like any other tool teachers use in their instruction, representations can scaffold student thinking *only if they are selected and used appropriately*. This selection and use of representations constitutes, as several scholars (Cai, 2005; Ball, 1993a; Lamon, 2001; Sfard, 2002) argue, a critical part of the work of teaching mathematics and an important skill that effective teachers possess. Effective mathematics teachers are also well aware that mathematical meaning does not reside within the representations they use, but rather is developed from the work that is done with and on these representations. Although this seems obvious, more often than not teachers' use of representations suggests that teachers *do* consider representations carriers of meaning. This situation usually arises simply because teachers themselves can see meaning in these representations. As Ball (1992) explains, because teachers "already have the conventional mathematical understandings [they] can 'see' correct ideas in the material

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<sup>46</sup> In the 1989 NCTM document *Curriculum and Evaluation Standards for School Mathematics* representations are presented and discussed as part of the *Communication* principle.

representations” (p. 17). This ascribing of meaning to the representations used in teaching often leads teachers to hold what Ball (1992) calls magical hopes that the use of the representations alone can support student learning. Alas, daily practice often defeats these hopes. To compensate, teachers use representations rather algorithmically, and in ways that cannot support higher level thinking and rich explorations of mathematics. The review of relevant studies points to at least three ways in which teachers’ ineffective use of representations constrains the learning environments they structure, and accordingly, impairs student learning.

Ball (1992) refers to two ways in which teachers’ selection and use of representations results in constraining students’ opportunities to experience rich and intellectually challenging mathematics. According to the first way, teachers often ask students to use highly structured representations that actually do the thinking for them. Consider, for instance, the case in which students are presented with prepartitioned models of fractions and are asked to simply identify a particular fraction. To make matters worse, even consider the case in which the representations used are equipartitioned in as many parts as those represented by the denominator of the given fraction. Such a representation not only requires a lower level of thinking on the part of students but can actually reinforce misconceptions or lead to overgeneralizations, especially when it is the only kind of representation employed during instruction.

The second way in which teachers can limit the potential of representations to forge high-level thinking relates to teachers’ presentation or use of these representations while working with their students. To ensure that students can use the employed representations, teachers often give students *explicit directions and rules* to follow.



Although students are largely able to follow these rules and eventually use the representations to arrive at the expected answers, they do not actually develop conceptual understanding of the relevant ideas, which is one of the main reasons for which representations are originally utilized. Ball (ibid) uses the following analogy to describe this situation:

Following the rules, [students] readily arrive at the correct answers. In a sense, the manipulatives [or more generally, the representations] are employed as “training wheels” for students’ mathematical thinking. However ... when the training wheels are removed ... [s]tudents, rather than riding their mathematical “bicycles” smoothly, fall off, reverting to [procedures that they had been using before the representations were employed]. Even with close controls over how students work in the concrete domain, there are no assurances about the robustness of what they are learning. These training wheels do not work magic. Seeing students work well within the manipulative context can mislead – and later disappoint – teachers about what their students know. (p. 18)

The third way in which teachers can use representations ineffectively is typified by the case of Ms. Daniels (Borko et al., 1992; also see Leinhardt, 2001) discussed in the previous section of this chapter. Attempting to explain why the division  $\frac{3}{4} \div \frac{1}{2}$  makes sense, Mr. Daniels first drew a rectangle, divided it into four equal parts and shaded  $\frac{1}{4}$  of it. Then, she split the part that was not shaded into two equal parts, but, unable to proceed from there, suggested that students use their rule of inverting and multiplying. Thus, Ms. Daniel’s approach exemplifies how teachers cannot only model inappropriate ways of using representations, but can also convey to their students the idea that representations are rather auxiliary tools for doing mathematics and, hence, what matters more is to learn and apply certain rules and formulae.

The three uses of representations just considered compared to effective uses of representations discussed in the third section of this chapter clearly show that teachers’ use of representations can shape the environments structured for student learning. This is why representations are often deemed a double-edge sword (cf. Leinhardt, 2001). This is

also the reason that several scholars (e.g., Ball, 1992, 1993; Leinhardt, 2001; Sfard, 2002) contend that to use representations effectively, teachers need knowledge that enables them to select among a wide array of available representations those that are more suitable for their instructional purposes and use them in ways that support student learning of worthwhile mathematics, while also respecting students' prior ideas and conceptions. This argument again points to the kind of specialized knowledge that teachers need to use representations to their potential. It is this specialized type of knowledge that helps teachers discriminate between useful and less useful representations, and more critically, identify *when* and *how* to actually use these representations to scaffold student learning. Sfard (2002) puts this very eloquently when she talks about the specialized content knowledge needed by a cardiologist to make meaning in electrocardiograms or the knowledge needed by an architect when interpreting blueprints. Along these lines, this study seeks to provide further insights into the association between the practice of using representations and this specialized teacher knowledge.

#### *Analyzing Students' Work and Contributions*

Mathematics teachers do not simply teach mathematics per se; they teach mathematics to *their students*. Hence, in addition to the preceding three practices – that merely focus on what teachers do during task selection and presentation – the two last practices the study considers attend to what teachers do *with their students* during task enactment.

A fundamental element of teacher's work is to analyze and build on their students' contributions and thinking. This work entails interpreting student explanations

and making sense of what students say; determining the mathematical validity of students' strategies and non-conventional problem solving approaches; figuring out what students know or do not know based on their written work (especially their errors); and identifying "holes" in student thinking or ideas that students have not fully developed (Lampert, 2001; Mathematics Teacher Preparation Content Workshop Program Steering Committee et al. [MTPCWPS], 2001).

The importance of closely attending to, analyzing, understanding, and building on student work and contributions was demonstrated in the previous section when discussing the case of Bill (Thompson & Thompson, 1994, 1996). Bill could structure rich and challenging learning environments for the student with whom he was interacting. However, by failing to attend to and analyze the student's work and thinking, Bill fell short of providing the student with access to the environment that he crafted. Lampert (2001), Ball (1993b), and Fran (Lubinski, 1993), on the other hand, painstakingly followed and analyzed their students' thinking and work and adapted the learning environments they built, when necessary, to ensure *both* that rich and challenging mathematical ideas were considered during their lessons and that students could access and work on these ideas.

Pursuing this twofold agenda, namely honoring the integrity of the mathematics one teaches and also being responsive to students – or as Ball (1993) eloquently puts it having one's "ears to the ground" to listen to one's students and one's eyes "focused on the mathematical horizon" (p. 376) – is by no means easy. However, by analyzing student work and thinking, teachers cannot only ensure their students' access to the rich and challenging environments they craft but can also gain insights into *how* to craft such

environments, especially by capitalizing on their students' errors and misconceptions. Previous research on students' errors or misconceptions (Ben-Zeev, 1995; Resnick, Nesher, Leonard, Magone, Omanson, et al., 1989; Smith III, diSessa, & Roschelle, 1993) suggests that students' incorrect responses are often *systematic* and *rational*, as opposed to random and illogical: they are not caused by students' lack of knowledge, but rather by students' incomplete or problematic understanding of the content. In other words, these errors are due to the *presence* rather than the *absence* of knowledge. To complicate matters more, these errors often arise from the instructional approaches pursued when presenting the content to students.

Therefore, identifying these errors and deciphering their causes can support teachers' attempts to build rich and challenging environments in at least two ways. First, students' errors can be used as instructional tasks themselves, or inform the development of rich instructional tasks. Presenting and discussing these errors can help the students who committed these errors revise their thinking; it can also offer students whose work on these tasks was correct the opportunity to solidify their understanding of the embedded mathematical ideas. In this way, students' errors become occasions for further learning. Second, the analysis of students' errors can inform future instructional approaches. Specifically, teachers can decide which tasks to select and how to sequence them so that their future instruction helps avoid rather than reinforces particular errors and misconceptions that appear to have stemmed from instruction itself.

The present study thus explores the association between teachers' MKT and their performance in analyzing student work and contributions, and particularly student errors. In so doing, the study aims at providing empirical evidence justifying the

conceptualization of MKT as a significant asset in the work associated with error analysis (cf. Ball et al., 2005).

### *Responding to Students' Direct or Indirect Requests for Help*

The last practice examined in the study pertains to responding to students' requests for help. These requests can be either explicit/direct or implicit/indirect. Specifically, when assigned a task, students might explicitly solicit teachers' guidance and support. Alternatively, teachers might perceive students' frustration, confusion, struggle, or momentary lack of progress when working on an assigned task as indirect requests for support and scaffolding.

Responding to students' requests for help – be they explicit or tacit – constitutes one of the biggest predicaments teachers face (Ball 1993b; Hiebert et al., 1997, pp. 29-30; Lampert, 2001, pp. 121-122), especially if these requests are intense (Skott, 2001). This predicament becomes even more intense when teachers assign tasks that challenge students' thinking. In this case, as Doyle (1992) suggests, attempting to negotiate with the teacher the demands of the tasks and reduce their apparent complexity, students press for help either by asking for hints or guidance or by appearing reluctant to engage with the assigned task. How should teachers then support and guide students to explore and understand the mathematical ideas and procedures embedded in or targeted by the task without assisting them too much and thus doing most of the thinking for them? In other words, how should teachers *support* student thinking without *replacing* it?

The first TIMSS video-study that compared instruction in the U.S., German, and Japan revealed that American teachers often tended to respond to students' requests for help by doing most of the thinking for them and thus denying their students the

opportunity to move beyond their momentary impasses by drawing on their own resources. As Stigler and Hiebert (1999) explain, perceiving student confusion and frustration as indications of insufficient instruction, American teachers often rushed to assist students by providing most or all the information needed to complete a task. These scholars have, in fact, identified the same litany of events occurring in most of the American lessons they explored in their study:

Teachers assign students seatwork problems and circulate around the room, tutoring and monitoring students' progress. Several students ask, in quick succession, about the same problem. Teachers interrupt the class and say, for example, "Number twenty three may be a little confusing. Remember to put all the  $x$ -terms on the one side of the equation and all the  $y$ -terms on the other, and then solve for  $y$ . That should give you the answer. (p. 92)

In contrast to their American counterparts, Japanese teachers rarely showed their students how to solve the problem midway through the lesson. Instead, they closely monitored students' work so that they could then facilitate the comparison of different solutions. Occasionally, they provided some hints to support students' progress.

One plausible way to resolve this unfavorable situation, one could argue, would be to ask teachers to refrain from giving any information to their students and, limit themselves to first monitoring students' work and then facilitating the sharing of solutions. Alas, exhorting teachers "not to tell" cannot resolve the problem either. As Chazan and Ball (1999) maintain, such a behavior might lead to other, at least equally unpleasant, situations. If the teacher lets students follow their own path of thinking and totally avoids injecting mathematical ideas, students may end up with mathematically incorrect assumptions. When sharing and discussing their solutions, students might also fail to reach consensus; even worse, feeling ownership about their contributions, students might obstinately support their ideas without actually reflecting on theirs and others' contributions, thus missing important opportunities to focus on the mathematics itself.

Whatever the situation might be, students might end up with either ideas that distort the mathematics or with no ideas at all about the mathematics that the particular task was intended to surface and serve.

Providing students all the information they need or limiting teachers' role to just monitoring and facilitating student work and discussion define the two ends of the spectrum. Hence, one could reasonably argue that to effectively respond to students' requests for help teachers can move somewhere between these two ends. Even further, one might advocate that, when faced with students' requests for help, instead of providing students with all the information they need to complete a task, teachers can pose questions to gradually guide their students to construct the mathematical ideas at hand. However, this might lead to yet another unpleasant situation captured by what Brousseau (1997) called the *Topaze effect*. In their attempt to gradually lead their students to the targeted outcome – be it constructing a mathematical idea, seeing meaning in a mathematical procedure or building connections among different mathematical notions – teachers might pose questions that ultimately dilute the challenge and the richness of the assigned task.

Consider, for instance, the following scenario. After figuring out the formula that gives the area of a rectangle ( $\text{area} = \text{base} \cdot \text{height}$ ), in the next lesson, the teacher asks students to figure out a way to calculate the area of a parallelogram. The students start working on this task but soon become confused; some students even start asking for help. Aware of the importance of *not* telling students how to solve the problem, the teacher starts asking questions: “Remember what we did yesterday?” The question, though, does not seem to help. Worried that students are not “getting there,” the teacher intervenes

with increasingly guided questions: “How does the rectangle we discussed yesterday relate to the parallelogram?” “Can you use your parallelogram and make a rectangle?” “What would the base of the parallelogram be?” “Can you find its height?” At the end, the students *do* figure out that they can use exactly the same formula to calculate the area of the parallelogram. Alas, the teacher has again done all the thinking for them; the students merely executed steps indirectly suggested by the teacher’s pointed and leading questions. This and other similar scenarios<sup>47</sup> are by no means hypothetical. For example, this was the prevailing mode of instruction in one of the schools considered in Boaler’s (2002) study (see second section of this chapter). As the reader might well recall, this type of instruction significantly affected the quality of student learning.

The three main situations discussed above – providing students all the information necessary for solving a task, refraining from giving any information at all, and posing very pointed and leading questions – represent ways in which teachers’ reactions to students’ requests for help diminish the richness and the cognitive complexity of the assigned tasks. Therefore, these three scenarios collectively showcase how teachers’ practices vis-à-vis responding to students’ appeals for help can influence the richness and the cognitive level at which the content is experienced in mathematics classes, hence justifying the decision to explore this practice in this study. Previous research has attributed the aforementioned ways in which teachers’ respond to students’

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<sup>47</sup> A similar scenario is that recounted by Kennedy (2005) in discussing the case of Ms. Toklisch teaching a lesson on comparing the area of different circles. In this lesson, Ms. Toklisch asked her students to compare the area of two 9-inch pizzas with the area of a 16-inch pizza. Although the students estimated the area of a 9-inch and a 16-inch pizzas quite accurately, they insisted that the two smaller pizzas would “provide more to eat,” because the sum of their diameters was greater than the diameter of the larger pizza. After asking a series of clarifying questions to lead her students to the correct answer, and discouraged by her students lack of progress, Ms. Toklisch suggested that they move to another task, because they had other content to cover.



plea for help to teachers' beliefs in particular and the *cultural script*<sup>48</sup> of teaching in general (Stigler & Hiebert, 1999) or to teachers' limited questioning techniques (Boaler, 2002; Stein et al., 2007; Weiss & Pasley, 2004). The present study extends this work by exploring associations between teachers' knowledge and the practice of responding to students' requests for help.

Thus far, I have situated the study in the bigger research picture and justified the first three design decisions made in the study to narrow its scope and facilitate the exploration of its inquiry. In the present section, in particular, I have justified the selection of the five teaching practices the study considers and explained how these practices contribute to building rich and intellectually challenging environments. In the next section of this chapter I discuss the fourth design decision of the study.

#### Challenges Inherent in Exploring the Association between Teacher Knowledge and Their Practices

The preceding four sections focused on just one factor: teacher knowledge. Focusing on this factor helped simplify the teaching complexity, and thus facilitated a more in-depth exploration of how teacher knowledge can influence teachers' instruction. However, teaching is much more complex than delineated in these four sections; teachers' decisions and actions are informed by several factors other than teacher knowledge. Therefore, in this section, I bring back into play the teaching complexity I have purposefully excluded thus far; I also consider methodological challenges in exploring the association between teacher knowledge and performance. In conjunction,

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<sup>48</sup> Stigler and Hiebert (1999) use the term *cultural script* to refer to a set of cultural beliefs about teaching and learning mathematics that nurture, shape, and legitimize certain instructional approaches used within each culture when teaching this subject.

these challenges help explain the fourth design decision of the study: to explore the association between *PSTs'* knowledge and their practices via a *teaching simulation*.

This section consists of three parts. In the first part, I discuss other factors that might affect teachers' decisions and actions with respect to the practices the study considers. In the second part, I consider methodological challenges inherent in investigating the association between teachers' knowledge and practices, and in the last part, I justify the fourth design decision of the study.

### *Challenges Related to a Net of Factors Potentially Informing Teachers' Practices*

The factors that potentially inform teachers' instructional practices, and consequently their establishment of rich and challenging learning environments, can be clustered into four main categories: teacher-related factors, student-related factors, classroom-related factors, and other within-school or out-of-school contextual factors. Figure 2.6 presents these factors as the rectangles surrounding the ellipse. The ellipse presented in the center encapsulates a replica of Figure 2.5. Thus, Figure 2.6 portrays a more complex picture of teaching and the factors that might affect teachers' instructional decisions and practices. In what follows, I briefly consider each of the four sets of factors; this review is by no means comprehensive. Rather, it aims at illustrating the complexity of teaching and the several factors that researchers should consider when exploring teachers' practices. Notably, the ensuing review disproportionately emphasizes the teacher-related factors. The priority on the teacher-related factors by no means implies that these factors are more important than the other factors; I prioritize these factors because they are included in my investigation of the association between teacher knowledge and their performance in the selected set of practices.

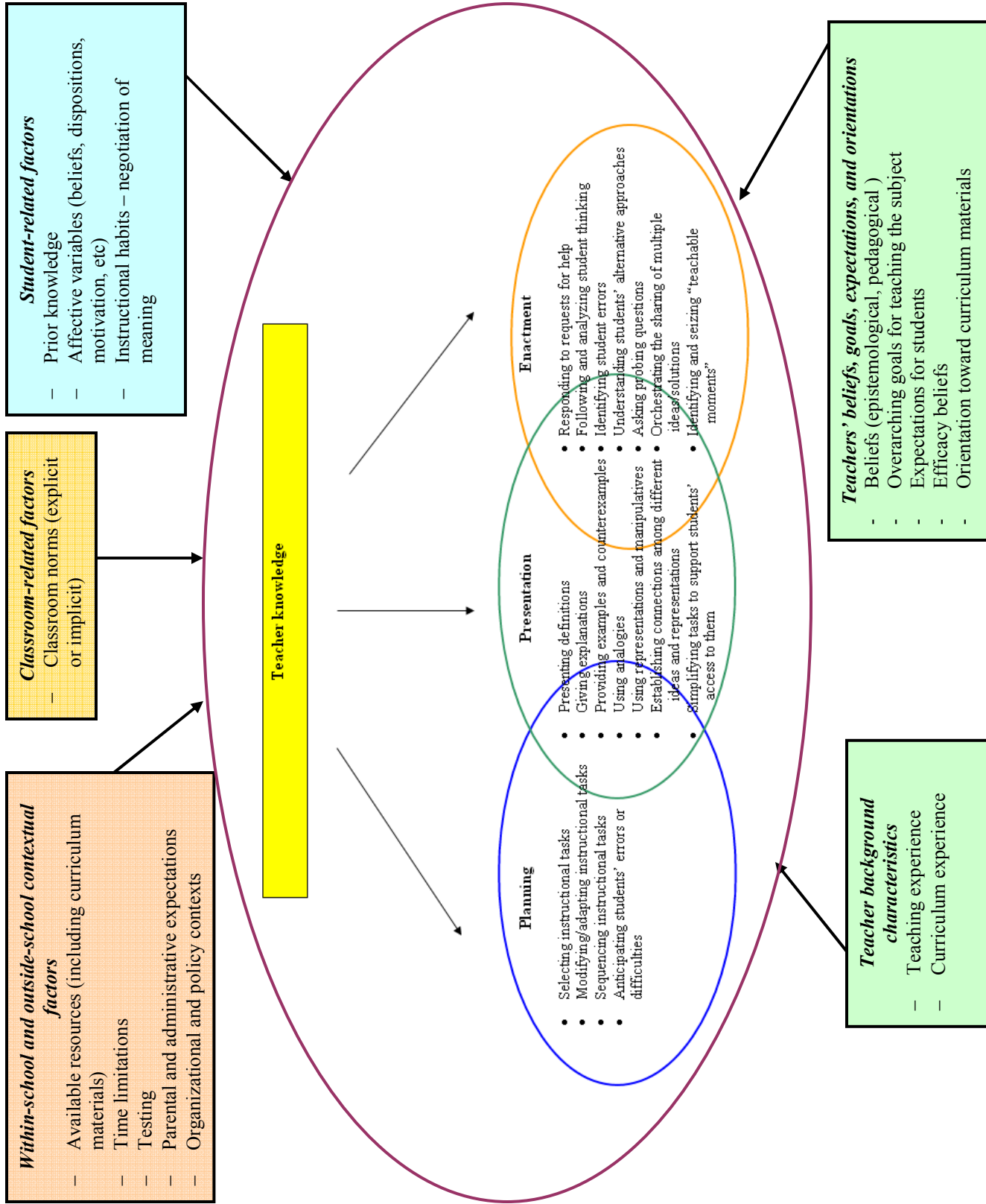


Figure 2.6. A complex net of potential mediators of the association between teacher knowledge and teaching performance.

### *Teacher-Related Factors*

Several studies conducted during the 1980s and the early 1990s (e.g., Berliner, 1988; Leinhardt, 1988; 1989; Borko & Livingstone, 1990) compared the practices of expert and novice teachers. These studies showed that compared to their novice counterparts, expert teachers are better decision-makers and can more easily improvise during instruction. Expert teachers were also found to provide better explanations, keep their lessons on track, and accomplish their lesson goals more frequently. Therefore, it appears that *teaching experience* plays a significant role in informing teachers' instructional decisions and actions. In fact, as several scholars (e.g., Feiman-Nemser, 1983; Lampert & Ball, 1998; Sherin, 2002) suggest, much of the learning of teaching might occur in the job: like other professionals, teachers can learn in and from their own practice. However, this learning *in* and *from* practice highly depends on the extent to which teachers closely attend to, analyze, and reflect on their practice. Hence, although teaching experience appears to influence teachers' practices, notable differences should be expected in the practices of teachers who share the same years of experience.

In addition to teachers' overall teaching experience, the analyst should also take into consideration teachers' *curriculum experience*, namely the period that teachers have been using the same curriculum. Previous studies that traced teachers' use of instructional materials at different points suggest that teachers' instructional approaches improve as teachers get more familiar with the curriculum materials (Remillard, 2000, 2005). However, a recent study by Silver and colleagues (Silver et al., in press) complicates matters. In this study, these scholars use the term *curriculum implementation plateau* to suggest that after implementing a curriculum for some years, teachers might reach a stage

during which their instructional approaches do not seem to improve any further. Despite their familiarity with the curriculum materials, teachers at this stage might still encounter significant instructional challenges, the more significant being how to support their students' thinking without replacing it. Thus, although curriculum experience appears to inform teachers' practices, the pattern of this association remains unclear.

Besides these teacher *background characteristics*, previous studies have pointed to a conglomerate of other *within-teacher factors* that influence their instructional decisions and actions. These include teachers' epistemological and pedagogical beliefs, their goals and relative emphases for teaching mathematics, their efficacy beliefs, and their orientations toward their school curriculum materials.

*Teachers' beliefs* about the nature of mathematics (i.e., epistemological beliefs) and the teaching and learning of mathematics (i.e., pedagogical beliefs) constitute key factors in research on teachers and their teaching practices, especially after the publication of Alba Thompson's (1992) seminal work, in which she elaborated the concept of beliefs and discussed the structure of teachers' beliefs systems. In this review paper, Thompson also recommended that researchers explore how both teachers' beliefs and knowledge influence teachers' practice, pointing out that "[t]o look at research on mathematics teachers' beliefs and conceptions in isolation from research on mathematics teachers' knowledge will necessarily result in an incomplete picture" (Thompson, 1992, p. 131). Ever since, several scholars have systematically explored or discussed the impact of teachers' beliefs as mediators (or even predictors) of teachers' instructional practices,<sup>49</sup>

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<sup>49</sup> Supporting evidence about the role of beliefs as *mediators* of the association between teacher knowledge and practices is presented in Skott (2001) and Wilkins (2008). In the first study, Skott documented how the same teacher, Christopher, responded to different students' requests for help in markedly different ways, partly due to his beliefs about teaching and learning and his dissimilar expectations for the students

including teachers' selection, adaptation, and presentation of curriculum tasks, their questioning techniques, their tolerance of student frustration, confusion, and errors, and the manner in which they support their students when the latter face momentary impasses (e.g., Collopy, 2003; Manouchehri & Goodman, 2000; Remillard, 1999, 2005; Ross et al., 2003; Skott, 2001; Speer, 2008; Stein et al., 2007; Stigler & Hiebert, 1999; Stipek, Givvin, Salmon, & MacGyvers, 2001; Wilkins, 2008; Wilson & Cooney, 2002; Wilson et al., 2005).<sup>50</sup> The accumulating body of research in this area notwithstanding, in his recent review of research on teacher beliefs, Philipp (2007) argued that the complex relationship among teachers' beliefs, knowledge, and practices still remains unclear (p. 257).

*Teachers' goals and the expectations* they hold for their students have also been found to inform teachers' instructional decisions and actions. For instance, think of Teresa, the elementary teacher described in Sztajn (2003). When teaching mathematics, she placed more emphasis on helping her students learn basic facts and develop procedural fluency, because she considered this type of instruction more suitable for her students of lower socioeconomic backgrounds, whom she perceived to possess low cognitive skills. Hence, even though she recognized that helping students to develop

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who solicited his help. In the second study, using structural equation modeling, Wilkins (2008) found that teachers' beliefs *mediated* the association between their content knowledge and the frequency with which they reported using certain practices in their teaching. Moving a step farther, in a recent study, Speer (2008) argued for and provided evidence supporting that teachers' beliefs can help explain and even predict teachers' moment-to-moment instructional decisions and actions. As she maintained, to achieve such an explanatory power, these beliefs should be sufficiently fine-grained. She also proposed the notion "collections of beliefs" as an instantiation of such a fine-grained analysis unit.

<sup>50</sup> Consider, for instance, the two teachers discussed in Remillard (1999), who both used the same curriculum. Convinced that mathematics is a collection of topics to be mastered and that students learn from being told or shown what to do and how to do it, Catherine paid attention to the drill-and-practice tasks included in the curriculum. Jackie, in contrast, believed that learning occurs when students are given the opportunity to engage in different problems, to invent solutions, and to explore different relationships among and between ideas. Therefore, in planning her lessons, Jackie drew more frequently on cognitively demanding tasks and adapted them to meet her students' needs and engage her students in high-level mathematical thinking.

higher order thinking skills is critical to their intellectual development, she mainly selected and implemented unchallenging tasks.

*Teachers' efficacy beliefs* – namely teachers' sense of ability to organize and execute teaching that promotes learning (cf. Bandura, 1997) – have also been considered in studies that explored factors influencing teachers' instruction. Pointing to the importance of including teachers' efficacy beliefs in investigations of their teaching, Bandura (1997) asserted that “[t]he self-assurance with which people approach and manage difficult tasks determines whether they make good or poor use of their capabilities. Insidious self-doubts can easily overrule the best of skills” (p. 35). Thus far, research has suggested that teachers with high efficacy beliefs set high goals for their instruction and are more willing to experiment with more demanding instructional approaches (Bandura, 1997; Tschannen-Moran, Hoy, & Hoy, 1998). Given that establishing rich and challenging learning environments imposes several demands on teachers and encompasses several risks (Doyle, 1992), one could reasonably expect that teachers might be more inclined to establish such environments when they teach topics in which they feel more competent. The case of Shey (Sowder et al., 1998), as discussed in the third section of this chapter, supports this claim: when Shey felt more confident himself in the area of multiplicative reasoning, he employed more complex tasks and maintained the cognitive demand of these tasks during their implementation.

Finally, Remillard and Bryan's study (2004) reviewed in the previous section suggests that teachers' instructional approaches can also be influenced by their *orientations toward their school curriculum materials*. In this study, teachers who were positively disposed toward their school curriculum followed the curriculum suggestions

more closely, compared to their colleagues who were not strong supporters of this curriculum. The differences found in teachers' instructional approaches and particularly the learning settings they furnished for their students were attributed to the teachers' dissimilar implementation of this curriculum.

Overall, the studies reviewed above collectively suggest how complex the analysis of teachers' decisions and actions during instruction is, even when considering only factors related to teachers themselves.<sup>51</sup> This complexity is further aggravated when student, classroom, and other contextual factors are brought into play.

### *Student-Related Factors*

Several student-related factors can potentially affect teachers' instructional decisions and practices at different phases, especially if teachers design their instruction to be responsive to their students' characteristics. These factors include students' prior knowledge and several dimensions of the affective domain: beliefs, values, emotions, dispositions, and motivation (Bransford, Brown, & Cocking, 2000; Chazan, 2000; Kilpatrick et al., 2001; Mack, 1995). Additionally, factors related to how students engage with the content, and particularly their instructional habits, should be considered when considering the decisions that teachers make and especially those related to establishing rich and intellectually demanding learning environments. As several scholars (Doyle, 1983, 1992; Stein et al., 1996; Henningsen & Stein, 1997) suggest, when teachers select demanding tasks for their instruction, students often press them to reduce the complexity

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<sup>51</sup> Schoenfeld (1998, 2008) proposed and used a theory to explain teachers' in-the-moment decision making. According to this theory, teachers' decisions are the result of a complex cost-benefit analysis, which is informed by teacher's *goals* (i.e., overarching goals, major instructional goals, "local" goals), *beliefs* (i.e., beliefs about teaching, learning, students, classroom environment, and mathematics), *knowledge* (i.e., subject matter knowledge, general pedagogical knowledge, pedagogical content knowledge, knowledge of the students, knowledge of the history of the classroom), and *decision-making mechanisms*.



of the work. As explained in the previous section of this chapter, to respond to such persistent calls, teachers often split the task into simpler steps, suggest procedures to follow, or “take over” the difficult parts of it. Christiansen (1997) also found that regardless of the cognitive level at which a task gets presented, while interacting with their classmates and attempting to understand the requirements of an assigned task, students might also negotiate and decrease its cognitive demands by proceduralizing its more challenging aspects.

### *Classroom-Related Factors*

The established spoken or tacit *classroom norms* also influence teachers’ instructional decisions and practices. Whereas it is easier to understand the effect of the explicit norms on teachers’ instruction (e.g., a norm that designates how students are expected to work when assigned a problem), the influence of the implicit norms is harder to discern and study. However, these unspoken norms appear at least equal in how they affect the teacher-student interactions. Brousseau (1997) coined the term *didactical contract* to capture these implicit classroom norms that govern the teacher-student interactions around the content and that define distinct roles for the teacher and students during instruction, and accordingly determine what and how something gets taught and learned. Building on this notion, in a recent study, Herbst (2006) illustrated how these norms might shape certain teacher decisions and actions. In this study, he explored how Megan, an experienced high-school teacher, worked with her students on a rich and intellectually demanding task. While enacting this task with her students, Megan deviated significantly from the implementation plan that Herbst and herself had developed; confronted with her students’ difficulties in arriving at the expected conjectures, Megan

provided students with guidance that, in essence, diminished its cognitive complexity. Yet, in so doing, she abided by the unspoken rules of the didactical contract that expects the teacher to structure her instruction so that it yields at least some student learning.

### *Within-School and Outside-of-School Contextual Factors*

Several studies have proposed a multitude of *contextual factors* operating within or outside school that appear to influence teachers' instructional decisions and actions (Cohen, Raudenbush, & Ball, 2003; Henningsen & Stein, 1997; Prawat, 1992; Price & Ball, 1997; Remillard, 2005; Stein et al., 2007; Wilson, 1990). These factors include the availability of certain instructional resources (e.g., manipulatives, curriculum materials); perceived or actual time constraints and pressures to cover the curriculum; parents' and administrators' expectations for student performance in general and on external examinations and standardized tests, in particular; and the organizational and policy context in which teachers' work (e.g., the extent to which teachers' instructional decisions and actions are endorsed by their school principal or the district superintendent).

The factors reviewed thus far are only a set of the potential contributors to teachers' decisions and actions during planning, task presentation, and enactment. In briefly discussing these factors, I sought to depict the complex net of interrelated forces associated with teachers' instructional decisions, and consequently identify the challenges inherent in exploring the association between teacher knowledge and practices. In what follows, I consider other methodological challenges related to investigating this association.

### *Methodological Challenges*

Assume, for the sake of argument, that a researcher could explore the association between teacher knowledge and the five practices the study considers using an *experimental design* with random assignment of teachers to student, classroom, and school conditions. In particular, imagine that this researcher could recruit both a sufficiently large random sample of the U.S. teacher population and a large number of classrooms and schools, randomly selected. Also imagine that the researcher could randomly assign teachers to students, and schools. Given the random selection of subjects and conditions as well as the random assignment of subjects to conditions, one might expect that this researcher could quite effectively explore the association under consideration, and even identify causal-effect relationships. Even if this design could be implemented, and also assuming that the researcher could obtain valid and reliable measures of the factors/constructs considered above, this researcher would still face several methodological challenges.

Take, for instance, the practice of analyzing student errors. This experimental design cannot ensure that teachers will be confronted with and expected to analyze certain student errors. Therefore, how can one explore the association between teacher knowledge and this particular practice, if it cannot be ensured that all teachers are exposed to the same student errors? Or take the practice of responding to students' requests for help. How likely is it that students in all these classrooms ask for help at the same juncture while solving a task? That they ask for the same type of help? Or to complicate matters even more, that they ask for the same type of help in exactly the same

manner and with the same intensity? Even were it possible to ask students to do so, how likely would it be that their requests for help would be adequately comparable?

Let us also assume that all teachers were asked to teach the same content in order to obtain comparable measures of their respective practices. Obtaining such measures would still remain difficult for at least two reasons. First, because the tasks teachers select might determine teachers' performance in the other four practices: their use of representations, the explanations they provide, the extent to which their students will solicit their help (and consequently how the teachers will respond to these requests), and the opportunities they have to consider certain student errors. Second, because teachers are not the only actors of what transpires during instruction. For example, as the case of Ms. Daniel (Borko et al., 1992) suggests, even if teachers might not plan to provide any explanations or use any representations, their students might press them to do so.

An additional methodological challenge relates to measurement issues. It was previously assumed that it is possible to obtain valid and reliable measures of the different factors depicted in Figure 2.6. But how realistic is this assumption? Take, for instance, the example of teachers' experience, be it their teaching or curriculum experience. Is it sufficient to measure this experience in terms of the years that teachers have clocked in schools or have used a certain curriculum? In other words, if the researcher recruits teachers who have used the same curriculum for the same number of years, would that be enough to capture teachers' experience with this curriculum? If we agree that the magnitude of teachers' experimentation with the available curriculum materials and their reflection on their own practice matter, then simply considering the *duration* of teachers' curriculum use is not adequate.

Instead of using a randomized experimental design to explore the inquiry of the present study – a design recently considered to be the gold standard of scientific research in education (cf. NRC, 2002; Schoenfeld, 2006) – the researcher could alternatively pursue an ethnographic approach (Eisenhart, 2001). Rather than identifying and “controlling” several noise factors – which constitutes the foundation of randomized experimental designs – this latter approach could help provide thick descriptions of teachers’ practices, based on which the analyst could then make assertions about the inquiry the present study entertains.<sup>52</sup> Its multiple strengths notwithstanding, this design has certain limitations, too. Specifically, if the researcher collects data merely through classroom observations, the researcher can make only *inferences* about teachers’ knowledge, just like the studies reviewed at the beginning of the third section of this chapter did. Additionally, given that the tasks teachers select might shape their performance with respect to the other four teaching practices, the ethnographic approach cannot ensure that teachers will engage in these practices. Consider, for example, the case of a teacher who largely selects and presents simple algorithmic tasks; the students of this teacher are less likely to press him or her for more explanations or support if they are assigned tasks that merely expect them to apply known procedures. Finally, to obtain thick descriptions of the phenomenon under exploration, the analyst might need to limit her or his investigation to a rather small sample of teachers, which, in turn, can restrict the variability of the phenomenon at hand.

The list of methodological challenges considered above is neither comprehensive nor exhaustive. It does indicate, though, several of the complexities associated with

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<sup>52</sup> In fact, some of the studies considered in the third section of this chapter (e.g., Lloyd & Wilson, 1998) could be considered ethnographic, since they followed teachers for rather an extensive period of time.

exploring the inquiry of the present study. Taking these complexities into consideration, I decided to investigate *preservice teachers' (PSTs')* knowledge and practices *via a teaching simulation*. In the last part of this section, I justify this dual design decision.

*Exploring PSTs' Knowledge and Practices via a Teaching Simulation:*

*Justifying the Fourth Design Decision*

The decision to focus on PSTs might seem counterintuitive. “How can one explore the knowledge *for* teaching or the teaching practices of PSTs, given that these teachers have not yet had the opportunity to teach and hence, develop such knowledge or practices?” the critical reader might ask. Although this concern is legitimate, it should be borne in mind that PSTs enter teacher education programs with quite strong images of mathematics and its teaching. As Lampert and Ball (1998) argue, in contrast to neophytes in other professions, PSTs bring with them well-established ideas about teaching and learning mathematics which they have developed through their long apprenticeship as students of mathematics. For example, although they might not have had opportunities to use representations *to teach* a mathematical idea or to support students when the latter solicit their help, PSTs *do* know what these practices look like. Similarly, the knowledge *for* teaching that teachers possess is not only developed *in* practice (i.e., while they work as teachers), but also through vicarious experiences: through their experiences as learners of mathematics and as observers of the practice of other teachers.

The critical reader, though, might still have other reservations: “Even if we assume that PSTs hold some images of these teaching practices, these images are largely *imitations* of their teachers' practices.”<sup>53</sup> Hence, by focusing on PSTs, in essence, one

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<sup>53</sup> See Ball (1988) for an elaboration of this argument.

explores an association between PSTs' knowledge and the practices that *other* teachers have developed and used." I argue that this "inconsistency" does not really hold. As the theorists of the *situated*-knowledge perspective argue, the knowledge that one holds is the by-product of the *activity, context, and culture* in which one's knowledge has been developed (Brown, Collins, & Duguid, 1989; Lave & Wenger, 1991). Hence, although it is reasonable to expect that the knowledge that PSTs possess encompasses, to some extent, features of the learning environment in which this knowledge has been constructed, this does not imply that this knowledge is simply a mere reflection of the practices of other teachers. Besides, it should not be ignored that like every learner, PSTs do not unwittingly adhere to just any image or model of teaching they experience. Rather it seems more reasonable to assume that they *filter* and *appropriate* the images and models of teaching they consider more effective and suitable for them.

The reader might then reasonably wonder: "Wouldn't it be more reasonable to focus on in-service teachers rather than on PSTs?" Although I do not dismiss the several benefits associated with focusing on in-service teachers, I do believe that there also exist significant benefits in focusing on PSTs. First, PSTs represent a somewhat "intact" teacher population, in the sense that, in contrast to in-service teachers, they most likely have not had any *substantial* personal teaching or curriculum experience. Additionally, they have not yet experienced – at least as teachers – the multitude of pressures stemming from several contextual factors considered in the previous section (e.g., testing, parental and administrative expectations, time pressure, pressure to cover the curriculum) that impinge on teachers' decisions and actions. Second, as previous research suggests (e.g., Ball, 1998; Crespo & Nicol, 2006; Lampert & Ball, 1998; Tirosh, 2000) these candidate

teachers often enter the teacher preparation programs with weak knowledge of the content and its teaching. This means that there is significant room for change and improvement of their knowledge. This is important for the purposes of the present study, which, incidentally, does not only seek to explore the association under consideration at a fixed point in time but also to investigate this association by considering changes in knowledge and practices. Finally, from a pragmatic perspective, it is often easier to recruit prospective rather than in-service teachers, especially when the researcher is interested in exploring their knowledge, practices, and changes thereof.

The critical reader might also point to certain limitations in exploring teachers' practices by using a teaching simulation, the most critical being that a teaching simulation constitutes an artificial environment. Due to its artificiality, the teaching simulation excludes several of the factors presented in the previous section that can inform teachers' decisions. Hence, the reader might argue that a better option would be to explore PSTs' practices *in-vivo*, for example, during their fieldwork placement. Although these concerns are legitimate, I argue that there are several advantages in exploring teachers' practices *in-vitro*.

First, the teaching simulation helps simplify the teaching complexity, since it excludes some of the contextual factors (e.g., classroom management problems, mentors' teaching style and expectations from PSTs) which could impinge on the decisions that PSTs make *in-vivo*.<sup>54</sup> In so doing, the teaching simulation allows for a more in-depth exploration of the association between teachers' knowledge and practices. I acknowledge, though, that teachers' perspectives about the subject and its teaching (e.g., their

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<sup>54</sup> See Walshaw (2004) for an example of how mentors/collaborating teachers limit PSTs' fieldwork decisions and actions.



epistemological and pedagogical beliefs, their efficacy beliefs, their perceptions about what is important to emphasize when teaching mathematics), and other background characteristics might still color their decisions and actions, even in an in-vitro situation. Hence, besides exploring the association between knowledge and practices, in this study, I also consider the potential mediating effects of such factors on the participants' performance in the five teaching practices.

Second, performing in-vitro ensures that all PSTs “teach” or consider the same lesson, which facilitates comparability. Even more critically, as explained in the next chapter, the simulation used in the present study immerses participants in all three MTF-phases (planning, task presentation, and task enactment); hence, it helps obtain comparable data with respect to all phases of task-unfolding.

Third, even if it were possible to ensure that all PSTs would teach the same lesson in an in-vivo situation, it would still not be realistic to expect that the students in the PSTs' classes would react in a comparable fashion, and particularly, in ways that call for making certain decisions that pertain to the five teaching practices examined in this study.

Fourth, exploring beginners' decisions and actions in simulated conditions has been a common practice in other fields. Consider, for example, the preparation of candidate pilots.<sup>55</sup> Before being asked to fly airplanes in real conditions, intended pilots are required to participate in simulated flights. As simulations, these flights expose pilots to a selected gamut of challenges they might encounter in real flights, thus facilitating the

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<sup>55</sup> Three parallels between teaching and flying an airplane informed the selection of this analogy. First, teaching and flying an airplane are both complex performances. Second, the complexities in both performances can somewhat be attenuated by the development and application of certain routines (consider, for example, the automatic pilot in the case of flying). Third, despite the development of these routines, both performances often require making decisions *in situ* which might critically affect the final outcome of these performances.

investigation of these candidates' potential to make those decisions and engage in those actions necessary for a secure flight. Grossman and colleagues (Grossman & McDonald, 2008; Grossman et al., in press) remind us that analogous simulations have also been used in medicine and law. These scholars also use the term *approximations of practice* to describe such simulated environments that furnish PSTs the opportunity to enact challenging practices in relatively safe and low-risk environments, to “lear[n] to kayak on calm waters,” to use their own words (Grossman et al., in press).<sup>56</sup> By simplifying certain aspects of teaching and aggravating others, these approximations, they argue, offer unique opportunities for novices to not only learn from *doing* but also learn from their own mistakes. Although mainly used as a research tool rather than as a learning site, the teaching simulation utilized in this study constitutes such an approximation of practice, since it immersed PSTs in a simulated environment and asked them to enact certain tasks pertinent to the five practices of interest. From this perspective, this teaching simulation can also be considered a response to Grossman and McDonalds' call (2008) to shift from *pedagogies of investigation* to *pedagogies of enactment*. As these researchers explain, for many years the dominant paradigm in research on teaching has been to solely focus on investigating several teacher characteristics, such as teachers' knowledge and beliefs, without actually examining what teachers can actually *do*. If we really want to understand how teaching can affect student learning, these scholars maintain, we need to also employ pedagogies of enactment, namely to examine what teachers can do in actual teaching settings or in approximations of teaching.

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<sup>56</sup> In this work, Grossman and colleagues also talk about *representations of teaching*. I avoid referring to the simulation utilized in this study by this term, because I use the term “representations” when referring to one of the practices examined in the study. Additionally, in contrast to the representations of teaching discussed in Grossman and colleagues' work, the simulation used in this study expected PSTs to engage in certain teaching practices, instead of simply asking them to discuss or reflect on these practices.

Fifth, in mathematics education, teachers' decisions and actions as well as the rationale thereof have been evoked and investigated through several tools, ranging from short classroom vignettes (e.g., Ball, 1988; Stecher, Vi-Nhuan, Hamilton, Ryan, Robyn, et al., 2006) to narrative cases (e.g., Stein et al., 2000; Silver, Clark, Ghouseini, Charalambous, & Sealy, 2007) to discussions of videotaped lessons (e.g., Jacobs & Morita, 2002; Sherin, 2002; Star & Strickland, 2008) to collections of different records of practice (e.g., Lampert & Ball, 1998) and, more recently, to animated movies and computer simulations (e.g., Herbst & Chazan, 2003, 2006; Herbst, Chazan, & Nachieli, 2007). Often carefully crafted to focus on certain challenges and dilemmas inherent in teaching mathematics, these representations of teaching aim at probing what informs the decisions that teachers make at different junctures of a lesson or explore their reactions to certain instructional issues or challenges. Although made in-vitro, these decisions and actions are not unrelated to those that teachers could undertake in real-classroom settings. Stecher and colleagues' study (2006), for example, showed moderate correlations between the decisions that a sample of approximately 40 teachers made while teaching and those they made while considering a set of classroom vignettes. Interestingly enough, these correlations were higher than those between teachers' actual practice and their responses to a relevant survey or an instructional log.

In sum, despite its artificiality, a teaching simulation appears to be a promising approach to explore PSTs' decisions and actions, for several reasons. It helps simplify the complexity of teaching and thus facilitates a better investigation of the association this study examines. It also exposes all participants to exactly the same stimuli and engages them in exactly the same teaching episodes and conditions. As an approximation of

teaching it expects PSTs to engage in certain teaching practices rather than simply contemplate them; additionally, as a designed environment, it lends itself to crafting instances that call for certain decisions and actions – such instances might not arise in real situations. Finally, similar designs have been employed in other fields to explore candidates' performances. Previous studies in mathematics education also suggest that the decisions that teachers make in such artificial environments are not unrelated to their reactions in similar real-classroom settings.

One last comment is in order here. Whereas the artificiality of this environment enhances the internal validity of the findings of the study, since it simplifies complexity to better study the knowledge-performance association, it errs on the side of the external validity – or to phrase it within the context of a qualitative paradigm, the applicability of the study findings. In other words, if a PST performs in a certain way during the teaching simulation, what might this tell us about his or her performance in a real-life situation? Although, as already mentioned previous research suggests that teachers' in-vitro performance is not unrelated to their in-vivo performance, one needs to interpret the results of this study with caution. The reader, however, is reminded that this study undertakes a first step in exploring the association between teachers' knowledge and practices. A future study might extend and complement the findings of the present study by comparing participants' performances in-vitro and in-vivo. I further elaborate upon these issues in the last section of the next chapter, where I contemplate the generalizability/applicability of the study findings.

## Focusing on the Division of Fractions: The Fifth Design Decision

The studies presented in the previous sections, and particularly those that combine an “affordance” and a “constraint” approach, suggest that teachers’ knowledge of different mathematical topics varies. In Fennema and Franke (1992), for instance, Ms. Jackson possessed a profound knowledge of arithmetic word problems, but her knowledge was weak in fractions; similarly, Ms. Lehava (in Kahan et al., 2003) was strong in geometry, but her knowledge of combinatorics was shaky. These differences in teachers’ knowledge call for an additional design decision: the researcher could either focus on a wide variety of topics but explore them in less depth, or focus on a single mathematical topic, but investigate it in more depth. Since the present study aims at examining how teachers’ knowledge informs their decisions and actions regarding a wide array of teaching practices, I decided to concentrate on a single topic, namely the *division of fractions*. This design decision, one might argue, limits the applicability of the study findings, particularly if one assumes that participants’ performance in the five practices explored in the study differs from topic to topic. Yet, even in this case, the findings of this study could provide the foundation for similar investigations in other topics.

Mathematical, pedagogical, theoretical, practical, and methodological considerations informed this design decision. From a *mathematical* perspective, the division of fractions lies at the intersection of division *and* fractions, both of which are rich mathematical topics on their own and central to school mathematics. The concept for division, for instance, is considered consequential for the learning of other mathematical topics such as place value, rational and irrational numbers, and even more complicated topics such as limits and powers (Ball, 1988, p. 60). Fractions, on the other hand, are

fundamental for the learning of rational and irrational numbers, proportional reasoning, and algebra (Lamon, 1999, NCTM, 2000; National Mathematics Advisory Panel, 2008). Although studying division and fractions separately is valuable because of the mathematical richness of these topics and their importance for school mathematics, studying the division of fractions carries additional value, because it offers a venue for considering the *confluence* of these two topics. In fact, as Vergnaud (1983, 1988) argues, division and fractions are mathematically interconnected concepts and fall within the same conceptual field, that of multiplicative structures. Hence, focusing on the division of fractions could facilitate exploration of the interconnectedness of division and fractions as well as of how the construction of each of them might contribute to understanding the other, and in turn, to a more solid understanding of the division of fractions. Such an understanding is critical both for teachers and students since the division of fractions comprises one of the hardest topics to teach and learn in the upper elementary and middle-school grades.

As several scholars (Ball, 1988, 1990; Ball & Wilson, 1990; Flores, 2002; Lubinski, Thomason, & Fox, 1998; Ma, 1999; MTPCWPS, 2001; Siebert, 2002) have pointed out, teaching the division of fractions requires a solid understanding of a multitude of concepts and processes. Teachers should know both the partitive and the quotitive/measurement interpretation of division (cf. Greer, 1992, pp. 276-277);<sup>57</sup> they

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<sup>57</sup> In the partitive interpretation, the divisor represents the numbers of shares to be made and the question is to figure out the size of these shares (e.g., “Four marbles are shared equally between two children. How many marbles does each get?”). In the quotitive/measurement interpretation of division the interest lies in figuring out how many shares of a specific size denoted by the divisor can be made from the quantity represented by the dividend (e.g., “How many times does 2 fit into 4?”). Since in the partitive interpretation the divisor represents the numbers of shares, it is harder to generate a problem in which the divisor is a fraction, although such problems can be generated (e.g., “If  $\frac{1}{4}$  of a pizza slice costs  $\frac{3}{4}$  of a dollar, how much does the whole pizza cost?”). Vergnaud (1983, pp. 129-133) considers these two types of division as constituent components of the first type of multiplicative-structure problems, that of

should understand the role of reference units (i.e., units defined by the dividend or the divisor) to appropriately interpret and explain both the whole-number part and the fractional part of the quotient;<sup>58</sup> they should be aware of the multiple interpretations of fractions, and primarily that fractions actually represent divisions; they should also have a solid understanding of inverse operations and what a reciprocal number represents. Teachers should also be able to explain why the “invert-and-multiply” rule works. Otherwise, when faced with students’ requests for conceptual explanations of the reciprocal rule – like Ms. Daniel in Borko et al. (1992) – they run the risk of not being able to offer such explanations. Additionally, teachers might benefit from being aware of non-conventional algorithms suitable for dividing fractions (e.g., divide the quotient of the numerators with the quotient of the denominators), especially given that such algorithms have made their way into curriculum materials.<sup>59</sup> In sum, the multiple components that contribute to a thorough understanding of the division of fractions provide a multitude of “entrance points” to explore PSTs’ MKT and, hence, increase the range of the potential variability in the knowledge under exploration. Both these aspects, in turn, support a better exploration of the inquiry the study undertakes.

From a *pedagogical* perspective, given its mathematical richness and its complexity, the topic of the division of fractions lends itself to better exploring teachers’ potential to generate mathematically rich and challenging learning environments. Such

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*isomorphism of measures*, which refers to problems that represent simple direct proportions between two measure spaces,  $M_1$  and  $M_2$ . He also postulates (1988, p. 90) that learners’ competence in solving simple proportional problems might form a hierarchy with understanding each of the partitive and measurement interpretations situated at the second level of this hierarchy and understanding both of them and their connections lying at its third level.

<sup>58</sup> Take, for instance, the division  $4 \div \frac{3}{4}$ . In this division, the reference unit is  $\frac{3}{4}$ . After taking five  $\frac{3}{4}$ -pieces away from the dividend, a piece of  $\frac{1}{4}$  is left. However, the remainder is  $\frac{1}{3}$  of the  $\frac{3}{4}$  piece, the reference unit in this division. This is why the final answer to this division is  $5 \frac{1}{3}$  and not  $5 \frac{1}{4}$ .

<sup>59</sup> As discussed in Chapter 3, such a non-conventional algorithm was included in the second textbook page utilized in this study.

environments can be established in many different ways: by paying attention to the underlying *meaning* of the division of fractions, by discussing the different interpretations of the quotient and the remainder in division problems, by explaining why the traditional invert-and-multiply algorithm works and why it makes sense, and by building connections to other topics such as the division of whole numbers or decimals. To craft such environments, however, teachers need a profound understanding of this topic. Recall, for example, the case of Tom as discussed in Sowder et al. (1998). Since Tom's own knowledge could not support him in engendering a rich learning environment, he suggested a gimmick that his students could follow in dividing fractions. Likewise, another participant in that study, Linda, argued: "If I don't really understand what the division of fractions means, I'll have a hard time conveying it to my kids" (p. 70). Tom and Linda, however, are not the exceptions to the rule; rather, they represent the norm, as several scholars argue (Flores, 2002; Ma, 1999; Tirosh, 2000). In sum, the division of fractions lends itself to creating rich and challenging environments. Nevertheless, as suggested by previous studies, the establishment of such environments is contingent on teachers' respective knowledge. Hence, the division of fractions appears to be a good fit for exploring the research questions of the study.

Specifically, because of its richness and complexity, the topic of the division of fractions creates a rich space for exploring teachers' potential to successfully engage in the practices considered in this study. Specifically, textbook presentations of this topic often range from a conceptual treatment of the topic (e.g., *Connected Mathematics*)<sup>60</sup> to an almost procedural treatment of it with almost no conceptual explanation at all (e.g.,

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<sup>60</sup> Lappan, Fey, Fitzgerald, Friel, & Phillips (2006). *Connected Mathematics 2: Bits and pieces II* (Teacher's Guide). Upper Saddle River, NJ: Pearson Prentice Hall. (see in particular, Investigation 4, pp. 89-119).



*Harcourt Math*,<sup>61</sup> *Houghton Mifflin Mathematics*<sup>62</sup>) and several shades of procedural and conceptual approaches in between (*Everyday Math*,<sup>63</sup> *Scott Foresman-Addison Wesley Mathematics*<sup>64</sup>). This variety in the presentation of the content gives the opportunity to explore teachers' selection and use/modification of the existing textbook tasks. Additionally, in teaching this topic, teachers often face the challenge of designing and delivering coherent and meaningful explanations as well as using appropriate representations to support their students' learning. Finally, when working on this topic, students often commit several types of errors (cf. Tirosh, 2000), including *algorithmic errors* (e.g., they invert the dividend instead of the divisor or invert both fractions); *intuitive errors* (e.g., they consider a division in which the dividend is smaller than the divisor impossible, because "the dividend should always be larger than the divisor"); and *errors stemming from overgeneralizations* of certain properties of operations (e.g., they consider  $1 \div \frac{1}{2}$  equivalent to  $\frac{1}{2} \div 1$ ). All these errors create opportunities to explore teachers' capacity to analyze, understand, and scaffold student thinking.

From a *theoretical* viewpoint, considering the topic of the division of fractions allows building on but also extending the work done in this area. So far, the extant studies in this area have largely reported that PSTs' knowledge of the division of fractions is algorithmic and fragmented.<sup>65</sup> However, it has not been systematically explored what

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<sup>61</sup> Maletsky, E. M., McLeod, J. C., Scheer, J. K., et al. (2002). *Harcourt Math* (California Teacher's Edition, Grade 6, Volume 1, pp. 210 -215). Orlando, FL: Harcourt.

<sup>62</sup> Greenes, C., Leiva, M. A., Vogeli, B. R., et al., (2002). *Houghton Mifflin Mathematics* (Teacher's Edition, Grade 5, Vol., 2, pp. 384-388). Morris Plains, NJ: Houghton Mifflin

<sup>63</sup>The University of Chicago School Mathematics Project (2007). *Everyday Mathematics (Teacher's Lesson Guide, Grade 5., Vol. 2, pp. 680-685)*. Chicago, IL: McGraw-Hill

<sup>64</sup> Charles, R. I., Crown, W., Fennell, F., et al., (2004). *Scott Foresman- Addison Wesley: Mathematics* (Teacher's Edition, Grade 5, Vol. 3, pp. 502-503 ). NJ: Pearson: Scott Foresman

<sup>65</sup> For example, in her exploration of 19 elementary and secondary PSTs' understanding of the division of fractions, Ball (1988) found that only 5 of those intending teachers could generate a correct problem for the division problem  $1 \frac{3}{4} \div \frac{1}{2}$ . Many of them confounded dividing *by* half with dividing *in* half and did

PSTs' knowledge of the division of fractions affords them or prevents them from *doing when teaching this topic*. For example, to what extent might their knowledge be associated with the tasks they select and the way they use them in developing a lesson on the division on fractions or with their crafting and delivering of explanations and their use of representations? Is there any association between prospective teachers' knowledge of the division of fractions and their analysis of respective student errors? This study seeks to make a first step in exploring such questions.

Addressing such questions could also have *practical* implications. Investigating what informs teachers' practices when teaching complex topics such as the division of fractions in particular, and fractions in general, could provide insights necessary for improving instruction in this area, which has been an enduring and pervasive challenge for teachers and teacher educators alike. The recently released report *Foundations for Success* of the National Advisory Panel (2008) clearly underscores that significant work is still needed in this area. The report recommends that developing K-8 students' proficiency in fractions be a major goal for mathematics education, because while proficiency in this area is foundational for algebra, it is severely underdeveloped among U.S. students, who are reported to have very poor preparation in rational numbers and operations involving fractions (cf. *ibid*, pp. xvii, 18, 28-29).

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not see any connections between the division of fractions and the operation of division in general. These findings led Ball to conclude that these PSTs' knowledge was rule-bound, compartmentalized (p. 63), and grounded in memorization rather than in conceptual understanding (p. 79). Ball reached analogous conclusions when she extended the sample of her study to include preservice and in-service teachers from 11 different education programs around the U.S. (cf. Ball & Wilson, 1990). These findings were corroborated by Simon's study (1993), which explored PSTs' understanding of division with fractions. Only 10 of the 33 participants in his study were able to propose a problem that correctly represented the division  $\frac{3}{4} \div \frac{1}{4}$  and only 5 participants were able to appropriately interpret the remainder in fraction divisions. Along the same lines, Piel and Green (1994) found that only 2 of the 39 PSTs participating in their study could identify what the unit in a division-of-fractions problem represented, whereas Lubinski, Thomason, and Fox's study (1998) outlined the several challenges faced by a candidate teacher while trying to develop in-depth knowledge in this area.

Finally, from a *methodological* perspective, the topic of the division of fractions was one of the content topics directly addressed in the intervention under examination (see Chapter 3), thus making it possible to explore the association between any changes in the study participants' MKT and instructional practices that might have been leveraged by the intervention (i.e., the second research question of the study). A natural question that arises is why not choose a less difficult topic also covered in the intervention. In selecting an easier topic and one in which PSTs have a decent knowledge even at the beginning of their teacher education, the researcher runs the risk that the changes in participants' knowledge resulting from the intervention are small and, hence, hard to detect, let alone study their correspondences to changes in PSTs' practices. Consider, for instance, the case of the addition of whole numbers. Although PSTs have a lot to learn to effectively teach addition of whole numbers, they nevertheless enter teacher preparation programs with a good grasp of addition, and perhaps a relatively decent understanding of its teaching. Although these programs might contribute to the enrichment of prospective teachers' "entrance" knowledge in this terrain, the respective changes might be small, and thus hard to identify and explore. In contrast, the division of fractions offers considerable space for improvement in PSTs' MKT – especially if one accepts that the candidate teachers of this study do not differ from the PSTs other studies have examined.

To summarize, the selection of the division of fractions is consonant with my overall inquiry and research questions. A particularly rich topic but also one that is hard to teach, the division of fractions offers the means to explore how teachers' MKT or changes thereof relate to their performance in five teaching practices considered supportive for engendering rich and challenging learning settings. In the next session, I

conclude my discussion of the design decisions of the study, by outlining how this study explored PSTs' performance in the five teaching practices under consideration.

#### Decomposing Teaching Performance to Facilitate Its Exploration: The Sixth Design Decision

In Section 5, I drew on Grossman and colleagues' work (2008, in press) and explained that the present study seeks to investigate PSTs' performance in five practices using an *approximation of teaching*. Grossman and colleagues' work also informed the sixth design decision of the study that relates to how PSTs' performance in the five teaching practices was explored.

Recently, Grossman and colleagues (in press) have talked about the need to *decompose* the complex work of teaching into its constituent components. Such decomposing, they have argued, is particularly pivotal to helping beginning and novice teachers closely attend to the work of teaching, analyze this work, and learn from and by reflecting upon it. By studying the teaching and learning of certain practices in other professions, such as the clergy and clinical psychologists, Grossman and colleagues concluded that such a careful parsing of practice can offer guidelines and inform curriculum decisions of teacher education programs. I argue that this idea of practice decomposition is not only consequential for instructional purposes, but can be particularly informative for research purposes, as well. In what follows, I detail how this idea of practice decomposition informed the design of this study, and particularly how it helped develop an analytic framework (see Figure 2.7) that guided the exploration of PSTs' teaching performance.

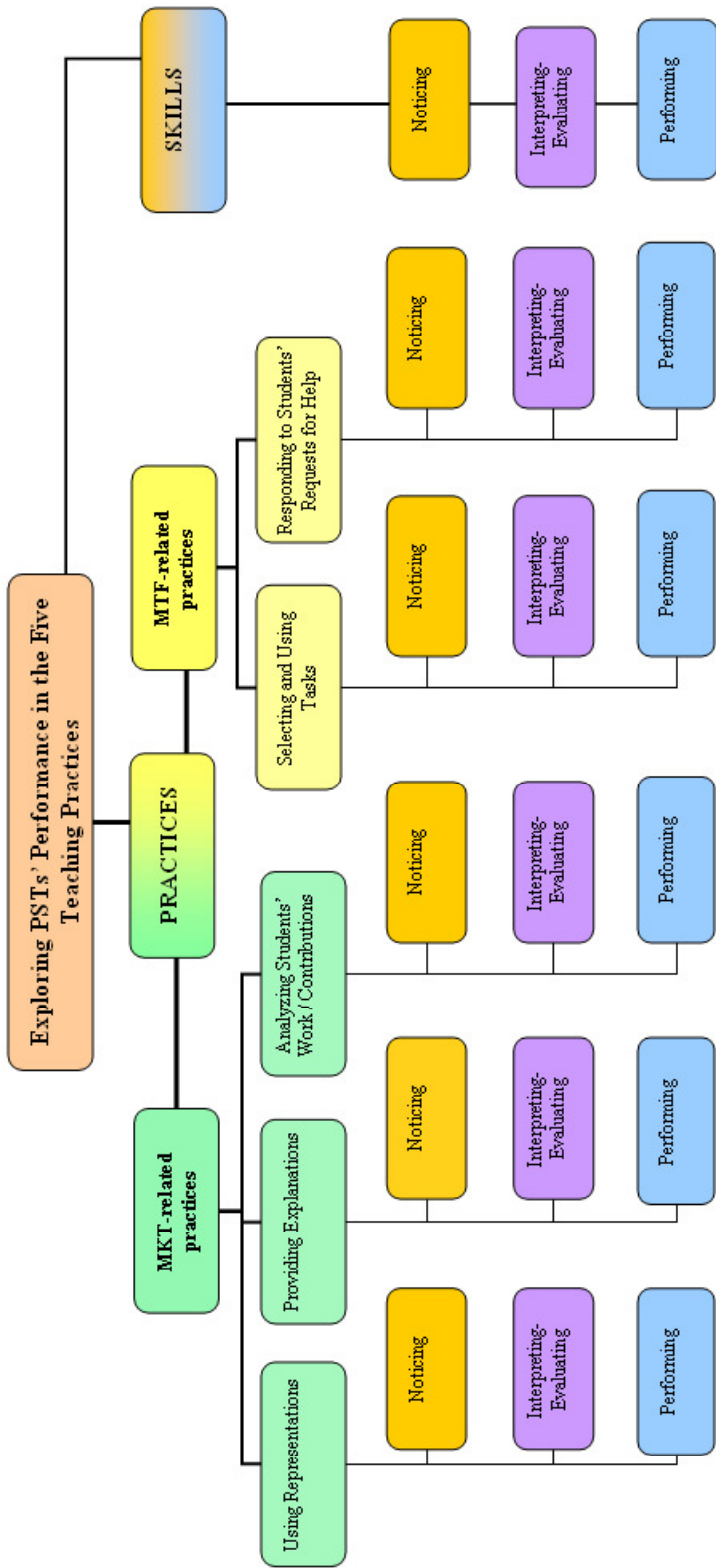


Figure 2.7. The analytic framework utilized in this study to explore PSTs' performance in five teaching practices.

As suggested in Figure 2.7, a first-level decomposition of the work of teaching pertains to focusing on distinct teaching practices. For reasons detailed in the third section of this chapter, this study decomposed teaching by attending to five practices, three of which are more pertinent to the work associated with MKT (MKT-related practices) and two of which are related to the work on selecting tasks and maintaining their cognitive level during enactment (MTF-related practices). This decomposition of teaching is by no means exhaustive; rather it is targeted toward the exploration of practices that are conducive to establishing the type of environments considered in this study.

The question that then arises is *how* to examine performance *within* each practice. One viable approach would be to simply ask PSTs to perform certain tasks related to each teaching practice. For instance, PSTs could be asked to select tasks for a lesson on the division of fractions, to provide an explanation on dividing fractions with or without using some sort of representations, to analyze students' work on dividing fractions and to explain how they would help students struggling with the content. However, prior studies that explored experts and novices' performances suggest that more can be learned by expanding the scope of analysis to examine not only how one *performs*, but also what one *notices* when observing a particular performance and how one *interprets* and *evaluates* this performance.

One of the earliest studies of this research paradigm was DeGroot's (1965) groundbreaking work on expert and novice chess players. Instead of examining these players' performance while actually playing chess, DeGroot presented them with examples of chess games and asked them to think aloud while observing these games.

The think-aloud method pursued in this study helped surface important differences between experts and novices. For example, compared to their novice counterparts, experts were found to notice and recall more events occurring during a chess game, mainly due to their ability to identify and integrate meaningful pieces of information together. They were also more adept at analyzing certain episodes of a chess game and more capable of thinking through all the possible countermoves a player could make.

In education, the think-aloud approach was utilized by asking expert, beginning, and novice teachers to view and comment on videotaped classroom lessons or slides with selected classroom episodes (Berliner, 1988; Sabers, Cushing, & Berliner, 1991). Like DeGroot's study, these studies showed notable differences in the performance of the three groups of teachers, particularly with respect to *what* they attended to and *how* they interpreted their observations. For instance, whereas expert teachers were more frequently seen focusing on events that could potentially have instructional significance, novices and beginning teachers provided step-by-step accounts of the observed events or classroom episodes as if they were reporting a football game (Berliner, 1988). Experts were also found to analyze and interpret the classroom events quite differently from their less experienced counterparts. Consider, for instance, two different comments drawn from Sabers et al.'s study, in which expert, beginning, and novice teachers were asked to attend to three monitors showing a high-school science lesson videotaped from three different angles (the left, the center, and the right of the classroom). The first comment was made by an expert teacher:

On the left monitor, the students' note taking indicates that they have seen sheets like this and have had presentations like this before; it's fairly efficient at this point because they're used to the format they are using. (p. 72)

Compare this comment with the following comment made by a beginning teacher: “In the right monitor, we have the teacher lecturing, students taking notes” (p. 73). Obviously, whereas both attended to students’ note-taking activity, the expert’s analysis and interpretation of this event was much more thorough than that of the beginning teacher.

Exploring what teachers notice when observing videotaped classroom episodes has been the focus of a series of more recent studies, as well (Sherin & van Es, 2005; Star & Strickland, 2008; van Es & Sherin, 2002, 2008). In their studies, Sherin and van Es developed and used what they call the *Learning to Notice* framework, which decomposes noticing into three components: (a) identifying the significant events in a teaching situation; (b) using knowledge from one’s context (i.e., knowledge of the subject matter, knowledge of how students think about the subject matter, as well as knowledge of the local context) to reason about these events; and (c) linking these events to more general teaching/learning principles (van Es & Sherin, 2008, pp. 245-246). Their studies showed quantitative and qualitative changes in the noticing performance of preservice and in-service teachers over time, as these teachers participated in video club meetings during which they analyzed video clips from their own lessons (for in-service teachers) or used a video and multimedia environment to analyze videotaped lessons (for PSTs).

Focusing on the first component of the *Learning to Notice* framework and utilizing the video-viewing approach, Star and Strickland (2008) also examined the noticing behavior of PSTs before and after a math methods course that had an explicit focus on improving PSTs’ ability to notice classroom events. This study showed that at the beginning of the course, prospective teachers mostly attended to classroom management issues but they were inattentive to the mathematical content covered in the



lesson and particularly to subtleties in the ways in which the teacher interacted with students around the content. The comparison of these PSTs' pre- and post- intervention noticing performance showed gains in all five noticing categories under consideration (i.e., classroom management, classroom environment, classroom activities, mathematical content, and communications); however, PSTs exhibited modest gains in the latter two categories that collectively represent the student-teacher and the student-student interactions around the mathematical content.

At this juncture, the question is unavoidable: why should one study what PSTs notice while watching classroom episodes or how they interpret these episodes, if the aforementioned studies suggest that prospective teachers are less astute observers of classroom practice compared to their more experienced counterparts? Three reasons justify this design decision, the latter two of which pertain to the two research questions of this study.

First, recall that the present study is not exploring PSTs' actual performance in the five teaching practices but their potential to engage in such practices. As the studies reviewed in this section collectively suggest there is value in exploring not only what PST can do, but also what they notice when considering classroom events and how they interpret and analyze these events. Although the latter skills do not directly translate into how one performs, they offer another venue for exploring one's potential to engage in a certain practice.

Second, although the lack of experience is considered a primary factor affecting the noticing performances of prospective teachers, the aforementioned studies directly or tacitly suggest that teachers' knowledge might also relate to what these teachers attend to

while observing episodes from classroom practice. Star and Strickland (2008), for example, hypothesize that the ability to closely attend to student-teacher interactions around the content might be associated with PSTs' MKT, particularly in the areas of choosing and using tasks and representations. Likewise, the second dimension of the *Learning to Notice* framework suggests that one's knowledge might be consequential for how one interprets a classroom event; however, such an association has yet to be examined. The present study can add to the aforementioned body of research by making a first step in exploring this association.

Third, as the reader might recall, in addition to its static perspective, this study explores the association between knowledge and teaching practices from a dynamic perspective, namely by comparing changes in PSTs' knowledge and teaching performance potentially leveraged by the prospective teachers' participation in an intervention. This offers a unique opportunity to investigate whether any potential changes in MKT are associated with changes in PSTs' noticing and interpreting performance.

Hence, as illustrated in Figure 2.7, each of the five practices considered in this study was further decomposed into three skill-subcomponents: noticing, interpreting-evaluating, and performing. The *noticing* subcomponent corresponds to the first dimension of the *Learning to Notice* framework and it seeks to capture what PSTs attend to without prompting while observing the teaching simulation. The second subcomponent, the *interpreting-evaluating* seeks to explore how PSTs interpret the classroom events depicted in the teaching simulation when explicitly asked to reflect on them. The inclusion of an evaluative aspect in this second subcomponent departs to some

extent from the *Learning to Notice* framework, in that the latter framework values interpretation more than evaluation.<sup>66</sup> However, this inclusion was envisioned to add to the exploration undertaken in this study since it could capture PSTs' *perspectives* with respect to actions and decisions related to the five practices under exploration. It is also important to highlight a significant difference between the present study and the studies reviewed above: instead of asking PSTs to consider actual classroom lessons, and consequently the practices of real teachers (something that might render teachers more reserved in undertaking an evaluative stance), the present study asked them to comment on a virtual teacher's decisions and actions. The third subcomponent explored in this study corresponds to the participants' in-vitro performance.

As illustrated in Figure 2.7, this two-level decomposition of practice resulting from cross-tabulating the five practices under consideration with the three abovementioned skill subcomponents provides multiple entrance points for exploring prospective teachers' performance in the teaching practices under consideration. Consider, for instance, the practice of providing explanations. Instead of simply asking PSTs to provide an explanation for the algorithm of dividing fractions, information about their performance in this practice can also be gleaned by considering what these PSTs notice when observing the virtual teacher providing such an explanation or by directly asking them to interpret and evaluate the virtual teacher's explanation. Figure 2.7 also indicates that, in addition to the 15 entrance points that result from this cross-tabulation of practices and skills, *aggregated* information can also be obtained regarding certain

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<sup>66</sup> In particular, van Es and Sherin (2008) argue: "While teaching certainly involves making judgments about what went well or poorly in a lesson, we believe it is critical for teachers to first notice what is significant in a classroom interaction, then interpret that event, and then use those interpretations to inform their pedagogical decisions" (p. 247).

practices or skills. The next chapter details how this analytic framework informed the design of the teaching simulation.

### Summary

In this chapter, I have outlined the theoretical underpinnings of the exploration the study undertakes and particularly the six design decisions I made to facilitate this exploration. I have also situated the study within the larger spectrum of research efforts that seek to understand the relationship between teacher knowledge and student learning. In the next chapter, I will elaborate upon the methodological approaches of the study.

## CHAPTER 3

### METHODOLOGY AND RESEARCH PROCEDURES

#### Initial Considerations

The present study explores the association between PSTs' MKT and their performance in five teaching practices considered conducive to creating mathematically rich and cognitively challenging learning environments: selecting and using instructional tasks, responding to students' direct or indirect requests for help, providing explanations, using representations, and analyzing students' work and contributions. The first two practices are associated with the MTF and the work concerning cognitively demanding tasks and maintaining focus on meaning-making; the latter three practices are more pertinent to the work concerning MKT. The study approaches this inquiry from two different perspectives, one *static* and one *dynamic*.

From a static perspective, the study seeks to understand the degree to which and means whereby PSTs' MKT at a fixed point in time relates to their performance in the five abovementioned practices. In exploring the potential ways in which MKT relates to PSTs' practices, I also consider other factors that might be associated with PSTs' respective performance. These factors include PSTs' beliefs about teaching and learning mathematics; their overarching goals for teaching this subject; their perceived efficacy in the content area examined in the study and the teaching practices under consideration; and certain background characteristics explored in studies on teacher knowledge and teacher effectiveness (i.e., courses taken and measures of general knowledge/aptitude).

From a dynamic perspective, the study explores the relationship between MKT and the five teaching practices in the context of an intervention designed to induce changes in the study participants' MKT. Specifically, the study examines the extent to which changes in the participants' MKT relate to changes in their performance in the five practices. The study does not focus on the intervention per se, but considers the intervention as a vehicle for influencing changes in the PSTs' MKT, thus facilitating further exploration of the relationship at hand.

Given the shortcomings of the studies reviewed in the previous chapter – especially those that pursued either a quantitative or a qualitative approach – to investigate the inquiry of the present study I use a mixed-methods approach (Borman, Clarke, Cortner, & Lee, 2006; Smith, 2006; Strauss & Corbin, 1998<sup>67</sup>). In adopting this approach, I agree with Smith (2006), who contends that

[e]ducational phenomena are far too complex to restrict the researcher to a single method, no matter how technically elegant or theoretically pure. To understand these phenomena and answer [pertinent] questions ... requires more than a concept of molar causality (p. 470). ... Multiple methodology [i.e., mixed methods] simply constitutes a more adequate science. (p. 473)

The mixed-methods approach is also consonant with the overall purpose of this study: to make a first step in exploring the complex relationship between teachers' knowledge and their performance in teaching practices that are supportive for establishing rich and challenging learning environments, a relationship the details of which remain a conundrum for scholars and educators alike. The mixed-methods approach also resonates with the exploratory character of this study and its envisioned outcomes: to generate working hypotheses that can be subjected to further inquiry.

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<sup>67</sup> Even renowned advocates of qualitative approaches acknowledge the value of using such mixed-method approaches. For instance, Strauss and Corbin (1998) emphasize that “a researcher should make use of any or every method at his or her disposal, keeping in mind that a true *interplay of methods* is necessary” (p. 33, emphasis in the original).

To explore this complex relationship the study employs different instruments and various approaches, some quantitative-oriented and other qualitative-oriented: it measures PSTs' knowledge using a multiple-choice test; it investigates their performance in the five teaching practices by using a teaching simulation; and it examines their beliefs and goals for teaching and learning mathematics via a survey.

This chapter, which consists of seven sections, details my methodological approaches. The first section outlines the context of the study, provides information about the study participants, and briefly describes the intervention. The next sections present the data collection instruments developed and used in the study (second section), the data collection procedures (third section), the data processing (fourth section), the measures developed from the collected data (fifth section), and the data analysis procedures employed in the study (sixth section). In the last section, I consider issues of validity, reliability, and generalizability.

### The Context of the Study

The study was conducted at the University of Michigan. Its participants were all prospective teachers who possessed an undergraduate degree and were enrolled in the *ELementary Masters of Art with Certification* (hereafter ELMAC) program, an intensive one-year full-time preparation program leading to initial teacher certification and a Masters of Arts degree in education. Three criteria informed the selection of this sample.

First, in contrast to undergraduate candidate teachers, these PSTs represented a more diverse group. This diversity was deemed important for the purposes of this study, because, as Yin (2006) and Borman et al. (2006) argue, examining diverse cases helps maximize the variation of the phenomenon under investigation, and consequently leads to

stronger findings than those resulting from exploring a single case or less diverse cases. Second, the PSTs of this study comprised a *cohort*, meaning that during their participation in the program they all took the same courses at the same time. Although they inevitably had different fieldwork experiences, due to their placement in different schools and grade levels, that they were all taking the same courses at each particular point of the program ensured that they had at least comparable coursework experiences. This comparability of experiences was considered important, given the multitude of factors that can inform teachers' decisions and actions, as discussed in Chapter 2. Finally, the study participants comprised a *convenience* sample, for all were enrolled in the two intervention courses taught by the author of this study.<sup>68</sup>

In what follows, I present the participants' background characteristics and briefly summarize the educational program in which they participated. Next, I focus on the two courses that constituted the MKT intervention the study considers.

### *The Participants of the Study*

The sample of this study comprised one of the two cohorts of the PSTs enrolled in the ELMAC program during the academic year Summer 2007 – Summer 2008. This cohort consisted of 20 prospective teachers who represented a diverse group in terms of race, mathematics content and methods courses taken in high school and during undergraduate studies, and prior teaching experience, as illustrated in Table 3.1. In

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<sup>68</sup> The fact that author of the study was also the instructor of the two intervention courses could be considered an additional limitation of the study. Yet, the reader is reminded that the study did not explore the effectiveness of the intervention per se, but rather treated this intervention both as context and as a vehicle to trigger changes in the PSTs' MKT and thus investigate the relationship between MKT and teaching performance from a dynamic perspective. It should also be noted that as the instructor of the intervention courses, the author remained blind to the participants' MKT score throughout both courses, in an attempt to ensure that he was impartial in his interaction with these candidate teachers.



particular, the cohort consisted of 16 female and 4 male PSTs. All the male and 12 female PSTs were white; 2 female PSTs were African American, and 2 were Asian-American.

With the exception of one PST, the study participants had taken at least three math content courses in high school; however, the highest level of these courses, as identified by the study participants themselves, differed remarkably. Four PSTs had completed an *Algebra II* and a *Trigonometry* class or a *Business Mathematics* class, at the most; nine had taken a *Precalculus* class, while five had taken a *Calculus* class; two of the study participants had taken more advanced math classes, such as *Multivariate Calculus*, *Linear Algebra*, and *Advanced Placement Calculus*. The number of the mathematics content courses these PSTs had taken during their undergraduate studies also differed remarkably: one had taken no math courses; five had completed one or two *Basic Mathematics* or *Statistics* courses; two had taken a *Precalculus* course, and six had completed a *Calculus I* course; the remaining six PSTs had completed at least two calculus courses. From this latter group of PSTs, one had taken several advanced content courses, including *Chaos and Fractals* and *Combinatorics*; two had a minor or major in mathematics, and one had a bachelors' degree in engineering. Two of the PSTs in this cohort had also taken two math methods courses. Finally, two of the study participants had worked as substitute teachers from two to five years, whereas another PST had worked with special-education children for one year.

Table 3.1

*Background Characteristics of the Study Participants*

<b>Name</b> <sup>1</sup>	<b>Gender</b>	<b>Race</b>	<b>Number of high-school math courses</b> <sup>2</sup>	<b>Undergraduate math content courses</b>	<b>Undergraduate math methods courses</b>	<b>Math Minor/Major</b>	<b>Prior teaching experience</b>
Barbara	Female	White	4 (Precalculus)	Calculus I	None	No	None
Bob	Male	White	5 (Algebra II - Trigonometry)	Precalculus, Calculus I	None	No	Worked with special education children for a year
Dana	Female	White	5 (Precalculus)	Required Math Basic Course	None	No	None
Deborah	Female	White	5 (Precalculus)	Basic Statistics course	None	No	None
Dorothy	Female	White	4 (Precalculus)	Precalculus	None	No	None
Frances	Female	White	5 (Precalculus)	Calculus I, Statistics	None	No	None
Habika	Female	African-American	4 (Algebra II - Trigonometry)	Basic Statistics Course	None	No	None
Kimberley	Female	White	7 (Calculus)	Calculus I and II, Statistics, Population Genetics, Geometry	None	Minor	Taught 8 <sup>th</sup> grade algebra for two years as a substitute teacher
Leo	Male	White	6 (Calculus)	Calculus I and II	Teaching Children Mathematics (I and II)	No	None
Lillian	Female	White	5 (Precalculus)	Precalculus, Statistics	None	No	None

*Notes.*

<sup>1</sup> All PSTs were assigned pseudonyms to preserve their anonymity.

<sup>2</sup> Listed in parentheses is the course that the study participants identified as the highest level high-school math content course they had taken.

Table 3.1

*Background Characteristics of the Study Participants (continued)*

Name <sup>1</sup>	Gender	Race	Number of high-school math courses <sup>2</sup>	Undergraduate math content courses	Undergraduate math methods courses	Math Minor/Major	Prior teaching experience
Nathan	Male	White	8 (Multivariate Calculus I and II, Linear Algebra)	Six courses (including Abstract Algebra, Chaos and Fractals, Series and Differential Equations, Combinatorics)	None	No	None
Nicole	Female	White	7 (Calculus)	Calculus I, II, III, Geometry, Statistics, Algebra-based Physics	Math Methods for Elementary grades, Math Methods for Secondary Grades	Major	Worked as a substitute elementary and middle-school teacher for 5 years
Nora	Female	White	6 (Precalculus)	2 Basic Math courses	None	No	None
Suzanne	Female	White	5 (Precalculus)	Calculus I	None	No	None
Teresa	Female	White	5 (Advanced Placement Calculus)	Eight courses (engineering major)	None	No	None
Tiffany	Female	White	5 (Algebra II /Trigonometry)	None	None	No	None
Travis	Male	White	6 (Calculus)	Calculus I and II, Differential Equations, Engineering Math	None	No	None
Vonda	Female	African-American	2 (Business Math)	Algebra I and II	None	No	None
Vui	Female	Asian-American	6 (Precalculus)	Calculus I	None	No	None
Ying	Female	Asian-American	8 (Calculus)	Precalculus, Calculus I	None	No	None

*Notes.*

<sup>1</sup> All PSTs were assigned pseudonyms to preserve their anonymity.

<sup>2</sup> Listed in parentheses is the course that the study participants identified as the highest level high-school math content course they had taken.

### *A Brief Overview of the ELMAC Program*

The educational preparation program in which the study participants were enrolled was designed to prepare them for teacher certification as generalists Grade K-8 teachers. The study participants were expected to complete 30-credit coursework over 12 months and work closely with an experienced cooperating teacher throughout the course of the program. From June to August, they took three university courses, including the math content course, which constitutes the first part of the intervention, and participated in site visits. From September to December, they enrolled in four university courses, including the math methods course, which represents the second part of the intervention; they also worked for several days a week in their field placement. Between January and March they took two courses and taught full-time for two months. From April to June they taught all subject matter areas for three to four weeks; they also took two more courses to complete the required coursework.

### *The MKT Intervention*

As already explained, this study does not seek to examine the effectiveness of the intervention itself. Rather, the intervention was expected to leverage changes in participants' MKT and respective MKT-related practices (i.e., using representations, giving explanations, and analyzing student work and contributions), and thus facilitate a further exploration of the association between MKT and the five practices under examination. To explicate how the intervention was expected to leverage such changes, this part provides a brief overview of the two constituent components of the intervention: the math content course (*MTH 485: Mathematics for Elementary and Middle School*

*Teachers*) and the math methods course (*ED 518: Teaching Children Mathematics*).<sup>69</sup> A more comprehensive description of the two courses appears in Appendix 1.

Both courses sought to help these prospective teachers develop mathematical skills, knowledge, and ways of reasoning necessary for teaching mathematics effectively. The content course aimed at helping the PSTs move from simply knowing mathematics as educated adults to knowing mathematics as teachers; the methods course was designed to help them move further along this continuum, and develop mathematical knowledge and skills necessary for supporting other people in learning mathematics.

### *The Math Content Course*

The math content course had a dual goal. First, it aimed at helping PSTs unpack and develop flexible understanding of important mathematical ideas and processes, largely within the realm of number theory and operations. Second, it sought to offer them opportunities to practice using representation, providing explanations, and analyzing others' thinking.

Four of the meetings of this course focused on fractions in general and the division of fractions in particular. During these meetings, the PSTs compared different representations and interpretations of fractions (i.e., part-whole, measure, operator, quotient, and ratio). They also dwell on the notion of units when considering fractions; this idea is particularly important in the division of fractions, given that this operation requires consideration of two different units: the unit defined by the dividend (what I call *absolute* unit) and the unit defined by the divisor (what I call *relative* unit). Additionally,

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<sup>69</sup> Both the math content course and the math methods course have been designed and refined over a decade by a diverse group including mathematics educators, mathematicians, practicing teachers, researchers, and graduate students both within and outside of the United States, all of whom profess interest in understanding and improving the teaching of mathematics.

the PSTs were introduced to the two interpretations of division (i.e., partitive and measurement) and asked to write word problems for different division-of-fractions sentences. Furthermore, they considered the interpretation of the remainder in fraction division using either absolute or relative units. Regarding the second goal of the course, the study participants had ample opportunities to use different forms of representations (symbols and different diagrams) to explain their thinking; they were also persistently prompted to elaborate their ideas and closely attend to and analyze others' thinking.

### *The Math Methods Course*

The math methods course extended the work pursued in the content course, but mostly focused on four domains: leading a whole class discussion; representing mathematical ideas; assessing students' mathematical knowledge, skills, and dispositions; and planning mathematics lessons. The focus on these domains afforded the study participants additional opportunities to hone their performance in the three MKT-related practices examined in this study. Several of the course activities required the study participants to map their use of representations and manipulatives on certain algorithms and clearly explain these correspondences; these activities sensitized the PSTs to the importance of linking these algorithms to their underlying meaning. The study participants were also engaged in guided observation and analysis of student thinking and work; they were also offered some opportunities to engage in the practice of responding to students' questions and requests for help. The course also targeted other teaching practices, such as posing questions, learning in and from practice, and selecting and using tasks; however, it did not address issues pertaining to maintaining the curriculum tasks' cognitive demand.

Regarding its mathematical content, the math methods course largely focused on place value and operations with whole numbers and decimals. Therefore, the PSTs were not offered the opportunity to further work on the division of fractions. Yet, they considered division of whole numbers and the meaning of remainders when working with whole or rational numbers.

Figure 3.1 summarizes the relative emphasis placed on each of the five practices this study considers in each of the two intervention courses.<sup>70</sup> As this figure suggests, using representations and providing explanations were the teaching practices mostly emphasized during the content course; analyzing students' thinking was moderately emphasized during the content course, mainly in the form of asking the PSTs to analyze their classmates' thinking. The methods course put a heavy emphasis on analyzing student thinking and using representations and manipulatives (when modeling different algorithms). Giving explanations was also a significant part of the math methods course, however to a smaller extent than the former two practices.

Regarding the two MTF-related practices, Figure 3.1 suggests that during the methods course, the PSTs had some opportunities to consider task selection and use without, though, entertaining issues related to the cognitive complexity of these tasks. Responding to students' requests for help was considered in one of the classes of the math methods course; however, no explicit connections were made to how teachers might lessen the cognitive challenge of a task by the manner in which they respond to their students' requests for help. Overall, neither the content nor the math methods course provided their participants opportunities to attend to issues concerning the cognitive

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<sup>70</sup> This figure was developed by considering the comprehensive description of the two courses as outlined in Appendix 1.

demands of instructional tasks and how certain teaching moves can result in maintaining or diminishing the cognitive demand of these tasks during enactment. However, the emphasis that both courses placed on linking different representations (i.e., arithmetic symbols, diagrams, verbal explanations) – especially when working on number operations – and providing meaningful and clear explanations alerted these PSTs to the importance of helping students connect the mathematical procedures under consideration to their underlying meaning.

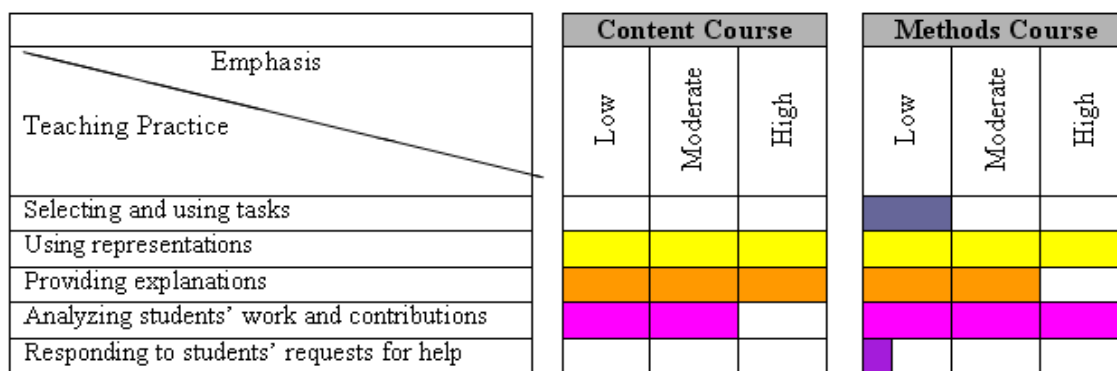


Figure 3.1. Relative emphasis placed on each of the five teaching practices during the math content and math methods courses.

Having presented the study participants and outlined how the intervention relates to the practices the study considers, I now turn to the data collection instruments.

### Data Collection Instruments

Addressing the study research questions required using various data collection instruments. Specifically, the following instruments were used: a paper-and-pencil test to tap the PSTs' MKT, developed by adapting an existing multiple-choice LMT test; a teaching simulation and an accompanying interview protocol to explore the participants' performance in the five practices this study considers; and a survey to investigate the participants' beliefs and goals about teaching and learning mathematics, as well as their



perceived competence in working on fractions and engaging in the MKT-related practices. I present each of these instruments in turn.

### *Measuring PSTs' MKT: The Adapted LMT Test*

To examine the PSTs' MKT in the division of fractions, I first reviewed previous studies which explored teachers' pertinent conceptions and/or detailed the concepts and skills teachers need to develop in order to teach this subject effectively (Ball, 1988, 1990; Ball & Wilson, 1990; Flores, 2002; Lubinski, et al., 1998; Ma, 1999, p. 77; MTPCWPC, 2001; Piel & Green, 1994; Siebert, 2002). This review helped me assemble a set of criteria – concepts and processes – considered necessary for teaching the division of fractions. I used these criteria to generate a specification table (see Table 3.2) that guided the selection and adaptation of the test used in the study.

Once this specification table was developed, I explored whether any of the existing LMT instruments designed to measure teachers' MKT could provide items for all the categories and subcategories identified in the specification table. From the gamut of the available LMT instruments, the one that best served the needs of the study was the *LMT Rational Numbers* form, which included 37 multiple-choice items. A close analysis of this instrument revealed that several of its items were not particularly relevant to the present study: these items concerned decimals and operations on them, percents, comparisons between rational and irrational numbers, and operations with the lowest common denominator. To calibrate the LMT test to the topic examined in this study, I excluded these items and retained those that focused on fractions in general and on the division of fractions in particular. In the specification Table 3.2, the retained items are

represented in regular-font characters; their numbering corresponds to the serial number with which they appeared on the test used in the study.

Table 3.2

*Specification Table for the Development of the Instrument Used to Measure PSTs' MKT*

<b>Concepts-processes related to the division of fractions</b>	<b>Questions</b>
<i>Division and other operations</i>	
Two interpretations of division	
Partitive	<b>5</b> , 11a, 11d, 14c, <b>15</b>
Measurement – division as repeated subtraction	11b, 14a, 14e, 19e, <b>15</b> , <b>25</b>
Multiplication and division as inverse operations	19b, 19c
<i>Fractions in general</i>	
The concept of the reference unit	1, 3, 12, <b>20</b>
The concept of fractions – different interpretations of fractions (part-whole, measurement, ratio/rate, and operator)	<u>9</u> , 24c, 24d
Fractions as quotients (and vice versa)	19a, 21, 24a, 24b
<i>Multiplicative operations of fractions</i>	
Multiplication of fractions	<u>4</u> , <u>7</u> , 8a, 8b, 11c, 14b, 14d, <u>18</u>
The quotient and the remainder in divisions on fractions	<b>17</b> , <b>22</b>
The rule of “invert and multiply” in divisions on fractions	----
Familiarity with other non-conventional division-of-fractions algorithms	10
<i>Operations on fractions other than division and multiplication</i>	
Addition/subtraction of fractions	8c, 8d, 13
Simplifying fractions	6
Comparing fractions	2, 16, 23

Along with the 19 original LMT items, the MKT instrument used in this study included six additional items (shown in bold characters in the specification table) to compensate for some concepts/processes not captured by the existing items (e.g., interpreting the remainder of division with fractions) and to increase the diversity of the items within each category. In developing these items, I worked closely with two members of the LMT group, experienced in writing and evaluating MKT items. The

*adjusted* LMT test that resulted from these modifications comprised 25 items,<sup>71</sup> five of which (i.e., 8, 11, 14, 19, and 24) were multiple-question items; thus, in total, the instrument included 41 questions,<sup>72</sup> spanning four different categories: questions pertinent to the concept of division; questions germane to fractions; questions on multiplicative operations on fractions; and questions on operations other than division or multiplication of fractions.

The first category included questions on the two interpretations of division as well as questions that addressed the inverse relationship between division and multiplication. Although several of these questions *did* include fractional numbers, they were clustered in this category because they were designed to explore PSTs' conceptual understanding of division. To put it differently, replacing the fractional numbers with whole numbers in these questions would not affect the concept these questions were designed to gauge. Because most of these questions required evaluating the correspondence between word problems and arithmetic operations (questions 11a-11d; 14a-14d; 19a-19e), I added two questions (questions 5 and 15) that investigated the two interpretations of division through the use of pictorial representations; one of these questions (question 15) explored both interpretations of division. Questions 19b and 19c examined participants' understanding of fraction multiplication as the inverse operation of fraction division.

The second category included questions that addressed the concept of fractions in general. Four of these questions pertained to the concept of unit; two questions (questions 1 and 20) explored this concept directly, whereas the exploration of this concept in

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<sup>71</sup> Because none of these questions has been released, instead of presenting the actual questions, Appendix 2 briefly describes each question (with the exception of question 5, which is presented as it appeared on the test). This appendix also presents a related released sample question from another LMT test.

<sup>72</sup> Although question 19d was part of the test, it was not considered for any of the analyses presented in this study, since it was considered irrelevant to any of the categories of Table 3.2.

questions 3 and 12 was intertwined with an examination of subtracting and comparing fractions. The remaining seven questions of this category pertained to different interpretations of fractions: the part-whole interpretation and the idea of equipartitioning (question 9), the notion of operator (question 24c), the notion of rate (question 24d); and the idea that fractions represent quotients, and vice versa, that quotients represent fractions (questions 19a, 21, 24a, 24b).

The third category pertained to the multiplicative operations on fractions. Questions 4, 7, and 18 were designed to investigate the PSTs' understanding and use of multiplication in problems involving different representations, whereas questions 8a, 8b, 11c, 14b, and 14d explored participants' performance in associating a word problem with the correct arithmetic operation. Question 10 was designed to examine the participants' familiarity with non-conventional algorithms on the division of fractions. Because there was no question gauging understanding of the different interpretations of the remainder in division with fractions, such a question was developed and added to the adapted LMT test form (question 17). Another question was also written and added to the LMT test to measure conceptual understanding of the operation under consideration; this question asked the study participants to estimate the quotient in a given division of fractions (question 22). As Table 3.2 suggests, no question was included to tap the participants' conceptual understanding of the invert-and-multiply algorithm of the division of fractions. This was due to two main reasons: first, to the inherent difficulties associated with developing a multiple-choice item to capture the richness of this topic, and second to the fact that this topic was directly addressed in the teaching simulation, as discussed in what follows. Finally, seven questions from the original LMT test were retained in the

adapted LMT test, because these questions represented operations on fractions (addition-subtraction, comparison, simplification) and, thus, were tangentially related to the topic under investigation.

Overall, the 41 questions of the adapted LMT test used in the present study represented an attempt to calibrate the MKT instrument to the area under examination (i.e., the division of fractions), on the premise that such a calibration is needed when exploring correspondences between knowledge and teaching performances. Recall that, as discussed in Chapter 2, the studies that used measures or proxies of teachers' knowledge that had little to do with the mathematical content and its teaching, resulted in low significant correlations or non significant correlations at all.

A final note is in order here. The 41 questions used in this test corresponded to only two of the four MKT domains. In particular, the majority of these items were thought to tap teachers' Specialized Content Knowledge (SCK = 36 questions), whereas only five questions were designed to gauge teachers' Common Content Knowledge (CCK).<sup>73</sup> In Table 3.2, these latter questions are underlined.

*Exploring PSTs' Performance in the Five Teaching Practices: The Teaching Simulation  
and Its Accompanying Protocol*

To explore the PSTs' performance in the five practices under examination, a teaching simulation was developed. The development of the teaching simulation consisted of five phases. First, the skeleton and format of the simulation were decided; then its content was determined; next, the *ThExpians-P* character set was used to develop the associated virtual lesson; finally, the accompanying lesson protocol was developed,

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<sup>73</sup> This distinction was proposed by a member of the LMT group experienced in generating and classifying items.

and both the lesson protocol and the teaching simulation was vetted and pilot-tested. I detail each of these phases below.

*Initial Steps in Developing the Simulation: Determining Its Skeleton and Format*

In developing the teaching simulation, I drew on and synthesized work in three areas. First, the MTF provided the *overall structure* for the teaching simulation. Following the MTF, the simulation was designed to unfold in three phases: the first phase related to analyzing curriculum materials to select instructional tasks for a lesson on the division of fractions; the second phase pertained to presenting these tasks to students; the third phase concerned the enactment of these tasks. Whereas it was easy to engage the PSTs in analyzing curriculum materials for a lesson on dividing fractions, it was hard to explore the PSTs' performance in the phases of task presentation and enactment. It was at this point that I turned to some ideas developed by Herbst and Chazan.

Herbst and Chazan (Herbst & Chazan, 2003, 2006; Herbst et al., 2007) have been using animated cartoon characters to represent teaching scenarios as a means to study in-service teachers' rationale for the decisions they make in different instructional situations. I made the assumption that representations using similar cartoon characters might be valid tools to explore *preservice* teachers' decisions and actions regarding the five practices under investigation. In contrast to Herbst and Chazan, I sought to understand not teachers' rationale for certain decisions and actions but merely how teachers' knowledge might play out in informing their decisions and actions in different instructional situations. As I explained in Chapter 2, I considered such a virtual environment appropriate for the purposes of the present study, since it could expose the PSTs to *purposefully designed teaching episodes* and hence facilitate exploration of their

performance in the practices of interest. Herbst and Chazan's work also provided the specific means for producing the representation. As detailed in the third part of this section, in developing this teaching simulation, I used the *ThExpians-P* character set created at the *Geometry, Reasoning, and Instructional Practices* (GRIP) Lab under the direction of Patricio Herbst.<sup>74</sup> I used this character set to produce a slide show on an introductory lesson on fraction division.

The next design decision concerned the teaching episodes to be included in the teaching simulation: how should these episodes look like to be close approximations of the work of teaching mathematics? Ball's work (1988, pp. 233-246), and particularly the vignettes she developed to explore PSTs' knowledge, provided useful guidelines. For instance, for the category of analyzing students' work and contributions, Ball (*ibid*) presented her study participants with artifacts of student work on a textbook task and asked them to analyze these artifacts and draw inferences about students' understanding or struggles. Drawing on this idea, the teaching simulation utilized in this study presented the PSTs with three student solutions on a division-of-fractions problem and asked them to analyze these solutions and make assertions about these students' understanding or lack thereof. Ball's work also provided ideas for developing other episodes used in the teaching simulation, such as the episodes on responding to students' requests for help and the episode on evaluating students' responses. However, in contrast to the vignettes that Ball used in her study, all the episodes used in the teaching simulation revolved around

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<sup>74</sup> For more information see <http://grip.umich.edu>. This set of two-dimensional cartoon characters supports representations of individual differences across several axes (race, gender, affiliation, etc.) and allows the depiction of many gestures, postures, and facial expressions that are common in classrooms, while not overcomplicating the anatomy of the characters. The reader is notified that when discussing the animations produced by this character set, the GRIP researchers use the term "representations of teaching" instead of the term "teaching simulation" used in the present study.

the same topic – the division of fractions – and were structured so that they be coherent and form a self-contained lesson on that topic.

The works reviewed above offered the *skeleton* and the *format* for the teaching simulation. To flesh out this simulation, I turned to the analytic framework presented in the previous chapter (see Figure 2.7) which defines 15 entrance points resulting from the cross-tabulation of the five practices with the three skills of noticing, interpreting-evaluating, and performing. The next step was to develop a coherent teaching simulation that would provide opportunities to explore the PSTs' performance in each of these 15 entrance points. It is to this issue that I now turn.

#### *Fleshing Out the Teaching Simulation: Determining Its Content*

The first step in determining the content of the teaching simulation was to develop a specification table that operationalized the 15 entrance points under examination (see Table 3.3). This specification table operationalized each of the 15 entrance points examined in the study by defining how the teaching simulation would be used to explore the PSTs' performance. Consider for instance, the operationalization of the teaching practice of *selecting and using tasks*. Table 3.3 suggests that the PSTs' performance in the noticing skill of this practice was examined by exploring whether the PSTs noticed (without any prompting) that the virtual teacher diminishes the cognitive complexity and the richness of the tasks she uses in her lesson by the manner in which she presents and enacts these tasks with her students. Similarly, according to this specification table, the PSTs' performance in the *interpreting-evaluating* skill of the same practice was examined by directly asking the PSTs to comment on and evaluate the virtual teacher's presentation and enactment of tasks. Finally, with respect to the *performing* skill, the



table shows that the PSTs' performance was examined by having them consider two textbook pages, one procedurally oriented and one conceptually oriented, and by asking them to (a) decide which of the two pages (if any) they would select for an introductory lesson on dividing fractions; (b) specify which tasks they would select for such a lesson; and (c) briefly outline how they would use these tasks in this lesson.

Having operationalized the 15 entrance points, the next steps in fleshing out the teaching simulation were to select the two textbook pages discussed above and then to develop the story for the virtual lesson of the teaching simulation. I detail each of these steps below. Before doing so, a clarification is necessary. The teaching simulation consisted of two parts. The first part, which I call the *lesson planning* part, corresponded to the planning phase; during this part the PSTs were presented with two textbook pages and were asked to design a lesson on dividing fractions. The second part, which I call the *virtual-lesson* part, was designed to capture the phases of task presentation and enactment; during this part the PSTs were asked to observe and discuss a virtual lesson.

Table 3.3

*Specification Table for the Development of the Teaching Simulation*

Skill Practice	Noticing	Interpreting-Evaluating	Performing
<b>Using representations (MKT-related practice)</b>	<p>Explore whether the PSTs notice that the virtual teacher uses representations in her lessons but these representations do not support understanding of the division of fractions, because:</p> <ul style="list-style-type: none"> <li>• in using these representations she emphasizes following certain steps rather making meaning</li> <li>• she does not discuss how the use of these representations corresponds to the algorithm of the division of fractions</li> </ul>	<p>The PSTs are asked to <i>comment on and evaluate</i> the virtual teacher's use of representations in the lesson; her use of representations is insufficient, because of:</p> <ul style="list-style-type: none"> <li>• her emphasis on following certain steps in using these representations rather than on building meaning</li> <li>• her not drawing any connections between these representations and the algorithm of the division of fractions</li> </ul>	<p>The PSTs are asked to <i>use a representation to explain</i>:</p> <ul style="list-style-type: none"> <li>• the quotient in a division-of-fractions problem that includes both a whole-number and a fractional-number part (explore whether interviewees can use the representation appropriately to identify the dividend, the divisor, and the two parts of the quotient)</li> <li>• the algorithm of the division of fractions</li> </ul>
<b>Providing explanations (MKT-related practice)</b>	<p>Explore whether the PSTs notice that the virtual teacher:</p> <ul style="list-style-type: none"> <li>• provides explanations that describe rather than explain the algorithm of the division of fractions</li> <li>• avoids providing explanations or resorts to the idea of “mathematics being a set of rules” when asked to explain the pertinent algorithm</li> </ul>	<p>The PSTs are asked to <i>comment on and evaluate</i> the virtual teacher's explanations. Her explanations are problematic, because they:</p> <ul style="list-style-type: none"> <li>• describe rather than explain the operation at hand</li> <li>• include inconsistencies or mistakes</li> <li>• include inappropriate analogies</li> <li>• have limited applicability</li> </ul>	<p>The PSTs are asked to <i>explain</i>:</p> <ul style="list-style-type: none"> <li>• a division-of-fractions problem that includes a fractional part in its quotient</li> <li>• why the rule “invert and multiply” works</li> </ul>
<b>Analyzing student work and contributions (MKT-related practice)</b>	<p>Explore whether the PSTs notice that the virtual students' work and contributions during the lesson are reflective of the following errors/misconceptions:</p> <ul style="list-style-type: none"> <li>• the dividend is always larger than the divisor (intuitive-based error)</li> <li>• the remainder is explained in absolute rather than in relative units (conceptual error)</li> <li>• a student inverts the dividend instead of the divisor (algorithmic error)</li> </ul>	<p>The PSTs are asked to <i>comment on and evaluate</i> the virtual teacher's responses to two student contributions. The virtual teacher fails to capture and work on the following errors-misconceptions:</p> <ul style="list-style-type: none"> <li>• the dividend is always larger than the divisor (intuitive-based error)</li> <li>• the remainder is explained in absolute rather than in relative units (conceptual error)</li> </ul>	<p>The PSTs are asked to <i>analyze three student solutions</i> of a division-of-fractions problem:</p> <ul style="list-style-type: none"> <li>• two algebraic solutions (a correct and an incorrect one)</li> <li>• a pictorial solution</li> </ul> <p>For each solution, the PSTs are asked to:</p> <ul style="list-style-type: none"> <li>• analyze the students' work</li> <li>• make assertions about the students' understanding or lack thereof.</li> </ul>

Table 3.3

*Specification Table for the Development of the Teaching Simulation (continued)*

Skill	Noticing	Interpreting-Evaluating	Performing
<b>Practice</b> <b>Selecting and using tasks (MTF-related practice)</b>	<p>Explore whether the PSTs notice that the virtual teacher diminishes the cognitive complexity and richness of the textbook tasks she selects and uses in her teaching by the manner in which she:</p> <ul style="list-style-type: none"> <li>• <i>presents</i> these tasks (e.g., she takes over the thinking, she explains how students should solve the tasks, etc)</li> <li>• <i>enacts</i> these tasks with her students (e.g., she places more emphasis on obtaining correct answers rather than on developing meaning and understanding, she outlines steps for students to follow, etc)</li> </ul>	<p>The PSTs are asked to <i>comment on and evaluate</i> the virtual teacher's presentation and enactment of tasks. The virtual teacher's work on presenting and enacting the selected tasks is problematic, because of the way in which she:</p> <ul style="list-style-type: none"> <li>• <i>presents</i> these tasks (e.g., she takes over the thinking, she explains how students should solve the tasks, etc)</li> <li>• <i>enacts</i> these tasks with her students (e.g., she places more emphasis on obtaining correct answers rather than on developing meaning and understanding, she outlines steps for students to follow, etc)</li> </ul>	<p>The PSTs are presented with two textbook pages (one procedurally oriented and one conceptually oriented) and are asked to <i>design an introductory lesson</i> on dividing fractions. Explored in this category are:</p> <ul style="list-style-type: none"> <li>• the page that the PSTs select (if any)</li> <li>• the tasks they decide to use in this lesson</li> <li>• <i>how</i> the PSTs would use the selected tasks during their lesson</li> </ul>
<b>Practice</b> <b>Responding to students' direct or indirect requests for help (MTF-related practice)</b>	<p>Explore whether the PSTs notice that the virtual teacher responds to students' direct or indirect requests for help in ways that diminish the complexity of these tasks and without actually supporting her students' understanding. Specifically:</p> <ul style="list-style-type: none"> <li>• the teacher leads a confused student to the correct answer without ensuring that this student understands the problem</li> <li>• as soon as a couple of students appear to struggle with a task, the teacher rushes to provide hints</li> </ul>	<p>The PSTs are asked to <i>comment on and evaluate</i> how the virtual teacher responds to students' direct or indirect requests for help. Specifically:</p> <ul style="list-style-type: none"> <li>• the teacher leads a confused student to the correct answer without ensuring that this student understands the problem (indirect request for help)</li> <li>• as soon as a couple of students appear to struggle with a task, the teacher rushes to provide help (direct request for help)</li> </ul>	<p>The PSTs are asked to <i>respond to students' direct or indirect requests for help</i>. In particular, the PSTs are asked to outline how they would:</p> <ul style="list-style-type: none"> <li>• support a confused student (indirect request for help)</li> <li>• respond to a couple of students who struggle with an assigned task and thus solicit the teacher's help (direct request for help)</li> </ul>

*Selection of two textbook pages.* The presentation of the division of fractions in several fifth- and sixth-grade math textbooks<sup>75</sup> was explored. In examining those textbooks, I was looking for two distinctly different treatments of the division of fractions: one that emphasizes the meaning of the operation and which is centered on rich and cognitively challenging tasks, and one that mostly aims at helping students learn and apply the algorithm of the division of fractions by engaging them in intellectually undemanding tasks. Providing the PSTs with textbook tasks that treat the content in these two different ways was envisioned to help investigate the PSTs' *selection* of tasks and to explore the extent to which they would degrade, maintain, or upgrade the richness and complexity of those tasks by the manner in which they would use them in a lesson.

For the *algorithmic treatment* of the content, I compiled two textbook pages from Houghton-Mifflin *Mathematics* (Greenes, Leiva, & Vogeli, 2002, Practice 8-8, p. 106, Reteach 8-8, p. 106) into one page. From the first page, I used the worked out example on the algorithm of the division of fractions;<sup>76</sup> from the second page, I used the exercises included to provide students with opportunities to practice this algorithm. The one page that resulted from this adaptation (see Figure 3.2) presents the division of fractions as a two-step process consisting of finding the reciprocal of the divisor and then multiplying this reciprocal with the dividend. Listed below this explanation are 16 division-of-fractions exercises (i.e., tasks 1-16). At its bottom, this page includes two problems on the division of fractions (i.e., tasks 17-18).

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<sup>75</sup> The textbooks considered were *Everyday Mathematics*, *Connected Mathematics II*, Scott Foresman-Addison Wesley's *Mathematics*, Houghton-Mifflin *Mathematics*, and Hartcourt's *Math*. The exact references of these textbooks appear in Chapter 2 (see footnotes 60-64).

<sup>76</sup> To facilitate the exploration of the PSTs' performance in explaining and representing the fractional part of the quotient in a division-of-fractions problem, I actually replaced the worked out example originally presented in the textbook page (i.e.,  $4 \div \frac{2}{3}$ ) with one that lends itself to such an exploration (i.e.,  $2 \div \frac{3}{4}$ ).

## Divide by a Fraction

Divide  $2 \div \frac{3}{4}$

**Step 1:** Find the reciprocal of the divisor.

**Step 2:** Multiply by the reciprocal of the divisor.

The reciprocal of  $\frac{3}{4}$  is  $\frac{4}{3}$ .

$$2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{2}{1} \times \frac{4}{3} = \frac{8}{3} = 2 \frac{2}{3}$$

1.  $\frac{2}{3} \div \frac{1}{3}$  \_\_\_\_\_

2.  $\frac{1}{3} \div \frac{3}{4}$  \_\_\_\_\_

3.  $\frac{5}{8} \div \frac{5}{6}$  \_\_\_\_\_

4.  $\frac{3}{5} \div \frac{2}{3}$  \_\_\_\_\_

5.  $\frac{3}{4} \div \frac{2}{3}$  \_\_\_\_\_

6.  $\frac{2}{3} \div \frac{4}{7}$  \_\_\_\_\_

7.  $\frac{1}{3} \div \frac{7}{10}$  \_\_\_\_\_

8.  $\frac{5}{3} \div \frac{1}{3}$  \_\_\_\_\_

9.  $\frac{5}{6} \div \frac{1}{3}$  \_\_\_\_\_

10.  $\frac{11}{12} \div \frac{1}{4}$  \_\_\_\_\_

11.  $\frac{1}{2} \div \frac{7}{8}$  \_\_\_\_\_

12.  $\frac{1}{4} \div \frac{1}{3}$  \_\_\_\_\_

13.  $2\frac{5}{8} \div \frac{1}{8}$  \_\_\_\_\_

14.  $\frac{3}{4} \div \frac{3}{8}$  \_\_\_\_\_

15.  $1\frac{1}{6} \div \frac{5}{8}$  \_\_\_\_\_

16.  $\frac{3}{5} \div \frac{4}{7}$  \_\_\_\_\_

### Problem Solving

17. One-fourth of the students in the fifth grade play baseball. If 30 students play baseball, how many students are in the fifth grade?

\_\_\_\_\_

18. Marvin is trying to finish a jigsaw puzzle. He has placed  $\frac{2}{3}$  of the pieces so far. If he has put in 60 pieces of the puzzle, how many pieces altogether are in the puzzle?

\_\_\_\_\_

Figure 3.2. The procedurally oriented textbook page used in the teaching simulation.

The selection of this adapted textbook page to represent the algorithmic treatment of the content was based on the fact that it treats the content in a strictly procedural manner: the notion of the reciprocal of a fraction is not defined; no explanations or representations are provided as to why the algorithm of “multiplying by the reciprocal” works or makes sense; and students are mainly expected to apply a given procedure to figure out the result of 16 similar exercises. Additionally, although this page *does* include two problems on the division of fractions, these problems pertain to the partitive interpretation of division, which is not as helpful for building meaning of the division of fractions as is the measurement interpretation of division. Given these features, this textbook page was considered representative of a poor and undemanding treatment of the content.

At the same time though, these deficiencies create opportunities to upgrade the demands of the included tasks. For instance, in using this adapted textbook page, a teacher might employ a suitable representation to explain why the algorithm of the division of fractions makes sense. This teacher might also emphasize the meaning of the operation at hand by judiciously organizing and presenting the available tasks. For instance, task 1 ( $\frac{2}{3} \div \frac{1}{3}$ ) and task 8 ( $\frac{5}{3} \div \frac{1}{3}$ ) are predominantly conducive to explicating the meaning of the division of fractions from a measurement perspective (e.g., How many one thirds are there in two thirds or in five thirds?). Tasks in which the divisor is larger than the dividend (e.g., tasks 2, 3, 4, 7, 11, and 12) clearly lend themselves to addressing one common student misconception (i.e., that the dividend should always be larger than the divisor). Additionally, tasks in which the denominators of the dividend and the divisor are multiples of each other (e.g., tasks 9, 10, 11, and 14) can be used to explore

what it means to divide fractions by turning the dividend and the divisor into fractions with similar denominators. Finally, a teacher might also consider modifying the two word problems that appear in this adapted textbook page (tasks 17 and 18) so that they correspond to the measurement interpretation of division, which is more amenable for understanding the division of fractions.

For the rich and intellectually complex treatment of the division of fractions, I selected a lesson from sixth-grade *Connected Mathematics II* (*Connected Mathematics II: Bits and Pieces I*; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006, pp. 52-53; see Figure 3.3). Three considerations informed this selection decision, all related to the richness and intellectual complexity of the tasks of this lesson. First, the lesson tasks revolve around a mathematical situation that requires that students explore and understand the measurement interpretation of the division of fractions (tasks *A* to *C*). Second, in solving these tasks, students are asked to use representations and provide explanations to explicate their reasoning, including an explanation for the fractional part of the quotient. Third, students are expected to *generate* the algorithm that applies to dividing a fraction by a fraction (task *D*). Thus, instead of spoon-feeding students with the algorithm of the division of fractions and then expecting them to use it to solve several similar exercises (like the previous textbook page does), this lesson expects students to figure out this algorithm by themselves by indentifying patterns in their solutions to tasks *A* through *C*.

For the purposes of the present study, the textbook pages of this lesson were adapted as follows. First, the picture accompanying the tasks was eliminated. Second, although tasks *A* through *D* were presented in two consecutive textbook pages, I merged them in a single page. Both these changes aimed at increasing the comparability of this

page to the procedurally oriented page discussed earlier. Finally, a fifth task originally included in the lesson was dropped to keep the task-selection phase of the teaching simulation within a reasonable amount of time.

**Dividing a Fraction by a Fraction**

Rasheed and Ananda have summer jobs at a ribbon company. Answer the questions below. Use written explanations or diagrams in each to show your reasoning. Write a number sentence to show your calculation(s).

**A.** Rasheed takes a customer order for ribbon badges. It takes  $\frac{1}{6}$  yard to make a ribbon for a badge. How many ribbon badges can he make from the given amounts of ribbon? Describe what each fractional part of an answer means.

1.  $\frac{1}{2}$  yard
2.  $\frac{3}{4}$  yard
3.  $2\frac{2}{3}$  yards (Remember  $2\frac{2}{3} = \frac{8}{3}$ .)

**B.** Ananda is working on an order for bows. She uses  $\frac{2}{3}$  yard of ribbon to make one bow. How many bows can Ananda make from each of the following amounts of ribbon?

1.  $\frac{4}{5}$  yard
2.  $1\frac{3}{4}$  yards
3.  $2\frac{1}{3}$  yards

**C.** Solve each of the following examples as if they were ribbon problems.

1.  $\frac{3}{4} \div \frac{2}{3}$
2.  $1\frac{3}{4} \div \frac{1}{2}$
3.  $2\frac{3}{4} \div \frac{3}{4}$

**D.** What algorithm makes sense for dividing any fraction by any fraction?

Figure 3.3. The conceptually oriented textbook page used in the teaching simulation.

The two textbook pages just discussed created a platform for exploring the participants' performance in selecting and using tasks. Furthermore, I used the second textbook page to develop the story of the virtual lesson of the teaching simulation. A brief description of this lesson is presented below. In presenting this story, I identify each of its teaching episodes by a particular name (presented in parentheses). The names of the



teaching episodes also appear in Figure 3.4. This figure is intended to serve as the linchpin between the analytic framework presented in Chapter 2 (see Figure 2.7) and the operationalization of the 15 entrance points of the specification Table 3.3.

*The story of the virtual lesson.* A sixth-grade teacher, Ms. Rebecca, decides to use the second textbook page presented above to teach an introductory lesson on the division of fractions. In previous lessons her class studied addition, subtraction, and multiplication of fractions. The lesson starts with Ms. Rebecca specifying that in today's lesson the class will consider the division of fractions. She then asks students to read the directions on the second textbook page which is used as the worksheet for this lesson. Once these directions are read, Ms. Rebecca asks her students to specify which numbers they should divide to solve task  $A_1$ ; this task asks for the number of one-sixth yard ribbons that can be made with half a yard (Presentation of task  $A_1$ ). Shaun responds that they should divide a half by one sixth; Ben wonders why they do not divide a sixth by a half. Ms. Rebecca elicits other students' ideas, to which June replies that the number order is as suggested by Shaun because one half is larger than one sixth. June's explanation is suggestive of a common misconception that the dividend is always larger than the divisor; furthermore, her explanation does not actually consider the problem at hand (June's explanation). The teacher accepts June's explanation, thanks her for her contribution (Reacting to June's explanation), and moves on to draw a diagram to help the students solve task  $A_1$ .

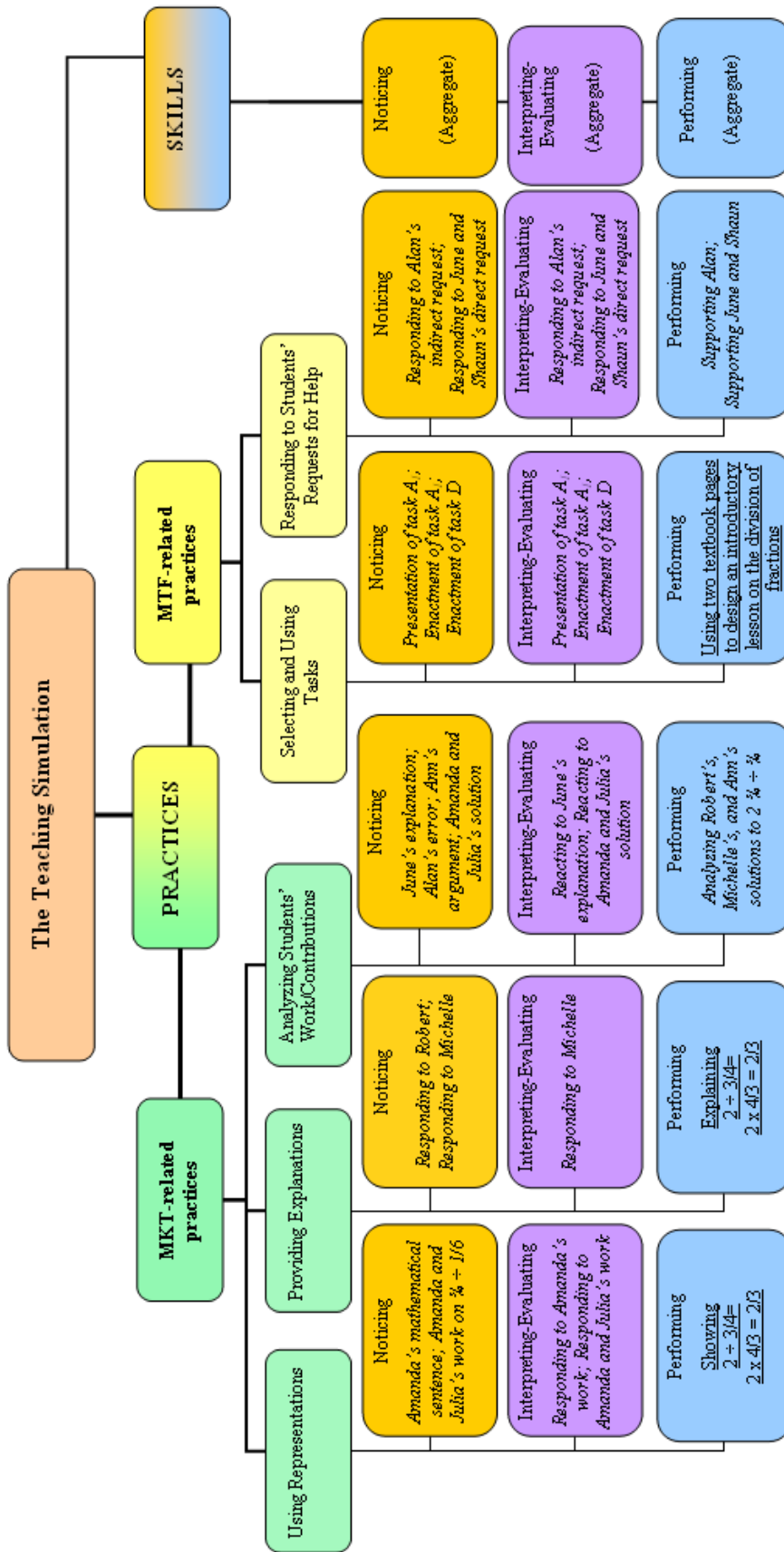
She first represents the half yard with a red line and then extends this line by drawing the other half yard in black. She then asks a student, Alan, to come to the board and use the diagram to solve the problem. Alan incorrectly divides the red line into six parts, instead of the whole yard, which is represented by the red and the black lines

together (Alan's error). In working on this problem, Alan also appears to be confused (Supporting Alan); as Stigler and Hiebert maintain (1999, pp. 91-93), U.S. teachers often perceive such student confusion as an indirect request for help.

To help Alan, Ms. Rebecca asks a very pointed question that directly leads Alan to correct his error (Responding to Alan's indirect request for help).<sup>77</sup> After Alan corrects his work, the teacher asks another student, Amanda, to come to the board and write the mathematical sentence for this problem. Amanda comes to the board and writes " $\frac{1}{2} \div \frac{1}{6} = 3$ " without discussing what the numbers of her mathematical sentence mean and without drawing any connections among the drawing shown on the board, her mathematical sentence, and the actual ribbon problem (Amanda's mathematical sentence). Without prompting Amanda to consider such connections, the teacher moves on to summarize the class' work on task A<sub>1</sub> (Responding to Amanda's work). This summary, co-constructed by the teacher and the students, overemphasizes following procedures and obtaining correct answers rather than making meaning of the division of fractions (Enactment of Task A<sub>1</sub>). During this episode, another student, Ann, argues that to figure out the dividend in a division-of-fractions problem one simply needs to select the larger of the given numbers, an argument that resonates with June's claim that the dividend is always larger than the divisor (Ann's argument). The teacher does not capture the misconception in this idea and moves on to assign tasks A<sub>2</sub> ( $\frac{3}{4} \div \frac{1}{6}$ ) and A<sub>3</sub> ( $2\frac{2}{3} \div \frac{1}{6}$ ).

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<sup>77</sup> The manner in which Ms. Rebecca supports Alan is reminiscent of the *Topaze* effect discussed by Brousseau (1997). According to this effect, the student gets the right answer to a task in virtue of the very narrow and pointed questions the teacher asks. Thus, the student's performance tells very little, if anything, about the student's actual understanding (for more on this issue, see Chapter 2, Section 4).



Note. Underlined characters: Entry points to explore the PSTs' performance during the lesson-planning part of the teaching simulation; *Italicized* characters: Entry points to explore the PSTs' performance during the virtual-lesson part of the teaching simulation.

Figure 3.4. Analysis of the teaching simulation.

While working on task A<sub>2</sub>, June and Shaun solicit the teacher's help (Supporting June and Shaun). As soon as the teacher realizes that two students are struggling with this task, she jumps in, and while addressing the whole class, she suggests that students use the common denominator of twelve to solve the problem.<sup>78</sup> However, she does not explain why this approach is relevant to solving this problem or how it can be used toward this end (Responding to June and Shaun's direct request for help).

After the students spend some time solving tasks A<sub>2</sub> and A<sub>3</sub>, Ms. Rebecca asks Amanda and Julia to share their work on task A<sub>2</sub>. Amanda and Julia come to the board, draw a line, divide it into 12 segments, color nine of them red, and then cluster these nine segments in four groups of two. Asked by the teacher to write a mathematical sentence that corresponds to this problem, the girls write  $\frac{3}{4} \div \frac{1}{6} = 4 \text{ R } \frac{1}{12}$ . Two things are important to notice in this episode. First, the girls draw no connections among their diagram, the numbers they used, and the actual word problem (Amanda and Julia's work on  $\frac{3}{4} \div \frac{1}{6}$ ). Second, their answer is incorrect because it confounds two different units: number four corresponds to the *ribbons* that can be made, while number  $\frac{1}{12}$  corresponds to the left-over *yards* (Amanda and Julia's solution). The teacher praises Amanda and Julia, and asks if there are any questions. Because no one has any questions, the teacher asks Michelle and Suzan to share their solution to task A<sub>3</sub> (Responding to Amanda and Julia's work/solution).

After Michelle and Suzan share their work, the teacher directs the students' attention to task D and suggests that they find a pattern to solve division-of-fraction problems. Without allowing any space for exploration, she writes the answers to tasks A<sub>1</sub>

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<sup>78</sup> This move is also consonant with the cultural script of teaching in U.S. as outlined in Stigler and Hiebert (1999, pp. 91-93).

and  $A_3$  on the board, and by eliciting the students' responses to very closed questions and manipulating the numbers involved in these two tasks, she arrives at the invert-and-multiply rule (Enactment of Task D). After that she has her students practice finding reciprocals; she then asks if the students have any questions. To that, Michelle responds that she still does not understand why the reciprocal works; Robert also observes that if one uses the invert-and-multiply algorithm to solve task  $A_2$ , one gets four and a half and not four and one twelfth, which was Amanda and Julia's solution to the problem. He thus wonders if the rule just presented by the teacher works for all problems.

Responding to Michelle, the teacher explains that "the reciprocal works because we invert the second fraction; and because we are using the reciprocal, we need to use the inverse operation." She also likens this rule to a similar situation, that of subtracting a positive number instead of adding its negative. The teacher's explanation to Michelle's question (Responding to Michelle) is problematic in several respects: it describes rather than explains the rule; it employs semantic rather than mathematical arguments (i.e., "we use a reciprocal operation because we use the reciprocal"); and it uses an inappropriate analogy. Ms. Rebecca postpones addressing Robert's question for the next day, but argues that the rule does work; "otherwise we wouldn't call it a rule." In doing so, she not only avoids providing an explanation, but she also seems to reinforce the conception of mathematics being a set of mindlessly followed rules (Responding to Robert's question). Near the end of the lesson, and without addressing Robert's question, which is important for solving task  $C_3$  (i.e.,  $2\frac{3}{4} \div \frac{3}{4}$ ), Ms. Rebecca assigns this task to her students. While circulating in the class, she observes that three students solved this task in different ways, as outlined in Figure 3.5. As shown in this figure, both Robert and Ann used the

traditional algorithm to solve this task; Ann however, inverted part of the dividend, instead of inverting the divisor.<sup>79</sup> Michelle solved the problem using the pictorial approach considered during this lesson but did not write a mathematical sentence to represent her work. At this point of the lesson, the PSTs are asked to analyze the three student solutions and make assertions about these students' understanding or lack thereof (Analyzing Robert's, Michelle's, and Ann's solutions to  $2\frac{3}{4} \div \frac{3}{4}$ ). The virtual lesson concludes with Ms. Rebecca asking Robert to share his work on the board.

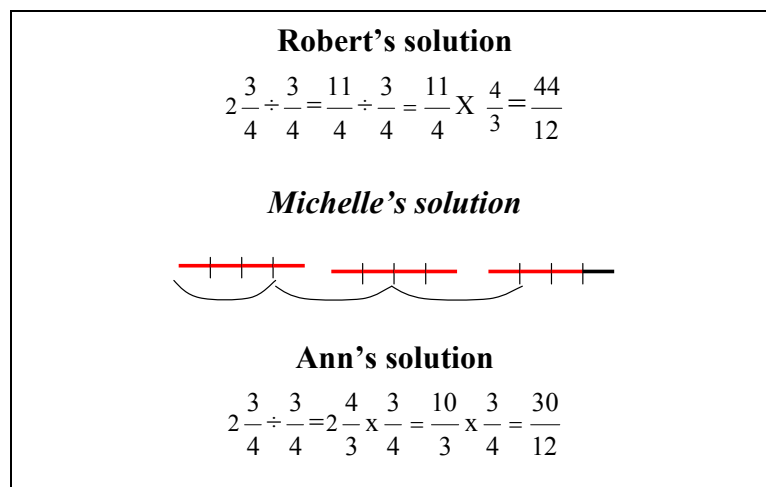


Figure 3.5. The three student solutions to the problem  $2\frac{3}{4} \div \frac{3}{4}$ .

Figure 3.4 shows that the virtual lesson created a rich arena for exploring the PSTs' performance in 12 out of the 15 entry points defined by the specification Table 3.3 (these entry points appear in italicized characters). The remaining three entry points (illustrated in underlined characters) concerned performing in selecting and using tasks, using representations, and providing explanations, and were all part of the lesson-planning portion of the teaching simulation. As already explained, the PSTs' task

<sup>79</sup> Ann's algorithmic error represents one of the three types of common student errors identified in the literature with respect to dividing fractions (cf. Tirosh, 2000; Piel & Green, 1994). The other two types of errors are captured in June's explanation and in Ann's argument (i.e., an intuitive-based error that the dividend is always larger than the divisor) and in Alan's and Amanda and Julia's work (i.e., a conceptual error emerging from lack of understanding of the relative and absolute units).

selection and use was examined by presenting them with the two textbook pages and asking them to design a lesson on dividing fractions. Their performance in using representations and providing explanations was examined by asking them to explain the worked out example presented on the first textbook page (see Figure 3.2) and by prompting them to use a representation in their explanation, if they did not choose to do so. The PSTs were, in particular, asked to explain the quotient of this example and why it makes sense to multiply  $2 \times \frac{4}{3}$  when one divides 2 by  $\frac{3}{4}$ .

Three clarifications are necessary regarding the story of the virtual lesson. First, in designing the virtual lesson, I decided to eliminate classroom-management issues and present a relatively well-behaved class. This decision was informed by considerations of both the purpose of the study and of recent research findings. Recall that this study was designed to explore the association between PSTs' MKT and their performance in five teaching practices. Although classroom-management issues are part of the complexity of teaching, their elimination from the virtual lesson simplified the complexity of teaching which, in turn, was envisioned to allow for better exploration of the association at hand. This simplification also allowed for accentuating the complexity of teaching regarding the *mathematical* requirements of the lesson and its teaching. Moreover, recent research findings (e.g., Star & Strickland, 2008) suggest that when asked to consider and discuss teaching episodes, PSTs mostly focus on classroom-management issues, rather than on the teacher-student interactions around the mathematical content. Hence, by eliminating such classroom-management issues, an attempt was made to direct the PSTs' attention to the mathematics of the lesson and its teaching. Like the other design decisions of the study, this design decision is taken into consideration when discussing the study findings.

Second, in designing this lesson, I made an effort to include instructional features that, on the surface level, are considered conducive to establishing rich and challenging learning environments. Such features include using representations, having students work in groups and then share their work, circulating and monitoring students while working individually, and emphasizing the importance of providing explanations. Although these features could theoretically help create rich and challenging learning settings, merely employing them in a mathematics lesson does not suffice to establish such settings. Consider, for instance, the use of representations. A teacher might very well use representations or manipulatives in her lessons, but still fail to use them to help students construct meaning.<sup>80</sup> Similarly, just having students share their work is not sufficient to establish rich and challenging environments since students can share their work, but still fail to draw any connections between them. Along the same lines, simply emphasizing the value of providing explanations cannot help build rich mathematics environments; it is by soliciting students' explanations or by the teacher's appropriate modeling of providing explanations that such environments could be established. By incorporating such features in the virtual lesson, I wanted to explore whether the PSTs would evaluate Ms. Rebecca's lesson based on these surface characteristics or, alternatively, if they would delve deeper into how her decisions and actions actually resulted in diluting the cognitive complexity and richness of the tasks she used in her lesson.

Third, to ensure that the PSTs would not judge the learning environment that Ms. Rebecca created by merely focusing on equity considerations, an attempt was made to

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<sup>80</sup> Such a situation is presented in *The Case of Fran Gorman and Kevin Cooper* (Stein et al., 2000, pp. 65-80). Although both teachers use manipulatives in their teaching, Fran shifts the emphasis from meaning, concepts, and understanding to merely using the manipulatives as a means to get the correct answer. In doing so, Fran fails to engage his students in considering rich and challenging mathematics.



show her treating all her students equitably, regardless of their background characteristics (i.e., race and gender). Additionally, all students in the virtual lesson were shown being engaged at least for the most part of the lesson and no student was shown dominating the classroom discussion.

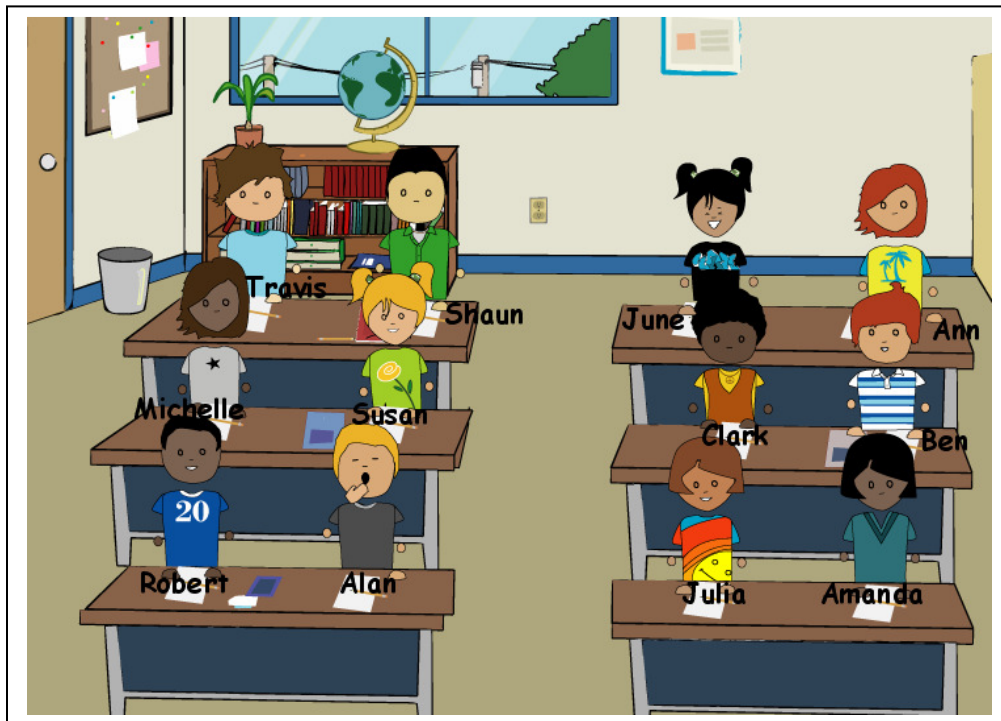
#### *Using the ThExprians-P Character Set to Develop the Virtual Lesson*

The story of the virtual lesson was used to develop the script for this lesson. This script detailed the utterances of the teacher and different students, their work in their notebooks and on the board, and their facial expressions (when necessary). Once this script was determined, I developed the PowerPoint slide animation in consultation with one of the members of the GRIP project.

The first step in generating the slides was to develop the characters of the story. For this purpose, I used the template of the *ThExprians-P* character set which consists of a collection of male and female faces of different complexity and shape; their corresponding hands and elbows; various hairstyles; different types of facial expressions that represent various emotions (e.g., confusion, boredom, surprise, anger), and a set of T-shirts. Using this template, I developed 12 sixth-grade student characters, six for boys and six for girls. Five of these students are Caucasian, four are African-American, two are Hispanic, and one is Asian-American (see Figure 3.6). In developing these characters, an attempt was made to simulate the average proportion of the different racial groups in the student population of the Detroit elementary and middle-grade public schools, where most of the study participants were expected to be placed during their field placement. Because the template of *ThExprians-P* was originally designed for high-school student characters, in developing the cartoon characters used in this study, I scaled down all their

constituent parts (e.g., face, hairstyles, T-shirts). Using the existing template, I also developed the cartoon character of Ms. Rebecca.

The next step in developing the virtual lesson was to design the classroom setting. To achieve this, I used the existing classroom setting that was developed by the members of the GRIP project, but adapted it to represent a sixth-grade classroom. Specifically, I arranged the student desks in three rows, consisting of two desks each, instead of having them arranged in groups of two. I also eliminated a poster from the wall that pertained to high-school science content. Finally, I designed chairs for students, for those slides that presented the classroom from its back side.



*Figure 3.6.* The student characters used in the virtual lesson.

After developing the characters and the classroom setting, I parsed the script into small chunks of text (utterances or story narration) that could fit into each single slide.

Once the slides and their content were determined, I used this basic template of

characters and the classroom setting to develop the PowerPoint animation; this process took about a month. When the first draft of the animation was developed, I revised it based on suggestions from members of the GRIP project. These suggestions included using the same type of speech bubbles consistently, using words to represent the numbers in the characters' utterances, and relocating the hands and elbows of the cartoon characters.

Finally, the virtual lesson was parsed into five large chunks.<sup>81</sup> Each chunk captures part of the virtual lesson and concludes by a slide entitled "Time for reflection." This slide signals the point during which the interviewees were asked to comment on the virtual lesson. The parsing of the lesson was strategic, in that each of these lesson segments concludes with an instance during which the virtual teacher has to make specific decisions related to the five practices under exploration. This created the opportunity for exploring the study participants' respective decisions and actions and then exposing them to the decisions and actions of the virtual teacher. For instance, the first lesson segment, which concludes with Alan's error, helped explore the PSTs' work on handling this error while simultaneously dealing with Alan's confusion.

#### *Developing the Accompanying Interview Protocol*

The need to conduct semi-structured interviews (in order to increase the comparability of the interview data) required that an interview protocol be developed to guide the interview process. The protocol used in this study was developed based on ideas presented in Ball (1988) and Kennedy, Ball and McDiarmid (1993). After developing an initial draft of this protocol, I worked closely with two other instructors

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<sup>81</sup> The final version of the script of the virtual lesson appears in Appendix 3.

involved in teaching the math content course, and revised the original draft to achieve greater focus and to ensure that the interview would be kept within a reasonable amount of time. In its final version (see Appendix 4), the interview protocol consisted of an opening statement and four parts.

The opening statement explained the purpose of the interview to the study participants and asked their permission to audiotape the discussion. The first part included a general question, directly drawn from Ball's (1988) interview protocol. Designed to establish some rapport between the interviewer and the interviewee, this question asked the PSTs to talk about their motivation to become K-8 teachers. The next two parts were designed to represent the three phases of a *teaching cycle*: planning a lesson, enacting the lesson, and reflecting on it.<sup>82</sup>

The second part of the interview protocol captured the planning phase of this cycle and asked the PSTs to discuss the two textbook pages presented in Figures 3.2 and 3.3. In particular, the PSTs were asked to identify the strengths and limitations of these pages and to explain how they would use them to design an introductory lesson on the division of fractions for sixth graders. Also included in this part was a set of more focused questions that probed the PSTs to discuss their selection and use of tasks for such a lesson and the way(s) in which they would explain the algorithm of the division of fractions (by referring to the worked out example of  $2 \div \frac{3}{4}$ ).

The third part of the interview protocol corresponded to the enactment and reflection phases. First, the protocol provided background information for the virtual

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<sup>82</sup> This three-phase structure of the interview has been considered suitable for exploring teachers' knowledge and the ways in which their knowledge can inform their teaching practices (cf. Wilson et al., 1987). Notice, though, that there is not an one-to-one correspondence between the phases of the teaching cycle and those of the MTF. In particular, the MTF does not include a reflection phase. On the other hand, the teaching cycle combines task presentation and enactment into one phase.

lesson and invited the PSTs to imagine that they worked in a school and one of their colleagues, Ms. Rebecca, asked them to observe her while teaching an introductory lesson on the division of fractions to a sixth-grade class; according to the protocol, Ms. Rebecca decided to use the second textbook page. The interview protocol also provided a brief description of the virtual students' background characteristics; it also informed the study participants that at the end of each of the five segments of the lesson they would be asked to: (a) discuss the segment they had observed, (b) describe and evaluate the virtual teacher's approach in certain teaching episodes, and (c) outline how they would perform in analogous instructional situations, if they had to teach the lesson. These three types of prompts correspond to the noticing, interpreting-evaluating, and the performing skills of the teaching performance explored in the study. The interview protocol also included a set of specific questions for each of the five lesson segments. The first question in each set asked the participants to describe what they noticed in each segment. Developed along the specifications of Table 3.3, the remaining questions sought to explore the participants' evaluation of the virtual teacher's decisions and actions and investigate what the PSTs would do when faced with similar instructional situations. The third part of the interview protocol concluded with two brief questions that aimed at investigating the PSTs' evaluation of the virtual lesson as a whole and Ms. Rebecca's use of the textbook page. Together, these two questions were designed to capture the reflection phase.

The last part of the interview included a concluding question designed to offer the PSTs the opportunity to discuss their experience in participating in the interview, to ask questions, and to communicate any concerns they might have had to the interviewer.

For the purposes of the post-intervention interviews the abovementioned protocol was slightly modified. The changes made to the protocol to meet the purposes of the post-intervention interviews concerned: (a) the introductory comments, where it was explained that the post-intervention interview had a similar format to that of the pre-intervention interview; (b) the question of the first part, in which the PSTs were asked to discuss their thinking about teaching and learning mathematics after the end of the two courses; and (c) the third part, in which two questions were added. The first question asked the PSTs to comment on another teacher's evaluation of Ms. Rebecca's approach, according to which Ms. Rebecca largely used the second textbook page procedurally. This question was designed to elicit comments regarding the virtual teacher's use of the textbook page. The second question asked the PSTs to compare their thinking in the pre-intervention interview with that in the post-intervention meeting; if the PSTs identified any differences, particularly with respect to their performance, they were additionally asked to define what might have contributed to these changes.

#### *Vetting and Pilot-Testing the Simulation and Its Accompanying Interview Protocol*

The last stage in the development of the simulation and its accompanying interview protocol consisted of vetting and pilot-testing this instrument. A group of LMT researchers participated in the vetting process. These researchers evaluated the simulation as an appropriate instrument for exploring how teachers' MKT might inform their teaching practices; they also made some editing suggestions.

After revising the simulation upon the suggestions made, I pilot-tested it with two PSTs who were enrolled in the ELMAC program during the previous academic year (Summer 2006 – Summer 2007). To this effect, I first sent an e-mail to all the PSTs in

this cohort and asked for volunteers to participate in pilot-testing the instrument. Five PSTs responded to this e-mail. From those volunteers, I selected two, who differed in their performance in the math content and methods courses, according to their instructors' comments. Both volunteers received a 30-dollar stipend for their participation in the pilot-testing phase of the instrument.

The PST identified as weak in mathematics mostly focused on the virtual teacher's pedagogical moves without actually exploring whether the teacher's moves distorted the mathematics. She also encountered significant difficulties using a representation to explain the algorithm of the division of fractions, which she admitted that she herself did not understand. The PST identified as strong at math was able to use a representation and partly explain the algorithm of the division of fractions; like her counterpart, she nonetheless found this activity challenging. This PST, however, attended more to the mathematics of the virtual lesson compared to her colleague. The comparison of these two PSTs' viewing of the virtual lesson and the differences identified in their performances further supported that the teaching simulation could support the exploration of the research questions of the study. This pilot testing also suggested that although the teaching simulation included challenging tasks, it offered a rich context for exploring PSTs' relative decisions and actions. The pilot-testing phase also helped eliminate some questions from the interview protocol that seemed to offer little insights into PSTs' thinking and performance regarding the study inquiry.

In addition to the two *main* instruments described above (the adapted LMT test and the teaching simulation), a survey was also employed to tap other aspects of the PSTs' traits potentially associated with their teaching performance.

*Capturing Other Factors Potentially Associated with PSTs' Teaching Performance:  
The Survey*

As explained in Chapter 2, besides teacher knowledge, several other factors might be associated with PSTs' performance in the practices this study examines. From the gamut of these factors, in this study, I decided to focus on teachers' beliefs about teaching and learning mathematics; their overall goals for teaching this subject; their efficacy beliefs about working on fractions and engaging in the MKT-related practices; and a set of background characteristics explored in studies on teacher knowledge and effectiveness (i.e., math content and math methods courses taken during undergraduate studies and having a major/minor in mathematics). To obtain relevant data, a survey was developed and used. In this part, I present this instrument and detail its development.<sup>83</sup>

*Beliefs about Teaching and Learning Mathematics*

The first part of the survey was designed to capture the PSTs' beliefs about teaching and learning mathematics. After an initial review of instruments developed to investigate teachers' respective beliefs (e.g., Ball, 1988; Kennedy et al., 1993; Ross, McDougall, Hogaboanm-Gray, & LeSage, 2003), I compiled a list of 43 candidate statements. Given the focus of the study on exploring practices supportive of establishing rich and challenging learning environments, I wanted to use only those statements that would help better discriminate between PSTs who were more and less inclined to build such environments, simply because of their beliefs. To do so, I tried to calibrate the set of 43 statements to capture the aforesaid beliefs. Central to this calibration process was the delineation of the portraits of two categories of teachers. These portraits, developed by building on the *Task Analysis Guide* (Stein et al., 2000, p. 16), are presented in Table 3.4.

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<sup>83</sup> The survey appears in Appendix 5.



Table 3.4

*Portraits of Teachers More or Less Inclined to Establish Rich and Challenging Environments*

<b>Type-A teacher</b>	<b>Type-B teacher</b>
<ul style="list-style-type: none"> <li>- Invests time and effort in having students explore and understand concepts, processes or relationships.</li> </ul>	<ul style="list-style-type: none"> <li>- Invests time and effort in having students reproduce previously learned facts, rules, formulae or committing facts, rules, formulae or definitions to memory.</li> </ul>
<ul style="list-style-type: none"> <li>- Usually assigns tasks that require complex and non algorithmic thinking or/and expects students to link the procedures at hand with their underlying meaning. These tasks require moderate to high cognitive effort on the part of students.</li> </ul>	<ul style="list-style-type: none"> <li>- Usually assigns algorithmic tasks that require students to reproduce previously learned material. These tasks require limited cognitive effort on the part of students because there is little ambiguity about what needs to be done and how to do it.</li> </ul>
<ul style="list-style-type: none"> <li>- Usually expects the class to solve a given task in more than one way and encourages students to make connections among multiple solutions.</li> </ul>	<ul style="list-style-type: none"> <li>- Usually does <i>not</i> invest time and effort in helping students connect a procedure to its underlying meaning.</li> </ul>
<ul style="list-style-type: none"> <li>- Is more concerned with students' developing understanding of mathematics concepts and ideas.</li> </ul>	<ul style="list-style-type: none"> <li>- Is more concerned with students' producing correct answers rather than with their developing of mathematical understanding.</li> </ul>
<ul style="list-style-type: none"> <li>- Requires and insists on students' finding patterns, making generalizations, explaining, and justifying their thinking.</li> </ul>	<ul style="list-style-type: none"> <li>- Requires students to provide explanations rather infrequently; when doing so, students are largely expected to <i>describe</i> the procedure used.</li> </ul>

Type-A teachers are those whose beliefs are conducive to planning and enacting rich and cognitively challenging math lessons. These teachers place more emphasis on helping students develop understanding of mathematical processes and concepts; they insist that students provide explanations and justify their answers; they value making connections between different ideas and solutions to problems; and they tend to select activities that engage students in exploring ideas and identifying patterns. Type-B

teachers, on the other hand, are less inclined to build such environments: they are more concerned with ensuring that students become proficient in learning and applying rules and formulae and getting correct answers; they infrequently ask students to explain and justify their answers, and they prefer to assign less complex tasks. In developing these bipolar portraits, I was not expecting the PSTs to be clustered in the one or the other category, but rather to be situated along one or more continua defined by the opposite ends designated by the subcomponents of these portraits.

To help sift through the 43 statements and identify only those that would better capture the polarity of these portraits, I sent the 43 statements along with Table 3.4 to five researchers, all of whom have used the MTF and the accompanying *Task Analysis Guide* in their work with teachers.<sup>84</sup> These experts were asked to evaluate whether each of the 43 statements could satisfactorily discriminate between the two types of teachers portrayed above. To do so, they were requested to use a 1-7 Likert scale (with 1 corresponding to “strongly disagree,” 4 representing the neutral point, and 7 corresponding to “strongly agree”) and consider how PSTs who hold beliefs similar to those of the two portrayed types of teachers would respond to each statement. They were also instructed to use number 8 to designate statements which they considered inappropriate for discriminating between the two types of teachers.

After receiving these experts’ responses, an average score was calculated for each of these statements as follows. First, for each statement, I calculated the difference between each expert’s answer for the type-A and type-B teacher. For instance, if an expert considered that a type-A teacher would strongly disagree with a statement (code 1) and type-B teacher would strongly agree with it (code 7), this statement was assigned a

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<sup>84</sup> The document sent to these experts appears in Appendix 6.

score of 6 for this particular expert. The statements that were assigned a code of 8 by these experts (i.e., were not considered appropriate for discriminating between the two types of teachers) were recoded as zero. I then calculated the average absolute score for each of these statements and finally, identified the statements that had an average absolute value greater than three. A threshold of three was considered sufficient to discriminate between the two types of teachers, on the premise that this difference ensures that the two types of teachers are positioned in opposite sites of the neutral point (i.e., code 4). This procedure led to the identification of the 18 statements presented in Figure 3.7.

1. It is confusing to see many different methods and explanations for the same idea.
2. A good mathematics teacher is someone who explains clearly and completely how each problem should be solved.
3. Teachers should not necessarily answer students' questions but let them puzzle things out themselves.
4. Students learn mathematics best if they have to figure things out for themselves instead of being told or shown.
5. When students can't solve problems, it is usually because they can't remember the right formula or rule.
6. When students solve the same math problem using two or more different strategies, the teacher should have them share their solutions.
7. It is important for pupils to master the basic computational skills before they tackle complex problems.
8. If students are having difficulty in math, a good approach is to give them more practice in the skills they lack.
9. To do well, students must learn facts, principles, and formulas in mathematics.
10. In learning math, students must master topics and skills at one level before going on.
11. Doing mathematics allows room for original thinking and creativity.
12. The most important issue is not whether the answer to any math problem is correct, but whether students can explain their answers.
13. Basic computational skill and a lot of patience are sufficient for teaching elementary school math.
14. Teachers should try to avoid telling.
15. Doing mathematics is usually a matter of working logically in a step-by-step fashion.
16. A lot of things in math must simply be accepted as true and remembered.
17. Students should never leave math class (or end the math period) feeling confused or stuck.
18. If students have unanswered questions or confusions when they leave class, they will be frustrated by the homework.

*Figure 3.7.* The statements used to explore the PSTs' beliefs about teaching and learning mathematics.

In this figure the statements are presented in descending order, with the first being that deemed the most appropriate for discriminating between the two types of teachers. These 18 statements were then randomly reordered and included in the first part of the survey. The study participants were asked to use a seven point Likert scale (1= “strongly disagree”; 7 = “strongly agree”) to indicate their level of agreement with each statement.

### *Overarching Goals for Teaching Mathematics*

The second part of the survey was designed to gain insights into the PSTs’ *general* goals for teaching mathematics. To achieve this, I used the five strands of mathematical proficiency proposed in the NRC report, *Adding it Up: Helping Children Learn Mathematics* (Kilpatrick et al., 2001, pp. 115-133). This report defines mathematical proficiency as consisting of five interwoven and interdependent strands:

- *conceptual understanding*, which refers to the comprehension of mathematical concepts, operations and relations
- *procedural fluency*, which captures the skill to carry out procedures flexibly, accurately, and efficiently
- *strategic competence*, which refers to the capacity to identify, formulate, represent, and solve problems
- *adaptive reasoning*, which captures the ability for logical thought, reflection, explanation, and justification
- *productive disposition*, which pertains to the learner’s inclination to see mathematics as sensible, value the activity of doing mathematics as useful and worthwhile, and conceive of oneself as capable of engaging in such activity

Although all five strands are important for enhancing students’ mathematical proficiency, a teacher might prioritize certain ones over others. Thus, to explore the PSTs’ overarching goals for teaching mathematics, the second part of the survey presented the five strands of mathematical proficiency as five plausible goals for teaching mathematics. In working on this part of the survey, the PSTs were asked to write a percent next to each goal to designate the emphasis they thought appropriate to be placed on each goal when teaching mathematics. They were also required to explain their percent allocation.

### *Efficacy Beliefs*

The third part of the survey was designed to provide information regarding the PSTs' efficacy beliefs (i.e., perceived competence) about working on fractions and engaging in the three MKT-related practices (i.e., using representations, providing explanations, and analyzing student work). Scholars working in the realm of efficacy beliefs (e.g., Bandura 1997; Pajares, 1996; Tschannen-Moran, Woolfolk-Hoy, & Hoy, 1998) have suggested measuring these beliefs using statements calibrated to the specific content area and practices under investigation. Following this suggestion, instead of using an existing instrument on efficacy beliefs, I developed a set of 17 statements that would help examine the PSTs' respective efficacy beliefs (see Figure 3.8).

- How well can you do the following?*

  1. Operate with whole numbers (addition, subtraction, multiplication, and division)
  2. Understand and use different interpretations of fractions:
    - a. Fraction as a part of the whole
    - b. Fractions as quotients
    - c. Fractions as operators
    - d. Fractions as numbers
  3. Identify the unit (the whole), when given a representation on fractions
  4. Represent fractions or decimals by words, numbers, or models
  5. Add and subtract fractions
  6. Multiply and divide fractions
  7. Understand the algorithm of the multiplication of fractions
  8. Understand the algorithm of the division of fractions
  9. Use multiple representations when solving a problem
  10. Make connections between different representations
  11. Map correspondences among different representations (e.g., between a picture and a set of symbols)
  12. Explain why a procedure or an algorithm works/makes sense
  13. Identify incorrect explanations
  14. Identify inappropriate explanations
  15. Evaluate whether a proposed solution to a problem is correct
  16. Use appropriate representations when solving a problem
  17. Use appropriate examples to explain a mathematical idea

*Figure 3.8.* The statements used to explore the study participants' efficacy beliefs.

Statements 1–8 of this instrument concerned PSTs' perceived competence in the area of numbers and operations and particularly in fractions. The remaining nine

statements (statements 9–17) pertained to using multiple representations and building connections among them, providing and evaluating explanations, and sizing up proposed solutions to a problem. The PSTs were expected to use a seven-point Likert scale to indicate their sense of efficacy to engage in the tasks defined by each statement.

### *Background Characteristics and Experiences in Learning Mathematics*

The survey also included a fourth part designed to collect information on the PSTs' background characteristics (i.e., math content and methods courses taken in high school and during undergraduate studies; having a major/minor in mathematics). The study participants were also asked to briefly reflect on their experiences in learning mathematics in the elementary and middle grades. Directly drawn from Ball's study (1988), this latter question was envisioned to help provide information about the PSTs' images of teaching mathematics as developed through their long apprenticeship in mathematics classes over their school years. This could serve as an additional source of information on the study participants' beliefs about teaching and learning mathematics.

### *Data Collection Instruments: Summary*

To summarize, three instruments were used to collect data for the purposes of this study: an adapted version of an existing LMT test to measure the participants' MKT, a teaching simulation for exploring the participants' teaching performance in the five practices, and a survey expected to yield information regarding the PSTs' beliefs and overarching goals for teaching mathematics, as well as some of their background characteristics. In the next section, I detail how these three instruments were used to collect the study data.

## Data Collection

Data collection began in late June 2007 and concluded in April 2008. A description of the data collection procedures associated with each study instrument is presented below. Figure 3.9 provides a timeline of the data collection procedures.

### *Administration of the Adapted LMT Test*

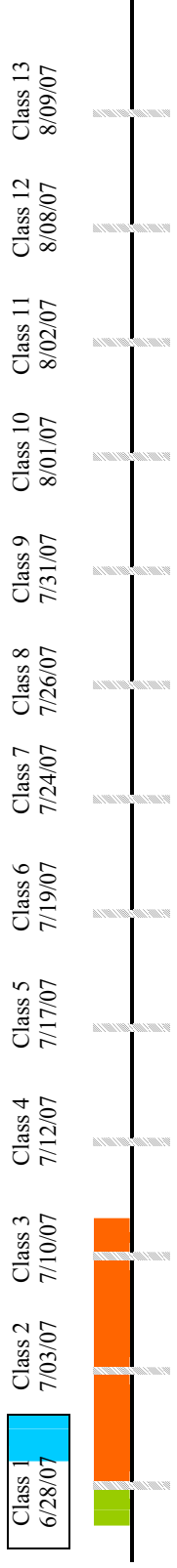
The adapted LMT test was administered twice. As illustrated in Figure 3.9, the first administration of the LMT test occurred during the first class of the math content course. During this class, the course instructor introduced the notion of MKT as a foundation for both the math content and the methods course and discussed how this knowledge constitutes an important asset when teaching mathematics. At the last third of this class, the course instructor administered the adapted LMT test and explained that the PSTs would also be asked to complete this test at the end of the math methods course. Given that this study fell under the umbrella of research efforts associated with understanding the work of teaching mathematics, the course instructor also explained that the comparison of the PSTs' pre-and post-test performance would inform the ongoing process of redesigning and improving both courses. The study participants did not write their name on the test, but they included their University of Michigan identity number and/or the four last digits of their Social Security Number.

The PSTs had approximately an hour to complete the test, during which no additional clarifications were provided. One of them finished early and submitted his test in half an hour. Most of the PSTs needed 45–55 minutes to finish the test, whereas two PSTs were given additional time (they worked for approximately 70–75 minutes).<sup>85</sup>

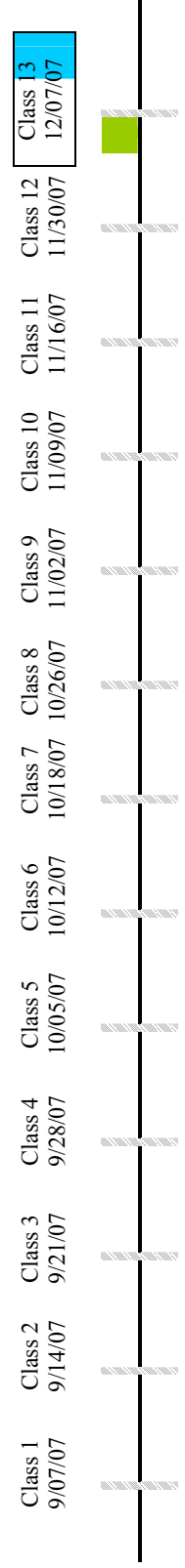
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<sup>85</sup> For comparative reasons, the test was administered to the PSTs of the second ELMAC cohort, as well.

### Summer Semester 2007 (Math Content Course)



### Fall Semester 2007 (Math Methods Course)



### Winter Semester 2008

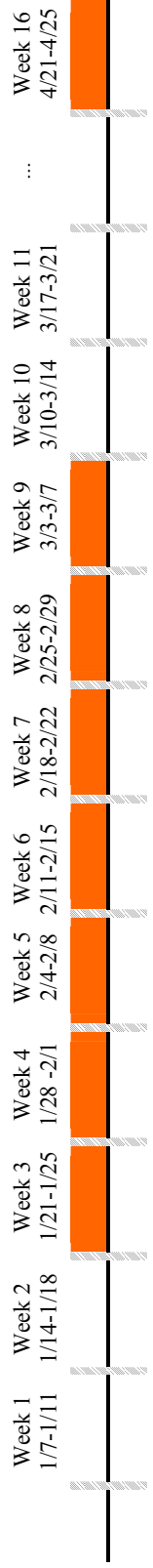


Figure 3.9. Data collection timeline.



A similar approach was followed during the last class of the math methods course (December 2007, see Figure 3.9). The course instructor administered the same LMT test during the last third of this class. Like what happened during the first administration of the test, some PSTs finished earlier than the allotted time (the first student submitted his test after 35 minutes the test was administered). The PSTs who needed more time to complete the test during its first administration also took more time to complete the test during its second administration (they worked for approximately 85-90 minutes).

### *The Interview Meeting*

During the first math content class, the course instructor explained to the PSTs that, along with the other instructors of the math content and methods courses, he was interested in understanding and improving the work of teaching mathematics. Also, it was important to the course instructor to get to know his students better and establish the rapport necessary for their future work. Thus, he asked them to schedule meetings with him during which each PSTs would engage in the teaching simulation outlined above; these meetings would also provide an arena for the PSTs to discuss any concerns or expectations they might have had for the course. As part of their first course assignment, the study participants were asked to review the two textbook pages presented in Figures 3.2 and 3.3, prepare to discuss the textbook pages' strengths and weaknesses, and explain how they would use those pages to build a lesson on the division of fractions; they were expected to complete this part of the assignment before coming to the interview meeting.

I conducted individual interviews with each of the study participants during the first third of July 2007.<sup>86</sup> All the interviews were conducted in my office, and each lasted

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<sup>86</sup> I met with ten PSTs between the first two classes, nine of them after the second class and before the third class, and one of them right after the third class.

anywhere from 80 to 135 minutes ( $\bar{x}=110.5$  min;  $SD=17.6$  min ). I started each meeting by clarifying that the main purpose of the interview was to understand the PSTs' thinking and the requirements entailed in teaching mathematics. As such, I reminded them that the interview should not be perceived as an examination. After asking permission to audiotape the interview, there followed an informal discussion about the PSTs' impression of the first class, which mostly served as an ice-breaker. This was succeeded by a discussion of what brought them to teaching, which also served to establish rapport. During this part of the interview, I tried to maintain a friendly tone that would help the interviewees open up and feel comfortable to share their ideas. In the next parts of the interview, I maintained this tone but I also tried to be as neutral as possible (both in my verbal and non-verbal behavior) when responding to the interviewees' thoughts, ideas, and answers. This ensured – to the extent that this was possible – that my interactions with them would not color their thinking.<sup>87</sup>

Next, I introduced the idea of a teaching cycle and asked the interviewees to share their thoughts on the two textbook pages. Using the interview protocol, I then asked more focused questions to elicit the interviewees' perspectives on using these two textbook pages for teaching an introductory lesson on the division of fractions. During this part of the interview, the interviewees were provided with pencils, color markers, and paper, and were asked to illustrate how they would introduce and explain the algorithm of the division of fractions to sixth graders. A clinical-approach method (Brenner, 2006; Koichu & Harel, 2007) was followed during this part of the interview. In particular, I often asked for more information and pressed for explanations, using prompts like: "How do you

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<sup>87</sup> Brenner (2006) considers establishing a neutral tone and avoiding praising specific answers particularly critical to obtaining high-quality interview data.

know that?”, “Why did you use this representation?”, “Could you explain this algorithm in a different way?”, especially if the interviewees’ explanations were simply a description of the steps of the algorithm. When the interviewees seemed puzzled, I probed them to clarify what was hard for them in explaining this algorithm; I also reassured them that during the two courses we would have ample opportunities to delve into the issues considered during the interview.

Once the “planning phase” was over, I introduced the activities associated with the virtual lesson (i.e., the “enacting” and “reflecting” phases). I first explained that we would go over a PowerPoint presentation and that at five points during this slide presentation, I would be asking them questions regarding their observations and their evaluations of the virtual-teacher’s decisions and actions. They would also be asked to outline how they themselves would react when faced with similar instructional situations to those depicted in the virtual lesson. The interviewees controlled the pace of viewing the PowerPoint presentation; if they wanted, they could also move back and forth to reconsider the content of particular slides. The software *Profcast*, installed on the computer used for viewing the virtual lesson, allowed recording the time the PSTs spent on each slide and their transitions between slides. While going over the PowerPoint presentation, the interviewees were encouraged to think out loud, which the majority of them did. They were also given as much time as they wished to consider certain slides. In addition to the questions included in the interview protocol, at several points during the interview, I encouraged the PSTs to elaborate on their ideas, especially if they found certain aspects of the virtual lesson intriguing or strange. When the virtual lesson was over, the interviewees were asked to reflect on the virtual teacher’s instruction as a

whole; they were also encouraged to comment on their experience participating in the interview and to voice any concerns or expectations they had about the course. The audio files and those generated by using *Profcast* were archived in digital folders separately for each interviewee. The written artifacts produced during the interview were scanned and also stored in these folders.

A similar approach was followed in January through April 2008 while conducting the post-intervention interviews. In particular, in the first half of January, 19 of the 20 study participants were sent an e-mail inviting them to participate in a post-course interview; one of the study participants (Travis) withdrew from the program. Sixteen of the remaining 19 PSTs (all the PSTs but Dana, Dorothy, and Lillian) replied to this e-mail. Those PSTs who replied to the e-mail were sent the two textbook pages they had to consider for the first interview meeting and were informed that the second meeting would have a similar format to that of the summer interview; they were thus encouraged to study the two textbook pages before this meeting. The meetings transpired at a time that was convenient to the study participants; the great majority of them occurred in January and February 2008.<sup>88</sup>

With the exception of one meeting, all other meetings lasted anywhere from 105 to 190 minutes ( $\bar{x}=146.0$  min;  $SD=23.5$  min ). One meeting lasted even longer (205 minutes) because it was interrupted in the middle and resumed two days later; hence, some time was spent on recollecting the ideas discussed during the first part of the interview. The post-intervention meetings were, obviously, longer than the pre-intervention meetings, a pattern that was consistent across *all* the study participants. On

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<sup>88</sup> Six interviews were conducted in January, eight in February, one in early March, and one in late April. This last interview was scheduled for early April but, due to unforeseen circumstances, it was postponed for later that same month.

average, the post-intervention meetings were approximately 36 minutes longer than the pre-intervention meetings, a difference that was statistically significant (Wilcoxon  $Z = 3.52, p < .001$ ).

During the post-intervention meetings, the PSTs were asked to consider almost the same questions as those asked during the first interview, with only a few exceptions. In particular, in the first part of the post-intervention interview, the PSTs were prompted to discuss their thinking about teaching and learning mathematics after having taken the two intervention courses and to compare this thinking with how they had thought about teaching and learning mathematics in the summer. If they identified any differences, they were asked to specify what might have contributed to these changes. Additionally, during the “reflection” phase, they were asked to comment on the critique that the virtual teacher taught the lesson rather procedurally. At the culmination of the interview, the PSTs were also prompted to consider whether they sensed any differences between their performances in the pre- and the post-intervention meetings. Each PST received a 30-dollar stipend for their participation in these meetings.

#### *Administration of the Survey*

The survey was sent by e-mail to the study participants two days before the beginning of the math content course; they were asked to complete and submit the survey before the course’s first session. In contrast to the LMT test, the surveys were not anonymous. The study participants received the same survey before the last class of the math methods course; on the second occasion, they were asked to complete and return the first three parts of the survey. Complete survey data sets were obtained from all study participants. For comparative purposes, the survey was also administered at the beginning

of the math content course to the PSTs of the second ELMAC cohort. The first part of the survey was also administered to two cohorts of undergraduates who enrolled in the math methods course in Fall 2007. As detailed in the next chapter, the data obtained from the three administrations of the survey (45 surveys from ELMAC students and 43 part-A surveys from undergraduate students) were used to explore the construct validity of the statements of the first and third part of the survey. They were also used to investigate whether these statements could be represented by a small number of underlying factors.

In the next section, I explain how the collected data were processed to facilitate data analysis.

### Data Processing

This section details the steps taken between data collection and data analysis, including calculating the PSTs' pre- and post-intervention MKT scores, transcribing and editing the interview data, quantifying the interview data, and compiling and preparing the survey data for analysis. The four parts of this section correspond to the aforesaid steps.

#### *Calculating MKT scores*

The participants' answers to the LMT test were entered on a spreadsheet. A score of 1 was assigned to each correct answer and a score of 0 to each incorrect answer; a score of 0 was also assigned to questions left unaddressed or questions for which participants answered that they were not sure of the correct answer. For each of the two administrations of the test, an *overall MKT-score* was then calculated for each participant based on their correct answers to the test. Given that the test included 41 questions, participants' overall MKT-score could range from 0 to 41. For each test administration,

two separate scores were also obtained for each participant: a Common-Content-Knowledge score (hereafter CCK score), which was calculated by summing up the participants' correct answers to the five corresponding test questions (see underlined questions in Table 3.3) and a Specialized-Content-Knowledge score (hereafter SCK score), which represented participants' correct answers to the remaining 36 test questions.

### *Transcribing and Editing the Interview Data*

The interview data comprised 20 pre-intervention interviews and 16 post-intervention interviews. All 36 interviews were audio-taped; as discussed in the previous section, separate *Profcast* files that provided information about the participants' navigation through the virtual lesson were also available. I randomly selected and transcribed three of these interviews to get first insights into the interview data; this process also helped me identify particular notation symbols needed for transcribing the interview data as faithfully as possible (e.g., notation for short or longer pauses; notation to denote the participants' emotional state or tone; and notation for transitions between the different phases of the interview). A document outlining these notation decisions and other guidelines for transcribing the interviews was developed and sent alongside the interview audio-files to three professional transcriptionists who transcribed the remaining 33 interviews.

Once returned, each interview transcript was edited to ensure that it represented each interviewee's exact words and tone (where such information was available) as faithfully as possible. The editing process also aimed at eliminating any inconsistencies in transcribing among the different transcriptionists. Finally, while editing the files, I incorporated the artifacts that the PSTs produced during or before the interviews in the

interview transcripts; I inserted these artifacts at the place they occurred in real time. This included the notes that the interviewees had taken on the two textbook pages before the interview meeting, drawings that they produced while explaining the division of fractions during the “planning” phase of the interview, and all their work (e.g., notes, drawings) while observing the virtual lesson. Using information from the *Profcast* files, I also included notes explaining the participants’ deictic words or other referents. Both the audio and the *Profcast* files provided information on the duration of different interview segments (i.e., the planning phase, each of the five segments of the virtual lesson, the reflection phase), which was also included in the interview transcripts. The interview transcripts were then carefully scrutinized and any potential identification information was eliminated (e.g., actual names were replaced by pseudonyms). Finally, the transcripts were line-numbered. The interview transcripts averaged 51 single-spaced pages (on average 2308 lines).

### *Quantifying the Interview Data*

As outlined above, the interview data were voluminous. Hence, a first step in exploring these data was to “quantify” them. Such quantification would help epitomize the data and allow for exploration of the research questions from a quantitative perspective. Chi’s (1997) practical guide for quantifying verbal data – and particularly the steps of developing coding schemes and defining what counts as evidence for each code – provided useful guidelines toward this end. The process of data quantification unfolded in four phases. First, separate coding schemes for each of the five teaching practices were developed and operationalized; second, these coding schemes were validated by three experts in the areas the study considers and subsequently refined; third,



the refined coding schemes were applied to code the interview data; finally, a sample of the interview data was coded by two independent coders for inter-coder reliability purposes. I discuss each of these phases in turn.

### *Developing the Coding Schemes*

Chi (1997) proposes that the development of the coding schemes be an interactive top-down and bottom-up process. She recommends that, when developing the coding schemes, the researcher first build up some tentative schemes based on his or her theoretical framework (i.e., top-down approach) and then refine the coding schemes following a more bottom-up approach, by “testing” the codes against the verbal data. This ensures that the coding schemes are sensitive to the interview data; it also helps retain the richness of the data, to the extent that this is possible.

The analytic framework for exploring the PSTs’ performance in the five teaching practices (Figure 2.7) provided the skeleton for developing the coding schemes. Grover’s (1989) approach to scoring teachers’ responses to semi-structured interviews helped flesh out this skeleton. Grover (1989) suggests that the researcher first identify the dimensions (i.e., general constructs) to be assessed in the teachers’ responses. The researcher should then identify the subcomponents that indicate competence in each dimension. Finally, each of those components should be assigned a score that represents a particular level of competence. For instance, as Grover proposes, a score of 0 could represent poor performance, a score of 1 moderate performance, and a score of 2 high performance. Grover also suggests creating a scoring matrix that defines how the subcomponents of each construct are scored. These subcomponents should be scored independently and then summed into a total score for the construct under exploration.

In this study, the five practices under exploration (selecting and using tasks; using representations; providing explanations; analyzing students' work and contributions; and responding to students' direct or indirect requests for help) constituted the five general constructs. Each construct was decomposed into three subcomponents-skills: noticing, interpreting-evaluating, and performing. For each of those subcomponents a zero-to-two scale was initially developed and performances for each score were determined. To exemplify this process, I discuss the development of the third coding scheme, which corresponds to the practice of providing explanations (see Table 3.5). A similar approach was pursued in developing the other four coding schemes.

The first step in developing the coding scheme for providing explanations was to identify the entry points related to each subcomponent-skill. These entry points, which appear in the second column of Table 3.5, were directly drawn from the specification Table 3.3 and Figure 3.4. For example, for the performing skill, Table 3.5 defines that the PSTs' performance be explored by examining their explanations for the quotient of the division problem  $2 \div \frac{3}{4}$  and their explanation for the invert-and-multiply algorithm with respect to this same numerical example. Following Grover's suggestion, I then developed three different scales to code performance in each skill. Initially, all the scales ranged from 0 to 2, with 0 representing the weakest performance and 2 representing the strongest performance. The next step concerned the operationalization of each code, namely determining what evidence from the interview transcripts would count toward each score. In determining the operationalization criteria, I followed Chi's suggestion to pursue both a top-down and a bottom-up process.

Table 3.5

*The Coding Scheme for the Practice of Providing Explanations*

<b>Skill</b>	<b>Entry Points</b>	<b>Score</b>	<b>Code Operationalization</b>
<i>Noticing</i>	A. Virtual teacher's problematic explanation for the reciprocal (Responding to Michelle)	<b>2</b>	The PST considers Ms. Rebecca's explanation or lack thereof problematic for <i>both</i> episodes.
	B. Virtual teacher's explanation (or lack thereof) for the remainder (Responding to Robert)	<b>1</b>	The PST considers Ms. Rebecca's explanation or lack thereof problematic for one of the two episodes.
		<b>0</b>	The PST does <i>not</i> notice anything with respect to Ms. Rebecca's explanations. OR The PST does <i>not</i> notice anything <i>problematic</i> in Ms. Rebecca's explanations or lack thereof.
<i>Interpreting-Evaluating</i>	The virtual teacher's explanation for the reciprocal (i.e., Responding to Michelle) is problematic	<b>2</b>	The PST considers Ms. Rebecca's explanation problematic <i>and</i> provides at least two reasons to justify his/her answer. These include, but are not limited to: <ul style="list-style-type: none"> <li>• The teacher <i>describes</i> rather than explains the rule.</li> <li>• Her explanation has limited applicability (does not apply to divisions with remainder)</li> <li>• The teacher's argument that "because we are using the reciprocal we should use a reciprocal operation" is not a mathematically valid argument</li> <li>• The teacher uses an inappropriate analogy to explain the rule (her reference to positive and negative numbers).</li> </ul> <p><i>Notes:</i> (1) The PSTs' comments should address the teacher's explanation for the reciprocal and not how she worked with her students to derive the algorithm of the reciprocal (this aspect of the teacher's work is considered in the first coding scheme); (2) General comments (e.g., "the teacher needs to explain more" should not count toward this category; such comments should only count toward the category of "noticing").</p>
		<b>1</b>	The PST considers Ms. Rebecca's explanation problematic <i>and</i> provides one reason to justify his/her answer (see reasons above).
		<b>0</b>	The PST does <i>not</i> consider Ms. Rebecca's explanation problematic. OR The PST considers Ms. Rebecca's explanation problematic but either cannot justify his/her answer or provides reasons unrelated to those listed above.

Table 3.5

## The Coding Scheme for the Practice of Providing Explanations (continued)

Skill	Entry Points	Score	Code Operationalization
Performing	A. The PST provides an explanation for the quotient of the division $2 \div \frac{3}{4}$	3	All criteria listed for score 2 are met; additionally the explanation is calibrated to an average sixth grade student (i.e., could make sense to such an average student).
	B. The PST explains the rule of “invert and multiply” in the division problem $2 \div \frac{3}{4}$	2	The PST provides an explanation for <i>both</i> cases under consideration. Both explanations are mathematically valid/correct and comply with the following criteria: <ul style="list-style-type: none"> <li>- They are not descriptions of the steps involved in the procedure (e.g., the explanation for the reciprocal does not simply describe the steps involved in algorithm)</li> <li>- They connect the procedure at hand to its underlying meaning</li> <li>- If the explanations make use of any of the following tools, these tools are suitable and are used appropriately: <ul style="list-style-type: none"> <li>o <i>Mathematical examples</i>: simpler examples are used to scaffold meaning (e.g., start from explaining <math>1 \div \frac{3}{4}</math>)</li> <li>o <i>Context</i>: the algorithm is connected to a real-life situation (word problem)</li> <li>o <i>Representations</i>: diagrams, pictures, or manipulatives.</li> </ul> </li> <li>- They are complete, in the sense that they do not omit critical steps/pieces of information: <ul style="list-style-type: none"> <li>o <i>For the quotient</i>: the explanation defines the two different types of units (absolute and relative) and the type of unit used to explain the fractional part of the answer; the left over part is explained in terms of relative units.</li> <li>o <i>For the reciprocal</i>: the explanation explicates both the reciprocal and the multiplication involved in the “invert and multiply” algorithm (e.g., explanation refers to how many relative units are included in each absolute unit). If an algebraic approach is followed, all involved steps should be explained.</li> </ul> </li> </ul>
		1	The PST provides an explanation for <i>one</i> of the two cases under consideration. The explanation is mathematically valid/correct and meets the criteria listed above. OR The PST provides an explanation for <i>both</i> cases under consideration. Both explanations are mathematically valid/correct, but do not satisfy at least <i>any two</i> of the above-listed criteria.
		0	The PST does not/cannot provide any explanation. OR The PST’s explanations are mathematically invalid/incorrect.

Specifically, I first developed a set of operationalization criteria by considering what the PSTs were asked to do during the interview. In general, this procedure was easier for the noticing and interpreting-evaluating skills and harder for the performing skill. For example, for the *noticing* skill as regards to providing explanations, the PSTs were expected to comment on two episodes (without any prompting by the interviewer): how the virtual teacher responded to Michelle's question about why the reciprocal works, and how she handled Robert's question about the quotient in the division  $\frac{3}{4} \div \frac{1}{6}$ . Noticing the insufficient manner in which the virtual teacher handled both questions received a score of 2; discussing only one of these two episodes received a score of 1, whereas if the PSTs did not talk about any of these episodes, they received a score of 0.

For the *interpreting-evaluating* skill, the interviewees were prompted to discuss and evaluate the explanation that the virtual teacher gave to Michelle. Consulting studies on instructional explanations (e.g., Leinhardt, 2001), four problematic features of the teacher's explanation were first identified. It was then determined that a score of 2 would be assigned to those PSTs who identified and discussed three of the four problematic aspects of the teacher's explanation; the PSTs who identified and talked about one or two problematic aspects of this explanation would be assigned a score of 1, and those who considered the teacher's explanation satisfactory would be assigned a score of 0. At this point, I turned to the interview transcripts to explore whether such a score allocation would be sensitive enough to capture differences in the PSTs' answers. I noticed that, overall, most interviewees identified one or two problematic aspects of the virtual teacher's explanation. Hence, it was decided that a better allocation of the scores would be to assign a score of 1 to the PSTs who identified and discussed one problematic aspect

of this explanation, and a score of 2 to the PSTs who discussed at least two of its problematic aspects.

The review of the relevant literature and other pertinent scoring rubrics (e.g., the *Video Codes* developed by the LMT group at the University of Michigan<sup>89</sup>; the *Scoring Rubrics* developed by Stein and colleagues at the University of Pittsburgh in the content of the QUASAR project) also helped identify criteria for scoring the PSTs' work in the *performing* tasks. The key criterion to evaluating the PSTs' performance was that their explanations be mathematically correct/valid. Other criteria included whether any representations, mathematical contexts, or simpler examples, when employed, were suitable for the explanation under consideration and were used appropriately; the explanations were not simply a description of the involved steps but built connections to the meaning of the underlying procedure; the explanations were complete; and the explanations were calibrated to the characteristics of their intended audience (i.e., sixth graders). These criteria were further operationalized to correspond to the specific explanations, which the PSTs were asked to offer during the interview (see the last column of Table 3.5).

Once these criteria were developed, I again "tested" them against the available data. I noticed that whereas the PSTs' explanations met most of the criteria under consideration, several of these explanations (especially those offered during the pre-intervention interviews) did not meet the criterion of calibration, that is, they were not appropriate for an average sixth grader. Such a criterion, however, could not be dropped since it was considered fundamental for evaluating the quality of an instructional explanation. Hence, I explored the possibility of extending the scale of the *performing*

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<sup>89</sup> See [http://sitemaker.umich.edu/lmt/files/LMT-QMI\\_Description\\_of\\_codes.doc](http://sitemaker.umich.edu/lmt/files/LMT-QMI_Description_of_codes.doc)

skill, to include an additional level (score 3) which would help discriminate an acceptable from an outstanding performance. Given that a similar distinction appeared to be appropriate for the performing tasks of the other four practices as well, I decided to use a zero-to-three scale to code the PSTs' performance in the performing skill and a zero-to-two scale to code their noticing and interpreting-evaluating performances.

Two final remarks are necessary here regarding the development of the coding schemes. First, given the research questions of the study, all five coding schemes were developed to capture the PSTs' performance in practices related to structuring and maintaining rich and intellectually challenging learning environments. As explained in Chapter 2, this did not imply that the PSTs needed to engage in the so-called student-centered approaches since direct instruction was deemed equally appropriate for maintaining focus on meaning and understanding. However, this explicit focus on making meaning and engaging students in higher level thinking meant that certain PSTs' performances were privileged over others. Consider, for instance, how the *noticing* performance was scored. During the interview, the PSTs were asked to discuss what captured their attention. Although several of the virtual teacher's actions impaired the quality of the mathematics considered in her lesson (e.g., that the virtual teacher is shown glossing over the error in Amanda and Julia's answer) – and hence were expected to capture the PSTs' attention – a PST could, alternatively, opt to discuss other redeeming features of the teacher's work during this episode: that the teacher engages students; that she asks students to share their work; that the students work collaboratively. No doubt, these features could help create a productive learning environment; however, the manner in which the mathematics is treated in this virtual lesson was considered more important

to notice and discuss than other features. Hence, noticing and discussing issues such as student engagement and collaboration, although important, were not credited with any score since they fell beyond the scope of this study.

Second, in developing the coding schemes pertaining to the PSTs' performance in the two MTF-practices (Coding Scheme 1: Selecting and Using Tasks and Coding Scheme 5: Responding to Students' Requests for Help), an initial attempt was made to clearly distinguish between two fundamental aspects of teaching associated with research on tasks and cognitive demand (cf. Stein et al., 2000): drawing connections to the underlying meaning of the procedures under consideration and supporting students' understanding without doing the thinking for them. This distinction appeared to further complicate the coding schemes; hence, it was only retained for the performing tasks of both Coding Schemes 1 and 5. Specifically, according to these schemes, if the PSTs' performance had the potential to support meaning-making, this performance was assigned a score of 2. If *additionally* the PSTs' performance did not only support making meaning but also allowed for more cognitive work on the part of the student, this performance was credited with a higher score (i.e., score 3). Therefore, Coding Schemes 1 and 5 included the only two cases in which a more exploratory work on the part of the students was preferred to a direct-instruction approach. This decision was purposefully made to capture the distinction made in the MTF between making connections to meaning (i.e., "procedures-with-connections" to tasks<sup>90</sup>) and additionally engaging students in high-level tasks (i.e., "doing mathematics" tasks).

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<sup>90</sup> In my perspective, the requirement that connections be made to meaning does not imply that these connections should necessarily be made by the students. As explained in Chapter 2, I consider a direct-instruction approach that helps students see and build such connections equally appropriate for



### *Validating and Revising the Coding Schemes*

Once the coding schemes were developed, they were content-validated by a Research Area Specialist Lead and two post-doctoral fellows (henceforth referred to as the validation group), whose different types of expertise and experiences were considered cardinal to this process.<sup>91</sup> The validation of the coding schemes had two goals. First, it aimed at determining whether the criteria outlined for each of the five practices and its subcomponents (i.e., the code operationalization of the coding schemes) were valid for exploring the PSTs' performance in these practices. Second, it sought to explore the soundness of the score allocation as outlined in these schemes. Five two-hour meetings were allotted to the validation process, each of which was spent on validating one coding scheme. Part of the first meeting was also spent on familiarizing the validation group with the design of the study and particularly with the decomposition of the PSTs' performance into the five teaching practices and the three skill subcomponents. The validation of each coding scheme unfolded roughly as follows.

First, all the episodes from the virtual lesson that corresponded to the noticing skill (for each coding scheme) were identified. The validation group was asked to evaluate whether these episodes were indeed worthy of the PSTs' attention and, if so, to

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supporting student understanding. To be sure, the degree to which this approach is effective can only be determined by considering actual student outcomes.

<sup>91</sup> At the time the study was conducted, the Research Area Specialist Lead completed 15 years of teaching as a fifth-grade mathematics teacher. During the fall semester of 2007, he also coordinated the instructors teaching of the Math Methods course, which is part of the intervention considered in this study; he was also the director of a project designed to develop, implement, and evaluate an integrated assessment system in elementary teacher education for mathematics and language arts. The first post-graduate fellow was a member of a project designed to scaffold middle-grade mathematics teachers in implementing complex mathematical tasks. Her involvement in this project familiarized her with the MTF and issues on maintaining the complexity of cognitively demanding tasks. The second post-graduate fellow, who was a former elementary and middle-grade teacher, had expertise in developing items to assess teachers' MKT and rubrics to evaluate the mathematics quality of mathematics lessons; she also worked on applying these rubrics to code videotaped lessons.

articulate what was particularly noteworthy in these episodes; this assisted me in evaluating the operationalization of the *Noticing* part of each coding scheme. Next, the validation group considered the criteria developed to evaluate the PSTs' *noticing* performance. Suggestions for revisions or refinements were discussed and adopted after reaching consensus.

Similar to the procedure followed above, the validation of the *interpreting-evaluating* part of each scheme commenced with identifying and discussing all the corresponding episodes from the teaching simulation. The validation group was asked to consider whether these episodes lent themselves to exploring the participants' interpretation and evaluation of the virtual teacher's moves. Each group member was also asked to interpret and evaluate the virtual teacher's moves in the corresponding episodes. All presented ideas were discussed and consensus was reached on an appropriate evaluation of the virtual teacher's corresponding performance. This evaluation was then compared with the evaluation criteria listed in the coding scheme and, if needed, revisions were made. The validation group also explored the allocation of scores to different performances and, occasionally, made suggestions for including additional categories, particularly with respect to the intermediate threshold (i.e., the score of 1).

A slightly different approach was pursued to validate the *performing* part of the coding schemes. First, the validation group was asked to discuss whether the tasks identified by the coding scheme to explore the PSTs' corresponding performance were suitable for the practice under consideration. Next, the operationalization criteria included in the coding scheme were discussed; when suggestions were made for revising or elaborating these criteria, these were adopted after reaching consensus. Because of the

difficulties associated with allocating scores to different performances, starting from the second meeting the validation of the score allocation outlined in the coding schemes was performed by considering excerpts from the interview data. These excerpts were purposefully selected to engage the validation group in discussion around the criteria outlined in the coding schemes. This process pointed to places where the code operationalization needed further unpacking; in a couple of cases it also pointed to criteria that were particularly hard to apply, and thus were dropped (e.g., the criterion of “clarity” for evaluating the PSTs’ explanations). After each meeting, the coding schemes were revised accordingly and the revisions made were briefly discussed at the beginning of the next meeting.<sup>92</sup>

#### *Applying the Coding Schemes to Code the Interview Data*

After the coding schemes were validated, 15 coding templates (i.e., 5 teaching practices x 3 skills) summarizing these codes were developed. These spreadsheets were used to code the interview data. The data coding unfolded in two phases. First, the data were coded per each interviewee. Specifically, each interview was scrutinized for evidence pertaining to each of the five practices and their three subcomponents. At the end of this process each interviewee was assigned 15 scores pertaining to the entry points illustrated in Figure 2.7; these scores ranged from 0 to 2 for the noticing and interpreting/evaluating categories and from 0 to 3 for the performing category. Short memos justifying each score were also prepared; these memos included references to particular number lines from the interview transcripts. Notes were also kept for some cases in which there was some ambiguity whether a performance should be assigned a

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<sup>92</sup> A more comprehensive report of the validation process, which outlines the specific suggestions made by the validation group, appears in Appendix 7.

particular score (i.e., when the performance at hand appeared to be on the cut-off point of two different scores). These ambiguities were resolved during the second phase of coding.

In the second phase, the interview transcripts were explored *across* interviewees; this was done for each of the 15 entry points under consideration. For instance, the performances of all interviewees in providing an explanation for the division  $2 \div \frac{3}{4}$  were compared to each other to decide whether performances assigned the same score were comparable; few adjustments were made, when necessary. This process also helped resolve the ambiguities that emerged from the previous phase since it provided yardsticks to decide whether a given performance should be assigned the one or the other score under consideration. Following Chi's (1997) suggestion for pursuing a top-down and a bottom-up approach in coding the data, at the end of this process, some additional notes were incorporated into the coding schemes to detail some subtleties in the coding that resulted from considering all 36 interview transcripts (e.g., see note 2 in the interpreting/evaluating category of Table 3.5). The revised version of the coding schemes that resulted from this process was used by two independent coders who were asked to code part of the interview transcripts for intercoder reliability purposes, an issue to which I now turn.

### *Exploring Intercoder Reliability*

To determine the reliability of my coding decisions, and consequently of the findings reported in the next chapters, two of the members of the validation group (the Research Area Specialist Lead and the first post-doctoral fellow) were asked to code part of the interview data. The selection of the coders was purposeful in that, having

participated in the validation process, these two coders were already familiar with the coding schemes; hence, they could use them without any additional training. Each coder was asked to code two different interview transcripts; thus the intercoder reliability reported herein is based on four of the 36 interview transcripts (11.11%). At the completion of this process, each coder received a 300-dollar stipend for their work.

The four interview transcripts used for the purpose of intercoder reliability constituted a stratified random sample with the strata defined by the following three criteria:

- Given the overarching question under exploration (i.e., whether MKT is related to PSTs' performance in the five teaching practices), it was decided that the four transcripts span different levels of the participants' MKT. To this end, the entrance MKT score of each interviewee was calculated and turned into a z-score; interviewees were then clustered into four groups (first group:  $\overline{X}_{\text{MKT}} > 1\text{SD}$ ; second group:  $0 \text{ SD} < \overline{X}_{\text{MKT}} < 1\text{SD}$ ; third group:  $-1 \text{ SD} < \overline{X}_{\text{MKT}} < 0 \text{ SD}$ ; fourth group:  $\overline{X}_{\text{MKT}} < -1 \text{ SD}$ ). Based on this classification, one interviewee was selected from each group.
- Two pre-intervention interviews and two post-intervention interviews were selected; this ensured an equal consideration of the pre- and the post-intervention data.
- The transcripts were of average length (about 50 pages each) to ensure that the coders' time investment was kept within reasonable limits.

After the four interview transcripts were selected, they were sent to the two coders along with the revised version of the coding schemes and the coding templates developed

to code the interview data. The two coders worked independently to code the interview transcripts and both were blind to the MKT paper-and-pencil score of the PSTs whose interview transcripts they were asked to code. When received, their decisions were compared to mine and all disagreements were identified. Instead of simply identifying disagreements in the scoring decisions (i.e., the scores assigned to each of the 15 entry points illustrated in Figure 2.7), I also explored disagreements in the *coding* decisions; that is, even in those cases in which there was agreement in the scoring, I additionally explored whether my coding was similar to that of the two coders. This comparison surfaced 18 disagreements out of the 60 decisions under consideration (i.e., 15 entry points x 4 interview transcripts); in two cases there was agreement in the scoring decisions but not in the coding. Hence, in total, 20 disagreements were identified. These disagreements were discussed in individual meetings with the two coders. This discussion helped distinguish the disagreements into two different types.

The first type of disagreements (n=13) stemmed from the fact that either the coders or the author did not notice supporting pieces of evidence from the transcripts or did not take into account certain criteria specified in the coding schemes. Specifically, in 12 of those cases the coders did not consider supporting evidence from the transcripts or certain aspects of the coding criteria; in one case, the author did not notice supporting evidence from the transcripts.<sup>93</sup> These disagreements were easily resolved by pointing to the supporting transcript evidence or to the criteria from the coding schemes that were overlooked. Because these disagreements did not represent discrepancies in interpretation but rather resulted from inattentiveness to parts of the transcripts or the coding schemes, I

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<sup>93</sup> In all these cases, the scores assigned by the author and the coders differed by only one point.

call them “attentiveness-based” disagreements.<sup>94</sup> These disagreements can be perceived as indicators of the *complexity* of the coding system and the coding process, in general; the frequency with which they were observed should also be considered reasonable, given that the coders had to code about a hundred pages of transcript by applying a multitude of criteria outlined in a lengthy coding glossary.

The second type of disagreements (n=7)<sup>95</sup> were due to different *interpretations* of the same pieces of evidence, and reflected what Chi (1997) identifies as one of the major complexities in coding verbal data: the ambiguity of interpretation. Hence, I call these cases “interpretation-based” disagreements. Specifically, disagreements in interpretation were identified in four cases pertaining to *performing* tasks (i.e., selecting and using tasks; using representations; and providing explanations) and three cases corresponding to *interpreting-evaluating* tasks (i.e., providing explanations; analyzing students’ work and contributions; and responding to students’ direct or indirect requests for help).<sup>96</sup> No such disagreements were identified for the *noticing* tasks; this was reasonable, given that noticing was a low-inference category since it only captured what the PSTs attended to

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<sup>94</sup> I borrow this term from Cronbach (1990, p. 586) who argued that one common source of disagreements is that the coders might not be fully attentive to certain sources of evidence.

<sup>95</sup> In four of these cases, the scores assigned by the author and the coders differed by only one point; in the remaining three cases, their score allocation differed by two points.

<sup>96</sup> I illustrate the nature of these disagreements with a sample instance (which is further elaborated in Chapter 5 when discussing Vonda’s case). This disagreement pertained to coding a PST’s performance in explaining the division  $2 \div \frac{3}{4} = 2 \cdot \frac{4}{3} = 2 \frac{2}{3}$ . At a surface level, this PST’s explanations seemed to be appropriate: she correctly identified the dividend and the divisor and explained that one could get two groups of  $\frac{3}{4}$  out of “two” [meaning the two wholes]. Up to this point, the author and the coder agreed that her performance was decent. However, after this point, the PST’s explanation of the fractional part of the quotient of this division and the rule of “invert and multiply” was ambiguous. Although some parts of her explanation seemed to be appropriate, at least at a surface level (i.e., she divided one whole into four parts, shaded three of these parts, and argued that she “had one and one third of it left” without defining, though, what the “it” corresponded to), other parts were mathematically invalid (e.g., in explaining the fractional part of the quotient, she drew two rectangles, divided each of them into thirds and shaded two parts of each; she then considered the shaded parts to represent the  $\frac{2}{3}$  and the non-shaded parts to correspond to the whole-number part of the quotient). Given the ambiguity of her explanation and the severe mathematical errors in some parts of her explanations, it was finally decided that her explanation be given a score of 0, instead of a score of 2 (cf. coding decisions of Coding Scheme 3, Table 3.5).

(or failed to attend) while going over the virtual lesson. The seven cases that represented “interpretation-based” disagreements were discussed extensively and consensus was reached on how they should be coded. In six cases, the final coding decisions aligned with those of the author; in one case they aligned with those of the coders. These discussions also pointed to a couple of criteria of the coding schemes (especially with respect to the *performing* tasks of Coding Scheme 1) that needed further clarification. The final version of the coding schemes that resulted from these minor modifications appears in Appendix 8.

Given that the first category of disagreements was considered to reflect the complexity of the coding process rather than the reliability of the derived scores, to calculate the intercoder reliability, I only considered the second category of disagreements (i.e., the “interpretation-based” disagreements). In particular, I calculated the intercoder reliability for the five teaching practices as a whole, the MKT-related and the MTF-related practices separately, each of the five practices separately, and the three subcomponents-skills of those practices (see Figure 3.10). The calculation of these indices was based on *Cohen’s kappa* ( $\kappa$ ) and was performed using the Crosstab function of SPSS.<sup>97, 98</sup>

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<sup>97</sup> *Cohen’s kappa* criterion allows for exploration of intercoder reliability when there are two coders and the variables under consideration have more than two categories. This criterion represents a function of the ratio of agreements to disagreements in relation to expected frequencies. That means that this criterion is not only sensitive to the number of disagreements, but also to the magnitude of these disagreements (i.e., how much the two coders’ scores differ in each disagreement); hence, it is more demanding than a simple intercoder reliability calculated as the ratio of agreements to the total number of observations (agreements and disagreements).

<sup>98</sup> Once the scores of the two coders are organized in a  $p \times p$  table (where  $p$  represents the number of categories under investigation), the  $\kappa$ -values can also be calculated manually using the formula  $\kappa = (a - e)/(n - e)$ , where  $a$  denotes the number of actual agreements (elements on the diagonal),  $n$  denotes the total number of ratings (observations) and  $e$  denotes the number of expected agreements on the diagonal. This latter value is calculated as  $[(\text{row total} * \text{column total})/n]$ , summed for each cell on the diagonal. Kappa is equal to the ratio of the surplus of agreements over expected agreements, divided by the number of expected disagreements (Garson, 2007).



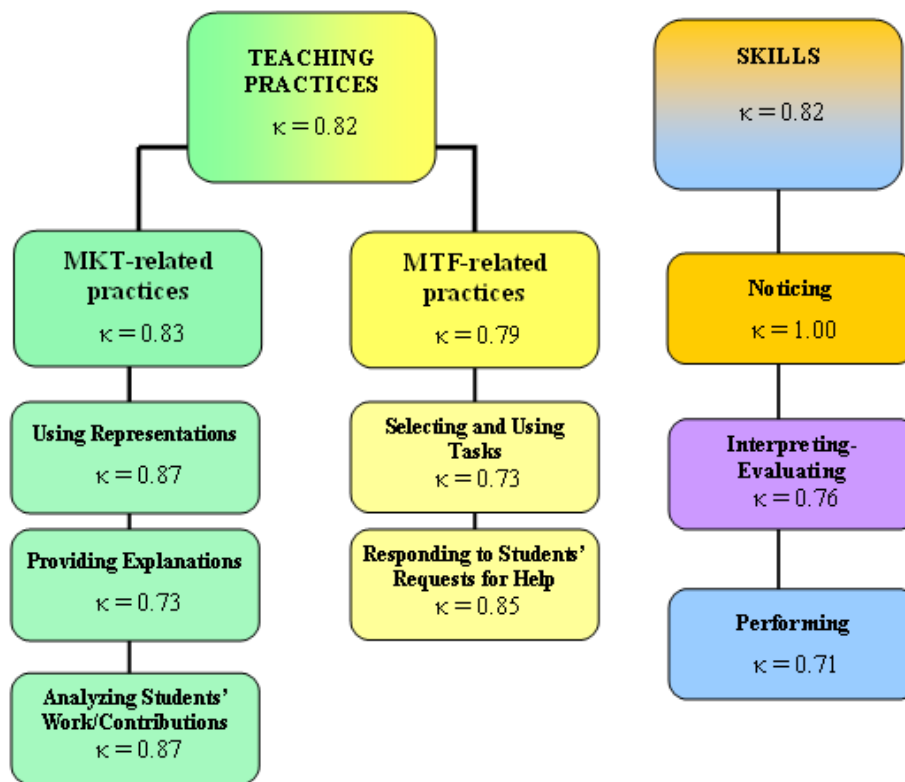


Figure 3.10. Kappa values for intercoder reliability (as a whole and per practice or skill).

Landis and Koch (1977, p. 165) propose that  $\kappa$ -values lower than 0.20 correspond to slight reliability/agreement, between 0.20 and 0.40 correspond to fair reliability,  $\kappa$ -values between 0.40 and 0.59 represent moderate reliability,  $\kappa$ -values in the range of 0.60 to 0.79 correspond to substantial reliability, and  $\kappa$ -values higher than 0.80 represent outstanding reliability; also, by convention,  $\kappa$ -values larger than 0.70 are considered acceptable. As evident in Figure 3.10, *all* the obtained kappa-values were higher than 0.70, meaning that there was acceptable agreement for all the categories under consideration. The reliability for the teaching practices as a whole (or alternatively for the skills as a whole) was outstanding ( $\kappa = 0.82$ ). Outstanding reliabilities were also obtained for the MKT-related practices (as a whole) ( $\kappa = 0.83$ ) and for three of the five practices under consideration (using representations:  $\kappa = 0.87$ ; analyzing students' work and

contributions:  $\kappa = 0.87$ ; and responding to students' requests for help:  $\kappa = 0.87$ ). The reliability of the MTF-related practices (as a whole) was on the verge of being outstanding ( $\kappa = 0.79$ ); the practices of providing explanations and selecting and using tasks had the lowest reliabilities from the five practices at hand; yet, these reliabilities were still substantial ( $\kappa = 0.73$  in both cases). Regarding the skills subcomponents, Figure 3.10 shows that there was perfect reliability for noticing ( $\kappa = 1.00$ ) and substantial reliabilities for the other two skills (interpreting-evaluating:  $\kappa = 0.76$ ; and performing:  $\kappa = 0.71$ ). Because all the obtained kappa-values were acceptable, I decided to not pursue another round of coding. However, I checked whether all my coding decisions resonated with the minor modifications made to the coding schemes upon the two coders' suggestions.

#### *Compiling and Preparing the Survey Data for Analysis*

The survey consisted of four parts, each of which corresponded to the following: beliefs about teaching and learning mathematics; overarching goals for teaching the subject; efficacy beliefs with respect to fractions and the MKT-practices explored in the study; and background characteristics. As already mentioned, in addition to collecting survey data from the study participants ( $n=20$ ), survey data were also obtained from 25 PSTs of the second ELMAC cohort and from 43 undergraduate students who enrolled in the methods course in fall 2007.<sup>99</sup> The latter group of students was asked to complete only the first part of the survey. Hence, for the first part of the survey, data were available

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<sup>99</sup> The undergraduate students comprised a slightly different population from the ELMAC PSTs considered in this study. However, given that the undergraduate students were also intending teachers, this group of students was the closest group to the PSTs considered in the study from which data could be obtained. Obtaining such data was deemed necessary to increase the power of the analyses performed to explore the construct validity of the survey scales and to classify the survey statements into different factors.

from 88 subjects; for the remaining three parts data were available from 45 subjects. Given the small sample size of the study, the data obtained from the non-study participants were mainly used to explore the construct validity and the internal reliability of the scales developed to capture the study participants' beliefs (i.e., beliefs about teaching and learning and efficacy beliefs, see Chapter 4). In what follows, I detail how the data for each part of the survey were compiled and organized for further analysis.

#### *Compiling and Organizing the Data from the First Part of the Survey*

Eighty-eight subjects responded to the 18 statements that comprised the first part of the survey during its first administration;<sup>100</sup> these responses were compiled in a database. To explore the degree to which these 18 statements measured the same construct – namely, beliefs about teaching and learning mathematics – I used *Cronbach's alpha* coefficient, whose value represents an indicator of the correlation between the persons' true variance in the trait the instrument is considered to measure and their observed variance in this measure (cf. Cronbach & Shavelson, 2004); this coefficient can be regarded as an index of the internal consistency of the instrument items.<sup>101</sup> The application of this criterion to the 18 statements yielded an alpha coefficient of 0.64, which did not meet the acceptable lower threshold of 0.70 (cf. Berends, 2006, p. 634). The elimination of five statements<sup>102</sup> raised the alpha-coefficient to 0.78. Exploratory factor analysis (Kim & Mueller, 1978; Kline, 1994) was then used to analyze the 88

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<sup>100</sup> For the ELMAC students the first administration of the survey occurred before the intervention (before the first math content course). The undergraduates were asked to complete the first part of the survey at the beginning of the math methods course (Fall 2007).

<sup>101</sup> For more on this issue, see Cronbach (1990, p. 105) or Cronbach and Shavelson (2004) for a more recent discussion of this statistic and its limitations.

<sup>102</sup> The statements were eliminated one at a time. The eliminated statements were 11, 5, 1, 17, and 12 (see Appendix 5, part A), and were dropped in that order. These statements correspond to numbers 3, 4, 6, 11, and 14 in Figure 3.7.

subjects' responses to the remaining 13 statements; the results of this analysis are reported in Chapter 4. The study participants' responses to these 13 statements during the second survey administration were used to explore their corresponding beliefs at the end of the intervention.

#### *Preparing the Data from the Second Part of the Survey*

In the second part of the survey, the study participants were presented with five goals and were asked to write a percent next to each of them to indicate the emphasis they thought appropriate to be placed on each goal during instruction. In most cases, the participants' allocation of percents added up to the expected total percent (100%). For those cases that the total percent was lower or greater than 100%, participants' percent allocation was readjusted by using the ratio of the sum of their allocated percents to a total of 100%.

#### *Compiling and Organizing the Data from the Third Part of the Survey*

Forty-four ELMAC students (20 study participants and 24 ELMAC students of the second cohort)<sup>103</sup> responded to the 17 statements of the third part of the survey during its first administration. Their responses were compiled into a single database. The application of Cronbach's alpha to these data suggested that the 17 statements were internally consistent (alpha =0.93). These 17 statements could not be subjected to factor analysis, because as Kline (1994) argues, in order to obtain stable results from factor analysis, the ratio of subjects to item/statements should be at least be 5:1. Hence, instead of subjecting these items to exploratory factor analysis, I used Ward's hierarchical cluster analysis to explore the construct validity of this part of the survey. The results from this

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<sup>103</sup> One of the students from the second ELMAC cohort did not complete this part of the survey.

analysis are reported in Chapter 4. The responses of the study participants to the same 17 statements during the second administration of the survey were used to explore their corresponding efficacy beliefs at the end of the intervention.

#### *Preparing the Data from the Fourth Part of the Survey*

The study participants' responses to the first question of this part (i.e., recollections from experiences in learning mathematics in elementary and middle school) were compiled into a file. These data informed the qualitative analysis of the cases considered in the next two chapters. The PSTs' responses to the second question, regarding the number of courses taken in high school, were used to develop the corresponding variable. The participants' responses to the third question of this part were used to develop two variables: whether they had received an undergraduate mathematics major, and whether they had had an undergraduate mathematics minor. The participants' responses to the fourth question of this part were used to develop two variables, one corresponding to the math content courses the study participants had taken at college or university, and the other pertaining to the math methods courses they had taken. The remaining three questions of part D aimed at collecting information for instructional purposes and were not used in this study.

#### Measures

Having described the collection and initial processing of the data, I now turn to the measures obtained from these data; these measures were used to explore the two research questions of the study from a quantitative perspective. This quantitative analysis, which was the first step to dissect the data and gain some initial insights into the research questions under exploration, was succeeded by a more in-depth qualitative analysis. In

presenting these measures, I first consider those associated with the PSTs' MKT and their performance in the five teaching practices. I then outline the measures pertaining to the PSTs' beliefs about teaching and learning mathematics, their perceived importance of certain instructional goals, and their efficacy beliefs. I conclude with considering the measures of the PSTs' background characteristics. Given that this study aimed at exploring the association between knowledge and practices from both a static and from a dynamic perspective (first and second research questions, correspondingly), the measures reported in this section are of two different types:

- *Entrance measures*: these measures, used to address the first research question, correspond to the PSTs' MKT, teaching performance, beliefs and goals at the beginning of the intervention, as well as to their background characteristics.
- *Measures of change*: these measures, considered when addressing the second research question, capture changes in the PSTs' MKT, teaching performance, beliefs, and goals.

### *MKT Measures*

#### *Entrance Measures*

Three different entrance MKT scores were used in this study: an *overall MKT*-score that corresponds to the PSTs' performance in the first administration of the LMT adjusted test, and two subscores, one corresponding to the PSTs' correct answers to the CCK questions of the test, and one capturing the PSTs' correct answers to the SCK questions of the test. All three scores were standardized using the mean and the standard deviation of the PSTs' respective scores. The same transformation was applied to all the measures reported below, unless otherwise specified.

### *Measures of Change*

Three different MKT scores were utilized to capture changes in the PSTs' performance in the adapted LMT test: an *overall MKT*-score, a *CCK* score, and an *SCK* score. All three scores represent differences in the PSTs' performances in the pertinent questions of the LMT test during its two administrations (before and after the intervention).

### *Teaching-Performance Measures*

#### *Entrance Measures*

Twenty seven different scores were calculated based on the PSTs' pre-intervention performance in the teaching simulation; these scores, which correspond to the different entry points illustrated in Figure 2.7, included: an overall teaching-practices score;<sup>104</sup> an MKT-related practices score and an MTF-related practices score; one score for each of the five teaching practices considered in the study; three scores for the three subcomponents of these practices; an overall-skills score (which is identical to the overall-practices score); and 15 scores which correspond to the cross-tabulation of the five practices with their three skill subcomponents. By and large, these measures followed a normal distribution and were not transformed any further.<sup>105</sup>

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<sup>104</sup> To explore whether it was reasonable to combine the PSTs' scores on each subcomponent to form an overall score for each practice, and then to combine the overall scores for each practice to form an overall teaching-practices score, I used the *Kendall Coefficient of Concordance W* test (see Siegel & Castellan, 1988, pp.262-272). This test, which explores associations among more than two variables, showed that these combinations were justifiable, since all the associations under consideration were positive (although not always significant).

<sup>105</sup> In addition to exploring the plot of each distribution, I considered the criteria of skewness and kurtosis. For both criteria, I divided the skewness and kurtosis statistics by their standard errors. Except for the *entrance* measure of "responding to students' requests for help/noticing" and the change measure of "providing explanations/evaluating," all other 52 measures (26 entrance measures and 26 change measures), these ratios were lower than the cut-off point of 1.96. Because the great majority of these distributions did not significantly depart from normality, I decided to not transform the two measures that departed from normality, for two reasons: (a) to achieve consistency across all the teaching

### *Measures of Change*

Twenty seven scores that reflect the changes in the PSTs' pre- and post-intervention performance in the teaching simulation were calculated; these scores correspond to the 27 entry points identified above.

### *Measures of Beliefs about Teaching and Learning Mathematics*

#### *Entrance Measures*

As explained in the previous section, 13 out of the 18 statements of the first part of the survey were utilized to explore the PSTs' beliefs about teaching and learning mathematics. The responses of 88 PSTs to these 13 statements were subjected to exploratory principal components factor analysis. This analysis, the results of which are reported in Chapter 4, yielded three factors. Hence, the PSTs' factor scores on each of these factors were calculated. Because the PSTs' responses to the first factor did not follow the normal distribution, the PSTs' responses to this factor were recoded into four categories: low (L), medium low ( $M^-$ ), medium high ( $M^+$ ), and high (H) with the "low" category representing the lower end of the scale, and the "high" category the upper end of the scale.<sup>106</sup> To achieve consistency in the beliefs measures, the PSTs' responses to the other two belief-factors were transformed accordingly.<sup>107</sup>

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performance measures; and (b) because both of the exceptions were at the highest level of decomposition (i.e., they resulted from cross-tabulating a teaching practice with a skill).

<sup>106</sup> The four categories were formed using the z-threshold of  $\pm 0.67$ , which partitions the distribution into four approximately equal groups. The four categories were (i) low (L): z-values  $\leq -0.67$ ; (ii) medium low ( $M^-$ ):  $-0.67 < z\text{-values} \leq 0$ ; (iii) medium high ( $M^+$ ):  $0 < z\text{-values} \leq 0.67$ ; and (iv) high (H): z-values  $> 0.67$ .

<sup>107</sup> The reader is cautioned that when I talk about the PSTs' beliefs about teaching and learning mathematics (in the presentation of the quantitative results), I refer to the PSTs' beliefs regarding the three factors that emerged from the factor analysis and not to their beliefs about teaching and learning mathematics in general.



### *Measures of Change*

Using the three-factor solution just discussed, the changes in the study participants' beliefs between the pre- and post-intervention administration of the survey were calculated. For the purposes of subsequent analyses, and to be consistent with the transformation applied to the entrance measures, these changes were classified into the following four categories: negative high ( $N^-$ ), negative (N), positive (P), and positive high ( $P^+$ ).<sup>108</sup>

### *Measures of Perceived Relative Importance of Instructional Goals*

#### *Entrance Measures*

In the second part of the survey, the study participants were asked to determine the relative importance of a set of five instructional goals for the teaching of mathematics; these goals reflected the five strands of mathematical proficiency outlined in the *Adding it Up* NRC report (Kilpatrick et al., 2001). The study participants' percentage allocation to these goals was hence considered to reflect their perceived relative importance of each instructional goal. Because the four latter goals departed notably from the normal distribution, to achieve consistency, the PSTs' responses to all five goals were recoded into four categories, similar to those created for the entrance measures of beliefs (i.e., low, medium low, medium high, and high).

### *Measures of Change*

These measures capture changes in the PSTs' perceived relative importance of each of the five aforementioned goals and were recoded into four categories, following the same approach pursued to transform the changes in the PSTs' beliefs.

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<sup>108</sup> The four categories were formed as follows:  $N^-$ : z-values  $\leq -0.67$ ; N:  $-0.67 < \text{z-values} \leq 0$ ; P:  $0 < \text{z-values} \leq 0.67$ ; and  $P^+$ : z-values  $> 0.67$ .

## *Measures of Efficacy Beliefs*

### *Entrance Measures*

Data on efficacy beliefs were obtained from 44 PSTs (from the 20 study participants and 24 ELMAC PSTs of the second cohort). These data were subjected to Ward's hierarchical cluster analysis to explore the extent to which they could be organized in homogenous clusters, which represented overarching factors. This analysis suggested that the participants' responses to the 17 statements of the third part of the survey be grouped into three clusters; hence the average score of each PST on the statements of each cluster was calculated. Because the PSTs' average scores on the first of these clusters departed notably from the normal distribution, to achieve consistency across the three clusters of efficacy beliefs, their scores to *all* three clusters were transformed into four categories (low, medium low, medium high, and high), following an approach similar to that described above.

### *Measures of Change*

These measures reflect pre- and post-intervention changes in the PSTs' perceived competence in the three clusters of efficacy beliefs. The approach pursued to transform the changes in the PSTs' beliefs and perceived goal importance was also followed to transform the cluster factors representing changes in the PSTs' efficacy beliefs.

## *Measures of PSTs' Background Characteristics*

### *Entrance measures*

Three different types of measures of the PSTs' background characteristics were considered: the courses they had taken; whether they had received a math major or minor in mathematics; and two measures of general knowledge/aptitude. Four different

measures were utilized for the courses that the study participants had taken: the number of math content courses the PSTs had taken during their high-school studies; the number of math content courses they had taken during their undergraduate studies; the number of math methods courses they had taken during their undergraduate studies; and whether they had taken a Calculus class.<sup>109</sup> The three first measures were captured using the following scale: 0 = No courses; 1 = One to two courses; 2 = Three to five courses; 3 = Six or more courses. The fourth measured was gauged as follows: 0 = No *Calculus* classes; 1 = A *Precalculus* class; 2 = At least a *Calculus I* class. Two other measures captured the extent to which the study participants had received a minor or major in mathematics, correspondingly. Finally, two measures of the PSTs' general knowledge/aptitude were also obtained for 18 of the study participants.<sup>110</sup> These were their *standardized* GRE-verbal and GRE-quantitative scores. A synopsis of all the measures utilized in the study is reported in Table 3.6.

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<sup>109</sup> As discussed in Chapter 2, the extent to which teachers had taken at least a Calculus class was among the variables explored as contributing factors to teacher effectiveness, and consequently to student learning (e.g., Begle, 1979).

<sup>110</sup> The academic records of two of the study participants did not include such information.

Table 3.6

*Synopsis of the Measures Utilized in the Study*

<b>Measure</b>	<b>Description</b>
<i>MKT entrance and change measures</i>	
Overall MKT	Overall MKT score on the LMT adapted test
CCK	Score on the five CCK questions of the LMT adapted test
SCK	Score on the 36 SCK questions of the LMT adapted test
<i>Teaching performance entrance and change measures</i>	
Practices (overall)	Overall teaching performance score
MKT-related practices	MKT-related practices score
MTF-related practices	MTF-related practices score
Using representations	Overall score on using representations
Noticing	Score on the noticing component of this practice
Interpreting-evaluating	Score on the interpreting-evaluating component of this practice
Performing	Score on the performing component of this practice
Providing explanations	Overall score on providing explanations
Noticing	Score on the noticing component of this practice
Interpreting-evaluating	Score on the interpreting-evaluating component of this practice
Performing	Score on the performing component of this practice
Analyzing students' work/contributions	Overall score on analyzing students' work and contributions
Noticing	Score on the noticing component of this practice
Interpreting-evaluating	Score on the interpreting-evaluating component of this practice
Performing	Score on the performing component of this practice
Selecting and using tasks	Overall score on selecting and using tasks
Noticing	Score on the noticing component of this practice
Interpreting-evaluating	Score on the interpreting-evaluating component of this practice
Performing	Score on the performing component of this practice
Responding to students' requests for help	Overall score on responding to students' requests for help
Noticing	Score on the noticing component of this practice
Interpreting-evaluating	Score on the interpreting-evaluating component of this practice
Performing	Score on the performing component of this practice
Skills (overall)	Overall teaching performance score
Noticing	Overall noticing score (across practices)
Interpreting-evaluating	Overall interpreting-evaluating score (across practices)
Performing	Overall performing score (across practices)

Table 3.6

*Synopsis of the Measures Utilized in the Study (continued)*

<b>Measure</b>	<b>Description</b>
<i>Entrance and change measures of beliefs about teaching and learning mathematics</i>	
Skill mastery and closure	Factor score of the study participants' beliefs on the first factor that emerged from the exploratory factor analysis
Teaching and learning mathematics as remembering and applying formulas	Factor score of the study participants' beliefs on the second factor that emerged from the exploratory factor analysis
Minimizing the complexity in teaching mathematics	Factor score of the study participants' beliefs on the third factor that emerged from the exploratory factor analysis
<i>Entrance and change measures of perceived relative importance of instructional goals</i>	
Conceptual understanding	Perceived importance of goal related to the first strand of mathematical proficiency
Procedural fluency	Perceived importance of goal related to the second strand of mathematical proficiency
Strategic competence	Perceived importance of goal related to the third strand of mathematical proficiency
Adaptive reasoning	Perceived importance of goal related to the fourth strand of mathematical proficiency
Productive disposition	Perceived importance of goal related to the fifth strand of mathematical proficiency
<i>Entrance and change measures of efficacy beliefs</i>	
Understanding and operating on fractions	Average score of the study participants' efficacy beliefs on the statements of the first cluster that emerged from Ward's hierarchical cluster analysis
Understanding the algorithms of fraction multiplication and division	Average score of the study participants' efficacy beliefs on the statements of the second cluster that emerged from Ward's hierarchical cluster analysis
Engaging in MKT-related practices with respect to fractions	Average score of the study participants' efficacy beliefs on the statements of the third cluster that emerged from Ward's hierarchical cluster analysis
<i>Entrance measures of background characteristics</i>	
High-school math content courses	Number of math content courses taken during high school
Undergraduate math content courses	Number of math content courses taken during undergraduate studies
Undergraduate math methods courses	Number of math methods courses taken during undergraduate studies
Calculus classes	Number of Calculus courses taken during high school or beyond
GRE-verbal <i>relative</i> score	Standardized performance on the GRE-verbal test
GRE-quantitative <i>relative</i> score	Standardized performance on the GRE-quantitative test

## Data Analyses

Different mixed-methods designs exist to analyze quantitative and qualitative data; the selection of the one design over the other should be guided by the research questions and the purpose of a study (Smith, 2006). Given its goal to make a first step in understanding the association between PSTs' MKT and their performance in five teaching practices, this study pursued a *two-phase sequential mixed-method* approach to analyze the collected data. In the first phase, each research question was addressed quantitatively using non-parametric statistics to explore the strength, direction, and robustness of the association between (changes in) the PSTs' MKT and (changes in) their performance in the five teaching practices. The quantitative analysis offered some structure to the volume of data collected and delineated a rough picture of the relationship between the participants' MKT and their performance in the five practices; it also helped explore the extent to which certain factors mediated the aforementioned association. In the second phase, qualitative analysis was pursued to build and elaborate explanations regarding how (changes in) MKT may relate to (changes in) the PSTs' performance in the five practices. In this section, I detail the two phases of the analyses pursued with respect to each of the two research questions the study aimed at addressing.

### *First Research Question: Exploring the Relationship between the PSTs' MKT and Their Performance in the Five Practices from a Static Perspective*

The first research question pertained to understanding the degree and the ways in which PSTs' MKT informs their performance in the five teaching practices considered conducive to engendering and sustaining rich and intellectually challenging environments. To explore the direction and strength of this relationship, the data were

first analyzed quantitatively (phase A). To this end, the association between the PSTs' MKT score and their score in the five practices was explored. Following that, in order to investigate whether certain factors mediated this association, other measures related to the PSTs' beliefs, goals, and certain background characteristics were employed. An overview of the exploration of the first research question from a quantitative perspective appears in Table 3.7. Next, a more fine-grained qualitative analysis was pursued to build and elaborate explanations regarding the ways in which MKT appears to manifest itself in the PSTs' performance in the five teaching practices (phase B). The process of developing these explanations was based on exploring *convergent* and *divergent* cases (Ericson, 1986, p. 140). All these cases helped identify patterns in the participants' performance and build, elaborate, and refine explanations for the association under consideration. They also pointed to other factors (in addition to those explored from a quantitative perspective) that appeared to relate to the PSTs' performance in the five practices. Table 3.8 presents an overview of the qualitative approaches pursued to address the first research question. Below, I detail each phase.

Table 3.7

*Overview of the Data Analysis Processes Pertaining to Addressing the First Research Question from a Quantitative Perspective*

<b>Research Question</b>	<b>Data Instrument</b>	<b>Measures</b>	<b>Data Analysis</b>
<p>1.1. To what extent does MKT relate to performance in the following teaching practices:</p> <ul style="list-style-type: none"> <li>• selecting and using tasks</li> <li>• using representations</li> <li>• providing explanations</li> <li>• analyzing students' work and contributions</li> <li>• responding to students' requests for help</li> </ul>	<p>Adapted LMT test</p> <p>Teaching simulation</p>	<p>Entrance MKT scores</p> <p>Entrance teaching performance</p>	<p><i>Quantitative analysis:</i></p> <p>Spearman's <math>r_s</math> criterion used to explore correlations between the PSTs' entrance MKT scores and their entrance teaching performance scores</p>
<p><i>Products of intermediate step: Direction and strength of the association between MKT and performance in the five practices</i></p>			
<p>1.2. To what extent do other factors mediate the aforesaid association?</p> <ul style="list-style-type: none"> <li>• Beliefs about teaching and learning mathematics</li> <li>• Goals about teaching mathematics</li> <li>• Efficacy beliefs</li> <li>• Background characteristics</li> </ul>	<p>Survey (Part A)</p> <p>Survey (Part B)</p> <p>Survey (Part C)</p> <p>Survey (Part D); Academic records</p>	<p>Entrance beliefs about teaching and learning</p> <p>Entrance perceived relative importance of instructional goals</p> <p>Entrance efficacy beliefs</p> <p>Math content and methods courses taken; major/minor in mathematics; measures of general knowledge</p>	<p><i>Quantitative analysis:</i></p> <p>(a) Spearman's <math>r_s</math> used to investigate correlations between these factors and (i) MKT, (ii) performance in the five practices;</p> <p>(b) Kendall's partial rank-order correlation criterion used to identify mediators of the association between MKT and performance in the five practices</p>
<p><i>Products of Phase A: Direction, strength, and robustness of the association between MKT and performance in the five practices</i></p>			



Table 3.8

*Overview of the Data Analysis Processes Pertaining to Addressing the First Research Question from a Qualitative Perspective*

<b>Research Question</b>	<b>Data Instrument</b>	<b>Pertinent Data</b>	<b>Data Analysis</b>
<p>1.3. How does MKT manifest itself in PSTs' performance in the five teaching practices under consideration?</p> <ul style="list-style-type: none"> <li>• selecting and using tasks</li> <li>• using representations</li> <li>• providing explanations</li> <li>• analyzing students' work and contributions</li> <li>• responding to students' requests for help</li> </ul>	<p>Adapted LMT test</p> <p>Teaching simulation</p>	<p>PSTs' work on the test questions (pre-intervention)</p> <p>Interview Transcripts (pre-intervention)</p>	<p><i>Qualitative analysis:</i></p> <p>Selection and scrutiny of four convergent and three divergent cases to:</p> <p>(a) identify patterns in the PSTs' performance relative to their MKT and (b) to elaborate and refine explanations about the association between the PSTs' MKT and their teaching performance</p>
<p>1.2. What <i>other</i> factors appear to mediate the relationship between the PSTs' MKT and their performance in the five teaching practices?</p>	<p>Teaching simulation</p> <p>Survey (Part D)</p>	<p>Interview Transcripts (pre-intervention)</p> <p>PSTs' responses to the open-ended question of Part D of the survey</p>	<p><i>Qualitative analysis:</i></p> <p>Scrutiny of seven selected cases to refine the patterns and explanations developed in the previous step</p>

*Products of Phase B: Refined patterns and explanations about the association between MKT and performance in the five teaching practices*

### *Exploring the First Research Question from a Quantitative Standpoint*

In this phase, the study data were analyzed quantitatively to gain initial insights into the net of possible associations among the PSTs' performance in the five teaching practices and their MKT, beliefs, goals, and other background characteristics. Because of the small sample size considered in the study ( $n=20$ ), all analyses reported below were performed by utilizing non-parametric criteria.<sup>111</sup> To gain initial insights into the association under exploration, the PSTs' entrance MKT score and their entrance teaching performance scores were utilized and associations between those scores were explored. This analysis was performed at five levels. At the first level, a correlation between the participants' overall MKT-score and their overall teaching-practices score was calculated. Following the decomposition of the teaching practices illustrated in Figure 2.7, I then explored the associations between the participants' overall MKT-score and their performance in each of the following: (a) the MKT-related practices and the MTF-related practices (second level); (b) each of the five practices (third level); (c) each of the 15 entry points depicted in Figure 2.7 (fourth level); (d) each of the three skill subcomponents examined in the study (fifth level). This five-level analysis was repeated for each of the two components of MKT (CCK and SCK). All these correlations were estimated using the non-parametric criterion *Spearman rank-order correlation coefficient* (i.e., Spearman's  $r_s$ ), which allows exploring associations between two variables

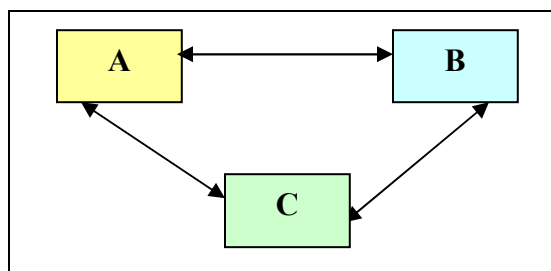
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<sup>111</sup> In addition to the criteria considered for exploring *associations* between the PSTs' knowledge and performance, which was the main focus of this study, two other non-parametric criteria were utilized to explore the significance of *differences* in PSTs' knowledge and teaching performance. These were the *Wilcoxon Signed Ranks Test* and *Kendall's Coefficient of Concordance W*. The first criterion can be used to compare the relative magnitude and the direction between differences in two paired samples (i.e., when comparing the PSTs' performance in two measures), whereas the second one pertains to exploring differences in more than two related samples (i.e., when comparing the PSTs' performance in three measures). The reader may find more information on these criteria in Siegel and Castellan (1988, pp. 87- 95, 262-271).

measured in at least an ordinal scale (Shavelson, 1996; pp. 164-169; Siegel & Castellan, 1988, p. 235).

When an association is observed between two variables, it is possible that another variable mediates this relationship. Hence, the next step in my analysis was to examine whether the association between the PSTs' overall MKT-score and their overall teaching performance was mediated by other variables. Before outlining this analysis, I briefly explicate how mediating variables can be identified, drawing on Baron and Kenney's (1986) pertinent discussion.

A condition necessary to explore mediating effects is to obtain a statistically significant association between the focal variables; in the diagram shown in Figure 3.11, such an association is assumed to exist between variables A and B. Variable C can be considered a mediator of the association between variables A and B if this variable is significantly associated with both variables A and B, as shown in Figure 3.11. Additionally, when the effect of this variable is controlled, the association between variables A and B is either no longer significant or, in the extreme case, it becomes zero.



*Figure 3.11.* Variable C mediates the association between variables A and B.

Hence, to establish that a variable C acts as a mediator of the association between A and B, the following three conditions should be met: (a) there is a significant association between variables A and B; (b) there is a significant association between variables A and

C; and (c) the association between variables A and B gets smaller (in absolute terms) when controlling for the effect of variable C.

In the present study, variables A and B were the PSTs' overall MKT score and their teaching performance in the five teaching practices (i.e., overall teaching-practices score). All other study measures were considered to function as potential mediating variables. These included: the PSTs' beliefs about teaching and learning mathematics, their efficacy beliefs, their perceived relative importance of instructional goals, the courses that the PSTs had taken in high school or during their undergraduate studies, whether they had received a major or minor in mathematics, and the two measures of their general knowledge. To explore which of these measures mediated the MKT-teaching performance association, I first examined the relationships between each of these measures and the PSTs' overall teaching-practices score, on the one hand, and their overall MKT-score on the other hand. Given that all aforementioned variables were measured on at least an ordinal scale,<sup>112</sup> all these analyses were performed using Spearman's  $r_s$ . Then, the factors found to be associated with both the teaching-performance score and the MKT score were statistically controlled by using *Kendall's partial rank-order correlation* criterion. This criterion holds constant the effect of a variable on a given relationship (Siegel & Castellan, 1988, p. 254-262), and thus allows exploring whether the focal relationship becomes smaller when controlling for the effect of a given variable.<sup>113, 114</sup>

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<sup>112</sup> As described in the previous section, after recoding, the variables representing the PSTs' background characteristics were also measured on a rank-ordered scale.

<sup>113</sup> To exemplify how this criterion works consider a simple example in which a four-level variable C mediates the relationship between variables A and B. Instead of exploring the association between A and B holistically, this criterion explores *partial correlations*, namely correlations *within* each group defined by the four levels of the mediating variable.

Overall, the quantitative analyses pursued in this first phase helped explore the direction, strength, and robustness of the association between the PSTs' MKT and their performance in the five teaching practices. The quantitative measures considered in these analyses also provided guidelines for the selection of the cases used to delve deeper into the association between MKT and performance in the five practices. I elaborate this issue in what follows.

### *Exploring the First Research Question from a Qualitative Standpoint*

To further explore the association between MKT and the PSTs' teaching performance, in the second phase of the analyses a number of cases were selected for further scrutiny. Utilizing a *multiple-case* approach instead of a single-case approach<sup>115</sup> was envisioned to help better understand the phenomenon under consideration, namely the relationship between the PSTs' MKT and their performance in the five teaching practices.

Decisions related to case selection are, as several scholars (e.g., Patton, 2002, Stake, 2000, Yin, 2006) pointed out, critical to understanding a phenomenon under investigation. Hence, in selecting the cases explored in this study, I followed two key criteria. First, I sampled both convergent and divergent cases (e.g., Erickson, 1986; Yin, 2003), the former representing cases that support the association between MKT and the performance in the five teaching practices and the latter reflecting cases which challenge

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<sup>114</sup> Because SPSS does not directly calculate this criterion, I calculated the values of this criterion using the following equation proposed by Kendall (cf. Siegel & Castellan, 1988, p. 257):  $T_{AB,C} = \frac{T_{AB} - T_{AC}T_{BC}}{\sqrt{(1-T_{AC}^2)(1-T_{BC}^2)}}$ .

In this equation, T represents the Kendall rank-order correlation coefficient, A and B are the variables of interest, and C is the controlling variable. The values of  $T_{AB}$ ,  $T_{AC}$ ,  $T_{BC}$  were calculated using SPSS.

<sup>115</sup> Different names are used for studies that utilize more than one case. For example, Stake (2000) refers to *collective case studies*, whereas Borman and colleagues (2006) talk about *cross-case analysis* and Yin (2006) refers to *multiple-case studies*.

this association. Second, I sampled diverse cases with respect to both the main constructs considered in the study (MKT and teaching performance) and other factors found to mediate the association between the aforesaid main constructs. Several researchers (Borman et al., 2006; Marshall & Rossman, 1999, pp. 68-69; Strauss & Corbin, 1998, pp. 201-215; Yin, 2006) suggested that considering such diverse cases can help maximize variation, which, in turn, can support better understanding of a phenomenon and the conditions under which it occurs; this variation is also considered critical to making, modifying, and testing explanations for the phenomenon at hand.

To identify convergent and divergent cases, I first plotted the PSTs' entrance overall MKT-score against their entrance overall teaching-performance score (both in standardized values). This allowed clustering the PSTs into four groups, according to their overall MKT-score and their overall teaching-performance scores. Based on this classification, four convergent cases (i.e., PSTs who were high or low in both scores) and three divergent cases were selected (i.e., PSTs whose scores on the MKT test were higher than their scores on the teaching simulation, or vice versa, their teaching-performance score was higher than their MKT score). The seven selected cases also met the second criterion, namely to represent a wide range of the factors under consideration: MKT, performance in the teaching practices, and the factors found to mediate the MKT and teaching performance association.

The seven identified cases were used to identify patterns and build explanations about the association between the PSTs' MKT and performance in the five practices. In considering these cases, I pursued the analytic approach proposed by Yin (2003). Yin recommends that the process of pattern development and generation of explanations start

with a guiding framework (e.g., some initial patterns-propositions);<sup>116</sup> when considering the first case, the researcher should refine the initial patterns to accommodate the case data, if needed. If discrepancies arise between the pattern and the case data, the researcher should develop plausible explanations to account for these discrepancies. This leads to a revised proposition/pattern, which is then “tested” against the data from the second case. This process, which iterates for each case under consideration, is eventually expected to yield more general and refined explanations.

To start this exploration, I used the results of the previous phase (i.e., the quantitative analysis) as the guiding framework of my investigation. In addition to this framework, my exploration was guided by a set of initial foci, which pertained to different aspects of each of the five practices considered in the study (see Table 3.9); these different aspects formed my tentative analytic categories. However, other foci emerged as the data were closely dissected, which led to either elaborating the properties and dimensions of these analytic categories or generating new categories (cf. Strauss & Corbin, 1998, pp. 116-119). Based on the elaboration of these categories, I then looked for *repeated patterns* across the seven cases; for each pattern, a memo was kept and was revised accordingly as the analysis unfolded.

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<sup>116</sup> This contradicts the grounded-theory approach which holds that the researcher enters the investigation without any preconceived list of foci or a guiding theoretical framework in order to allow the categories/patterns/explanations to emerge merely out of the data (cf. Strauss & Corbin, 1998, pp. 27-34). In this study, I utilized the first rather than the second approach for practical and methodological reasons. From a practical perspective, it was envisioned that pursuing a totally grounded-theory approach would engender difficulties in managing the volume of study data. From a methodological standpoint, I wanted to elaborate the analytic framework outlined in Figure 2.7 based on which the whole study was designed and developed.

Table 3.9

*Initial Foci for Exploring the Association between MKT and Teaching Performance*

<b>Teaching Practice</b>	<b>Initial Foci for Exploring Similarities/Differences among the Seven Cases</b>
Selecting and using tasks	<ul style="list-style-type: none"> <li>- what PSTs attended to when comparing the two textbook pages</li> <li>- PSTs' task selection and rationale for this selection</li> <li>- PSTs' proposed task presentation enactment and rationale for these decisions</li> </ul>
Using representations	<ul style="list-style-type: none"> <li>- the type of representations PSTs selected</li> <li>- how PSTs used these representations to support their explanations</li> <li>- PSTs' building of connections (or lack thereof) between these representations and the algorithm of the division of fractions</li> </ul>
Providing explanations	<ul style="list-style-type: none"> <li>- the <i>quality</i> of the explanations offered regarding the following:               <ul style="list-style-type: none"> <li>• mathematical validity</li> <li>• use of appropriate examples, representations, and analogies</li> <li>• calibration to the intended student population</li> </ul> </li> </ul>
Analyzing student work and contributions	<ul style="list-style-type: none"> <li>- PSTs' attention to problematic aspects of students' work and contributions</li> <li>- the appropriateness and depth of PSTs' analysis of students' work</li> <li>- the appropriateness of PSTs' assertions about students' understanding (or lack thereof) and their justifications for these assertions</li> </ul>
Responding to students' requests for help	<ul style="list-style-type: none"> <li>- the manner in which PSTs would respond to requests for help and their rationale for their decisions/actions (i.e. doing the thinking for students vs. scaffolding students without doing the thinking for them)</li> <li>- the type of support PSTs would provide to students</li> </ul>

As outlined in Table 3.8, I first considered the four convergent cases and then explored the three divergent cases. Exploring the cases in this order was deliberate, because it was envisioned that discrepancies would emerge more frequently when considering the latter rather than the former cases. The consideration of the discrepant cases was also envisioned to help develop competing explanations about the PSTs' performance in the five teaching practices and help identify other potential mediating factors of the association between MKT and teaching performance. For each case under exploration, I scrutinized all relevant sources of information. This included the PSTs' interview transcripts, their answers to the first question of the fourth part of the survey



(i.e., their experiences as learners of mathematics in elementary and middle grades), and their answers to the adapted LMT test. Instead of following a line-by-line analysis, I coded the interview transcripts per teaching practice. Although less fine-grained, this approach is considered suitable for coding interview data when seeking to identify categories and generate patterns (cf. Strauss & Corbin, 1998, pp. 119-120). Additionally, I scrutinized these documents for evidence of factors other than MKT impinging on the PSTs' performance. An approach similar to the latter was pursued when considering the PSTs' answers to the open-ended survey question. Finally, I compared the questions that the PSTs got right and those that they got wrong on the LMT test, trying to identify any potential similarities and differences across the seven cases.

*Second Research Question: Exploring the Relationship between the PSTs' MKT and Their Performance in the Five Practices from a Dynamic Perspective*

The second research explored the association between MKT and performance in the five teaching practices from a dynamic perspective, namely by investigating whether *changes in MKT* related to *changes in performance in the five teaching practices*.

Because the intervention was also designed to hone the participants' MKT-related practices (i.e., using representations, providing explanations, and analyzing student work and contributions), I considered two groups of practices, one encompassing the aforementioned three MKT-related practices and the other related to the MTF-related practices (i.e., selecting and using tasks; responding to students' requests for help) (see Table 3.10). Similar to the approach pursued to explore the association between MKT and the five teaching practices from a static viewpoint, I approached the second research question first from a quantitative perspective and then from a qualitative standpoint.

### *Exploring the Second Research Question from a Quantitative Standpoint*

From a quantitative perspective, I explored whether there was a significant association between the *changes* in the participants' MKT and the *changes* in their MTF/MKT-related practices, using the measures of change discussed in the previous section of this chapter. Similar to the approach pursued to explore associations between the PSTs' entrance MKT and teaching performance, this investigation unfolded in a series of steps. First, I explored the association between changes in the PSTs' overall MKT and their overall teaching performance. Following the decomposition of the teaching practices in MKT-related and MTF-related practices, I then explored associations between changes in the participants' overall MKT and changes in their performance in each group of practices. This was followed by an exploration of the association between changes in the PSTs' MKT and changes in their performance in (a) each of the five teaching practices; (b) the 15 entry points depicted in Figure 2.7; and (c) the three subcomponents of noticing, interpreting-evaluating, and performing for the MKT- and the MTF-related practices *separately*. This multi-step analysis was repeated for each of the two components of MKT the study explores (CCK and SCK).

Given the focus of the intervention, before investigating factors that could potentially act as mediators of the association between changes in the PSTs' MKT and changes in their teaching performances, I first explored if there was any association between the changes in the PSTs' MKT-related practices and the changes in their MTF-related practices. The direction and strength of this association helped understand whether the changes in one set of practices were dependent on the changes in the other set of practices. All aforesaid associations were estimated using the Spearman's  $r_s$ .

Table 3.10

*Overview of the Data Analysis Processes Pertaining to Addressing the Second Research Question from a Quantitative Perspective*

<b>Research Question</b>	<b>Data Instrument</b>	<b>Measures</b>	<b>Data Analysis</b>
<p>2.1.1. To what extent are changes in the PSTs' MKT associated with changes in their teaching performance:</p> <ul style="list-style-type: none"> <li>• as a whole</li> <li>• in the MKT-related practices</li> <li>• in the MTF-related practices</li> </ul> <p>2.1.2. To what extent are changes in the PSTs' performance in the MKT-related practices associated with changes in their performance in the MTF-related practices?</p>	<p>Adapted LMT test</p> <p>Teaching simulation</p>	<p>Changes in the PSTs' MKT</p> <p>Changes in the PSTs' teaching performance</p>	<p><i>Quantitative analysis:</i></p> <p>Spearman's <math>r_s</math> criterion used to explore correlations between:</p> <p>(a) changes in the PSTs' MKT and changes in their performance as a whole and in the MKT/MTF-related practices</p> <p>(b) changes in MKT-related practices and changes in MTF-related practices</p>
<p><i>Products of intermediate step: Direction and strength of the associations between (a) changes in MKT and changes in performance as a whole and in the MKT/MTF-related practices and (b) changes in MKT- and changes in MTF-related practices</i></p>			
<p>2.2. To what extent are the aforementioned associations mediated by the following:</p> <ul style="list-style-type: none"> <li>• Beliefs about teaching and learning mathematics</li> <li>• Goals about teaching mathematics</li> <li>• Efficacy beliefs</li> <li>• Background characteristics</li> </ul>	<p>Survey (Part A)</p> <p>Survey (Part B)</p> <p>Survey (Part C)</p> <p>Survey (Part D)- Academic records</p>	<p>Measures of change</p> <p>Measures of change</p> <p>Measures of change</p> <p>Courses taken; major/minor in math; general knowledge.</p>	<p><i>Quantitative analysis:</i></p> <p>(a) Spearman's <math>r_s</math> criterion used to investigate correlations between these factors and (i) changes in MKT, (ii) changes in teaching performance</p> <p>(b) Kendall's partial rank-order correlation criterion used to identify mediators of the associations explored in the previous step</p>
<p><i>Products of Phase A: Direction, strength, and robustness of the associations between (a) changes in MKT and changes in performance as a whole and in the MKT/MTF-related practices and (b) changes in MKT-related practices and changes in MTF-related practices</i></p>			

Next, I explored whether any of the quantitative measures of change (beliefs about teaching and learning, perceived relative goal importance, efficacy beliefs) as well as the PSTs' entrance characteristics (courses taken, major/minor in mathematics, and the two measures of general knowledge explored in the previous research question) mediated those of the following associations found to be statistically significant: (a) the association between changes in MKT (overall) and changes in teaching performance (overall); (b) the association between changes in MKT (overall) and changes in MKT-related practices; (c) the association between changes in MKT (overall) and changes in MTF-related practices; and (b) the association between changes in MKT-related practices and changes in MTF-related practices. This exploration was performed in a similar manner to that pursued in considering mediating effects from a static viewpoint (i.e., in addressing Research Question 1.2). In conjunction, as Table 3.10 shows, these analyses helped explore the direction, strength, and robustness of the aforesaid associations.

#### *Exploring the Second Research Question from a Qualitative Standpoint*

To further explore the relationship between changes in the PSTs' MKT and changes in their performance in the five teaching practices, in the second phase, I worked with the same seven cases that were selected to address the first research question. As discussed in Chapter 5, these seven cases met both criteria previously discussed: (a) they represented convergent and divergent cases of the main associations under exploration; and (b) they were representative of different levels of the factors found to contribute to these associations. These cases also represented different patterns of the changes in their MKT-related and the MTF-related practices. An overview of the processes undertaken to analyze the case data is presented in Table 3.11.

Table 3.11

*Overview of the Data Analysis Processes Pertaining to Addressing the Second Research Question from a Qualitative Perspective*

Research Question	Data Instrument	Pertinent Data	Data Analysis
<p>2.3. How do changes in the PSTs' MKT appear to play out in changes in their performance in the:</p> <ul style="list-style-type: none"> <li>• MKT-related practices:               <ul style="list-style-type: none"> <li>- using representations</li> <li>- providing explanations</li> <li>- analyzing students' work and contributions</li> </ul> </li> <li>• MTF-related practices:               <ul style="list-style-type: none"> <li>- selecting and using tasks</li> <li>- responding to students' requests for help</li> </ul> </li> </ul>	<p>Adapted LMT test</p> <p>Teaching simulation</p>	<p>PSTs' work on the test questions (pre- and post-intervention)</p> <p>Interview Transcripts (pre- and post-intervention)</p>	<p><i>Qualitative analysis:</i></p> <p>(a) Within-case exploration to identify changes <i>per case</i></p> <p>(b) Between-case exploration of these changes (convergent and divergent cases)</p>
<p>2.2. What <i>other</i> factors appear to mediate the associations between changes in (a) MKT and teaching performance (overall, MKT-related, MTF-related) and (b) MKT-related performance and MTF-related performance</p>	<p>Teaching simulation</p>	<p>Interview Transcripts (pre- and post-intervention)</p> <p>PSTs' responses to the open-ended question of Part D of the survey</p>	<p><i>Qualitative analysis:</i></p> <p>(a) Within-case exploration to identify changes <i>per case</i></p> <p>(b) Between-case exploration of these changes (convergent and divergent cases)</p>

*Products of Phase B: Refined patterns and explanations about the associations between (a) changes in MKT and changes in teaching performance as a whole and in the MKT/MTF-related practices and (b) changes in performance in the MKT-related practices and changes in performance in the MTF-related practices*

As Table 3.11 shows, the processes undertaken to analyze the case data was almost identical to those undertaken in addressing the first research question with the only difference (and the most critical one) being that a *within-case* analysis was first employed for each case. This within-case analysis helped identify changes in the participants' MKT, teaching performance, and other related factors that could mediate the association between the two aforementioned traits. In particular, for each case, I used both pre- and post-intervention sources of data (i.e., the PSTs' work on the LMT adapted test and interview transcripts). I also compared these PSTs' experiences as learners of mathematics in elementary- and middle-school grades (as they described them in the fourth part of the survey) with their experiences as learners of mathematics in the two intervention courses. By considering both convergent and divergent cases (as outlined above), I also developed patterns and explanations for the associations under consideration, which I then refined by considering competing explanations.

#### Validity, Reliability, and Generalizability Considerations

In this last section of this chapter, I consider issues of validity, reliability, and generalizability. Although I have already discussed some issues pertaining to validity and reliability, in this section I revisit and elaborate on them. In particular, I outline the different types of validity and reliability considered in the study and describe actions taken to establish the validity and the reliability of the findings of the study. I also consider how generalizable/transferable the findings reported in the next chapters are, given the several design decisions made in this study.

## *Validity*

As already explained, a mixed-methods approach was employed to collect and analyze the data. This approach requires that issues of validity, both from a quantitative and a qualitative perspective, be considered. From a quantitative perspective, I considered two types of validity: content and construct validity. From a qualitative perspective, I considered a series of criteria identified by several scholars working in this area.

*Content validity* refers to the extent to which the tools developed to measure the constructs under consideration do so satisfactorily, according to *expert opinion*. This type of validity is typically established by having a group of experts review the data collection and analysis tools to determine their relevance and representativeness of the trait(s) under consideration (cf. Crocker, 2006, p. 379; Cronbach, 1990, pp. 157-158). In this study, this approach was followed in three instances. First, the teaching simulation developed to capture the PSTs' performance in the five teaching practices was vetted by scholars, experts in the field of MKT, who ascertained that this instrument could be used to capture and explore the PSTs' teaching performance. Second, the first part of the survey was content-validated by scholars, experts in the field of MTF. These experts helped me select a set of statements considered suitable for identifying the PSTs who differed in their beliefs apropos establishing rich and challenging environments. Third, the coding schemes developed to "quantify" the PSTs' performance in the five teaching practices were content validated by a group of scholars with different types of expertise (teaching; assessment; issues of cognitive demand, task selection, and implementation; and issues related to studying and measuring MKT).

*Construct validity*, on the other hand, refers to the extent to which the data collection tools actually capture the underlying traits they were supposed to be measuring (cf. Crocker, 2006, p. 379; Cronbach, 1990, pp. 159-160). Contrary to content validity, construct validity is determined by empirical evidence. In this study, the construct validity of the data collection tools was considered in two cases. First, an exploratory factor analysis was utilized to explore whether a subset of the 18 statements of the first part of the survey could be grouped into meaningful groups that echoed the design considerations made in developing this instrument. This exploratory factor analysis resulted into factors that represented beliefs that resonated more with the type-B teacher (namely, the teacher who places more emphasis on procedures and reduces the complexity of teaching; see Table 3.4). Hence, a lower score on these three factors corresponded to a type-A teacher, namely a teacher who pays more attention to meaning and who builds rich and challenging environments for her students. Therefore, these three factors met the expectations for developing continua along which the PSTs could be positioned based on their pertinent beliefs about teaching and learning mathematics. Second, Ward's hierarchical cluster analysis was performed to explore whether the PSTs' answers to the statements of the third part of the survey reflected the underlying traits that this part of the survey was designed to gauge. The cluster analysis suggested that the statements under consideration be clustered into a three-cluster solution, which was consonant with the theoretical classification of these statements. It should be kept in mind, however, that this scale was validated using a small sample of PSTs; hence, further validation is warranted.



Several criteria have been proposed to evaluate the validity of studies that employ *qualitative approaches* in regard to data collection and analysis. Scholars seem to have reached consensus that the following two are central among these criteria: first, the extent to which the analyst provides *thick and rich descriptions* of the study design and the data collection and analysis methods; and second, the degree to which the analyst adequately illustrates the connection between the data and the conclusions of the study (Borman et al., 2006; Eisenhart, 2006; Erickson, 1986; Freeman, deMarrais, Preissle, Roulston, & St. Pierre, 2007; Lincoln & Cuba, 1985). Providing rich and thick descriptions of the data collection processes enables evaluating the quality of the data upon which the study's conclusions are drawn; a detailed and comprehensive discussion of how the study data are used to identify patterns, provide explanations, and make inferences helps evaluate the trustworthiness of the study's conclusions. Following these criteria, in this chapter, I have attempted to provide a systematic and careful documentation of the data collection instruments, the data collection and processing procedures, the measures used in the study, and the data analyses processes. In addition to providing a careful documentation of the instruments used and the procedures employed, I have explicated several decisions made either during the design of the study (i.e., the six design decisions reported in the second chapter), and during the collection and analysis of the data (e.g., decisions related to inter-coder reliability and decisions pertaining to the selection of cases for further scrutiny).

Additional criteria to evaluate (and establish) the validity of a study include triangulation of data, data sources, and findings (Smith, 2006); testing rival hypotheses (Yin, 2003, 2006); providing confirming and disconfirming evidence (Erickson, 1986);

member checking (Brenner, 2006; Freeman et al., 2007; Patton, 2002); and a clear articulation of the researcher's biases and the limitations of the study (Brenner et al., 2006; Denzin & Lincoln, 2000; Freeman et al., 2007). In this study, attempts to establish triangulation included: (a) employing quantitative and qualitative approaches to data analysis; (b) using both quantitative and qualitative data to identify mediating factors of the associations under consideration; and (c) examining diverse cases to generate patterns and explanations. As detailed in the section on data analysis, in developing these explanations, competing explanations were also taken into consideration, by examining both convergent and divergent cases, and by searching for other factors that could mediate the associations of interest. Finally, in interpreting the findings reported in the subsequent two chapters, I was aware of my personal biases and the study limitations, which I discuss below.

Regarding my personal biases, as earlier discussed, both the teaching simulation developed to explore the PSTs' performance in the five teaching practices and the coding schemes developed to "quantify" this performance were driven by a conceptualization of teaching that focuses on meaning-making and on supporting students' understanding without doing the thinking for them. This means that other aspects of the PSTs' performance that are also critical to teaching and learning mathematics (e.g., the extent to which a teacher engages the majority of students in her teaching and other considerations of equity) were not captured or credited.<sup>117</sup> It is important, however, to underscore that although in this study I focused on what I call instruction that supports the structuring and engendering of rich and challenging learning environments, by and large, I did not

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<sup>117</sup> In fact, while considering the teaching simulation, several of the PSTs also attended to such issues; however, exploring issues that the PSTs noticed or discussed which were not captured in the coding schemes was beyond the scope of this study.

endorse any particular type of instruction that promotes these types of environments. As I explained in both Chapters 2 and 3, a teacher might support student thinking not only by pursuing an inquiry-based approach but also by a more direct-instruction approach that successfully supports meaning-making and understanding. In fact, this latter belief represents an additional personal bias.

Several limitations have been identified in Chapter 1. Additional design and methodological limitations include the following:

- Some PSTs might have correctly answered some of the questions on the LMT test merely by guessing, which I address when considering the case of Suzanne.
- The study participants might have prepared differently and in various degrees for the teaching simulation both before and after the intervention.
- Although the study participants were asked to not discuss the content of the teaching simulation with their classmates until all interviews were over,<sup>118</sup> I cannot exclude the possibility that some of them might have discussed their interview experiences with their counterparts. However, to the extent that this occurred, it seems reasonable to assume that it affected the findings of the study in the opposite direction (i.e., reduced the strength of the association between MKT and teaching performance). This is because there was a dissimilar distribution of high-MKT and low-MKT PSTs being interviewed at the beginning and the end of the time period during which the pre-intervention interviews were conducted, with the former group concentrating more at the beginning of this time period and the latter group concentrating more at the end of

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<sup>118</sup> In addition to their research purpose, the interviews also served instructional purposes since the information yielded from them was also used to calibrate the courses – to the extent that this was possible – to the particular needs of the study participants. Hence, it was explained to the study participants that it was to their benefit to help the course instructor form a more accurate picture of their entrance thinking and difficulties.

- it.<sup>119</sup> If one assumes that a diffusion of information about the simulation did occur, it seems more reasonable to postulate that this might have been more intense toward the end of the interview period, as these PSTs were getting to know each other better.
- Because the interviews were semi-structured, I cannot dismiss the likelihood that slight modifications in the wording and sequencing of some questions might have affected the PSTs' responses. However, I painstakingly attempted to closely follow the interview protocol, to the extent that this was possible.
  - During the teaching simulation some PSTs might have been more verbose than others; to minimize the impact of the PSTs' verbosity, I followed Chi's (1997) suggestion and coded not how much the PSTs said but whether what they said captured the criteria identified in the coding schemes. However, it is likely that some PSTs addressed more criteria of those identified in the coding schemes than their counterparts, simply because they were more talkative and they analyzed the teaching simulation more meticulously. I take up this issue when considering Tiffany's case.
  - The performance of the study participants in the pre- and post-intervention administrations of the LMT test and the teaching simulation might have been affected by the PSTs' different levels of test-anxiety. To reduce the effect of this factor, I tried to create a friendly and relaxed environment even during the first administration of both instruments by constantly emphasizing that the PSTs' thinking was at least as important as their getting the "correct" answers. However, I cannot exclude the possibility that some study participants might have still been more stressed than their

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<sup>119</sup> The pre-intervention interviews spanned a period of 12 business and non-business days. Three high-MKT PSTs, one medium-MKT PST, and one low-MKT PST were interviewed during the first three days of this period. Nine PSTs, three from each group, were interviewed during the next three days. In the last six days interviews were conducted with three low-MKT PSTs, and one PST from each of the other two groups.

counterparts when taking the LMT test and while participating in the teaching simulation. Nor can I exclude the fact that some of these PSTs might have felt more comfortable participating in the post-intervention meeting than they felt participating in the pre-intervention meeting, simply because they were more familiar with the content of the simulation and hence, they knew what to expect; additionally they were more familiar with the interviewer who, by that point, had been their instructor for approximately five months.

- Although there was a considerable time interval between the two administrations of the study instruments (5.5 months for the LMT test and the survey and at least 6.5 months for the survey), I cannot exclude the possibility of test-retest effects (cf. Shadish, Cook, & Campbell, 2002, p. 60). However, some anecdotal evidence suggests that to the degree that this happened, it was not intense.<sup>120</sup>
- The intervention might have rendered the study participants better viewers and discussants of records of practice, thus affecting their post-intervention performance in the teaching simulation, which is itself a record of practice. However, as the reader will notice in the next chapters, I deliberately avoided making definite causal claims, since several aspects of the intervention might have affected the PSTs' performance in both the LMT test and the teaching simulation.

All these limitations, along with those listed in the first chapter, were considered when discussing the study findings.

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<sup>120</sup> At the end of the post-intervention interview, I asked the study participants to compare their thinking during the pre- and the post-intervention meetings. Some of the PSTs wondered whether the simulation used in the pre-intervention meeting was identical to that of the post-intervention meeting. Also, only a few participants (e.g., Kimberley) commented that they “kind of remembered” certain teaching episodes from the teaching simulation; however, even in these cases, they mostly remembered one or two episodes they found particularly challenging to discuss during the pre-intervention meeting.

## *Reliability*

Reliability pertains to the degree to which a construct is consistently measured (cf. Cronbach, 1990, p. 705). From a quantitative perspective, two different types of reliability were considered in this study: intercoder reliability and internal reliability. From a qualitative perspective, I considered the notion of dependability (Lincoln & Guba, 1985).

*Intercoder reliability* captures the homogeneity with which (a subset of) the data are coded by different coders; this reliability index reflects a measure of consistency in data coding by independent coders (Cronbach, 1990, pp. 583-595). In the present study, satisfactory intercoder reliability values were obtained by having about 10% of the interview data coded by two independent coders. As already explained, the several “attentiveness-based” disagreements that emerged were considered to reflect the complexity of the coding schemes (and hence a limitation of these schemes) rather than being regarded as evidence of low intercoder agreement.

*Internal consistency* represents a measure of precision of an instrument used to measure a particular trait. Specifically, it represents how well each of the items of this instrument relates to the rest of the instrument items (Cronbach, 1990, pp. 202-206). In this study internal consistency was considered in two cases: when using factor analysis to explore the PSTs’ responses to the 18 statements of the first part of the survey, and when employing cluster analysis to investigate the PSTs’ responses to the 17 statements of the third part of the survey. As reported in the next chapter, with only one exception, the measures that emerged from both analyses were internally consistent, as suggested by the obtained Cronbach’s alpha values.

From a qualitative perspective, Lincoln and Guba (1985, p. 299) identified *dependability* as the analog of (the quantitative notion of) reliability. This notion captures not only the idea of reliability from a conventional sense, but also other shifts or changes the researcher deliberately introduces as the design of the study inquiry unfolds. To explore whether a study meets the dependability criterion, these researchers suggest using what they called an *audit trail*, which allows exploring both the *process* and the *products* of inquiry for absence of bias (ibid, pp. 317-321). Central to this audit trail is what they dubbed a *reflexive journal* (p.327), in which the researcher keeps different types of logs (i.e., a log of evolving ideas, a log of day-to-day procedures and personal introspections, a log of methodological decision points, and a log for developing insights and hypotheses). Using this reflexive journal, the auditor – a professional peer – can then determine the dependability of the study process and products. Although in this study I did not follow the audit-trial approach – for practical and financial reasons – I drew on Lincoln and Guba’s suggestions and kept notes on the procedures followed to collect, process, and analyze the data; I also kept a memo of evolving ideas and thoughts. These notes helped me provide what these scholars call *thick descriptions* of both the processes pursued (i.e., the design of the study, the data collection and processing, the development of the measures, the data analyses procedures, and the criteria that informed the case selection) and the final products (i.e., the ideas and themes around which the cases were built and the working hypotheses and explanations that were developed). By providing these thick descriptions of both the study processes (this chapter) and products (Chapters 4 and 5), I envisioned that the readers of the study might assume – to some extent – the role of the auditor and determine the dependability of the findings reported herein.

## *Generalizability*

Generalizability refers to the degree to which the findings and the conclusions of a study can be extrapolated to a larger population, which the study sample is considered to represent. In quantitative research, in addition to considerations of the participants of the study and how representative these are of a larger population, generalizability considerations also include three other dimensions: *settings*, *treatment variables*, and *outcomes* (Schofield, 2002; Shadish et al., 2002, pp. 83-90). In qualitative research, generalizability has initially received less attention than reliability and validity (cf. Firestone, 1993; Patton, 2002, pp. 581-584; Schofield, 2002). In fact, some qualitative researchers went as far as to argue that generalizability should not be a goal for the qualitative researcher, because qualitative studies often employ small samples, thus making any extrapolations difficult, if not impossible. Other scholars, however, proposed analogs of generalizability for qualitative studies. For instance, Lincoln and Guba (1985) advanced the notions of *transferability* and *fittingness* to talk about the extent to which the findings of a qualitative study can be extrapolated in other settings. These researchers posited that “the degree of transferability is a direct function of the similarity between the two contexts” (p. 124). They went on to define fittingness as the degree of congruence between two contexts. Such congruency, they argued, is necessary for exploring the applicability of the findings of a qualitative study. Along with other researchers working in qualitative research, these scholars also contended that such considerations can be facilitated by providing thick data descriptions and making explicit how the study data were used to inform its findings and conclusions.



In this study, I approached the issue of generalizability drawing on both the qualitative and the quantitative perspective of generalizability. From a qualitative perspective, I concurred with Guba that the findings and the conclusions of a qualitative study should not be considered definitive but rather *working hypotheses for future testing* (cited in Patton, 2002, p. 583). From this perspective, the findings of this study –and especially those pertaining to the manifestations of the PSTs’ MKT in their teaching performance – could be regarded as working hypotheses warranting further exploration. The locus of applicability of these working hypotheses<sup>121</sup> could also be considered to be constrained by the six design decisions presented in the second chapter. To discuss how these decisions limit the scope and applicability of the study findings, and consequently the working hypotheses generated from them, I draw on the dimensions of populations, settings, treatment variables, and outcomes. These four dimensions used in quantitative studies provide a useful framework for considering how transferable and applicable the study findings and its conclusions could be – even as working hypotheses – in other situations and settings. I consider each dimension in turn.

Regarding the dimension of *populations*, the reader is reminded that the data utilized in this study were drawn from a sample of PSTs. This means that the study findings and the explanations offered in subsequent chapters might not apply to in-service teachers. This is because, when confronted with an instructional situation or dilemma, practicing teachers, in contrast to PSTs, can draw not only on their MKT, but also on other personal resources, the most obvious being their teaching experiences. The study

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<sup>121</sup> Here I use the mathematical rather than the everyday meaning of the term locus. According to the *American Heritage Dictionary* (2006), in mathematics, the term locus is used to describe “a set or configuration of all points *whose coordinates satisfy a given set of conditions*” (p. 1027, emphasis added).

sample should also not be considered representative of the wider PST population. In contrast to PSTs admitted to other institutions or even the undergraduate PSTs studying at the University of Michigan, the study participants, like all the ELMAC students who have been admitted to the University of Michigan over the past ten years, had bachelors' or masters' degrees in other subject areas. All of them had to show that they had successful work experiences in some kind of setting and that they had significant experiences with children (e.g., as parents, school librarians, tutors, coaches, etc). All of them also had to demonstrate at least satisfactory academic competence through a GPA of 3.0 or above and/or GRE scores at the mean. Obviously, these requirements are more rigorous than the requirements for the undergraduate PSTs accepted at the same university and more rigorous than those of teaching education programs in several other institutions. In addition to these requirements, the study participants, like most of the ELMAC students, also showed a keen interest in teaching.

That said, there is no reason to believe that the study participants represent an atypical sample of ELMAC students, especially if one considers the notable diversity in their background characteristics. Recall, for example, that there was a wide variety among these PSTs in terms of the *number* of high-school and college math content courses they had taken and the higher *level* of these courses: whereas some of these PSTs had only taken basic math content courses, others had taken more advanced courses such *Abstract Algebra* and *Chaos and Fractals*. Also whereas some of the study participants had a major or minor in mathematics or other mathematics-related undergraduate degrees (e.g., one PST had a bachelor's in engineering), others had taken none or only a few math content courses during their undergraduate studies. In terms of their entrance MKT – the

basic trait considered in this study – these PSTs were also found to represent a heterogeneous group, as suggested by the value of the standard deviation of their entrance performance in the LMT adapted test (see Table 4.1). These PSTs were also not atypical ELMAC students, given that their performance in that test was not significantly different from that of the ELMAC students of the second cohort.<sup>122</sup>

Before shifting to the next dimension, another remark is in order here. Three of the study participants opted to not participate in the post-intervention interviews (the fourth non-participant withdrew from the program). Based on the classification proposed in this chapter, these three non-participants demonstrated either low or medium performances in the LMT test and the teaching simulation.<sup>123</sup> However, other study participants with similar or even lower performances in both aforesaid measures opted to participate in the post-intervention interviews. Hence, there is no reason to believe that the findings related to the second research question are not representative of the whole cohort of the study participants.

Regarding the dimensions of *settings*, the reader is reminded that to explore the PSTs' performance in the five teaching practices this study utilized a teaching simulation. Although an approximation of teaching, this simulation differed in significant respects from a real teaching setting. First, as explained in Chapters 2 and 3, this teaching simulation simplified certain aspects of the teaching complexity, the most notable being

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<sup>122</sup> The two ELMAC cohorts did not differ significantly in their initial MKT. In particular, in a scale from zero to 41, the mean score of the cohort under consideration was 22.75 with a standard deviation equal to 7.30. The mean score of the second ELMAC cohort was 23.71 with a standard deviation of 6.51. A *t*-test of independent samples suggested that the difference between the performance of the two cohorts was not statistically significant ( $t=.40$ ,  $df=42$ ,  $p=.70$ ). Hence, it could be argued that the study participants did not constitute a special group of preservice ELMAC teachers, at least in terms of their MKT scores.

<sup>123</sup> The first non-participant scored low on both the LMT test and the teaching simulation, the second participant scored low on the LMT test but average on the teaching simulation, whereas the third participants scored low on the teaching simulation but average on the LMT test.

classroom management. Second, in contrast to a real-classroom setting in which teachers often have to make decisions on the spot, in this in-vitro environment the study participants could ponder different alternatives without the time pressure. Third, like other approximations of teaching, this simulation immersed the study participants in what Grossman and colleagues call a low-risk situation (Grossman et al., in press): the decisions that these PSTs were asked to make regarding certain instructional situations had no real cost, either for them or for real students. Therefore, although for issues of simplicity in this study I talk about *teaching performance*, in no way should the performance of these PSTs in the teaching simulation be considered reflective of how these they would perform in real-classroom settings, when confronted with real students and when having to work under certain accountability pressures and time constraints. If anything, the performance these PSTs exhibited in the context of the teaching simulation could be considered suggestive of their *potential* to successfully engage in the five practices under consideration.

The third dimension of generalizability considerations pertains to issues of *treatment*. The reader is reminded that although a particular intervention was utilized, this study was not designed to explore the effectiveness of this “treatment,” but rather to use this intervention as a context to leverage changes in the PSTs’ MKT and their teaching practices. This, in turn, was envisioned to facilitate the exploration of the association between MKT and performance in the five teaching practices from a dynamic perspective. Nor was this study designed to explore causal relationships between MKT and performance in these practices. Even so, treatment considerations are still warranted when contemplating the results of this study. As earlier explained, the author of this study

had a dual role: he was the course instructor and the interviewer. Additionally, the intervention considered in this study had an explicit focus on enhancing the PSTs' MKT in the area of number sense and operations by affording its participants systematic opportunities to engage in the practices of providing explanations, using representations, and analyzing student work. Hence, it is open to further investigation whether the findings reported in Chapter 5 regarding the associations between changes in the PSTs' MKT and changes in their teaching performance would differ if a different intervention was employed. For instance, would the results be different if the interviewer was not the course instructor? Would the strength of the relationships reported in Chapter 5 be different if this intervention immersed these prospective teachers in a dissimilar set of practices? Would these relationships be different if the intervention was not practice-based (e.g., if the study participants were asked to write a term paper on how instructional explanations might affect student learning instead of actually engaging in providing and evaluating explanations)? These and other similar questions suggest that, even though the present study treated the intervention merely as a *context* to facilitate changes in the PSTs' MKT, treatment considerations are still warranted when discussing the study results and its conclusions.

The fourth dimension of generalizability pertains to *outcome* considerations, namely the extent to which the findings reported in a study can be generalized over different outcomes. Five of the design decisions presented in Chapter 2 seem relevant in this discussion. This study focused on a particular type of knowledge, MKT (first design decision), and explored whether this knowledge relates to teaching performance in five teaching practices (third design decision) considered conducive to structuring rich and

challenging learning environments (second design decision). To explore this association, this study focused on a challenging mathematical topic – the division of fractions (fifth design decision) – and studied performance as a composite of three skills: noticing, interpreting-evaluating, and performing (sixth design decision). All these decisions limit the scope of the transferability of the study findings. For instance, it is open to further inquiry whether the study findings would be the same if the study focused on a less challenging mathematical topic or if it decomposed performance differently. The reader is also cautioned that the study findings might not apply to a different type of knowledge or to a different set of practices.

In sum, the four dimensions of generalizability considered above significantly narrow the locus of transferability and applicability of the study findings. At this juncture, the question is unavoidable: How generalizable are the results reported in the next two chapters beyond the study sample, the type of knowledge and practices examined in the study, the setting in which the PSTs' performance was investigated, and the mathematical topic in which this performance was explored? As I discuss in the concluding chapter, the findings of this study offer some insights into an association between a certain type of knowledge and PSTs' *potential* to engage in five teaching practices; the study also yields some working hypotheses about this association (cf. Yin, 2003, pp. 120-122). Future studies could further explore these hypotheses and thus expand their locus of applicability and transferability.

### Summary

In this chapter, I have detailed the methodological approaches of the study. In particular, I have outlined the context of the study and provided information about its

participants and the intervention the study considers. I have then presented the data instruments the study utilized, outlined the data collection and processing procedures, presented the measures utilized in the study and detailed the data analysis processes pursued to answer the two research questions of the study. I have concluded this chapter by considering issues of validity, reliability, and generalizability. The following two chapters present the findings of the study by research question. Chapter 4 addresses the first research question and Chapter 5 takes up the second research question.

## CHAPTER 4

### EXPLORING THE ASSOCIATION BETWEEN MKT AND TEACHING PRACTICES FROM A STATIC VIEWPOINT: RESULTS

#### Overview

This chapter presents the study results pertaining to the first research question, which aimed at exploring the relation between the PSTs' MKT and their performance in the five teaching practices from a *static* perspective. The chapter consists of three sections: the first section pertaining to addressing the first research question from a quantitative standpoint, and the second and third sections associated with addressing this question from a qualitative standpoint.

In particular, in the first section of this chapter, I explore the *direction* and *strength* of the association of interest using the entrance measures considered in the previous chapter. I then explore whether other factors mediated this relation. Thus, the first section of this chapter addresses the two subordinate questions of the first research question, namely, the extent to which there is an association between the PSTs' MKT and their performance in the five teaching practices (SRQ1.1), and if such an association exists whether it is robust or it is mediated by other factors (SRQ1.2)

In the second and third sections of this chapter, I approach the first research question from a qualitative perspective. The second section presents a case-by-case analysis of seven cases selected for further scrutiny. Starting from the convergent cases and then moving to the divergent cases, in this section I outline these PSTs' teaching



performance as gauged by the teaching simulation and consider factors that seem to have informed their performance in the five practices under consideration. In the third section, I present a cross-case analysis of the seven PSTs, in which (a) I identify patterns in the seven PSTs' MKT and teaching performance; (b) I point to two factors that appear to have informed these PSTs' teaching performance, in addition to their knowledge; and (c) I outline a set of propositions about the ways in which the PSTs' knowledge seems to have played out in their teaching performance. Together, the second and third sections of this chapter provide further evidence regarding the association under consideration (SRQ 1.1) and offer insights into how these PSTs' knowledge was manifested in their teaching performance (SRQ1.3). By considering factors impinging on the PSTs' performance, these sections also contribute to exploring the SRQ1.2.

Conventionally, the *Results* chapters include only the presentation of the study findings and postpone their interpretation for subsequent chapters. In this dissertation, I depart from this convention: I report and interpret the study findings. I do so, because postponing the interpretation of all the results reported in Chapters 4 and 5 for the discussion chapter (Chapter 6), would not only impair the readability of the dissertation but it would also make it harder to associated certain arguments made in Chapter 6 with the supporting evidence presented in the previous two chapters.

#### Exploring the Association between the PSTs' Entrance MKT and Performance in the Five Teaching Practices from a Quantitative Standpoint

In this section, I report the results pertaining to exploring the first research question from a quantitative viewpoint. I organize this report along the two subordinate questions of the first research question. First, I consider the *direction* and *strength* of the

association between the PSTs' MKT and their performance in the five teaching practices (SQ 1.1). I do so by briefly discussing the PSTs' entrance MKT and teaching performance; I then consider correlations between MKT (and its CCK and SCK components) and teaching performance in the five practices and its components (i.e., performance in the MKT-related practices and the MTF-related practices; performance in each of the five practices and their subcomponents; and performance in the three skill-subcomponents of noticing, interpreting-evaluating, and performing). Following the same pattern, to address SRQ1.2, I first report on the factor and cluster analyses that allowed grouping the PSTs' responses to the survey statements into a small set of underlying factors/clusters. Next, I consider the descriptive statistics related to all the potential mediating factors under investigation (i.e., beliefs about teaching and learning mathematics, overarching goals, efficacy beliefs, and background characteristics). Following that, I consider the association between these factors and the PSTs' MKT and teaching performance. This section concludes with an exploration of the factors mediating the association between MKT and teaching performance.

*Exploring the Direction and Strength of the Association between the PSTs' MKT and  
Their Teaching Performance*

*A First Glance at the PSTs' MKT and Teaching Performance*

The means and the standard deviations of the PSTs' scores on the entrance measures of knowledge and teaching performance are presented in Table 4.1. The figures reported in this table suggest that, by and large, at their entrance to the ELMAC program, the study participants performed at or below the means of the scales used to measure their

knowledge and teaching performance. From the statistics reported in this table only those pertaining to the subcomponents of the five practices are comparable, since they were measured on the same scales (0-2 for noticing and interpreting-evaluating, and 0-3 for performing). Hence, I first consider the PSTs' performance on these subcomponents, and then discuss their performance on the remaining subcomponents after transforming these results to the same scale.

Looking across the PSTs' *noticing* performance in the five practices suggests that the study participants performed the best in the practice of providing explanations and the worst in the practice of responding to students' requests for help. Their performance in the other three practices was comparable. The use of Kendall's  $W$  test for dependent samples suggested that these differences were statistically significant (Kendall's  $W = 0.21, p < .01$ ). If one looks across the means reported for the *interpreting-evaluating* tasks of each practice, one observes that the PSTs' respective performances were comparable; the slight differences observed between their performance in the practices of using representations and responding to students' requests for help were not large enough to render the PSTs' performance in the five practices statistically significant (Kendall's  $W = 0.03, p > .10$ ). In contrast, the PSTs' performance on the *performing* tasks of the five practices were notably different: the PSTs performed much better on the practices of analyzing students' work and contributions and using representations than how they performed on the practice of providing explanations; these differences were statistically significant (Kendall's  $W = 0.33, p < .001$ )

Table 4.1

*Means and Standard Deviations of the PSTs' Entrance MKT and Teaching Performance*

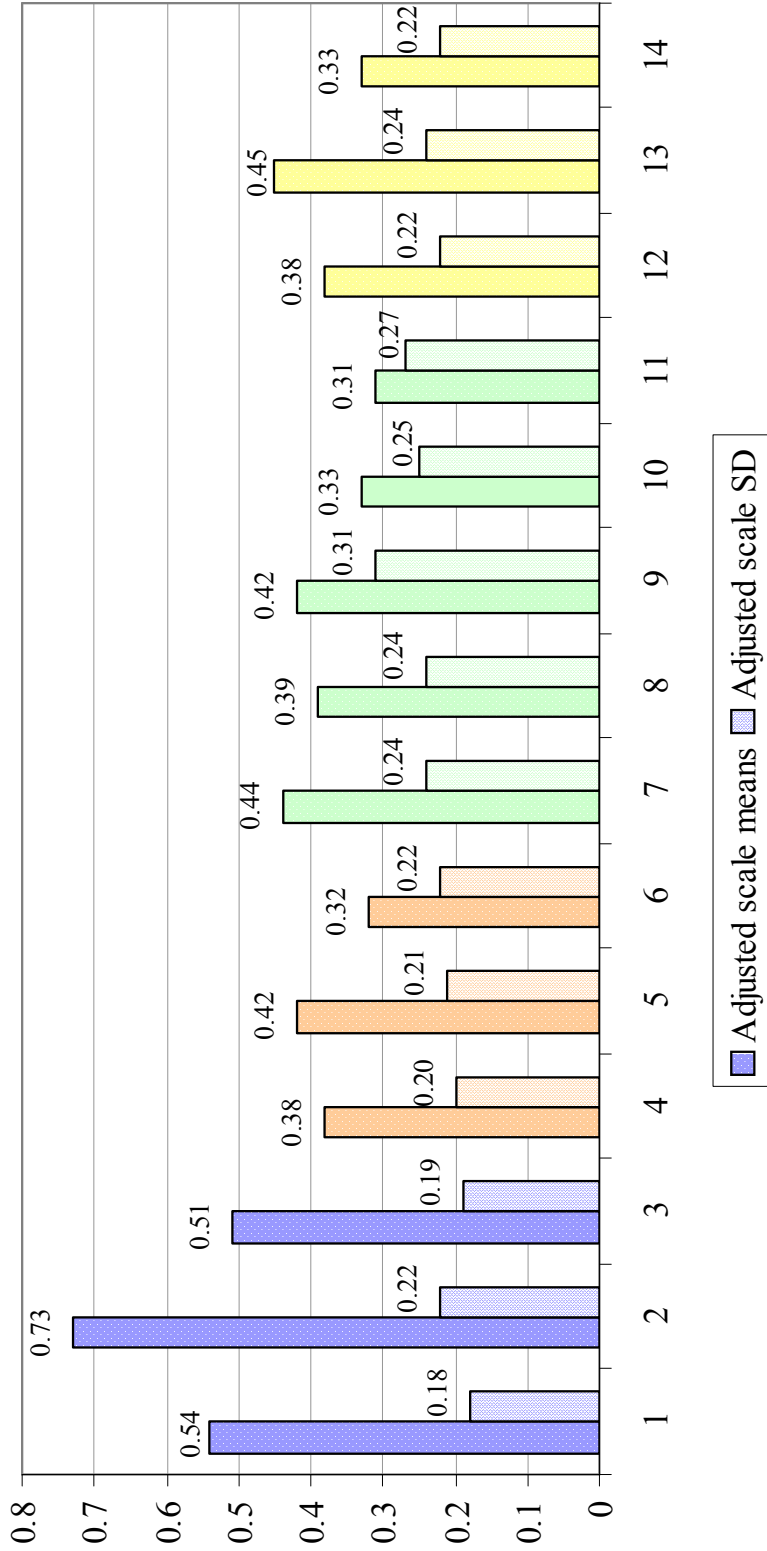
<b>Measure</b>	<b>M*</b>	<b>SD</b>
<i>MKT measures</i>		
Overall MKT (0-41)**	22.10	7.35
CCK (0-5)	3.65	1.09
SCK (0-36)	18.45	6.69
<i>Teaching performance measures</i>		
Practices (overall) (0-35)	13.20	6.86
MKT-related practices (0-21)	8.75	4.39
MTF-related practices (0-14)	4.45	3.09
Using representations (0-7)	3.10	1.68
Noticing (0-2)	0.80	0.70
Interpreting-evaluating (0-2)	1.05	0.76
Performing (0-3)	1.25	1.02
Providing explanations (0-7)	2.70	1.66
Noticing (0-2)	1.20	0.70
Interpreting-evaluating (0-2)	0.95	0.83
Performing (0-3)	0.55	0.69
Analyzing students' work and contributions (0-7)	2.95	2.19
Noticing (0-2)	0.70	0.86
Interpreting-evaluating (0-2)	0.90	0.85
Performing (0-3)	1.35	0.81
Selecting and using tasks (0-7)	2.30	1.72
Noticing (0-2)	0.70	0.66
Interpreting-evaluating (0-2)	0.85	0.75
Performing (0-3)	0.75	0.72
Responding to students' requests for help (0-7)	2.15	1.90
Noticing (0-2)	0.40	0.68
Interpreting-evaluating (0-2)	0.70	0.73
Performing (0-3)	1.05	0.83
Skills (overall) (0-35)	13.20	6.86
Noticing (0-10)	3.80	2.19
Interpreting-evaluating (0-10)	4.45	2.42
Performing (0-15)	4.95	3.33

*Notes.* \* n=20; \*\*The numbers in parentheses correspond to the scale used to measure each score.

To facilitate the interpretation of the remaining descriptive statistics reported in Table 4.1, these figures were adjusted to a scale from one to zero, by dividing each value reported in this table by the highest value of its corresponding scale. The adjusted means and standard deviations for the knowledge and the performance measures are shown in Figure 4.1.

The first three categories of this figure correspond to the means and standard deviations of the MKT measures; the next three categories represent the means and standard deviations of the PSTs' overall teaching performance and their performance on the two sets of practices (MKT/MTF-related practices). Categories 7 to 11 correspond to the participants' performance on each of the five practices under consideration, while the last three categories (i.e., categories 12 to 14) correspond to the PSTs' performance on the composites of the three skills-subcomponents (i.e., noticing, interpreting-evaluating, and performing).

As shown in Figure 4.1, the PSTs' entrance overall MKT-score was above the scale mean (0.54) while their overall teaching performance was notably below the scale mean (0.38). This difference was statistically significant, as suggested by the *Wilcoxon Signed Rank test* (Wilcoxon  $Z= 3.93$ ,  $p<.001$ ). The PSTs' performance on the SCK component of the MKT was also noticeably lower than their performance on the CCK component of MKT (0.51 and 0.73, correspondingly), a difference that was again statistically significant (Wilcoxon  $Z= 3.92$ ,  $p<.001$ ). This result was not surprising, given that CCK represents the *basic knowledge* that one needs to possess to be able to teach the content; hence, it seems reasonable that at their entrance to the program, these PSTs scored higher on the CCK measure than on the SCK measure, which corresponds to not simply knowing the content, but knowing it *for the purposes of teaching it*.



**Legend.** 1. MKT (overall); 2. CCK; 3. SCK; 4. Teaching performance (overall); 5. Performance on the MKT-related practices; 6. Performance on the MTF-related practices; 7. Using representations; 8. Providing explanations; 9. Analyzing students' work and contributions; 10. Selecting and using tasks; 11. Responding to students' direct or indirect requests for help; 12. Noticing (overall); 13. Interpreting-evaluating (overall); 14. Performing (overall).

*Figure 4.1.* The PSTs' MKT and teaching performance after adjusting the pertinent measures to a zero-to-one scale (n=20).

Figure 4.1 also shows that the study participants' scores on the teaching-performance measures were below the scale mean for all measures under consideration. The PSTs' average scores ranged from 0.39 to 0.44 for the measures of the MKT-related practices and were much lower for the MTF-related practices (0.31 to 0.33). The use of the *Wilcoxon Signed Rank test* suggested that at their entrance to the program, the study participants performed significantly better on the MKT-related practices than on the MTF-related practices (Wilcoxon  $Z= 3.82$ ,  $p<.001$ ). At their entrance to the program, these PSTs also performed better on interpreting and evaluating the virtual teacher's actions and decisions (0.45) compared to their noticing performance (0.38) and their performance in the performing tasks of the simulation (0.33). As a whole, these differences were not statistically significant (Kendall's  $W = 0.03$ ,  $p >.10$ ); only the difference between the PSTs' performance on the noticing and the performing tasks was significant at level 0.10 (Wilcoxon  $Z=1.79$ ,  $p<.10$ ).

The descriptive statistics just reported provide some first insights into the inquiry under exploration. Specifically, they show that at their entrance to the program these PSTs performed better on the MKT test rather than on the teaching simulation; they exhibited higher CCK scores than SCK scores; they performed better on the MKT-related practices than on the MTF-related practices; and their teaching performance across the five practices differed for the tasks of noticing and performing (e.g., whereas they performed relatively well on the noticing tasks of the practice of providing explanations, their corresponding performance on the performing tasks of this practice was low). With the exception of the CCK scores, all other scores pertaining to their MKT and their performance on the teaching simulation were at or below the scale means.

### *Exploring Associations between the PSTs' MKT and Their Teaching Performance*

*Spearman's*  $r_s$  was used to explore the strength and direction of the association between the PSTs' MKT scores and their teaching-performance scores. Before exploring these associations, I first searched for *outliers* and *influential observations*. Although no outliers were found,<sup>124</sup> one PST, Nathan, appeared quite frequently as an influential observation whose scores were *inflating* the correlation coefficients for several of the associations under exploration.<sup>125</sup> Hence, he was dropped from the analyses reported in this section. The *Spearman's* correlation coefficients that resulted from the analysis of the data of the remaining 19 PSTs appear in Figure 4.2. Because of the small sample utilized in this study, I considered significant all the correlations at or below the significance level of 0.10. I discuss the correlation coefficients reported in Figure 4.2, by first considering the correlations between the PSTs' *overall* MKT-score and their teaching performance; I then shift to the correlations between each of the MKT subcomponents (CCK and SCK) and the PSTs' teaching performance.

The first immediate observation from this figure is that the PSTs' MKT was moderately correlated with their overall performance in the five teaching practices ( $r_s =$

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<sup>124</sup> To identify potential outliers, I first inspected all the scatterplots of the measures of knowledge (MKT-overall, SCK, CCK) against the measures of performance (all the 27 measures of performance considered in Table 3.6). For the cases identified as potential outliers, I calculated their studentized residual, which resulted from regressing the specific measure of teaching performance on the specific measure of knowledge. To determine whether a potential outlier was indeed an outlier, I then used the Bonferroni critical value of 3.54, which I calculated using  $R$  (see Faraway, 2005, pp. 66-69 for more on this issue); in the case under consideration the Bonferroni critical value was estimated for 20 participants and 17 degrees of freedom. Because none of the studentized residuals exceeded this value, it was decided that none of the 20 study participants was an outlier for any of the associations between the entrance measures of knowledge and the entrance measures of performance.

<sup>125</sup> To identify influential observations, I first inspected all the scatterplots mentioned in the previous footnote. In those cases that there appeared to be a potential influential observation, I obtained the values for the *Cook distance* statistic for the knowledge/teaching performance regression and identified the PST who was associated with the highest value (in absolute terms). An examination of all these values showed that Nathan was an influential observation for 12 out of the 50 cases in which a significant correlation (at  $p < .10$ ) was identified.



0.56).<sup>126</sup> The PSTs' overall MKT-score was also moderately correlated with their performance on the MKT-related practices and the MTF-related practices ( $r_s = 0.51$  and  $r_s = 0.52$ , respectively). The PSTs' MKT performance was significantly related to their performance in only two of the five practices, namely analyzing students' work and contributions ( $r_s = 0.43$ ) and responding to students' requests for help ( $r_s = 0.47$ ). Interestingly enough, the PSTs' MKT score was not related to their noticing and interpreting-evaluating performance for *any* of the practices under consideration. Instead, it was significantly related to their performance on the performing tasks of the teaching simulation for four out of the five practices under consideration: for all the MKT-related practices (i.e., providing explanations, using representations, and analyzing students' work and contributions) and for the MTF-related practice of responding to students' requests for help; these correlations were moderate in the first two cases and marked in the latter two. A similar pattern was also observed for the *aggregates* of the three skill subcomponents: although positive, the correlations between the PSTs' overall MKT-score and their noticing and interpreting-evaluating performances were non-significant. In contrast, a statistically significant and marked correlation ( $r_s = 0.68$ ) was found between the PSTs' overall MKT-score and their overall performance on the performing tasks of the simulation.

The only negative correlation found between the PSTs' MKT and their teaching performance, which concerns the case of using representation/noticing, is also worthy of attention. Although numerically negligible, this correlation suggests that at least some of

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<sup>126</sup> I use the following classification to characterize the strength of the correlations reported above: *no or negligible* correlations for  $r_s$  ranging from zero to 0.20 (in absolute values); *low* correlations for  $r_s$  ranging from 0.20 to 0.40; *moderate* correlations for  $r_s$  ranging from 0.40 to 0.60; *marked* correlations for  $r_s$  ranging from 0.60 to 0.80; and *high* correlations for  $r_s$  ranging from 0.80 to 1.00 (cf. Franzblau, 1958, p.81; Shavelson, 1995; p. 155).

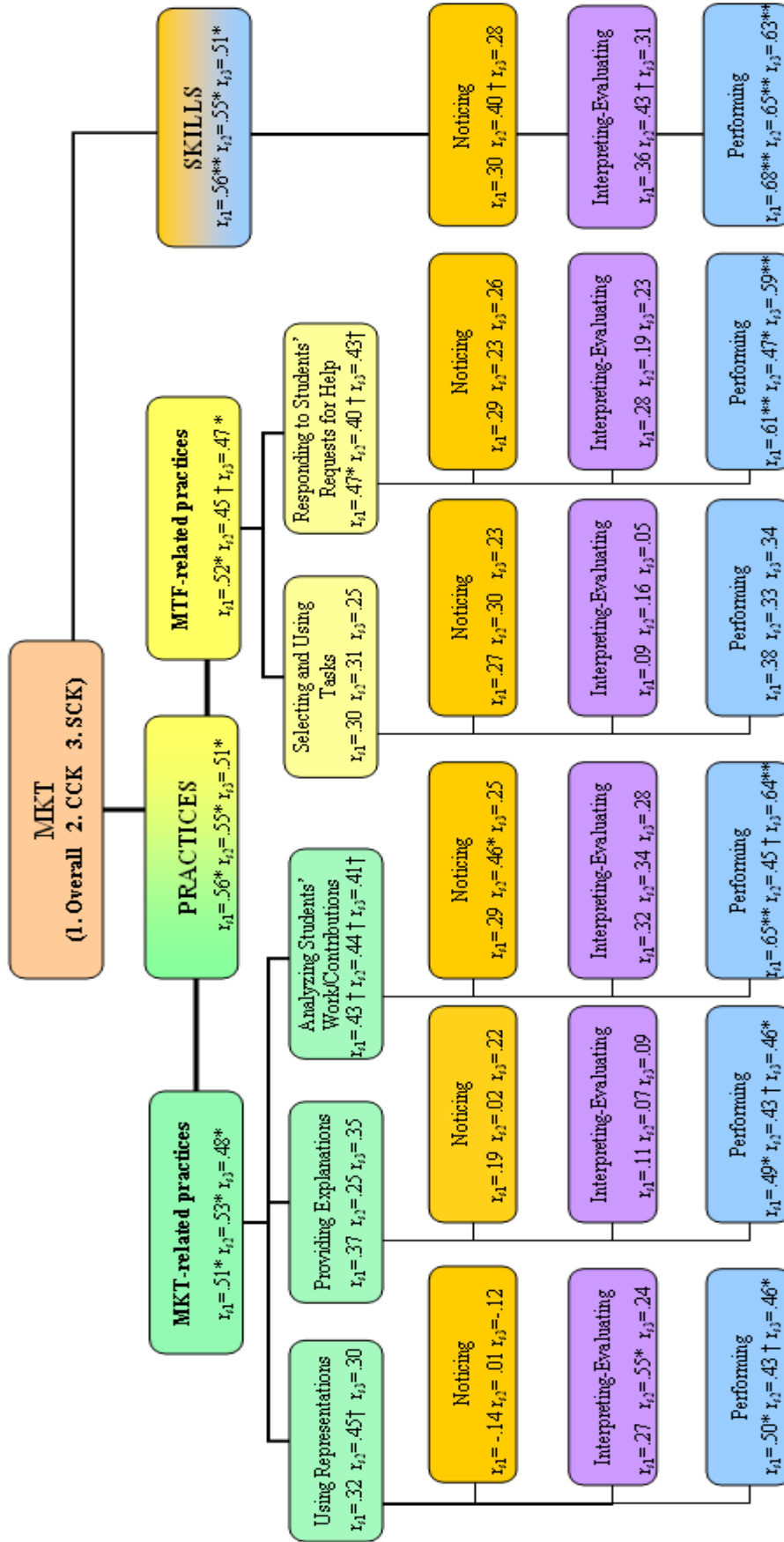
the high MKT-performers did not notice problematic aspects of the virtual teacher's use of representations. I revisit and expound this finding when discussing the case of Nicole (see second section of this chapter).

Figure 4.2 also shows that with only two notable exceptions,<sup>127</sup> the correlations between the PSTs' SCK-scores and CCK-scores and their performance on the teaching simulation were similar to those just reported for their overall MKT-performance. It is also interesting to notice that the correlations between the PSTs' MKT and their performances on the *performing* tasks were higher for the SCK component than for the CCK component, especially for the practices of analyzing students' work and contributions and responding to students' requests for help. This finding resonates with the theoretical framework of MKT, which posits that common knowledge of the content does not suffice to perform certain tasks of teaching.

In sum, the findings reported above show that overall there was a *positive moderate association* between the PSTs' MKT score and their performance in the five teaching practices. This result was mainly due to the moderate to marked correlations observed between the PSTs' MKT score and their performance on the *performing* tasks of the teaching simulation. Similar correlations were found between the PSTs' teaching performance and each of the two components of MKT. In the next part, I explore the robustness of the moderate association between the PSTs' overall MKT and their overall teaching performance.

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<sup>127</sup> The two exceptions were: (a) a moderate correlation was found between the PSTs' CCK-score and their performance on using representations; for the overall MKT-score this correlation was not significant; and (b) a similar pattern was observed for the noticing tasks of the practice of analyzing students' work and contributions.



**Legend.** \*\*  $p < .01$ ; \*  $p < .05$ ;  $^\dagger p < .10$

Figure 4.2. Spearman's  $r_s$  correlation coefficients between *entrance* measures of knowledge and teaching performance (n=19).

*Exploring the Robustness of the Association between the PSTs' MKT and Their Teaching Performance*

Several factors were utilized to explore mediating effects of the association between the PSTs' MKT and their teaching performance. Among these factors were the PSTs' beliefs about teaching and learning mathematics (with a particular focus on structuring and maintaining rich and cognitively challenging environments) and the PSTs' perceived importance of a set of instructional goals representing the five strands of mathematical proficiency as defined in the *Adding it Up* report (Kilpatrick et al., 2001). Additional factors included the PSTs' efficacy beliefs to work on fractions and engage in MKT-related practices in this area, and a set of background characteristics (courses taken, having a minor or major in mathematics, and two measures of general knowledge /aptitude).

This part is organized in four subdivisions. In the first two subdivisions, I report the results of the exploratory factor analysis and the hierarchical cluster analysis that allowed grouping the statements of the first and the third parts of the survey into a smaller set of factors/clusters. In the third subdivision, I briefly consider the PSTs' entrance characteristics with respect to the aforesaid factors. In the fourth subdivision, I explore the extent to which these factors mediated the association found between the PSTs' MKT and their teaching performance.

*PSTs' Entrance Beliefs about Teaching and Learning Mathematics*

As explained in Chapter 3, 13 out of the 18 statements of the first part of the survey were utilized to explore the PSTs' beliefs about teaching and learning mathematics. The responses of 88 PSTs to these statements were subjected to exploratory

principal components factor analysis with varimax rotation. This analysis helped explore the construct validity of the first part of the survey; it also helped reduce these statements into three underlying factors, as shown in Table 4.2. Specifically, six of these statements had high loadings on the first factor. Altogether, these statements reflected the PSTs' belief that the teacher of mathematics should afford students ample opportunities for skill mastery and strive to reach closure to minimize student frustration; hence, I called this factor "skill mastery and closure." Three of the statements, which had high loading on the second factor, reflected the belief that remembering and applying formulas is critical to teaching and learning mathematics; hence, I named this factor "teaching and learning mathematics as remembering and applying formulas." Finally, the remaining four statements had high loadings on the third factor. These statements captured the PSTs' belief that the mathematics teacher should pursue any of the following approaches to minimize complexity and support student learning: considering only one method or explanation, attending to the final product rather than the process,<sup>128</sup> accepting mathematical ideas without challenging them, and following a (predetermined) sequence of steps. Therefore, the third factor reflected beliefs about minimizing the complexity in teaching mathematics.

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<sup>128</sup> Statement 16 which corresponded to this aspect was recoded, since it was negatively related to Factor 3.

Table 4.2

*Factor Loadings of the Three Factors against the Statements Associated with the PSTs' Beliefs about Teaching and Learning Mathematics*

No <sup>1</sup>	Statements	Factors <sup>2</sup>			
		I	II	III	h <sup>2</sup>
7.	It is important for pupils to master the basic computational skills before they tackle complex problems	0.77	0.09	0.05	0.60
9.	If students have unanswered questions or confusions when they leave class, they will be frustrated by the homework.	0.74	-0.08	0.10	0.56
13.	If students are having difficulty in math, a good approach is to give them more practice in the skills they lack.	0.69	0.19	0.12	0.53
8.	In learning math, students must master topics and skills at one level before going on.	0.65	0.25	0.18	0.52
10.	A good mathematics teacher is someone who explains clearly and completely how each problem should be solved.	0.58	0.06	0.21	0.39
15.	Students should never leave math class (or end the math period) feeling confused and puzzled.	0.56	0.40	0.06	0.47
18.	Basic computational skill and a lot of patience are sufficient for teaching elementary school math.	0.06	0.80	-0.16	0.67
6.	When students can't solve problems, it is usually because they can't remember the right formula or rule.	0.07	0.70	0.32	0.60
14.	To do well, the most important things students must learn are facts, principles, and formulas in mathematics.	0.23	0.69	0.31	0.62
3.	A lot of things in math must simply be accepted as true and remembered.	0.32	0.06	0.78	0.71
4.	It is confusing to see many different methods and explanations for the same idea.	0.10	0.17	0.61	0.41
2.	Doing mathematics is usually a matter of working logically in a step-by-step fashion.	0.28	0.00	0.53	0.36
16.	The most important issue is not whether the answer to any math problem is correct, but whether students can explain their answers.	0.40	-0.12	-0.52	0.45
	Eigenvalues	3.08	1.92	1.86	
	Percentage of variance explained	23.72	14.80	14.27	
	Cumulative percentage of explained variance	23.72	38.52	52.79	

Notes.

<sup>1</sup> This is the serial number of statements as they appear on the survey (see Appendix 5).

<sup>2</sup> Factor I: Skill mastery and closure; Factor II: Teaching and learning mathematics as remembering and applying formulas; Factor III: Minimizing the complexity in teaching mathematics.

The three-factor solution yielded from this analysis was satisfactory given that:

- a) It explained about 53% of the variance in the PSTs' answers (see last row of Table 4.2).
- b) It explained at least 40% of the variance in the PSTs' answers to each statement, as indicated by the communality values shown in the last column of Table 4.2. The only exceptions were statements 2 and 10. However, even for these statements, the explained variance was in the range of 0.30 to 0.40; additionally, the inclusion of these statements did not affect the interpretability of the factors, which would have suggested that these statements be removed.
- c) The value of the KMO (Kaiser-Meyer-Olkin) criterion, which provides an index for the appropriateness of applying factor analysis to a given set of data, was 0.78. This value was very close to KMO threshold of meritorious values (i.e. 0.80) (cf. Kline, 1994).
- d) The internal consistency of the first two factors was higher than the lenient cut-off point of 0.60 (Factor I:  $\alpha = 0.80$ ; Factor II:  $\alpha = 0.66$ ). However, for the third factor this value was lower but close to this threshold (i.e., Factor III:  $\alpha = 0.54$ ). It should be borne in mind, though, that the value of Cronbach's alpha increases as the number of items included in a scale increases (see Cronbach, 1990, pp. 204, 207); hence, the internal consistency of the statements of this factor could have been higher had it included more statements.

After the extraction of the three-factor solution, the factor scores of the study participants' responses to each factor were calculated. Given that the 13 statements were measured on a scale ranging from 1 to 7, with 1 corresponding to "strongly disagree" and

7 to “strongly agree” with a given statement, higher values of factor scores were taken to represent higher levels of agreement with the beliefs encapsulated in each factor. The means and standard deviations of these factor scores are reported in Table 4.3.

#### *PSTs’ Entrance Efficacy Beliefs*

As explained in Chapter 3, data on efficacy beliefs were obtained from 44 PSTs. Because of the small number of the obtained responses, instead of submitting these data to exploratory factor analysis, I used Ward’s hierarchical cluster analysis to explore if the participants’ responses could be organized in homogenous clusters. This analysis suggested that the participants’ responses to the 17 statements of the third part of the survey be grouped into three clusters since, as indicated by the Agglomeration schedule, there was a fairly large increase in the value of the distance measure from a two-cluster solution (606.90) to a three-cluster solution (777.07). The three-cluster solution explained 51% of the variance in the PSTs’ responses, as determined by the ratio of the distance value of the agglomeration schedule for the three-cluster solution to the distance value of a single-cluster solution.

The first cluster comprised statements 1-6 and corresponded to the PSTs’ perceived efficacy to understand and operate on fractions. The second cluster consisted of statements 7-8 and reflected the PSTs’ perceived efficacy to understand the algorithms of multiplication and the division of fractions. The last cluster encompassed all the statements related to the PSTs’ perceived efficacy to engage in using representations, providing explanations, and analyzing student work when dealing with fractions (statements 9-17); hence, this latter group was considered to reflect the PSTs’ efficacy to engage in MKT-related practices when working on fractions. The internal consistency of



the statements clustered in each group was very high (Cronbach's alpha =0.88, 0.99, and 0.94, for the three groups respectively). After clustering the statements in three groups, the study participants' average perceived competence for each cluster was calculated. The means and the standards deviations of their efficacy beliefs are reported in Table 4.3.

*PSTs' Entrance Characteristics on the Measures Representing Potential Mediators of the Relationship Explored in the Study*

Table 4.3 presents the means and the standard deviations of the PSTs' beliefs about teaching and learning mathematics, their perceived relative importance of a set of instructional goals, and their efficacy beliefs about working on fractions and engaging in the MKT-related practices examined in this study. Since Nathan was excluded from subsequent analyses (i.e., recall that he was found to be an influential observation), this table reports statistics for the whole sample of the study participants (out of brackets) and the subsample of the 19 study participants considered for subsequent analyses (inside brackets). With the exception of two cases, there were no notable differences in the statistics reported for the whole sample and for the subsample of the 19 PSTs.<sup>129</sup>

*Beliefs about teaching and learning mathematics.* The PSTs' beliefs about teaching and learning mathematics were measured on a 1-7 Likert scale, with 1 representing strong disagreement and 7 corresponding to strong agreement. Although, overall, the study participants tended to slightly disagree with the idea that teaching and learning mathematics is solely a matter of remembering and applying formulae and rules,

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<sup>129</sup> The statistics reported in Table 4.3 suggest that after excluding the case of Nathan, there was a notable increase in the PSTs' mean score on the first beliefs factor (i.e., skill mastery and closure) and the goal of productive disposition. Although both changes were small to be statistically significant, they suggest that Nathan strongly disagreed with the idea of emphasizing skill mastery and closure when teaching mathematics; he also placed less emphasis on the goal of productive disposition than the average emphasis placed on this goal by his classmates.

they seemed to be somewhat positively disposed to the idea that the math teacher should strive for skill mastery and closure and should minimize complexity to facilitate student understanding (given that the mean values of the PSTs' responses to both these categories were above the scale neutral point of 4). Although on the surface these beliefs might seem contradictory, they are not since teachers can aim at more than mere memorization of facts and rules, but still place a lot of emphasis on skill mastery and closure and avoid complexity to support student learning. The use of Kendall's  $W$  test suggested that the differences shown in Table 4.3 in the strength of the PSTs' agreement with the three beliefs factors were statistically significant (Kendall's  $W=0.58$ ,  $p < .001$ ).

Table 4.3

*Means and Standard Deviations of the PSTs' Entrance Characteristics on the Potential Mediating Factors Explored in the Study*

<b>Measure</b>	<b>M*</b>	<b>SD</b>
<i>Entrance beliefs about teaching and learning mathematics</i>		
Skill mastery and closure (1-7)	5.02 [5.15]	1.35 [1.25]
Teaching and learning mathematics as remembering and applying formulas (1-7)	3.46 [3.52]	1.19 [1.19]
Minimizing the complexity in teaching mathematics (1-7)	4.39 [4.47]	0.98 [0.93]
<i>Entrance perceived relative importance of instructional goals</i>		
Conceptual understanding (0-100)	19.40 [19.47]	8.25 [8.47]
Procedural fluency (0-100)	15.40 [15.26]	4.74 [4.83]
Strategic competence (0-100)	18.47 [18.23]	6.46 [6.54]
Adaptive reasoning (0-100)	19.50 [19.31]	5.37 [5.45]
Productive disposition (0-100)	27.75 [28.26]	14.05 [14.24]
<i>Entrance efficacy beliefs</i>		
Understanding and operating on fractions (1-7)	5.94 [5.89]	0.78 [0.77]
Understanding the algorithms of fraction multiplication and division (1-7)	4.98 [4.97]	1.32 [1.36]
Engaging in MKT-related practices with respect to fractions (1-7)	5.14 [5.12]	0.89 [0.91]

Notes. \* n=20 [n=19]; \*\*The numbers in parentheses correspond to the scale used to measure each score.

*Perceived relative importance of instructional goal.* At their entrance to the program, the study participants were somewhat negatively disposed to the idea of remembering and applying rules when teaching and learning mathematics, as suggested by their low percentage allocation to the goal of procedural fluency. They considered the goals of conceptual understanding, strategic competence, and adaptive reasoning somewhat more important than the aforementioned goal; for these PSTs, the most important goal was that of helping students develop a productive stance toward the subject. These differences were statistically significant (Kendall's  $W=0.14$ ,  $p < .05$ ).

*Efficacy beliefs.* The study participants reported feeling very competent in understanding and working on fractions; they reported feeling less competent in grasping the algorithms of multiplication and division of fractions and in engaging in the MKT-related practices on fractions, a difference that was statistically significant (Kendall's  $W = 0.21$ ,  $p < .05$ ). However, even in the latter two cases their reported competency was quite high, as indicated by the mean values of their responses (4.98 and 5.14, respectively).

*Background characteristics.* As shown in Table 3.1, during their high-school studies, one PST has taken at most two math courses, 11 PSTs had taken three to five courses, and eight PSTs had taken at least six courses; the excluded case of Nathan was clustered in the latter category. During their undergraduate studies, one PST had not taken any math content course, 14 PSTs had taken one to two math content courses, one PST had taken three to five content courses, and six PSTs (including Nathan) had taken at least six content courses. Twelve of the study participants had taken at least a *Calculus I* class (including Nathan); five PSTs had only taken a *Precalculus* class, whereas three PSTs had taken no calculus classes at all. Two of the study participants had also taken

one to two math methods courses; one PST had a major in mathematics and a classmate of her had a minor in mathematics.<sup>130</sup>

### *Exploring Mediators of the Association between MKT and Teaching Performance*

To investigate mediating effects of the association between the PSTs' MKT and their performance on the five teaching practices, I explored the relationships of the factors just discussed with (a) the PSTs' MKT scores and (b) their teaching-performance score. The *Spearman's* correlation coefficients of this exploration are reported in Columns I and II of Table 4.4, respectively. The effects of the factors that were found to correlate with both MKT and teaching performance were then controlled using *Kendall's partial rank-order* correlation criterion; the correlation coefficients *after* controlling for these effects are reported in Column III. In what follows, I first discuss some notable associations and then consider the mediating effects suggested by the figures of this table.

*Associations between the potential mediators and the PSTs' MKT and teaching performance.* Several figures reported in Table 4.4 are noteworthy. First, the correlation between the PSTs' MKT and their GRE-quantitative performance was 0.78. Given that both measures represent aspects of PSTs' knowledge, one could legitimately wonder whether this marked correlation suggests that these two measures can be used interchangeably. To address this question, one needs to consider the *coefficient of determination* (i.e., the squared correlation coefficient) of these two measures of knowledge.<sup>131</sup> This coefficient represents the percentage of variance in one variable accounted for by the other variable. In the case at hand, this coefficient was 0.61 (i.e.,  $r_s^2 = 0.78^2$ ); this implies that the variance in the PSTs' GRE performance could explain

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<sup>130</sup> Because I was granted access to the PSTs' *standardized* GRE scores, no descriptive statistics regarding these measures are discussed here.

<sup>131</sup> For more on this coefficient, see Shavelson (1996, pp. 162-163).

61% in the variance in their MKT performance. In turn, this suggests that the two measures of knowledge cannot be regarded as interchangeable. In fact, the PSTs considered in the next section for which a high difference between the two measures of knowledge was found further corroborate this argument. A similar argument applies to the correlation found between the number of calculus classes the study participants had taken and their MKT. As Table 4.4 suggests, the number of calculus classes taken was moderately related with the PSTs' MKT score ( $r_s = 0.76$ ). That means that having taken a *Precalculus/Calculus* class explained about 58% of the variance in these PSTs' performance on the MKT test. In conjunction, both the aforesaid figures suggest that the PSTs' entrance performance on the MKT test was moderately correlated with two measures that can be considered indicators of the PSTs' general mathematical knowledge. It is important to notice, however, that only the first measure was also found to relate to their teaching performance.

Second, a marked correlation was found between the PSTs' MKT and their reported efficacy to understand and work on fractions. The strength of this association was not surprising given that both the efficacy and the MKT measures were calibrated to capture *perceived* and *actual* competence in the area of fractions, respectively. Third, of the goals examined in the study, the PSTs' perceived importance of the goal of strategic competence was found to be moderately related to their MKT; this goal also had a low yet positive association with their performance on the teaching simulation.

Table 4.4

*Exploring Mediators of the Association between MKT and Teaching Performance*

Factor <sup>2</sup>	Correlation coefficients and changes thereof after controlling for mediating effects <sup>1</sup>		
	I <sup>3</sup>	II <sup>3</sup>	III <sup>3</sup>
<i>Beliefs about teaching and learning mathematics</i>			
Skill mastery and closure	- 0.45†	- 0.11	--
Teaching and learning math as remembering and applying formulas	- 0.08	- 0.02	--
Minimizing the complexity in teaching mathematics	- 0.37	- 0.23	--
<i>Perceived goal importance</i>			
Conceptual understanding	0.08	- 0.04	--
Procedural fluency	- 0.27	- 0.44†	--
Strategic competence	0.50*	0.36	--
Adaptive reasoning	0.28	0.31	--
Productive disposition	- 0.33	- 0.18	--
<i>Efficacy beliefs</i>			
Understanding and operating on fractions	0.75**	0.55 *	0.20
Understanding the algorithms of fraction multiplication and division	0.30	0.12	--
Engaging in MKT-related practices with respect to fractions	0.18	- 0.07	--
<i>Background characteristics</i>			
High-school math content courses	0.48 *	0.39 †	0.33 *
Undergraduate content courses	0.54 *	0.12	--
Calculus classes taken	0.76**	0.30	--
Math methods courses	0.47 *	0.53 *	0.29 †
Undergraduate math major	0.35	0.35	--
Undergraduate math minor	0.24	- 0.13	--
GRE verbal performance (n=17)	0.62**	0.34	--
GRE quantitative performance (n=17)	0.78**	0.59 *	0.19

Notes.

<sup>1</sup> I. Spearman correlation coefficient of the focal factor with MKT; II. Spearman correlation coefficient of the focal factor with teaching performance; III. Kendall's partial rank-order correlation coefficient for the association between MKT and teaching performance after controlling for the focal factor.

<sup>2</sup> n=19 (unless otherwise specified)

<sup>3</sup> \*\* p < .01; \* p < .05; † p < .10 (two-tailed test)

Fourth, a number of negative associations were found; among them, the highest three are worth noticing since they depict some interesting trends. Table 4.4 shows a negative moderate correlation between MKT and the PSTs' belief about skill mastery and closure in mathematics; it also shows a negative low correlation between the PSTs' MKT and their belief about minimizing the complexity when teaching mathematics. Both aforementioned negative associations suggest that the PSTs who performed higher on the LMT test tended to disagree with a type of instruction that emphasizes skill mastery and seeks to minimize complexity when teaching the subject. The negative low correlation between the PSTs' perception of the goal of procedural fluency and their performance on the teaching simulation suggests that the PSTs who tended to highly value the goal of procedural fluency did not do very well on the teaching simulation. This finding resonates with the philosophy undergirding the design of the teaching simulation (i.e., emphasis was placed more on meaning and understanding rather than on mindlessly executing certain procedures). From this respect, this negative correlation provides evidence in favor of the construct of the "teaching performance geared toward supporting student understanding" measured in this study.

Fifth, the math content and math methods courses that the PSTs had taken during their high-school and undergraduate studies were moderately related to their MKT. Of these measures, the number of high-school math content courses and the number of math methods courses these PSTs had taken during their undergraduate studies were also found to be significantly associated with their teaching performance. Interestingly enough, having a major or minor in mathematics was significantly related to neither the PSTs' MKT score nor their teaching-performance score.

*Exploring mediating effects.* Table 4.4 shows that from the factors considered in this study, only four were found to be significantly associated with *both* the PSTs' MKT and their teaching performance. These factors were the number of high-school math content courses these PSTs had taken, the extent to which they had taken math methods courses during their undergraduate studies, their reported efficacy in grasping and working on fractions, and their quantitative GRE-score. After controlling for each of those factors, the correlation between the PSTs' MKT and their teaching performance became smaller, suggesting that each of these factors mediated the aforesaid association. In the former two cases, the association between the PSTs' MKT and their teaching performance remained significant (at level  $p < 0.10$ , for a two-tailed test), even after controlling for the mediating effects. In the latter two cases, however, this association became non significant, suggesting that the PSTs' efficacy beliefs to work on fractions and their quantitative GRE-performance were strong mediators of the focal association.

To further explore the mediating effect of the latter two factors, I considered the association between the PSTs' MKT and the *performing* component of their performance, which, as the reader might recall, had the highest correlation with their MKT. After controlling for these factors, this correlation remained significant at level 0.10 (for a two-tailed test) for both the PSTs' efficacy beliefs (*Kendall's coefficient* = 0.28) and their GRE performance (*Kendall's coefficient* = 0.28). This suggests that the *correlation between the PSTs' MKT and their performance in the performing tasks of the simulation was robust to all the factors considered in this study at the significance level of 0.10.*

After providing a brief summary of the quantitative results reported above, in the next two sections I explore the association at hand from a qualitative standpoint.



### *Summary of the Quantitative Findings*

The quantitative analysis showed positive moderate correlations between the PSTs' MKT and their teaching performance on the five teaching practices. Moderate correlations were also found between their MKT and each of the two sets of practices the study examines (MKT-related practices and MTF-related practices). These correlations were mostly due to the moderate to marked associations found between the PSTs' MKT and their performance on the *performing* tasks of the teaching simulation. Similar associations, although with some discrepancies, were also found between each of the two components of MKT (CCK and SCK) and the PSTs' overall teaching performance. The main association that the study examines, namely PSTs' MKT and their performance in the five teaching practices, was mediated by four factors, three related to the PSTs' background characteristics (i.e., high-school math content courses taken, math methods courses taken, and GRE-quantitative performance) and the fourth pertaining to their efficacy beliefs to understand and work on fractions. The association of interest was robust to two of the four factors (those related to the courses taken); the mediating effects of the PSTs' efficacy beliefs and their GRE-quantitative performance rendered this association low or negligible. Nevertheless, the association between the PSTs' MKT and their performance in the *performing* tasks of the simulation was robust to even the PSTs' efficacy beliefs and their GRE-quantitative score (at level 0.10).

## Exploring the Association between the PSTs' Entrance MKT and Performance in the Five Teaching Practices from a Qualitative Standpoint: Case Analysis

This section is organized in eight parts. In the first part, I present the seven cases that were selected for further scrutiny and explain the criteria that inform their selection. Each of the remaining parts corresponds to a case under investigation; these parts share the same structure. After providing some background, I discuss the PST's performance on each of the five practices under consideration: selecting and using tasks; providing explanations; using representations, analyzing students' work and contributions, and responding to students' requests for help. In presenting each PST's performance on these practices, I follow the structure of Figure 3.4, which details the teaching simulation. To facilitate readability, for each practice under consideration, I first focus on the *performing* tasks and then move to the noticing and interpreting-evaluating tasks, which I consider in tandem. This is followed by a brief discussion of each PST's performance on the LMT adapted test. The presentation of each case concludes with an analytical commentary, where I synthesize the findings reported for each case by contemplating the two subordinate questions that the qualitative analysis aimed at exploring: (a) How does MKT manifest itself in the PSTs' performance in the five teaching practices under consideration? and (b) What factors appear to mediate the association between the PSTs' MKT and their performance in the five practices?

### *The Selection of the Seven Cases*

As discussed in Chapter 3, two key criteria informed the selection of the seven cases considered in this study: including both convergent and divergent cases and selecting cases that represented various manifestations of the mediating factors identified

in the previous section of this chapter. To identify convergent and divergent cases, I first plotted the PSTs' entrance overall MKT-score against their entrance overall teaching-performance score (both in standardized values). This resulted in clustering the PSTs in four groups represented by the quadrants of the plot shown in Figure 4.3.<sup>132</sup> Participants clustered in the first and third quadrants represented convergent cases. Those clustered in the first quadrant (i.e., Leo, Nathan, Nicole, Teresa, and Ying) performed well on both MKT and teaching performance; conversely, those presented in the third quadrant (i.e., Barbara, Dana, Deborah, Habika, Lillian, Nora, Vonda, and Vui) exhibited a low performance relative to their fellow-students on both the LMT test and the teaching simulation. The remaining PSTs represented divergent cases. Those clustered in the second quadrant (i.e., Bob, Dorothy, and Tiffany) performed better on the teaching simulation than how they would have performed had there been a perfect positive association between MKT and teaching performance; the opposite pattern was true for the PSTs clustered in the fourth quadrant (i.e., Frances, Kimberley, Suzanne, and Travis).

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<sup>132</sup> The numbers shown in this diagram correspond to the serial number of the PSTs' pseudonyms shown in the legend on the right, when read across columns (e.g., Deborah is number 4 and Leo number 9.)

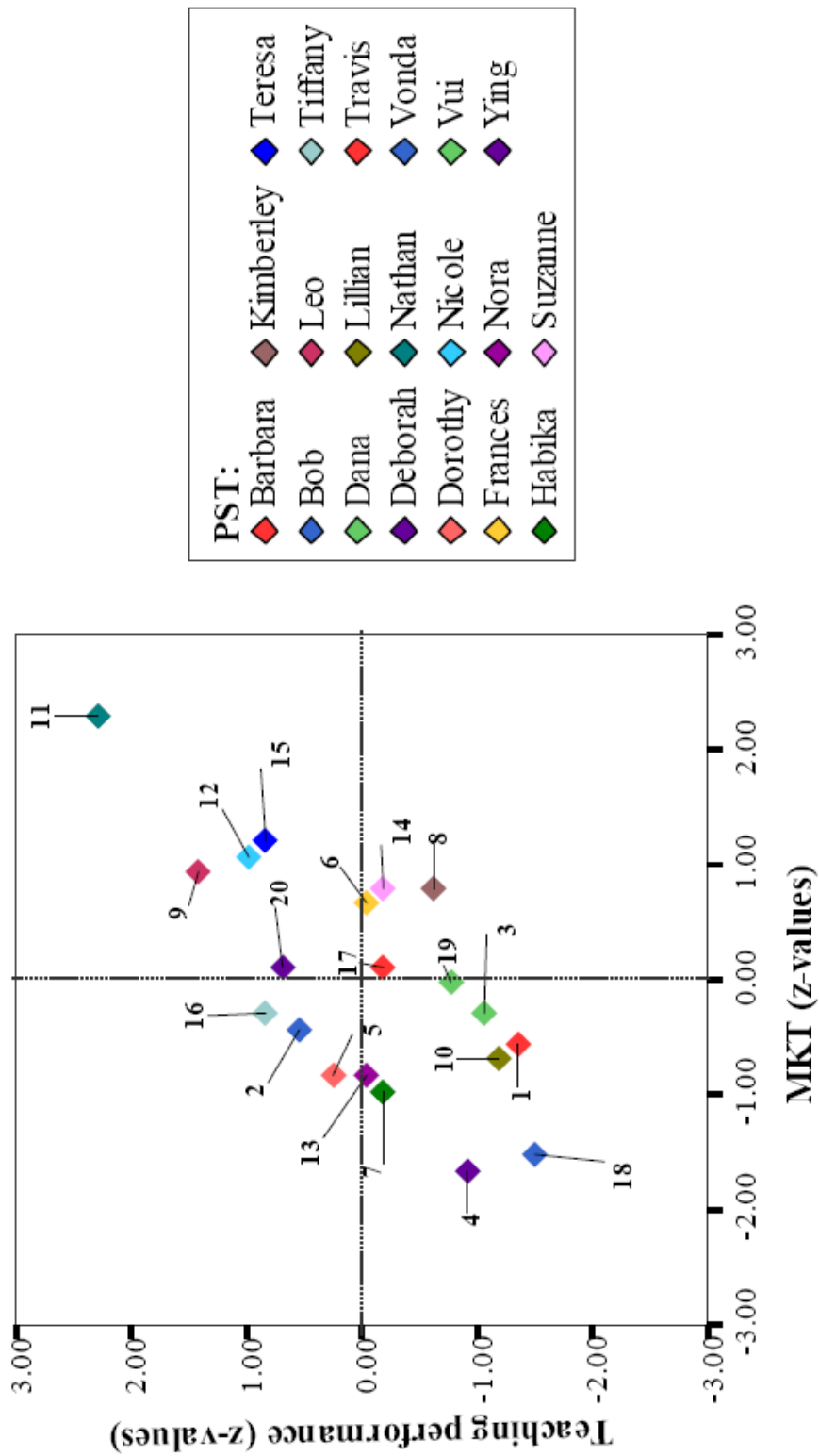


Figure 4.3. Scatterplot of the PSTs' entrance overall MKT score against their entrance overall teaching-performance score.

As reported above, four factors were found to mediate the association between the PSTs' MKT and their teaching performance: the number of math content courses the PSTs had taken during high school; the number of math methods courses they had taken during their undergraduate studies; the PSTs' perceived efficacy to understand and work on fractions; and their GRE-quantitative performance. As shown in Table 4.5, the seven cases selected spanned the range of the plausible values for each of those factors.<sup>133, 134</sup> In addition to including divergent and convergent cases and cases that represented a wide variety of the values of the mediating factors, I also decided to sample cases for which complete data sets (both pre-and post-intervention data) were available. This allowed using the same cases for addressing the two overarching research questions of the study. I further elaborate this issue in Chapter 5.

Table 4.5 presents the seven selected cases with respect to the criteria under consideration. I selected four convergent cases (Nathan, Nicole, Deborah, and Vonda) and three divergent cases (Kimberley, Suzanne, and Tiffany). I oversampled convergent cases to facilitate the identification of patterns and the development of explanations regarding the association at hand. I included one divergent case from the second quadrant and two cases from the fourth quadrant (see Figure 4.3); the oversampling from the latter quadrant was due to the fact that, although falling into the same quadrant, Kimberley and Suzanne exhibited notable differences in the factors under consideration. The criteria

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<sup>133</sup> The PSTs' entrance MKT scores and entrance teaching-performance score were categorized as low (L), medium low ( $M^-$ ), medium high ( $M^+$ ), and high (H), using the z-threshold of  $\pm 0.67$ . A similar transformation was applied to the mediating factor of perceived competence to work on fractions.

<sup>134</sup> Instead of grouping the study participants according to their standardized GRE scores, I calculated the difference between their standardized GRE score and their standardized entrance MKT overall score. This was done for two reasons. First, it was an additional step to protect the participants' anonymity; second, this difference was considered more informative for case selection. The values reported in Table 4.5 (category IV) correspond to the standardized values of this difference. These values were recoded as follows: negative high ( $N^-$ ):  $z \leq -0.67$ ; negative (N):  $-0.67 < z \leq 0$ ; positive (P):  $0 < z \leq 0.67$ ; positive high ( $P^+$ ):  $z > 0.67$ .

listed in the last five columns of this table are discussed in Chapter 5 when considering the case selection for addressing the second research question (since the same seven cases were used for addressing both research questions).

Having explained the selection of the seven cases, I now present each case in turn. I start with the four convergent cases – Nathan, Nicole, Deborah, and Vonda – and then move to the divergent cases – Tiffany, Kimberley, and Suzanne. In discussing the first case, the case of Nathan, I briefly outline what the PSTs were expected to do with respect to the three subcomponents of noticing, interpreting-evaluating, and performing for each teaching practice. Hence, although the cases may be read in non sequential order, it is recommended that the reader refer to the short introductory comments included at the beginning of each practice when presenting Nathan’s case.

Table 4.5

*Portrait of the Seven Cases Selected for In-Depth Exploration of the Two Research Questions of the Study*

Names	Criteria <sup>1</sup>										
	I <sup>2</sup>	II <sup>3</sup>	III <sup>3</sup>	IV <sup>4</sup>	V	VI	VII <sup>3</sup>	VIII <sup>4</sup>	IX <sup>4</sup>	X <sup>4</sup>	XI <sup>4</sup>
Nathan	C/C	H	H	P	6 or more	None	H	N <sup>-</sup>	N <sup>-</sup>	N <sup>-</sup>	N <sup>-</sup>
Nicole	C/C	H	H	N <sup>-</sup>	6 or more	Two	H	N	N <sup>-</sup>	N	N <sup>-</sup>
Deborah	C/C	L	L	P <sup>+</sup>	3 to 5	None	L	P <sup>+</sup>	P <sup>+</sup>	P <sup>+</sup>	P <sup>+</sup>
Vonda	C/D	L	L	NI <sup>5</sup>	1 to 2	None	L	N <sup>-</sup>	P	N <sup>-</sup>	P <sup>+</sup>
Tiffany	D/D	M <sup>-</sup>	H	P	3 to 5	None	L	P	N	P	N <sup>-</sup>
Kimberley	D/C	H	M <sup>-</sup>	P <sup>+</sup>	6 or more	None	H	P <sup>+</sup>	P	P	N
Suzanne	D/D	H	M <sup>-</sup>	N <sup>-</sup>	3 to 5	None	M <sup>-</sup>	N <sup>-</sup>	P <sup>+</sup>	P <sup>+</sup>	P

*Notes.*

<sup>1</sup> Criteria: **I**. Type of case (convergent/divergent); **II**. Entrance MKT; **III**. Entrance teaching performance; **IV**. Difference between standardized GRE score and standardized entrance MKT overall-score. **V**. Number of high-school math content courses taken; **VI**. Number of math methods courses taken; **VII**. Perceived efficacy to work on fractions; **VIII**. Changes in overall MKT-score; **IX**. Changes in teaching performance (overall); **X**. Changes in performance in the MKT-related practices; **XI**. Changes in performance in the MTF-related practices.

<sup>2</sup> C: convergent case; D: divergent case. The first letter reflects whether the case was convergent or divergent based on the *entrance* measures (i.e., first research question) and the second whether it was convergent or divergent based on the measures of change (i.e., second research question).

<sup>3</sup> L: Low; M<sup>-</sup>: Medium low; M<sup>+</sup>: Medium high; H: High.

<sup>4</sup> N<sup>-</sup>: Negative high; N: Negative; P: Positive; P<sup>+</sup>: Positive high. These values should be interpreted as *between-case* indices rather than as *within-case* indices. That means that a PST's score (or change thereof) was considered N<sup>-</sup>, N, P, P<sup>+</sup>, *relative* to the corresponding scores (or changes thereof) in the performance of the other study participants.

<sup>5</sup> NI: No available information.

### *The Case of Nathan: Attending to the Mathematical Motivations of Dividing Fractions*

Nathan, the first convergent case considered in this study, scored high on both the LMT test and the teaching simulation. His high scores on both performances rendered him an influential observation, thus he was excluded from subsequent analysis. This finding in itself made Nathan an interesting case to explore. An additional reason for scrutinizing his case was that he had not taken any math methods courses, yet he had taken many high-level math content courses, such as *Abstract Algebra*, *Chaos and Fractals*, and *Combinatorics*. His case helps illustrate the strong association between the PSTs' MKT and their performance on the teaching-simulation tasks, particularly the *performing* tasks. His case also provides insights into how PSTs' knowledge (or limitations thereof) might support them in (or constrain them from) attending to the meaning underlying the procedures considered during a math lesson.

#### *Background*

A son and grandson of educators, Nathan joined the program with no substantive teaching experience other than having spent a couple of weeks in summer schools. Teaching, however, “had always been an important value in [his] family.” This, coupled with his desire to offer students equal opportunities to learn, motivated him to join the ELMAC program (PE.I.82-86).<sup>135</sup> Based on his recollections, most of his elementary-grade math experiences were “basically memorizing and practicing facts and algorithms.” In middle school, however, he had the opportunity to delve deeper into the concepts,

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<sup>135</sup> I use the following notation to denote the data source: the first two letters (PE or PO) denote whether the data correspond to pre-intervention or post-intervention data sources; the third letter specifies the data source (I: interview, S: survey, T: LMT adapted test). For the interview data, I also specify the transcript's lines.



especially starting from pre-algebra; he also worked collaboratively with his classmates, an activity that helped him appreciate figuring out multiple solutions to a single problem.

### *Selecting and Using Tasks*

As discussed in Chapter 3, for the performing part of this practice, the study participants were asked to consider two textbook pages and design an introductory lesson on the division of fractions. For the noticing and the interpreting-evaluating parts, I explored whether the study participants attended to and discussed the virtual teacher's presentation and enactment of task A<sub>1</sub>, as well as her enactment of task D. These tasks, both drawn from the second textbook page, are presented in Figure 3.3. Task A<sub>1</sub> corresponds to a word problem solved by using division  $\frac{1}{2} \div \frac{1}{6}$ , while task D corresponds to figuring out an algorithm for the division of fractions.

*Performing.* Nathan first noticed that neither textbook page included any pictures; he considered pictures an important scaffold for student learning. He justified this idea by referring to a videoclip discussed during the second math content class.<sup>136</sup> He then argued that to design an introductory division-of-fractions lesson, the teacher needs to know students' pertinent prior concepts and skills. He identified the following as such prerequisites: knowing what the reciprocal and the divisor represent; understanding that multiplication and division are inverse operations; knowing that fractions are representations of division; and being able to multiply fractions and convert mixed numbers into improper fractions. He considered the first couple of ideas pivotal for proving why the algorithm of “invert and multiply” works. Building on these two ideas,

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<sup>136</sup> Nathan was interviewed after the second math content class. In the videoclip under discussion, different manipulatives and pictures were used to support students' problem-solving activities.

he proved this algorithm, an issue to which I return later when discussing his performance in providing explanations.

Nathan identified three distinct sections in the first textbook page:<sup>137</sup> the algorithm section, the section including the exercises 1-16,<sup>138</sup> and the section consisting of the two word problems. He argued that the algorithm was presented very abstractly and without any explanations as to why it works; in his words, it had no “conceptual foundation” and mainly showed that the division of fractions is simply an issue of “rearranging numbers” (PE.I. 179-181). If he were to use this algorithm, he would ground it in the idea that multiplication and division are inverse operations.

Nathan maintained that assigning all the exercises included on the page’s second section would be cumbersome:

If the student gets the first couple [of exercises] right ... are sixteen [exercises] really necessary? And if they get the first one wrong because they don’t understand it, then that also makes me wonder if sixteen [exercises] are really necessary. (PE.I. 354-359)

When asked to identify which of those exercises he would assign, he remarked that some of them lend themselves to understanding the division of fractions more conceptually. He noted, for example, that one could solve exercises 1 and 8 “on one’s head” because in both exercises, the dividend and the divisor are in thirds. For him, exercise 8 (i.e.,  $\frac{5}{3} \div \frac{1}{3}$ ) meant “five divided by one in units of thirds” (PE.I. 345-347). Nathan also considered the two aforesaid exercises the easiest to solve using a drawing. He considered exercises in which the divisor is larger than the dividend harder to draw and less conducive to helping students grasp the division of fractions conceptually. As he remarked, if the students were to think about division as making bows of a given amount of ribbon, they would have

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<sup>137</sup> I refer to the procedurally oriented textbook page shown in Figure 3.2 as the *first page* and to the conceptually oriented page shown in Figure 3.3 as the *second page*.

<sup>138</sup> I distinguish between the numeric problems (1-16) and the word problems (17-18) included in this page by referring to the former as *exercises* and to the latter as *word problems*.

difficulties understanding exercise 11 (i.e.,  $\frac{1}{2} \div \frac{7}{8}$ ) since in this exercise “you can only make a fraction of a bow” (PE.I. 784-785). He also claimed that exercises in which the dividend and divisor are not in common denominators lend themselves better to applying the algorithm proposed on this page. Nathan also considered using some of these exercises to organize a problem-posing activity. He explained,

[T]hat might help me understand if they really know what a practical application of this problem would be; if they can do it. Starting with the word problem, going into the equation, getting the answer, or doing it in the opposite direction: starting with the equation, getting the word problem, and then the answer, might just show how fluent they are with the idea. (PE.I. 383-389)

To help students understand the applications of the division of fractions, he would start with the two word problems that appear in what he considered the third section of this page. Some students, he argued, might approach these problems using multiplication whereas others might use division. Not only did he consider both approaches appropriate, but he argued that having students approach these problems in both ways could help reinforce the idea of multiplication and division being inverse operations. He noticed, however, that these two problems lend themselves more to multiplication than division. In contrast, he considered the second page’s “ribbon problems” more suitable for teaching the division of fractions, since “one can think of how many things can fit into something else, and that concept is just how I think of division” (PE.I. 451-452). Drawing on discussions during his first *Social Studies* class and on his reading of Liping Ma’s Chapter 1,<sup>139</sup> he argued that if he were to start with a word problem, he would not walk the students through it; instead, he would ask the students to solve it and then discuss the different solution strategies they proposed.

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<sup>139</sup> As Figure A1 shows (see Appendix 1), this was an optional reading assigned to the study participants before the commencement of the first math content class.

Nathan preferred the second page to the first one, for two reasons. First, he thought that he could use the second page to teach the meaning of the division of fractions by having students cut strips of given lengths from a given strip of ribbon. Second, he preferred the second page's introduction of the pertinent algorithm over that of the first page. He justified his argument as follows:

[J]ust giving the algorithm first without any -- there is no reason for it to work. So why would a kid first of all pay attention to it? And, then, it creates an idea that math is arbitrary when it's really the opposite. I don't think it's a good idea. (P.E.I. 889-894)

For Nathan, the algorithm needed to not only make sense but also to help students perceive mathematics as concrete and sensible. Hence, he thought that presenting the algorithm first might not engage students since the procedure under consideration might not make sense to them. In contrast, asking students to derive the algorithm after having worked on related problems – an approach supported by the second page – resonated with Nathan's idea that students should not mindlessly apply a given algorithm.

Considering both pages in tandem, Nathan concluded that if he were to design an introductory lesson on the division of fractions, he would not use either page as is. In contrast, he would draw on the affordances of each page to help students understand this process. Prompted to explain how this lesson would look, he clarified that he would start with the second page's "ribbon" problems to help students begin thinking about division. After having students explore these problems, he would then engage them in a discussion of the algorithm of dividing fractions, during which he would explain that division and multiplication are inverse operations.

In short, Nathan evaluated the two pages by using his own understanding of division in general and the division of fractions in particular. He noticed that by being grounded in the idea of fitting divisor units into the dividend, the second page lends itself

better to designing a lesson on the division of fractions. As discussed in Chapter 2, this idea corresponds to the measurement interpretation of division, which is more appropriate for teaching the division of fractions. This idea also helped Nathan see some of the exercises of the first page through a more conceptual lens: he considered exercises with same-denominator fractions easier because they correspond to the same unit. On the contrary, he considered the exercises in which the dividend is smaller than the divisor harder because one can only fit a fractional part of the divisor into the dividend.

In structuring a lesson on the division of fractions, Nathan appeared to also draw on other resources besides his knowledge. These included discussions he attended during the first classes of the ELMAC program as well as pertinent readings. His performance on selecting and using tasks also appeared to have been informed by his beliefs about mathematics and its teaching, especially that mathematics is not arbitrary and that students need to see the applicability and the reasoning behind certain procedures instead of mindlessly following them.

*Noticing and Interpreting-Evaluating.* In considering the virtual teacher's presentation of task  $A_1$ , Nathan wondered if it would have been better had the teacher stopped after she introduced the task and checked whether students really understood what the problem was asking them to do. This, he explained, would have ensured that the students did not misunderstand the problem. Nathan was also concerned with the enactment of this task because, as he argued, more emphasis was placed on getting correct answers than on developing understanding. He explained that, in working with her students, the virtual teacher did not provide any mathematical motivation for justifying her suggestions:

So [Ms. Rebecca's] not really referring to any kind of mathematical motivation for dividing up the entire line into six pieces instead of just the half. She's just saying, "Don't do that because it is not correct." And so, how are the kids supposed to know the next time a similar situation comes up or an analogous situation? How are they supposed to know to apply this? I don't think they really will be able to. (P.E.I. 1236-1241)

He also observed that two of the teacher's moves while solving this task – using a representation and summarizing the work on this task – could have potentially supported students' understanding, if enacted appropriately:

When she [Ms. Rebecca] summarizes the activity, the students repeated exactly what she said in the first part, which was "let's draw a line; draw a line, and use the whole line".... So if that is all [a] student remembers, "use a line," then it might not really be too helpful. (P.E.I. 1257-1263)

Nathan also criticized the virtual teacher's enactment of task D. He claimed that the teacher did not actually afford students the opportunity to find a pattern. "She just kind of gave them the answer. ... They didn't even know what a reciprocal was. So how could they have discovered that pattern?" (P.E.I. 1504-1507). From his perspective, the teacher derived the algorithm for her students and did so rather awkwardly. In addition, she appeared to magically transform the numbers to get the algorithm:

So it looks like a lot of trickery there probably to some of those students. Like she's just magically transformed [the numbers in the equation used to derive the algorithm]. ... I bet it still seems random to some of those kids who were confused.... So why did they get this? She just said ... "We don't have a six here," and then she switched gears and said, "Well imagine that we do have a six." (P.E.I. 1474-1488)

To summarize, Nathan discussed the virtual teacher's presentation and enactment of the two tasks through the spectrum that mathematics needs to make sense to students. Being able to closely follow and analyze each step of the teacher's work in deriving the algorithm, he observed that the teacher did not actually scaffold students' thinking. Instead of attending to whether students construed *what* was done, let alone understand *why* something was done, the teacher, as Nathan contended, placed more emphasis on getting correct answers and showing that something works.

### *Providing Explanations*

As discussed in Chapter 3, for the performing tasks of this practice, the study participants were expected to explain why the division-of-fractions algorithm works by referring to the worked example provided in the first page (i.e.,  $2 \div \frac{3}{4} = 2 \times \frac{4}{3} = 2\frac{2}{3}$ ). Specifically, they were required to provide two explanations: one for the quotient of this division and another for the reciprocal and the use of multiplication when dividing fractions. For the noticing and the interpreting-evaluating tasks, the study participants were expected to attend to several deficiencies in the teacher's explanation for the reciprocal. For the noticing tasks, they were additionally expected to attend to how the teacher handled Robert's question. Given the discrepancy he identified between his answer and Amanda and Julia's answer to the division problem  $\frac{3}{4} \div \frac{1}{6}$ , Robert asked if the "reciprocal rule works all the time".

*Performing.* Nathan characterized the first page's presentation of the division-of-fractions algorithm as arbitrary; he argued that when using this page, the teacher should explain the mathematical motivation behind this algorithm. Provided that his students had some basic algebraic background, he would help them see that the algorithm works by "proving it algebraically." Drawing on the idea that multiplication and division are inverse operations and that fractions represent divisions, he proved the algorithm as illustrated in Figure 4.4.

Specifically, after writing the equation shown on the first line of this figure, he argued that because multiplication and division are inverse operations, this equation could be transformed by multiplying the term  $c/d$  by the unknown quantity on the right-hand side of the equation (see second row of Figure 4.4). He justified the transformations

he pursued in the third, fourth, and fifth rows of Figure 4.4 by pointing out that fractions actually represent divisions. The final transformation presented on the last row of this figure showed, according to Nathan, why the algorithm works. In hindsight, however, he remarked that he could have moved directly from the second step to the last step by multiplying both parts of the equation by the fraction  $c/d$ .

The image shows a handwritten sequence of mathematical steps:
   
1.  $\frac{a}{b} = x$ 
  
2.  $a = x \cdot b$ 
  
3.  $1 = x \cdot \frac{b}{a}$ 
  
4.  $\frac{a}{b} \cdot \frac{c}{d} = x \cdot \frac{c}{d}$ 
  
5.  $\frac{a}{b} \cdot \frac{c}{d} = x$

Figure 4.4. Nathan’s proof of the invert-and-multiply algorithm.

Although Nathan attempted to give meaning and justify the steps he followed in the abovementioned proof, his justification of why the algorithm works included several gaps. First, he appeared to take for granted that  $x$  represents the unknown quantity without explaining this to students. Second, the idea that multiplication and division are inverse operations does not suffice to justify why one can move from the first step of his proof to the second step. A better justification would have been to also point out that when one multiplies both terms of the equation with the same number (in this case with the term  $c/d$ ), the equation still holds. Third, the idea that multiplication and division are inverse operations needed more unpacking; Nathan should have pointed out that when a given number is multiplied and divided by the same number, its value remains unchanged. Fourth, similar justifications were also warranted for justifying the



intermediate steps between the second and the last step of this equation. Finally, Nathan appeared to take for granted that, because the right-hand side of the first and the last step of his proof were both equal to  $X$ , they were also equal to each other. However, he needed to make this idea explicit.

In reconsidering his algebraic explanation, Nathan pointed out that it might not be suitable for sixth graders, especially if the students did not have any exposure to algebraic concepts. Thus, he wondered if using a drawing would be a better way to explain this algorithm. Although initially reluctant to offer an explanation using such a drawing, because he was not sure if he conceptually understood the reciprocal, he eventually attempted a pictorial explanation for the division  $2 \div \frac{3}{4}$  (see Figure 4.5).

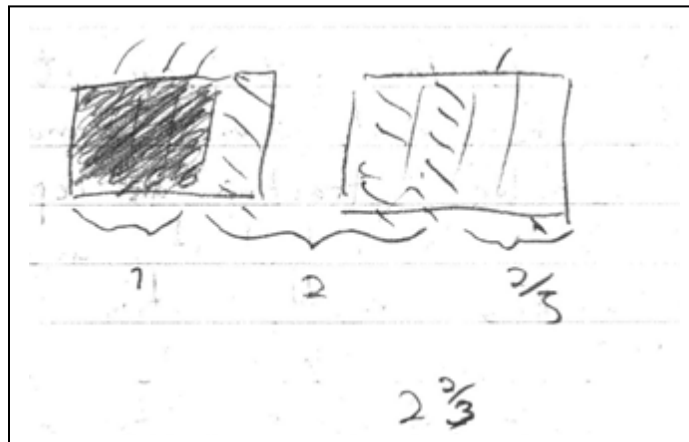


Figure 4.5. Nathan’s illustration of the division problem  $2 \div \frac{3}{4}$ .

In explaining this division, he first clarified that division meant “how many three fourths one can fit into two”; thus he drew two rectangles to represent the dividend. He then divided the two rectangles into fourths and shaded in two sets of three fourths, as illustrated in Figure 4.5; he denoted these two sets of three fourths with the numbers 1 and 2. Next, he remarked that a two-fourths piece is left over. However, as he pointed out, “If there’s [sic] three in a group, that [pointing to the leftover fourths] is two thirds ...

so two thirds of three fourths fit inside these [again pointing to the leftover fourths]” (PE.I. 586-588).

Using the same diagram, Nathan then attempted to explain the reciprocal. He observed that the first rectangle also represents four thirds; however, he was quite unsure about the extent to which this four thirds was actually the reciprocal of three fourths:

I mean it’s a representation of ... four thirds, but I’m not sure if it’s the four thirds that is the reciprocal in this problem, you know what I mean? But, anyway ... instead of considering all four of those [points to the four fourths in the first rectangle] as one, and then broken up into fourths, you could think of these first three fourths [points to the  $\frac{3}{4}$ -portion shaded in black], you could consider them as one instead. Then you’ve got one, and then another little piece which would be one third so that would be four thirds right there [in the first rectangle]. (PE.I. 604-610; emphasis in the original)

Yet, Nathan was concerned that his explanation did not account for the multiplication  $2 \times \frac{4}{3}$ . Considering multiplication mainly from an array perspective, (i.e., “three by two ... is three across and two down, or two down and three across,” PE.I. 635-637), he struggled to explain what the  $2 \times \frac{4}{3}$  corresponded to. Eventually, drawing on his work in Figure 4.5, he revised his thinking and identified the two four thirds as following: the first comprising the portion of the first rectangle shaded in black and one of the remaining fourths of the leftover part in the second rectangle; the second consisting of the remaining parts of the first and the second rectangles.

Both Nathan’s explanation and proof were driven by his conceptual understanding of the content or lack thereof. Consider, for instance, how he started his first explanation by pointing out that to him, division meant fitting divisor-units into the dividend. This idea was also crucial to explaining the fractional part of the quotient of the division at hand. Regarding his explanation for the reciprocal, although he correctly speculated that there are four-thirds divisor units in the first rectangle, he could not build on this idea to offer a justifiable explanation; this was because his grasp of the notion of the reciprocal

was not solid, as he himself admitted. His difficulties in offering such an explanation were accentuated by his viewing of multiplication as an array model. Hence, although he correctly speculated what the reciprocal means, his final explanation did not reflect the idea of the reciprocal being the number of relative units to be fit in *every* dividend unit.

I call his first explanation (for the quotient) a *conceptually driven* explanation. I pit such explanations against what I call *numerically driven* explanations, which lack any conceptual foundations and are mainly informed by the numbers involved in an algorithm. Typical examples of such explanations are discussed when considering the remaining cases. I do not characterize Nathan's second explanation as a typical numerically driven explanation because it did have some glimmers of the idea of the reciprocal. However, on a continuum from conceptually driven to numerically driven explanations, Nathan's second explanation was more numerically driven.

As already explained, his proof was also lacking several steps necessary to make it comprehensible to a sixth grader. His explanation of the quotient also needed more unpacking, especially with respect to why the remaining part can be considered two thirds of a  $\frac{3}{4}$ -unit. Nathan was correct in pointing out that "if there is three in a group, [the left over part] is two thirds." Yet, he needed to explain why this was the case; to do so, he could have drawn, for example, on the part-whole concept of fractions. In Chapter 5, I take up this issue again when considering his explanations for the quotient of the same division problem.

*Noticing and Interpreting-Evaluating.* Without any prompting, Nathan noticed that the teacher's explanation of the reciprocal was problematic for three reasons. First, it did not explain why the "invert-and-multiply" rule works. Second, as he argued, she used

semantic and linguistic arguments rather than mathematical arguments. Third, her analogy of subtracting a positive number instead of adding its negative could hardly be connected to the idea of the reciprocals in divisions of fractions:

She's saying, "We invert the second fraction, and *because* we are using the reciprocal, we should use a reciprocal operation, which is multiplication." That's really using a semantic argument. Reciprocal and reciprocal? That's not really a mathematical argument. ... So now she's ... saying negative numbers and subtraction, but ... I guess that's not really the same as talking about addition and subtraction as inverse. Actually this might not have anything to do at all with what she's trying to say. (PE.I. 1548-1561; emphasis in the original)

Nathan also disapproved of how the teacher handled Robert's question. He was particularly concerned with the teacher's argument that "if the invert-and-multiply rule did not work, we would not call it a rule." To Nathan, this argument was again a semantic and linguistic argument rather than a mathematical one.

### *Using Representations*

For the performing part of this practice, I explored the representations that the PSTs used in their explanations considered in the previous practice; if they initially refrained from using representations in their explanations, I prompted them to do so. For the noticing and interpreting-evaluating part of this practice, the study participants were expected to comment on two episodes. In the first episode, Amanda comes to the board and writes the mathematical sentence for division  $\frac{1}{2} \div \frac{1}{6} = 3$  without making any connections to the drawing shown on the board or the word problem under consideration. In the second episode, Amanda and Julia use a linear representation to show how they solved the division problem  $\frac{3}{4} \div \frac{1}{6}$ . They split their line into twelve parts, color nine of them red, identify four groups of two red pieces each, and, given that one red piece is left over, conclude that the answer is "4 and remainder  $\frac{1}{12}$ ," which they write on the board. The teacher does not press the two girls to draw connections among (a) their

representation, (b) the mathematical sentence they came up with, and (c) the ribbon problem they were asked to solve.

*Performing.* Nathan's performance in using representations has already been considered when discussing his explanations for the division of  $2 \div \frac{3}{4}$ , but three additional remarks are in order. First, he chose a suitable representation for explaining the division of fractions (i.e., a rectangular representation, see Figure 4.5) and used it appropriately, by first identifying that the dividend units should be partitioned in portions equal to the divisor. Furthermore, he labeled the different  $\frac{3}{4}$ -portions he was referring to by using the numbers 1, 2, and the fraction  $\frac{2}{3}$  to denote that the last part was two thirds of a three-fourths piece. Thus, like his explanation, his use of representations can also be characterized as conceptually driven. By this I do not imply that his use of representations could not be further improved since his demarcation of the last piece as " $\frac{2}{3}$ " could be potentially confusing to students, given that in actuality this leftover piece represents two fourths dividend units.

Second, Nathan's representation focused on the structural instead of the superficial characteristics of the mathematical situation at hand (cf. Diezmann & English, 2001): he clearly presented the dividend units in two rectangles and he then showed the divisor units by identifying portions of three fourths *within* the dividend units. As will become clear when considering other cases (e.g., Deborah), Nathan's use of representation was in sharp contrast to an approach that focuses more on the superficial characteristics of a division-of-fractions situation.

Third, even if Nathan did not provide an appropriate explanation of the reciprocal, he used the representation he drew quite decently. For instance, in contrast to the other

PSTs considered in this chapter, Nathan insisted on using the representation he had originally drawn, exploring whether there was a way to perceive his drawing from a different angle: rather than focusing on the palpable dividend units (i.e., two rectangles), he tried to focus on the less visible divisor units.

*Noticing and Interpreting-Evaluating.* In considering Amanda's episode, Nathan noticed that she neither showed her work nor she explained how she got her answer. He argued that such an explanation might have supported her classmates' understanding. Despite the legitimacy of his claims, his arguments did not clearly address more substantive issues, and particularly the lack of the three types of connections identified in the introductory paragraph describing the tasks designed for this practice. Nathan initially made analogous general remarks when considering Amanda and Julia's work in the second episode. Yet, building on a discussion that transpired in the second math content class, he remarked,

Oh! One thing that [the instructor] say[s] ... all the time is how to map the graphical representation to the equation, which is what they have not done. ... So, they're explaining the diagram as they draw, but they don't really explain how that ... maps to the equation. (PE.I. 1413-1420)

In short, although Nathan noticed that in both episodes the students did not explain their work, he could not specify what the problematic aspects of the students' work were in terms of their use of representations. The second class discussions appeared to have offered him a lens through which to start looking at students' work and the language necessary to talk about how their work could have been improved. However, his comments were still generic and needed more unpacking.

### *Analyzing Students' Work and Contributions*

For this practice, the PSTs were asked to analyze three student solutions to the problem  $2 \frac{3}{4} \div \frac{3}{4}$  (see Figure 3.5). They were also required to make assertions about the three students' understanding or lack thereof. For the noticing and interpreting-evaluating tasks of this practice, the PSTs were expected to attend to four students' contributions that included debatable ideas or errors: June's explanation as to why one needs to divide one half by one sixth and not vice versa (when solving task A<sub>1</sub>); Alan's erroneous partitioning of half of the line instead of the whole line into six parts; Ann's argument that in division-of-fractions one needs to examine how many times the smaller part goes into the larger part; and Amanda and Julia's answer "4 remainder  $\frac{1}{12}$ " to the division problem  $\frac{3}{4} \div \frac{1}{6}$ . As discussed in Chapter 3, June's and Ann's arguments reflect a common misconception that the dividend is always larger than the divisor; June's explanation was also disconnected from the context of the problem considered in task A<sub>1</sub>. Alan's error, on the other hand, reflects a misunderstanding of what the unit is in division-of-fractions problems. Although the problem asked for dividing half a yard into pieces of one-sixth yards, Alan needed to divide the whole yard into six pieces in order to represent the sixths. By dividing half the yard into six pieces, he was dividing one half *into six* instead of one half *by one sixth*. Finally, Amanda and Julia's answer confounded the dividend (i.e., absolute) units with the divisor (i.e., relative) units.

*Performing.* Instead of first solving the problem to figure out whether the three student solutions to this problem were correct, Nathan tried to follow the students' thinking. In considering Robert's work, he first followed each of the steps of his solution, which allowed him ascertain its correctness. He noticed, however, that Robert might still

have not understood why the algorithm works or if it works all the time. As he observed, it was Robert who asked the teacher whether the algorithm works all the time, a question that the teacher postponed until the next day. Thus, Nathan asserted that Robert just followed the rule presented in the lesson without really having understood it conceptually.

In considering Michelle's work, Nathan first mapped the problem's numbers into her representation: he noticed that the dividend  $2\frac{3}{4}$  was represented by what Michelle colored in red. Building on his knowledge of division, he then observed that Michelle was using bracketed lines to denote the pieces of the divisor that she could take away from the dividend. He concluded that her work was correct and that she applied the approach the teacher had illustrated earlier in the lesson. He therefore asserted that Michelle seemed to have understood the graphical approach of solving division problems discussed during the lesson; however, the fact that she did not write any numerical sentence to represent her work led him to question whether Michelle really understood fraction division and especially the quotient in such divisions.

Finally, Nathan noticed that Ann's work was incorrect because she took the reciprocal of part of the dividend and not the divisor. He concluded that Ann seemed to have understood that she needed to take the reciprocal of "something" and multiply it, but she did not know what to take the reciprocal of. Nathan considered the instruction as partly accountable for Ann's confusion, on the premise that during the lesson both the algorithm and the notion of the reciprocal were presented very mechanically. He also observed that during the lesson, the students were not asked to work on divisions in



which the dividend was a mixed number; this, he asserted, might have also caused Ann's confusion and error.

Nathan's performance in analyzing the three student solutions was notable in many respects. To start, instead of solving the problem and using his answer as a yardstick against which to gauge the correctness of the three solutions, he preferred to analyze the three solutions, attempting to figure out if there were any missteps in them. Thus, I call his approach *non answer driven*. Second, he was quick to identify the correctness of these solutions and their errors. Plausibly drawing on his knowledge of division as "fitting divisor units into the dividend," he was also capable of following Michelle's solution and he appropriately concluded that, although correct, this solution did not reveal much about Michelle's understanding. Third, while making assertions about each student's understanding or lack thereof, Nathan also considered student solutions in the context of the instruction given. Thus he concluded that Robert's work could not reveal much about this student's conceptual understanding of the division of fractions, especially since Robert was the student who posed the question about the algorithm's applicability. Similarly, in considering Ann's work, Nathan entertained the idea that her errors reflected the insufficient ways in which the algorithm was introduced and worked on during the lesson.

*Noticing and Interpreting-Evaluating.* Nathan identified all but Ann's problematic contributions without any prompting. Right after reading June's explanation, he remarked that although correct, June's contribution was not sufficient as an explanation. As he noticed, the fact that one half is larger than one sixth was an inadequate argument for justifying why one half was the dividend and one sixth the divisor; additionally, he

observed that this explanation was not connected to the ribbon problem. He was also quick to capture the error in Alan's work. In fact, while reading the slide in which the teacher suggested that Alan "divide the line into six parts," he speculated that Alan would probably divide half the line instead of the whole line, partly because of the teacher's vague prompt. Noticing that Alan drew "one sixth of one half," he argued that this student might have not committed this error had the teacher used two separate lines to show the dividend and the divisor.

Nathan was also quick in identifying Amanda and Julia's error:

So, four and one twelfth is left. Well, one twelfth *is* left, but it's one twelfth of the yard, which is one half of the sixth yard that they are dividing by. ... So, [the] remainder [of] one twelfth should be one half of the one sixth, right? And not the one twelfth of the whole thing. It seems like that remainder is not, might not be correct. (PE.I. 1332-1339)

To validate his argument, he solved the problem using the algorithm, which helped him confirm that the answer to the problem should have been  $4\frac{1}{2}$  instead of  $4\text{ and } \frac{1}{2}$ . He then explored whether the girls' solution could somehow be correct, namely whether it was reasonable to argue that the remainder was one twelfth. Admitting that he had not used remainders for long and that he had never actually seen someone using a remainder in the context of fractions, he drew on his knowledge to reason through the girls' argument:

[W]hen I think of a remainder it's got to be less than the divisor. If I divide something by three, I should come up with a remainder of zero, or one, or two, for instance. So I guess it would make sense if you are dividing by one sixth, you could get a remainder of one twelfth. But, you wouldn't get a remainder of one half." (PE.I. 1397-1402)

In this statement, Nathan resorted to the *concept* of the remainder to analyze the students' work. This concept gave him a yardstick against which to explore the appropriateness of the girls' work. Because he did not have any experience in working with remainders in the set of rational numbers, he also considered a simple example from the set of whole numbers. Therefore, lacking pertinent knowledge that would allow for direct evaluation

of the students' work and thinking, Nathan drew on two other equally important resources: his conceptual understanding of remainders and the heuristic of using an easier example (cf. Polya, 1957).

To summarize, Nathan was not only quick in noticing the problematic aspects in the students' contributions, but he also considered instruction as a plausible source of students' errors, which accords with how he analyzed the students' work in the performing tasks. Moreover, his assessment of Amanda and Julia's solution is suggestive of his reasoning when his knowledge and experiences could not offer him direct scaffolds for exploring the appropriateness of a mathematical argument.

#### *Responding to Students' Direct or Indirect Requests for Help*

As discussed in Chapter 3, for the performing part of this practice, the PSTs were asked to outline how they would react to two teaching episodes, the first pertaining to supporting Alan and the second related to supporting June and Shaun. In the first episode, Alan works on dividing a linear representation into sixths. Not only does he seem confused (which can be perceived as an indirect request for help) but he also splits half the line, instead of the whole line, into six parts. The PSTs were expected to explain how they would support Alan in correcting this error. In the second episode, while working on the division problem  $\frac{3}{4} \div \frac{1}{6}$ , June and Shaun encounter a momentary impasse since the sixths and the fourths in their drawings do not match up. Hence, they directly solicit the teacher's help. The study participants were asked to describe how they would support June and Shaun if they were teaching the lesson. For the noticing and interpreting-evaluating tasks of this practice, the PSTs were expected to comment on how the virtual teacher reacted to the aforesaid requests for help. In particular, they were expected to

notice that the virtual teacher places more emphasis on procedures instead of supporting her students' understanding (e.g., in the second episode she suggests that students use the common multiple of twelve without explaining why this idea is applicable or how it can be used to solve the problem). Additionally, she appears to do most of the thinking for her students by outlining steps for them to follow and by posing leading questions.

*Performing.* Nathan quickly noticed that Alan divided half the line into six parts, which yielded twelfths instead of sixths. To support Alan in correcting his error, Nathan would first ask him to describe what he did. If that would not help, he would ask Alan to consider whether the pieces that resulted from how he partitioned the line represented sixths of the *entire* line. If Alan continued being confused, Nathan would ask him to draw a different line and show the sixths on that line. He alternatively thought asking Alan and his classmates to use the approach earlier displayed by June, namely to draw two different lines, one to show the half yard (i.e., the available amount of ribbon) and a second one to show the sixth of a yard (i.e., the length needed to make a bow). He would then ask the students to compare the two lines to figure out how many bows they could make out of the given amount of ribbon.

In considering the episode with June and Shaun, Nathan disliked how the teacher supported these two students. He argued that if he were teaching the lesson, he would not simply give students the hint to use the common multiple of twelve but would explain why it made sense to use this common multiple in solving this particular problem. Specifically, he would first discuss the difficulty that the two students encountered, namely that fourths and sixths are not commensurate. He would then organize a discussion on why the common multiple of twelfths is handy. He mentioned,

So, maybe if ... we could talk about how you could compare two fractions that have different denominators, then they might be able to come up with that idea on their own and they'll have some mathematical reasoning behind it. (PE.I 1290-1293)

Nathan envisioned this approach as helping students see meaning in what they were told to do. He also proposed an alternative plan: instead of having students deal with the common multiple of twelve, the students “could just measure and see how many of the sixths could fit into the three fourths” (PE.I. 1303-1304).

In both episodes, Nathan proposed mathematically sound approaches that could support students' understanding of the mathematical ideas at stake. For Nathan, it was also important that students see the mathematical reasoning behind their work rather than simply follow a set of steps and get the right answer. Additionally, Nathan was flexible enough to propose alternative plans in case his original plans were not particularly effective; in proposing these plans he was also seen building on some of the students' contributions in earlier teaching episodes.

*Noticing and Interpreting-Evaluating.* In considering how the teacher supported the students in both episodes, Nathan argued that the teacher provided them with hints without ensuring that they understood what they were doing. For instance, while discussing Alan's episode, he commented that Alan seemed to be “using the [teacher] cues ... without necessarily thinking about the mathematics of what he's doing” (PE.I. 1166-1167). He also thought that the teacher was more concerned with students' getting correct answers rather than developing understanding of the ideas discussed. However, he did not notice the ways in which the teacher's approach impaired the cognitive level at which the content was considered. For instance, as soon as the teacher noticed that two students faced difficulties, she stopped the whole class and gave them a hint as to how they could solve the problem; that hint might have not been necessary for some students.

In addition, in supporting Alan, the teacher gave away the answer to her question by her intonation (i.e., “Is it *only* the half line that you should divide into sixths?”).

### *Performance on the LMT Test: A Closer Look*

Nathan’s performance on the LMT test was, to a large extent, consonant with his performance on the teaching simulation. He correctly answered all but two of the test questions. The first question pertained to a non-conventional algorithm for dividing fractions (question 10, see Appendix 2); the second question pertained to identifying two problems reflecting the measurement interpretation of division (question 25). Given his performance on the teaching simulation, one would expect that he would have also answered the latter question correctly. Nevertheless, he did answer correctly all the other questions related to the mathematical ideas explored in the simulation.

### *Analytical Commentary*

Nathan’s performance as discussed above substantiates the strong association between the PSTs’ MKT and their teaching performance yielded from the quantitative analysis. Additionally, because of his strong knowledge, Nathan attended to the *concepts* involved in the operation of dividing fractions rather than getting carried away by the numbers involved in this operation. His attention to the pertinent concepts was manifested in a multitude of ways in his performance in all five teaching practices explored in this study. That said, there were limits in what Nathan’s strong knowledge supported him in doing, especially when it came to unpacking the ideas he was explaining to make them comprehensible to the intended audience of sixth graders. Below I explore how the case of Nathan supports the strong association between knowledge and teaching performance yielded from the quantitative analysis. I also use

his case to identify certain ways in which strong knowledge supports PSTs in structuring rich and challenging learning environments, by *attending to the meaning* of the operation(s) at hand. Following the structure of Figure 4.6, which outlines Nathan's profile,<sup>140</sup> I first consider Nathan's performance in the two MTF-related practices and then turn to his performance in the three MKT-related practices. I conclude with discussing other factors that appear to have informed his teaching performance.

Nathan's analysis of the two textbook pages was driven by the core concept of division of fractions from a measurement perspective (i.e., exploring how many divisor units can be fit in the given dividend units). He also drew on other supporting concepts such as that division and multiplication are inverse operations and that fractions represent divisions. This allowed Nathan to consider and evaluate the textbook tasks from a conceptual perspective. For example, he noticed that exercises in which the dividend is smaller than the divisor do not easily lend themselves to exploring the division of fraction from a measurement standpoint. Similarly, he maintained that the second page's word problems are more conducive to teaching the division of fractions than the first page's approach. His conceptual lens also enabled him to explore the potential of certain tasks to reinforce particular mathematical ideas. For instance, in considering the first page's two word problems, he remarked that he could let students solve these problems in different ways; he could then ask students to share their different solutions to reinforce the idea that multiplication and division are inverse operations. In sum, Nathan's performance in analyzing the first textbook page echoes what Ball and colleagues (Ball, Bass, & Hill,

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<sup>140</sup> The reader will notice that in this figure, and the others used to sketch the portrait of the remaining six cases, I refer to the *mathematical knowledge of the content and its teaching* instead of referring to MKT. I do so, because I consider the PSTs' knowledge not only as suggested by their performance on the LMT test, but also as implied by their performance in the teaching simulation itself.

2004) called *restructuring tasks*, namely a teacher's ability to scale up easy tasks (such as those included in the first page) or to scale down difficult tasks. More so, his performance suggests that a strong understanding of the content supports the teacher in restructuring tasks to address their limitations and to create more opportunities for students to focus on the meaning of the operations at hand rather than mindlessly carry out given algorithms.

Nathan's knowledge also supported him in responding to students' requests for help in mathematically sound ways. His responses also had the potential to maintain emphasis on conceptual understanding rather than shifting emphasis to a set of meaningless steps to follow. Nathan was also seen "hearing" students' contributions and offering alternative action plans, often drawing on these contributions. Thus, Nathan's case suggests that with a strong conceptual understanding of the content, a teacher can respond to students' requests in a manner that is mathematically valid *and* supports emphasis on meaning and understanding.

Nathan's performance in providing explanations demonstrates another way in which the strength of PSTs' knowledge can support or limit them in attending to the concepts underlying the operations to be taught. When asked to provide an explanation, instead of *describing* the steps involved in the algorithm under consideration (which was the case with other PSTs with a weaker MKT, as discussed next), he offered a *conceptually* driven explanation for both the whole-number part and the fractional part of the quotient. Yet, because of limitations in his conceptual understanding of the notion of the reciprocal, his respective explanation was more numerically driven compared to his explanation of the quotient; thus, Nathan represents a convergent case both from an affordance and a constraint perspective, in that he shows a strong association between a



teacher's conceptual understanding of the content and this teacher's capacity to provide conceptually driven explanations of the content at hand. Yet, the case of Nathan also suggests that a strong understanding of the content does not suffice to provide sufficiently unpacked explanations, since his explanations (along with his proof) warranted further unpacking to be comprehensible to sixth graders.

Nathan was also seen carefully following and analyzing the mathematical quality of the virtual teacher's explanations. This close analysis helped him distill the main deficiencies of this teacher's explanation. I argue that Nathan would have not been able to do so without a deep knowledge of the content. This knowledge seemed to have enabled him to distill the mathematical substance in the teacher's explanation and decide whether the arguments she used in this explanation were valid and sufficient. In short, Nathan's case suggests that thorough mathematics knowledge supports the providing of conceptually based explanations and the identifying of the mathematical limitations in others' explanations. Nevertheless, his case also suggests that a strong knowledge of the content and its teaching does not suffice to provide explanations sufficiently unpacked and understandable by a specific student population. I return to this hypothesis when I revisit his case in the next chapter.

Nathan's case also suggests that a profound understanding of the content allows selecting suitable representations and using them appropriately by focusing on their structural characteristics and by capitalizing on these characteristics to surface and discuss the mathematical concepts under consideration. Nathan selected a suitable representation and used it appropriately to show the quotient of a division of fractions. He rightfully tried to use this *same* representation to explain the reciprocal and the associated

“invert-and-multiply” rule. Like his explanations, his use of representations was, for the most part, conceptually driven. In addition, his drawing clearly represented the structural characteristics of the situation under consideration. At the same time, however, Nathan’s lack of close attendance to issues of mapping between these representations and the algorithm or the word problem(s) at hand, as manifested in his performance and his comments on the virtual lesson, suggests that knowledge alone is not sufficient to use representations in ways that can scaffold students’ learning.

Finally, Nathan’s performance in analyzing students’ work and contributions corroborates the quantitative findings, according to which the PSTs’ MKT score was more strongly associated with their performance in this practice than with their performance in the other MKT-related practices. His case suggests five ways in which a strong knowledge base might support a teacher while analyzing students’ work and contributions. First, Nathan was able to closely attend to and follow students’ work and thinking both while analyzing students’ written work and while commenting on the virtual lesson. Second, he was quick in figuring out whether a student’s solution was correct or incorrect. Third, for the most part, his evaluation of the students’ contributions was based on a close step-by-step analysis of the students’ work and thinking rather than on the extent to which the students answered the problem correctly. Fourth, his analysis of the students’ work was meticulous and included considerations of the instruction’s potential negative effects on student thinking. Fifth, when his knowledge could not support him in evaluating certain aspects of the students’ work, Nathan was flexible enough in drawing on other personal resources (i.e., other skills and heuristics, such as using easier examples) to reason through the legitimacy of the students’ arguments.

Nathan's case, nevertheless, also points to other factors that might inform one's performance in the practices this study examines. All these factors constitute competing explanations about the association between MKT and teaching performance as delineated above. I distinguish between two types of factors: factors representing more external resources that can inform teachers' decisions and actions and factors related to more internal teacher characteristics, like teachers' beliefs (shown at the top and bottom of Figure 4.6, respectively).

Nathan was seen drawing on several resources apart from his knowledge to either justify his decisions or to analyze the virtual lesson. These resources included pertinent readings and class discussions, which Nathan explicitly referenced during the interview. These resources appeared to work synergistically with his knowledge in informing his decisions and actions, thus, the addition sign (+) shown in Figure 4.6. His performance might also have been informed by two other resources: his school experiences and his family background. Although the evidence supporting this argument is scarce, it seems reasonable to assume that Nathan's experiences in exploring different solutions to a given problem while working collaboratively might have informed his idea of asking students to consider different solutions when solving the word problem of the first textbook page. That his parents and grandparents were educators could have also informed his performance (at least implicitly). For instance, it seems reasonable to assume that Nathan might have vicariously experienced his parents or grandparents providing explanations; he might also have been asked by his parents or grandparents to explain his own thinking when working on mathematical problems.

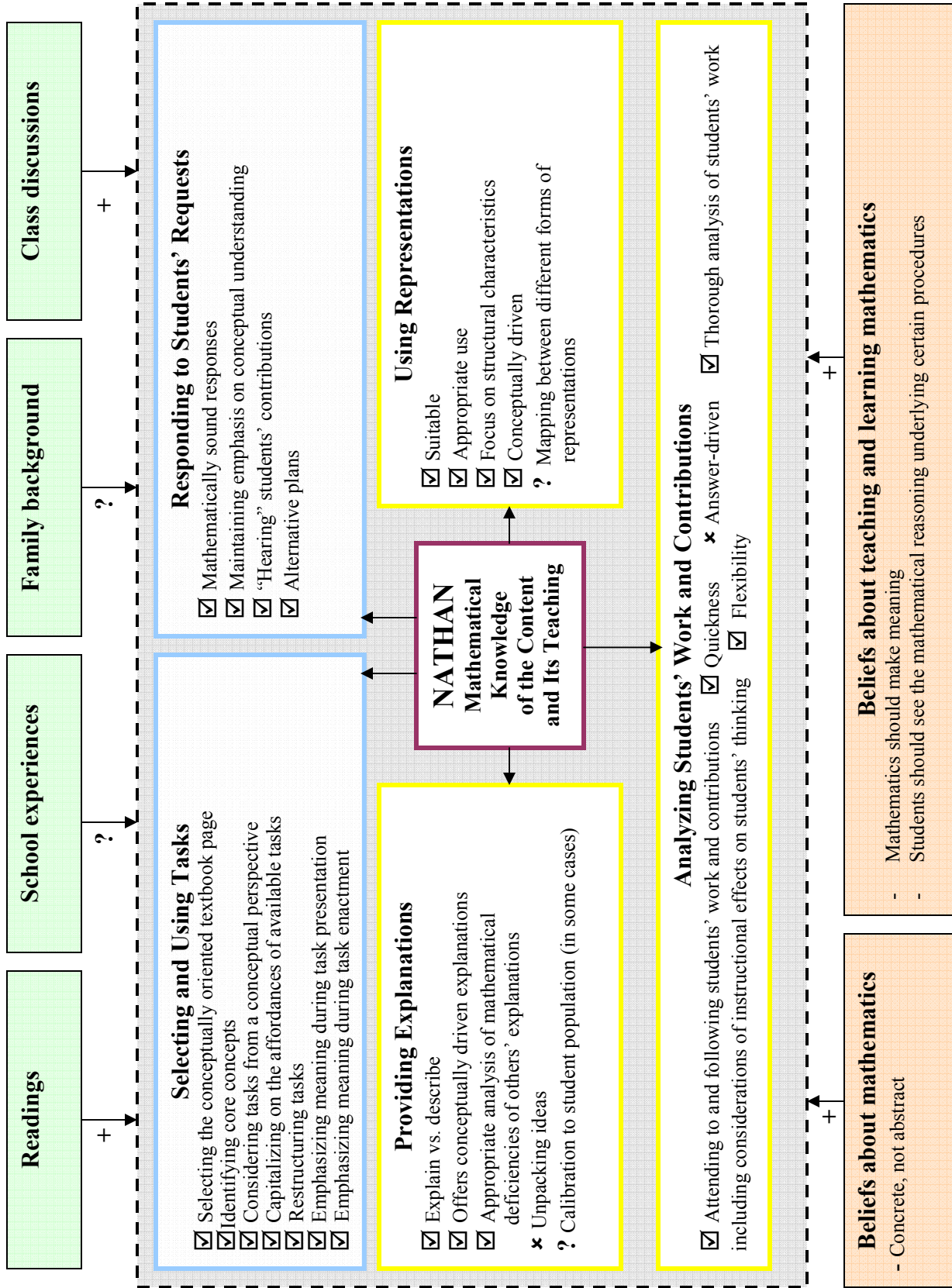


Figure 4.6. Considering the association between knowledge and teaching performance through Nathan's profile.

Finally, Nathan's beliefs about mathematics and its teaching should also be credited for his performance. Nathan believed that mathematics is concrete and should be presented as such to students. He also thought that students should be supported in seeing and understanding the mathematical reasoning underlying certain procedures. At several points during the interview, these beliefs appeared to have informed Nathan's performance. It seems reasonable to argue, though, that these beliefs worked synergistically with this knowledge. What would have happened had Nathan's knowledge been less profound? Would his beliefs suffice for the kind of performance he exhibited? Or alternatively, what would have happened if his beliefs worked antithetically to his knowledge, namely if he did not so strongly endorse making meaning and understanding? Other cases considered in the remaining of this chapter offer some plausible answers to these questions.

*The Case of Nicole: Engaging in a Cost-Benefit Analysis of Teaching Situations*

Nicole is the second convergent case considered in this study. As outlined in Table 4.5, compared to her fellow students, she scored high on both the LMT test and the teaching simulation; she was also one of the two PSTs who had taken several math content courses and two math methods courses. In contrast to Leo (who resembled Nicole in all the aforesaid features), Nicole performed better on the LMT test than on the standardized GRE test. Nicole's case is informative in at least three respects. First, it supports the strong association between the PSTs' MKT and their performance in the teaching simulation yielded from the quantitative analysis; like Nathan's case, her case does so from merely an affordance perspective. Second, because her performance in the performing tasks of the teaching simulation was better than that of her counterparts with

higher GRE scores (for example see Kimberley’s case below), Nicole’s case corroborates the quantitative analysis which showed the association between the PSTs’ knowledge and their performance in the performing tasks to be marginally robust to the PSTs’ GRE-quantitative scores. Third, her performance in the noticing and interpreting-evaluating tasks of the teaching simulation points to other factors that mediate the association between knowledge and teaching performance; in doing so, it supports the quantitative findings that showed low to negligible correlations between the PSTs’ MKT and their performance in the tasks of noticing and interpreting-evaluating.

### *Background*

Nicole had a positive attitude toward mathematics, which was fueled by some “great mathematics teachers” she had who “were excited about math and willing to take time to work with [her and her classmates]” (PE.S.). Her love for the subject and her keen interest in explaining math ideas to other people were her main motivations for becoming a teacher. She explained,

[S]tarting at the fifth grade, people started asking me for help with math problems. And through high school we didn’t really have a regular study group, but a lot of times I would meet with other people in study hall time and explain the math to them. And I just enjoyed helping people learn. (PE.I. 56-61)

Although she held an undergraduate degree in elementary education, Nicole had never worked as a teacher since she got married and was devoted to raising her kids. Yet, she spent a lot of time in school as a parent volunteer and as a substitute teacher. As she recollected, once her children started going to school, “I practically lived at [their] school; I helped in the classrooms. I got to a point where people said, ‘you can get paid for this.’ I started substitute teaching” (PE.I. 65-68). Despite these informal and formal

teaching experiences, Nicole felt that she needed more work to “see things from a student’s point of view” (PE.S.).

### *Selecting and Using Tasks*

*Performing.* When asked to talk about the strengths and the limitations of the two textbook pages, Nicole first outlined the basic knowledge and skills that students need to have developed before being introduced to a division-of-fractions lesson. She listed the concepts of fractions, divisor, and reciprocal as well as the skill of multiplying two fractions. She also remarked that, while working on dividing fractions, students will realize that the quotient might be bigger than the dividend (contrary to what happens in whole-number division).

Although Nicole acknowledged that the first page was effective in presenting the *steps* involved in the algorithm of dividing fractions, she considered it insufficient to explain the division of fractions. She justified her argument by pointing out that this page should have included a picture to support students’ understanding because “division of fractions is a very difficult concept to teach” (PE.I. 142-143). Her preference toward using pictures appears to have been informed by Nicole considering herself a visual learner.

Nicole would rather not use the first page for an introductory division-of-fractions lesson; instead, she would use it to offer her students more practice in solving division-of-fractions problems since “kids need to practice doing what they’ve learned” (PE.I. 194). If she had to use it for an introductory lesson, however, she would start with the worked example at the top of the page and go over the first step several times so that students understand the concept of the reciprocal:

I'd go over the reciprocal several times, to make sure that they understood that concept. Especially since the whole thing is key. You know, if you don't understand reciprocal you're not going to get any of it [the algorithm]. So that's really important to nail that down. ... I guess I'm not sure how much to elaborate on reciprocal, because it seems easy enough to just flip it over. It's like, you know, you've got a definition and you follow the definition of what it means: flip it upside down, that's all. And, like, you would just practice it until you get used to the idea. (PE.I. 336-345)

She would then move on to the second step (i.e., multiplying the dividend by the reciprocal of the divisor), which she considered quite easy to teach. After having students work on the two steps of the algorithm, she would shift to the 16 exercises. She would work with her students on a selected sample of them, each of which would represent different challenges that students might face when dividing fractions; this would ensure that students would “have an example to follow when they do their own practice” (PE.I. 391-392). For instance, she would start with exercises involving the same denominator so that students would understand how these denominators cancel out during the second step of the algorithm; she would also have her students consider exercises involving “reducing.” Finally, she would use exercises involving conversions between mixed numbers and improper fractions. She considered this last type of exercises the hardest, because, as she remarked, one needs to “remember to change [the dividend or the divisor] into an improper fraction before you do the multiplication or before you take the reciprocal” (PE.I. 438-439).

Nicole thought that the first page's two word problems were related more to multiplication than to fraction division. In fact, when she initially tried to solve the first problem using a division-of-fractions sentence, she solved it incorrectly, using the mathematical sentence  $\frac{1}{4} \div 30$ ; yet she soon realized that this was wrong because it was “gonna come out to a smaller number instead of a bigger number” (PE.I. 219-220). If she had to use this word problem in her teaching, she would point the students' attention to



the word “of,” which she associated with fraction multiplication. She would also use a picture to support students’ understanding. In particular, she would draw a rectangle to denote the whole number of fifth graders and divide it into four parts. She would then associate one of these four parts with the thirty students, to designate the students who play baseball. From there, she would help the students see that the whole quantity is four times 30. Numerically, she would represent this as  $\frac{1}{4} \times \underline{\quad} = 30$ , which she would then turn into  $30 \div \frac{1}{4}$  and  $30 \times \frac{4}{1}$  (see Figure 4.7).

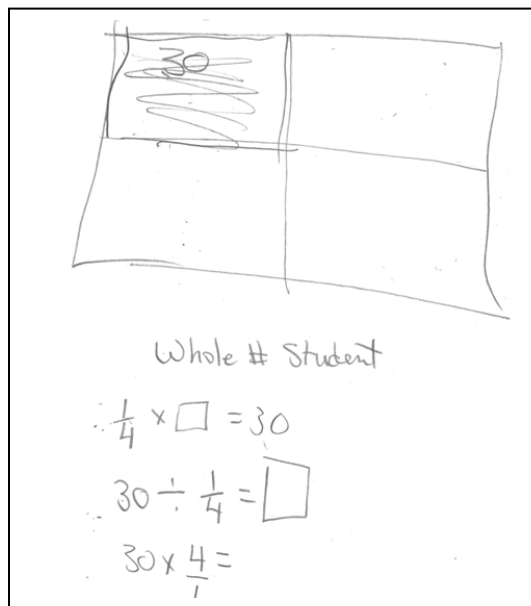


Figure 4.7. Nicole’s work on the first word problem of the first textbook page.

In discussing this problem, she would emphasize two important relationships: the part-to-whole relationship and the relationship between multiplication and division. With respect to the first relationship, she would help her students see that their final answer needs to be bigger than thirty because this number represents only a part of the whole. She would use the second relationship to justify moving from the first equation to the second: because multiplication and division are inverse operations, “you can take either one of these two factors [the  $\frac{1}{4}$  or the unknown number] and divide it by the product [i.e.,

30] to get the other factor” (PE.I. 308-309). In explaining how she would move from the second to the third equation, Nicole remarked:

[W]hen you take the reciprocal, thirty times four over one, you can see that -- and to me, looking back at the picture -- you can see that one fourth, that you need to take four of those parts to get the whole. (PE.I. 324-327)

This last remark encompasses the fundamental idea of the reciprocal in fraction division: that the reciprocal represents the number of times one can fit the divisor unit into a dividend unit. Nicole revisited this idea during a subsequent part of the teaching simulation, while discussing the division  $\frac{1}{2} \div \frac{1}{6}$  (see below).

In sum, Nicole appeared to have the potential to use the first page from a conceptual perspective, as suggested by at least three pieces of evidence. First, she wanted to ground her explanations in “visuals” instead of simply listing the steps of the algorithm (and she was successful in providing such an explanation, as discussed below); second, she identified one key idea in dividing fractions, namely that the quotient is larger than the dividend; third, in explaining how she would work with her students on the first word problem, she addressed a fundamental idea related to the reciprocal – that it represents the number of relative units that can be fit into an absolute unit. Yet, her work on explaining the two steps of the algorithm and on using the 16 exercises were both geared toward honing students’ procedural fluency rather than on enhancing their understanding of the procedure’s underlying meaning. Hence, it is unclear whether Nicole would draw on these ideas to teach fraction division from a more conceptual perspective. A case in point here is her argument that she would have her students practice the reciprocal until they get used to the idea of “flipping it upside down.” As will become obvious, this algorithmic approach seemed to have been fueled – at least partly – by her own experiences as a learner of mathematics.

The second page was more appealing to Nicole for teaching an introductory lesson on fraction division. She justified this preference by pointing out that this page, in contrast to the first one, included “real-life” word problems:

I *really* liked the idea of ... finding a real-life situation. That’s another thing I picked up from my physics class, because my teacher was really good at comparing [a situation] to real life. ... [T]o understand that it’s not just a page full of problems -- you look at this [the first textbook page, and you say], “Oh, it’s math,” you know, “a bunch of work to do, tedious.”... I really think that relating it to real life makes ... [students] want to do it and understand it, instead of just [thinking], “Oh, it’s another math lesson.” (PE.I. 754-766, emphasis in the original)

Obviously, in making this decision, Nicole was drawing on her own experiences as a learner of mathematics and on considering her teachers’ instructional approaches and styles. As discussed below, her decision, however, was not solely informed by the real-life nature of the problems included in the second page.

Nicole also noticed that, in contrast to the previous page, this page was asking students to present their answer in three different ways: in written explanations, diagrams, and number sentences. She noted that if she were to teach this lesson, she would use a lot of drawings because they help make the ideas at hand more transparent. Nicole also considered the requirement that students explain the fractional part of their answers a significant strength of this page. Referring to task A<sub>2</sub> (i.e.,  $\frac{3}{4} \div \frac{1}{6} = 4 \frac{1}{2}$ ), she remarked:

**Nicole:** [B]ecause you can look at [task A<sub>2</sub>] and say, “Oh, there’s ... something left over.”... And, I think, it’s good to recognize that it doesn’t mean a half a yard, it means half of a badge; and a badge is a sixth yard, so you have a twelfth of a yard left over.

**Charalambos:** Why do you think that this is important to do?

**Nicole:** Because I’m a sewer (laughs) and I know that it’s important to understand what the leftover means.... I don’t know how to explain it but ... if you want to use the extra for something else, you want to know how much you have. And that it’s not half a yard, it’s half of a sixth; it’s a big difference. (PE.I. 631-646)

Nicole’s explanation of the fractional part of the quotient was not strictly mathematical: she did not talk about the relationship between the remainder and the divisor, as Nathan did when analyzing Amanda and Julia’s work. Yet, her out-of-school experiences as a

sewer, in which calculating the amount of fabric available and remaining is important, seemed to have alerted her to the importance of addressing the fractional part of the quotient in division-of-fractions problems.

When prompted to discuss the sequencing of the tasks on this page, and particularly that task D, which pertains to the division-of-fractions algorithm, is presented last, she noted:

I do think that if you have a chance to see it and understand where the ideas come from, then when you get to the algorithm it makes more sense, instead of just -- I'm pretty sure most of the time, when I was in school, they just gave you the formula and said "practice it." And I was pretty good at picking up the ideas and understanding where [they] came from, and I wasn't afraid to use [them]. But I also know from practice ... from getting into the higher mathematics and into the physics, I know that the more you understand you can make-up your own formulas, instead of just relying on what you remember. (PE.I. 732-742)

Apparently, Nicole's preference for postponing the consideration of the algorithm to the end of the lesson contradicted her school experiences of learning mathematics. As the above excerpt suggests, as a learner of mathematics, she was often asked to apply a given algorithm without really understanding it. While taking more advanced math and science classes, however, she came to appreciate the value of understanding the mathematical procedures at hand instead of mindlessly following them. As discussed below (see *Providing Explanations*), this inclination to help students understand the algorithm of the division of fractions was not just wishful thinking since Nicole did propose a way of conceptualizing the reciprocal in the division  $\frac{1}{2} \div \frac{1}{6}$ .

Overall, Nicole's discussion of the second page suggests that she was able to identify the affordances of this page for scaffolding students' understanding of the procedure of dividing fractions. The findings presented above also suggest several sources that appear to have informed her analysis of this page and possibly how she would enact this page if she were asked to teach an introductory lesson on the division of

fractions. Central among these factors appear to have been her reported propensity to use drawings when doing mathematics and her in-school and out-of-school experiences.

*Noticing and Interpreting-Evaluating.* While watching the simulation, Nicole did not comment on the virtual teacher's introduction of task A<sub>1</sub>; instead she focused on the teacher's calling on and involving different students. When directly asked to comment on the teacher's introduction of this task, she argued that if she were teaching the lesson, she would have preferred to "give more at the beginning" (PE.I. 967). Although she saw value in letting students think about the problem themselves, she would have drawn a diagram on the board to represent the problem; she would then have explained the problem to them. She justified this move by pointing out that "that's the way it was always done for [her]" (PE.I. 972). Compared to the virtual teacher's approach, the approach that Nicole proposed to introduce this task could support students in constructing meaning, especially if she drew explicit connections between the word problem and the drawing. Yet, her approach would most likely diminish the task's cognitive demand since she would "give more" to the students. Nicole's evaluation of the enactment of this task was also consonant with her inclination to give more. She interpreted the teacher's pointed questions as clarifying questions aimed at helping the students understand what the problem was asking them to do. She was even somewhat annoyed with what she perceived as an attempt on the teacher's part to avoid giving answers to the students; she mentioned, "some of this – we don't get to tell [students] the answers – gets frustrating" (PE.I. 1106-1107).

Despite endorsing the virtual teacher's introduction and presentation of task A<sub>1</sub>, Nicole disapproved of Ms. Rebecca's enactment of task D. She thought that during this

episode the teacher was simply “playing with numbers” without helping her students understand the meaning of the division-of-fractions algorithm. If Nicole were teaching this lesson, she would have drawn a diagram on the board and helped the students associate the different numbers involved in the algorithm with their corresponding parts in the drawing. She would have done so by using the division problem  $A_1$  (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ), whose quotient does not involve a fractional part. Her choice of this problem is worth noticing, since this simpler division problem – compared to the division problem  $2 \div \frac{3}{4}$  outlined in the worked example of the first page – would render explaining the reciprocal a less convoluted task, as her performance below suggests.

Nicole’s performance in all the noticing and interpreting-evaluating episodes of this practice was consistent. She was attentive to whether instruction helped students construct meaning of the procedure under consideration without necessarily being concerned about whether the teacher was doing most of the thinking for her students. This approach seems to have been informed by the images of teaching she apparently formed as a learner of mathematics.

### *Providing Explanations*

*Performing.* Consistent with her propensity to “use visuals,” when asked to explain the division  $2 \div \frac{3}{4}$ , Nicole started with drawing two big rectangles, one attached to the other, to show the dividend (see Figure 4.8a). Focusing on the divisor, she then explained that she needed to partition the two rectangles into fourths. After dividing both rectangles into fourths, she shaded in two  $\frac{3}{4}$ -portions, as illustrated in Figure 4.8a. Once done, she clarified that she had two whole sets of three fourths and she wrote number 2 underneath her drawing. She paused for a while and then continued:

And then, I have two left over [she wrote  $\frac{2}{4}$  underneath her drawing next to number 2; see Figure 4.8a]. But, I think, I'm gonna be confused on this. I've got -- it looks like two fourths left over. ... But that's not right, because it's [pause]. Well, that's interesting! Because ... the two left over are two out of three fourths. So it's two out of three that's left over. So it seems like you would need to somehow draw another picture that ... somehow you need to understand that you're comparing it to three so that when you have a two left over, you know, it's two out of three [she crosses out the  $\frac{2}{4}$  in  $2\frac{2}{4}$  and writes  $2\frac{2}{3}$ ; see Figure 4.8b].

After figuring how to explain the fractional part of the quotient, Nicole repeated her explanation:

So, I'm dividing by three fourths. So I'm dividing by a set of three; I've got this set of three over here [she draws another rectangle and divides it into three parts to show the  $\frac{3}{4}$ -part; see Figure 4.8b.] And I want to find out how many of those [the  $\frac{3}{4}$ -portion] are in these two [the two big rectangles]. So ... here's three [pointing to the first  $\frac{3}{4}$ -portion shaded in the first rectangle] and there's another three here [pointing to the remaining fourth in the first rectangle and to two fourths in the second rectangle]; and there's two left over [pointing to the non-shaded part in the second rectangle]. So I've got two, two whole sets of three fourths, and then two left over. But that two is out of three [pointing to the  $\frac{3}{4}$ -portion on the right of the two big rectangles], because I was looking for three and I've two left that don't quite make three. (PE.I. 516-527)

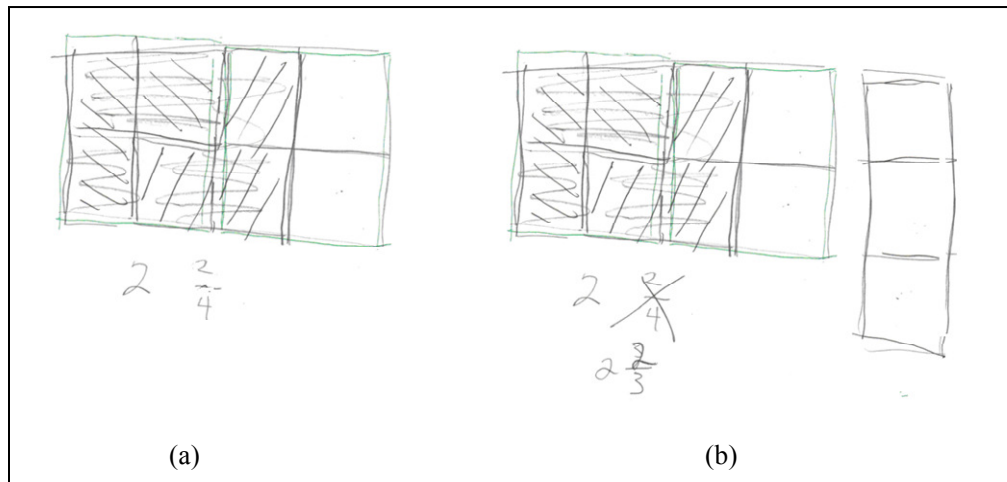


Figure 4.8. Nicole's work on explaining the division problem  $2 \div \frac{3}{4}$  (a: first steps; b: her work at the completion of her explanation).

The two excerpts just considered suggest that Nicole was not very familiar with explaining the fractional part of the quotient in terms of relative (i.e., divisor) units, something that she confirmed at a subsequent point in the interview. Yet, she was quick in identifying the "inconsistency" between her drawing and the answer one gets by using the traditional algorithm and flexible enough to resolve this inconsistency. Her

explanation as outlined in the last quotation could be characterized as conceptually driven since it was grounded in the concept of division as the number of divisor units which can be fit into the dividend. Unquestionably, her explanation warranted further polishing. For example, instead of talking about “threes,” Nicole could have explicitly referred to three fourths; similarly, her drawing on the right might have been confusing to some students since it was equal in area but different in shape than the  $\frac{3}{4}$ -portions she was shading in the two big rectangles. Nonetheless, it seems reasonable to argue that, overall, her explanation would be comprehensible to an average sixth grader.

Nicole, however, was not equally successful in explaining the division-of-fractions algorithm using this same division problem. In attempting to provide such an explanation, she started with drawing two big rectangles to show the dividend (see Figure 4.9). After pausing for awhile, she figured out the numerical value of the reciprocal of the given division (i.e.,  $\frac{4}{3}$ ) and its quotient (see the left-hand side of Figure 4.9). Because the quotient involved thirds, she thought of dividing each rectangle into thirds, which she did by using horizontal lines. She again paused for awhile and then drew three vertical lines in each of the two original rectangles to partition each rectangle into four parts. Next she shaded in the squares of the first row in the first rectangle, thinking that this shaded part represented the reciprocal. She soon realized that this was not the case and stopped.

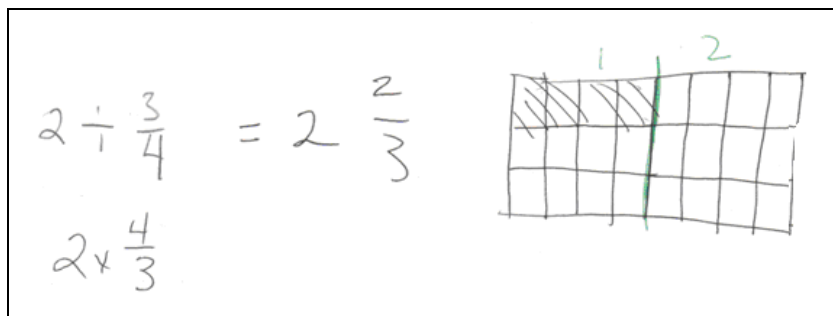


Figure 4.9. Nicole’s work on explaining the division-of-fractions algorithm using the division problem  $2 \div \frac{3}{4}$ .



Nicole's explanation was not only incomplete but was also a typical example of what I call a numerically driven explanation. Since the reciprocal under consideration involved thirds, she first partitioned the two rectangles in thirds. Such a partitioning, however, did not allow for showing the four thirds. To overcome this obstacle, Nicole entertained the idea of also partitioning each rectangle into fourths. Indeed, this new partitioning resulted in four little squares in each third. This quantity, however, was by no means equal to four thirds; instead it represented four squares, which, when taken together, they still formed a third.

Nicole's difficulties in explaining the reciprocal might have stemmed from her lack of understanding of the concept of the reciprocal and/or from the fact that the reciprocal in the division  $2 \div \frac{3}{4}$  was not a whole number. Her work on explaining the reciprocal of the division-of-fractions algorithm during a subsequent point of the interview does not allow dismissing either hypothesis.

After she found Ms. Rebecca's explanation for the reciprocal in the division problem  $\frac{1}{2} \div \frac{1}{6}$  inappropriate, Nicole attempted a second explanation for the reciprocal. To do so, she drew a line, divided it into six equal parts, and identified the resulting parts as sixths of a yard. She then showed that three sixths corresponded to half a yard and wrote the first equation shown on the right-hand side of Figure 4.10. Next, she explained that the answer to this equation can also be obtained by considering " $\frac{1}{2}$  of 6," which she represented with a multiplication sentence (see the two last equations in Figure 4.10). She concluded by pointing out that there are six parts in a whole yard and because the problem asks for half the yard, one needs to consider only three sixths of the yard. This last comment speaks to the idea of the reciprocal. Nicole, however, was still not solid on

the idea that the reciprocal represents the divisor units that can be fit into a dividend unit; notice, for example, that she did not explicitly refer to six sixths in a whole. Also, after she provided this explanation, she commented that she was not sure whether her explanation was correct, something that suggests that she was still in the process of developing the concept of the reciprocal and/or that her explanation was driven by the numbers involved in the division problem  $\frac{1}{2} \div \frac{1}{6}$ .

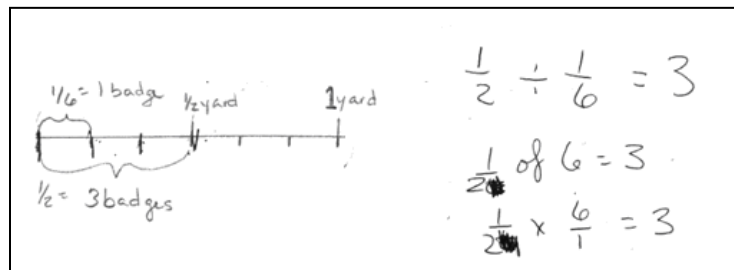


Figure 4.10. Nicole’s work on explaining the division-of-fractions algorithm using the division problem  $\frac{1}{2} \div \frac{1}{6}$ .

In conjunction, Nicole’s explanations were grounded in pertinent concepts when her knowledge supported her in this respect. When her understanding of the content could not scaffold such conceptual explanations, she appeared to ground her explanations in the numbers involved in the algorithm she was trying to explain.

*Noticing and Interpreting-Evaluating.* While considering both episodes designed to capture the PSTs’ noticing performance in this practice, Nicole was dissatisfied with the teacher’s moves. In particular, while going over the episode in which the virtual teacher responded to Michelle’s question, Nicole noticed that Ms. Rebecca tried to connect the procedure under consideration to the addition and subtraction of negative numbers. However, she was not sure that this analogy was appropriate; nor was she sure that the teacher’s argument that “we are using the reciprocal because we are using the reciprocal operation” was suitable. Yet, when explicitly asked to evaluate the teacher’s

explanation, Nicole appeared to be trying to fill in the gaps. She thought that the teacher's analogy "makes sense," because addition and subtraction are inverse operations, just like multiplication and division are. Similarly, she found the teacher's argument about using a reciprocal operation reasonable. Two explanations seem to account for Nicole's performance in evaluating the virtual teacher's explanations. First, she might have tried to avoid criticizing the teacher; second, her understanding of the concept of the reciprocal might have not been solid enough to support her in thoroughly analyzing the teacher's explanation, identifying its deficiencies, and foremost, justifying why she considered the teacher's explanation deficient. Given that Nicole did criticize the teacher's approach in other episodes, and given her difficulties in explaining the reciprocal as outlined above, the second explanation seems more plausible.

### *Using Representations*

*Performing.* Whenever Nicole used representations to explain a mathematical idea, she selected suitable representations. For instance, although she used area representations to explain the division  $2 \div \frac{3}{4}$ , in explaining the division  $\frac{1}{2} \div \frac{1}{6}$  she switched to a linear representation, since this division pertained to dividing yards. However, she did not always use these representations appropriately, especially when her understanding of the content was not robust. In those cases, her representations, like her explanations, were numerically driven. On the contrary, when her understanding was more solid, she was seen drawing connections between the representations she was using and the algorithm under consideration, as her work in explaining the quotient of  $2 \div \frac{3}{4}$  suggests. Finally, even when using a drawing to represent a real-life situation (e.g., making bows from given lengths of yards), she was representing the structural

characteristics of the situation at hand rather than its more superficial characteristics (see Figure 4.10).

Nicole's case suggests that a strong inclination to use representations to explain the content does not suffice to use representations appropriately. At several points during the interview, she explained that because she considered herself a visual learner, she would like to ground her work and explanations in "visuals." She was able to do so successfully when her knowledge supported her in this endeavor; when it did not, she merely used the representations to *show* how the numbers involved in the division-of-fractions algorithm play out instead of actually explaining why this algorithm works.

*Noticing and Interpreting-Evaluating.* In neither of the episodes under consideration did Nicole comment on the lack of connections among the representations being used during the lesson, the algorithms, or the word problems these drawings were supposed to represent. When specifically asked to consider Amanda's use of representations, she made a general comment about the importance of using representations in a mathematics lesson:

The more the senses you use the more [the students] can understand; and you involve people that learn different ways. By drawing it, you know, you're using tactile; and by having a picture you've got visual, and by talking about it you have the audio understanding, oral, whatever that is. And, so it involves different senses. (P.E.I. 1174-1180)

Nicole's argument was valid: during this episode, the students were indeed using visuals and were talking about the problem. However, she did not notice that these different modes of presenting information were not connected to each other in ways that could support student learning. In fact, as judged by her comment that Amanda was "translating" the drawing into a mathematical sentence, Nicole appeared to be thinking that such connections were actually made.

In the second episode, Nicole filled in the gaps in Amanda and Julia's use of representations when solving the problem  $\frac{3}{4} \div \frac{1}{6}$ :

So, [Amanda and Julia] drew the whole yard, and then nine pieces was the three quarter yard, okay. And then [Amanda] is ... counting how many sixths there are in that piece. So, she came up with four pieces, four sets of one sixth, and then half a piece left over, which is actually one twelfth of the whole yard.

Nicole was right in observing that the nine pieces that the two girls drew corresponded to the three quarters of a yard and that the two twelfths that Amanda was counting corresponded to one sixth of a yard. However, when asked whether she would react to the two students' work differently from how the virtual teacher reacted, Nicole did not see a reason for doing so, thus missing an opportunity to identify gaps in the girls' work. At least two explanations might account for her performance in this episode.

First, Nicole, like other high-MKT students (e.g., Teresa), might have taken for granted that the correspondences between the drawing and the numbers the girls were using were apparent to the students, just like they were apparent to her. To the extent that this explanation holds, it also accounts for the negative (yet non-significant) correlation found between the PSTs' MKT-performance and their noticing performance in this practice. In other words, high-MKT PSTs might have filled in the gaps in the virtual students' use of representations, and hence, these PSTs did not see any need for making the correspondences identified above explicit. Alternatively, Nicole might have considered her comments while observing this episode to be pointing to the lack of connections between the mathematical sentence and the drawing the two girls used in their work. Hence, when specifically asked to articulate how she would react to the girls' work, she preferred to focus on the girls' incorrect answer.

Although both explanations appear to be legitimate, Nicole's comments on Amanda's work during the first episode suggest that she might have not considered that the connections she could see might have not been equally transparent to the students. At first glance, such a finding would contradict the fact that Nicole was able to make such connections when explaining the quotient of division  $2 \div \frac{3}{4}$ . Yet, being capable of drawing such connections when explaining an algorithm might not suffice to identify the lack of such connections in students' work.

### *Analyzing Students' Work and Contributions*

*Performing.* Nicole closely attended to and analyzed the three students' work and made appropriate assertions about the students' understanding.

In analyzing Robert's work, she followed each step of his work and concluded that his answer was correct. Based on this analysis, she also argued that Robert was able to apply the algorithm that was presented during the lesson:

Robert took two and three fourths divided by three fourths, he changed the two and three fourths into an improper fraction, so he has eleven fourths divided by three fourths. He took the reciprocal of three fourths – it's four thirds – and then multiplied eleven fourths times four thirds is forty-four twelfths. (P.E.I. 1595-1599) Robert seems to understand that you take the reciprocal of the number you're dividing by, the divisor. He knew how to do the improper fraction, he knew how to take the reciprocal; he followed the steps that [the teacher] gave them. (P.E.I. 1624-1628)

Nicole's assertions about Robert's work were meticulous. After parsing Robert's work into a sequence of steps (e.g., finding the reciprocal, converting a mixed number into an improper fraction), she drew accurate assertions about what he was able to do. At the same time, she avoided making assertions that could not be supported by Robert's work. For instance, she did not make any assertions about Robert's conceptual understanding of the division of fractions, nor did she make any inferences about his ability to convert an

improper fraction into a mixed number, given that Robert did not convert his final answer into a mixed number.

In analyzing Michelle's work, Nicole first identified the dividend and the divisor in this student's drawing. She then noticed that after taking away three  $\frac{3}{4}$ -segments, Michelle was left with two pieces. She argued that, in addition to her drawing, Michelle needed to present her work in numbers because this would allow for making inferences about her grasp of the fractional part of the quotient. Similar to her assertions when considering Robert's work, her assertions about Michelle's understanding were precise:

Michelle understands the general concept from the diagram of how many pieces there are and how to divide them. But I don't know for sure that she understands the answer. And you can't really tell if she's using a reciprocal, since it's not written. (PE.I. 1628-1632)

Nicole was also quite quick in noticing the error in Ann's work. Considering this student's work as a whole, she asserted that Ann understood that she needed to take the reciprocal of a number but that she was not clear as to which number this should be. Hence, she inferred that Ann "probably doesn't understand why [the algorithm works]" (PE.I. 1634). As the reader might notice, when talking about Ann's work, Nicole went a step farther and argued about this student's conceptual understanding. However, she was careful not to make definite statements about this student's understanding, probably because Ann's work did not allow for drawing such definite inferences.

*Noticing and Interpreting-Evaluating.* Of the four faulty student contributions, Nicole identified and talked about two (i.e., June's contribution and Amanda and Julia's work) without any prompting; she also commented on the third instance, Alan's error, when explicitly asked to comment on his work.<sup>141</sup>

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<sup>141</sup> The interview protocol did not include any prompt for directing the PSTs' attention to Ann's contribution. This instance was considered only for assessing the PSTs' noticing performance.

Nicole immediately noticed the misconception in June's explanation. While going over the slides in which June argues that the one half is the divisor simply because the second fraction is smaller than the first one, Nicole remarked, "But that's not ... always going to be that way," and added: "[Y]ou don't always divide a smaller fraction by a bigger fraction" (PE.I. 905-906, 919-920). The immediacy with which Nicole reacted to June's explanation could be attributed to her understanding of the content. As discussed when considering her performance in selecting and using tasks, Nicole was mindful of the relative sizes of dividends, divisors, and quotients in division-of-fractions problems. This knowledge appears to have scaffolded her in capturing the mathematical error in June's explanation.

When asked to evaluate June's explanation, however, Nicole argued that it made sense when seen in the context of the problem under consideration; hence, she would not follow up to address the misconception she identified in June's contribution. Several explanations could account for Nicole's decision to not further elaborate upon June's misconception; below I consider one possible explanation after discussing her performance in the next episode.

Nicole's knowledge also supported her in quickly identifying the error in Amanda and Julia's work. As soon as she watched the slides in which the girls presented their solution using a numerical sentence and the teacher accepted their work and moved on to the next problem, Nicole commented:

**Nicole:** The question was "How many badges can you make?" And [Amanda and Julia] got four badges and then one twelfth of a yard left over. But that's half a badge. So [the teacher] didn't go into the part, where it's four and a half badges... It seems like [the students] would need to know that.

**Charalambos:** So if you were the teacher, what would you do at this point?

**Nicole:** I think I would ask them, "What were you dividing by?" and go back and clarify what the original question was. You have three fourths of a yard and you're dividing by one sixth



of a yard. And they can see visually that one sixth is two [one twelfth] pieces. I think I would go back and ask, "How much of a sixth is that piece that's left over?" and have them understand it that it's half a piece. (P.E.I. 1315-1334)

**Nicole:** I am surprised that [the teacher] didn't go back and talk about that one twelfth, because that would be important when you go to find a pattern.

**Charalambos:** In what sense would it be important?

**Nicole:** Because you need to know, because the answer needs to be in the same terms of the factor that you started with, that you divided by. (P.E.I. 1367-1374)

The excerpts just cited provide at least two insights about the relationship between knowledge and teaching performance. First, knowledge appears to scaffold teachers in identifying flaws in students' work and resolving them in ways that can potentially support student learning. Nicole, for example, did not only capture the error in the girls' solution, but she also proposed two ways in which she would support the students in seeing and resolving the error. Because the error might have stemmed from the two girls' not attending to the problem's question, she would first direct their attention to what the problem was asking them to figure out. She would then ask a more pointed question to help the two girls understand that the remainder needs to be expressed in relative units.

Second, the latter excerpt suggests that teachers' decisions and actions at any point of a lesson are informed by factors beyond their knowledge. In considering Amanda and Julia's solution, Nicole appeared to have perceived their error not as an isolated opportunity to support student thinking and understanding but as a relay element in the wider context of the lesson. In other words, for Nicole, addressing this error was not only important for helping the students understand the concept of the remainder in division-of-fractions problems, but was also critical for the flow of the lesson: the students would not have been able to identify patterns and consequently derive the division-of-fractions algorithm without obtaining correct solutions to all problems under consideration. Hence, Nicole sensed that there was an additional reason for addressing the error in Amanda and

Julia's work. In contrast, in the previous episode with June's explanation, she might have not sensed such a need since June's explanation appeared to be legitimate in the context of *the particular problem at hand*. Viewed from this perspective, Nicole's analysis of both teaching episodes resonates with what Schoenfeld (1998, 2008) considers a cost-benefit analysis of a teaching situation, an issue to which I return in Chapter 6.

Nicole did not directly identify the error in Alan's work, but when prompted to discuss Alan's solution, she noticed the error, which she interpreted in two different ways. First, she argued that Alan considered twelfths instead of sixths because he took one sixth of half a yard instead of one sixth of the whole yard; second, she observed that Alan divided by six instead of dividing by one sixth. Like Nathan, Nicole also considered Ms. Rebecca's instruction as a potential source for Alan's error. She argued that the teacher did not label the representation she drew on the board, which might have led Alan to think that the red portion of the representation was the whole line and not just half of it. Nicole concluded by saying that she was surprised that she had not originally attended to such a striking error.

Taking Nicole's last comment into consideration, one could argue that, by and large, Nicole was capable of attending to and following student work and contributions and was relatively quick in capturing the errors in their work. Her analysis of the students' work was quite meticulous, as suggested both by her discussion of Alan's error but also by her dissection of Robert's, Michelle's, and Ann's solutions. Her analysis of Alan's work also suggests that she could consider the implications of certain instructional moves on students' thinking. Finally, her analysis of Amanda and Julia's work implies that her decisions with respect to addressing a student error were informed by

considerations of the implications that addressing such an error would have for the flow of the whole lesson.

*Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Nicole successfully analyzed the mathematics at stake in each of the two episodes designed for this practice. Her response to the students' requests differed across the two episodes, largely due to her considerations of the situation at hand.

As already discussed, Nicole captured the error in Alan's contribution after she was explicitly asked to comment on his work. To handle Alan's confusion and error, Nicole proposed two alternative plans. First, thinking that Alan's confusion might have stemmed from the teacher's not labeling the representation used to solve this problem, she proposed that a first step in addressing Alan's error would have been to label this representation to specify the portion that corresponded to the whole yard. Second, if this move did not help Alan understand and correct his error, Nicole would have directed his attention to the question of the problem. She would have asked him to explicate what the one sixth mentioned in this question represented to help him understand that the problem was asking for one sixth of the whole yard. In short, in this episode, Nicole would have tried to first eliminate factors that might have caused Alan's confusion and then scaffold him to correct his answer without necessarily doing the thinking for him.

In the second episode, Nicole drew on her thinking while solving task A<sub>2</sub> to consider alternative ways to respond to June's and Shaun's requests for help. She remarked that, like June and Shaun, she initially grappled with fitting sixths into three fourths, but she then figured out two ways of resolving this difficulty: by using a common denominator of four and six, one could divide both fourths and sixths into commensurate

pieces; alternatively, one could take each fourth and divide it into sixths, which would result in  $\frac{1}{24}$ -yard pieces. Although both approaches yield pieces “small enough so that [they] could come out even” (PE.I. 1126-1127), she considered the latter approach more convoluted because it involved more work and it resulted in smaller pieces, which she considered difficult to handle.

Despite this analysis, Nicole concurred with how the teacher responded to the two students’ request for help. She argued that if she were teaching the lesson, she would have followed a similar approach to that pursued by the teacher:

**Nicole:** [The teacher] relates another concept that [the students] already know to what they’re working on now. I don’t know if there’s another way to make them think it through more on their own. But it seems, like, at this point, she’s feeding them the idea of common multiples. But it is something they already know and it seems like it would take a lot of unnecessary time to come up with that idea, when it’s something that they’re already strong in; and they’re working with fractions, so that’s a related topic. And she’s saying, “Oh, look. Here’s a tool that we already know, let’s use it.” And it works here. So, she’s giving them, reminding them of a tool that they already have, that can help them.

...

**Charalambos:** If you were the teacher would you do something different?

**Nicole:** Not at this point. (PE.I. 1233-1252)

Nicole’s idea that the teacher was “feeding” students an answer suggests that she was not oblivious to the virtual teacher’s taking over and doing some thinking for students when responding to the two students’ request for help. However, she again seemed to engage in what was identified before as a cost-benefit analysis of the teaching situation at hand. She argued that if it would have taken a lot of time for students to come up with the idea of common multiples, then the teacher’s decision to offer students a tool to overcome the impasse was reasonable; she would thus have followed a similar approach if she were teaching the lesson. One could reasonably argue that at this juncture Nicole was making some unwarranted inferences about students’ understanding of the concept at hand, particularly given that only a couple of students were shown responding

to the teacher's questions; hence, her assertion that the idea of common multiples "is something that [the students] already know" might have been an overstatement of what the students were able to do at this point of the lesson. Even so, one needs to acknowledge the legitimacy of her argument that in certain situations – and depending on the teacher's agenda and goals – it is justifiable for the teacher to even feed the students with certain ideas to scaffold their thinking.

*Noticing and Interpreting-Evaluating.* While watching both episodes designed to capture the PSTs' noticing and interpreting-evaluating performance in this practice, Nicole endorsed the virtual teacher's decisions and actions. In considering how Ms. Rebecca supported Alan, Nicole credited the teacher for not "putting Alan down" and for posing "clarifying" questions that helped him understand what the problem was asking him to do. Similarly, she approved of the teacher's approach to first let the students explore the second task for a while and then step in to offer them some guidance. She felt that the teacher's move to discuss June's and Shaun's difficulty with the whole class and offer them a tool for overcoming their difficulty was justifiable because in doing so, the teacher was both recognizing the difficulty that students encountered and validating their question. Nicole argued that this, in turn, would encourage students to ask more questions when struggling with the content.

Nicole's performance in this practice was consistent with her performance in the practice of selecting and using tasks. For Nicole, it seemed important that students be helped to understand the mathematics at stake, regardless of the extent to which the teacher's interventions were reducing students' opportunities for thinking. That said, Nicole should not be considered unmindful of how the teacher's interventions resulted in

minimizing the thinking in which students were expected to engage, as her comment in the last episode implies. In contrast, her decisions and actions in this practice appear to have been informed by several factors, including her understanding of the content and her own experiences as a learner of mathematics. Given her experiences as a substitute teacher, it seems reasonable to add another factor to the mix: her awareness of the limitations that contextual factors impose on the teacher. I revisit this claim in awhile.

### *Performance on the LMT Test: A Closer Look*

Nicole's performance on the LMT test was consistent with her performance in the teaching simulation in three respects. First, according to her overall score on the LMT test, she was identified as a high-MKT teacher. Several aspects of her performance in the teaching simulation corroborated the picture that her LMT score painted about her understanding of the content and its teaching: she was able to analyze the content of the two textbook pages and identify their affordances; she was capable of providing a satisfactory explanation for the quotient in division  $2 \div \frac{3}{4}$ , and she was on the right track when proposing an explanation for the reciprocal in division  $\frac{1}{2} \div \frac{1}{6}$ . More often than not, Nicole also closely followed and analyzed students' contributions. Second, she answered correctly most of the test questions that were squarely relevant to the division of fractions: the questions pertaining to interpreting division situations presented either in word problems or in mathematical sentences (i.e., questions 11, 14, 19, and 24); the question on interpreting the remainder in a division-of-fractions problem (i.e., question 17); the question on figuring out the answer to a fraction division (question 22); and one of the questions pertaining to using drawings to represent division of fractions (question 5). Her performance on these questions was aligned with her performance in the teaching

simulation, where she was seen addressing similar ideas. Third, the questions that Nicole did not correctly answer (e.g., questions 3, 4, 7, 21, and 23) pertained either to ideas peripheral to the topic of division of fractions examined in the teaching simulation or to ideas closely related to the division-of-fractions, which were not necessary for a good performance in the teaching simulation (e.g., question 10, which pertained to an alternative algorithm for dividing fractions).

Nicole's performance on four of the test questions (questions 1, 15, 20, and 25), however, was not consistent with her performance on the teaching simulation. Questions 1 and 20 pertained to the idea of units in fraction problems and questions 15 and 25 were used to capture the PSTs' understanding of the measurement interpretation of division (of fractions); Nicole answered all four questions incorrectly. In including these questions in the LMT test, it was assumed that one needs to have a flexible understanding of units to be able to interpret the quotient in division of fractions appropriately; likewise, it was assumed that an understanding of the measurement interpretation of division would support the PSTs' work in "teaching" a division-of-fractions lesson.

That Nicole was seen to passably navigate the division-of-fractions terrain without correctly answering these questions challenges both assumptions. However, it might be the case that both the context of the simulation and the thinking that Nicole was seen doing while addressing the simulation tasks enhanced her understanding of the ideas explored therein. For instance, in explaining the quotient in the division  $2 \div \frac{3}{4}$ , Nicole admitted that she had never thought about the quotient from the perspective of relative units. Her difficulties in handling relative and absolute units when explaining the reciprocal in division  $2 \div \frac{3}{4}$  also suggest that her understanding of units was not solid and

flexible enough to support her when engaging this task. Similarly, Nicole's idea of "how many divisor units fit into the dividend units" might have compensated for her weak understanding (as suggested by her work on the LMT test) of the measurement interpretation of division. From this perspective, Nicole's performance in all four aforesaid questions was not inconsistent with her performance in the teaching simulation.

### *Analytical Commentary*

Like Nathan's case, Nicole's case corroborates the strong association between MKT and PSTs' teaching performance yielded from the quantitative analysis; her performance in the performing tasks of the teaching simulation also shows how strong knowledge can aid teachers in structuring learning environments that support attention to the meaning of an operation under consideration. In her case, a conglomerate of factors also appeared to inform her decisions and actions, especially with respect to the noticing and the interpreting-evaluating tasks of the teaching simulation. In this analytic commentary, I first consider examples from her performance in each teaching practice which collectively substantiate the strong association between knowledge and teaching performance. I then turn to the other factors that seem to have informed her performance.

Nicole's performance in providing explanations and using representations provides both an affordance example (as demonstrated in her explanation of the quotient) and a constraint example (as demonstrated in her difficulties to explain the reciprocal) of how teacher knowledge can help the teacher emphasize the underlying meaning of the procedures under consideration. When asked to explain the quotient in division  $2 \div \frac{3}{4}$ , Nicole drew on her knowledge of division as the number of divisor units that can be fit into the dividend units and appropriately explained the whole-number part of the



division's quotient. Explaining the fractional part of the quotient was a challenge for Nicole since, as she confessed, she had never thought about fractional parts in division-of-fractions quotients. Yet, capitalizing on the aforementioned fundamental understanding of division,<sup>142</sup> she quite flexibly resolved the discrepancy that emerged from explaining the fractional part of the quotient in absolute and relative terms. This enabled her to provide a conceptually driven explanation and to use the representations she employed in this explanation accordingly.

In contrast, her understanding of the concept of the reciprocal was not solid; additionally, her performance on the LMT test provided some evidence about her restricted understanding of units when dealing with fractions. Both these limitations in her understanding of the content appeared to play out when she attempted to provide an explanation for the reciprocal in division  $2 \div \frac{3}{4}$ . Without a firm conceptual understanding of reciprocals, Nicole merely offered a numerically driven explanation, as typified by how she attempted to represent the four thirds (i.e., the reciprocal) as four squares that occupied a third of the rectangle. At the same time, she offered a decent explanation for a division whose reciprocal was a whole number (when considering the division problem  $\frac{1}{2} \div \frac{1}{6}$ ); this explanation might have been guided by a preliminary understanding of the concept of the reciprocal. This hypothesis is also supported by her work on the first problem of the first textbook page, which was also reminiscent of such a nascent (and largely implicit) understanding of the notion of the reciprocal.

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<sup>142</sup> In explaining the fractional part of the quotient Nicole also drew on her out-of-school experiences as a sewer. According to Nicole, these experiences helped her understand what the leftover pieces mean, especially in relation to the length of the piece being used (the dividend) and the piece needed to sew something (the divisor). Thus, in Figure 4.11, these out-of-school experiences are positively linked to her teaching performance.

Nicole also attempted to draw connections between the diagrams and the numbers she was using. This was particularly obvious when she was explaining the quotient in the division  $2 \div \frac{3}{4}$ . Unlike Nathan, Nicole showed the  $\frac{3}{4}$ -portion in a separate picture and was explicit about the correspondences between each of the numbers of this division and the different parts of her drawings. Her representations were also suitable and focused on the structural characteristics of the situation under consideration, even when Nicole was representing real-life situations. Although it is reasonable to assume that her understanding of the content and its teaching played a key role in her performance as outlined above, it is equally reasonable to postulate that these features of her performance might have also been informed by other factors, the most prominent being her formal and informal teaching experiences, whose Nathan was apparently lacking.

Nicole's performance in analyzing students' work and contributions provides additional support to the association between knowledge and teaching performance. Nicole was able to closely follow and make sense of the students' solutions, she was quick in identifying errors and misconceptions in their work, and she made precise assertions about the students' contributions, based on their work. Like Nathan, she explored the potential effects that instruction itself might have had on the errors that students committed and the misconceptions they seemed to hold. Although other factors might have informed her performance in this practice, I argue that her knowledge of the content and its teaching was key to her performance. Consider, for instance, her analysis of Michelle's solution to the division  $2\frac{3}{4} \div \frac{3}{4}$ . Without being solid on the content, Nicole would not have been able to understand what Michelle was doing, let alone identify that Michelle might have not been clear on the fractional part of the quotient in this division.

Similarly, without knowing the relative sizes of dividends, divisors, and quotients in divisions of fractions, it seems unlikely that Nicole would have immediately spotted the misconception in June's explanation.

Consistent with the quantitative findings, Nicole's performance in the tasks of noticing and interpreting-evaluating was not always aligned with her knowledge. For instance, while Nicole was seen drawing connections between numbers and diagrams in the performing tasks, she was not equally concerned with the lack of such connections in the virtual teacher's and students' performances. Similarly, Nicole's evaluation of the virtual teacher's reactions to June's contribution and to Amanda and Julia's solution was informed by factors other than her knowledge. Although Nicole appropriately analyzed these students' contributions from a mathematical perspective, she evaluated the teacher's reaction to these contributions in notably different ways. Endorsing the teacher's move to not further probe June, Nicole argued that June's misconception was not worth further addressing; she perceived this student's explanation acceptable in the context of the particular problem considered at that moment. In contrast, she critiqued the teacher for not addressing the error in Amanda and Julia's work; according to Nicole, addressing this error was pivotal to ensuring a smooth flow of the lesson.

Her evaluation of the teacher's work in these two episodes was suggestive of Nicole's engagement in a cost-benefit analysis of the virtual teacher's decisions and actions, thus the name given to her case (given that she was the only PST seen engaging in such an analysis).<sup>143</sup> Aware of certain contextual limitations in teaching, Nicole argued

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<sup>143</sup> That Nicole was the only PST seen explicitly engaging in such an analysis should not be divorced from her formal and informal teaching experiences. For example, she might have had first-hand experiences of such analysis when working as a substitute teacher. Alternatively, she might have witnessed or discussed issues pertaining to such an analysis with the teachers in whose classes she was parent

that the teacher often has to balance between withholding information from students at the expense of time-consuming student explorations and providing them with more information in order to save time. Hence, the case of Nicole brings back to the picture some of the contextual factors that this teaching simulation attempted to “control” by immersing the PSTs in an in-vitro teaching environment.

In addition to this cost-benefit analysis, like Nathan’s case, Nicole’s case also points to other factors, besides her knowledge, that informed her teaching performance. These factors include school experiences as a learner of mathematics, enrollment in a teacher education program, and formal and informal experiences as a teacher. All these experiences appeared to have informed her beliefs and the *images* she formed about teaching mathematics,<sup>144</sup> which, in turn, apparently informed her teaching performance in the simulation.

Nicole’s experiences as a learner of mathematics can be clustered into two categories which are not necessarily mutually exclusive: people and activities. In the first category, were Nicole’s mathematics teachers whom Nicole presented as dedicated to their job and invested in helping their students learn. In this category one could also include her physics teacher, who was reported helping his or her students draw connections between physics and real-life situations. The second category, which pertains to Nicole’s activities in learning the content, includes experiences such as remembering and mechanically applying formulas as well as being “given a lot of information” when introduced to a problem or when faced with difficulties; this category also includes other

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volunteering. From this perspective, it seems reasonable to argue that although working in vitro, Nicole was also drawing on her in-vivo experiences.

<sup>144</sup> I use this term as defined by Ball (1988), drawing on Lortie’s (1975) analysis. Ball (ibid, pp. 160-162) argues that PSTs’ apprenticeship in mathematics classes as learners of the subject provides them with practices to imitate, which are often limited and limiting.

experiences such as explaining ideas to classmates and peers, often by using “visuals,” and trying to decipher the underlying meaning of formulas and algorithms (especially in more advanced mathematics and physics classes). In conjunction, these experiences appeared to have informed Nicole’s beliefs and led her form certain images of teaching.

Some of these images seemed to militate against the type of instruction endorsed in this simulation, namely, the structuring of rich and challenging learning environments; others were more consistent with it. An example of the images of the first type was Nicole’s propensity to “give more” when introducing a problem or when supporting students struggling with a mathematical task, an inclination she appeared to have formed by her being in classes in which she was told and shown what to do and how to do it. An example of the images of the second type was her inclination to emphasize meaning, which appears to have stemmed from her appraisal of the aforesaid type of instruction as inadequate for supporting student understanding. Interestingly enough, the teaching simulation itself appeared to offer Nicole some *alternative images of teaching* that contradicted her own experiences of being shown and told exactly what to do. This was evident in Nicole’s appraisal of the virtual teacher’s lesson:

I really like the idea of having to think about why [the algorithm] works before you just do a bunch of problems. ... It looks like you’re finding out how much they understand before you give them the practice problems to do on their own. So, you’re having a chance to let them work through it, figure out where the glitches are before you assign the practice work. (PE.I. 1724-1733)

The influence of such alternative images of teaching on the PSTs’ performance is taken up in the remaining cases and is further discussed in the next chapter.

In contrast to all other PSTs considered in this study, Nicole received an undergraduate degree as a generalist teacher. The courses she had taken in this program seem to have had some impact on her performance. This influence was most obvious in

her discussion of Amanda's use of representations. In analyzing this episode, she drew on the general teaching principle that the more senses involved in teaching, the greater the learning is. This general principle, however, did not support Nicole in delving deeper into the episode under consideration and noticing the lack of connections in the student's work. From this perspective, one could argue that the knowledge that Nicole gleaned from these courses did not provide her with enough scaffolding when working on the teaching simulation. However, because the relevant interview data are scarce, in Figure 4.11, I present the effect of this factor with a question mark.

In addition to her formal training as a teacher, Nicole also had other formal and informal teaching experiences, the former associated with her serving as a parent volunteer and the latter with her work as a substitute teacher. Again, the available data do not support drawing any conclusive inferences about the effects of these experiences on Nicole's performance. Yet, it seems reasonable to argue that these experiences might have informed the images of teaching that Nicole had developed.

The interview data also suggest that Nicole's beliefs – both about teaching and learning the subject and her efficacy beliefs – were in play and contributed to her performance in the teaching simulation. Some of these beliefs (e.g., that mathematics should make sense) appeared to have positively informed Nicole's noticing and interpreting-evaluating performance, particularly in those instances that the virtual teacher plainly shifted emphasis from meaning and understanding to manipulating numbers. Other beliefs (e.g., learning by being shown and told and by practicing and repeating procedures) appeared to have impinged on Nicole's performance, particularly with respect to the MTF-related practices. Additionally, as already explained, whereas

Nicole's beliefs appear to have sensitized her to the importance of supporting students in making meaning, in other cases they appeared to have rendered her less sensitive to issues of supporting student thinking without replacing it. Thus, in Figure 4.11, which presents Nicole's pre-intervention portrait, I capture the influence of Nicole's beliefs on her performance with both a positive and a negative sign. In contrast, I indicate the potential influence of her efficacy beliefs on her performance with a positive sign because Nicole was seen to be confident in her capacity to work on and explain mathematical ideas.

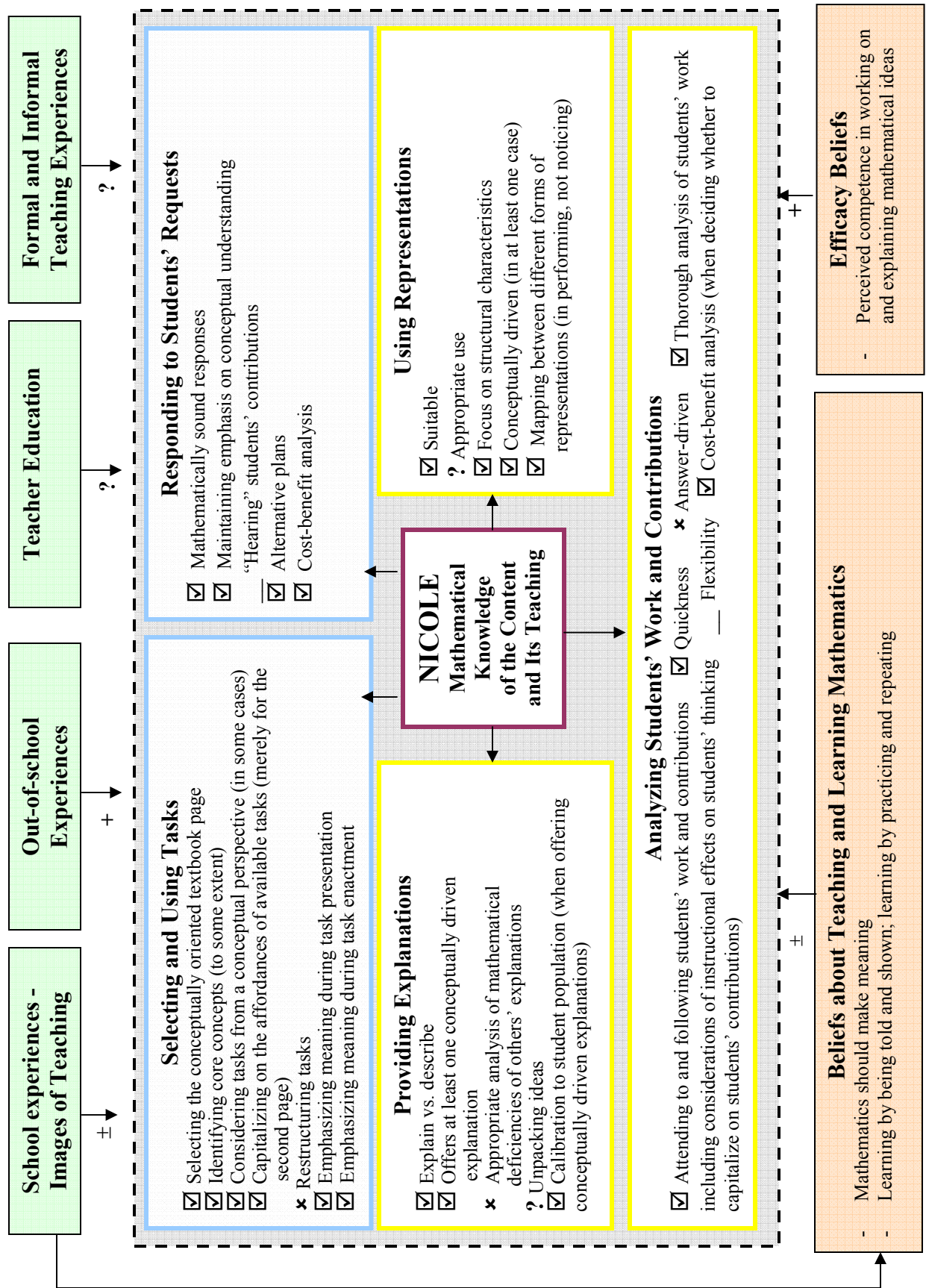


Figure 4.1.1. Considering the association between knowledge and teaching performance through Nicole's profile.



### *The Case of Deborah: When There is a Will but Not a Way*

Deborah is the third convergent case considered in this study. As shown in Table 4.5, compared to her counterparts, she scored low on both the LMT test and the teaching simulation; moreover, her ranking based on her GRE-quantitative score was higher than her ranking based on her score in the LMT test. These quantitative results motivated her selection for further scrutiny. Unlike the two cases considered above, Deborah's case supports the relationship between knowledge and teaching performance merely from a constraint perspective. More so, it shows that favorable beliefs are not adequate to support teachers in building the rich and challenging environments considered in this study.

#### *Background*

Deborah joined the program with a keen interest in becoming a teacher. Before entering the program she had worked as a reading tutor and as a teacher aide for first and third graders. She had also worked with toddlers as a babysitter and in day care. From her perspective, she had “worked with children like pretty much [her] whole life” (PE.I.56-57). She admitted, though, that “mathematics was not [her] strength,” and she had not studied this subject for a long time (PE.S.). Being a visual learner, in elementary school she struggled with the “oral explanations” her mathematics teachers had provided. Her middle-school experiences were not favorable either, since emphasis was merely placed on obtaining right answers: “[I]t seemed that there was a ‘right’ way to do things and a ‘wrong’ way. The *one* ‘right’ way was emphasized and there was not much room for interpretation or individual thought” (PE.S., emphasis added). Coupled with that was an overemphasis on practicing rather than on understanding:

I think that in my previous educational background we really just were like doing problems and problems and problems, but there were a lot of us who didn't really understand what it was that we were doing. (PE.I. 606-609)

These negative images of teaching constituted avoidance models for Deborah, as discussed below.

### *Selecting and Using Tasks*

*Performing.* In considering the two textbook pages, Deborah thought that the second page was more complex than the first because “it makes you think, rather than just doing something mindlessly” (PE.I. 852-853). She explained that the first page simply asked students to perform certain calculations that “really have no meaning,” while the second page expected them “to understand what it is that they are doing” (PE.I. 868-870). Hence, if she had to teach a lesson on division of fractions, she would use the second rather than the first page:

**Deborah:** So I don't know if it's better as a teacher to have kids know the meaning before they actually can do the calculations or while they're doing their calculations. Or if it's better to have them do the calculations and then analyze the meaning. ... But, I think it's good to have both in conjunction; and I think that that's what the second [page] does.

**Charalambos:** ... Which approach do you think is more appropriate, if there's an appropriate approach?

**Deborah:** I mean that's a tough call. But I think it's good to do both simultaneously, you know, if possible. ... I think I would probably do both simultaneously. ... If you can work out how to do the calculations while understanding what it is that you're doing and why you're doing it, then I think at the end of it you will have a better sense of what you're doing as a whole. (PE.I. 892-917)

This excerpt reveals Deborah's preference for a type of instruction that promotes both conceptual understanding and procedural fluency. Her preference notwithstanding, the interview data suggested that she was rather unprepared to fully capitalize on the affordances of the second page to structure a lesson that would concurrently promote students' understanding and mastery of the procedure at hand.

To start, she admitted that she both liked *and* disliked the second page: she liked it because of its affordances, but she disliked it because she found it hard to enact. Second,

she confessed that she herself found some of the directions on this page hard to understand: “[W]hen [the page] said, ‘Describe what each fractional part of an answer means,’ that I was like, ‘I’m gonna have to ask you’” (PE.I. 757-759). Her difficulty in understanding the page’s direction appeared to stem from her struggle with the content. Consider, for instance, her work on task A<sub>2</sub>, which pertains to the division  $\frac{3}{4} \div \frac{1}{6}$ . In considering this division, Deborah drew two separate circles, divided the first one into fourths, and shaded in three fourths of it. She then divided the second one into sixths, colored in one sixth (see Figure 4.12), and estimated that the answer would be “greater than three,” given that a sixth is smaller than one fourth and that she had three fourths. However, she could not move any farther to explain how she would use her drawings to figure out the actual quotient of the division at hand. Instead, she carried out the algorithm and found four and one half, which verified her initial estimation. Third, although she figured out that in task D students are expected to come up with the algorithm of multiplying the dividend by the reciprocal of the divisor, she was unclear as to how she could lead them toward discovering this algorithm. In sum, despite her preference for the second textbook page, the interview data suggest that she would face significant difficulties capitalizing on the affordances of this page if asked to use it for an introductory division-of-fractions lesson.

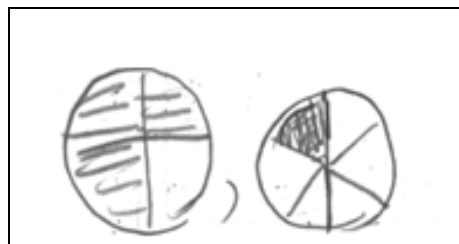


Figure 4.12. Deborah’s work on the division problem  $\frac{3}{4} \div \frac{1}{6}$ .

Deborah thought that the first page could be used only for bell work or to test students' fluency in carrying out division of fractions. When prompted to articulate how she would use this page to teach a lesson on fraction division, she said that she would start by explaining the two steps illustrated at the top of this page. She would explain that when dividing by a fraction, one needs to multiply by the reciprocal of the second fraction. She would also clarify that the reciprocal is the inverse of a given fraction and remind students to "simplify their answers" (PE.I. 206), a term she used to describe converting improper fractions into mixed numbers. When asked to clarify how she would use the 16 exercises, she said that she would assign only a subset of them. However, she could not decide which exercises to exclude; all exercises seemed similar to her:

To me they were just calculations. So I think that I was in that mindset while I was doing them: that this is just another calculation rather than taking a step back from it and looking at it. Well, even when I tried to take a step back from it, I really couldn't see it. ... I didn't find any of them more complicated than the others. ... I was trying to look through like the lens of a teacher, but I couldn't. I was still, you know, [looking] through the lens of a student. (PE.I. 498-515)

Deborah's comment that she could not see these exercises through the lens of the teacher is interesting in and of its own; it also suggests that because she conceived of these exercises simply from a procedural perspective, namely as sites for practicing the division of fractions, she could not restructure these tasks. This restructuring would have allowed for scaling up some of these tasks to render them more challenging and less repetitive for students, as Nathan did. However, the only criterion she thought of as differentiating these exercises was the extra step of converting mixed numbers into improper fractions, which, as she admitted, would not actually make a difference for the procedurally fluent students.

Deborah regarded the two word problems at the end of the first page as opportunities for more substantive work. Even so, she was not sure if and how these

problems could be perceived as division problems. To her, both problems could be solved using multiplication (e.g., problem 17 could be solved by multiplying 30 by  $\frac{1}{4}$ , after transforming the equation  $\frac{1}{4}x = 30$  students). The only reason she saw for using division to solve these problems was that they appeared on a division-of-fractions lesson:

“[S]ticking those [problems] at the end of the worksheet leads you to believe that you’re supposed to [be] ... dividing fractions” (PE.I.721-723). To connect problem 17 to fraction division, she thought of dividing the first part of the equation she used to solve this problem (see first row of Figure 4.13) by one fourth and then using the reciprocal to solve this division. In doing so, she merely applied the given algorithm without explaining how this approach would help students see the underlying meaning of the algorithm.

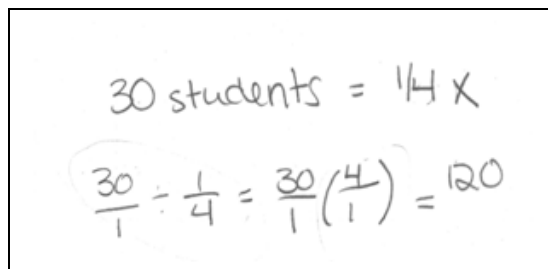

$$30 \text{ students} = \frac{1}{4}x$$
$$\frac{30}{1} \div \frac{1}{4} = \frac{30}{1} \left( \frac{4}{1} \right) = 120$$

Figure 4.13. Deborah’s work on solving problem 17 of the first page.

To summarize, even though Deborah wanted to use the textbook pages to support students’ competence in the process of dividing fractions as well as to help them understand the underlying meaning of this procedure, she seemed unprepared to do so; she could neither capitalize on the affordances of the second page nor could she restructure the tasks of the first page to teach the type of lesson she endorsed.

*Noticing and Interpreting-Evaluating.* When considering how the teacher introduced task  $A_1$  to her students, Deborah focused on the teacher calling on specific students to read the directions out loud. When later prompted to specifically consider the teacher’s presentation of this task, she remarked that if she were teaching the lesson, she

would have not moved directly into the worksheet. Thinking that the teacher presented the task as isolated from the associated word problem, she argued that a better way to introduce this task would have been to use a picture to help students link the word problem to the division of fractions. Drawing on her personal experiences as a learner of mathematics, she also opined that the teacher should not have taken for granted that students had already known the concept of fractions as part of the whole because “kids could be hit over the head with that [concept] a couple times before they got it” (PE.I. 1403-1405). In considering the enactment of this task, Deborah liked how the teacher created a space for her students to be “actively engaged in the problem solving [process]” and “take responsibility of their learning” while simultaneously skillfully guiding them as needed. She also found the enactment of task D quite decent. She thought that the teacher “walked the students through the steps [of getting the algorithm of the division of fractions] pretty well” (PE.I. 2214-2215) and that, overall, she was effective in helping her students figure out this algorithm: “it wasn’t just ... leaving the kids hanging for them to try and figure it” (PE.I. 2226-2227).

Deborah’s noticing and interpreting-evaluating performance suggests that she was more concerned with the introduction rather than with the enactment of these tasks. Because of her own experiences as a learner of mathematics, she thought that the virtual teacher should have talked about the concept of fractions and used a picture to connect the introduced problem to the procedure of division of fractions. Her analysis of the enactment of the tasks under consideration shows that she did not notice the several ways in which the virtual teacher shifted attention from meaning and understanding to following steps; nor did she appear to attend to that the teacher degraded the demands of

these tasks by posing leading questions and doing the thinking for students on the tasks' most difficult aspects. Instead, to Deborah, the teacher's work seemed more like scaffolding students. It could even be assumed that this might have been the type of scaffolding that Deborah would have liked to receive as a learner of mathematics.

### *Providing Explanations*

*Performing.* When Deborah was asked to outline how she would explain the division of fractions to her students, she mentioned that she would like to use a picture to make her explanation more comprehensible. She started explaining the division  $2 \div \frac{3}{4}$  by pointing out that first she needed to define the unit. She then drew two circles to represent the dividend, put a comma, and drew a third circle. After shading in three fourths of the last circle to represent the divisor (see left panel of Figure 4.14), she stopped. The following discussion ensued:

**Deborah:** I don't know how I'd go about doing divided by three fourths. I would sort of be stumped as to do that or. ... Yeah, I really wouldn't know.

**Charalambos:** So what makes it hard for you?

**Deborah:** I think it doesn't step out like intuitively at me. And, you know, despite that it's been quite a long time, I think just, six divided by two or something like that, to me it makes sense. ... [S]ix divided by two makes sense, ten divided by five, etcetera, etcetera. But when you start dealing with like what is actually occurring with the fractions, then I find that to be more difficult articulating.

**Charalambos:** Okay, so if somebody were to ask you to say, "What does this mean?" What would you say that two divided by three fourths means?

**Deborah:** Me? Right now? I really wouldn't know what to say to a student. Because I think I'm still at the stage myself when, you know, it's just a calculation. (PE.I.257-285)

....

**Charalambos:** What does division mean for you?

**Deborah:** Um, I don't know. I should be using like "How many times one unit or something goes into another?" or um [pause].

**Charalambos:** Okay. So if you were given the division ... ten divided by two, can you make a problem [for this division]?

**Deborah:** Sure! You have ten children in two classrooms, how many children are in each?

**Charalambos:** Okay, can you make a different one? Ten divided by two.

**Deborah:** Divided by two? ... I don't know, like if ten candies and only two friends, how many candies do [sic] each of them get? (PE.I. 347-379)

These excerpts suggest that Deborah was not lacking a concept for division. Although her understanding of division might have not been that solid (given her uncertainty to answer the question, “What does division mean for you?”), she still conceptualized division as the number of times “one unit can go into another unit.” This conceptualization could have laid the foundation for explaining the division  $2 \div \frac{3}{4}$  since by drawing on it she could have thought of this division as how many times three fourths goes into two. However, she could not apply this piece of knowledge to explain the given division. What might have been causing these difficulties? Below I consider three plausible reasons which are not necessarily mutually exclusive.

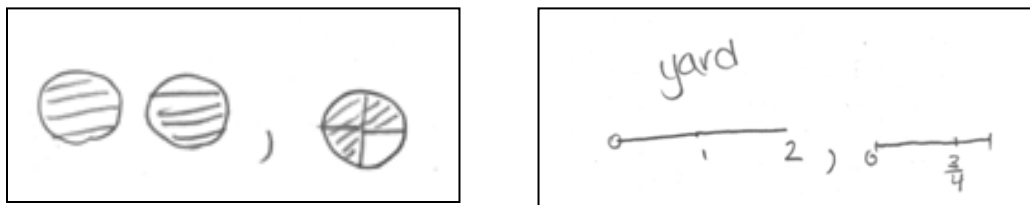


Figure 4.14. Deborah’s attempts to explain the division problem  $2 \div \frac{3}{4}$ .

First, Deborah’s understanding of division seemed to have been dominated by the partitive rather than the measurement interpretation of division. For example, when asked to propose two problems for a whole-number division (i.e.,  $10 \div 2$ ), both the problems she proposed echoed the partitive interpretation of division. As discussed in Chapter 2, this interpretation is not particularly conducive to understanding the division of fractions.

Second, Deborah appeared to be thinking about fraction division merely from a procedural perspective. As she argued, whereas division of whole numbers made sense to her, she considered the division of fractions merely as a calculation. Her procedural understanding was also implied by her representation of the dividend and the divisor: the dividend and the divisor were presented as *separated* entities with no obvious



connections between them. In fact, the comma she put in her representation between the dividend units and the divisor units corroborates this argument.

One could argue, however, that her struggle with explaining this division might have stemmed from her being asked to explain the given division out of context. Yet, even when encouraged to put the given division into a context, she again faced notable difficulties. In particular, drawing on the second page, she first thought of the given division as dividing a two-yard ribbon into bows of three fourths of a yard. She then represented this problem as illustrated on the right panel of Figure 4.14. Her drawing was consistent with her earlier representation since she again represented the dividend and the divisor units separately from each other and with a comma in between. Even thinking in this context did not provide her with additional insights into fraction division; she admitted being stuck and felt like “there’s ... a step missing for [her]” (PE.I.337-338).

A third explanation is also plausible. At a later point in the interview, and while considering task A<sub>1</sub> of the second page ( $\frac{1}{2} \div \frac{1}{6}$ ), Deborah used a drawing quite decently to show this division (see Figure 4.15). She drew one circle and shaded in half of it. She then wrote a comma, drew another circle, and divided the second circle into sixths. She shaded in one sixth and contended that in this case, one could get the answer to the problem right from the picture:

Okay, so this one at least I think someone could get from the picture itself. You know, so if it takes one sixth of a yard of ribbon to make one bow ... and you have one half, how many is that equal? And you can tell by the picture that, you know, it’s three. [Shading in three sixths in the second circle:] one, two, three. (PE.I. 966-970)

Prompted to elaborate upon her thinking, she then divided the shaded half of the first circle into three parts, arguing that you can think about this division either by considering the first or the second circle. Deborah’s work on this division suggests that her thinking

might have been driven by the *answer* to the division problems she was asked to consider. Hence, while she could quite easily show the quotient of the problem  $\frac{1}{2} \div \frac{1}{6}$  since it did not involve any fractional parts, the quotient of the division  $2 \div \frac{3}{4}$  might have complicated matters for her. To the extent that this third explanation holds, it also seems reasonable to argue that her thinking and explanations were merely numerically driven.

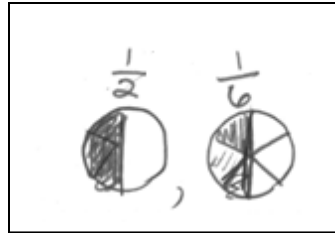


Figure 4.15. Deborah's work on the division problem  $\frac{1}{2} \div \frac{1}{6}$ .

Deborah did not even try to explain the reciprocal for the division  $2 \div \frac{3}{4}$ , arguing that she herself did not know why the reciprocal works. As she mentioned, she just knew that “a fraction times its inverse equals one” (PE.I. 426-427). Interestingly enough, when asked how she would respond to Michelle's question about why the reciprocal works, she answered that “I think I would just do it multiple times and, you know, show that it actually does work” (PE.I. 2263-2265). In other words, she thought that showing that an algorithm works in several cases explains why this algorithm works.

To summarize, Deborah wrestled with providing an explanation for the quotient of the division  $2 \div \frac{3}{4}$  and did not even try to provide an explanation for the reciprocal. Her struggles with providing these explanations appeared to stem from her own difficulties with the content: she appeared to perceive division merely from a partitive perspective; although she also thought of whole-number division as “how many times one unit goes into another unit,” she could not transfer this idea to dividing fractions;

moreover, explaining the fractional part of quotients appeared to complicate matters for her, given that she could not ground her explanation in the quotients' numerical value.

*Noticing and Interpreting-Evaluating.* Deborah did not originally comment on the quality of the virtual teacher's explanation as to why the reciprocal works (i.e., Michelle's question) or that the teacher did not actually answer Robert's question. Prompted to discuss the teacher's explanation to Michelle's question, she argued that she did not identify anything problematic in the teacher's explanation. However, she felt that the explanation was lacking more "substance" and that Ms. Rebecca described rather than explained why the rule works. In Deborah's words, "She's like, 'The reciprocal works because ... you invert the second fraction.' I mean that's the definition of the reciprocal. ... I think that that's not answering the question sufficiently" (PE.I. 2678-2691). Deborah also found the teacher's analogy of adding and subtracting negative numbers difficult to understand. However, she did not necessarily consider it deficient. Instead, she wondered how much more the teacher needed to tell students to sufficiently respond to Michelle's question: "She could probably talk for a very long time about why [the reciprocal] works, but I don't know if that's sufficient or enough for them to get" (PE.I. 2713-2714). Her final remark was legitimate, in that lacking actual teaching experience, she could not decide how elaborate an explanation should be for sixth graders. However, considered as a whole, Deborah's evaluation did not address the mathematical substance of the teacher's explanation. For instance, Deborah did not comment on that the teacher was using semantic instead of mathematical arguments to explain the reciprocal or that the analogy she employed was actually inappropriate.

### *Using Representations*

*Performing.* Deborah's use of representations has already been discussed when considering her work on providing explanations; yet three additional comments are in order. First, despite her difficulties in using these representations to explain the quotient and the reciprocal in a given division, she was a strong proponent of representations as scaffolds of student thinking. This stance had apparently been informed by her experiences as a learner of mathematics. As the reader might recall, Deborah recollected that as a visual learner, she tended to struggle with the "oral" explanations she was given and wished that her teachers had accompanied their explanations with pictures.

Second, in explaining the quotient of the division of fractions, Deborah used two representations: a circular and a linear representation. Although both representations were suitable for explaining the division  $2 \div \frac{3}{4}$ , one would expect her to have used a linear representation when considering the second page's ribbon problems; however, in two cases (see Figures 4.12 and 4.15), she used circular rather than linear representations. This was not haphazard; as she acknowledged, she was more accustomed to using "pies" when dealing with fractions. In fact, when prompted to consider the virtual teacher's use of a linear representation instead of a circular one, Deborah remarked that the teacher was justified to use a linear representation because she was dealing with ribbons and not pizza. This last comment seems to suggest that Deborah tended to associate work on fractions with circular pizza models.

Third, in accordance with the argument made about her performance in providing explanations, it seems reasonable to also suggest that her use of representations was to a large extent *numerically* driven. Even when Deborah used a drawing to show the quotient

for the division  $\frac{1}{2} \div \frac{1}{6}$  (see Figure 4.15), she did not explicitly talk about fitting divisor units into the dividend units. As already discussed, her separation of the dividend units from the divisor units by the use of a comma can also be considered indicative of her numerically driven use of representations; Deborah seemed to simply be *illustrating the numbers* included in a division of fractions.

*Noticing and Interpreting-Evaluating.* After considering Amanda's use of the linear representation on the board, Deborah argued that if she were the teacher she would have stopped Amanda and asked her to explicitly outline how she used the traditional algorithm to solve the problem. Considering the student's work at this point from a more procedural perspective, Deborah did not entertain the idea that Amanda got her answer from the drawing shown on the board. It should not be surprising, then, that Deborah did not comment on the absence of any connections among the representation, the word problem, and the mathematical sentence considered in this episode.

While going over the second episode, Deborah had a hard time understanding what Amanda and Julia were doing on the board. She wondered why they divided their line into twelve pieces, why they colored nine of them red, and what the remaining three blue pieces represented. She also wondered why the two girls were grouping the red pieces into pairs. Despite the time it took her to figure out the answers to all these questions, when asked whether she would have reacted to the girls' work differently from how the virtual teacher did, Deborah did not see any reason for doing so, probably because of not seeing any shortcoming in the students' use of representations. Two explanations can be offered for her performance at this point. First, the difficulties she encountered understanding the girls' representation might have not alerted her to the fact

that neither the two girls nor the teacher drew explicit connections between their representation and the mathematical sentence written on the board. Alternatively, being more concerned with understanding the error in Amanda and Julia's solution (as discussed below) she underestimated their inadequate use of representations. However, the fact that she neither attended to Amanda's use of representation in the first episode renders the first explanation more plausible.

### *Analyzing Students' Work and Contributions*

*Performing.* In considering the three student solutions, Deborah quickly identified that Robert's solution was correct and concluded that "from all appearances, [Robert] seems to get it" (PE.I.2641-2642). In considering Ann's solution, Deborah initially thought that Ann took the reciprocal of part of the dividend. On a second thought, she figured out that Ann did not actually take the reciprocal of the divisor, as she was expected to do. Hence, Deborah concluded that Ann did not understand what she was doing or why she was doing it. For Deborah, Ann appeared to have only understood that she had to "flip" numbers.

Contrary to her fluency in analyzing Robert's and Ann's work, Deborah had a hard time understanding Michelle's solution. Not only did she take a long time to figure out how Michelle represented the dividend and the divisor in her work, but she also incorrectly concluded that Michelle's answer was two and three fourths. When asked to elaborate upon this idea, Deborah confessed not knowing what Michelle was actually doing; she sensed, however, that there might have been an error in her work similar to that in Amanda's and Julia's solution (see below). Because of her difficulties in understanding Michelle's solution, Deborah resorted to the division-of-fractions

algorithm to figure out the answer to the problem. Alas, this answer did not provide her with any additional insights:

I mean I understand ... that she divided them [the three lines that she used to represent 2 and  $\frac{3}{4}$ ] into four pieces [each]; I understand what the eleven fourths was. But ... I would have a hard time helping her out without telling her to abandon her picture, which, I think, can be useful; I just don't know how to use what she's drawn. (PE.I.2650-2655)

In sum, Deborah was quick enough to analyze Robert's work; however, her assertions about what Robert understood were rather thin since she did not entertain the idea that Robert might mindlessly have followed the algorithm discussed in the lesson. Her struggle with analyzing Michelle's work should not be dissociated from her own difficulties in explaining the fractional part of the quotient in division-of-fractions problems. Although she correctly identified the dividend and the divisor in Michelle's work and understood that Michelle was trying to show how many times the divisor units fit into the dividend units, the fact that the divisor units did not fit evenly in the dividend units prevented Deborah from making any further progress in unpacking Michelle's work. Even finding the numerical answer to the problem did not appear to scaffold her thinking. Hence, albeit being a keen supporter of representations, Deborah considered recommending that Michelle abandon her picture and revert to a more algorithmic approach to solve this problem. This finding highlights another way in which a teacher's narrow grasp of the content might impose limitations on what she can do while working with her students.

*Noticing and Interpreting-Evaluating.* Deborah did not notice any of the four problematic student contributions considered for this practice (i.e., June's explanation, Ann's argument, Alan's error, and Amanda and Julia's work on dividing  $\frac{3}{4} \div \frac{1}{6}$ ).

When considering June's explanation, Deborah just argued that June seemed to have understood that one sixth is smaller than one half and that she was "thinking critically" (PE.I.1275). When later prompted to reconsider June's explanation, drawing on her own struggles with grasping this explanation, she argued that June was not articulating her ideas clearly enough; later on, however, she remarked that June did not situate her explanation in the context of the ribbon problem, which would have been helpful for supporting her classmates' understanding. Although this remark was correct, Deborah still did not capture the misconception underlying June's explanation (that the dividend is always larger than the divisor). Overall, Deborah's evaluation of June's explanation should not be divorced from her own understanding of the content. As she mentioned after appraising June's work,

I mean, I know the problem is one half divided by one sixth and so it's, you know, how many times does one sixth go into one half? But like I said, in terms of me actually being able to communicate that concept or that sort of thinking, I don't -- I wouldn't know what to do. (PE.I.1488-1493)

As her comment suggests, Deborah's understanding of the content was not solid enough to support her in providing an explanation, let alone analyzing and evaluating the explanation that June offered.

When prompted to reconsider Alan's work, Deborah identified his error, pointing out that Alan divided half of the line instead of the whole line: "It's really six sixths equals one; you know, it's one unit in and of itself. So, I think this whole line should be divided into six [parts]" (PE.I.1556-1557). Even though she appropriately analyzed Alan's work, she was not very confident in her analysis. For instance, at one point, she argued, "I say this with hesitation, but [Alan] is only operating on one half of the line, and it should be across the entire line" (PE.I. 1562-1564).



When prompted to reconsider Amanda and Julia's solution, Deborah reverted to the division-of-fractions algorithm to figure out whether the girls' answer of "4 and remainder  $\frac{1}{2}$ " was correct. Although the answer of four and one half she got from applying the algorithm helped her determine the correctness of the two students' solutions, it did not provide her with any scaffold to identify the discrepancy between the girls' solution and hers: "Three fourths divided by one sixth is not equal to four and one twelfth. But I really don't know if that one twelve is like something else that they're trying [to do]. I don't know" (PE.I. 2083-2086). Even after having gone over the students' work for a second time, Deborah could not find anything wrong in the girls' work: the students correctly identified nine twelfths as the dividend and they took groups of two twelfths, which corresponded to the divisor unit. And yet, they got a different answer to the problem than she got by using the algorithm. To Deborah, this was a conundrum: "I don't know where they went awry. I didn't quite understand what was going on" (PE.I. 2432-2425).

The discrepancy she identified between her answer and that of the two girls reminded her of an analogous puzzlement she experienced while answering question 17 on the LMT test:

It was something to do with like you have X number of -- you know, you have like two thirds of a cup of flour and you can make pies and then you have six cups of flour, how many pies can you make? And one kid said "It's like nine and one sixth of a pie," and someone else said, "It's nine pies and one sixth of a cup of flour." And that didn't seem right to me, but I was like "Wait a minute!" you know and I was like "No, it's not." But I don't know; that sort of messed me up. (PE.I. 801-808)

This test question, which she answered incorrectly, was actually designed to explore the PSTs' understanding of the fractional part of quotients. The connection that she made between her two experiences supports the association between knowledge and teaching

performance examined in this study. It is also indicative of how gaps in a teacher's knowledge might limit her in dealing with unexpected students' solutions:

Honestly, I really don't know what I would do in that situation. 'Cause I think I would have to understand what it was that they were trying to do and where they went wrong. And right now I don't. So I'm glad I'm not in that position right now (laughs). (PE.I. 2166-2170)

To summarize, Deborah did not attend to any of the problematic aspects of the students' contributions. The only student contribution she appropriately analyzed after being prompted was Alan's, and even in this case she was hesitant as to whether her analysis was correct. Her performance in this practice, and especially the connections she made to her experiences in working on the LMT test, corroborate the association between knowledge, as measured by this test, and performance in this practice, as tapped by the teaching simulation.

#### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Deborah's performance in this practice differed across the two episodes under consideration; these differences were largely due to her understanding of the mathematics at stake in each episode.

As already discussed, Deborah correctly identified the error in Alan's work, which helped her come up with some ideas to support him in correcting his error:

**Charalambos:** Okay. So if you were the teacher, what would you do at this point [when Alan divided half the line into six parts]?

**Deborah:** Um either ask him a question -- I don't quite know how I would phrase that question, but I'd try and get at his thinking rather than saying that he's wrong or that he's right at this juncture. But if he said that he's only looking at, you know, one [part of the line; only half the yard] then I think that I'd probably redraw or ask him to redraw [the line] so it's open to the class. But I think you have to ask him what he's thinking.

**Charalambos:** Um hum. So I'll press you a little bit to model the question that you will ask him; to say a little bit more. What would you do with him? So ... at this moment Alan is on the board, he draws what he draws, and you're the teacher.

**Deborah:** Yeah. (Laughs.) Um [pause] I think I'd ask him why he chose to draw what he drew. ... I don't think he drew the whole thing. So, I think I'd ask him like what he was thinking. ...

**Charalambos:** Okay, I'll try to be Alan. ... Okay you're asking me what I was thinking, and I say, "Well, the problem asks me to consider half of the line. Isn't it what the problem's asking me?"

So, I took half of the line, and the problem says that I need to consider one sixth. So, I took half of the line and I split it into six pieces.”

**Deborah:** Right. (Laughs.) I don’t know. I would say that “I see where you’re going”; you know, try to validate him before I shut him down. And then, you know, explain what the unit like -- that you’re not looking at one half; you’re looking at one sixth and you have to draw two separate things rather than drawing it on the actual one-half line. (PE.I.1589-1626)

Deborah’s plan to support Alan was pedagogically appropriate: not only would she avoid shutting Alan down by pointing out that he was wrong, but she would also try to engage him in a discussion to help him understand his error. However, her approach lacked mathematical substance. For instance, although she wanted to phrase a question that would support and not disillusion Alan, she was not clear what this question should be or how it should be phrased. Similarly, even though she would prompt Alan to revise his work by redrawing the diagram, it is not clear how this move would support Alan in fathoming his error. In fact, simply asking Alan to correct his drawing is exactly what the virtual teacher did, a move that Deborah appeared to endorse, as discussed below.

Being pushed to clarify how she would scaffold Alan, Deborah started thinking more about the mathematical substance of Alan’s error and identified the notion of unit as a key idea to be addressed when supporting Alan. Her comments were in the right space, but they needed more unpacking to really support Alan. For instance, instead of saying, “you are not looking at one half, you are looking at one sixth,” Deborah needed to clarify that the problem was asking Alan to consider sixths of a yard. Hence, he needed to divide the whole yard (i.e., the whole line) into six parts. By simply saying that Alan should not be looking at one half but at one sixth instead, Deborah could aggravate Alan’s confusion because, in fact, the problem was asking him to consider both halves and sixths. The idea of showing the one half and the one sixth in separate lines that Deborah then entertained also had some potential to support Alan. To be functional, however, it should have been

accompanied by a pertinent mathematical justification. For instance, Deborah could have pointed out that drawing two separate lines allows better representing division as fitting divisor units into the dividend units.

In the second episode, Deborah confessed that she had never thought of using common multiples when dividing fractions. Hence, she regarded the teacher's suggestion that the students use a common multiple "wise" (PE.I. 1703) on the premise that it would help the students visually see "how much [the divisor] fit [into the dividend]" (PE.I.1702-1703). This comment suggests that Deborah had *a general* understanding of why using a common multiple could help the students overcome the difficulty they had in dividing the two non-commensurate fractions. However, she could not articulate how using the common multiple of twelve would support students. For example, when asked how she would react in a similar situation, she acknowledged that she was still not clear how the teacher's suggestion would scaffold students' work; she mentioned, "I kind of want to know what [the teacher] is gonna do" (PE.I. 1716-1717) and elsewhere, "I was clueless as to how to explain it [dividing three fourths by one sixth]" (PE.I.1722-1723).

To summarize, Deborah's grasp of the mathematics involved in June's and Shaun's episode was weaker compared to her understanding of the mathematics in Alan's episode. This narrow understanding of the content appears to have narrowed her options for supporting June and Shaun.

*Noticing and Interpreting-Evaluating.* When directly asked to discuss the virtual teacher's work in supporting Alan (in the first episode) and June and Shaun (in the second episode), Deborah found the teacher's approach satisfactory. In particular, in discussing Alan's episode, she did not attend to the very pointed questions the teacher

asked Alan, nor did she comment on that the teacher was doing most of the thinking for him. Instead, she thought that the teacher was very deliberate in the questions she was posing:

**Deborah:** I think [Ms. Rebecca] knew what it was that [Alan] was doing before he knew what he was doing. And I think she knew what question to ask to get him to see what he had done. And so she's always like five steps ahead of the child. And I think that that's evident in the questions she asks.

**Charalambos:** And how can you tell that she's five steps ahead of the child?

**Deborah:** Because she anticipates their questions, she anticipates what they're going to do, and she knows what she would do if that situation -- I mean I don't know, but I'm assuming that she knows what would happen if that situation arose. (P.E.I. 1785-1798)

From Deborah's perspective, the teacher's questioning was meritorious: she was "five steps ahead of the students," she could anticipate their responses, and she knew what to do in each situation.

Deborah's evaluation of the teacher's work in the second episode was similar to that in the previous episode. She argued that it was nice that the teacher did not directly give students an answer and that she instead tried to engage them in a discussion.

Deborah also considered the teacher's asking students to recall what they did in previous lessons a remarkable move, because in doing so, the teacher was helping them use a prior piece of knowledge "cumulatively in what they [were] doing [during this episode]." This, in turn, would help them "realize that [they were] not learning distinct things ... [but] something that applies to what [they] were doing" (P.E.I.1747-1753). On the surface, Deborah was right: the teacher did try to build connections to previous lessons. However, Deborah did not entertain the idea that the teacher's recommendation to use the common multiple of twelve might have not made any sense to the students, just like it did not make sense to Deborah.

### *Performance on the LMT Test: A Closer Look*

Deborah's performance on the LMT test was in accord with her performance in the teaching simulation since the questions she correctly answered did not require deep thinking of the mathematics and its teaching. In particular, she correctly answered three of the CCK questions (i.e., questions 4, 9, 22) and some SCK questions which could be easily answered by first figuring out the numerical answer to the word problems presented in these questions (e.g., questions 2, 8c, 8d, 11a-11d). In contrast, she answered incorrectly SCK questions that warranted a deeper analysis of the mathematics at stake and its teaching (e.g., questions 14, 15, 17, 19, and 25). For instance, based on her written work on the LMT test, in solving questions 14a-14e, she first calculated the quotient of the division  $\frac{1}{6} \div \frac{2}{3}$ . Figuring out this quotient, however, did not seem to have helped her to determine whether the interpretations of this division were valid. Similarly, in question 15, although she figured out the quotients of the given divisions (i.e.,  $4 \div 3$  and  $4 \div \frac{4}{3}$ ), she could not determine whether the given representation matched any of these divisions. The questions that Deborah answered incorrectly also targeted ideas with which Deborah struggled during the teaching simulation (e.g., questions 1 and 20 that addressed the concept of unit; question 17 that addressed the fractional part of quotients, etc).

### *Analytical Commentary*

Unlike the previous two cases, Deborah's case supports the strong association between MKT and teaching performance from a constraint perspective; at the same time it points to other factors that appear to have informed the PSTs' teaching performance. What makes Deborah's case unique, though, is that it helps better examine whether these other factors suffice for the performances explored in this study. Therefore, in discussing

her case, I consider the knowledge and performance association *after* considering how other factors appeared to have informed her teaching performance.

As Figure 4.16 shows, Deborah's performance seemed to have been informed by her school experiences and a gamut of beliefs, including her efficacy beliefs and her beliefs about teaching and learning math. Although for analytic purposes I separate those factors, in reality they cannot be clearly distinguished since Deborah's school experiences appear to have been a critical contributor to the formation of her beliefs.

Deborah's reported school experiences were not particularly positive: emphasis was placed on getting correct answers and not on thinking and understanding; accordingly, "wrong" answers were not validated and did not comprise sites for further thinking and learning. These negative experiences appeared to constitute avoidance models for Deborah. Due to her own difficulties with learning and understanding mathematics in the aforesaid contexts, Deborah believed that mathematics should make sense and that emphasis should be placed on helping students develop both conceptual understanding and procedural fluency. Convinced that mathematics should not be perceived as a discipline of "right" and "wrong" answers, she also thought that the teacher of mathematics should capitalize on students' incorrect answers. Moreover, being a visual learner herself, she regarded pictures and drawings as useful tools for supporting student learning and thinking. At the same time, she appeared to believe that repetition fosters learning, as suggested by her comment that students should be "hit with the same concept a couple of times before they get it."<sup>145</sup> Taken together, Deborah's beliefs can be considered conducive to creating the rich environments considered in this study. Thus in

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<sup>145</sup> Her endorsement of the idea of repetition as a means to support student learning was not necessarily antithetical to her other beliefs since a teacher can support student learning by recurrently revisiting and emphasizing the meaning of certain ideas.

Figure 4.16, their positive influence is presented with the addition sign. Deborah's case, however, suggests that *beliefs alone do not suffice to build the type of the learning environments the study considers*. Before justifying this argument, I first consider two other factors that appeared to have a negative impact on her teaching performance: her school experiences and her efficacy beliefs.

By arguing that Deborah's school experiences constituted avoidance models, I maintained that these experiences appeared to be productive for her own teaching performance. At the same time, though, these experiences could also have had negative entailments on her performance, because they might have restricted the images for teaching that Deborah had at her disposal. To put it differently, Deborah might have wanted to teach for learning and understanding, but her own images of teaching might have not supported her in this endeavor. I will return to and elaborate upon this argument in Chapter 5, while considering the changes in Deborah's knowledge and performance. For the time being, I illustrate the impact that Deborah's limited images of teaching appear to have had on her performance with a negative sign (-). Given that these images of teaching might have also helped Deborah form avoidance models (as explained above), in Figure 4.16 her school experiences and her corresponding images of teaching are presented with both a positive and a negative sign.

Deborah's doubts about her own efficacy in mathematics should also not be underestimated. Besides, as the reader might recall, the quantitative analysis showed this factor to be a strong mediator of the knowledge and performance association. An example that speaks to how her efficacy beliefs manifested themselves in her performance in the teaching simulation pertains to the doubts she was seen having about



her analysis of the students' work and contributions (e.g., when analyzing Alan's work). These doubts should not be dissociated from her own efficacy beliefs about understanding mathematics and working in this realm.

With these two additional factors in mind, I return to my claim that productive beliefs are not sufficient to build rich and challenging environments. I support this argument by considering Deborah's performance in each of the five practices examined in this study. I start with the practices of selecting and using tasks and responding to students' requests for help, and then move to the three MKT-related practices.

When asked to select a textbook page to build a lesson on division of fractions, Deborah displayed a clear preference for the conceptually oriented page, thinking that this page could support more meaning-making. Her selection notwithstanding, she appeared to face significant challenges in designing a lesson that would indeed forge students' conceptual understanding. First, she did not understand the concept of the fractional part of a quotient, which is pivotal when teaching the division of fractions. Second, her own understanding of division in general and the division of fractions in particular was not thorough enough, as suggested by her performance on several of the teaching simulation tasks. Third, as she confessed, she could not see the division of fractions in any way other than simply considering it as a meaningless calculation. Therefore, despite her desire to design a lesson that would enhance both students' conceptual understanding and procedural fluency, her own understanding of the content did not support her in this endeavor. On the one hand, she was unprepared to capitalize on the multiple affordances of the second page to design such a lesson; on the other hand, even if she wanted to, she could not restructure the tasks of the first page to use them in a

manner that would scaffold students' understanding of the content. This was even more obvious when she considered the two word problems on this page. Although she thought that these problems could support student learning, she was not clear on how to use them to this respect.

Deborah's dissimilar performance in the two episodes of the practice of responding to students' requests for help is also suggestive of the association between knowledge and teaching performance explored in this study. In the first episode, Deborah captured the error in Alan's work and correctly identified its source (i.e., the idea of units). Hence, when considering plausible ways in which she could scaffold Alan's work, she rightly referred to helping Alan understand the idea of unit. Despite the shortcomings of her approach, her performance in this episode was better than that in the second episode, in which she admitted being clueless as to why one could use common multiples when solving division-of-fractions problems, thus, she did not know how to respond to June's and Shaun's request for help. Her performance in this latter episode is consistent with the commonsensical idea that the teacher cannot actually support students in making meaning if she herself cannot see the meaning underlying the processes at hand.

Deborah's work in providing explanations is also suggestive of the strong association between knowledge and teaching performance. Having considered the division of fractions from merely a procedural standpoint, by largely understanding division from the partitive perspective and without grasping the fractional part of the quotient in relative units, Deborah could not provide a conceptual explanation for the quotient in a division-of-fractions problem; the only explanation she gave was incomplete and numerically driven; thus, it could hardly make sense to an average sixth grader.

Additionally, she did not even attempt to provide an explanation for the reciprocal. In fact, as she argued, if she were pressed by her students to provide such an explanation, she would use several examples to show them that the algorithm works rather than explaining why it works. This latter idea, which is also telling of Deborah's understanding of what comprises an appropriate explanation, should be associated not only with her difficulties with the content but also with her lack of alternative images as to what constitutes a suitable explanation. Two pieces of evidence support this argument. First, recall that she complained about her own teachers' "oral" explanations; second, when asked to evaluate Ms. Rebecca's explanation, she pointed out that she was not in a place to determine how much of an explanation is adequate for students.

Deborah's difficulties with the content did not prevent her from identifying that the virtual teacher's explanation was insufficient, largely because the teacher appeared to describe rather than explain the rule of the reciprocal. However, her difficulties with the content appear to have hindered her from a more in-depth appraisal of Ms. Rebecca's explanation. She engaged in a *preliminary* evaluation of the virtual teacher's explanation, an evaluation that a student of mathematics could also offer when arguing that "this explanation does not make sense to me or does not help me understand." Moving from this preliminary evaluation to a deeper analysis of the teacher's explanation required more thorough knowledge, which would allow for noticing both the problematic analogy and the semantic rather than mathematical arguments in the virtual teacher's explanation. Deborah's case suggests that without profound mathematical knowledge, it is rather unlikely that the teacher moves beyond a superficial appraisal of the mathematical quality of an instructional explanation.

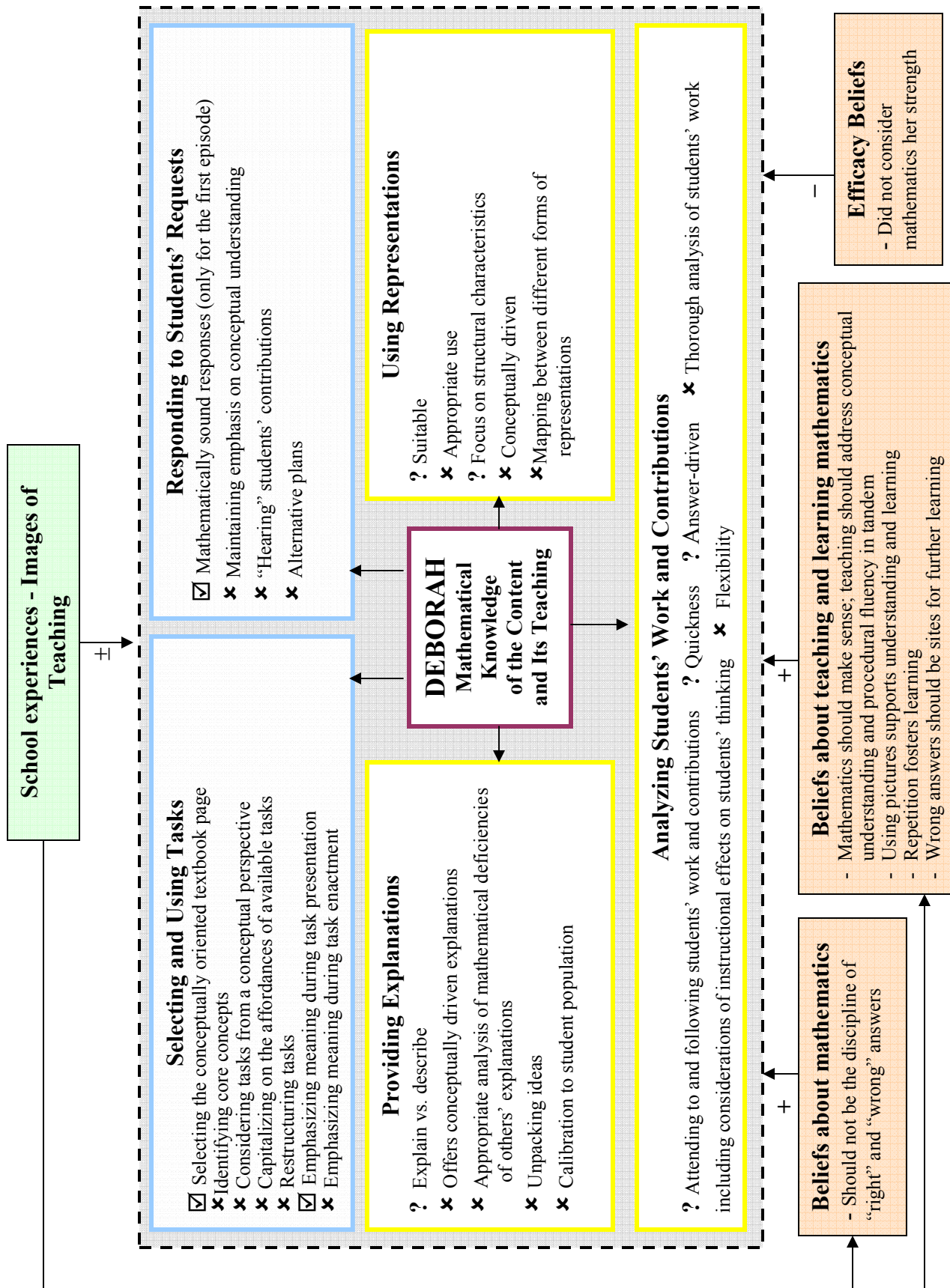


Figure 4.16. Considering the association between knowledge and teaching performance through Deborah's profile.

Perhaps the strongest piece of evidence suggesting that good intentions are not sufficient for effective teaching comes from Deborah's performance in using representations. At several points during the teaching simulation, Deborah made it crystal clear that she liked grounding her explanations in "pictures" to support student understanding. However, her difficulties with the content did not impede her from doing so but were also reflected in the way she was using the representations she selected. For instance, her representation of the dividend and the divisor as two separate and rather unrelated entities should not be divorced from her procedural understanding of division of fractions and from her difficulties in transferring her understanding of whole-number division to the division of fractions. Likewise, her rather numerically driven use of representations should be seen in light of her struggles with the fractional part of the quotient in divisions of fractions. To this, one could also add her not always appropriate selection of representations.<sup>146</sup>

Deborah's performance in analyzing students' work and contributions also supports the association between knowledge and teaching performance. It does so in two different ways: by what her knowledge enabled her to do and by how the limitations in her knowledge constrained her work. From an affordance perspective, one could consider Deborah's performance in analyzing Robert's and Ann's solutions. Being quite proficient in using the algorithm of the division of fractions, she quickly determined that Robert's solution was correct; she was also relatively fluent in figuring out the error in Ann's

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<sup>146</sup> Alternative explanations for her performance in using representations should also be entertained. For example, the fact that Deborah did not attend to the extent to which the representations being used in the virtual lesson were mapped to the algorithm or the word problems under consideration might have stemmed from her limited images of teaching. If during her school years she had not had substantial experiences with using representations, then simply seeing the virtual teacher using such representations in her lesson was definitely a better alternative to what Deborah reported having experienced.

solution. From a constraint perspective, one could refer to Deborah's work on analyzing Michelle's solution to the same problem. Because she herself encountered significant challenges in understanding the fractional part of a quotient, she could not follow and analyze Michelle's work. Confronted with these difficulties, she figured out the numerical answer of the problem, but alas, this answer did not provide her with any additional insights. Without having any additional resources upon which to draw to understand and appraise Michelle's work, Deborah suggested that Michelle abandon her pictorial approach and solve the problem using the algorithm. This suggestion, which was in sharp contrast to her beliefs about using representations when teaching mathematics, is indicative of how good will might not be sufficient for teaching, especially if it is not undergirded by the corresponding mathematical knowledge.

Finally, Deborah's case also provides some direct evidence about the association of the PSTs' MKT as measured by the LMT test and their performance on the teaching simulation. When during the simulation she encountered difficulties explaining the fractional part of the quotient, Deborah confessed having had analogous difficulties in solving the pertinent LMT test question. This confession in and of itself provides additional evidence of the relationship between the mathematical knowledge measured by this test and the knowledge needed for several teaching tasks explored in the teaching simulation (especially those associated with the MKT-related practices). This finding also provides a plausible explanation of why Deborah's ranking according to her MKT score was more consonant with her performance in the teaching simulation compared with her ranking according to her GRE performance: unlike the GRE test, the LMT test was more calibrated to the tasks of teaching examined in the teaching simulation.

### *The Case of Vonda: Knowing and Applying Steps*

Vonda is the fourth convergent case examined in this study. As shown in Table 4.5, relative to her counterparts, she scored low both on the LMT test and the teaching simulation. Vonda took only a couple of math content courses in high school and during her undergraduate studies, with the highest level course being *Algebra II* (cf. Table 3.1). Thus, it was interesting to explore how the association between knowledge and teaching performance was manifested in her case. Vonda's case, like Deborah's case, supports the strong association between MKT and teaching performance from a constraint perspective. Her case also shows how limitations in knowledge – alongside unproductive teacher beliefs and limited images of teaching – can hinder teachers from building learning environments that support students' conceptual understanding of the content. Because of her struggles with the content even from a procedural perspective, Vonda's case also suggests that, although not sufficient for teaching, a basic understanding of the content is indispensable for passable instruction.

#### *Background*

A strong desire to teach and to make an impact on students' lives motivated Vonda's decision to join the ELMAC program, after she had worked for several years in a big private company. As she recollected upon her entrance to the program, in elementary school she had good experiences with mathematics, which had been her "strong subject" (P.E.S.). In middle school, however, she had some negative experiences, which, as she argued, did not mitigate her enthusiasm for the subject.

### *Selecting and Using Tasks*

*Performing.* Vonda considered both textbook pages inappropriate for teaching a lesson on fraction division on the premise that neither of them was explanatory enough, in the sense of providing *step-by-step* instructions for dividing fractions. Even though the first page did provide a worked out example for this operation, she considered this example inadequate; she was even more concerned with the second page, which included no such examples at all. Additionally, for Vonda, neither of the two pages explained why one has to find the reciprocal of the divisor and then multiply it by the dividend. She also considered the two pages comparable in terms of their difficulty.

If she had to use the second page to design a division-of-fractions lesson, she would use its first task to walk the students step-by-step through it, “to give them a better understanding of the process” (PE.I. 610). If given the option, however, she would like to use the first rather than the second page, because she thought that the sequencing of tasks in the first page was more appropriate than that of the second page. She justified her argument pointing out that the word problems on the first page, which she considered harder than the page’s numeric exercises, were presented last.

When using the first page, she would take the worked out example (i.e.,  $2 \div \frac{3}{4}$ ) and “walk the students through the entire steps on the board, step-by-step,” explaining why the divisor “has to be flipped” and then multiplied by the dividend (PE.I.140-144). Yet, her explanation would simply be a somewhat more unpacked description of the steps outlined in the worked out example (see her performance in *Providing Explanations*). After explaining the steps, she would move to the 16 exercises and ask students to solve all of them; she would then have them share their work on the board. She explained: “I



would go through each [exercise]. I would call upon a student to work [exercise] one, [exercise] two, all the way through each [exercise]” (PE.I. 386-388). She further explicated:

I’m going to write [exercise] one on the board. Two thirds divided by one third and then I’m going to ask student number one to come up to the board. Can you solve this [exercise] for me? And walk the class through the steps. (PE.I. 414-417)

Believing that experience (in the sense of practice) “is the best teacher” (PE.I.363), she would follow exactly the same approach to sharing the solution of each exercise; she anticipated that doing so would help students “solidify the processes involved in [the] division of fractions” (PE.I. 129-130).

After having students work on these exercises, she would move to the two word problems. She would read the first problem and ask the students to write the “formula” (i.e., a mathematical equation) for it; she would repeat the same approach in solving the second problem. To solve the first problem, she would expect students to write the formula “ $30 \div \frac{1}{4} = X$ .” Although this equation was correct, when asked to elaborate why she considered it appropriate for solving this word problem, she revised her thinking and wrote “ $X \div \frac{1}{4} = 30$ ” (see first row in Figure 4.17). The following discussion ensued:

**Vonda:** Because, I don’t know how many kids are in the class, and I’m trying to find that out. So  $X$  represents the total number of kids, and one fourth of those kids play baseball. So  $X$  divided by one fourth is going to equal to thirty.

**Charalambos:** [Pointing to the equation  $X \div \frac{1}{4} = 30$ :] And why are you dividing by one fourth here?

**Vonda:** [Hesitantly:] Because I am trying to find out what  $X$  is. And so I do that. I am not sure. ... I’m thinking one fourth of  $X$  is going to give me thirty.

**Charalambos:** One fourth of  $X$  is going to give you thirty. And is this what you have here [still pointing to the equation  $X \div \frac{1}{4} = 30$ ]?

**Vonda:** Um-hmm.

**Charalambos:** Okay. And how are you going to solve it?

**Vonda:** [While talking, she writes the steps shown in the second and third rows of Figure 4.17:] If I take  $X$  over one times four over one equals four  $X$  [pause]. I am going to get [pause]. I have no idea where I’m going with this. Then you take, then you get  $X$  to one side, so  $X$  equals thirty, it would be a hundred and twenty kids.

**Charalambos:** How do you know that’s thirty times four?

**Vonda:** I’m calling on what I used to know. (PE.I. 495-524)

Figure 4.17. Vonda’s work on solving problem 17 of the first page.

Vonda’s work on solving this problem is interesting in many respects. First, although she correctly associated the numbers and the variable she was using with the problem data, her equation was incorrect. In fact, as suggested by her answers to the interviewer’s follow-up questions, she was not sure why it was appropriate to use division in solving this problem. Her response that the equation  $X \div \frac{1}{4} = 30$  represented “one fourth of  $X$  [equals] thirty,” implies that she might have been thinking about this problem largely from a multiplication perspective (i.e.,  $\frac{1}{4} \cdot X = 30$ ); alternatively, she might have been confounding division *into* four with division *by* one fourth, since the equation  $X \div 4 = 30$  can be interpreted as “one fourth of  $X$  equals thirty.” Vonda’s transformation of the equation  $X \div \frac{1}{4} = 30$  to get the answer of 120 was incorrect because when multiplied by  $\frac{4}{1}$ , the left-hand side of her equation yields  $16X$  and not  $X$ , as she concluded. Apart from this mistake, it is also important to notice that she used the same symbol to represent both a variable and the operation of multiplication.

Overall, six aspects of Vonda’s performance in selecting and using tasks are noteworthy. First, Vonda did not identify the different affordances of the two pages for teaching a lesson on the division of fractions. For example, she completely overlooked that the second page asks students to consider the fractional part of their answer and to explain their thinking using written explanations or representations; nor did she consider the affordances of having the students discover an algorithm for the division of fractions

instead of actually spoon-feeding them with such an algorithm. Second, her proposed presentation and enactment of the first task of the second page (i.e., “translating” the word problem into a division-of-fractions equation and then walking students through the steps of this equation) suggests that if she had to use this textbook page she would probably diminish its cognitive potential by simply having students follow a sequence of steps. Third, her selection of the first page for a division-of-fractions lesson seems to have been informed by a latent belief that in teaching mathematics the teacher needs to organize instruction from simpler to more complex tasks; it also appeared to be informed by her conviction that students should be explicitly presented with steps to follow. Fourth, although she criticized both pages for not providing sufficient explanations as to why the algorithm works, as will be later discussed, her own approach in presenting this algorithm did not actually clarify the mathematical meaning of this algorithm. Fifth, her proposed enactment of the 16 exercises of the first page suggests that she would have her students work on these exercises by merely applying a given formula. Sixth, both her description of how she would use the two word problems of this page and her work on solving one of these problems imply that Vonda would work on these tasks from a procedural perspective and rather awkwardly, thus missing an opportunity to help her students associate the operation of interest with its underlying meaning. Taken together, both her selection and use of tasks, as described above, suggest that she would merely seek to enhance her students’ procedural fluency of the division-of-fractions operation.

*Noticing and Interpreting-Evaluating.* Vonda noticed nothing problematic in the teacher’s presentation and enactment of tasks  $A_1$  and D. When directly prompted to discuss the virtual teacher’s presentation and enactment of these tasks, she considered

Ms. Rebecca's work effective. For instance, she endorsed the teacher's presentation of task A<sub>1</sub>, because instead of simply lecturing, the teacher engaged her students by asking them questions, thus giving them an opportunity to articulate their thinking and then get feedback from the teacher. She argued:

If I'm walking through something with you, as a student, telling you how I think this should be solved or what my thought patterns are, and then you confirm it, that makes me, lets me know that I'm on point. You know, I understand what's being told to me. (PE.I. 778-782)

Similarly, in appraising the teacher's enactment of this task, she claimed that the teacher did a good job, because her students appeared to be engaged. In both cases, what mostly captured Vonda's attention was students' participation and the opportunities made available to them to voice their thinking and get feedback from the teacher. These aspects are unquestionably important to consider when teaching mathematics; yet, equally, or even more importantly, is the level at which the content is considered during a mathematics lesson. Vonda did not appear to be concerned with such issues. Instead, she was satisfied with seeing the teacher assuming a more active role and clearly walking her students through a sequence of steps, as suggested by her comments when considering the enactment of task D: "[The teacher] did go back and do some of the things I had suggested earlier: walked [students] through the actual steps that they did. So I was glad to see that" (PE.I. 1351-1353).

To summarize, Vonda's evaluation of the virtual teacher's approach in presenting and enacting both tasks was consonant with her own performance in using the two textbook pages. A strong supporter of the idea of walking students through a sequence of steps, she did not identify any problematic aspects in the teacher's presentation and enactment of the tasks under consideration.

At least three explanations can be offered for Vonda's performance in the practice of selecting and using tasks. First, her beliefs about what she considered effective teaching might have informed both her decisions in the performing tasks of this practice and her evaluation of the virtual teacher's work. Second, one could also argue that her endorsed approach of walking students through a sequence of steps was probably informed by the images of teaching mathematics she had developed as a learner of the subject. I argue, though, that a third explanation is also possible, in particular with respect to her decisions in the performing tasks of this practice: that her knowledge also played a pivotal role in the tasks she selected to use and the way she proposed using them. As both her work on solving one of the word problems of the first page and her performance in the other practices examined below suggest, her difficulties with the content might have prevented her both from seeing the different potentials of the two pages and from using the first page to support more meaning-making. In the analytical commentary, I revisit and further elaborate upon this argument.

### *Providing Explanations*

*Performing.* In considering the first textbook page, Vonda argued that an explanation was warranted to support student understanding of the algorithm at hand. Prompted to clarify how this explanation would look, she thought necessary to first point out that she had not worked on fractions for years. As she clarified later on during the interview, all she knew was that "when you get the reciprocal, you have to flip the fraction and then you multiply it" (PE.I. 1388-1389). Even so, she was willing to offer such an explanation. Drawing on the worked example of  $2 \div \frac{3}{4}$ , and assuming the role of the teacher, she offered the following explanation:

So in order to divide two divided by three fourths [she writes  $2 \div \frac{3}{4}$ , see Figure 4.18] you have to find the reciprocal of three fourths; because you have to divide a fraction by a fraction. [Then pointing to number 2:] And you have a whole number here. And you have to have a common denominator. So, a reciprocal of three fourths is four thirds [she writes  $\frac{4}{3}$ ]; and if you're gonna get a fraction with the whole number two, you're gonna put a one underneath it [writes  $\frac{2}{1}$ ]. And then you're gonna multiply it. Then you're gonna -- your answer's gonna be two times four is eight, and one times three is three [she writes  $\frac{8}{3}$ , and then converts it into a mixed number]. So, your answer will be two and two thirds. Yeah. Now, that is not a full step-by-step process, but that's the best I can do right now. (P.E.I. 167-177)

Obviously, instead of explaining why the algorithm of dividing fractions works, Vonda simply provided a somewhat more unpacked *description* of the worked out example: she mentioned that one needs to “put a one underneath the two,” and that after taking the reciprocal of the divisor, one needs to multiply across. Additionally, her explanation was fragile in two other respects. First, she did not actually explain the mathematical rationale for the steps she mentioned, even though she expressed interest in doing so. For example, she justified using the reciprocal of  $\frac{3}{4}$  by simply mentioning that this is what one needs to do when dividing fractions. Similarly, she did not provide any rationale for “put[ting] a one” underneath the dividend. Second, in providing this explanation she did not use the mathematical terms dividend and divisor. Instead, her explanation merely focused on manipulating numbers. From this perspective, it is debatable if her explanation would be more effective than the “explanation” provided by the worked out example on the first page, which at least uses the term “divisor.”

$$\frac{2}{1} \div \frac{3}{4} = \frac{2}{1} \times \frac{4}{3} = \frac{8}{3} = 2 \frac{2}{3}$$

Figure 4.18. Vonda's work on explaining the division-of-fractions algorithm.

Also interesting in Vonda's explanation was her idea that when dividing fractions, one needs to have a common denominator. Although this could have been perceived as

merely a slippage of language, the discussion that ensued and her overall performance during the rest of the simulation provided evidence on the contrary:

**Charalambos:** You mentioned something about common denominators, and I wasn't getting that.

**Vonda:** Well, in order to divide a fraction, you have to have a common denominator. The bottom number is your denominator. Is that correct?

**Charalambos:** Um-hmm.

**Vonda:** And here you have a four and you have a one [pointing to  $\frac{2}{4}$  and  $\frac{3}{4}$ ]. So those two are not common. ... So when you flip the fraction to get the reciprocal you get four over three. And then that's when you added your one [referring to the denominator  $\frac{2}{4}$ ]. So now you got three. That's the same as saying two times four over three, 'cause one times three is three. Does that make sense?

**Charalambos:** What is the common denominator here?

**Vonda:** Is one times three is three. I guess we could use -- if we looked at number two here [refers to exercise 2 on the first page] you had one third divided by three fourths [she writes  $\frac{1}{3} \div \frac{3}{4}$ , see Figure 4.19 below], you have to get a common denominator before you can divide that fraction.

**Charalambos:** Why?

**Vonda:** I'm not sure at this moment. It's just what I recall from back when I was taught how to divide fractions. So, in order to get these common denominators, something that both numbers would then divide into equally, I think. And in order to do that, the common denominator is going to be twelve in this instance [she writes  $\frac{1}{12}$ ] because -- well, I know that. It's twelve, because four goes into twelve three times. Three times one is three, and three goes into twelve four times, four times three is twelve. So it'll be one and three twelfths.

...

**Charalambos:** Okay. I got this, but I wasn't sure why you didn't do this same thing here with the first one [he points to  $2 \div \frac{3}{4}$ ].

**Vonda:** Because we're only basically dealing with one fraction here. Two in theory is really two over one. Because one goes into two, two times. So if you want to make a fraction out of that [the dividend] you just put the one there [refers to writing the dividend as  $\frac{2}{1}$ ]. It just makes it easier go through the process.

In explaining the algorithm under consideration, Vonda was drawing on what she could remember from her experiences as a learner of mathematics. If these experiences were similar to the type of instruction that she appeared to endorse (i.e., walking students through a sequence of steps) then it is not surprising that Vonda could remember certain processes but she was not sure why these processes are applicable, let alone, under which circumstances they are applicable and what the actual steps involved in them are. The process of finding common denominators to which she referred is usually used in adding or subtracting fractions with dissimilar denominators; although it can also be used as an

alternative algorithm to the traditional division-of-fractions algorithm,<sup>147</sup> it is rather unlikely that Vonda was referring to this algorithm at this point. Instead, she appeared to inappropriately transfer this algorithm from the addition to the division of fractions. Probably having forgotten the exact steps of this algorithm, she also mistakenly multiplied each denominator with its numerator, instead of multiplying the numerator of the dividend with the denominator of the divisor and the numerator of the divisor with the denominator of the dividend. To make matters worse, her explanation as to when this common-denominators algorithm is applicable was even more troublesome: she argued that because the worked example involves only one fraction, employing the aforesaid algorithm is not necessary. Yet, if that were the case, then after writing the dividend as a fraction (i.e.,  $\frac{2}{1}$ ) one should still use the common-denominators approach and not the traditional algorithm. Overall, probably being what Lampert (1986, p. 340) calls “a rememberer and a forgetter” of meaningless mathematical procedures, Vonda provided an explanation that was procedurally oriented, void of any meaning, and even worse, incorrect.

$$\frac{1}{3} \div \frac{3}{4} = \frac{3+12}{12} = \frac{15}{12} = 1 \frac{3}{12}$$

$$\frac{1}{3} \div \frac{3}{4} \quad \frac{\quad}{12}$$

Figure 4.19. Vonda’s use of the common-denominators approach.

A final remark is also in order here regarding Vonda’s performance. In her second utterance in the dialogue outlined above, Vonda argued that in writing the dividend of the

<sup>147</sup> According to this algorithm, one first turns the dividend and the divisor into similar fractions. Because the resulting fractions have the same denominator, they represent same-unit quantities. Hence, one can then find the quotient to the original division by simply dividing the numerators of the resulting fractions.



division  $2 \div \frac{3}{4}$  as a fraction “you add a one.” This clarification is potentially confusing, because, it suggests that when “adding one” the value of the fraction does not change, an idea that contradicts the addition of whole or fractional numbers.

At a subsequent point in the interview, Vonda was presented with the scenario that after she had provided the abovementioned explanation, a student was still puzzled and argued that he “did not get the meaning of this procedure.” Hence, she was encouraged to use some sort of drawing to help this student understand the meaning of the division  $2 \div \frac{3}{4}$ . Despite her initial reservation to do so, she drew a rectangle and divided it into two parts to represent the dividend (see Figure 4.20). She then drew another rectangle, attached to the first one, and approximately equal in size to the original rectangle. To represent the divisor she divided this second rectangle in four parts and shaded in three of them. She then paused and argued that she could not use a picture to explain this operation; she identified the three fourths as the main source of her struggle.



Figure 4.20. Vonda’s drawing for explaining the division problem  $2 \div \frac{3}{4}$ .

*Noticing and Interpreting-Evaluating.* Vonda quickly noticed that the virtual teacher did not address Robert’s question. She considered the teacher’s decision to defer Robert’s question to the next day inappropriate, because his question would have not been “fresh in [students’] minds” the next day (PE.I. 1554). Although this was a valid justification, more importantly, addressing Robert’s question was key for supporting students’ work on the next assigned task, something that Vonda did not notice.

Vonda neither attended to any problematic aspects of the teacher's explanation for why the reciprocal works. When prompted to discuss the quality of the teacher's explanation, she remarked that by offering the analogy of adding and subtracting negative numbers, the virtual teacher was trying to build some connections to previous lessons. On a second thought, however, she found this analogy unrelated to the division of fractions: "I don't know what the positive and negative numbers have to do with [finding the reciprocal in the division of fractions]. ... That seems to be off-lesson to me" (PE.I. 1595-1599). Although not particularly elaborated, Vonda's evaluation of the analogy that the teacher used in her explanation was legitimate. Interestingly enough, the PSTs who scored better than Vonda on both the LMT test and on the teaching simulation did not capture this problematic aspect in the teacher's explanation. This might have been due to their attempt to force a connection between the process of finding a reciprocal and the process of adding and subtracting positive and negative numbers; for Vonda such a connection apparently did not make any sense.

### *Using Representations*

*Performing.* As already discussed, Vonda attempted to use a drawing to explain the quotient in the division  $2 \div \frac{3}{4}$  after she had directly been prompted to do so; she nonetheless could not use her drawing to explain this division (see Figure 4.20). This drawing, however, is interesting in three respects. First, like Deborah, Vonda presented the dividend units and the divisor units as different entities without any obvious connections between them. Second, she did not represent the divisor units appropriately; notice that the rectangle she drew to represent the three fourths was approximately equal in size to the rectangle she used to represent the quantity of the dividend. In other words,

Vonda did not even correctly represent the divisor in *absolute* units. Hence, had she tried to fit the divisor units she drew into the dividend units, she would have arrived at an incorrect answer. Third, her representation was largely numerically driven, thus resonating with her procedural understanding of (fraction) division.

*Noticing and Interpreting-Evaluating.* Vonda commented on the first episode designed for this practice only after being prompted. Consistent with her performance in the previous practices, she remarked that Amanda did not show the intermediate steps of her work. Like Deborah, she expected Amanda to use an algorithm to solve the problem; she did not entertain the idea that Amanda could just have gotten the answer to the problem by simply capitalizing on the representation shown on the board. Thus, she criticized Ms. Rebecca for accepting Amanda's work and not "walk[ing the students] through the steps" (PE.I. 1131). By again focusing on the steps that the students had to follow to get a correct answer, Vonda did not attend to the lack of connections among the representation, the mathematical sentence, and the word problem being used in this episode.

Similarly, in considering the second episode, Vonda focused on that Amanda and Julia did not "show the steps involved in solving the problem" (PE.I. 1155-1156). She argued that if she were the teacher, she would have asked another pair of students to come up and "explain what actually happened, the steps that they did" (PE.I. 1164-1165). Prompted to clarify what she meant when referring to "steps," she explained that she was thinking of the steps involved in the process of dividing fractions. Consequently, the fact that she did not pay any attention to issues of mapping between different representations

could partly be due to her concentrating on following and applying algorithms. However, two alternative explanations are also plausible.

First, it is likely that Vonda had no alternative images of teaching upon which to draw and evaluate the teacher's work in this episode. If her own school experiences as a learner of mathematics were limited to simply following and applying steps, it is highly unlikely that she could see and talk about the lack of connections, since she was probably not aware that such connections needed to be built. Second, it is also plausible that Vonda's own difficulties with the content did not support her in analyzing the episode under consideration in a more nuanced way; this latter explanation is more applicable to the second episode. In this episode Vonda could not see the relationship between the divisor of the one sixth the two girls used in their mathematical sentence and the two-twelfths segments which Amanda and Julia used in their drawing. Hence, it is reasonable to assume that some basic understanding of the mathematics at hand is prerequisite for a more nuanced analysis of a teaching episode (e.g., for observing if representations are used to their full potential to support students' understanding).

### *Analyzing Students' Work and Contributions*

*Performing.* Vonda's analysis of the three students' solutions exemplifies how limitations in a teacher's knowledge can constrain her in understanding and analyzing student work. Consider, for example, her analysis of Robert's work:

Robert is multiplying four times two is eight and then adding three to get eleven fourths, I think. And then he kept his divisor the same. And then here, he got the reciprocal of his divisor here; and then did the multiplication. (PE.I. 1608-1611)

Vonda closely followed each step of Robert's solution. Yet, having learned and forgotten another procedure, she could not decide whether his solution was correct, simply because

she could not determine whether Robert's conversion of the mixed number into an improper fraction was correct.

Vonda's difficulties were even more palpable when analyzing Michelle's work:

Michelle... took two wholes and three fourths. And then she counted, one, two, three, four, five, six, seven, eight, nine. She counted nine equal parts, but I don't know what that represents. (PE.I. 1611-1614) ... I'm not sure what Michelle's representing here. I don't see any of the numbers there. (PE.I. 1650-1651)

These difficulties should not be dissociated from Vonda's shaky understanding of the concept of division in general. Had she been more solid on this concept, she might at least have been able to decipher that Michelle was trying to fit three-fourths pieces (i.e., divisor units) into the line segment representing the dividend, as Deborah was seen doing. However, as the reader might recall, Deborah, unlike Vonda, appeared to have such a rudimentary conceptual understanding of division (i.e., she thought of division as the number of divisor units that can be fit into the dividend units).

Vonda initially thought that Ann was closer to the correct answer than the other two students, because "of the process she used" (PE.I. 1628). When asked to elaborate this idea, she revisited Ann's work, and after going over this student's solution for a second time, she figured out that Ann took the reciprocal of "the first three fourths" (PE.I. 1652) instead of the reciprocal of the divisor. From this perspective, her analysis of Ann's work was correct. Yet, she did not comment on the fact that Ann took the reciprocal of only part of the dividend. Hence, it is questionable how Vonda would have appraised Ann's work if Ann had been asked to solve an exercise in which the mixed number had appeared in the divisor instead of the dividend and if Ann had taken the reciprocal of only part of divisor (i.e.,  $\frac{3}{4} \div 2\frac{3}{4} = \frac{3}{4} \cdot 2\frac{4}{3}$ ).

Vonda's difficulties in analyzing the three students' work appeared to have prevented her from making substantive assertions about these students' understanding. She vaguely argued that the three students "understand some of the concepts, but not all of them" (PE.I. 1666-1667), but she could not clarify which those concepts were.

Overall, despite her attempt to closely follow the students' solutions, Vonda could not understand much of their work. Her procedural understanding of the division of fractions appeared to support her only in examining whether these students followed the steps of the procedure at hand. Yet, it could not support her in determining the correctness of the students' work, especially in those cases that the students employed other procedures which she could not remember (e.g., converting mixed numbers into improper fractions) or they solved the problem without using the traditional algorithm.

*Noticing and Interpreting-Evaluating.* Vonda overlooked all four problematic student contributions under consideration. When directly asked to consider these contributions, she encountered difficulties appraising the mathematical substance of students' ideas.

Consider, for instance, her evaluation of Alan's work. Initially, she claimed that Alan did a good job in "getting one sixth of something" (PE.I. 835-836). When encouraged to expound her argument, the following discussion ensued:

**Charalambos:** Okay, can you say more about that?

**Vonda:** I mean that [the word problem] is basically saying one -- what was it? One half divided by one sixth. So this black line represents one half and then he came and took one sixth of that, I think.

**Charalambos:** Oh what?

**Vonda:** Of the half.

**Charalambos:** [Pause.] Okay. So, what do you think about his work?

**Vonda:** I thought it was a good example of getting one sixth of something.

**Charalambos:** Okay, so if you were the teacher what would you do next?

**Vonda:** Um [Pause.] I think I would write up the problem, the formula itself.  
**Charalambos:** Um-hmm, which is what?  
**Vonda:** One half divided by one sixth.  
 ...  
**Charalambos:** And what are you expecting to get from this formula? What will the answer be?  
**Vonda:** Um, the answer is going to be [murmurs while writing the equation shown in Figure 4.21:] one sixth is going to [pause] how many [pause]. I don't know. (P.E.I. 848-886)

On the surface, Vonda's argument that Alan was taking one sixth of one half was correct. By splitting half of the line into six parts, Alan was indeed taking sixths of half of the yard, or differently put, twelfths of the whole yard. Had the discussion stopped at this point, Vonda's appraisal of Alan's work, although not thorough enough, could be acceptable. However, Vonda interpretation of Alan's work did not align with the mathematical sentence she proposed to represent his work: whereas she initially talked about Alan's work from a multiplication perspective, she proposed a division-of-fractions sentence. The question then arises as to what might account for this inconsistency.

The image shows handwritten mathematical work. On the left, there is a division problem:  $\frac{1}{2} \div \frac{1}{6}$ . To the right of this, there is a calculation:  $\frac{1 \times 3}{6} = \frac{3}{6} \frac{1}{2}$ . The work is written in a cursive, handwritten style.

*Figure 4.21.* Vonda's work on figuring out the answer to the division problem  $\frac{1}{2} \div \frac{1}{6}$ .

It is unlikely that Vonda got the "formula" one half divided by one sixth by chance. In fact, earlier in the interview, when discussing how she would use the second textbook page, she stated that the task under consideration is solved by using the division sentence  $\frac{1}{2} \div \frac{1}{6}$ . Consequently, it is likely that she could not see that her two ways of representing Alan's work – her verbal and her numerical representations – were incompatible.

At this point it was envisioned that figuring out the numerical answer to this problem might have supported Vonda in deciphering the error in Alan's work. Alas, by using the common-denominators approach (and again in a rather esoteric and flawed way) Vonda could not figure out the correct answer to this problem. Hence, she was left with no criteria against which she could determine the accuracy of Alan's contribution.

Consider also her work in analyzing Amanda and Julia's solution. To examine the correctness of the two girls' answer to the problem  $\frac{3}{4} \div \frac{1}{6}$ , Vonda again used her common-denominators approach, which yielded a totally different answer from the girls' (i.e.,  $\frac{9}{12}$  versus 4 remainder  $\frac{1}{12}$ ). Because of the huge discrepancy between the two solutions, her answer did not create a productive context for her in which to examine the girls' work. Such a context was created when later during the virtual lesson Robert pointed out that his answer to this problem (i.e.,  $4\frac{1}{2}$ ) was different from Amanda and Julia's. Alas, even Robert's comment and numerical answer did not support Vonda in moving farther in her analysis of Amada and Julia's work: she speculated that the contradiction between Robert's and the girls' answers stemmed from the fact that in division  $\frac{3}{4} \div \frac{1}{6}$ , "six is not divisible by four" (PE.I. 1534-1535). Although on the surface her speculation seems reasonable, it does not address the source of the discrepancy between two different answers. In fact, this discrepancy would still arise even when solving division problems in which the denominators of the dividend and the divisor are multiples of each other (e.g.,  $\frac{5}{12} \div \frac{1}{6}$ ).

In short, in all episodes under consideration Vonda was not successful at analyzing the students' work. Often without obtaining correct numerical answers to the division problems at hand, she did not have at least some rudimentary criteria against



which she could appraise students' contributions. Even when obtaining such answers, her analysis still did not address the actual problematic aspects of students' work. Hence, Vonda's analysis of the students' work and contributions in the episodes just discussed suggests that limitations in her understanding of and fluency in the division of fractions (both from a conceptual and a procedural perspective) imposed significant constraints on what she noticed in the students' work, how she analyzed their work, and the inferences she made about students' understanding.

*Responding to Students' Direct or Indirect Requests for Help*

*Performing.* As already discussed, Vonda did not identify the error in Alan's work; therefore, to her, the next logical step after Alan presented his work was to ask him to write the corresponding formula on the board. In the second episode under consideration, Vonda attributed the difficulty that June and Shaun had in solving the division problem  $\frac{3}{4} \div \frac{1}{6}$  to students' earlier work on solving the previous division problem. She explained,

**Vonda:** Solving one half divided by one sixth kind of confused them on the subsequent problems. Because they had one sixth in their minds.

**Charalambos:** In what way did it confuse them?

**Vonda:** 'Cause they were dealing with a different fraction of -- with two different denominators, three fourths and one sixth. (P.E.I. 912-919)

To a certain extent, Vonda was right because in the first problem ( $\frac{1}{2} \div \frac{1}{6}$ ) the denominators of the two fractions were "friendlier," in that they did not require any further transformation. That said, however, it is not clear why "having one sixth in their mind," would confuse students when dealing with the second problem.

When asked to clarify how she would support June and Alan if she were in Ms. Rebecca's place, Vonda thought that, overall, the virtual teacher did a decent job.

Accordingly, she thought that she would follow a similar approach to that of Ms. Rebecca. She would try, however, to “make it a little more simpler [sic] and explanatory” for the students (PE.I. 970-971). She would do so by writing the division problem  $\frac{3}{4} \div \frac{1}{6}$  on the board and by then walking students through the steps of finding the common denominator of twelve. From there, she would follow the common-denominators approach discussed earlier. Vonda’s description of how she would respond to June’s and Shaun’s request for help is consistent with her overall style of walking the students through the steps of a given procedure. I argue, however, that limitations in her knowledge might have left Vonda with no better alternatives. Her comment “I go way back when I learned this kind of stuff” (PE.I. 975-976) when observing this episode is indicative of the difficulties she experienced with the content under consideration. Presumably not knowing how the students could have used their drawings to figure out the answer to the given division, Vonda resorted to outlining a sequence of steps for students to follow, not simply because this approach resonated with her endorsed instructional mode, but also because it might have been the only approach she was able to pursue in response to June’s and Shaun’s request for help.

*Noticing and Interpreting-Evaluating.* Vonda considered the manner in which the teacher responded to both requests for help (i.e., Alan’s indirect request for help and June and Shaun’s direct request for help) appropriate. Commenting on the episode in which the teacher supported Alan, she remarked that it was nice that the teacher helped him see that he should have used the whole line. If she were in Ms. Rebecca’s shoes, she would have “added to what the teacher said to solidify it for students” (PE.I. 1049). She clarified that she would have liked to use a formula and to add some more step-by-step work.

Obviously, Vonda was not troubled with the teacher's diminishing of the cognitive complexity of the task considered in this episode; this should not be surprising since she would probably have proceduralized this task even further.

Vonda's appraisal of the teacher's response to June's and Shaun's request for help was similar to that just discussed. Immediately after watching the pertinent episode, she noticed that the students were confused and that it was nice that the teacher stopped the class, attempting to bring "some clarity" (PE.I. 928-929). According to Vonda, the teacher's intervention also ensured that all students were on task and were thinking the same question. Vonda also endorsed the teacher's referring back to previous lessons and reminding students of the idea of common multiples. She remarked that "hopefully [this idea] will get embedded into [the students'] *memories* as they go forward" (PE.I. 1032-1033, emphasis added).

Taken together, Vonda's appraisal of the teacher's responses to the students' requests for help appeared to be informed by her endorsed instructional style but also by her beliefs about the role that confusion and clarity on one hand, and remembering and applying procedures, on the other hand, have in learning the subject. Vonda interpreted June's and Shaun's questions as evidence of confusion. Hence, she approved the teacher's move to immediately stop the class, which she interpreted as an attempt to bring some clarity. This suggests that apparently Vonda did not see value in having students struggle with an assigned task, an argument corroborated by her responses to three pertinent survey statements. On a scale from 1 to 7, with 1 representing "strongly disagree" and 7 corresponding to "strongly agree," Vonda used number 6 to show her agreement with the statement, "Students should never leave math class (or end the math

period) feeling confused or puzzled.” In contrast, she strongly disagreed with the statement, “Teachers should not necessarily answer students’ questions but let them puzzle things out themselves.” Vonda’s endorsement of the virtual teacher’s approach to remind the students of the idea of common multiples “to help get it engrained into their memories” also resonates with her strong endorsement of the survey statement, “When students can’t solve problems, it is usually because they can’t remember the right formula or rule” (and hence, the teacher’s role is to help them remember such formulas and rules).

### *Performance on the LMT Test: A Closer Look*

As already discussed, Vonda’s score on the LMT test was low and thus consonant with her overall performance in the teaching simulation. A closer examination of the questions that Vonda got right on this test revealed both consistencies and inconsistencies with her performance in the teaching simulation.

Vonda answered incorrectly most of the CCK questions (four out of five); these questions could largely be solved by simply following a procedure. This finding is consistent with the difficulties she encountered during the simulation even in remembering and applying certain procedures. Equally consistent with her performance in the teaching simulation was that she answered incorrectly several questions on the division of fractions. These included question 17 on explaining the fractional part of a quotient and question 22, which could be answered by simply figuring out the quotient to the given division-of-fractions problem. She also answered incorrectly questions that addressed concepts fundamental to the division of fractions (e.g., question 1, which pertained to using different units to denote the whole).

Vonda's answers to some of the questions of the LMT test were, however, not in accord with her performance in the teaching simulation. For example, she correctly answered questions 14a, 14d, 19c, and 19e, all of which pertained to interpreting division-of-fractions problems. Her correct answer to question 14a (i.e., "How many  $\frac{2}{3}$ s are in  $\frac{1}{6}$ ?" for interpreting the division problem  $\frac{2}{3} \div \frac{1}{6}$ ) was particularly puzzling given that her overall performance in the teaching simulation suggested that she could not see division from this perspective. Equally puzzling was that she answered correctly questions 5 and 20, both of which were designed to tap knowledge of concepts necessary for division in general and the division of fractions in particular. Yet, in the latter two questions it is more than likely that she obtained the correct answer simply by chance. This can be inferred by a pattern governing Vonda's answers to the test questions. In six out of the seven questions that included the option "None of the given answers is correct," or "All the given answers are correct," she selected this option. For questions 5 and 20, this option was, in fact, the correct one. This pattern in her test-taking behavior resonates with an analogous pattern observed in a recent study in which an LMT multiple-choice test was used to capture teachers' Knowledge of the Content and Students (Hill, Ball, & Schilling, 2008). Like Vonda, many of the teachers in this study were more prone to select the "all of the above" response when such response was given, as indicated by the fact that this response was the most frequently observed incorrect answer.

### *Analytical Commentary*

The preceding analysis suggests that Vonda's performance in the teaching simulation was constrained by limitations in her knowledge, her unproductive beliefs

about teaching and learning the subject, and her rather limited images of teaching. A cursory interpretation of her performance could suggest that, regardless of the robustness of her knowledge, without productive beliefs and images of teaching, Vonda could not perform well on the teaching simulation. Without dismissing the impact that her beliefs and images of teaching had on her performance, in this analytical commentary I turn the foregoing argument on its head and argue that even with productive beliefs, Vonda would have not been able to perform much better in the teaching simulation – and especially on its performing tasks – because of the limitations in her knowledge. I justify this argument after considering the impact of her beliefs and images of teaching on her performance.

As shown in Figure 4.22, which outlines Vonda's portrait, her beliefs about teaching and learning mathematics were inconsistent with the type of instruction endorsed by the teaching simulation, and consequently, with how the PSTs' performance in this simulation was gauged. Central among Vonda's beliefs was the idea that teaching and learning mathematics means following and applying a sequence of steps. In fact, this idea was Vonda's refrain throughout the whole interview. Supportive of this belief was also a conglomerate of other beliefs expressed during the interview either openly or more covertly. For example, Vonda believed that the teacher should transmit knowledge by often walking the students through a procedure in a step-by-step fashion and, as the sole source of knowledge, ascertain the validity of students' thinking and ideas. Her work on selecting and sequencing tasks and her performance in responding to students' requests for help also revealed that she believed that the mathematics teacher should help students move from easier to more complex tasks and should strive to maximize clarity and minimize confusion. To solidify students' understanding, the teacher of mathematics

should also afford students ample opportunities to practice the procedures introduced in the lesson. Altogether, such beliefs left little room for her to create the rich and intellectually challenging environments this study considers. Hence, in Figure 4.22, I illustrate the role of her beliefs with the negative sign next to the arrow originating from the “beliefs” rectangle.

In addition to her beliefs about teaching and learning mathematics, Vonda’s comments during the interview were also suggestive of her weak efficacy beliefs. In a couple of instances, when asked to perform certain tasks, she thought it important to clarify that she had not worked with fractions for several years, a comment which implicitly suggested that she was not particularly confident in working in this terrain. Hence, it is reasonable to assume that her efficacy beliefs were also in play; however, there was not strong evidence suggesting how these beliefs mediated – to the extent that they did – the association under consideration. Thus, in Figure 4.22 this rather unclear role of Vonda’s efficacy beliefs in the association under exploration is displayed by the question mark next to the arrow originating from the efficacy-beliefs rectangle.

Vonda’s performance in the teaching simulation was also suggestive of her limited images of teaching, which were largely informed by her school experiences as a learner of mathematics. Consider, for example, her idea that after having students individually work on the 16 exercises of the first page, she would have them go to the board and present their solutions, “starting from student number one” on the student roster. This approach might have been the dominant type of instruction that she had experienced as a student of mathematics. Vonda’s overemphasis on following steps, minimizing confusion and maximizing clarity might have also been a byproduct of her

experiences as a learner of mathematics, especially if one considers Stigler and Hiebert's (1999) argument that the aforesaid characteristics are key aspects of the U.S. cultural script of teaching. Consequently, it seems reasonable to argue that Vonda's school experiences had equipped her with certain images of teaching the subject which, in turn, explicitly or more tacitly informed her decisions and actions during the teaching simulation. Because these images of teaching contradicted the type of environments this study explores, in Figure 4.22, I illustrate the mediating effect of her school experiences on the knowledge-performance association by a negative sign.

If Vonda's beliefs and school experiences allegedly had such a pivotal role in informing her performance, her case could challenge the association between knowledge and teaching performance considered thus far. To illustrate that Vonda's case supports rather than negates this association, I ask: Would Vonda's performance have been different, if she had more productive beliefs and images of teaching but the same knowledge as exhibited in the teaching simulation and as measured by the LMT test? I address this question by considering her performance in each of the five practices in turn. I conclude with discussing how the case of Vonda adds to the conversation about the association between knowledge and teaching performance.

Regarding the practice of selecting and using tasks, Vonda selected the procedurally oriented page for a division-of-fractions lesson and intended to use this page as is, by first walking the students step-by-step through the invert-and-multiply algorithm and then having them solve all the page's tasks. Even when using the second page, her approach would still fail to help students understand the meaning of fraction division. Would Vonda's performance have been different had she held more productive beliefs



and images of teaching? Although it is reasonable to assume that she could have selected the conceptually oriented page for her lesson, like Deborah did, it is questionable whether she would have been able to capitalize on the affordances of this page. Vonda's knowledge of the content as gauged throughout the teaching simulation and the LMT test and as she herself alluded to by mentioning that she had not worked on fractions for years was rather weak. As such, it imposed certain limitations on her decisions and actions. For instance, how could she have organized a discussion around the quotients' fractional part if she herself encountered difficulties even with remembering and applying the division-of-fractions algorithm? Would she have been able to effectively lead her students to discover an algorithm for division of fractions, if the algorithm she had in mind was flawed? It seems quite unlikely that she would have been able to do so without a relatively deep background of pertinent concepts and procedures. Such a background would have enabled her to first to identify the affordances and the limitations of the two pages, and then to capitalize on them to develop a lesson geared toward making meaning.

Similar arguments apply to Vonda's performance in the practice of responding to students' requests for help. Vonda largely endorsed the virtual teacher's approaches to support Alan as well as June and Shaun; she did not attend to how the teacher degraded the demands of the assigned tasks by posing leading questions and taking over the thinking on the tasks' more complex parts. It is reasonable to assume that if Vonda had alternative images of teaching which sensitized her to the importance of maintaining focus on meaning and upholding the tasks' cognitive level, she might have been less supportive of the virtual teacher's work during the episodes under consideration. Yet, this sensitivity to issues of cognitive demand and meaning-making would have not been

adequate to support her in responding to the students' requests for help in mathematically valid ways. In fact, without having a solid understanding of the mathematics considered in Alan's episode, Vonda did not identify the error in his work, let alone proposed an approach that could support Alan in grasping the fundamental concept of unit. Similarly, without being clear on the mathematics discussed in June's and Shaun's episode she could not help these students overcome the impasse they encountered.

Regarding the practices of providing explanations and using representations, Vonda was seen describing rather than explaining the division-of-fractions algorithm. Her description was exclusively procedural and drew no connections to meaning. As such, it could mainly support students in becoming more procedurally fluent in dividing fractions, even though the ambiguous language she used (e.g., "add one" to transform a whole number to a fractional number)<sup>148</sup> might have created more confusion instead of helping students construct meaning. To provide more conceptually driven explanations and to use the representations she selected in a conceptually driven manner, Vonda would have needed a more solid understanding of the content.

In Vonda's case, the strongest piece of evidence about the association between knowledge and performance pertains to the practice of analyzing students' work and contributions. Even though Vonda was trying to closely attend to the students' ideas, her own difficulties with the content constrained her in several ways. First, she could not always follow and understand students' work, especially if their work deviated significantly from her own understanding of the content. Recall, for example, the difficulties she had in understanding Michelle's solution or in making sense of Amanda

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<sup>148</sup> Vonda's failure to attend to issues of language suggests that teachers with a weaker knowledge base might offer more ambiguous explanations. Yet, this hypothesis warrants further investigation.

and Julia's work on the board. Second, some gaps in her own knowledge did not allow her to determine the correctness of students' contributions. A case in point here was her uncertainty about the correctness of Robert's work, simply because she could not decide whether the procedure he followed in converting a mixed number into an improper fraction was correct. Third, by applying an incorrect procedure or applying a correct procedure inappropriately, in some cases she got incorrect numerical answers to the problems under consideration. This, in turn, left her without any criteria against which she could determine the correctness of students' work. All these difficulties also appear to have prevented her from making substantive inferences about students' understanding or lack thereof. A typical case in point was her vague comment that Robert, Michelle, and Ann understood "some concepts."

In sum, Vonda's difficulties with the content, even from a procedural perspective, reminds us what is often taken for granted when discussing issues around MKT and its potential impact on teaching: *although not sufficient for effective teaching, CCK is necessary* for such teaching. Or to echo Cohen et al. (2003), "[k]nowledge of a subject is a necessary condition [for effective teaching], for teachers cannot use knowledge they do not have" (p. 132).

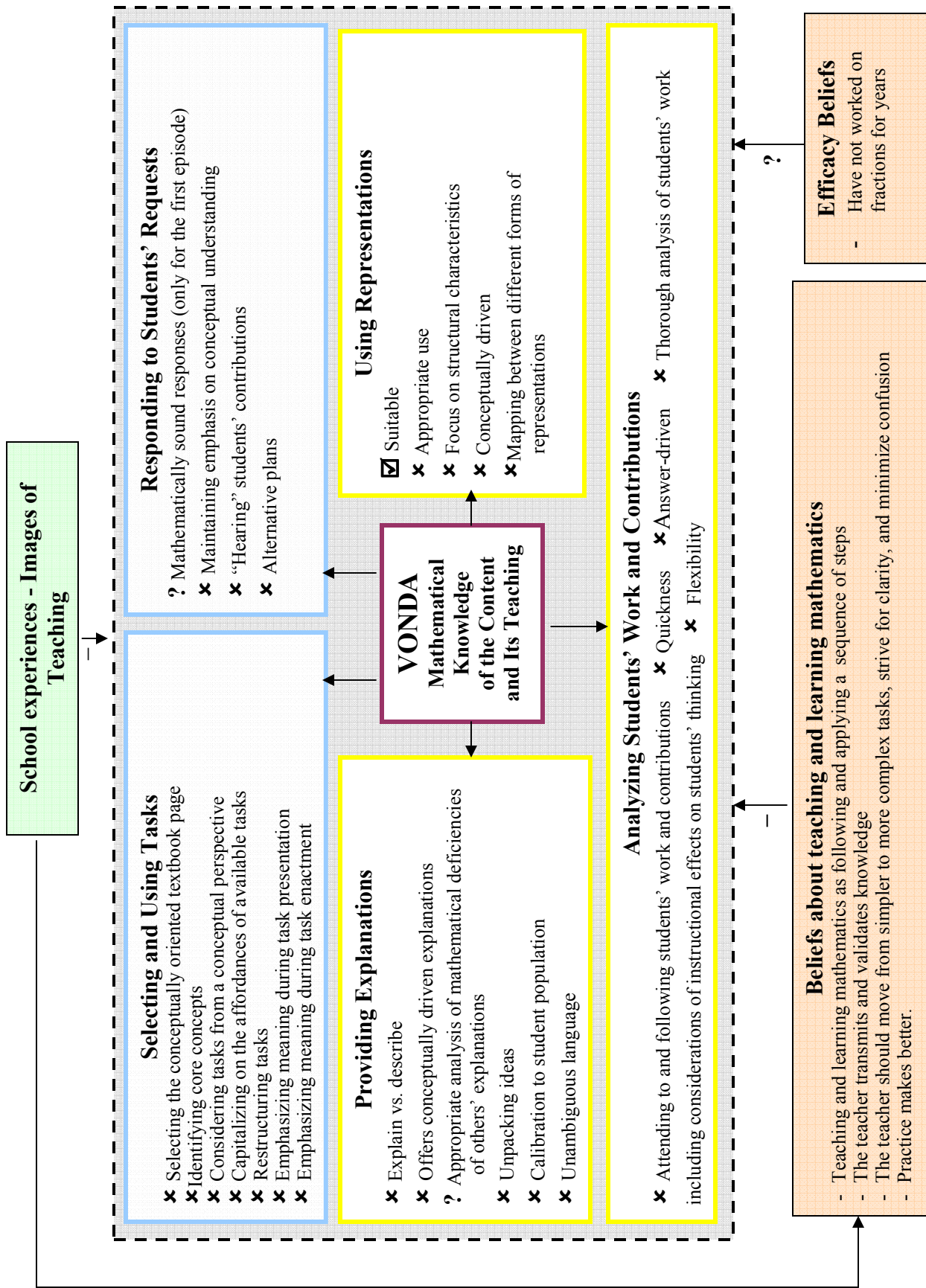


Figure 4.22. Considering the association between knowledge and teaching performance through Vonda's profile.

*The Case of Tiffany: “I Know How to Divide Fractions But I Don’t Have the Link”*

Tiffany is the fifth case considered in the study and the first divergent case. As shown in Table 4.5, compared to her counterparts, she performed better on the teaching simulation than on the LMT test. Tiffany had not taken any math content or methods courses during her undergraduate studies; the highest level math content courses she had ever taken were *Algebra* and *Trigonometry* (see Table 3.1). Given her low LMT score and her limited background of math coursework, her performance in the teaching simulation was quite surprising. Yet the scrutiny of her case actually supported the quantitative findings reported in the previous section of this chapter since it showed that her productive beliefs scaffolded her in the noticing and interpreting-evaluating tasks of the teaching simulation but not in the performing tasks.

*Background*

At the time Tiffany joined elementary school her brother had been working as a high-school history teacher and her sister as an elementary teacher. Growing up in what she called an “environment of teachers” (PE.I. 78) helped her appreciate teaching and learning. This, coupled with “a chain of some great teachers” she had in elementary school instilled within her the desire to become a teacher. She was particularly appreciative of her first-grade teacher, a caring and supportive man who used to stay with his students even after classes to help them.

During her elementary school years she had a positive attitude toward learning and doing mathematics, which mostly meant remembering and applying rules and formulas with speed and accuracy. She eloquently described these experiences:

Learn [a formula] and then practice it. ... That’s how I was taught. I was taught that bam, bam, bam; that’s what you do. And that’s what you’re always gonna do. (PE.I. 867-870)

Despite her struggles with carrying out her multiplication and division problems with speed, she still found the subject intriguing. She recalled “loving the subject” and “sitting for hours doing [her] math homework, [her] cheeks red with thought” (PE.S.).

Similar to her elementary-grade experiences, learning mathematics in middle and high school was again a remember-and-apply enterprise. She recollected her *Algebra* teacher outlining a set of steps for students to follow: “[You] need to do this, this, this, this, and this.” ...This is actually how the math teacher taught and I couldn’t stand it” (PE. I. 386-388). Her initial positive attitude toward the subject gradually deteriorated, as the content became harder. On entering the ELMAC program, she was concerned that her lack of speed and proficiency with the subject would impinge on her ability to teach the subject. She hoped that the program would help her become less scared of the subject and equip her with some teaching “tricks” (PE.S.).

### *Selecting and Using Tasks*

*Performing.* In discussing the strengths and the limitations of the first page, Tiffany focused mostly on its non-mathematical features. She argued that the title should have been placed at the center of the page and that the worked out example should have been presented with bigger font-size characters. She also disapproved the presentation of the second step of the worked out example as a series of equations all listed on the same line (i.e.,  $2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{2}{1} \times \frac{4}{3} = \frac{8}{3} = 2\frac{2}{3}$ ). She argued that because of the way these equations were outlined on this page, she could not easily follow them; she thus suggested presenting each of these equations on separate lines. She thought that this presentation would better support student learning because, as a learner herself, she often struggled

with such a sequencing of equations and spent a lot of time trying to understand how to move from one equation to the other.

If she had to use the first page for an introductory lesson on division of fractions, she would start with the two word problems listed at the bottom of this page, because she wanted the students to understand the applicability of fraction division.<sup>149</sup> Asked to explicate how she would use the two word problems to introduce her students to the division of fractions, Tiffany admitted that as a student she would have never solved either of these problems using division. To her, both problems lent themselves better to multiplication. For instance, to solve the first problem (exercise 17) she would simply calculate “four times thirty,” because thirty students represented one fourth of the class, and “in order to get that whole class ... [she] needed to have a full number” (PE.I. 1022-1024). She would solve this problem as a division problem only because it was included in a division-of-fractions page. Still, she was uncertain which mathematical sentence ( $30 \div \frac{1}{4}$  or  $\frac{1}{4} \div 30$ ) was appropriate for solving this problem. She figured this out only after having calculated the answer to each of these sentences. She explained that if she was actually solving the problem using a division-of-fractions sentence, she “would have done ... a little guess-and-check in her head” (PE.I. 1052, 1075).

After working on the two word problems, Tiffany would shift to the worked out example listed at the top of this page. She would ask students to rewrite the second step of this example to represent each of its equations in a different row. After considering the worked out example, she would move to the 16 exercises. Instead of having students

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<sup>149</sup> To motivate the learning of this operation, she alternatively proposed asking students to pose word problems for some of the exercises 1-16 included on this page.

simply solve these exercises, she would ask them to describe each step of the procedure they followed to ensure that they were not just plugging in numbers:

[I would] say, “Okay, can you show me step-by-step what that problem is? And then next to it, in words, say what you’re doing.”... You know, saying “finding the reciprocal” and then here saying, “I’m setting up a cross multiplication problem, cross-multiplied, I simplified, it was an improper fraction.” So just going through step-by-step, so that you, as a teacher, could look at it and say, “Okay, they got the words here, too. They didn’t just plug it into a calculator or do [sic] another way.” (PE.I. 243-253)

Tiffany did not refer to cross multiplication by accident. While solving the division problem listed in the worked out example (i.e.,  $2 \div \frac{3}{4}$ ), she wrote the dividend as a fraction and multiplied it by the reciprocal of the divisor (i.e.,  $\frac{2}{1} \cdot \frac{4}{3}$ ). She then stopped and wondered whether she should “multiply across” or “cross multiply.” She had some nebulous recollections from her elementary grades that division of fractions involves cross multiplication, but she could not remember where in the whole sequence of the division-of-fractions algorithm this cross-multiplication step gets applied or how it is applied. She initially thought that it was reasonable to cross multiply after converting the division into multiplication, because “cross multiplying refers to multiplication.” Yet, she also remembered that in multiplying fractions, one simply “multiplies across.” After consulting the worked out example, she figured out that she just needed to multiply across, but wondered: “Well, why do I call it cross multiplying if I’m really just multiplying straight across?” (PE.I. 616-617).

Tiffany’s confusion obviously stemmed from mixing two different algorithms for dividing fractions: the one presented in the worked out example on the first textbook page (see Figure 3.2) and the cross-multiply gimmick used to help students remember how to divide fractions (see Tom’s method in Figure 2.4 or the case of Suzanne in this chapter). She confessed experiencing a similar impasse while solving the LMT test questions



which involved division of fractions: “I couldn’t remember if I cross [multiply] ... or if I multiply straight across. ... I drew a blank on” (PE.I. 598-601).

Instead of having her students work on all 16 exercises, Tiffany would “select a few that might be more challenging rather than just doing routine” (PE.I. 214-215). Yet, she could not decide how to make this selection:

If I did what my teachers in elementary school did, they would just say, “Do the evens or odds.”... So, again, because of that, I right now don’t know what a good problem and a bad problem is. ... Because I can’t figure out a different way that a kid could try these problems right now, which is what I’m trying to do, I don’t know what would be a good problem or a bad problem. (PE.I. 852-856, 861-864)

As her comment suggests, although she intended to be more selective in using the 16 exercises, she did not have any criteria against which to make this selection. To her, all the exercises seemed almost identical; their only difference was that some involved more steps than others (e.g., “reducing,” converting a mixed number into an improper fraction). But was that a suitable criterion for selecting a subset of exercises to assess students’ understanding? Left with no other devices, she would revert to a familiar image of teaching, which she formed as a learner of mathematics: “select the evens or the odds.”

Given the option to use either page, Tiffany would start with the second one:

**Tiffany:** I would for sure, for sure, for sure, use [the second] one.... What I would do is start with [the second] page, have the kids reason this stuff out before they even knew anything about writing -- dividing a fractions page. You know, so have them turn this [page] into me and then they could look, you know, and say, “Okay, well what did you do?” You know, and if they said, “Oh, I drew out pictures,” “I did this,” “I did this,” and then to say, “Okay, well did you know there really is an easier way to do this?” And then that’s when I would introduce this [the algorithm of the first textbook page].

**Charalambos:** And why would you follow this sequence?

**Tiffany:** Why would I follow this sequence? Because I think it’s important to see the logic ... of this, “dividing of a fraction” [pointing to the title of the second textbook page]. I would take [the title] out though.

**Charalambos:** Why?

**Tiffany:** Because right there I don’t even think you need to know that. I think to throw in “dividing of a fraction” right there, the kids don’t even need to know that at first. Saying, “We’re gonna learn this new concept, a mathematical concept, but I’m gonna give you some story problems first.”... You know, that idea has to come later, I think, or else you’ll end up just like me and know how to do it and not why (laughs). (PE.I. 1198-1240)

Apparently, Tiffany had a strong inclination to use the second page for an introductory lesson on the division of fractions. Because as a student of mathematics she learned and then forgot how to divide fractions, she wanted her students to “understand the logic” behind this operation rather than mechanically follow a given algorithm. Hence, she would refrain from presenting the algorithm first and would have her students solve this page’s word problems [tasks A and B]. She would then introduce the concept of the division of fractions and eventually introduce the algorithm as a shortcut.

Overall, Tiffany’s approach in using the second page was pedagogically sound and aligned with a mode of instruction geared toward making meaning and understanding. Yet, a closer examination of how she considered using this page raises legitimate concerns whether her approach would really support student understanding.

To begin, as suggested by the excerpt listed above, it was not clear how she would support students while working on these tasks. Simply asking the students to use drawings to solve the word problem does not suffice to help them grasp a difficult concept, such as that of fraction division. Second, students could solve these problems by using repeated subtraction (or even repeated addition). In this case, how would she help them see that these problems could be solved by using division? Third, simply introducing the division-of-fractions algorithm after having students solve a set of pertinent problems does not ensure that students would see any reason for employing this algorithm when later asked to solve similar problems. In fact, it is debatable whether her proposed introduction of the algorithm would help connect this algorithm to the meaning of division of fractions and hence render it less cryptic. Fourth, Tiffany would not actually capitalize on the affordances of this page to support her students’ understanding

of the procedure under consideration. As she mentioned while going over this page, she did not understand what the “fractional part of an answer” means: “I don’t even understand what that question’s asking now that I’m looking at it” (PE.I. 1187-1188). Fifth, she had difficulties representing even problems that did not involve such fractional parts in their quotient. As she remarked, when solving such problems she would often figure out the numeric solution to the problem and then draw a representation to depict the situation at hand. For example, referring to how she solved task A<sub>1</sub> (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ), she remarked:

Once I figured out the answer, I set up the problem. So again that shows right there that I don’t personally understand what this concept of dividing fractions is. ... I know *how* to divide fractions, but I don’t have the link. (PE.I. 1156-1160; emphasis in the original)

In short, Tiffany’s performance in selecting and using tasks suggests that despite her propensity to teach a lesson that would help students see the underlying meaning of dividing fractions, constraints in her knowledge could limit her in this endeavor: on the one hand, she was not seen capitalizing on the affordances of the second page to teach such a lesson; on the other hand, her reordering of the tasks of the first page would not suffice to scaffold students’ understanding.

*Noticing and Interpreting-Evaluating.* Before outlining Tiffany’s noticing and interpreting-evaluating performance, it is worth mentioning that she was very meticulous in analyzing nearly all the teaching simulation episodes. She would go over the slides rather slowly and move back and forth, as needed, to reconsider the virtual teacher’s work and the students’ contributions.

Tiffany had several issues with how the virtual teacher introduced the first task. She thought that the teacher should have not introduced the first task as a division-of-fractions problem; she justified her argument as follows:

I still wouldn't have introduced that assignment as dividing by fractions. And I think that's just my way of wanting to teach, to have it be much, like more exploratory on the kids' part. So even asking the kids after they read this, "What are you doing with these fractions?" in order to introduce the lesson, instead of "Today we're learning about dividing fractions." (PE.I. 1363-1369)

An opponent of the show-and-tell mode of instruction, Tiffany would have placed more emphasis on having her students figure out how to solve the problem instead of telling them – directly or more implicitly – how the problem could be solved, as the virtual teacher did. In particular, Tiffany would have not told the students that the day's lesson was on the division of fractions. After having them read the problem, she would have posed a different question from Ms. Rebecca's introductory question. Rather than asking, "What are we going to divide in this case?" she would have asked a vaguer question to elicit students' thinking: "What are you doing with these fractions?" Instead of simply manipulating numbers, as the teacher was presented doing after she had introduced the task, Tiffany would have brought in ribbons and had the students experiment with this material to figure out the solution to the problem. She justified this move by referring to her own experiences as a learner of mathematics: "I am a visual learner. ... And I haven't really had teachers who ever [taught tasks like the first problem] visually" (PE.I. 1538-1542). This last comment implies that Tiffany's difficulties with the content as a student of mathematics, coupled with her dissatisfaction with the mode of teaching she experienced in elementary grades,<sup>150</sup> led her to endorse a type of instruction that differed remarkably from the virtual teacher's approach.

Tiffany was even more concerned with how the first task was enacted in the virtual lesson. She noticed that whereas the teacher started with a word problem, she soon

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<sup>150</sup> Earlier in this part, Tiffany was presented arguing that she had been fortunate to have several good elementary teachers. This argument does not contradict her claim about certain insufficient aspects of her mathematics teachers' instruction, because her former argument was grounded in more general features of her teachers' personality and instruction (e.g., being caring).

shifted emphasis to manipulating numbers; similarly, whereas she used a representation to scaffold students' thinking, she ended up using this representation out of the context of the problem. As Tiffany put it, “[Ms. Rebecca] is talking about [the representation] in terms of a line, not in terms of a yard” (PE.I. 1650), and elsewhere, “If you’re using a story problem, you should continue your explanation congruent to the story problem and not switch [to numbers]” (PE.I. 1771-1773). Even at the end of the interview, when asked to consider the virtual teacher’s lesson as a whole, she revisited this episode and claimed that she was not satisfied with the teacher’s approach, because “math is there for a reason and not just to compute numbers” (PE.I. 2555-2556).

Tiffany had similar concerns with the teacher’s enactment of task D, the most critical being that the students’ work on this task was totally devoid of any meaning since no attempts were made to connect this task to the word problems of the second page. According to Tiffany, Ms. Rebecca was simply manipulating numbers without helping her students see how the numbers involved in the algorithm related to the ribbon problems. Such an approach, she claimed, would have been justifiable only had the teacher used the first instead of the second page.

In short, without any particular prompting, Tiffany noticed and criticized the several ways in which the virtual teacher shifted emphasis from meaning and understanding to manipulating numbers and representations. Central in her analysis and evaluation of the teacher’s work was her conviction that mathematics should make sense. Her own difficulties with the content as a student of mathematics might have also rendered her more perceptive to the instances in which the virtual teacher’s instructional approach impaired rather than supported students’ thinking and learning.

### *Providing Explanations*

*Performing.* When asked to explain the division problem  $2 \div \frac{3}{4}$ , Tiffany remarked that she could not remember how this operation was explained to her. Thus, she was not sure how to provide such an explanation, apart from thinking that she wanted to use a “visual,” because she considered herself a visual learner. Although she was not able to provide an explanation for either the quotient or the reciprocal, it is informative to consider how she worked toward providing such explanations.

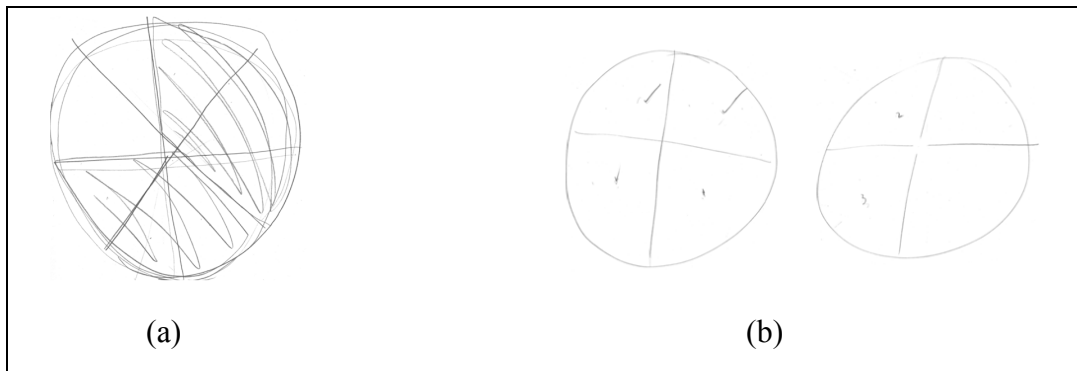
To scaffold her work, Tiffany first resorted to a simple whole-number division. She wrote “ $2 \div 6$ ” and thought that this meant “two goes into six how many times?”, obviously thinking of division “ $6 \div 2$ .” Yet, this was not a careless mistake as suggested by her difficulties in using symbolic expressions to represent mathematical ideas.<sup>151</sup> Even after correcting this mistake, she was still unclear about if and how the notion of fitting divisor units into dividend units also applied to the division of fractions: “Here, you’re saying three quarters goes into two; are you saying that, though?” (PE.I. 351-352). To explore whether this was the case, she used repeated addition. Adding three fourths three times, she ended up with nine fourths, which she interpreted as “going over the eight [fourths]” that represented the two whole units (i.e., the value of the dividend). After converting this improper fraction into a mixed number, Tiffany compared her answer (i.e.,  $2 \frac{1}{4}$ ) with the quotient of the worked out example of the first textbook page. Obviously, the two answers differed: that of the worked out example (i.e.,  $2\frac{2}{3}$ ) represented the number of times the divisor units could fit into the dividend; her repeated-addition answer suggested that she could fit the divisor into the dividend three times, if

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<sup>151</sup> She encountered analogous difficulties in other instances throughout the interview. For example, she originally interpreted the division  $2 \div \frac{3}{4}$  as “how many times two goes into three fourths” (PE.I. 327).

the dividend were larger by an additional fourth. Because the two answers did not match, she did not capitalize on the result of her repeated addition to make sense of this division.

Reverting to her initial idea of using a visual, she drew a circle and divided it into eight parts. Reminding herself that the division  $2 \div \frac{3}{4}$  meant “how many three quarters are in two,” she then shaded in three fourths of this circle, as shown in Figure 4.23a. At this point, Tiffany was apparently using units inconsistently. By using a single circle to represent the dividend (i.e., 2), she was, in essence, representing the absolute unit of one by half a circle; to show the divisor (i.e.,  $\frac{3}{4}$ ) she shaded in three fourths of the whole circle instead of half of it. Even if she had used this representation correctly to provide an explanation for the division at hand, her explanation would have been convoluted and rather difficult to understand, due to an additional, and perhaps unnecessary, layer of difficulty: that the absolute unit was represented by half instead by a whole circle.



*Figure 4.23.* Tiffany’s work on explaining the division problem  $2 \div \frac{3}{4}$  (a: initial steps; b: final explanation).

Probably realizing the complexity inherent in providing an explanation using this representation, Tiffany switched to a different representation (see Figure 4.23b). She drew two circles, partitioned them in fourths, and marked three fourths in the first circle. Next, she identified another  $\frac{3}{4}$ -portion, by putting numbers 1, 2, and 3 in the bottom right

fourth of the first circle and the upper and lower left fourths of the second circle. Because half of the second circle was not used, she concluded that the answer was “two and a half;” yet, as she observed, this answer was different from the answer she got by using the algorithm. She thus inferred that the concept of whole-number division does not apply to dividing fractions: “If a kid really understands division, they’re not gonna be able to apply the same concept, as far as I’m concerned, unless I’m just completely missing a big thing” (PE.I. 516-518). Tiffany was indeed missing “a big thing” because she was trying to interpret the leftover part in absolute rather than in relative units. To resolve the discrepancy between the two different answers she obtained for this division, she entertained the idea of reverting to her initial drawing, in which she represented the eight fourths in a single circle. After some further non-productive experimentation with this idea, she argued that she could not proceed any farther.

Explaining the reciprocal was an even more convoluted task for Tiffany. As she confessed, the only two pertinent pieces of knowledge she had available about this concept was that the reciprocal is the inverse of a number and that division and multiplication are “opposites” (i.e., inverse operations). Without knowing “the logic behind the reciprocal” (PE.I. 740), she could not provide any explanation as to why the division-of-fractions algorithm works.

In short, while attempting to provide the required explanations, Tiffany was seen employing two helpful strategies: using an easier example and drawing on repeated addition to figure out how many times the divisor fits into the dividend. However, limitations in her knowledge appear to have prevented her from providing such explanations. For instance, although she had some good recollections of what division



means, she was unsure whether the concept of whole-number division was also applicable to dividing fractions. In addition, her understanding of what a unit is was apparently not very solid, as suggested by her work shown in Figure 4.23a. Without a firm grasp of these two fundamental concepts – division and units – Tiffany had few resources upon which to draw to explain the quotient in the division problem  $2 \div \frac{3}{4}$ , let alone explain the reciprocal.

*Noticing and Interpreting-Evaluating.* Without any prompting, Tiffany noticed that the teacher’s explanation as to why the reciprocal works was deficient in two respects. First, it was totally decontextualized: the teacher explained the reciprocal without referring to the ribbon problem under consideration. Second, she found Ms. Rebecca’s analogy of adding and subtracting negative numbers inappropriate. From Tiffany’s perspective, although Ms. Rebecca’s analogy appeared to address an idea similar to inverting and multiplying when dividing fractions, it represented “a big jump” from the idea of the reciprocal. A closer look at Tiffany’s evaluation suggests that, although she identified important limitations of the teacher’s explanation, her analysis was informed by pedagogical rather than mathematical considerations. Tiffany was right that there seemed to be a gap between adding and subtracting negative numbers and inverting and multiplying; however, she could not explain this gap from a more mathematical perspective (e.g., by identifying the differences between the two situations at hand). Additionally, although her comment that the teacher needed to contextualize her explanation was valid, decontextualization was not the only deficiency of Ms. Rebecca’s explanation. For instance, Tiffany did not notice that the teacher was actually describing a rule rather than providing an explanation; Ms. Rebecca was also using linguistic rather

than mathematical arguments to explain the inverse-and-multiply rule (e.g., “because you use the reciprocal, you need to use the reciprocal operation”).

### *Using Representations*

*Performing.* Like Deborah, Tiffany was a strong proponent of using what she called “visuals” to explain mathematical ideas. Considering herself a visual learner who had been unfortunate to sit in classes in which such representations were rarely used, Tiffany had a strong desire to use such representations in her teaching. However, as her work on explaining the division  $2 \div \frac{3}{4}$  suggests, limitations in her knowledge did not allow her to use representations appropriately in her explanations. For instance, in explaining the quotient of this division, although she picked a suitable representation, in using this representation she did not retain the relative size of the dividend and the divisor. Even though she then resorted to a different representation that maintained the relative sizes of these quantities, attempting to resolve the discrepancy between the two different answers she got from her drawing and from applying the algorithm, she contemplated reverting to her original inconsistent representation. In the case of the reciprocal, she did not even try to use a representation, claiming that “I don’t even know what I’m trying to figure out” (PE.I. 839-840).

It is also informative to consider the type of representations she used in her explanation. Her use of “pies” was not coincidental, for as she explained, these representations were her favorite ones when dealing with fractions. Her recollections of her school experiences justify her preference for circular representations. Although her teachers rarely used visuals to teach fractions, when they did they often used “pies” to scaffold their students’ learning:

[W]hen I was younger in my seventh-grade math we used paper plates and that was our way of learning. So we started out the paper plate and then had it divided up so that you could have this divided in half, and then in quarters, and then in sixteenths. So that you could actually like pick up the plate and move it around. (PE.I. 281-285)

These images of teaching appeared to inform, and perhaps limit, Tiffany's selection of representations. If her school experiences were restricted to considering a single "pie" divided into different parts (halves, quarters, etc), then her difficulties in representing the dividend using two separate "pies" should not be surprising.

*Noticing and Interpreting-Evaluating.* Despite her meticulous analysis of the teaching simulation, Tiffany did not originally attend to the lack of connections between the representations used in the virtual lesson and the algorithm or the word problems that these drawings were supposed to represent. Yet when prompted to reconsider Amanda's work, she noticed that neither Amanda nor the teacher talked about the meaning of the quotient in the equation  $\frac{1}{2} \div \frac{1}{6} = 3$ . For Tiffany, to solidify students' understanding of the problem, the teacher should have guided Amanda to connect her answer to the word problem. Although Tiffany's analysis of this episode was correct, it was incomplete. Even if the teacher had pressed Amanda to clarify what the answer of three represented, more work was warranted: Amanda would have also needed to map the *whole* division equation to the representation drawn on the board and to the word problem at hand. Consistent with her analysis of the first episode, in analyzing the second episode, Tiffany again focused on Amanda and Julia's answer to the problem  $\frac{3}{4} \div \frac{1}{6}$ . In doing so, she did not notice that the two girls did not clarify how their visual representation actually mapped onto the equation they wrote on the board.

Looking across Tiffany's performance in the two episodes, one could argue that her sensitivity to issues of making meaning and understanding helped her see some gaps

in the students' and the teacher's use of representations. Yet, this sensitivity alone did not appear to support her in seeing that certain connections – among the representations, the algorithms, and the word problems – were never made in this lesson.

### *Analyzing Students' Work and Contributions*

*Performing.* Tiffany's performance in analyzing the three students' work, particularly Michelle's solution, exemplifies how limitations in a teacher's knowledge might prevent her from closely following and understanding student work.

Before analyzing the three student solutions to the division problem  $2\frac{3}{4} \div \frac{3}{4}$ , Tiffany first figured out the answer to this problem and used this answer to make sense of the students' work. In considering Michelle's solution, Tiffany initially associated the three lines in Michelle's drawing (see Figure 3.5) with the number of bows that could be made of the available yard length. Because Michelle colored two and three fourths of these lines in red (to show the dividend), Tiffany argued that this student's work was incorrect: Michelle needed to color three and two thirds in red, which represented the numerical answer to this problem. After some long pauses in which she reconsidered Michelle's solution, she realized that what Michelle colored in red represented the dividend and not the quotient. Tiffany contended that what confused her was that whenever a drawing was employed in the virtual lesson the quotient was always represented in red. Although this argument was not valid, it revealed that her analysis of the students' work was driven by the surface rather than the more substantive features of their work. Tiffany then figured out that Michelle's drawing represented the number of the  $\frac{3}{4}$ -segments that could be made out of the available  $2\frac{3}{4}$ -segment. At this point, she stopped and argued that Michelle's solution was wrong. She explained,

- Tiffany:** [Michelle] is actually doing what I started to do at the beginning [when she was explaining the division  $2 \div \frac{3}{4}$ ]. My first step when [I] said, “Okay how many times does three fourths go into two?” Which is -- that’s the same idea that we were talking about that she’s using the normal concept of division for that.
- Charalambos:** What concept should she use then?
- Tiffany:** She doesn’t have the concept of reciprocal but again, I don’t think that was taught well in the lesson.
- Charalambos:** So what so you think about what Michelle understands or doesn’t understand?
- Tiffany:** The idea that dividing fractions doesn’t have the same annotation as dividing whole numbers.
- Charalambos:** In what sense?
- Tiffany:** That -- I mean just that sentence. The sentence that you put in your head when you read a division problem. (PE. 2491-2509)

Tiffany projected her own difficulties in working with division  $2 \div \frac{3}{4}$  onto analyzing Michelle’s work. In other words, what Tiffany could see in this student’s work and how she interpreted it was contingent on her own understanding of the content. To her, whole-number division and division of fractions were two distinct fields, with the latter being associated more with the notion of the reciprocal. Hence, Michelle’s work could not be right: on the one hand, Michelle was not using any reciprocals; on the other hand, she appeared to be using the concept of whole-number division, a concept with which Tiffany experimented at the beginning of the interview only to figure out that it was not applicable to the division of fractions. Interestingly enough, when Tiffany was asked to explain what she would do after having seen the three student solutions, she argued: “You want to at least tell Michelle that she’s not doing it right. ... So that [she does not] think that that’s right and let that get engrained overnight” (PE.I. 2534-2538).

Robert’s and Ann’s solutions were easier for Tiffany to analyze because they complied with the standard invert-and-multiply algorithm. After going over the steps in Robert’s work, Tiffany concluded that his solution was correct; she argued though, that Robert needed to “reduce” his final answer, probably implying that he needed to convert the improper fraction in his final answer into a mixed number. In analyzing Ann’s work,

she originally thought that Ann made an error in converting the mixed number of the dividend into an improper fraction. After consulting her own solution, Tiffany figured out that Ann took the reciprocal of part of the dividend instead of taking the reciprocal of the divisor. Based on this analysis, she concluded that whereas Robert understood the problem, Ann did not, because she inverted the wrong number. Tiffany's assertion about Robert's understanding of the problem was quite surprising. Whereas throughout the whole interview she was particularly attentive to issues of meaning-making, in analyzing Robert's work she did not appear to even entertain the idea that Robert might have understood nothing more but the sequence of steps involved in the traditional division-of-fractions algorithm.

To summarize, in analyzing the three students' work, Tiffany could only follow the conventional student solutions that resonated with her own understanding of the content. Her analysis could also be characterized as *answer-driven* since she largely determined the correctness of the students' work based on the answer she obtained by applying the traditional algorithm. Finally, even when she could follow and fathom students' work she made inappropriate assertions about their understanding.

*Noticing and Interpreting-Evaluating.* With the exception of Ann's contribution, Tiffany identified all errors in the four episodes at hand without any prompting.

When reading June's explanation, she claimed that June's argument that the dividend needs to be bigger than the divisor was incorrect because "you don't always have to divide a bigger number by a smaller number" (PE.I. 1407-1408). Using a whole-number example, she explained that one can divide six by two or two by six. She also

argued that June's explanation did not really explicate why one needs to divide one half by one sixth and not the other way around:

No, her explanation doesn't explain which fraction goes on top or bottom [i.e., is the dividend or the divisor]. Her explanation explains why if you put one sixth on the top [and] one half on the bottom why it doesn't work logically. But that has no reasoning why you couldn't put that number on top or on bottom. (PE.I. 1557-1561)

Tiffany was also quick in noticing the error in Alan's work. She correctly observed that Alan was dividing "half the line into sixths;" hence, instead of taking sixths, he was taking twelfths of a yard. She also criticized how the teacher supported Alan in this episode by advocating that he divide the line into six parts without ensuring that he really understood what a sixth meant. In doing so, Tiffany alluded to Ms. Rebecca's instructional moves being a potential source of Alan's confusion and error.

In the third episode under consideration, Tiffany was again quick in noticing the error in Amanda and Julia's solution; yet, she struggled with resolving this error. While going over the two girls' solution, she remarked that they were using a totally different approach from how she learned to solve division-of-fractions problems: "So as a teacher, using like the framework and the base that I have, that's not the [solution] that I would have thought of" (PE.I. 1926-1928). She nonetheless rightfully argued that the teacher should have prompted the girls to explain what the remainder in their solution meant: "So, you've got this extra piece left; well, what does it mean? You know ... you just can't make a full ribbon out of it" (PE.I. 1950-1952). Although she provided an appropriate explanation for what the remainder represented, she wrestled with figuring out its meaning in terms of ribbon badges. Eventually, she figured out that she needed to express this remainder in sixths (i.e., the length of a ribbon badge). To do so, she set up the equation  $\frac{1}{2} = \frac{x}{6}$ . Yet it took her a long time to figure out the unknown in this equation

because, as she explained, “she never learned [her] multiplication by twelves in school” (PE.I. 1995). Even though the answer she eventually got resonated with the answer she originally obtained by solving the division problem  $\frac{3}{4} \div \frac{1}{6}$ , she was still uncertain about her work: “I still don’t know if I am right or wrong” (PE.I. 2105). After a couple of long pauses of thinking, she was eventually convinced that the girls’ answer was wrong.

In short, Tiffany was meticulous and flexible when analyzing students’ work only in those cases in which her knowledge supported her in doing so. In contrast, in the last episode, although she sensed that there was a flaw in the students’ work, she could not easily resolve the discrepancy that emerged from explaining the remainder in absolute rather than relative units. Her struggles in this episode were consistent with the difficulties she encountered explaining that same concept when working on  $2 \div \frac{3}{4}$ . This episode preceded the episode in which Tiffany was asked to analyze Michelle’s solution to the problem  $2 \frac{3}{4} \div \frac{3}{4}$ , which also required an understanding of relative units (see above). Hence, it seems that even after resolving this discrepancy, Tiffany’s grasp of the quotient in terms of relative units was still not solid to support her in appropriately analyzing Michelle’s work.

#### *Responding to Students’ Direct or Indirect Requests for Help*

*Performing.* Tiffany’s performance in the two episodes of this practice – handling Alan’s confusion and responding to June’s and Shaun’s direct request for help – was contingent on her understanding of the content.

As already explained, in considering Alan’s episode, Tiffany did not only identify the error in his work but also postulated that Alan was not solid on what a sixth is. To handle his confusion, she would give a mini-lecture. Addressing the whole class, she



would first describe Alan's error, namely that Alan divided half instead of the whole yard into six parts; she would also clarify the concept of sixths. To be more convincing, she would even use a yardstick and measure the different parts in Alan's drawing. Provided that his representation was drawn to scale, doing so, as she suggested, would help Alan and his classmates realize that he showed twelfths instead of sixths. After this mini-lecture, she would elicit students' ideas as to how Alan's diagram could be corrected. Tiffany's intervention in this episode, as just described, had the potential to support students' understanding of Alan's error. Additionally, instead of doing all the thinking for them, she would give students enough information to help them correct the error for themselves. Her approach in this episode was remarkably different from that in the second episode.

In considering June's and Shaun's episode, Tiffany identified the source of the students' difficulty: the one sixth did not fit evenly in the three-fourths piece. To explore whether there was a way to "fit" sixths into the three-fourths segment, she drew two lines, equal in size, one below the other (see Figure 4.24). She divided the first line into four approximately equal-sized parts and labeled the first of them a fourth; she repeated the same approach for the second line. She then contemplated following the teacher's suggestion to use twelfths, although, as she clarified, she had never solved division-of-fractions problems by finding the common multiple of the dividend and the divisor:

I don't ever [sic] use twelfths. ... So, I don't know why logically you would have [students use twelfths], to find the common multiple of twelve and use that. It doesn't make sense to me. (P.E.I. 1756-1760)

Although the teacher's suggestion seemed cryptic to Tiffany, she experimented with creating twelfths. Still working on the second line, she divided the first, second, and fourth fourths into half, and the third fourth into three parts; she labeled the first segment

resulting from this new partitioning a sixth. She soon realized, however, that her partitioning of the second line represented neither sixths nor twelfths; thus, she concluded that her diagram “did not work.” She attributed her difficulty in getting twelfths to that she did not draw the two lines to scale. Even though her claim was partly correct, it is questionable whether she would have been able to figure out the relationship between fourths and sixths had she drawn the two lines to scale. This is because she not only had difficulties splitting her lines into twelfths, but she also appeared to not understand how employing the idea of common multiples would result in commensurate pieces.

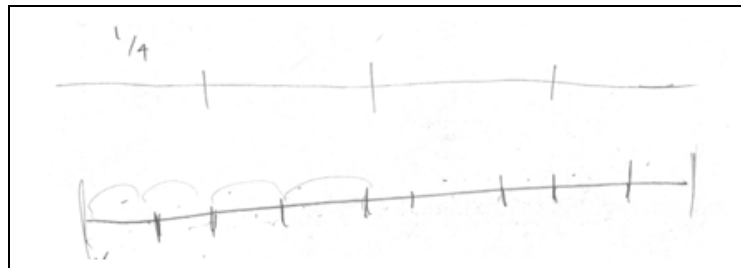


Figure 4.24. Tiffany’s work on solving the division problem  $\frac{3}{4} \div \frac{1}{6}$ .

Given all her difficulties with the content, it is not surprising that Tiffany would have responded to the two students’ request for help in pedagogically sound yet mathematically nebulous ways. In particular, after exploring whether other students also wrestled with this problem, she would have gone to the board and offered a better explanation rather than simply asking students to split their lines into twelfths, as the teacher did. When asked to expound how this explanation would look, she said that she was not clear and that she would have tried to use a drawing to help her students understand why the idea of common multiples was applicable to solving this particular problem. Overall, her performance in this latter episode suggests that without a strong background of the mathematics at play, it is questionable whether Tiffany could have

responded to the students' requests for help in a manner that would support them in understanding the mathematical ideas at stake.

*Noticing and Interpreting-Evaluating.* Immediately after going over the slides pertaining to how the virtual teacher supported Alan to overcome his confusion, Tiffany noticed the teacher's pointed questions. She speculated that Alan probably corrected his error not because of having understood the concept at hand, but mainly because of being able to appropriately interpret the teacher's questions:

[L]ittle Alan is just learning how to read this teacher. ... He didn't understand; he just knew what his teacher was questioning him. And maybe he was like, "Oh! Hum. If she said, 'Wait a little bit,' uh-oh, I must have done something wrong, I'm gonna say the answer of what the opposite is."  
(PE. I. 1632-1646)

She also criticized the teacher's move to suggest that the students use the "whole line and not part of it." From Tiffany's perspective, the teacher's suggestion was not actually supporting Alan's understanding; instead it was giving him another step to follow.

Tiffany's analysis of the second episode was less thorough. She commented that the teacher's suggestion that students use the common multiple of twelve was not sufficient to support students' work and understanding. She argued that instead of merely reminding students of the idea of common multiples, the teacher should have explained how this idea gets applied to the problem under consideration.

#### *Performance on the LMT Test: A Closer Look*

Although Tiffany represents a divergent case, her performance on the LMT test was consistent with the difficulties she encountered during the teaching simulation. Specifically, she incorrectly answered the two questions on units (questions 1 and 20), the question on the interpretation of the quotient when it involves a fractional part (question 17), four of the five questions that required interpreting the meaning of the

division of fractions (questions 14a to 14 d), and the two questions related to the measurement interpretation of division (questions 15 and 25); these were the ideas with which she struggled the most during the teaching simulation.

Yet, given her struggles with the division of fractions as captured by her performance in the teaching simulation, one would expect that she would not have correctly answered questions 11a-11d and 19a-19e, which pertained to division-of-fractions word problems, and question 5, which concerned using a representation in a division-of-fractions context. Her notes on the test suggest a plausible reason for her performance in questions 11a-11d: she first figured out the numerical answer to the division-of-fractions involved in these questions and then explored whether this answer matched the given word problems. She probably followed a similar approach when solving questions 19a-19e.

Based on this analysis, the question is unavoidable: Why is Tiffany considered a divergent case if her performance on the LMT test was aligned with her struggles during the teaching simulation? The analytical commentary offers some plausible explanations.

#### *Analytical Commentary*

To better understand Tiffany's case, a closer look at her teaching-simulation score is in order. Although Tiffany performed relatively well on the noticing and interpreting-evaluating aspects of the teaching simulation (i.e., 60% and 90%, correspondingly), her score on the performing tasks of the simulation was low (26%). She also performed better on the tasks pertaining to the two MTF-related practices (64%) than on those pertaining to the MKT-related practices (48%). Collectively, these figures suggest that her relatively high performance on the teaching simulation was largely due to her close attendance to

and analysis and interpretation of the virtual lesson rather than to what she could *do* with respect to the five practices at hand. In fact, if one considers only her performing score, Tiffany no longer constitutes a divergent case since this score aligns well with her score on the LMT test. Hence Tiffany's case helps explain why the correlation between the PSTs' MKT score and their score on the performing tasks was higher than the correlation between their MKT score and their overall teaching-performance score. The consistency between the limitations in her knowledge and her low performance on the performing tasks of the simulation also substantiates the association between knowledge and teaching performance considered thus far. Even more, Tiffany's case shows that her productive beliefs and her images of teaching – the two main factors that appear to have informed her performance besides her knowledge – could not compensate for limitations in her knowledge, particularly with respect to the performing tasks of the teaching simulation. I justify this argument by considering examples from her performance first in the MTF-related practices and then in the MKT-related practices.

A closer look at Tiffany's performance in the MTF-related tasks suggests that her respective decisions and actions were informed by her beliefs about teaching and learning mathematics, and particularly her conviction that students should be scaffolded to see the underlying meaning of the procedures they are asked to follow. The interview data suggest that her school experiences were consequential for (in)forming these beliefs. As a learner of mathematics, Tiffany reported being exposed to a rather algorithmic type of instruction: she was shown certain procedures and she was asked to apply them with accuracy and speed. This resulted in her learning certain algorithms without seeing or understanding their conceptual underpinnings; hence, she ended up being a “rememberer

and forgetter” of these rules, as her work on the cross-multiplication rule suggests. It is not surprising, then, that she was particularly attentive to issues of making meaning and supporting students’ understanding.

Apparently, this sensitivity helped Tiffany attend to the different ways in which the virtual teacher shifted emphasis from meaning and understanding to following procedures and manipulating numbers. She noticed that Ms. Rebecca decontextualized the first task even during its introduction; she observed that Ms. Rebecca was merely concerned with helping her students get right answers; she identified the instances in which the teacher appeared to be doing the thinking for her students by either posing leading questions or giving unnecessary hints; she even detected cases in which the teacher’s scaffolding was insufficient to support students’ understanding. In short, this sensitivity helped Tiffany perform well on the noticing and interpreting-evaluating tasks of the two MTF-related practices.

Tiffany’s orientation toward a type of instruction that promotes understanding also provided her with a head start in the performing tasks associated with the MTF-related practices. For instance, regarding the practice of selecting tasks, Tiffany selected the second textbook page, which was more appropriate for her endorsed type of instruction. She also had some good ideas for using the first textbook page in ways that would support students in “understand[ing] the logic” of the division of fractions. These included reordering the tasks of this page to start with the word problems, asking students to pose word problems for given numerical sentences, and assigning exercises selectively to avoid having students engage in repetitive and meaningless work. Similarly, with respect to the practice of responding to students’ requests for help, she was inclined to

expound the idea of common multiples to help June and Shaun understand why and how this idea was applicable to the problem they were trying to solve. At this point, however, the limitations in her knowledge entered the picture.

The LMT test and the teaching simulation both provided converging evidence about the limitations in Tiffany's knowledge. She knew the "hows" of the procedure of dividing fractions – although with some glitches, since her recollections of the cross-multiplication rule interfered with her procedural fluency in the invert-and-multiply algorithm. Yet, as she very eloquently put it, she did not "have the link," she did not "understand the logic." This lack of understanding was manifested in at least three ways: she tended to perceive the division of fractions as distinct from the division of whole numbers; she did not appear to have a solid understanding of the meaning of division of fractions; and her understanding of the different instantiations of units was not flexible. All these difficulties were reflected in Tiffany's performance in the performing tasks of the MTF-related practices. Although she wanted to use the second textbook page, she could not capitalize on its affordances. Even though she wanted to assign only a subset of the exercises of the first page, she did not have the criteria that would inform such choices. She wanted to introduce the division of fractions through word problems but it is questionable whether she could build connections between these word problems and the algorithm at hand. Similarly, while she would have liked to support June and Shaun in understanding why and how the idea of common multiples becomes handy when considering non-commensurate fractions, she did not appear to have the means to do so.

Tiffany's performance in the MTF-related practices is not only informative about how limitations in teachers' knowledge might constrain their instructional decisions and

actions but it also offers some insights into how teachers may compensate for these limitations. Two examples are particularly telling in this respect. First, lacking any substantive criteria that would allow her decide which exercises to assign, Tiffany resorted to a familiar image of teaching: she talked about assigning the “odds” or the “evens.” Second, probably because she could not see the connection of the second page’s word problems to the algorithm of the division of fractions, she considered introducing this algorithm as a shortcut to follow. Both examples imply that to compensate for limitations in their knowledge, teachers – and especially the novices – might resort to familiar images of teaching, which they had form during their school years, even if these images might be antithetical to the type of instruction they endorse. These examples, however, suggest that neither productive beliefs nor previous images of teaching can adequately compensate for limitations in PSTs’ knowledge.

Tiffany’s difficulties with the content and its teaching were also reflected in her performance in providing explanations and using representations. Admittedly, she had both the intention and some good tools to develop and offer worthwhile explanations. For instance, she knew that representations can help make opaque mathematical ideas more transparent to students. Similarly, she was seen using easier examples to scaffold her work, a strategy which can be particularly handy in developing and giving mathematical explanations. Yet, the limitations in her knowledge discussed above appeared to constrain what she could do. Without (substantive) conceptual understanding of the notion of the reciprocal, she did not even know how to begin building a germane explanation. Without a solid understanding of division (of fractions) and of the idea of relative units – and also confronted with difficulties even representing a division problem with a mathematical



sentence<sup>152</sup> – Tiffany found it hard to explain the quotient in a division-of-fractions problem. These difficulties were also reflected in her use of representations. Particularly telling in this respect was the inappropriate way in which she represented the dividend and the divisor of the division problem  $2 \div \frac{3}{4}$  without retaining their relative sizes.

Tiffany’s case also exemplifies how a teacher’s knowledge can support or constrain her when analyzing students’ work and contributions. When Tiffany was familiar with the content, she was quick in capturing errors in students’ contributions. In these cases, her analysis of students’ work was relatively thorough and she could even speculate about how instruction itself might have led students to commit these errors. In contrast, when confronted with unconventional student solutions or ideas with which she struggled, she could not follow the students’ work. In those cases, she focused on more superficial characteristics of students’ work (e.g., the colors they used in their solutions); she used the numerical answer to a problem as the main criterion against which to evaluate the students’ productions; and she interpreted the students’ work through the spectrum of her own difficulties with the content. Tiffany’s case, however, suggests that one’s knowledge of the content does not suffice to make appropriate assertions about students’ understanding. For instance, regardless of her appropriate analysis of Robert’s work, she made invalid assertions about his understanding.

Unlike her performance in the MTF-related practices, in the case of her performance in the MKT-practices the interview data suggested that no factors compensated for the limitations in her knowledge not only for the performing tasks, but

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<sup>152</sup> These difficulties should not be underestimated since they reflect a fundamental *syntactical* difference in how the idea of division as fitting divisor units into the dividend gets represented mathematically and linguistically. Specifically, the mathematical syntax is exactly the opposite of the linguistic syntax, which requires that the divisor appear first and be followed by the dividend (e.g.,  $6 \div 2$  is expressed as “how many times two goes into six?” and not the other way around).

also for the tasks of noticing and interpreting-evaluating. For example, although she was able to analyze the teacher's explanation of the reciprocal from a pedagogical perspective, she did not delve deeper into the mathematics of this explanation.<sup>153</sup> Similarly, she did not notice that the representations being used in the virtual lesson were not mapped onto the algorithm or the word problems at hand. Her inattentiveness to the lack of such connections was not unrelated to her facility with the content. As she admitted, because of her difficulties with the content she would use a representation not to *derive* the answer to a division-of-fractions problem, but merely to *illustrate* the answer she would get by applying the traditional algorithm. Hence, it is not surprising that she was not concerned with whether the representations being used in the virtual lesson were connected to the dividend or the divisor of the problems considered.

Some additional clarifications are necessary with respect to Tiffany's profile, as illustrated in Figure 4.25. Tiffany's school experiences seemed to have informed her performance both positively and negatively. On the one hand, they offered her avoidance models (e.g., rote learning and memorization); on the other hand, they seemed to constrain what she could do during the simulation and how she analyzed its content (e.g., the example of assigning the "even" or the "odd" exercises). The role of her family background in informing her performance was less clear. In contrast, the interview data provided ample evidence that Tiffany's beliefs about teaching and learning math aided her considerably in identifying deficiencies in the virtual teacher's instruction; these

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<sup>153</sup> This example points to a potential limitation of the coding schemes used to evaluate the PSTs' performance in the interpreting-evaluating tasks. Tiffany performed relatively well in the interpreting-evaluating tasks of the practice of providing explanations, even though her analysis did not go into much depth. This suggests that if higher thresholds had been used to evaluate the PSTs' respective performances, their MKT score might have been more aligned with their overall teaching-performance score and particularly with their interpreting-evaluating performance.

beliefs appear to have been informed by her school experiences and the images of teaching she formed as a learner of the subject. Finally, even though in the survey Tiffany explicitly mentioned that she did not feel confident in mathematics, the interview data did not provide evidence about the role of her efficacy beliefs in informing her performance.

A final remark is necessary before leaving Tiffany's case. In discussing the limitations of the study in Chapter 3, I pointed out that the PSTs' differed in their verbosity and the meticulousness with which they analyzed the teaching simulation. Tiffany was amongst the PSTs who analyzed the virtual lesson very meticulously, by paying close attention to the teacher's moves and the students' ideas and contributions. Given that she scored high on the noticing and the interpreting-evaluating tasks of the simulation, one could claim that her high score was merely due to her thorough analysis of the visual lesson. I argue that such a thorough analysis would not suffice for a high performance on the visual lesson, even for only its noticing and interpreting-evaluating tasks. Tiffany's performance in the practice of using representations corroborates this argument. In particular, despite her close analysis and dissection of the teaching simulation, she did not notice that neither the teacher nor the students were drawing connections among the representations being used, the mathematical sentences written on the board, and the word problems under consideration. Even when directly prompted to discuss the teacher's and the students' use of representations, Tiffany merely focused on the final answer and did not discuss the absence of connections just described. Her performance – along with that of other PSTs (e.g., Habika's) – suggests that without being aware of and sensitive to the issues examined in the virtual lesson, the PSTs could

not perform well on the noticing and interpreting-evaluating tasks simply by virtue of their methodical analysis of the simulation.

Along the same lines, being verbose in analyzing the virtual lesson did not suffice to obtain a high score on the teaching simulation. This was suggested by the non-statistically significant correlation (*Kendall's rank-order partial correlation coefficient* = 0.12,  $p > 0.10$ ) between the number of lines in each PST's transcript (i.e., an index of their verbosity) and their overall performance on the teaching simulation after controlling for the effect of their overall MKT-performance (as tapped by the LMT test).

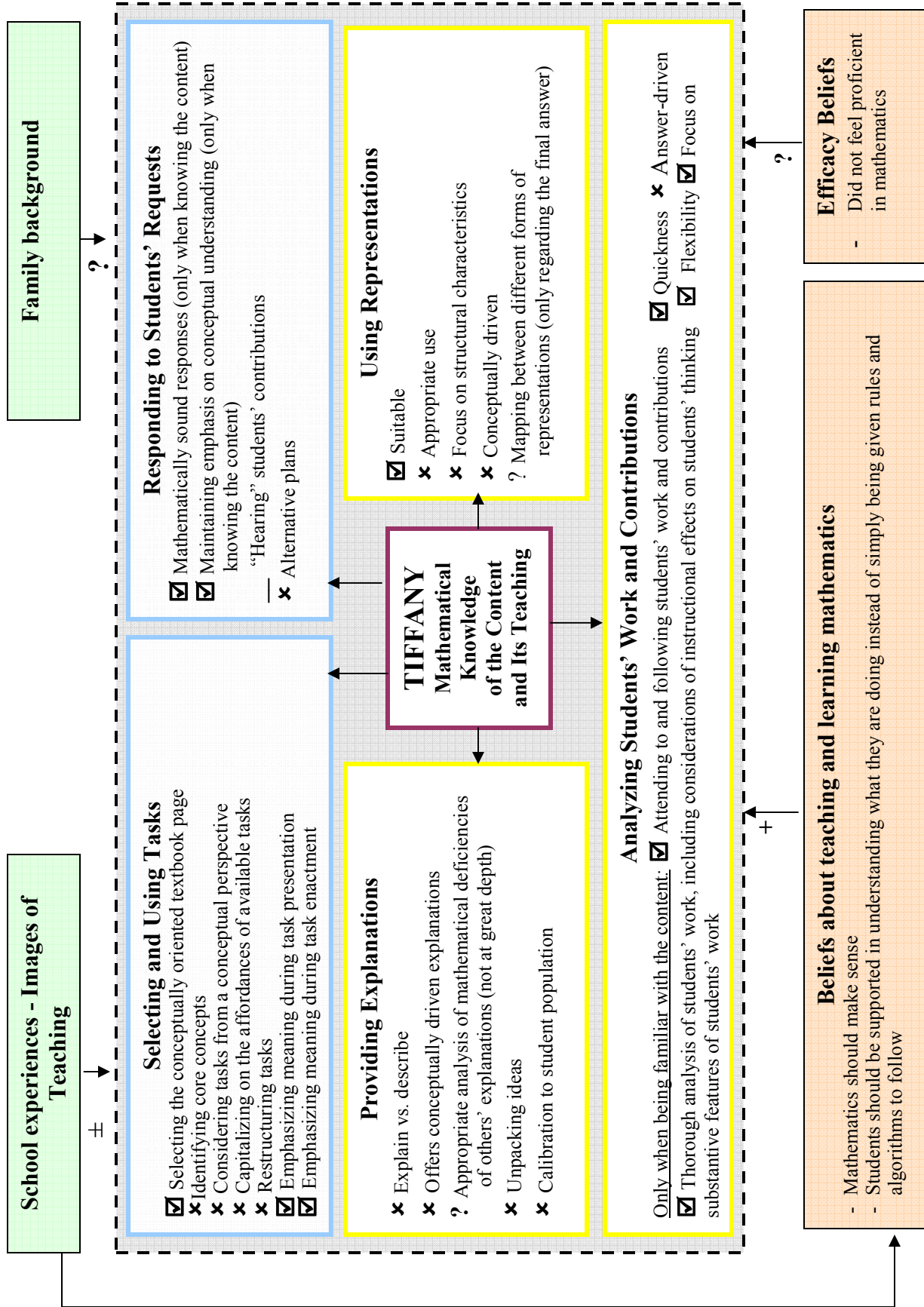


Figure 4.25. Considering the association between knowledge and teaching performance through Tiffany's profile.

*The Case of Kimberley: “You Can’t Explain Something That You Don’t Know.”*

Kimberley represents the second divergent case and the sixth case considered in the study. She performed similarly to Nicole on the LMT test but worse than her on the teaching simulation. As Table 4.5 suggests, she also performed better on the standardized GRE-quantitative test than on the LMT test. Two additional considerations motivated her selection for further scrutiny: she had a strong background of math coursework and had taught 8<sup>th</sup> grade Algebra as a substitute teacher for two years using *Everyday Mathematics*. In addition to providing both affordance and constraint examples that corroborate the association between knowledge and teaching performance, Kimberley’s case is informative in two other respects. First, it shows how certain unproductive beliefs and images of teaching might mediate the effect of knowledge on teaching performance. Second, her teaching performance (especially in the performing tasks) was more aligned with her LMT score rather than with her GRE score. Consequently, her case further supports the findings of the quantitative analysis which showed the relationship between knowledge and teaching performance to be marginally robust to the PSTs’ GRE scores when considering the performing tasks of the teaching simulation.

*Background*

Mathematics was Kimberley’s favorite subject throughout her school years. In elementary school, she found mathematics “fun, easy, [and] very tactile,” although “repetitive in many ways” (PE.S.). In middle-school grades, mathematics continued being easy and fun for her. Her positive attitude toward the subject was not the only reason that informed her decision to join the ELMAC program. She also had a keen interest in teaching. She “literally started teaching at the age of fifteen, when [she] was teaching at

[her] local Sunday school and pretty much never stopped” (PE.I. 58-59). For Kimberley, seeing students making progress over time and experiencing “aha moments” were the most fascinating and rewording aspects of teaching. At her entrance to the program, she expected to get acquainted with “current techniques for teaching mathematics” and even to “re-learn mathematics” (PE.S.).

### *Selecting and Using Tasks*

*Performing.* Kimberley considered the first textbook page highly procedural, in contrast to the second one, which she found more conceptual. Referring to the first page, she remarked that not only was there “little conceptual meat to this [the first] page,” but also that this page was too crowded, leaving no room for students to draw diagrams or to experiment with alternative strategies for solving the division exercises. These limitations notwithstanding, she thought that this page included a nice combination of numerical exercises, thus affording students the opportunity to practice different skills associated with dividing fractions (e.g., simplifying fractions, turning mixed numbers into improper fractions, etc). She also liked that this page included two word problems, which she thought appropriate for helping students see the practical applications of the division of fractions. She argued, however, that a different context (e.g., sharing pizza) might have been more relevant to sixth graders than the context of these two word problems.

If she had to use this page for an introductory division-of-fractions lesson, she would start by explaining the utility of dividing fractions. She would then explain the “mechanics” involved and have students figure out the reciprocal of several fractions:

**Kimberley:** [Assuming the role of the teacher:] “So, today we are going to divide using a fraction. So, we use the division of fractions in a bunch of different ways. Let me give you, guys, some examples of that. Let’s say, for example, that you had half of a pizza left, and there are three of you, and you want to make sure that everybody gets the same number of pieces. Right? You don’t want your buddy to get an extra piece more than you did. So, we are

going to take that as an example. So, if we have half of a pizza, and we are going to divide it by -- how many people do we have?"

**Charalambos:** [Assuming the role of the student:] "Three."

**Kimberley:** "Three, right. So, we are going to divide that half pizza by three. So, how would we write that problem up on the board? Does anybody wanna try?" And then, after somebody made an attempt, I would explain how we take the reciprocals. [I would say:] "In order to divide a fraction by another fraction, it is very important that you take the reciprocal of a fraction. Can everybody say 'reciprocal?'" [Student voice:] "Reciprocal." [Then again using teacher voice:] "Okay. Let's say that we have the fraction three fourths. The reciprocal of three fourths is worth four thirds. Okay. The reciprocal of two fifths is...?" [Student voice:] "Five halves." [Teacher voice:] "Okay, let's try another one. I'm gonna write three sevenths. Who can tell me what the reciprocal of three sevenths is?" [I would] get an answer. "Okay. Let's do another one. Who can tell me what the reciprocal of thirteen halves is?"

**Charalambos:** Okay. So, you would do several examples.

**Kimberley:** Yeah, I would do a lot of that. I think repetition is really important. So, you know what we are doing. ... And then, I would go through, you know, I'd simplify and then go from there. (PE.I. 310-343)

Before asking students to work on the 16 exercises, Kimberley would explain all the associated terms – dividend, divisor, reciprocal – and ensure that students were familiar with other procedures, such as simplifying fractions and converting mixed numbers into improper fractions. She would then have the students work on the first four exercises as a group and then individually on the remaining exercises. She determined the level of difficulty of these exercises based on whether they required manipulating large numbers and if they involved "extra steps" (e.g., simplifying, converting a mixed number into an improper fraction).

Although she originally considered working on the first page's word problems, after scrutinizing them, she claimed that conceptually both lent themselves to multiplication rather than to division. According to Kimberley, to conceptually consider them as division problems, one needed to have already developed the idea that "division makes something larger." She explained:

Because, most of us are taught conceptually that to divide makes things smaller. And this is a case where division is making something larger. And that's the nature of fractions, when you divide by it, it makes things larger. So, I haven't thought about it before, but that's actually a very hard thing to explain that dividing by a fraction makes something -- I mean, obviously you have to develop



that concept that dividing something by a fraction makes something bigger rather than smaller. (PE.I. 261-268)

To give a more conceptual flavor to this page, she proposed having students pose word problems for some of its exercises. This would help the students “flesh out the numbers and make them representative of something else” (PE.I. 632-634).

From the description outlined above, it is clear that Kimberley was not very satisfied with having students simply practice the pertinent procedure. Hence, she would start with a word problem to help them understand the applicability of the division of fractions. However, the problem she would use – dividing half a pizza among three friends – did not actually pertain to division *by a fraction* but to division of a fraction by a whole number. Even though at a later point in the interview she identified this difference, she argued that she would find it hard to show the meaning and the applicability of dividing by fractions; she was also unable to propose a sample dividing-by-fractions problem.

Kimberley, unlike many other PSTs, did identify a conceptual difference between whole-number division and division by fractions. She noticed that in the latter case the quotient is bigger than the dividend. Even so, she did not capitalize on this idea to develop a more conceptually oriented lesson. Instead, she thought of using this page’s worked example to offer students ample practice with finding the reciprocal of given fractions on the premise that “repetition helps one understand” what he or she is doing. In the same vein, she would use the 16 exercises to merely offer her students opportunities to practice the algorithm of dividing fractions and other related procedures (e.g., converting mixed numbers into improper fractions). In essence, then, she would not significantly depart from the philosophy of the first page, regardless of her critique of this

page as not being conceptually oriented. Even though she would try to help her students see the applicability of fraction division, her enactment of this page would mainly maintain its procedural orientation.

If she were given the option to choose between the two pages, she would use the second page for an introductory lesson on division of fractions. She explained that this page “shows more the functionality of division by a fraction” (PE.I. 811-812) and “sets up things more conceptually” (PE.I. 934), which she considered important for introducing a new concept. However, she regarded the page’s sequencing of tasks strange:

[T]he way that this worksheet is set up [i.e., starting from task A] ... it almost assumes that they already know how to divide a fraction by a fraction. And yet, then it goes to [task] C, which if they know how to do A, for sure they already know how to do C at this point. So, that’s why the structure seemed a little strange to me. (PE.I. 858-862)

For Kimberley, task C, which includes numerical exercises, appeared to be much easier than the two preceding tasks, which pertain to solving word problems. Therefore, she initially thought reversing the order of these tasks to start with task C to help students practice the “mechanics” before solving the word problems. She later revised her thinking, arguing that there was also value in starting with task A. If she started with this task, she would have students “take some time to think conceptually” about the division of fractions (PE.I. 863-864). She would then write the word problem on the board, and after eliciting student ideas on how to solve it, she would walk them through this problem, “showing them how to do it” (PE.I. 847). She would then move to task C and “giv[e] them some examples in terms of the mechanics of [dividing by fractions]” (PE.I. 847-848). Eventually, she would use task B to check whether her students had understood this procedure. She even considered using some of the first page’s 16 exercises to assess students’ understanding of the mechanics.

Kimberley's outlined enactment of the second page does not actually differ from her proposed enactment of the first page. She would start with a word problem to motivate the operation of dividing fractions, and although she would elicit different student ideas, she would then show her students how to solve the problem. Next, she would shift to task C to familiarize her students with the mechanics involved in dividing fractions; she would offer them opportunities to practice these mechanics by working on task B and/or on some of the exercises of the first page. Apparently, Kimberley ignored task D, which pertains to figuring out an algorithm for dividing fractions. When I drew her attention to task D, she remarked that having students find the algorithm on their own represented an even more conceptually oriented approach, but "it was not necessarily what [she] had in mind when [she] gave [her] little mini lesson-plan" (PE.I. 915-916).

Overall, Kimberley would not use the second page to its full potential to scaffold students' understanding of the meaning underlying the procedure of dividing fractions. Although she would start with task A, she would merely use this task as a segue to introduce her students to the mechanics of this operation (i.e., task C), thus missing opportunities to capitalize on other affordances of this task. For instance, she did not notice that task A expects students to explain the fractional part of their answers and to present their reasoning in words and diagrams.

*Noticing and Interpreting-Evaluating.* Kimberley found the virtual teacher's introduction to the first task very effective. She liked that Ms. Rebecca let the students "sort of pull it out on their own ... [and] puzzle it out together," instead of feeding them the answer (PE.I. 1036-1037). She was intrigued by how the teacher enacted the task with her students, and particularly with having the students work conceptually on the task,

before giving them the mechanics. She only criticized the teacher for not prompting the students to connect their final answer to the context of the word problem:

I think I would've done one thing differently, when [the teacher] was finishing up the last problem, problem A. I think I would've had the kids count out, you know, one, two, three, for this number of ribbons. I don't think she appropriately fleshed that part out. (P.E.I. 1268-1272)

Kimberley was even more fascinated with the enactment of task D. She thought that Ms. Rebecca did a very good job leading her students to the shortcut and the mechanics involved in dividing fractions. She commented:

[T]he way that she ... unveiled the idea of the reciprocal, half way through, is cute. Because the kids are excited about it: "Oh, hey, there is a trick, there is a shortcut." So, that's a cute way of doing it. And I can see how that actually would generate excitement in the class. So, I think that that was well done. ... I think it almost builds a little suspense for actually learning the mechanics. You know, you start conceptually and then give them the mechanics, which is ... obviously the best way to do it. Because it gives the kids a chance to understand what they are really doing. (P.E.I. 1383-1389, 1405-1409)

In considering the teacher's introduction and enactment of these two tasks, Kimberley apparently focused more on the pedagogical rather than the mathematical aspects of the teacher's approach: engaging students, using representations, working on a problem conceptually before moving into the pertinent algorithm. In doing so, she did not appear to notice the manifold of the ways in which the teacher was actually diminishing the demands of these tasks: by outlining steps for students to follow, by paying more attention to right answers instead of helping them understand the meaning of the division of fractions, and by walking them through the mechanics of the algorithm without helping them see its underlying meaning.

Kimberley's comments at the end of the interview gave insight into what might have informed her evaluation of Ms. Rebecca's work. As Kimberley remarked, the teacher's introduction and enactment of these tasks was different from the teaching she

experienced as a student; she considered the teacher’s approach totally “new to her”. It is not surprising, then, that she found the virtual lesson appealing in many respects:

[The lesson] was well done. [Ms. Rebecca] clearly started out conceptually and gave the kids a chance to draw some visual representations of what they were going to be doing before they started, you know, working out the algorithm and then practicing the mechanics of it. I think that that’s an appropriate order in terms of introducing a new concept. (PE.I. 1551-1556)

Ms. Rebecca’s approach was, in essence, offering Kimberley an alternative image of how the content could be taught. Since on the surface the virtual lesson targeted both conceptual understanding and procedural fluency, it represented an intriguing image of teaching, especially for Kimberley whose mathematical experiences were “very heavy on the mechanics, not on the conceptual” (PE.I. 1602-1603). It is understandable, then, why Kimberley largely focused on the superficial features of Ms. Rebecca’s approach.

### *Providing Explanations*

*Performing.* Explaining the division of fractions from a conceptual perspective was challenging for Kimberley, even though she felt “very comfortable with the mechanics of dividing fractions” (PE.I. 1575-1576). She attributed her difficulties to having never been exposed to any conceptual explanation for this or any other similar procedures.

Rather than moving directly to explaining the division problem  $2 \div \frac{3}{4}$ , Kimberley preferred to start with an easier problem. She initially proposed  $\frac{3}{4} \div \frac{1}{4}$  as such but then thought using the division problem  $\frac{3}{2} \div \frac{1}{4}$ . She argued that the easiest way for her to conceptually explain this division was to use a number line. She thus drew a number line from zero to two, divided it into fourths, and labeled it as shown in the upper panel of Figure 4.26. Next, she explained that if she were teaching, she would first ask students to

tell her what this division means, expecting to elicit answers such as “we are trying to find out how many times one quarter goes into three halves” (PE.I. 468-469).

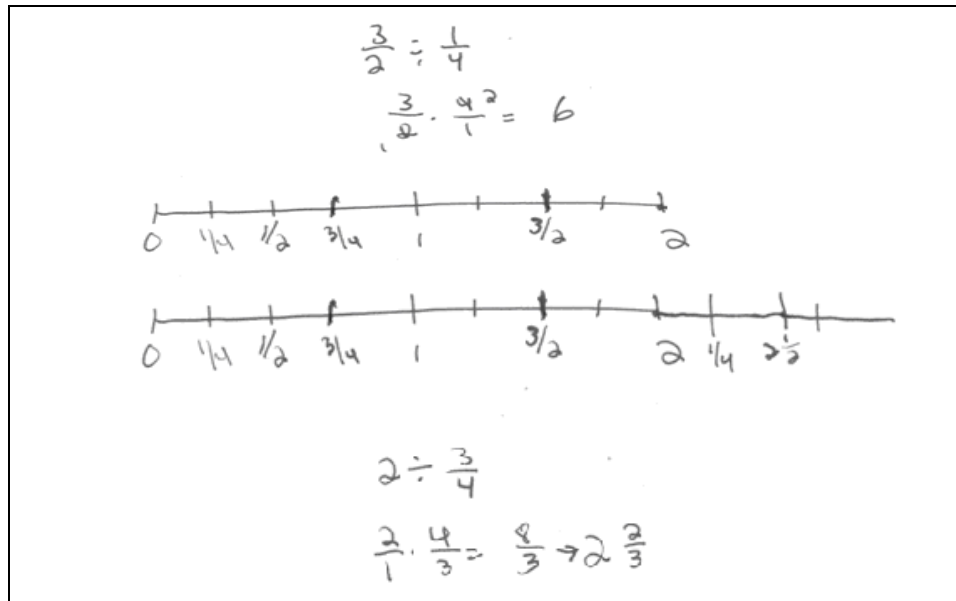


Figure 4.26. Kimberley’s work on explaining fraction division (upper panel: explaining the division  $\frac{3}{2} \div \frac{1}{4}$ ; lower panel: explaining the division  $2 \div \frac{3}{4}$ ).

Building on such answers, she would then prompt her students to count the number of quarters included in the segment defined by the zero and the three halves:

Then, I’ll say, “Let’s count: Here is one quarter [pointing to the first quarter], right? If we have another quarter where do we get to? Here [pointing to number  $\frac{3}{4}$ ], right? And then, we count across: one, two three, four, five, six [pointing to the corresponding pieces on the number line].” (PE.I. 468-472)

After helping her students use the number line to figure out the answer to this problem, she would show them that the traditional algorithm yields the same answer (see her work above the first number line in Figure 4.26).

Kimberley would explain the division  $2 \div \frac{3}{4}$  using the same repeated-addition approach. She would first ask students to show the number  $\frac{3}{4}$  on the number line and ask them to consider where they would land when adding another three fourths. Extending the number line to accommodate a third three fourths (see lower panel of Figure 4.26),

she would then encourage students to predict the result of adding another three fourths. She would validate the students' predictions by using the traditional algorithm to figure out the quotient of this division. Yet, applying this algorithm yielded a quotient that Kimberley found particularly confusing:

- Kimberley:** It just does not work out quite as neatly [as the previous example]. [Pause.] I mean, honestly, I wouldn't use that example because I think that you will end up confusing the kids, adding that extra point on the number line. So, I really wouldn't use that example, given this method. I would use that example for another method, maybe, but not this one, not the number line.
- Charalambos:** What is the extra point that you have to add?
- Kimberley:** Two and two thirds.
- Charalambos:** Two and two thirds?
- Kimberley:** I am currently doing this number line by quarters, not thirds.
- Charalambos:** And why would you have to add another point?
- Kimberley:** Because I broke up my number line by quarters, not thirds.
- Charalambos:** Okay? I am still not getting that.
- Kimberley:** So, I'm trying to show [the students] with the counting where we end up on the number line. Right?
- Charalambos:** Um-hmm.
- Kimberley:** I broke this number line down by *quarters*, not thirds. So, I have to create a new line on here for two and two thirds. It would be here-ish [showing a point between  $2\frac{1}{4}$  and  $2\frac{1}{2}$ ]. ... And I think that's why I wouldn't use this example on the number line. Because the other one works out more neatly, because you already have all the points marked on the number line. (PE.I. 494-532, emphasis in the original)

Kimberley was right that unlike the second problem, the first division problem worked out nicely; because it did not include a fractional part in its quotient, the first problem was easy to show by following a repeated-addition approach. Hence, in working on this division, Kimberley used the number line as a means to get the answer to problem; she then used the algorithm to verify that this answer was correct. From this respect, her explanation for the first division problem could be characterized as conceptually driven since her work was driven by the concept of division as fitting divisor-units into the dividend. In the second division, she followed a similar conceptually driven approach up to the point that she could not fit any more whole

divisor-units into the dividend. After this point, her explanation could be characterized as numerically driven since it was guided by the numerical answer to the problem. Notice, in particular, that she figured out the answer to the division problem using the traditional algorithm and then used the number line to simply *show* the obtained answer. In Kimberley's viewpoint, showing this answer required that a fraction with a different denominator (i.e., thirds and not fourths) be placed on the number line. Troubled with this anomaly, she argued that she would not use this division problem to explain the division of fractions.

When asked to explain the reciprocal in the division  $2 \div \frac{3}{4}$ , Kimberley claimed that she could not offer such an explanation because she had never been taught what the reciprocal is or why it works. As she eloquently put it, "You can't explain something that you don't know" (PE.I. 588). At a later point during the interview, when prompted to consider what explanation the virtual teacher might offer to her students to explain the reciprocal, Kimberley speculated that "there must be some visual way" to show the reciprocal; yet, because as a student she had never learned "this conceptual step" (PE.I. 1363, 1366), she did not have any idea how she or the virtual teacher could provide such an explanation.

*Noticing and Interpreting-Evaluating.* Kimberley found the virtual teacher's explanation for the reciprocal appropriate. She particularly liked that the teacher used an analogy to relate the reciprocal to adding and subtracting positive and negative numbers, a topic considered in a previous lesson: "She is using this information to scaffold their new learning, which is nice" (PE.I. 1496-1497). Although she acknowledged that Ms. Rebecca's explanation mostly described the mechanics of using a reciprocal, she was



satisfied with it and concluded: “If there’s something else conceptually, you can do it. But like I said, I don’t know.” (PE.I. 1504-1505). Her performance in the noticing and interpreting-evaluating tasks of providing explanations was consistent with her work in the performing tasks of this practice. Without a strong conceptual understanding of the notion of the reciprocal, she did not closely attend to and decipher the deficiencies in the teacher’s explanation. Although she correctly identified that the teacher’s explanation described rather than explained why the algorithm works, she did not notice that neither the analogy the teacher used in this explanation nor her argument about using a “reciprocal operation” was mathematically sound.

### *Using Representations*

*Performing.* Kimberley was the only PST who used a number line to provide an explanation for fraction division. When asked whether there was a particular reason for selecting this representation, she clarified that this was the first to come in mind and that she could alternatively have used “a pie or a box” (PE.I. 549). It then appears that Kimberley had a gamut of suitable representations from which to draw to provide an explanation. As her performance in the previous practice suggests, her use of representations was conceptually driven as long as her understanding of the content could support her in this respect. When lacking such conceptual underpinnings, Kimberley employed the representation she selected merely in a numerically driven manner: she used it to *illustrate* the answer rather than as a tool to derive the quotient to the given division. It is also worth highlighting that when explaining the division problem  $\frac{3}{2} \div \frac{1}{4}$  Kimberley was quite successful in building connections among the three different representational models she was using (i.e., drawings, numerical symbols, and words).

For example, in considering the number of fourths she could fit into the three-half segment, she counted six  $\frac{1}{4}$ -segments (i.e., words), while simultaneously pointing to those segments on her number line (i.e., drawing) and to the numerical label associated with the end points of each of those segments (i.e., numerical symbols).

*Noticing and Interpreting-Evaluating.* Kimberley identified two deficiencies in the teacher's and students' use of representations in the first of the two episodes designed for this practice. She noticed that Amanda, the student who was asked to write the mathematical sentence on the board, did not show how her final answer related to the drawing on the board. She also remarked that the teacher should have prompted her students to connect this answer both to the word problem under consideration and to the drawing. She clarified that if she were teaching the lesson, she would have had the students "count out how many ribbons [could be made], rather than just having [Amanda] write out that the answer is three" (PE.I. 1291-1293). As an alternative, she proposed asking each student to figure out the answer and then discuss their thinking in their groups. Kimberley's analysis of this episode was quite thorough since she pointed to the need that connections be built among three different types of representations: the diagram used to solve the problem, the word problem itself, and the numerical symbol representing the answer to this problem. She did not notice, however, that such connections also needed to be built for the dividend and the divisor shown in the mathematical sentence that Amanda wrote on the board.

In the second episode, Kimberley was quite satisfied with Amanda and Julia's use of representations. She observed that the two students first showed how they used their representation to get the answer and then wrote out the mathematical sentence

corresponding to this problem. Although Kimberley's comment was correct, she did not notice that several of the intermediate steps in the two students' work were never clarified. For instance, although the two girls talked about nine twelfths and two twelfths, they numerically represented their work as  $\frac{3}{4} \div \frac{1}{6}$ . When prompted to evaluate the teacher's reaction to the two students' work, Kimberley mentioned that the teacher could have asked other students to share their work or to come to the board and explain what the two girls did. Although this alternative move might have helped build the missing connections, it is quite unlikely that this is what Kimberley had in mind in proposing this move.

Kimberley's performance in the two abovementioned episodes was comparable. In both episodes she overlooked that no connections were built among the representations, the numerical sentences, and the word problems under consideration with regard to the dividend and the divisor of those problems. In the first episode, however, she did notice the absence of such connections with respect to the quotient. This may have occurred because the quotients considered in the two episodes were different: unlike the quotient in the first episode, the quotient considered in the second episode included a fractional part. Given Kimberley's difficulties in understanding and explaining quotients with fractional parts, it seems reasonable to argue that she was more attentive to the lack of such connections when she was solid on the content.

#### *Analyzing Students' Work and Contributions*

*Performing.* In considering the three student solutions to the division problem  $2\frac{3}{4} \div \frac{3}{4}$ , Kimberley mostly attended to whether the students appropriately used the algorithm and other associated procedures. She accepted Robert's work as correct,

although she noticed that he should have simplified the “two fours” in the expression  $1\frac{1}{4} \cdot \frac{4}{3}$  before multiplying across. If she had the opportunity to interact with Robert, she would alert him to how this simplification would have made his work easier.

In analyzing Ann’s work, Kimberley first commented that Ann made a computational error in the dividend because instead of getting eleven fourths, she got ten thirds; Kimberley also thought that Ann “forgot to take the reciprocal” (PE.I. 1444). After a closer examination of Ann’s work, she realized that the error in her solution was that she took the reciprocal of the first number instead of the second. Although she quickly identified the actual problem in Ann’s solution, her initial analysis seems to suggest that instead of closely following and analyzing this student’s work, Kimberley was exploring whether Ann’s work matched the intermediate steps of the algorithm. To put it differently, probably having in mind that converting the dividend’s mixed number into an improper fraction yields eleven fourths, on seeing the ten thirds in Ann’s work, Kimberley speculated that Ann made a computational error. Ann’s error, however, was not simply due to computation, but probably stemmed from her confusion as to which number she needed to invert.

Based on her final analysis, Kimberley correctly asserted that Ann simply understood that one has to take the reciprocal of *some* number. To help Ann correct her error, she would “show her how to do it” (PE.I. 1473). If other students made similar errors, she would review some “concepts” with the whole class:

I’d wanna do a little bit of review, and, you know, it’s clearly, just by looking at the [students’ solutions], that we need to review how to take the reciprocal of the divisor (laughs). So, I would review a few of these concepts, just by having kids you know, “Everybody *remembers* how to...?” You know, by have some of the kids raise their hands and share. (PE.I. 1516-1521, emphasis added)

Kimberley's use of the verb "remember" suggests that she presumably thought that Ann simply forgot which number to invert. Thus, Kimberley would support Ann and her classmates by showing them "how to do it" and by reminding them the details of the algorithm.

After analyzing Michelle's work, Kimberley argued that this student understood how to solve the problem visually but she probably had not developed the algorithm. Kimberley would thus accept Michelle's solution as correct, but she would ask her to explain what her answer was. She would also have Michelle share her work with her classmates and write the mathematical sentence that went along with her visual representation. Kimberley's analysis of Michelle's work was correct, yet incomplete. In addition to the pedagogical warrants for asking Michelle's to show her work in a numerical sentence, mathematical considerations also necessitated such a move. The quotient in this division problem included a fractional part; hence, it was imperative that Michelle also present her work with a numerical sentence. This would have allowed the teacher to determine whether she really understood how to express the remainder in divisions of fractions. That Kimberley mentioned nothing about this fractional part, along with the difficulties she herself encountered in a pertinent task which included such a fractional part (see below), casts doubts about whether she was really aware of the mathematical necessity of having Michelle share and discuss her work.

*Noticing and Interpreting-Evaluating.* Kimberley attended to none of the flaws in the four student contributions under consideration. Thus, according to the interview protocol, she was directly prompted to comment on three of these contributions: June's explanation, Alan's work, and Amanda and Julia's solution. Even with this direct

prompting, she did not see any shortcomings in June's explanation; it also took her a long time to identify the error in Alan's work, whereas the error in Amanda and Julia's work remained a conundrum to her even at the end of the interview.

In the virtual lesson, June incorrectly argues that the first task should be solved by dividing one half by one sixth and not the other way around because the half is bigger than the sixth; June also uses a visual representation to make her argument clearer.

Kimberley thought that June's explanation was "great." She explicated:

I think that's a great way of explaining it. I think that would make sense for a lot of the kids. It presupposes that each kid is going to understand that a sixth is smaller than a half. But as long as they understand that, that's a great explanation. Because they will be able to look at any problem in the future and figure out which number should be the divisor. So, I think that's a really good technique. (P.E.I. 1084-1090)

Kimberley's justification implies that she probably had a similar misconception to June's: that the divisor is always smaller than the dividend. It was not surprising, then, that she thought that June's explanation offered the other students a handy rule to determine which fraction corresponds to the dividend and which to the divisor.

In considering Alan's episode, Kimberley thought that by splitting half the line into six parts, Alan was indeed figuring out the number of one-sixth yards he could make out of half a yard. Hence, if she were the teacher, after Alan presented his work on the board, she would have another student use Alan's drawing to figure out the answer to the problem. To clarify what she expected this student to do, Kimberley first reproduced Alan's diagram: she drew a line, divided it into half, split the first half into six pieces, and labeled these pieces with numbers 1-6 (see Figure 4.27a). The following discussion ensued:

**Kimberley:** So, I would ask somebody "What are we trying to figure out here?" and lead them to the answer, you know, that we are trying to figure out how many one sixths there are in a half. And then, I will bring someone else up to count, and then we'll literally number them. Right? And we'll determine that the answer was six.

**Charalambos:** So, how many sixths are there in a half?

**Kimberley:** How many sixths are there in a half? There would be six. [Pause.] Right? No, hang on a second. Now, I'm starting to question my answer. [Pause.] There's [sic] three. [Pause.] So, how did I divide ... now, I am completely confused.

**Charalambos:** You can take some time and think about it.

**Kimberley:** [Long pause.] Right. Okay. Going back to the problem. I needed to look at the problem for a few minutes. All right, so, we are trying to figure out how many ribbon badges he can make from the given amount of ribbon. And if they got half a yard and it takes a sixth of a yard to make -- so, we are trying to figure out how many ribbons. So, we really need to count the number of ribbons here. [Pause.] And, so, the next thing we have to figure out is how to show a sixth of a yard. (P.E.I. 1178-1208)

As this excerpt suggests, Kimberley not only considered Alan's work correct but was also misguided by his drawing and thought that one could make six ribbons out of half a yard. Only after she was directly prompted to reconsider how many sixths there were in half a yard did she question her answer to this problem. Even when she figured out the correct answer to the problem, it took her some time to decide how to modify her representation to match the problem's answer. To do so, she went back to the word problem, which gave her a hint about the error in Alan's work: Alan was not taking sixths of a *yard* but rather was dividing his yard into six parts. Building on this idea, Kimberley corrected her diagram by showing that (a) the whole line needed to be divided into six pieces and (b) the pieces she and Alan originally created were twelfths, and hence, one needed to consider pairs of such pieces to figure out the answer to the problem (see Figure 4.27b).

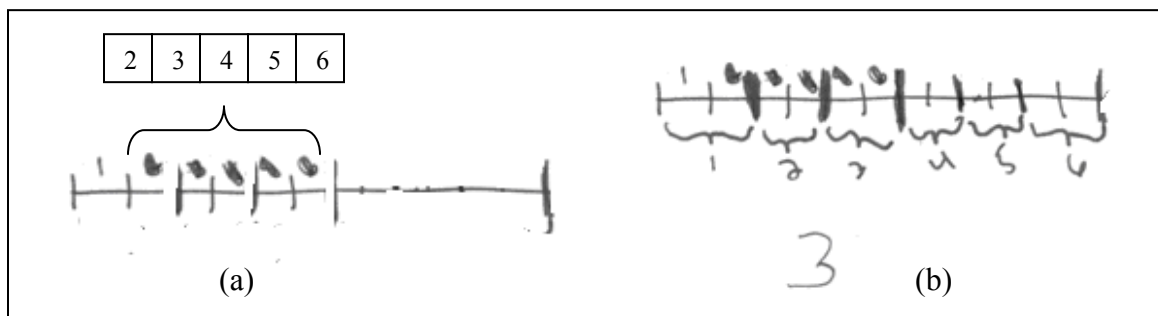


Figure 4.27. Kimberley's work on the division problem  $\frac{1}{2} \div \frac{1}{6}$  (a: initial work; b: work after identifying the error).

Kimberley was not as successful at identifying and correcting the error in Amanda and Julia's work. Even when asked to comment on their work, she argued that the two girls correctly showed that they could make four ribbons and that there "was a little bit left over" (PE.I. 1337). When later on Robert pointed out that his solution contradicted the two girls' solution, Kimberley was offered another opportunity to revisit the girls' solution. Using the traditional algorithm, she figured out the numerical answer to the problem. Although she spent a considerable amount of time trying to resolve the discrepancy between her answer and that of the girls, she eventually gave up, commenting: "I don't know. I'm mystified by that" (PE.I. 1430).

To summarize, Kimberley was not seen closely attending to and analyzing students' work. When considering solutions in which students used the traditional algorithm, she mainly explored whether the students' work matched the intermediate steps and the final answer of the algorithm. Her incomplete analysis of Michelle's solution aligned with her difficulty in explaining the error in Amanda and Julia's work and with her struggles with explaining the fractional part of a quotient. This finding suggests that without a solid grasp of pertinent ideas, Kimberley could not see potential flaws in students' work, let alone identify the source of these errors and suggest ways to correct them. A similar argument applies to her analysis of June's explanation, in which Kimberley appeared to have the same misconception that undergirded June's thinking.

#### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* In both episodes under investigation, Kimberley suggested or endorsed instructional moves that could support student learning; yet, she appeared to be doing most of the thinking for students, especially on the tasks' more demanding parts.



After identifying the error in Alan's work, as outlined above, Kimberley said that if she had to support Alan and his classmates, she would direct their attention to the source of the error, namely that they needed to divide the entire yard into sixths. To do so, she would erase Alan's work and re-divide the line into six parts. Finally, she would ask the students to count the number of sixths that could be made from the whole yard. Obviously, by simply asking students to count these pieces, Kimberley would leave little room for student thinking.

In considering June's and Shaun's episode, Kimberley endorsed the teacher's move to give students a "trick" to help them divide their lines. Arguing that some teachers would have chosen to give students this trick from the start instead of first letting them puzzle about the problem, Kimberley speculated that the virtual teacher had a good reason in mind for not presenting this information from the outset: "[She] gives [the students] a chance to try and figure it out before she feeds them the next step essentially. And she still let[s] them kind of sort it out on their own" (PE.I. 1256-1259). If she encountered a similar situation, she would pursue an analogous approach to that displayed in the teaching simulation: after having students explore this task for a while, she would feed them the next step.

*Noticing and Interpreting-Evaluating.* Kimberley's performance in the noticing and interpreting-evaluating tasks was consistent with her performance just considered. She commented that the teacher "fed Alan the questions that [were] going to advance his thought process" and that she did so "very effectively" (PE.I. 1110-1114). She also considered the teacher's approach in supporting Alan "perfect" (PE.I. 1306). Likewise, she endorsed how Ms. Rebecca handled June's and Shaun's questions by first letting

them struggle a bit with the task and then giving them the clue of finding a common multiple. In none of these episodes was Kimberley concerned with the teacher's lessening the cognitive demand of the assigned tasks; instead, Kimberley tended to think that by pursuing what she perceived to be an exploratory approach, the teacher created ample opportunities for students to investigate pertinent mathematical ideas.

*Performance on the LMT Test: A Closer Look*

In light of her performance in the teaching simulation, the analysis of Kimberley's answers to the LMT test showed both consistencies and inconsistencies. I first consider the consistencies. As one would speculate based on Kimberley's strong mathematical background, she correctly answered all the CCK questions. Conversely, she incorrectly answered questions that pertained to ideas with which she struggled during the simulation. These included question 1 (which concerned different manifestations of units when working on fractions), question 5 (which pertained to representing divisions in a drawing), questions 14c-14e (which dealt with different interpretations of the meaning of division of fractions), and question 25 (which corresponded to the measurement interpretation of division). Instead, she answered correctly question 14a, which pertained to the meaning of division as fitting divisor units into the dividend; as the reader might recall, this was a basic idea on which Kimberley drew to explain the quotient in fraction divisions.

The main inconsistencies in her performance pertained to three questions: question 15, designed to capture the PSTs' grasp of the two interpretations of division; question 17, which directly assessed understanding of a quotient's fractional part; and question 20, which aimed at gauging understanding of different manifestations of units in

fraction problems. Given the difficulties that Kimberley encountered during the simulation, particularly with explaining the fractional part of a quotient, one would expect her to answer these questions incorrectly. One could also consider her success in answering questions 11a-11d and 24a-24d, which explored conceptual understanding of fraction division, as somewhat inconsistent with her performance in the interview, since Kimberley felt quite facile with the mechanics but not with the conceptual underpinnings of division of fractions. The question then arises as to what might explain these inconsistencies.

One plausible explanation is that the context of these questions and/or their multiple-choice format aided Kimberley in answering them correctly, as suggested by a written comment she included at the end of the test:

For questions such as #8, #11, and #24, I figured out the answer to the problem and then checked to see which word problems matched my answer. I'm not sure if that was the method you intended us to use. I felt like perhaps I was circumventing the question's purpose. (P.E.T.)

This last comment, which I revisit in Chapter 6, raises issues about how the PSTs' knowledge was measured in this study.

### *Analytical Commentary*

Even though a divergent case, Kimberley supports the association between knowledge and teaching performance in two different ways: by what she was able to do when her understanding was solid and by what she was *not* able to do because of limitations in her knowledge. Kimberley's case also provides some examples of how her beliefs and her images of teaching mediated the impact of her knowledge in her performance. Moreover, the better alignment of her teaching performance with her LMT score rather than with her GRE score resonates with the quantitative findings that showed

the GRE scores to marginally mediate the association between knowledge and performance (for the performing tasks). I discuss each idea in turn.

When Kimberley's knowledge of the content (and its teaching) was strong, as suggested by her performance on the LMT test<sup>154</sup> and the teaching simulation, she was relatively effective in providing explanations and using representations. A case in point here is the explanation she provided for a division of fractions whose quotients did not include a fractional part. In this case, she selected a suitable representation and used it appropriately. Her explanation was conceptually driven and could be comprehensible to its intended audience. Additionally, she was seen building connections among different representational models (e.g., the numerical sentence, the visual drawing, and the arithmetic symbols included in her drawing).

In contrast, when Kimberley's understanding of the content was weak, she could either offer no explanations at all or the explanations she gave were numerically driven and by and large mathematically invalid (e.g., her explanation of divisions whose quotient included a fractional part). In those latter cases, she also appeared incapable of closely following and thoroughly analyzing others' instructional explanations and students' work and contributions. For example, she did not attend to several of the mathematical deficiencies of Ms. Rebecca's explanation for the reciprocal. Similarly, in analyzing Michelle's solution, she did not refer to the mathematical motivations for having this student present her answer in a numerical sentence.

In some cases, however, Kimberley *did* have the knowledge necessary to propose instructional moves conducive to a more conceptually oriented instruction; yet, she was

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<sup>154</sup> In making these arguments, I do not consider Kimberley's "inconsistent" performance (see discussion in the previous part) because, as she admitted, she answered some of these questions by simply drawing on their context.

seen making decisions and proposing instructional approaches that echoed a more procedurally oriented type of teaching. For these cases (mostly related to her performance in the MTF-related practices), it seems legitimate to assume that other factors, such as her school experiences and her limited images of teaching, as well as her beliefs mediated her decisions and actions. Before outlining some examples which capture this mediating role of the aforementioned factors, a brief delineation of these factors is in order.

Besides her knowledge, Kimberley's school experiences – and consequently the images of teaching she formed – seemed to be the most pervasive factor informing her teaching performance. As she mentioned throughout the interview, she largely experienced mathematics as a matter of learning and applying rules – as learning “the mechanics,” but not the conceptual underpinnings of these mechanics. At several times during the interview she admitted rarely having had experienced a type of instruction that emphasized the learning of the conceptual underpinnings of mathematical procedures or the use of visual representations when dealing with such procedures. These limited – and thus limiting experiences – appeared to color her beliefs about teaching and learning mathematics. Kimberley believed that repetition is important in learning the content. She also thought that the teacher should show and tell and, as needed, provide students with tricks and shortcuts. From her perspective, the teacher should also structure her lessons to minimize student struggle with the content and intervene when such struggles arise. Her school experiences, and consequently her images of teaching and her beliefs, were counter to the type of instruction endorsed by the teaching simulation (thus, the negative sign accompanying both her school experiences and her beliefs in Figure 4.28).

Kimberley also had the opportunity to teach Grade-8 Algebra as a substitute teacher. Although the interview data do not support drawing explicit connections between these latter experiences and her teaching performance in the simulation, some plausible connections can still be entertained. First, if when teaching Kimberley replicated the type of instruction she had experienced as a student of mathematics and her students were successful, then it is reasonable to assume that Kimberley's teaching experiences solidified her beliefs about the effectiveness of show-and-tell teaching methods. Second, Kimberley's reference to what other teachers believe about letting students struggle before giving them some hints (see *Responding to Students' Direct or Indirect Requests for Help*) might have been informed by her being exposed to such ideas while working as a substitute teacher. Third, that Kimberley wanted to add some conceptual flavor to her teaching of mathematics, despite her own procedurally oriented school experiences, might have resulted from her exposure to a somewhat more conceptual type of teaching, as exemplified in the curriculum materials she was using (i.e., *Everyday Mathematics*). Because the interview data allow only for making speculations about the role of her teaching experiences in her performance, in Figure 4.28 this factor is presented along with a question mark, suggesting that its role cannot be clearly determined.

Two examples, both drawn from Kimberley's performance in the MTF-related practices are particularly telling of the role of the foregoing factors in mediating the association between knowledge and teaching performance. First, in analyzing the first textbook page Kimberley clearly referred to the idea that in dividing fractions, the quotient is bigger than the dividend. Although she could have capitalized on this idea to develop a more conceptually oriented lesson, she did not depart from the procedural

orientation of this page. Second, even though she eventually identified the error in Alan's work, to help him correct this error she would largely follow a show-and-tell approach, doing most of the thinking for him; this instructional approach apparently mirrored the type of teaching she experienced as a learner of mathematics. Due to the mediating role of these factors, in Figure 4.28 I use question marks to link her knowledge of the content and its teaching to her performance in the MTF-related practices.

Other features of Kimberley's performance can be less directly connected to her knowledge (or lack thereof) and more directly associated with her school and teaching experiences, her beliefs, and her images of teaching. For example, in analyzing the virtual teacher's task introduction and enactment as well as the way she supported her students when the latter solicited her help, Kimberley did not attend to how the teacher shifted emphasis from meaning and understanding to simply manipulating numbers, nor did she attend to the virtual teacher doing most of the thinking for her students. I argue that with her long apprenticeship in classes in which the emphasis was "heavy on the mechanics," Kimberley could hardly attend to these issues even in those cases in which her own understanding of the content could support her in doing so. Similarly, in considering students' contributions, Kimberley was not seen engaging in thorough and meticulous analysis of the students' ideas and work; in contrast, her analysis was largely answer-driven, guided by the answers to the problems under consideration and by the extent to which the students were correctly applying all the steps of a given algorithm. Although Kimberley's procedural understanding of the content might also have had a role in her performance in this practice, her performance most likely reflected the type of instruction she herself experienced in school.

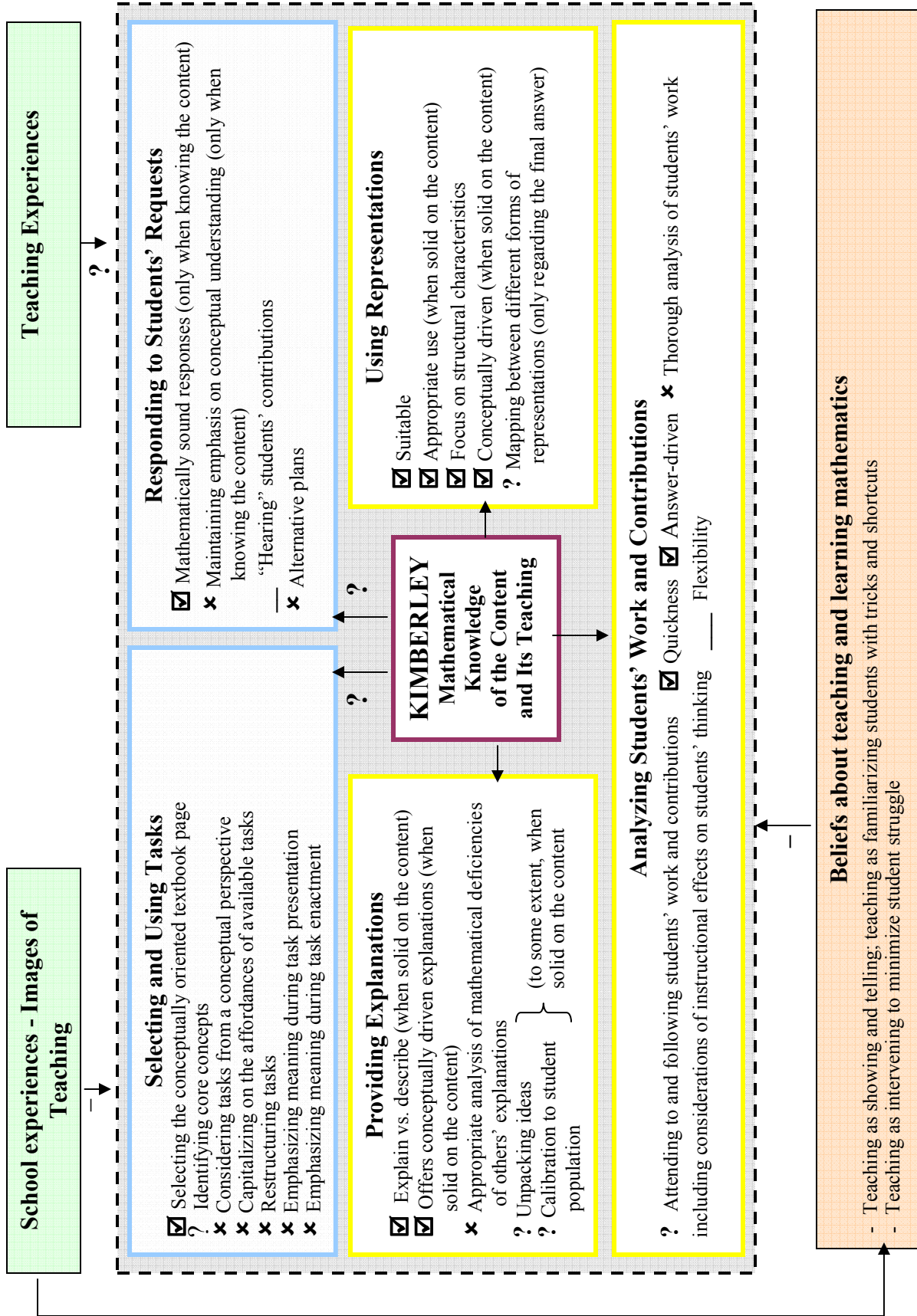


Figure 4.28. Considering the association between knowledge and teaching performance through Kimberley's profile.



One of the reasons that motivated Kimberley's selection was that, compared to her counterparts, she performed much better on the GRE-quantitative test relative to her MKT performance. The preceding analysis suggests that her MKT performance was more aligned with her performance in the teaching simulation (and particularly in the performing tasks of the simulation) than with the GRE score. Thus, her case challenges a cursory conclusion that one might draw from the marked correlation found between the two latter scores: that the GRE-quantitative score and the MKT score can be used interchangeably when exploring correlations between teacher knowledge and performance. In addition, the alignment between her MKT score and her performance in the performing tasks of the simulation comports with the quantitative analysis reported in the previous section of this chapter, which showed the association between the PSTs' MKT score and their teaching performance in the performing tasks to be robust to the mediating effect of the PSTs' GRE-quantitative score.

*The Case of Suzanne: "I Don't Know How to [Explain] It ... I Can't See It in My Brain."*

Suzanne is the third divergent case and the last case considered in the study. Like Kimberley, she performed better on the LMT test than on the teaching simulation; her teaching performance, however, was more aligned with her score on the GRE-quantitative test than with her score on the LMT test. Additionally, Suzanne did not have as strong a mathematical background as Kimberley: Suzanne had taken fewer math content courses during her high-school and undergraduate studies and had not had a minor in mathematics. These features informed her selection for further exploration. The case of Suzanne, in contrast to the cases considered thus far, challenges rather than

supports the association under exploration. Her case also raises issues of measurement, which I briefly consider in the analytical commentary and in Chapter 6.

### *Background*

Suzanne recalled enjoying the higher level math courses she had taken because she “could understand logic;” yet, she did not like the lower level ones because she was taught certain operations without being given adequate explanations about “the whys” (PE.S.). In elementary school, she could hardly memorize her multiplication tables beyond five, which often led her to commit computational errors, when solving mathematics problems. She characterized herself as capable of getting the “big picture” and understanding the principles but rather “sloppy” at executing the “parts that make up the whole” (PE.S.). Before joining the ELMAC program, she had worked as a middle-school librarian for four years. Not limiting herself to her typical library duties, she frequently interacted with teenagers through several school-wide activities she organized to promote reading. She joined the ELMAC program wanting to make an impact on teenagers’ lives. She expected the two intervention courses of the program to help her learn different methods of teaching mathematical concepts and support her understanding of what it means to learn mathematical concepts for the first time.

### *Selecting and Using Tasks*

*Performing.* The first textbook page reminded Suzanne of the type of mathematics she experienced as a student: learning and applying procedures without any motivation for using them and without knowing why they work. She mentioned,

This is the type of math that I was taught: here’s the formula, now you do it. Here’s an example, now you do it. ... [But] I don’t know why this works, I don’t know what I am trying to find. I don’t know what I would ever use this for. All I know is this is a procedure and this is how I follow the procedure. (PE.I. 133-135, 148-149)

Hence, Suzanne thought using this page to simply assess students' competence in carrying out this operation. If she had to use it for an introductory lesson on the division of fractions, she would reorder its tasks. She would start with the two word problems to "give meaning to why you would want to divide fractions" (PE.I. 164-165). She would then explain the pertinent concepts to students and have them solve some of the exercises 1-16. As a general plan, this approach had the potential to support more meaning-making compared to an approach based on simply following the tasks as outlined on the textbook page. Alas, Suzanne appeared to encounter several difficulties in realizing this plan.

The word problems could help students see the utility and applicability of the division of fractions if seen and solved as division-of-fractions problems. Yet in considering these problems, Suzanne could not see how this could be done. For her, both were merely multiplication problems. For example, to explain the first one (i.e., exercise 17), she would draw four sets of thirty players and find the answer by multiplying four by thirty. She even thought using "pies": "Oh, I know how I could do it! When I'm doing fractions it's always useful for me to think in terms of a pie" (PE.I. 312-313). She drew a circle, divided it into fourths and wrote "30" in each of the quadrants (see Figure 4.29). Yet, the only connection she could see to division was that "the circle [is] divided into four pieces" (PE.I. 335). She would not even use division of fractions to represent this problem mathematically; instead, she would present it either as a whole-number division (i.e.,  $120 \div 4=30$ ) or as a multiplication (i.e.,  $\frac{1}{4}$  of  $120=30$ ;  $\frac{1}{4} \cdot 120=30$ ). If she were forced to use a division-of-fractions equation, she would write the following:  $\frac{1}{4} \div 30 = 120$ . Yet, this latter sentence did not make any sense to her. She commented,

"I just lost myself; when I went into the fraction I lost myself. When I'm in whole numbers I understand it. ... But when I switch it around, it doesn't look right to me" (PE.I. 380-381, 386-387).

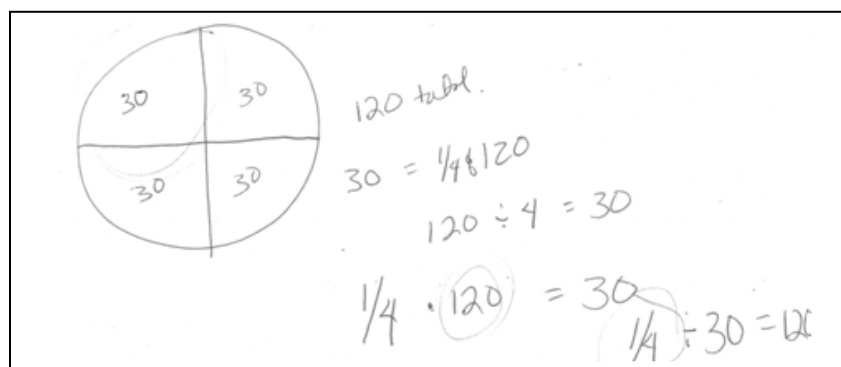


Figure 4.29. Suzanne's work on the first word problem of the first page.

Apparently, Suzanne would find it hard to help students see the applicability of the operation under consideration if she used this problem. But how about explaining the concepts underlying this operation?

Considering herself a visual learner, she would try to explain the worked out example of the first page (i.e.,  $2 \div \frac{3}{4}$ ) using a visual representation. After thinking awhile about how she would go about doing that, she gave up:

**Suzanne:** I don't know how [to] visually represent dividing by a fraction. ... I can't see it in my brain.

**Charalambos:** Okay. Um. [Pause.] So if you were to explain it without, you know, drawing it, if you were to explain what this two divided by three fourths represents, what would you say?

**Suzanne:** No concept; that's where I can't see it. I don't know. I don't know how you can divide by a fraction. ... I can divide by a whole number. I can multiply a fraction by a whole number.

**Charalambos:** What makes it easier for multiplication?

**Suzanne:** Um, because multiplication is adding a number of times. And division, I've just got a block, right there. (P.E.I. 402-425)

Despite her inclination to offer a conceptual explanation for fraction division, Suzanne felt that she did not have the resources for doing so, as clearly suggested by her assertion that she did not have any conceptual understanding of this operation. Whereas she could understand multiplication as repeated addition (i.e., as "adding a number of times"), an idea that applies both to whole numbers and fractions, she did not seem to have a similar idea upon which to draw to even start an explanation for dividing fractions.

Left with no devices that would allow her to develop a more conceptually oriented lesson, Suzanne would merely teach for procedural competence. Interestingly enough, she even contemplated not using the traditional algorithm as outlined in the worked out example of this page; instead, she would use the cross-multiplication rule which she learned as a student:

Well, if I think of the way that I learned how to do it – I guess that’s, you know, you’re gonna teach the way that you learned – and I remember this, I remember taking a fraction like the first one [exercise 1], two thirds divided by one third and I remember making the little arrows (see Figure 4.30). You multiply three times one is three, you multiply two times three is six. Is that right, by the way? (PE.I. 435-440)

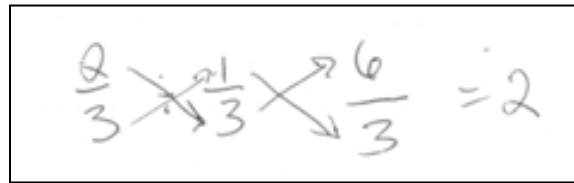

$$\frac{2}{3} \div \frac{1}{3} = \frac{6}{3} = 2$$

Figure 4.30. Suzanne’s work on dividing  $\frac{2}{3} \div \frac{1}{3}$  using the cross-multiplication rule.

Two ideas are important to highlight in this quotation. First, her argument that “you’re gonna teach the way that you learned” is suggestive of how (preservice) teachers’ school experiences and the images of teaching this subject they had formed as learners of mathematics might inform their own teaching practices. Second, her question at the end of this quotation corroborates the argument about “rememberers and forgetters” made in earlier cases. As a student, Suzanne had probably learned the cross-multiplication rule as a shortcut for dividing fractions. Yet, without any conceptual underpinnings, she was uncertain if she remembered this rule correctly. What is more, for Suzanne this rule had limited applicability: she would not use it to solve exercises involving mixed numbers because “the formula of going zigzag is ... easier without mixed numbers” (PE.I. 469-470). For those latter exercises she would use the algorithm proposed on the first page,

despite her finding it rather confusing (especially its second step, which involved a number of equations).

Suzanne thought that the second page was more suitable for conducting an introductory lesson on the division of fractions. She argued that if she had been exposed to pages like this as a student, she would have understood mathematics better. She particularly liked that this page started with a real-life situation and used the same scenario throughout instead of forcing students to jump from one scenario to another – in her own words, from “how many cookies mom made ... [to] how many bolts of cloth Mrs. Jones needs” (PE.I. 548-549). Among the strengths of this page she also listed that students are encouraged to discover a formula for solving the presented problems instead of being given a formula to learn and apply mechanically. She considered this aspect particularly important because, as she argued, students are helped to first learn the concept and then the formula; as she put it, this page moves students from “big to small,” helping them first see the big picture and then the details (PE.I. 597-598).

A closer look at Suzanne’s analysis of this page suggests that she merely focused on its more surface affordances without attending to some of its more substantive assets: she did not comment on this page’s requirement that students explain their answers in words and in diagrams, nor did she appear to notice that students are expected to explain the fractional parts of their answers, a key idea in dividing fractions. Even for the affordances that she did notice, she admitted having no idea how she could capitalize on them to support students’ learning. For example, although she liked that this page asks students to identify patterns and then propose an algorithm for the division of fractions,

when asked to articulate how she would enact this task if she were to use this page, she answered, “Clueless. I am absolutely clueless” (PE.I. 1230).

Overall, Suzanne appeared to have a clear orientation toward teaching for conceptual understanding. Limitations in her knowledge and her understanding of the content, however, seemed to constrain what she could do with either page. If she were to use the first page she would simply reorder its tasks to start with the word problems, seeking to help students see the applicability of the division of fractions. Yet, it is questionable whether she would be able to do so given that she saw these problems as multiplication rather than as division problems. Using the second page’s word problems could support her work on discussing the applicability of division of fractions. It is debatable, however, whether she would be able to capitalize on the affordances of this page given her own struggles with the content.

*Noticing and Interpreting-Evaluating.* Suzanne commented on none of the teaching episodes designed to gauge the PSTs’ performance in this practice. When directly asked to comment on the teacher’s introduction and enactment of tasks A<sub>1</sub> and D, she found Ms. Rebecca’s approach satisfactory. She liked that the teacher let the students solve the first problem instead of “jumping” in and solving it for them and she appreciated that she was scaffolding the students throughout, reminding them of important ideas, and forewarning them of potential mistakes. She also endorsed the teacher’s “let[ting] her students find the pattern” instead of simply presenting them with it. A comment that she made at the end of the interview summarizes her perspective of how the teacher worked with her students on these tasks:

I really liked how [Ms. Rebecca] made the students the active participants in their learning. She made *them* come up with the solutions rather than her just giving them the formula and giving

them the answer. I liked how she made it more of a discovery and I liked how she had the students explain what they did in their own words. (PE.I. 1535-1540, emphasis in the original)

Suzanne was right in arguing that the virtual lesson was exploratory: indeed, the students were asked to explain their thinking and the teacher was seen posing questions instead of simply feeding the students answers. Yet, if one moved beyond these surface characteristics, one could find evidence that the lesson was not really supporting students' understanding of the content. Although Suzanne's own images of teaching might have prevented her from delving deeper into these aspects, her difficulties with the content were also a factor that limited what she could see and how she interpreted it. The following comment speaks to this idea:

So it was really hard for me to evaluate how [Ms. Rebecca] did when -- I mean, if I knew the concept better, then I could really focus on what she's doing. But I was along with the students trying to learn how to do it, because I don't ... know why reciprocation works. (PE.I. 1570-1575)

I revisit and reconsider this comment in the next chapter when discussing changes in Suzanne's performance.

### *Providing Explanations*

*Performing.* As already discussed, Suzanne did not even try to provide an explanation for the division of fractions, arguing that she did not have the concept in her brain that would allow her offer such an explanation. When later in the interview she was again asked to reconsider providing an explanation to address Michelle's question as to why the reciprocal works, she replied that she did not know why this algorithm works; she simply knew that it works. Her performance in both these instances is indicative of how the lack of pertinent knowledge might impair teachers not only in providing appropriate explanations but even in attempting to provide explanations.



*Noticing and Interpreting-Evaluating.* Suzanne did not comment on either of the two episodes designed to capture the PSTs' performance in this practice. When prompted to evaluate the virtual teacher's explanation, she appraised it as poor. She argued that if Ms. Rebecca's explanation had been adequate, it would have made sense to her and it would have helped her understand why the reciprocal works. She particularly criticized the analogy that the teacher used in her explanation because she could not see how adding and subtracting negative and positive numbers was similar to taking the reciprocal of a number. She also pointed out that the teacher's argument about using the inverse operation was rather esoteric. Her evaluation of the teacher's explanation was valid, even though she did not go into much depth to justify the deficiencies of this explanation. In fact, Suzanne was not evaluating this explanation from the perspective of a teacher who knows the content and, consequently, can analyze the explanation from a mathematical and pedagogical viewpoint. She was rather viewing this explanation from the perspective of the student who, even after having been provided with an explanation, is still puzzled about why the division-of-fractions algorithm works.

#### *Using Representations*

*Performing.* The interview data provided only scarce information about Suzanne's use of representations since she did not attempt to offer any explanation when asked to do so. Even so, in considering her performance in using representations, it is informative to notice her comment that she liked using "pies" when working on fractions. This limited focus on just one type of representation might have stemmed from how she herself experienced learning fractions.

*Noticing and Interpreting-Evaluating.* Suzanne approached the two episodes designed for this practice in noticeably different ways. When discussing Amanda's episode, she dismissed that no connections were built between the representation drawn on the board and the corresponding mathematical equation. On the contrary, when discussing Amanda and Julia's solution to the second task (i.e.,  $\frac{3}{4} \div \frac{1}{6}$ ), she noticed all the missing steps in the students' work: that the students divided their line into twelve pieces without explaining why; that they colored nine of them red but were not explicit about the correspondence of the nine twelfths to the dividend of the problem; and that they were taking pairs of twelfths without explaining that these pairs corresponded to pieces of one sixth, which was the problem's divisor.

To explain the differences in Suzanne's performance in the two episodes, it is informative to consider that before analyzing the two students' work in the last episode, Suzanne spent a considerable amount of time trying to figure out how to solve the problem considered in this episode (see more in the next two practices). The difficulties she encountered in figuring out how to solve this problem probably sensitized her to the importance of making the connections discussed above clear. From this perspective, her performance in the latter episode could again be seen as Suzanne experiencing the lesson as a student rather than as a teacher.

### *Analyzing Students' Work and Contributions*

*Performing.* Of the three student solutions under investigation, Suzanne analyzed correctly only Robert's. She noticed that Robert first converted the dividend's mixed number into an improper fraction, then took the reciprocal of the divisor, and finally multiplied "across." Using his work as a yardstick, she then noticed that there was an

error in Ann's solution. She thought that this error stemmed from Ann's reversing the order of the steps involved in the algorithm: instead of first turning the mixed number into an improper fraction and *then* taking the reciprocal of the resulting fraction, Ann first took the reciprocal of part of the mixed number. Hence, although Suzanne correctly evaluated Ann's work as incorrect, she did not identify the actual source of her error.

Michelle's solution was the one that Suzanne liked the most because it was visual. She noticed that Michelle showed the dividend with the red line and then took portions of three fourths, which corresponded to the divisor. Although she correctly associated the whole-number part of the quotient with Michelle's drawing, she was not equally successful at explaining what the remainder meant. She argued that the leftover piece was two twelfths and concluded that Michelle's answer was correct. Suzanne's explanation of the remainder as two twelfths was not random. Indeed, after taking away the three  $\frac{3}{4}$ -portions (i.e., the whole-number part of the quotient), Michelle was left with two pieces (see Figure 3.5), yet those two pieces represented fourths, not twelfths. The only way in which one could perceive them as twelfths was to consider the three straight lines in Michelle's drawing as one unit instead of three distinct units, as Suzanne acknowledged doing. After being prompted to clarify why she considered the leftover part as two twelfths, Suzanne revised her analysis and argued that Michelle could not have had twelfths in her answer because she was dealing with fourths. Yet, she could not determine what the remainder in this student's solution represented. She thus concluded that she was not sure whether Michelle's work was right or wrong.

Based on her analysis, Suzanne made some general assertions about the three students' understanding. She asserted that they understood that they needed to take the

reciprocal, except for Ann, who was not carrying out this step correctly; she also noticed that the students knew how to multiply by the reciprocal. Regarding Michelle, in particular, she argued that this student seemed to know “how to take a portion of a portion, because she’s got her two and three quarters and then she’s checking to see how many three quarters fit into that” (PE.I. 1423-1424); she also remarked that Michelle did not seem able to translate her work into a mathematical sentence.

Her arguments about the three students’ understanding were partly correct. Robert and Ann seemed to be able to multiply by the reciprocal. Michelle was able to show a portion of a number (i.e., to show  $2\frac{3}{4}$  using a set of three lines) and then to take portions of three fourths out of this bigger part; Suzanne was also justified in assuming that Michelle might have not been able to present her work with a numerical sentence. Nevertheless, without being able to identify the real source of Ann’s error, Suzanne could not make accurate arguments about this student’s understanding. Similarly, without recognizing whether Michelle’s solution was correct – and particularly what the remainder in Michelle’s work represented – Suzanne’s analysis of this student’s work was rather generic.

*Noticing and Interpreting-Evaluating.* Suzanne did not attend to any of the four instances in which Ms. Rebecca’s students contributed contestable ideas. When directly prompted to consider three of these episodes (i.e., June’s explanation, Alan’s work, and Amanda and Julia’s solution), Suzanne identified the problem only in Alan’s work.

Suzanne argued that June’s explanation made sense to her since it was true that to make pieces of one sixth, one needed to have a longer piece of ribbon. She did not notice, though, that implicit in this student’s argument was the idea that the dividend needs to

always be larger than the divisor, nor did she attend to that June's explanation was not answering Ben's question since it was not connected to the word problem under investigation at that point.

After reconsidering Alan's work, Suzanne quickly noticed that Alan was, in fact, finding sixths of a yard instead of sixths of the whole yard, which was what the word problem was asking for. She considered the teacher's questions – and particular her inquiry, “How many pieces are you going to divide your line into?” – partly responsible for Alan's confusion.

While reviewing Amanda and Julia's work, Suzanne argued that she had to replicate their work to understand what exactly they were doing. She first drew a line, then divided it into fourths, and then identified a three-fourths portion (see Figure 4.31). Next, she drew another line approximately equal in size to the first one, divided it into sixths, and showed a piece of one sixth. To show the twelfths, she drew a third line equal in size to the first two and divided this line into fourths; she then divided each fourth into three parts. She put a dot under the first nine twelfths to illustrate the dividend and drew a bracket to show that the two twelfths corresponded to one sixth. Using this bracket as a yardstick, she concluded that the two girls' solution was correct: they could make four ribbons and they would be left with one-twelfth yards. To verify the correctness of this solution, Suzanne first turned the quotient into an improper fraction ( $4 \frac{1}{12} = \frac{49}{12}$ ), multiplied this fraction with the divisor, and argued that the resulting fraction (i.e.,  $\frac{49}{72}$ ) would be equal to the dividend. This argument, however, was incorrect: her validation approach would have yielded the dividend only if she had expressed the reciprocal in terms of ribbons (i.e., sixths).

Her work on replicating and validating the two students' solution revealed that she dismissed the error in the two students' work: that their quotient included two different types of units. When later in the interview she had an opportunity to reconsider the two girls' solution (when Robert questioned whether their work was correct since it did not match his solution), Suzanne speculated that either the two girls or Robert made an arithmetic error.

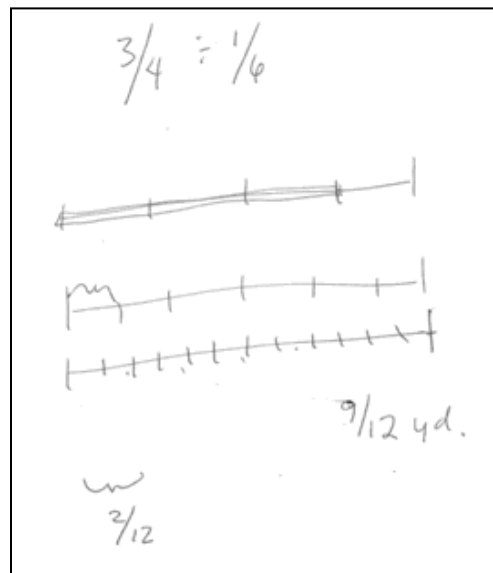


Figure 4.31. Suzanne's work on replicating Amanda and Julia's solution.

To summarize, Suzanne was not very successful at analyzing the students' work, both regarding the performing tasks and the tasks of noticing and interpreting-evaluating. In most cases (especially when analyzing Michelle's solution, June's explanation, and Amanda and Julia's solution), her analysis was mainly driven by the numbers involved in the students' contributions.

#### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Suzanne performed better in this practice than in the preceding four practices, both with respect to the performing tasks and the tasks of noticing and

interpreting-evaluating. In the performing tasks, she proposed mathematically sound ways to support Alan (in the first episode) and June and Shaun (in the second episode) after having spent some time to figure out the mathematics involved in the episodes under consideration.

As already discussed, after reconsidering Alan's work, Suzanne noticed that he incorrectly divided half the line into six parts instead of the whole line. To support Alan in identifying and correcting his error, Suzanne would build on June's earlier contribution. She would use the two lines June drew on the board to remind Alan of the sizes of half a yard and one-sixth yard. She would then discuss how one could get these two lengths: by dividing a whole line into two parts (in the first case) or into six parts (in the second case). From there, she would help Alan see the relationship between a sixth and a half yard. In general, her approach had the potential to support Alan's understanding since she clearly addressed the mathematical concept of unit, which was fundamental for correctly representing the yard sizes under consideration; she would, however, do most of the thinking for Alan.

In the second episode, Suzanne had a hard time understanding why the teacher recommended that the students use common multiples – and especially the common multiple of twelve – to solve the division problem under consideration (i.e.,  $\frac{3}{4} \div \frac{1}{6}$ ). She argued that to appropriately respond to June's and Shaun's requests for help, she first needed to understand how the idea of common multiples applied to this problem. She thus drew a line and partitioned it into four parts to show the dividend. She then divided each fourth into two parts, hoping that this would result in sixths. This experimentation helped her realize June and Shaun's apparent struggles with solving the problem:

So, their question is how do you split this three quarters into sixths, because it doesn't do it equally, doesn't do it evenly, which they didn't really cover previously, because [Ms. Rebecca's] first example [i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ] was very straightforward, it did it evenly. (PE.I. 960-964)

Assuming the role of the student, Suzanne then experimented with using the idea of common multiples the teacher proposed. She first figured out that this idea could help divide the three fourths and the sixths into commensurate parts, which was key to solving this problem. Because of her own struggles with grasping how the idea of common multiples becomes handy in this problem, she argued that if she were teaching the lesson she would not simply put this idea on the table. Instead, she would try to scaffold the students' work step-by step:

I wouldn't just ... assume that everybody ... understands that concept, just because one person did. ... So, I would ... take that line and show them: these are the hash marks, the sixths; this is dividing one yard into six. This one's dividing one yard into fourths, and those lines don't meet up. So how can we make it so that all those lines meet up? And then they would choose to refer to the lesson or wherever to find multiples and divide that into twelfths. Then I would go back to my original idea of the line, the yard, and go to the three fourths ... [and] say, "But the three fourths, as you know, is the same as how many twelfths?" Um, it'd be nine twelfths. And one sixth would be the same as two twelfths, and then do it from there. ... I would do it more, I would show it ... or had the students show [it on] the diagram. (PE.I. 997-1021)

Suzanne's approach was more unpacked than the teacher's in many respects. Instead of simply proposing a rule to follow (i.e., "split your line into twelfths"), Suzanne would first help her students understand why they needed to do so. Drawing on her insights from working on this task from the student perspective, she would use two lines to show the fourths and the sixths and illustrate that these pieces were not commensurate. From there, she would elicit students' ideas about how to make the hash marks denoting the fourths and the sixths line up, hoping that this would help students see the connection to the notion of common multiples. To ensure that students would then be able to build on this idea, she would scaffold them to show the problem's dividend and divisor in twelfths.



Obviously, her approach as outlined above had more potential to support students' work and understanding than that pursued by the virtual teacher. I argue that Suzanne was able to propose such an approach not only because she eventually figured out the mathematics involved in this episode, but also because of her struggles with solving this task. These struggles sensitized her to aspects of this problem that warranted unpacking when scaffolding student work and thinking.

*Noticing and Interpreting-Evaluating.* Suzanne identified several debatable aspects of the teacher's work without necessarily criticizing them. For example, she observed that in supporting Alan the teacher was posing a leading question; she also noticed that the teacher's intonation in posing this question gave away the expected answer. Yet, she evaluated the teacher's approach as better than hers, claiming that whereas she would directly tell Alan why his work was wrong, Ms. Rebecca led him to figure out the error himself. Although Suzanne's remark was valid, it is debatable which of the two approaches, the teacher's or hers, would be more effective for supporting Alan's understanding, especially given that the teacher did not address the mathematical idea of units that presumably caused Alan's confusion and error.

As already discussed, in the second episode Suzanne criticized the teacher's approach to simply propose the idea of common multiples without helping the students either understand the rationale for using common multiples or figure out how they could actually use this idea to solve the problem. At the same time, Suzanne applauded the teacher for quickly capturing the students' struggle and for trying to help them because, as she claimed, had the teacher not, the students would have "continued on and did the wrong way." This, in turn, could have resulted in "imprint[ing] the wrong way to do it" in

their minds (PE.I. 937-938). Interestingly enough, although during this episode only two students are presented soliciting the teacher's help, while going over the pertinent part of the simulation, Suzanne noticed a "trend," namely that several students did not understand how to solve the problem.

At this juncture it is informative to consider how Suzanne replied to two of the survey's statements that pertained to issues captured in this episode. Statements 11 and 15 of the first part of the survey were used to address the PSTs' beliefs about supporting students when struggling with a mathematical task. Suzanne strongly agreed with the first statement, which maintained that the teacher should not necessarily answer students' questions but let them puzzle things out for themselves. In contrast, she strongly disagreed with the second statement, which held that students should never leave a math lesson feeling confused or puzzled. Both her responses to these statements were in stark contrast to her evaluations of Ms. Rebecca's instructional moves as outlined above. This discrepancy raises concerns about gauging teachers' beliefs using questionnaires instead of using more interactive approaches such as the teaching simulation used in this study. I revisit this idea in Chapter 6, when discussing issues of measurement.

#### *Performance on the LMT Test: A Closer Look*

Given Suzanne's performance in the teaching simulation, the examination of her answers to the LMT test yielded more inconsistencies rather than consistencies. For example, given her difficulties in understanding and explaining division (of fractions) and the fractional part in those divisions, one would expect that she would incorrectly answer questions pertaining to the meaning of division (of fractions) (e.g., questions 11b, 14a-14e, 19a-19e, 25), questions related to the notion of unit (e.g., question 1), and questions

associating representations with division sentences (e.g., question 5). Yet, she answered all those questions correctly.

Some notes she took on the test while answering these questions suggest that in at least two cases Suzanne answered these questions correctly despite her wrong thinking. For instance, she associated the representation shown in question 5 with an incorrect division (i.e.  $\frac{1}{4} \div 3$ ). Because the multiple-choice answers did not include this mathematical sentence, she chose the answer “none of the above is mathematically correct,” which happened to be the correct answer.<sup>155</sup> Similarly, in question 25, although she identified two of the given options as correct, she circled the one that was indeed the correct answer.

Because she did not show her work in most of the remaining questions, it is not clear what guided her thinking in answering those questions. However, it seems reasonable to argue that she answered some questions correctly by chance. This argument is supported by a comparative analysis of all the PSTs’ responses to the two administrations of the test. This analysis showed that Suzanne was the PST with the highest number of questions (i.e., six) answered correctly during the first administration of the test but not during the second. If one accepts that the intervention enhanced the PSTs’ understanding of the ideas captured on the test – something that Suzanne’s post-intervention performance in the simulation suggests – then it seems unreasonable that a PST would correctly answer the test questions in the first but not in the second test administration. A plausible explanation for such a pattern could be that during the first

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<sup>155</sup> In a discussion that I had with her in July 2008, Suzanne validated this argument. She also clarified that she answered the first question correctly, although for the wrong reason. In this discussion, Suzanne additionally pointed out that she answered some other questions correctly by simply considering key words (e.g., in question 14b she associated the word “of” with multiplication).

test administration, this PST answered some questions correctly by chance or for the wrong reasons.

Consistent with her work in the teaching simulation, Suzanne incorrectly answered question 15 on the two interpretations of division, question 17 on interpreting the fractional part of a quotient, and question 20 on the different interpretations of unit. I consider her performance on those questions as well as the inconsistencies identified above in Chapter 6 when discussing issues of measurement.

### *Analytical Commentary*

Although to some extent Suzanne's performance in the simulation resonates with the patterns discussed in previous cases, in contrast to the preceding six cases, her case primarily raises questions. Hence, in presenting Suzanne's portrait, I first outline some of the questions that her case appears to pose and then outline patterns in her performance that comport with those considered in the previous cases.

Suzanne's case helps bring to the fore issues of measurement. The cross-analysis of her answers to the LMT test, her responses to the survey statements, and her performance in the teaching simulation pointed to some remarkable discrepancies. For example, whereas she correctly answered the LMT test questions pertaining to the meaning of (fraction) division – particularly questions 14a-14e – in the teaching simulation, she admitted not having any concept of division upon which to draw to provide an explanation for the division problem  $2 \div \frac{3}{4}$ . Similarly, her answers to some survey questions presented her as a proponent of “giving-less” and letting students struggle to investigate different mathematical ideas; yet, her performance in the teaching simulation painted a different picture: Suzanne applauded the teacher for intervening,

thereby saving students from struggling and from having wrong answers imprinted in their minds. These and other examples raise the following questions: What might be causing these discrepancies? How accurately did the LMT test measure the PSTs' knowledge of the content and its teaching? What might account for the PSTs' success in answering the test's questions, even in the absence of pertinent knowledge?<sup>156</sup> To what extent did the teaching simulation accurately capture what these PSTs knew and were able to do? How accurately did the survey used in this study tap the PSTs' beliefs about teaching and learning mathematics? Are these beliefs stable/generic or context-dependent? All these questions are taken up in Chapter 6, where I reflect on issues pertaining to measuring teachers' knowledge, beliefs, and teaching performance.

In considering the previous cases, I have treated knowledge from a static perspective: as something that the PSTs *possess* to a smaller or larger extent. Suzanne's case helps us consider knowledge in a more dynamic perspective. In the episode on responding to June's and Shaun's request for help, Suzanne was seen constructing the knowledge at stake in situ. During this construction, she appeared to struggle with two questions, not from the perspective of the teacher but largely from the perspective of the student who was told to "split her line into twelfths". These questions were: (a) why is the notion of common multiples (and particularly the common multiple of twelve) applicable to the problem at hand; and (b) how can one build on this idea to solve this problem? After spending a considerable amount of time figuring out the answers to both

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<sup>156</sup> As the reader might recall, an attempt was made to calibrate the test to the content examined in the teaching simulation. Yet, in the case of Suzanne, her performance on the LMT test was less aligned with her performance in the teaching simulation compared to the alignment between her performance on the GRE-quantitative test (a more generic test than the LMT test) and her teaching performance. Hence Suzanne's case clearly points to the need that questions such as those listed above be carefully considered in studies exploring the association between knowledge and teaching performance.

questions, Suzanne proposed an instructional intervention that built on those answers: she wanted her students to understand the rationale for using the idea of common multiples in this problem but also to get a better sense of how this idea applied to the context of this particular problem. Hence, the difficulties she encountered and particularly how she resolved them, merely as a learner of the content, reflected on her proposed instructional approach. From this perspective, Suzanne's case seems to suggest that in exploring the association between knowledge and teaching performance, one needs to consider not only the knowledge that teachers hold but also how they came to construct this knowledge.

Despite the issues that the case of Suzanne raises, in some respects, her case corroborates the patterns discussed in the previous six cases. Those patterns pertain both to the associations between knowledge and performance as well as to other factors that appear to have informed and/or mediated this association. I discuss both issues in turn.

Suzanne's case provides compelling evidence about the association between teaching performance and knowledge (as gauged through her work during the teaching simulation and as delineated by her performance on *specific* LMT test questions). During the simulation, she very eloquently confessed that she could not "see the concept of division (of fractions) in her brain." In addition, she also incorrectly answered the LMT test question on interpreting the fractional part of a quotient. These limitations in her knowledge were reflected on what she could do (or was not able to do) during the teaching simulation. To start, although she wanted to use the two word problems of the first textbook page to launch a conceptually oriented lesson on the division of fractions, she had difficulties seeing how these problems could be used to this end. Second, although she recognized that the second page was more conducive to conducting a lesson

geared toward making meaning, she was not seen building on the affordances of this page. Third, despite wanting to use a visual representation to explain the “whys” of dividing fractions, she neither used a representation nor did she provide an explanation. Fourth, she could not determine whether Michelle’s solution to the last problem was correct since she herself had difficulties interpreting the fractional part of quotients; she encountered analogous difficulties when trying to resolve the discrepancy between Robert’s work and Amanda and Julia’s solution.

There is, nonetheless, the “affordance” side of the coin; two examples that illustrate the positive side of the association between knowledge and performance are Suzanne’s work in both episodes associated with the practice of responding to students’ direct or indirect requests for help. When Suzanne identified the potential source of Alan’s confusion and error, she proposed an intervention that had the potential to support him in identifying and correcting his error. Similarly, after having figured out how the idea of common multiples applied to the second problem, she outlined an instructional intervention that could help students understand why and how to apply this idea to solve this problem.

Consistent with the previous cases, Suzanne’s case also points to two other factors that seem to have informed her decisions and actions: her school experiences as a learner of mathematics (and her corresponding images of teaching)<sup>157</sup> and her beliefs about teaching and learning mathematics. If one were to identify the study’s strongest piece of evidence illustrating the effect of one’s school experiences on one’s teaching

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<sup>157</sup> Her informal teaching experiences as a middle-school librarian might also have informed her performance in the teaching simulation; yet the interview data do not provide substantive relative evidence. Thus, Figure 4.32, which outlines Suzanne’s portrait, presents the path linking her informal teaching experiences to her knowledge and performance with a question mark.

performance, this would presumably be Suzanne's argument that "I guess you're gonna teach the way that you learned." Suzanne's mostly experienced mathematics as memorizing and applying formulas and rules without learning the "whys." Because of her own struggles with the content, she was aware of the deficiencies of this type of instruction, especially with respect to helping students see meaning in what they are doing instead of mechanically following procedures. It is not surprising, then, that she wanted to pursue a more conceptually oriented type of instruction than the type she had experienced as a learner of mathematics. Yet, when her knowledge could not support her in this respect, she was seen resorting to images of teaching she had formed as a student of mathematics. A typical example was the rule Suzanne would present to students for dividing fractions: not only would she not teach division conceptually, but she would introduce her students to the cross-multiplication rule, which was even more esoteric than the traditional "invert-and-multiply" algorithm.

In Suzanne's case, one could also talk about two different types of beliefs: professed beliefs and implicit beliefs. The first type corresponds to the beliefs she explicitly presented during the simulation or on the survey; the second pertains to latent beliefs, which one could infer from her performance in the simulation. In the first set of beliefs, one could list Suzanne's conviction that mathematics should make sense and that it should not be a matter of memorizing and applying rules. In the second cluster, one could list beliefs about teachers' role in supporting their students, especially those who struggle. Despite her professed beliefs as delineated by her survey responses regarding student struggle and confusion, in the teaching simulation Suzanne endorsed the teacher's move to intervene right after seeing the two struggling students.



The portrait of Suzanne developed based on the preceding analysis is presented in Figure 4.32, but some clarifications are in order. First, I illustrated the effect of her school experiences and the images of teaching she formed with both a positive and a negative sign. This is because Suzanne's images of teaching sometimes appeared to operate as avoidance models; at other times, and especially when she lacked alternative images of teaching or when her knowledge of the content was not solid enough, Suzanne resorted to those images of teaching, despite their being antithetical to her endorsed type of instruction. Second, the list of her beliefs is not exhaustive; rather, I listed two different types of beliefs to illustrate that some of them appeared to function synergistically to building rich and challenging learning environments and others worked antithetically to promoting these types of environments. Third, Suzanne's case helps enrich considerations of the practice of responding to students' requests for help by adding the element of unpacking ideas first for oneself and then communicating them to students. Fourth, I largely avoided characterizing Suzanne's performance in providing explanations and using representations because she did not provide any explanation or use representations. Finally, the interview data provide mixed findings about her performance in attending to and following students' contributions; thus the use of a question mark for this practice.

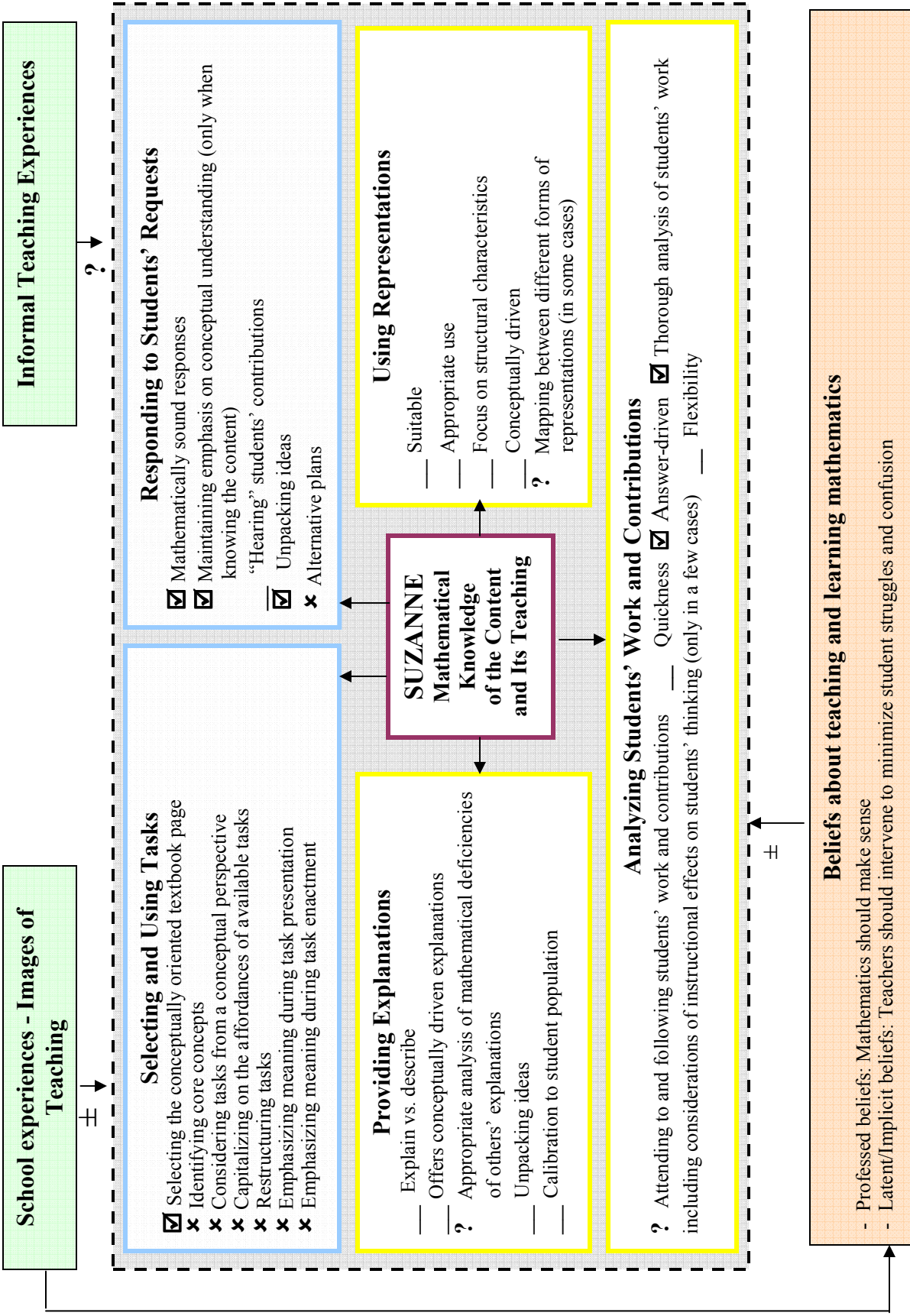


Figure 4.32. Considering the association between knowledge and teaching performance through Suzanne's profile.

## Exploring the Association between the PSTs' Entrance MKT and Performance in the Five Teaching Practices from a Qualitative Standpoint: A Cross-Case Analysis

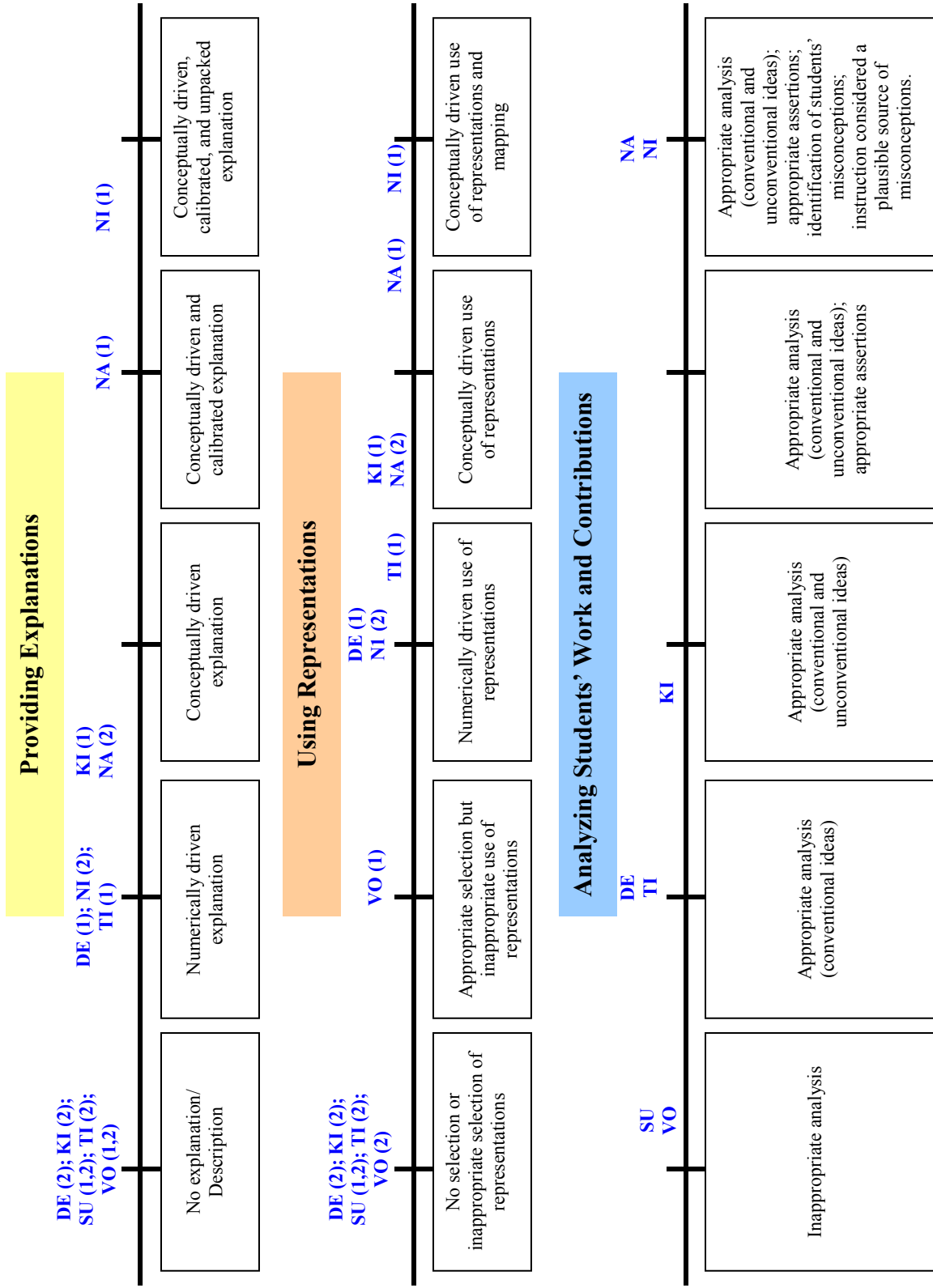
The first research question of this study (RQ1) explores the association between the PSTs' entrance MKT and their teaching performance as gauged by a teaching simulation. To this end, a quantitative analysis was first pursued, the results of which have been reported in the first section of this chapter. This analysis was followed by a qualitative, in-depth, exploration of the association by considering a purposive sample of seven cases. The last section of this chapter presents the results of a cross-case analysis, in which I bring together the seven cases, seeking to understand *if* there is a relationship between the PSTs' MKT and their teaching performance, and if so, *how* does this knowledge manifest itself in the PSTs' performance in the five practices under investigation.

This section consists of three parts. In the first part, I synthesize the results presented in the previous section of this chapter by considering five classification schemes, one for each of the study's five practices. This synthesis provides the foundation for addressing the *if* and *how* questions listed above, which are considered in the second and third parts of this section, respectively. In addressing the *if* question, I explore two themes: (a) the association between knowledge and teaching performance and (b) other factors that appeared to inform the study participants' teaching performance; these two themes correspond to the first and second subordinate questions of RQ1. In investigating the *how* question, I use the seven cases to build and elaborate plausible explanations for how the knowledge of the seven PSTs manifested itself in their teaching performance, thus addressing the third subordinate question of RQ1. Because

these explanations are tentative and warrant further exploration, following Ball's approach (1988), I present them as *propositions*. In Chapter 5, I examine these propositions in light of the findings pertaining to addressing the study's second research question.

*Bringing the Seven Cases Together: The Five Performance Classification Schemes*

The classification schemes presented in this part were developed by drawing on the seven PSTs' performance in each of the five practices. The PSTs' performance in the performing tasks contributed significantly more to the development of these schemes than did the PSTs' performance in the noticing and interpreting-evaluating tasks, which, for the most part, assumed a rather subordinate role. The decision to prioritize the PSTs' performance in the former rather than the latter tasks was also informed by the results of the quantitative and the qualitative analyses reported in the previous two sections, which both suggested a stronger association between knowledge and the performing than the noticing and interpreting-evaluating tasks. Each classification scheme consists of five categories, with each subsequent category representing an improved performance compared to the performance captured by its preceding category. In what follows, I present and explain the schemes' categories and justify the classification of the seven cases in those schemes. I first consider the MKT-related practices (see Figure 4.33) and then outline the MTF-related practices (see Figure 4.34).

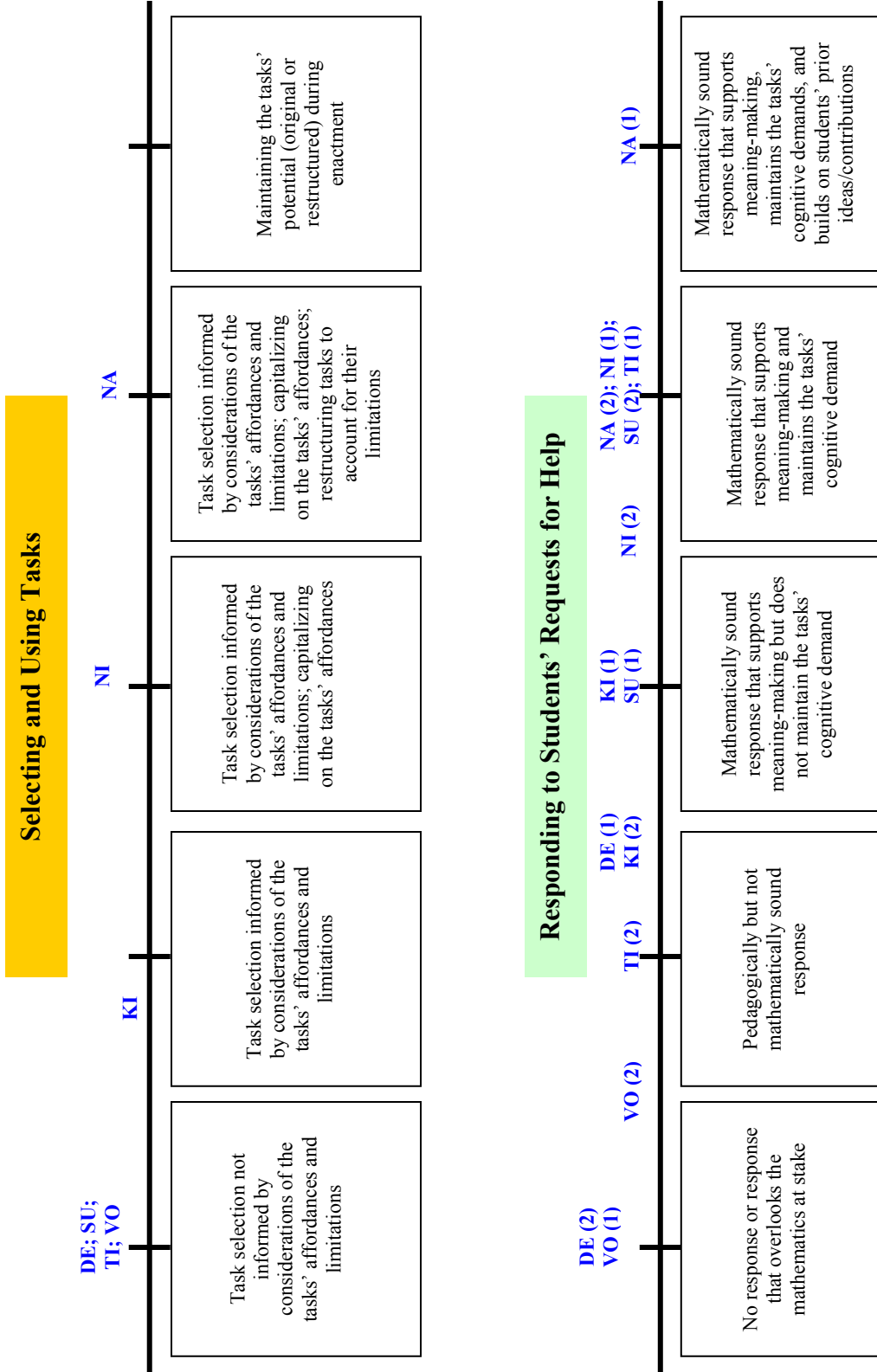


**Legend:**

**DE:** Deborah; **KI:** Kimberley; **NA:** Nathan; **NI:** Nicole; **SU:** Suzanne; **TI:** Tiffany; **VO:** Vonda

1. Quotient; 2 Invert-and-multiply algorithm (only for the practices of providing explanations and using representations)

Figure 4.33. The PSTs' classification in the schemes of the MKT-related practices (pre-intervention performance).



**Legend:**  
 DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda  
 1. Alan's episode; 2. June's and Shaun's episode (only for the practice of Responding to Students' Requests for Help)

Figure 4.34. The PSTs' classification in the schemes of the MTF-related practices (pre-intervention performance).

## *Providing Explanations*

The five categories of this classification scheme are:

1. *No explanation/Description*: No explanation is provided or the “explanation” provided describes rather than explains the procedure under consideration. Typical examples are Suzanne’s non-providing of any explanation and Vonda’s describing rather than explaining the invert-and-multiply algorithm.
2. *Numerically driven explanation*: The explanation is driven by the numbers involved in the procedure and particularly the final answer. Nicole’s explanation of the invert-and-multiply algorithm represents a typical example of this category.
3. *Conceptually driven explanation*: The explanation is grounded in the underlying meaning of the procedure under consideration; yet, it is not calibrated to its intended population. An example of this category is Nathan’s “proof” of the invert-and-multiply algorithm.
4. *Conceptually driven and calibrated explanation*: The explanation provided is grounded in the underlying meaning of the procedure under consideration and it is somewhat calibrated to its intended audience. An example of this is Nathan’s explanation for the quotient.
5. *Conceptually driven, calibrated, and unpacked explanation*. The explanation provided is grounded in the meaning of the procedure under consideration; calibrated to its intended audience; and adequately unpacked.<sup>158</sup> An example that is the closest

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<sup>158</sup> An alternative classification would be to switch the criteria of unpackedness and calibration, and include the first rather than the second criterion in the fourth category. Such a classification could account, for example, for sufficiently unpacked explanations, yet not calibrated to the intended audience (e.g., the explanation that a mathematician might give to a second grader). The classification presented above was preferred to the aforementioned because it was empirically supported by the findings reported in the previous section.

to this category is Nicole's explanation for the quotient.

The seven PSTs considered in the previous section were asked to provide two explanations, one for the quotient in the division problem  $2 \div \frac{3}{4}$  and another for the invert-and-multiply algorithm for the same division problem; hence, in the upper panel of Figure 4.33, which corresponds to the practice of providing explanations, each case is presented twice. Suzanne and Vonda are clustered in the first category for both explanations, because they either provided no explanations or their explanations were descriptions of the steps involved in the invert-and-multiply algorithm. Similarly, Deborah, Kimberley, and Tiffany gave no explanation for this same algorithm; thus, they are also clustered in the first category. Deborah's and Tiffany's explanation for the quotient and Nicole's explanation for the invert-and-multiply algorithm were driven by the numbers included in the division problem under consideration; thus, they are in the second category. In contrast, Kimberley's explanation for the quotient and Nathan's explanation for the pertinent algorithm were hybrids of a numerically driven and a conceptually driven explanation: they were partly grounded in one or more concepts, but they were also driven by the numbers included in the operation; thus, these two PSTs are situated between the second and third categories. Nathan's explanation for the quotient was conceptually oriented and could be relatively understood by a sixth grader, but it needed more unpacking; therefore, it is situated in the fourth category. Because Nicole's explanation was more unpacked than Nathan's, but still not sufficiently unpacked, it falls between the fourth and the fifth categories.

### *Using Representations*

The five categories of this classification scheme are the following:



1. *No representation selection or inappropriate selection of representations*: This category encompasses all the performances in which the study participants could not use any representation for the explanation they offered or they contemplated using inappropriate representations. Kimberley's performance on explaining the division-of-fractions algorithm is an example of this category.
2. *Appropriate selection but inappropriate use of representations*: This category includes performances in which the study participants selected appropriate representations, but used them inappropriately. One example is Vonda's use of the rectangular representation to explain the quotient. Although this representation was appropriate, it was used inappropriately since it did not maintain the absolute size of the divisor. A similar example is Tiffany's representation of the dividend and the divisor in a single circle, which failed to maintain their relative size.
3. *Numerically driven use of representations*: This category includes performances in which the study participants selected appropriate representations, but in using them they largely focused on the numbers involved in the procedure at hand and particularly on the procedure's final answer as opposed to focusing on the concepts themselves. An example of this is how Nicole tried to partition her representation in "four parts" and "thirds" to show the reciprocal of the given division (i.e.,  $\frac{4}{3}$ ).
4. *Conceptually driven use of representations*: This category includes those performances in which the study participants selected appropriate representations for their explanations and their use of these representations was based on the concept(s) underlying the procedure at hand. The performance that came close to meeting the criteria for inclusion in this category was Nathan's explanation of the quotient.

(During this performance Nathan was also seen engaging in some mapping, which pertains to the next category.)

5. *Conceptually driven use of representations and mapping*: This category includes the performances described in the previous category, which additionally paid attention to issues of mapping. In other words, the study participants made connections among the drawings they used, the mathematical sentences, and/or the word problems these drawings were supposed to represent. The example closest to this category was Nicole's use of a rectangular representation to explain the quotient. Kimberley's explanation for the quotient would have also been an example in this category had she not offered a numerically driven explanation at the end.

The middle panel of Figure 4.33 corresponds to the *Using Representations* scheme. The first category of this scheme includes Suzanne's lack of representations when explaining the quotient and all participants,' except for Nicole's and Nathan's, use of representations when explaining the invert-and-multiply algorithm. In explaining the algorithm, Vonda used a suitable representation inappropriately; thus, her performance occupies the second category. Deborah's and Nicole's use of representations when explaining the quotient and the algorithm (respectively) was merely driven by the numbers included in the pertinent procedure; thus, their performances were classified under the third category of this scheme. In contrast, Tiffany's and Kimberley's use of representations when explaining the quotient and Nathan's use of representations when explaining the invert-and-multiply algorithm were not totally numerically driven, but rather were informed, at least to some extent, by pertinent concepts; thus, their performances are situated between the third and fourth categories. Finally, when

explaining the algorithm, both Nathan and Nicole grounded their use of representations in the related concepts; Nicole was also more successful than Nathan at building connections between her diagram and the mathematical sentence of the given division problem. This explains why Nicole's performance is presented to the right of Nathan's performance.

*Analyzing Students' Work and Contributions.*

In contrast to the previous two classification schemes, the classification scheme on analyzing students' work and contributions emerged from the study participants' performance in both the performing and the noticing/interpreting-evaluating tasks of this practice. The categories for this classification scheme are not mere reflections of the criteria listed in the PSTs' portraits presented in the previous section, but represent more general categories. This was deemed necessary in order to represent a more holistic picture of the participants' performance in this practice. The five categories of this classification scheme are as follows:

1. *Inappropriate analysis of students' ideas:*<sup>159</sup> This category includes performances in which the study participants inappropriately analyzed even the conventional student ideas (e.g., Robert's and Ann's solutions to the problem  $2\frac{3}{4} \div \frac{3}{4}$ ).
2. *Appropriate analysis of students' conventional ideas:* This category encompasses performances in which the study participants' appropriately analyzed only the conventional student ideas.
3. *Appropriate analysis of students' conventional and unconventional ideas:* This category includes performances in which the study participants analyzed

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<sup>159</sup> For brevity's sake, I use the term "ideas" to refer to both students' work and contributions.

appropriately both conventional and unconventional student ideas. As an example of an unconventional idea one could consider Michelle's solutions to the problem  $2\frac{3}{4} \div \frac{3}{4}$  or Amanda and Julia's work on the division problem  $\frac{3}{4} \div \frac{1}{6}$ . As the reader might recall, Michelle used a drawing to solve the former problem, while Amanda and Julia used a common-multiples approach to solve the latter.

4. *Appropriate analysis of students' conventional and unconventional ideas;*

*appropriate assertions:* In addition to analyzing appropriately both conventional and unconventional student ideas, for a performance to be included in this category an additional criterion had to be met: the assertions made about the students' understanding had to be suitable. Such assertions might, for instance, refer to what students did or did not understand.

5. *Appropriate analysis of students' conventional and unconventional ideas;*

*appropriate assertions; identification of students' misconceptions and considerations of instruction as a plausible source of students' misconceptions/errors:* This category is reserved for higher levels of performance in this practice. The study participants clustered in this category analyzed conventional and unconventional student ideas appropriately, made valid assertions about students' understanding or lack thereof, and most importantly, identified potential misconceptions in students' ideas and explored the likelihood that the instruction itself caused some of students' misconceptions and errors.

For this practice, a composite performance was considered for each of the seven cases. As the lower panel of Figure 4.33 shows, Suzanne and Vonda are clustered in the first category because they both analyzed one of the two conventional student solutions –

Robert's and Ann's – inappropriately; Suzanne analyzed Ann's solution incorrectly, while Vonda was not sure if Robert's solution was correct. Deborah and Tiffany occupy the second category, since they appropriately analyzed only Robert's and Ann's solutions. Kimberley is placed between the second and third categories because her analysis of Michelle's solution was incomplete. Finally, Nathan and Nicole are in the fifth category, since both of them appropriately analyzed all three students' solutions to the division problem  $2\frac{3}{4} \div \frac{3}{4}$ , they made appropriate assertions about these students' understanding, they identified the misconception in June's contribution, they successfully explained the error in Alan's and Amanda and Julia's work, and they identified certain instructional moves that Ms. Rebecca made as potential sources of student error and misconception.

### *Selecting and Using Tasks*

This classification scheme consists of the following five categories:

1. *Task selection not informed by considerations of the tasks' affordances and limitations:*<sup>160</sup> This category includes performances in which the PSTs' selection of tasks did not draw on considerations of the tasks' affordances and/or limitations, either because the PSTs did not engage in such considerations or due to other reasons informing or mediating the PSTs' task selection decisions.
2. *Task selection informed by considerations of the tasks' affordances and limitations:*  
This category includes those performances in which the PSTs' selection of tasks drew on both the affordances and the limitations of the available tasks. Unlike the previous category, a key criterion for classifying the PSTs' performance in this category was

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<sup>160</sup> These limitations and affordances should be seen through the spectrum of the orientation of this study toward building rich and cognitively challenging learning environments.

- their appropriate analysis of the tasks' affordances and limitations.
3. *Task selection informed by considerations of the tasks' affordances and limitations; capitalizing on the tasks' affordances:* In addition to the criterion mentioned in the previous category, this category required that the PSTs capitalize on the affordances of the curriculum tasks in their proposed plan for using these tasks in their teaching
  4. *Task selection informed by considerations of the tasks' affordances and limitations; capitalizing on the tasks' affordances; and restructuring tasks to account for their limitations:* This category encompasses the criteria mentioned for the two preceding categories; for inclusion, the PSTs must have also proposed one or more ways for restructuring the limitations of the available curriculum tasks. Following Ball and colleagues (2004), I use the term restructuring to denote either upgrading a task because of its low cognitive demand or modifying the task to make it more accessible to students.
  5. *Maintaining the tasks' potential (original or restructured) during enactment:* This category pertains more to the enactment of the selected tasks and examines the extent to which the selected tasks' potential— be it their potential based on their original affordances or that engendered after their restructuring – is maintained during task enactment. The inclusion of this category facilitated the task of discriminating among the PSTs' performances, since some of the PSTs' plans for using the selected tasks also included considerations that could support preserving the tasks' potential during enactment. This issue will become clearer in Chapter 5, where I use the same classification scheme to consider the PSTs' post-intervention performance in this practice.

The upper panel of Figure 4.34 outlines the clustering of the seven cases in the categories of the *Selecting and Using Tasks* scheme. Deborah, Suzanne, Tiffany, and Vonda were clustered in the first category since they did not identify the affordances of the second textbook page; nor did they identify some conceptual ideas on which they could build to address some of the limitations of the first page's tasks. Kimberley, in contrast, did talk about such ideas when considering the first page, but did not actually capitalize on them in her proposed plan for using the first textbook page. Because she did not identify key affordances of the second page, she is situated between the first and second categories. In considering the second page, Nicole attended to the affordances of its tasks and built on them to develop an introductory division-of-fractions lesson; Nathan, in addition, proposed ways in which the first page's tasks could be restructured to upgrade their cognitive complexity. Hence, these two latter cases are situated under the third and fourth categories, correspondingly.

#### *Responding to Students' Direct and Indirect Requests for Help*

This classification scheme consists of the following five categories:

1. *No response or the response offered overlooks the mathematics at stake:*

Performances included in this category pertain to not responding to the students' requests for help or being unclear as to how to respond to students; such non-responses might be due to the PSTs' difficulties with the content. Also included in this category are performances that totally overlook the mathematics at stake. Typical examples of such responses are Deborah's performance in responding to June and Shaun's requests for help, where Deborah confessed that she was not clear as to how she would respond to these two students, and Vonda's performance in Alan's episode,

- where she totally overlooked Alan's error.
2. *Pedagogically but not mathematically valid response*: This category encompasses the responses that are pedagogically but not mathematically valid. These performances are based on general pedagogical principles (e.g., involving students, presenting ideas on the board) but do not (successfully) address the mathematics at stake. Whereas the PSTs in this category might have identified some of the mathematical issues warranting attention, they either did not address them or they addressed them inappropriately. One example of this is how Tiffany considered responding to June's and Shaun's request for help; although she identified the source of the two students' confusion, because she herself was not clear on the content, she proposed a pedagogically valid approach for supporting the two students. Yet, her approach did not address the mathematics at stake.
  3. *Mathematically sound response that supports meaning-making, but does not maintain the tasks' cognitive demand*: Responses clustered this category pertain to appropriately addressing the mathematics at stake and in ways that could support students' understanding. However, the teacher is doing most of the thinking for the students. Examples of this category are Suzanne's and Kimberley's approaches to support Alan. Both these PSTs identified the mathematics at stake and addressed them appropriately in their proposed plan for interacting with Alan; yet, both were seen doing most of the thinking for him.
  4. *Mathematically sound response that supports meaning-making and maintains the tasks' cognitive demand*: Unlike the previous category, performances included in this category address the mathematical issues at stake, potentially support meaning-



making, and maintain the cognitive demand for students. In other words, the teacher is not doing most of the thinking for the students, but engages them in a discussion or an exploration that helps them identify the mathematical ideas at stake and/or correct their error. Unlike the second category, this category encompasses interactions with students that could successfully focus attention on and address the mathematical ideas under consideration. One example of this category is how Nicole proposed supporting Alan to correct his error.

5. *Mathematically sound response that supports meaning-making, maintains the tasks' cognitive demand, and builds on students' prior ideas and contributions:* Apart from the criteria listed in the previous category, performances clustered in this category meet an additional criterion: in responding to students, the teacher is seen building on their prior ideas and contributions. An example for this category is how Nathan proposed addressing Alan's error by building on June's earlier work on representing the one sixth and the one half of a yard.

The performances of the seven cases in each of the two episodes pertaining to this practice are illustrated in the lower panel of Figure 4.34. Given that several of these performances were already considered while the categories of this scheme were being typified, the focus here is on only those not previously discussed. Vonda's performance in the second episode (i.e., responding to June and Shaun) is situated between the first two categories, because she had some preliminary ideas about the mathematics being at stake but she was not clear what mathematical ideas needed to be addressed. Deborah's performance in the first episode (i.e., supporting Alan) and Kimberley's performance in the second episode were located between the second and third categories, because they

both identified the mathematics at stake, but their proposed approaches were too general to support students' understanding of the mathematics considered in those episodes. Nicole's performance in the second episode rests between the third and fourth categories, because Nicole agreed with the teacher's approach to remind students of the idea of common multiples, only if this idea had been sufficiently discussed in previous lessons. Finally, Tiffany's performance in the first episode and Nathan's and Suzanne's performance in the second episode met the criteria of the fourth category, since to a smaller or larger extent these PSTs proposed posing questions and engaging students in discussions that could support but not reduce students' thinking.

Having outlined the five classification schemes, I now turn to the two main questions that the study addresses: whether there is an association between MKT and teaching performance and how MKT appears to manifest itself in the PSTs' performance. These questions are considered in the following two parts.

*Exploring the Knowledge and Teaching-Performance Association: Insights from the  
Cross-Case Analysis*

The five classification schemes outlined in the previous part provide some insights about the association between knowledge and performance in the five practices. Here I consider the classification of the seven cases in those schemes from two different perspectives: first, based on their clustering into different MKT categories, as presented in Column II of Table 4.5; second, by considering their knowledge of mathematics and its teaching as discussed in the previous section. I then consider other factors that appear to have informed the PSTs' performance besides their knowledge.

*Considering the Association between Knowledge and Teaching Performance through the Five Classification Schemes*

On the basis of their LMT scores, Nathan, Nicole, Suzanne, and Kimberley were clustered in the high MKT category; Tiffany in the medium-low category; and Deborah and Vonda in the low category. Figure 4.35 replicates Figures 4.33 and 4.34, but presents the participants of each of the three MKT-categories in different colors. If one excludes the case of Suzanne and if one considers the PSTs' performance in each of the two tasks of the practices of providing explanations, using representations, and responding to students' requests for help separately, Figure 4.35 shows a pattern across all five classification schemes. With only one exception, the three high-MKT participants (presented in red) are always positioned in the same or higher categories on the schemes relative to the medium-low MKT participant (presented in blue), who, in turn, is at the same or higher points of the schemes relative to the two low-MKT participants (presented in green); the only exception to this pattern concerns the first task of the practice of responding to students' requests for help, in which Tiffany appears at a higher category than Kimberley does. Given that the schemes were developed to represent increasing levels of performance when moving from their lower to their upper categories, this pattern corroborates the quantitative findings reported in the first section of this chapter: that there exists a positive association between the PSTs' MKT scores and their performance in the teaching simulation.

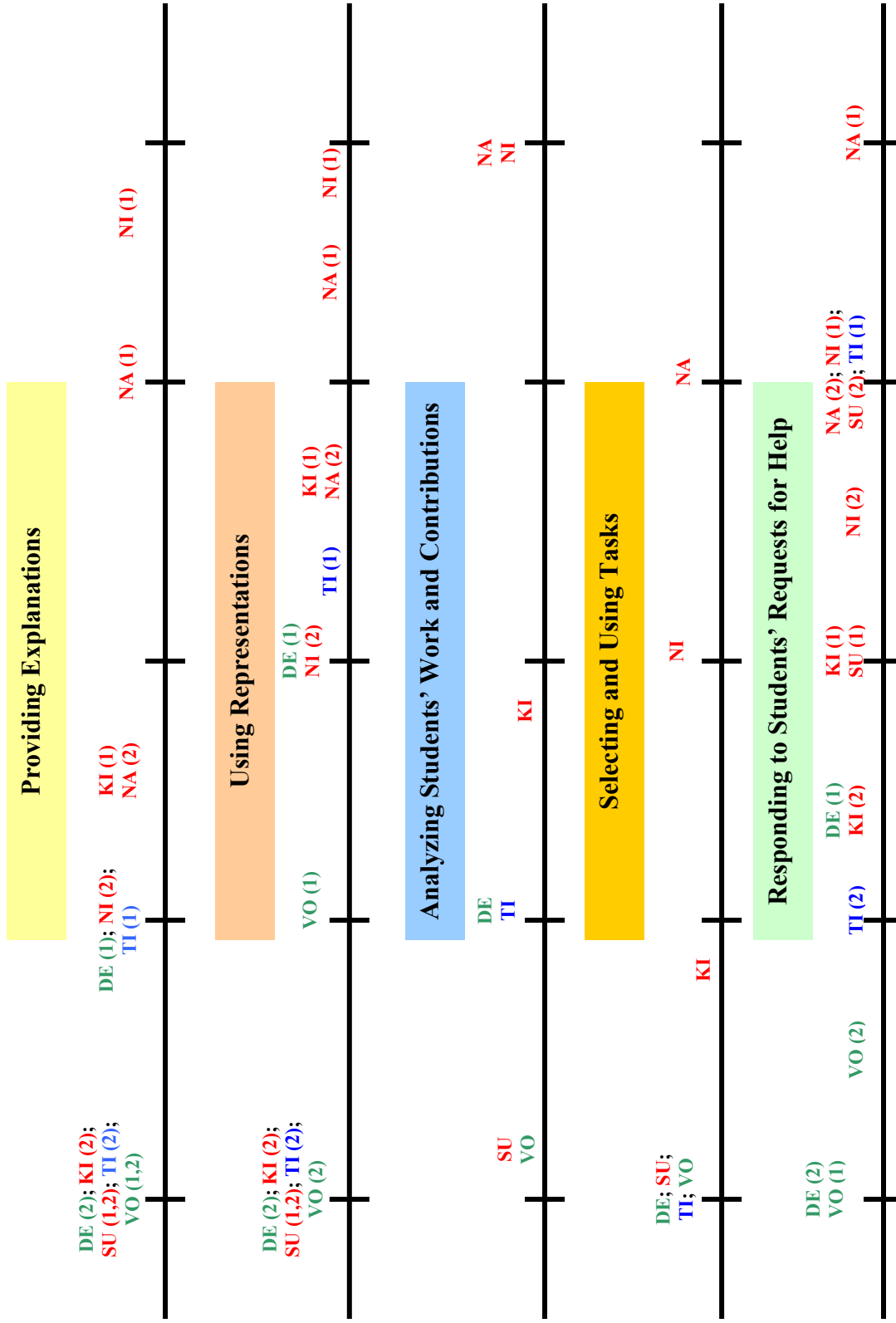


Figure 4.35. The PSTs' clustering in the five classification schemes based on their pre-intervention MKT scores.

The pattern just discussed is constrained in two respects: excluding Suzanne and considering as separate the two tasks for three of the classification schemes. Considering the PSTs' knowledge of mathematics and its teaching, as discussed in the previous section, could account for both constraints. Based on their knowledge as exemplified in the teaching simulation, the seven cases could be clustered in four groups. The first group comprises Suzanne and Vonda, who both had difficulties even in understanding division as the divisor units that can be fit into the dividend. The second group consists of Deborah and Tiffany, who both understood division as described above, but they were uncertain as to whether this idea applies to the division of fractions. The third "group" includes only Kimberley, who had solid knowledge about the idea of division, but encountered difficulties understanding relative and absolute units and, consequently, in interpreting the fractional part of the division-of-fractions quotients. Finally, in the fourth group, one could cluster Nathan and Nicole, who were not seen encountering any of the difficulties mentioned above, but struggled with the idea of the reciprocal. The participants of these four categories are presented with different colors in Figure 4.36, which again replicates Figures 4.33 and 4.34.

Based on this grouping, there seems to be more consistency between the PSTs' knowledge and their performance in the five teaching practices than that previously reported. Consider, for example, the two classification schemes of *Selecting and Using Tasks* and *Analyzing Students' Work and Contributions*. In the former scheme the members of the first two groups (presented in green and blue, respectively) are all clustered at the first category; those of the fourth group (presented in red) appear in the third and fourth categories and Kimberley (presented in purple) lies somewhere in

between. This pattern is even clearer for the latter classification scheme, for which the participants of each knowledge category are always presented in upper performance categories than the participants of lower knowledge categories. For the classification schemes of *Providing Explanations* and *Using Representations* this same pattern holds for each of the two tasks under consideration. Notice, for example, that for the first task (i.e., pertaining to the quotient), the distribution of the seven cases along both schemes is very similar to that discussed for the *Analyzing Students' Work and Contributions* scheme. For the second task, the PSTs in the first three groups are all clustered in the first category of the schemes of *Providing Explanations* and *Using Representations*. Notice also that for this second task the two cases of the fourth group appear at lower categories of these schemes compared with their classification for the first task. This is consistent with their knowledge as delineated above: Nathan's and Nicole's understanding of the reciprocal – a fundamental idea for performing well on the second task – was weaker than their understanding of the concepts addressed in the first task.

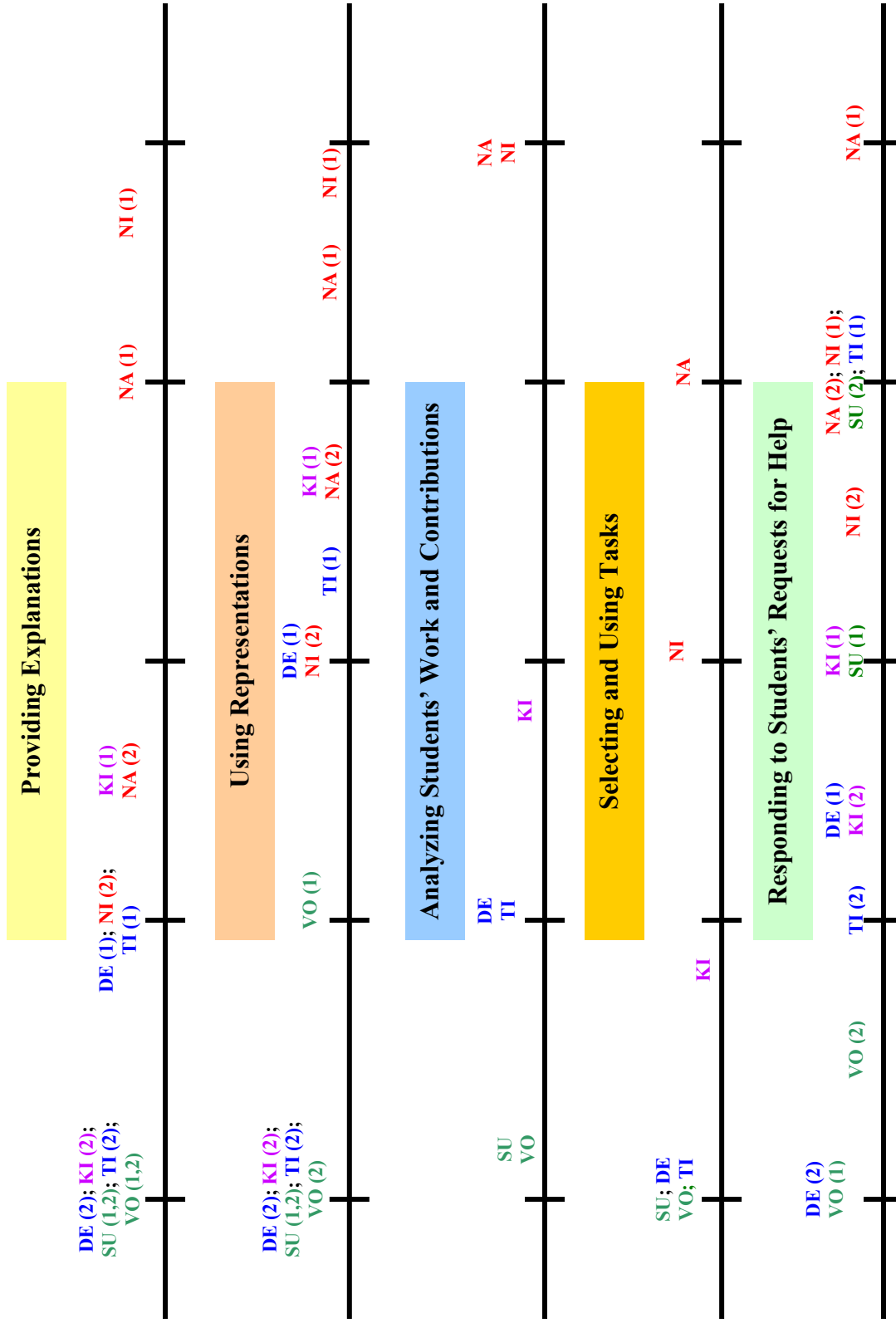
The picture, however, is more complicated for the scheme of *Responding to Students' Requests for Help*. The participants of the fourth group again are found at the upper two categories of this scheme; Kimberley is clustered in the middle; and Vonda is situated in the first category for the first task. Yet, Tiffany's and Suzanne's performance is not consistent with their group clustering. For the second task, Vonda also appears at a higher category than Deborah, which might also be considered inconsistent according to her group clustering. It is important to remember, however, that Tiffany was sensitive to issues of making meaning; hence, when she was knowledgeable about the ideas under consideration, her outlined plan for responding to the students' requests had the potential

to not only help students make meaning but also maintain the tasks' cognitive complexity – thus, her clustering in the fourth category (for the first task). Also recall that Suzanne was able to construct *in situ* the knowledge needed for the second task; hence, her respective clustering in the fourth category, as well. Overall, for this scheme, the PSTs' performance is more aligned with their MKT-grouping rather than with their grouping based on their knowledge of mathematics and its teaching.

### *Other Factors Informing the PSTs' Performance*

Besides PSTs' knowledge – be it their MKT as measured by the LMT test or their knowledge as manifested during the simulation – the cross-case analysis of the seven cases pointed to two additional factors that appeared to inform the PSTs' decisions and actions: their images of teaching and their beliefs.

The PSTs' images of teaching are manifest in one way or another in all seven portraits outlined in the previous section. These images of teaching were (in)formed by several sources, central among them being how the PSTs experienced the subject as learners of mathematics. Recall, for example, the “remember-and-apply” image of teaching that Deborah, Suzanne, and Tiffany discussed when referring to their school experiences or Nicole's “give-more” image. Other activities in which the PSTs engaged shaped these images of teaching, including parent volunteering (e.g., the case of Nicole), informal and formal teaching experiences (e.g., the cases of Kimberley and Nicole), and participation in teacher education programs (e.g., the case of Nathan, who drew on some of the course readings and discussions). Even the teaching simulation itself could have provided the PSTs with an (alternative) image of teaching.



**Legend:**

DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda

**Classification according to knowledge of mathematics and its teaching:** Green: Low; Blue: Medium-low; Purple: Medium-high; Red: High

For the practices of *Providing Explanations* and *Using Representations*: **1**. Quotient; **2** Invert-and-multiply algorithm

For the practice of *Responding to Students' Requests for Help*: **1**. Alan's episode; **2** June and Shaun's episode

Figure 4.36. The PSTs' clustering in the five classification schemes based on their pre-intervention knowledge of the content and its teaching.



The cross-case analysis showed the PSTs' teaching performance to also be informed by their beliefs.<sup>161</sup> These beliefs captured issues such as what mathematics is, what it means to teach and learn mathematics, how students should be supported when learning the content, what role the teacher should assume when encountering struggling students, and so on. Regardless of their manifestation in each PST, these beliefs appeared to inform performance, sometimes contributing positively and other times contributing negatively.

In sum, the cross-case analysis showed a primary triad of factors that informed the PSTs' performance in the teaching simulation: knowledge, images of teaching, and beliefs. This analysis also suggested at least two different patterns of interactions among the three factors: *synergistic* patterns, in which the three factors appeared to work in concert to influence the PSTs' performance, and *competing* patterns, in which the three factors appeared to inform the PSTs' performance in competing ways.

*Synergistic Patterns.* The two typical examples of the synergistic patterns come from Nathan and Vonda, whose knowledge, beliefs, and images of teaching were compatible. As described in the previous section, Nathan had a strong knowledge base and believed that students should be supported in understanding the meaning of the procedures they are asked to apply; some of his readings and early class discussions also enriched his images of teaching. Thus, in several of the teaching simulation tasks the three factors under consideration appeared to be working synergistically to inform Nathan's decisions and actions. This same synergistic pattern was also true for Vonda, but yet in the opposite direction. Vonda's knowledge of the ideas explored in the

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<sup>161</sup> This finding is not inconsistent with the quantitative data. Notice, for example, that although low, the negative correlation found between the PSTs' teaching performance and their beliefs about minimizing complexity when teaching mathematics (see Column II, Table 4.4), was in the right direction.

simulation was rather weak; her images of teaching and her beliefs were also consonant with a type of instruction that emphasizes memorization and application of procedures. As reflected by her performance in the teaching simulation, her knowledge did not support decisions and actions geared toward helping students make meaning; she also seemed to be more concerned with promoting students' procedural fluency rather than with enhancing their conceptual understanding of the procedure(s) under consideration.

*Competing Patterns.* The cross-case analysis also yielded at least two competing patterns. The first one concerned instances in which the study participants wished to build a learning environment that promoted students' conceptual understanding of the procedure of fraction division but both their knowledge base and their images of teaching did not support them in this endeavor. Consider, for instance, three examples from three different cases. The first example concerns Deborah, who was a strong proponent of using representations. Yet because she did not understand Michelle's solution, she suggested that Michelle abandon her visual-solution approach and solve the problem using the traditional algorithm. Similarly, although Suzanne wanted to restructure the first textbook page to support more meaning-making, she had insufficient knowledge to do so. Hence, she ended up presenting the division-of-fractions algorithm very mechanically, merely imitating a familiar, yet not endorsed, image of teaching. Tiffany also wanted to make a deliberate selection of exercises instead of assigning all 16 of them to students. Because her knowledge did not support her in seeing these exercises from a perspective other than simply considering them as a set of algorithmic division-of-fraction problems, she confessed that lacking any other criteria she would resort to a familiar image of teaching: assigning the "odds" or the "even" exercises. In conjunction,

these three examples suggest that lacking the necessary knowledge base, preservice teachers are likely to resort to their images of teaching, regardless of whether these images accord well with their professed beliefs about teaching and learning the content. This hypothesis certainly warrants more investigation.

The second manifestation of competing patterns concerns having a strong knowledge base that could allow teaching the content in a manner that maintains the cognitive demand for students, but not doing so because of one's beliefs or images of teaching. Recall, for example, the case of Nicole, who, adhering to her images of teaching and her beliefs, proposed "giving more" to students, instead of letting them work ideas out on their own. Similarly, in supporting Alan, Kimberley was seen doing most of the thinking for the students, clarifying almost all the ideas for them and merely asking them to count out the number of pieces into which a line was partitioned.

Taken together, both the synergistic and the competing patterns support the correlations between the PSTs' MKT and their teaching performance yielded from the quantitative analysis. Specifically, the negligible or low correlations for the noticing and interpreting tasks can be explained by considering that what the PSTs attended to and how they explained it was not only informed by their knowledge but also by their images of teaching and their beliefs. Additionally, the correlations between knowledge and performance were stronger for the performing than the noticing and interpreting-evaluating tasks, but not high. This could also be attributed to those other factors which informed the PSTs' decisions and actions. But the question arises as to why those correlations were higher for the performing tasks than the noticing and interpreting-evaluating tasks. The patterns examined above provide a plausible explanation. Whereas,

overall, the PSTs' beliefs and their images of teaching supported them in attending to aspects of the teacher's work and commenting on them in ways that resonated with the type of the learning environments this study endorses, those beliefs and images did *not* appear to compensate for limitations in the PSTs' knowledge when the latter were asked to perform certain activities, as Tiffany's case aptly suggests.<sup>162</sup>

*Manifestations of the PSTs' Knowledge in Their Performance:*

*Insights from the Cross-Case Analysis*

The preceding analysis largely resonated with the strong association between the PSTs' MKT and their teaching performance yielded from the quantitative analysis; it also showed some factors, which, besides knowledge, appeared to inform the PSTs' decisions and actions. In this part, I move a step farther and explore *how* the PSTs' knowledge<sup>163</sup> appeared to manifest itself in their teaching performance. I address this question by searching for patterns in the PSTs' performance in each of the five practices, as outlined in the previous section of this chapter. The patterns yielded from this cross-case analysis are summarized in Table 4.6 and are organized by the five teaching practices explored in the study. These patterns are presented as *propositions* since they do not constitute conclusive evidence about how knowledge plays out in PSTs' performance; providing such evidence was beyond the scope of this study. Rather, following Yin's suggestion (2003), I present these propositions as ideas warranting further exploration and whose goal is "not to conclude a study but to develop ideas for further study" (p. 120).

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<sup>162</sup> The PSTs' beliefs and images of teaching could not compensate for limitations in their knowledge even for some instances of noticing and interpreting-evaluating tasks. For example, regardless of their beliefs and images of teaching, the PSTs did not notice the misconception in June's explanation if they were not aware of it. Similarly, they did not go into much depth when interpreting and evaluating Ms. Rebecca's explanation for the reciprocal if their knowledge did not support them in this endeavor.

<sup>163</sup> In this part, when I refer to *knowledge*, I consider the PSTs' MKT, as measured by the LMT test, but also their knowledge of the content and its teaching, as gauged by the teaching simulation.

Three focal questions guided the development of these propositions: (a) How did the presence of knowledge appear to support the PSTs' decisions and actions? (b) How did the absence or the limitations of knowledge appear to constrain their decisions and actions? And (c) in what instances was knowledge insufficient to make certain decisions and engage in certain actions? The development of these propositions followed an iterative approach as outlined in Yin (2003, pp. 120-122) and was mostly grounded in the seven analytical commentaries of the previous section. Nathan's analytical commentary helped develop the initial propositions about the manifestations of PSTs' knowledge in their performance. Each initial statement yielded from his case was iteratively compared against the data of the subsequent six cases and revised accordingly. Rival explanations (in the form of other factors contributing to PSTs' teaching performance) were also entertained.

In what follows, I briefly elaborate each proposition, starting with those associated with the three MKT-related practices (propositions A, B, and C) and then moving to those associated with the MTF-related practices (propositions D and E).

Table 4.6

*Manifestations of Knowledge (or Lack Thereof) in the PSTs' Teaching Performance:  
A Set of Propositions*

<b>(A) Providing Explanations</b>	
A <sub>1</sub>	PSTs' knowledge enables them to provide conceptually driven explanations which clarify rather than describe a procedure at hand.
A <sub>2</sub>	Lacking sufficient knowledge, PSTs might provide numerically driven explanations or switch from a conceptually driven explanation to a numerically driven one.
A <sub>3</sub>	A strong knowledge base enables PSTs to flexibly navigate the mathematical terrain and build on several tools (e.g., other concepts, other related mathematical situations, easier examples), when providing explanations.
A <sub>4</sub>	A strong knowledge base does not suffice to provide adequately unpacked explanations; PSTs need opportunities to practice providing such explanations.
<b>(B) Using Representations</b>	
B <sub>1</sub>	PSTs' knowledge supports them in selecting appropriate representations and using them appropriately when explaining mathematical ideas/procedures.
B <sub>2</sub>	A profound knowledge base allows PSTs to use representations in a conceptually driven manner (i.e., as tools for <i>explaining</i> an algorithm) rather than in a procedurally driven fashion (as means for <i>illustrating</i> the numbers involved in an algorithm).
B <sub>3</sub>	PSTs' knowledge alone does not suffice to build connections among different types of representations (e.g., drawings, mathematical sentences, word problems); PSTs need to be sensitized to the importance of building such connections in order to pursue them in their teaching.
<b>(C) Analyzing Students' Work and Contributions</b>	
C <sub>1</sub>	A strong knowledge base allows for appropriate and quick analysis of conventional student ideas.
C <sub>2</sub>	A strong and connected knowledge base offers PSTs resources upon which to draw when exploring unconventional or flawed student ideas; such resources include other related concepts or strategies/skills (e.g., working with an easier example, testing an idea in a different set of numbers).
C <sub>3</sub>	When lacking a strong knowledge base, PSTs may analyze students' work by pursuing a number-driven approach, namely by using the numbers involved in an algorithm (and particularly the final answer) as yardsticks against which to evaluate students' work and ideas.
C <sub>4</sub>	A strong understanding of the content does not suffice to identify latent or overt misconceptions in students' contributions; instead, familiarity with misconceptions renders PSTs more perceptive to misconceptions in students' work.
C <sub>5</sub>	PSTs' knowledge alone does not suffice to make accurate assertions about students' understanding or lack thereof; PSTs need opportunities to practice making such assertions.

Table 4.6 (continued)

*Manifestations of Knowledge (or Lack Thereof) in the PSTs' Teaching Performance:  
A Set of Propositions*

<b>(D) Selecting and Using Tasks</b>	
D <sub>1</sub>	PSTs' knowledge supports them in identifying the mathematical affordances of the curriculum tasks.
D <sub>2</sub>	A strong knowledge base enables PSTs to identify ways to restructure curriculum tasks that do not satisfactorily promote understanding of the ideas under consideration.
D <sub>3</sub>	Other factors beyond knowledge inform PSTs' task presentation and enactment of tasks; these include their images of teaching, their beliefs, and their own difficulties with the content.
D <sub>4</sub>	Although strong knowledge could help maintain focus on important mathematical ideas during task presentation and enactment, it might not be adequate for maintaining the tasks' cognitive demand; instead, PSTs need to be sensitized to issues of maintaining the tasks' cognitive demand during task presentation and enactment to start considering such issues in their instruction.
<b>(E) Responding to Students' Direct or Indirect Requests for Help</b>	
E <sub>1</sub>	Knowledge of the content could support PSTs' identification and targeting of key mathematical ideas (in more than one way) when addressing students' requests for help.
E <sub>2</sub>	The manner in which PSTs respond to students' requests for help is contingent not only on their knowledge but also on their images of teaching, their beliefs about teaching and learning mathematics, and their analysis of the instructional situation at hand.

*Propositions Related to the Practice of Providing Explanations*

*A<sub>1</sub>. PSTs' knowledge enables them to provide conceptually driven explanations which clarify rather than describe a procedure at hand.* When the seven PSTs' understanding of the content was deep, they provided what I called *conceptually driven explanations*, namely explanations undergirded by key mathematical ideas pertinent to the situation at hand (e.g., Nathan's and Nicole's explanation of the quotient). In contrast, when their understanding of the content was weak, the PSTs either provided no explanation or described the steps involved in the algorithm. Take for example, Vonda's

and Suzanne's explanation for the reciprocal or Kimberley's claim that the most she could do was to tell her students "to flip the divisor and multiply it" by the dividend.

*A<sub>2</sub>. Lacking sufficient knowledge, PSTs might provide numerically driven explanations or switch from a conceptually driven explanation to a numerically driven one.* In the absence of substantive knowledge, PSTs might provide what I called *numerically driven explanations*. These explanations are largely driven by the numbers involved in a procedure rather than by the underlying meaning of these procedures. Examples of such numerically driven explanations were Nicole's explanation for the reciprocal or Deborah's explanation for the quotient. When their knowledge did not support them for a full-fledged conceptually driven explanation, the PSTs were also seen starting with a conceptually driven explanation and then switching to a numerically driven one. A characteristic example of such switching is how Kimberley started explaining the quotient from a conceptual perspective and then, confronted with difficulties in explaining the fractional part of the quotient, she concluded her explanation by shifting emphasis from the concept of division to the numbers involved in the division under consideration.

*A<sub>3</sub>. A strong knowledge base enables PSTs to flexibly navigate the mathematical terrain and build on several tools (e.g., other concepts, other related mathematical situations, easier examples) when providing explanations.* The PSTs with deeper knowledge appeared to have a more connected knowledge base. This enabled them to draw on other concepts, related mathematical situations, and easier examples when they had difficulties providing an explanation. In contrast, due to their rather compartmentalized knowledge, the PSTs with less profound understanding of the content



less frequently made use of such resources; when they did, they could not sufficiently capitalize on them. The cross-case analysis provides several examples substantiating this proposition. Nathan, for example, started his explanation by outlining what division meant in the context of whole numbers. Similarly, Nicole drew on her knowledge of the remainders to explain the fractional part of the quotient in  $2 \div \frac{3}{4}$ . Despite her difficulties in explaining the reciprocal for this same division problem, she also provided an appropriate explanation for the reciprocal for an easier division-of-fractions problem (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ). Similarly, Kimberley used an easier example (i.e.,  $\frac{3}{2} \div \frac{1}{4}$ ) to initially explain the quotient. Tiffany and Deborah, on the other hand, were not able to build on such resources even when they considered resorting to them. Recall, for example, that although Tiffany contemplated transferring the idea of whole-number division to explaining the division problem  $2 \div \frac{3}{4}$ , she soon dismissed this idea, claiming that the concept of whole-number division does not actually apply to the division of fractions. Deborah was equally uncertain whether she could build on the idea of fitting divisor units into the dividend when explaining the same division problem.

*A<sub>4</sub>. A strong knowledge base does not suffice to provide adequately unpacked explanations; PSTs need opportunities to practice providing such explanations.* A strong understanding of the content can support a PST in providing mathematically valid and meaningful explanations. However, these explanations might not be sufficiently unpacked to be comprehensible to students of a certain age. Two examples supporting this proposition is Nathan's and, to a lesser degree, Nicole's explanation for the quotient. Both of these PSTs drew on the idea of relative units and successfully explained the fractional part of the quotient in division  $2 \div \frac{3}{4}$ . However, their explanations, and

particularly Nathan's, were not sufficiently unpacked. This suggests that PSTs might need to be sensitized to the importance of providing more unpacked explanations and be afforded opportunities to practice providing such explanations.

*Propositions Related to the Practice of Using Representations*

*B<sub>1</sub>. PSTs' knowledge supports them in selecting appropriate representations and using them appropriately when explaining mathematical ideas/procedures.* The cross-analysis of the seven cases showed differences in the representations of the PSTs explanations. Kimberley, for example, referred to a gamut of representations upon which she could draw to provide an explanation. In contrast, Tiffany and Deborah argued that they associated fractions merely with "pies" (i.e., circular representations). Deborah even used a circular drawing when representing yards. In addition to selecting appropriate representations, the PSTs differed in how appropriately they used these representations in their explanations. Vonda and Tiffany, for example, represented the dividend and the divisor in their representations without maintaining their relative size. Although Tiffany corrected her representation, when she had difficulties explaining the fractional part of the quotient, she entertained the idea of reverting to her initial representation in which she represented the dividend and the divisor inappropriately.

*B<sub>2</sub>. A profound knowledge base allows PSTs to use representations in a conceptually driven manner (i.e., as tools for explaining an algorithm) rather than in a procedurally driven fashion (as means for illustrating the numbers involved in an algorithm).* The seven PSTs differed in how they capitalized on the representations they used to explain the quotient and the invert-and-multiply algorithm in the division problem  $2 \div \frac{3}{4}$ . When their understanding of the content was thorough enough, they used these

representations in a conceptually driven manner, namely the representations were vehicles to exemplify the underlying meaning of the procedure/algorithm at hand. In contrast, lacking a knowledge base that would allow for a more conceptually driven explanation, the PSTs used their representations to simply illustrate the numbers involved in the procedure/algorithm. Two examples of this latter situation are how Deborah used the representation to show the quotient and how Nicole attempted to explain the reciprocal. In the first example, Deborah represented the dividend and the divisor as two totally distinct entities having no relationship. In the second example, lacking a substantive understanding of the concept of reciprocal, Nicole attempted to show four thirds, by first dividing her drawing into thirds and by then partitioning the resulting thirds into fourths. In contrast, because of her strong understanding of the fractional part of the quotient, her pertinent explanation for the quotient was conceptually driven.

*B<sub>3</sub>. PSTs' knowledge alone does not suffice to build connections among different types of representations (e.g., drawings, mathematical sentences, word problems); PSTs need to be sensitized to the importance of building such connections in order to pursue them in their teaching.* Even the PSTs who provided conceptually driven explanations (e.g., Nathan, Nicole) were not very successful at building connections among different types of representations. Additionally, even the PSTs who noticed the lack of such connections in the virtual lesson (e.g., Kimberley, Tiffany) merely focused on the absence of such connections with respect to the final answer of a given procedure but did not pay equal attention to the absence of these connections for the other numbers involved in this procedure. For some of the study participants (e.g., Nicole, Vonda) the mere fact that such representations were used in the virtual lesson seemed to suffice to

scaffold student understanding. These results suggest that PSTs need to be sensitized to the importance of making such connections when teaching; they also need to be afforded opportunities to practice building such connections in their teaching.

*Propositions Related to the Practice of Analyzing Students' Work and Contributions*

*C<sub>1</sub>. A strong knowledge base allows for appropriate and quick analysis of conventional student ideas.* Without exception, the PSTs with a strong knowledge background (e.g., Nathan, Nicole, and Kimberley) appropriately and quickly analyzed Robert's and Ann's conventional solutions. In contrast, the PSTs with weaker knowledge encountered difficulties analyzing even the conventional student solutions. Recall, for example, that Suzanne analyzed Ann's work inappropriately and Vonda could not decide if Robert's work was correct. Although the interview data revealed that more often than not the PSTs with stronger knowledge appropriately analyzed both the conventional and the unconventional student ideas, I avoided generalizing this finding because none of the seven PSTs correctly answered question 10 of the LMT test, which focused on another unconventional student approach to fraction division. This result suggests that regardless of its depth, PSTs' knowledge can support them in understanding certain unconventional solutions, but not others that deviate remarkably from the norm (such as that tapped by question 10). The next proposition outlines a way in which PSTs with different levels of knowledge might differ in their analysis of unconventional solutions.

*C<sub>2</sub>. A strong and connected knowledge base offers PSTs resources upon which to draw when exploring unconventional or flawed student ideas; such resources include other related concepts or strategies/skills (e.g., working with an easier example, testing an idea in a different set of numbers).* Although exploring unconventional student ideas

was harder than exploring conventional ones for all seven PSTs, the PSTs with stronger (and more connected) knowledge capitalized on other resources when exploring the appropriateness of unconventional student ideas. Recall, for example, that in considering Amanda and Julia's answer "4 remainder  $\frac{1}{2}$ ," Nathan drew on his understanding of the concept of the remainder and used an easier example from the set of whole numbers to explore whether the girls' solution was correct. Contrast this performance to Tiffany's work on analyzing Michelle's unconventional solution to the problem  $2\frac{3}{4} \div \frac{3}{4}$ . Because of her own difficulties with transferring the idea of whole-number division to the division of fractions, Tiffany initially focused on superficial characteristics of Michelle's work (e.g., how Michelle colored the lines corresponding to the dividend). Projecting her own difficulties to explaining Michelle's work, Tiffany then decided that this student's solution was wrong because (a) the student did not use the reciprocal and (b) she used the idea of whole-number division although she was working on fractions.

*C<sub>3</sub>. When lacking a strong knowledge base, PSTs may analyze students' work by pursuing a number-driven approach, namely by using the numbers involved in an algorithm (and particularly the final answer) as yardsticks against which to evaluate students' work and ideas.* Differences were identified in how the seven PSTs analyzed students' work and contributions. When considering the three student solutions, Nathan and Nicole, for example, first attempted to make sense of the students' work instead of first figuring out the answer to the division problem and then evaluating the student response against that answer. In contrast, other PSTs, such as Tiffany, first figured out the answer to the division problem under consideration and *then* tried to make sense of the students' work. When confronted with difficulties in following Michelle's work, Deborah

also figured out the answer to the division problem, hoping that this answer would allow her to understand this student's answer. Vonda and Suzanne also evaluated Ann's solution by comparing it to Robert's work, which represented an appropriate application of the invert-and-multiply algorithm.

*C<sub>4</sub>. A strong understanding of the content does not suffice to identify latent or overt misconceptions in students' contributions; instead, familiarity with misconceptions renders PSTs more perceptive to misconceptions in students' work.* The most telling piece of evidence regarding this proposition stems from the PSTs' performance in identifying the misconception in June's explanation. The cross-analysis of the seven cases suggests that Kimberley, who was relatively strong on the content, not only failed to identify this misconception, but also appeared to hold the same misconception targeted in the teaching episode. In contrast, the PSTs who were aware that the dividend can also be smaller than the divisor (e.g., Tiffany) were quick to identify the misconception in June's explanation. To the extent that this proposition holds, it suggests that PSTs need to be familiarized with frequently observed student misconceptions, if they are to identify and address them in their teaching.

*C<sub>5</sub>. PSTs' knowledge alone does not suffice to make accurate assertions about students' understanding or lack thereof; PSTs need opportunities to practice making such assertions.* Although both Nathan and Nicole made appropriate assertions about students' understanding based on their analysis of the students' work, other PSTs who appropriately analyzed students' solutions did not make similarly accurate assertions. For instance, Deborah, Kimberley, and Tiffany, all analyzed Robert's and Ann's solutions appropriately; yet they overstated students' understanding. This suggests that PSTs might

need guidance and opportunities to offer precise and evidence-based assertions about student understanding.

*Propositions Related to the Practice of Selecting and Using Tasks*

*D<sub>1</sub>. PSTs' knowledge supports them in identifying the mathematical affordances of the curriculum tasks.* The cross-analysis of the seven PSTs' performance in selecting and using tasks showed that the PSTs with deeper knowledge were more successful at identifying the mathematical affordances of the two textbook pages, particularly the second one. Recall that both Nathan and Nicole saw value in the second textbook page's direction that asked students to explain the fractional part of their answers. They also talked about the importance of having students explain their answers in words and diagrams. The other five PSTs either did not discuss these affordances or explicitly mentioned that they could not understand the directions on the second page. An example is Deborah, who acknowledged her struggles understanding the direction regarding the fractional part of a quotient. Tiffany's analysis of the two textbook pages also corroborates this argument. When considering the first page, Tiffany mostly focused on the layout; when discussing the second page she valued the real-life scenarios given on this page but did not attend to other features, which were more substantive for teaching fraction division.

*D<sub>2</sub>. A strong knowledge base enables PSTs to identify ways to restructure curriculum tasks that do not satisfactorily promote understanding of the ideas under consideration.* Of the seven PSTs considered in the previous section, only Nathan, whose understanding of the content was more robust than that of his counterparts, considered ways to restructure the 16 exercises of the first page. In contrast to all of the other PSTs

who merely approached these exercises from a procedural perspective, Nathan grouped these exercises according to more conceptual characteristics (e.g., whether the dividend was larger than the divisor; if the dividend and the divisor corresponded to similar fractions). Deborah and Tiffany also wanted to approach these exercises from a more conceptual perspective; however, limitations in their knowledge prevented them from doing so. Recall, for example, Deborah's argument that each of these exercises was of equal difficulty, because for her, they all merely represented computational exercises. In addition to the PSTs' performance in restructuring the 16 exercises, this proposition was also informed by the PSTs' work on the two word problems of the first page. While Nathan and Nicole built connections between these problems and (fraction) division, the PSTs with a weaker knowledge base (e.g., Deborah, Suzanne, Tiffany, and Vonda) could not do so.

*D<sub>3</sub>. Other factors beyond knowledge inform PSTs' task presentation and enactment of tasks; these include their images of teaching, their beliefs, and their own difficulties with the content.* Although PSTs' knowledge can support them in identifying the affordances of certain curriculum tasks and in restructuring other tasks, this knowledge alone is insufficient to inform PSTs' task presentation and enactment decisions. For example, Kimberley noticed that in fraction division the quotient might be larger than the dividend. Although this idea could support her in presenting and enacting the tasks of the first textbook page in a more conceptually oriented manner than that implied by the first page itself, she did not build on this idea to restructure the page's tasks. Instead, her proposed plan for enacting the first page largely drew on the procedural aspects of this page. On the other hand, when considering how the virtual



teacher presented the first task to her students, Deborah noticed that the teacher's approach was devoid of meaning. Building on her own struggles with grasping the content, she argued that if she were introducing the first task, she would have built more connections to the word problem to help her students understand the pertinent ideas. In the same vein, because of the difficulties she had encountered as a student of mathematics, Tiffany would have brought ribbon to give her students the opportunity to deal with something tangible rather than simply manipulating numbers.

*D<sub>4</sub>. Although strong knowledge could help maintain focus on important mathematical ideas during task presentation and enactment, it might not be adequate for maintaining the tasks' cognitive demand; instead, PSTs need to be sensitized to issues of maintaining the tasks' cognitive demand during task presentation and enactment to start considering such issues in their instruction.* The cross-case analysis of the seven PSTs also suggests that a strong mathematical knowledge can help them maintain emphasis on important mathematical ideas during task presentation and enactment. Yet, this knowledge does not ensure that PSTs will necessarily attend to issues of maintaining emphasis on meaning and understanding when presenting and enacting tasks (hence the "could" modifier in the proposition). For example, although both Nicole and Nathan attended to such issues – recall their critique of the virtual teacher's enactment of task D – Kimberley who was also relatively strong on the content was not equally concerned with issues of making meaning. On the other hand, despite her difficulties with the content, Tiffany was particularly attentive to whether the virtual teacher's moves and decisions shifted emphasis from meaning and understanding to manipulating numbers. The cross-analysis of the seven PSTs also showed that, regardless of the depth of their

knowledge and irrespective of their attention to issues of meaning-making, the seven PSTs were not equally attentive to issues of maintaining the tasks' cognitive demands. Most of them did not attend to how the teacher in several instances took over and did most of the thinking for her students. Additionally, even participants who attended to issues of meaning-making, were inclined to "give more" to students (e.g., Nicole). Taken together, these findings suggest that although a strong knowledge base might sensitize PSTs to the importance of maintaining emphasis on meaning and understanding it cannot equally alert them to maintaining the tasks' cognitive demand during presentation and enactment. Hence, it is reasonable to argue that, regardless of the depth of their knowledge, PSTs need to be familiarized with issues pertaining to task complexity and discuss strategies to sustain the tasks' cognitive level and demand during instruction.

*Propositions Related to the Practice of Responding to Students' Requests for Help*

*E<sub>1</sub>. Knowledge of the content could support PSTs' identification and targeting of key mathematical ideas (in more than one way) when addressing students' requests for help.* When asked to respond to students' requests for help, the PSTs did so in mathematically valid ways when they were knowledgeable of the content under consideration. Their knowledge of the content allowed them to target key mathematical ideas in their responses. For example, in considering both episodes designed for this practice, Nathan identified and addressed key mathematical ideas in his responses in more than one way. Similarly, Nicole came up with two mathematically valid ways in which she could address the students' difficulty with the idea of common multiples when solving the second task. In contrast, the PSTs whose understanding of the content was not solid enough did not respond to the students' requests for help in ways that could help

students construe the mathematical ideas under consideration. For example, Deborah acknowledged that she was “clueless” as to how to respond to June’s and Shaun’s questions. Similarly, Tiffany proposed a pedagogically appropriate yet mathematically imprecise response to address the two students’ questions.

Although necessary, knowledge is insufficient to target key mathematical ideas when responding to students’ requests for help, as Kimberley’s performance in the second episode designed for this practice suggests; yet, PSTs’ understanding of the content is necessary for at least *identifying* the key mathematical ideas at stake in a given teaching episode. Suzanne’s work on this same task also suggests that PSTs can effectively respond to students’ questions not only if they possess the necessary knowledge, but also if they construct it while they struggle with the same ideas that students might wrestle with when working on a task. In fact, Suzanne’s work on the second task appears to imply that such struggles on the part of the teacher are productive for identifying the key mathematical ideas to be addressed and, consequently, unpacking these ideas to support student learning.

*E<sub>2</sub>. The manner in which PSTs respond to students’ requests for help is contingent not only on their knowledge but also on their images of teaching, their beliefs about teaching and learning mathematics, and their analysis of the instructional situation at hand.* The cross-case analysis pointed to at least two factors which, besides knowledge, appear to have informed the PSTs’ responses to students’ requests for help. These factors were the PSTs’ images of teaching and their beliefs about what it means to teach and learn mathematics and what the role of the teacher is when students struggle with the content. Nicole’s case also introduced considerations of other contextual factors. It

suggested that in deciding how to respond to students' requests for help, teachers might engage in a cost-benefit analysis of the instructional situation at hand; such an analysis can entertain and weigh different factors (e.g., time constraints, student familiarity with the content).

The 18 aforementioned propositions should be considered as tentative hypotheses that warrant further investigation. In the next chapter, I revisit these propositions in light of the findings pertaining to the changes in the PSTs' knowledge and teaching performance.

### Summary

This chapter has presented quantitative and qualitative findings related to the study's first research question, which explored the association between PSTs' MKT and their teaching performance from a static viewpoint. The qualitative analysis showed positive moderate correlations between the PSTs' MKT and their performance in the five teaching practices; these correlations were mainly due to the PSTs' performance in the performing tasks of the teaching simulation. Four factors were found to mediate the association between the PSTs' overall MKT and their overall teaching performance: the number of the math content courses the PSTs had taken in high school, the number of the math methods courses they had taken during their undergraduate studies, their GRE-quantitative test score, and their efficacy beliefs about understanding and working on fractions. The association of interest was robust to the former two factors, but not the latter two factors. Yet, the association between the PSTs' MKT and their performance on the *performing* tasks was robust to all the mediating factors the study explored.

Seven PSTs representing four convergent and three divergent cases were then scrutinized to better understand the association at hand. The cross-case analysis of these PSTs' performance in the teaching simulation helped develop five classification schemes representing PSTs' performance in the five practices. The classification of the seven PSTs in these schemes corroborated the findings of the quantitative analysis since it showed the PSTs' performance in the five practices to largely be aligned with their knowledge. Additionally, the cross-case analysis pointed to a duet of factors (i.e., PSTs' images of teaching and their beliefs) which, besides knowledge, appeared to inform the PSTs' instructional decisions and actions. To some extent, these factors seemed to compensate for limitations in the PSTs' knowledge but only for the tasks of noticing and interpreting-evaluating; they did not appear to scaffold the PSTs in the performing tasks of the teaching simulation. Looking across the seven PSTs' performance in the teaching simulation several patterns were also identified that helped form 18 propositions about the ways in which PSTs' knowledge (or lack thereof) appears to manifest itself in PSTs' teaching performance; I presented these propositions as hypotheses warranting further investigation.

PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING  
AND THEIR PERFORMANCE IN SELECTED TEACHING PRACTICES:  
EXPLORING A COMPLEX RELATIONSHIP

(VOLUME II)

by

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## CHAPTER 5

### EXPLORING ASSOCIATIONS BETWEEN MKT AND TEACHING PRACTICES FROM A DYNAMIC VIEWPOINT: RESULTS

#### Overview

In the previous chapter, I presented results pertaining to the exploration of the association between the PSTs' MKT and their performance in the five teaching practices from a static viewpoint, hence addressing the study's first research question. In this chapter, I take up the second research question and explore the aforesaid association from a dynamic perspective by considering changes in the PSTs' MKT and performance.

Following a structure similar to that of the previous chapter, this chapter is organized in three sections. The first section addresses the first two subordinate questions of the second research question (SRQ 2.1 and SRQ 2.2) from a quantitative perspective; namely, it presents the results of the quantitative analyses pertaining to exploring the direction, strength, and robustness of the association of interest using the measures of change reported in Chapter 3. The second section presents a within-case analysis of the changes in the performance of seven PSTs. Building on the results of this within-case analysis, in the third section, I look across the seven cases and identify patterns of changes in the PSTs' knowledge and performance. In light of this analysis, I also reconsider the 18 propositions outlined in the previous chapter about the manifestations of the PSTs' knowledge in their teaching performance. Together, the second and the third sections provide further evidence regarding the association under consideration (SRQ

2.1) and help address the third subordinate question (SRQ2.3) pertaining to how changes in the PSTs' MKT appear to play out in changes in their teaching performance. By also exploring other factors that appeared to inform the PSTs' teaching performance, these two sections additionally address the second subordinate question (SRQ 2.2).

#### Exploring the Association between the Changes in the PSTs' MKT and the Changes in Their Performance in the Five Teaching Practices from a Quantitative Standpoint

This section consists of two parts that correspond to SRQ 2.1 and SRQ 2.2. In the first part, I consider changes in the PSTs' knowledge and teaching performance. I then explore the direction and strength of the associations between the changes in the PSTs' MKT and the changes in their performance in the five teaching practices as a whole and separately for each of the two main sets of practices (MKT/MTF-related practices) (SRQ 2.1). Exploring associations for MKT- and MTF-related practices separately was necessitated by the nature of the intervention which, in addition to advancing the PSTs' MKT, also aimed at honing their performance in the MKT-related practices. Addressing SRQ2.2, the second part first outlines the changes in the PSTs' beliefs and perceived importance of a set of instructional goals. It then explores whether these changes, alongside the PSTs' background characteristics, mediated the aforesaid associations.

#### *Exploring the Direction and Strength of the Associations between the Changes in the PSTs' MKT and the Changes in Their Teaching Performance*

##### *A First Glance at Changes in the PSTs' MKT and Teaching Performance*

The means and standard deviations of the changes in the PSTs' performance on the LMT test and the teaching simulation are reported in Table 5.1. Including no negative



values, this table shows that the study participants experienced *gains* in all the measures under consideration. All these gains were statistically significant, as suggested by the use of the *Wilcoxon Signed ranks test* for paired samples.<sup>164</sup> The gains reported in Table 5.1 varied across the two components of knowledge and the components of teaching performance examined in the study. However, a direct comparison is legitimate only across the three components of the five teaching practices (i.e., noticing, interpreting-evaluating, and performing) in which a common scale was used. Consider for instance, the gains in the PSTs' *noticing* performance across the five practices. Table 5.1 suggests that the PSTs experienced the greatest *noticing* gains in the practice of responding to students' requests for help, and the lowest gains in the practice of providing explanations. The same pattern was true for the PSTs' gains in the skill of interpreting and evaluating the virtual teacher's decisions and actions; in addition, in this skill, the changes in the PSTs' performance in both the MTF-related practices were higher than those in their performance in the three MKT-related practices. In contrast, for the performing skill, this pattern was reversed: the study participants exhibited the greatest gains in the practices of providing explanations and using representations and the lowest gains in the two MTF-related practices (i.e., selecting and using tasks and responding to students' requests for help). The use of *Kendall's W* test for independent samples showed that all aforementioned differences were statistically significant, at least at level 0.10 (noticing: Kendall's  $W = 0.16$ ,  $p < .05$ ; interpreting-evaluating: Kendall's  $W = 0.13$ ,  $p < .10$ ; and performing: Kendall's  $W = 0.18$ ,  $p < .05$ ).

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<sup>164</sup> The use of *Wilcoxon signed ranks test* for paired samples showed that in *actual* values all the pre-and post-intervention changes were statistically significant at level  $p = 0.01$  or lower (the values of Wilcoxon  $Z$  ranged from 2.99 to 3.54).

Table 5.1

*Sample Means and Standard Deviations of the PSTs' Gains in MKT and Teaching Performance*

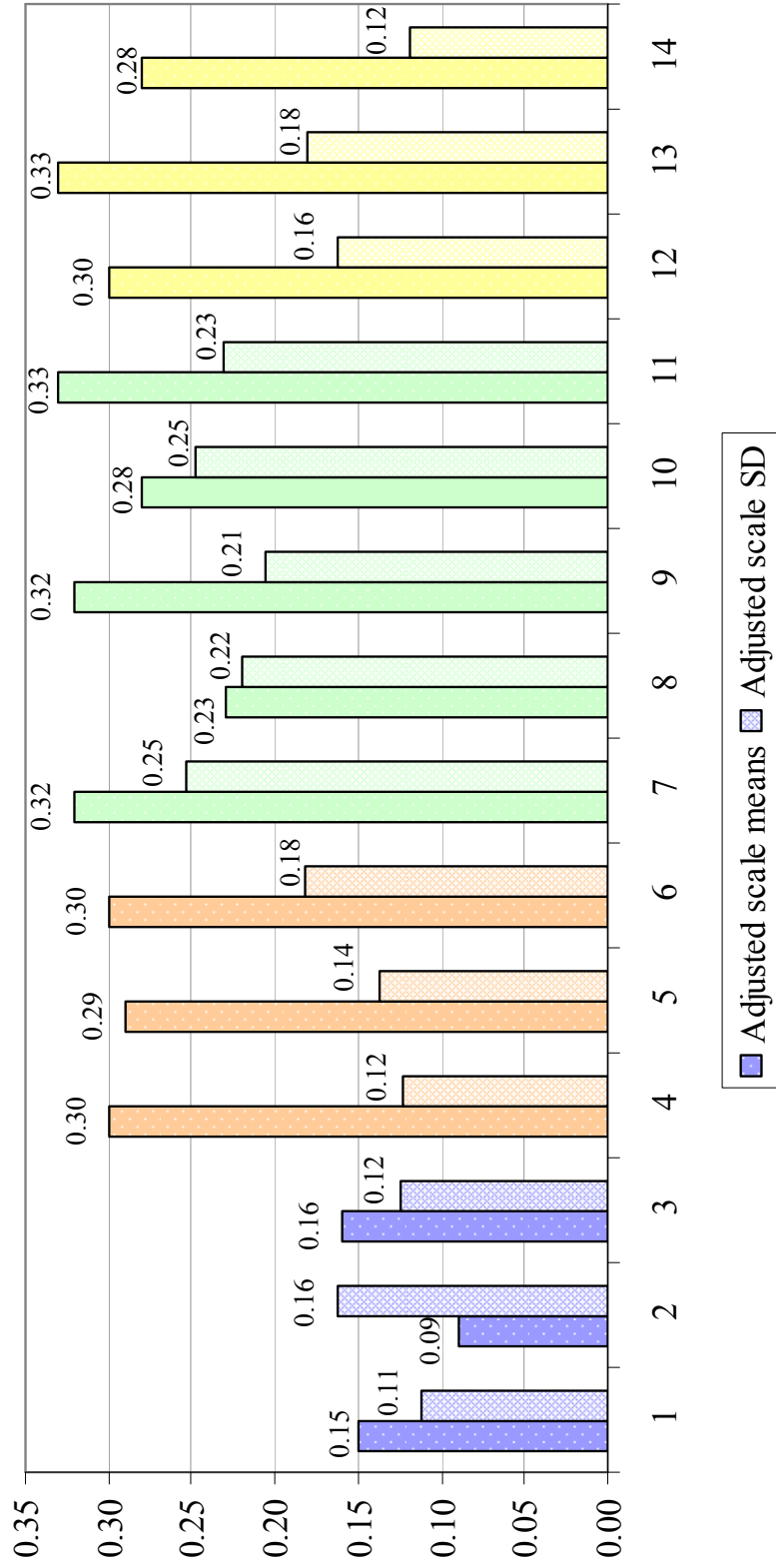
<b>Measure</b>	<b>M*</b>	<b>SD</b>
<i>Gains in MKT</i>		
Overall MKT **	6.31	4.59
CCK	0.44	0.81
SCK	5.88	4.47
<i>Gains in teaching performance</i>		
Practices (overall)	10.38	4.32
MKT-related practices	6.13	2.87
MTF-related practices	4.25	2.54
Using representations	2.25	1.77
Noticing	0.44	0.73
Interpreting-evaluating	0.63	0.81
Performing	1.19	0.98
Providing explanations	1.63	1.54
Noticing	0.25	0.58
Interpreting-evaluating	0.25	0.77
Performing	1.13	0.72
Analyzing students' work/contributions	2.25	1.44
Noticing	0.75	0.86
Interpreting-evaluating	0.63	0.50
Performing	0.88	0.62
Selecting and using tasks	1.94	1.73
Noticing	0.50	0.82
Interpreting-evaluating	0.81	0.98
Performing	0.63	0.50
Responding to students' requests for help	2.31	1.62
Noticing	1.06	0.68
Interpreting-evaluating	0.94	0.77
Performing	0.31	0.79
Skills (overall)	10.38	4.32
Noticing	3.00	1.63
Interpreting-evaluating	3.25	1.81
Performing	4.13	1.78

*Notes.* \* n=16; \*\*Given that theoretically the study participants could have received the highest score on one of the two administrations of the data collection instrument and the lowest score on the other, the range of the scale of each measure reported in this table is  $\pm$  the corresponding maximum score shown in Table 4.1.

To facilitate comparability of the other figures shown in Table 5.1, the descriptive statistics reported for the gains in the PSTs' knowledge and the aggregated gains in their teaching performance (per practice and skill) were adjusted to a zero-to-one scale. The means and standard deviations of these adjusted scores appear in Figure 5.1.

The first immediate observation from this figure is that the gains in the PSTs' performance in the teaching simulation outweighed those in their MKT. Specifically, on a zero-to-one scale, the average increase in the PSTs' MKT performance was 0.15 and twice as large for their overall teaching performance (see columns 1 and 4); this difference was statistically significant (Wilcoxon  $Z= 3.46$ ,  $p < 0.01$ ). Even though the PSTs gained more in their SCK (0.16) than in their CCK (0.09), this difference was not statistically significant (Wilcoxon  $Z= 1.17$ ,  $p = 0.24$ ), a result that can be attributed to the comparatively high standard deviation of the change in the PSTs' CCK.

This figure also suggests that the PSTs exhibited comparable gains in their performance in the MKT- and the MTF-related practices (see columns 5 and 6). They exhibited greater performance gains in the practices of using representations (0.32), analyzing students' work and contributions (0.32), and responding to students' requests for help (0.33), and lower gains in the practices of selecting and using tasks (0.28) and providing explanations (0.23); yet, these differences were not statistically significant (Kendall's  $W= 0.03$ ,  $df = 4$ ,  $p= 0.77$ ). This finding should not be surprising given the two different patterns undergirding the PSTs' gains in these two sets of practices: whereas for the skills of noticing and interpreting-evaluating these PSTs gained more in the MTF-related practices, this pattern was reversed for the performing skill.



**Legend.** 1. MKT (overall); 2. CCK; 3. SCK; 4. Teaching performance (overall); 5. Performance in the MKT-related practices; 6. Performance in the MTF-related practices; 7. Using representations; 8. Providing explanations; 9. Analyzing students' work and contributions; 10. Selecting and using tasks; 11. Responding to students' direct or indirect requests for help; 12. Noticing (overall); 13. Interpreting-evaluating (overall); 14. Performing (overall).

Figure 5.1. Changes in the PSTs' MKT and their teaching performance based on a zero-to-one scale (n=16).

The study participants' gains in the three skills under consideration ranged from 0.28 to 0.33, with the greatest gains observed for the skill of interpreting and evaluating the virtual teacher's actions and the lowest gains for the skill of performing; however, these differences were again not statistically significant (Kendall's  $W= 0.08$ ,  $df= 2$ ,  $p= 0.27$ ), a finding that again can be attributed to the two different patterns just discussed.

#### *Exploring Associations between Changes in the PSTs' MKT and Teaching Performance*

A necessary step before investigating associations between changes in the PSTs' MKT and changes in their teaching performance was to examine if certain cases were outliers or influential observations and hence needed to be removed. These diagnostic tests suggested that none of the 16 PSTs considered for these analyses was an outlier.<sup>165</sup> However, two PSTs, Deborah and Suzanne, constituted influential observations for several of the associations explored, with Deborah *inflating* several of the correlation coefficients and Suzanne *deflating* others;<sup>166</sup> hence, these two cases were dropped from subsequent analyses. Consequently, the results reported below are based on only 14 PSTs. The *Spearman rank-order correlation coefficients* ( $r_s$ ) between the measures of change in the PSTs' knowledge and the measures of change in their teaching performance are presented in Figure 5.2. The reader is notified that the value of these

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<sup>165</sup> An approach similar to that described in Chapter 4 was pursued to identify outliers. Because a smaller sample was considered for these analyses ( $n=16$ ), a smaller cut-off Bonferroni critical value was utilized (i.e., 3.44) to determine whether any of the study participants was an outlier. None of the studentized residuals of all regressions of the measures of change in teaching performance on the measures of change in knowledge exceeded the value of 3.44, thus suggesting that none of the study participants was an outlier.

<sup>166</sup> For this exploration, I used the highest values of the *Cook's distance* statistic yielded from regressing the measures of change in the PSTs' teaching performance on the measures of change in their knowledge. In most cases, these highest values were associated with Deborah or Suzanne. After excluding Deborah, only 3 out of the 15 statistically significant associations remained significant. In contrast, excluding Suzanne resulted in 11 non-significant associations becoming significant, two significant associations becoming non-significant, and ten associations changing their level of significance. All these changes suggested that both Deborah and Suzanne should be excluded from subsequent analyses.

coefficients depends on the number of the cases considered.<sup>167</sup> This means that the correlations reported in Figure 5.2 cannot be directly compared with those reported in Figure 4.2 since in the latter figure the correlations were based on 19 instead of 14 PSTs. This difference is not negligible for small samples such that used in this study.

Figure 5.2 shows that there was a positive marked correlation between the changes in the PSTs' *overall* MKT and the changes in their teaching performance ( $r_s=0.66$ ). This correlation was mainly due to their gains in the MKT-related practices rather than to those observed in the MTF-related practices: while the correlation between the gains in the PSTs' MKT and the gains in their performance in the MKT-related practices was high ( $r_s=0.85$ ), it was negligible for the MTF-related practices ( $r_s=0.17$ ). This finding resonates not only with the focus of the intervention under consideration but also with the nature of the knowledge examined in the study (i.e., as explained in Chapters 2 and 3, this type of knowledge is more aligned with the former rather than with the latter set of practices). From the three MKT-related practices examined in the study, only the changes in the PSTs' performance in the practice of providing explanations were moderately related to the changes in their MKT ( $r_s=0.50$ ); for the other two practices these correlations were positive but low. Yet, the PSTs' MKT gains were markedly correlated with their gains in the *performing* tasks of the teaching simulation for both the composite of the three MKT-related practices ( $r_s=0.73$ ) and for each of the practices of using representations ( $r_s=0.66$ ) and providing explanations ( $r_s=0.49$ ) separately. In

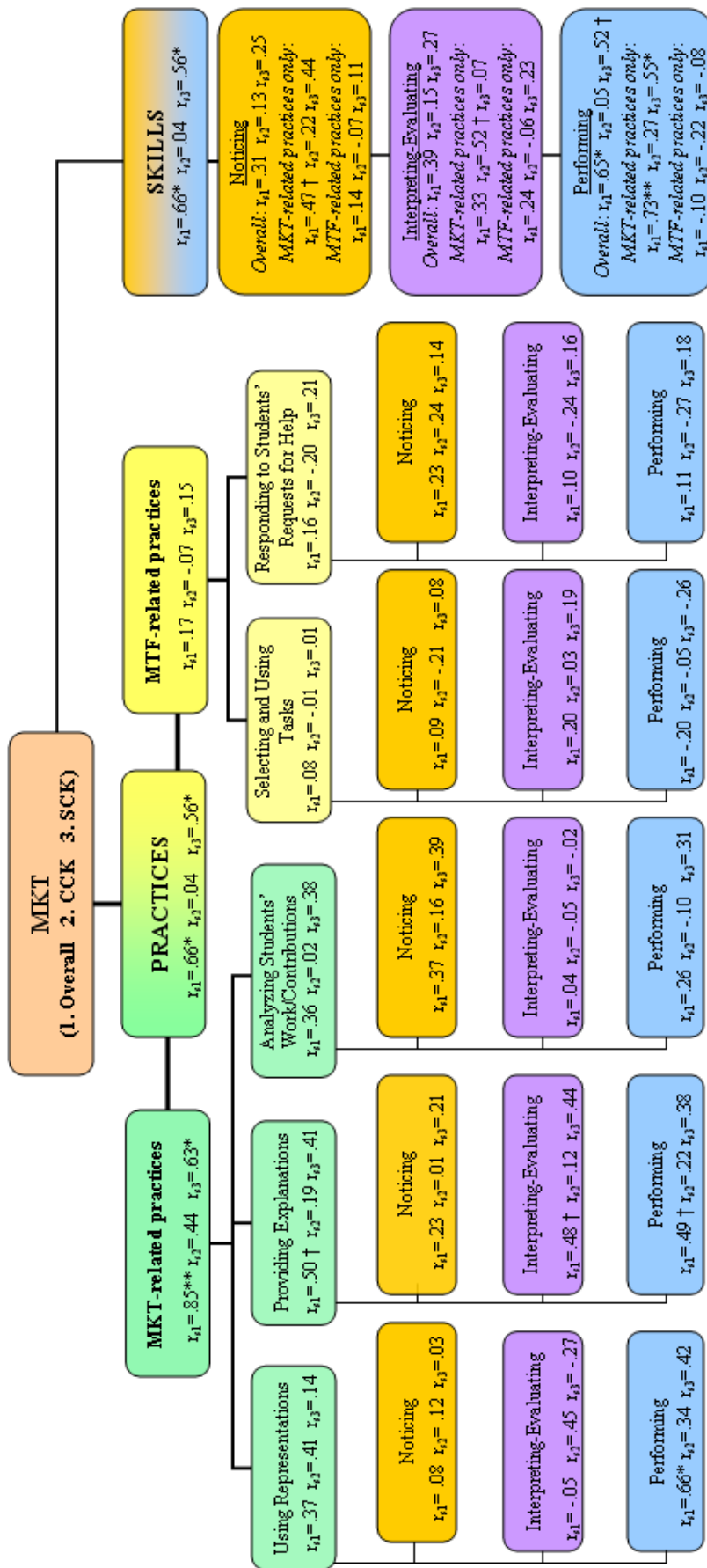
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<sup>167</sup> The Spearman coefficient is calculated using the formula  $r_s = 1 - \frac{6 \sum_{i=1}^N d_i^2}{N^3 - N}$ , where N represents the number of cases/participants. The difference ( $d_i$ ) represents the disparity between the rankings of each participant in each of the two focal measures (the two focal measures in this analysis were the changes in the PSTs' MKT and the changes in their teaching performance).

contrast, no significant correlations were observed between the PSTs' MKT gains and their performance gains in either of the MTF-related practices, even when considering only the performing tasks.

Figure 5.2 also shows that in most cases the correlations between the gains in the PSTs' knowledge and their teaching performance were stronger for the SCK rather than the CCK. Specifically, the correlation coefficients for SCK were higher than those for CCK in 26 out of the 32 distinct entry points considered in Figure 5.2. The gains in the PSTs' SCK were moderately associated with the gains in their overall performance ( $r_s=0.56$ ), their performance in the MKT-related practices ( $r_s=0.63$ ), their performance in all the performing tasks of the teaching simulation ( $r_s=0.52$ ), and their performance in the performing tasks of the MKT-related practices ( $r_s=0.55$ ); the correlations between these measures and the gains in the PSTs' CCK were not significant.

The negative correlations reported in Figure 5.2 are also worth noticing. Although none of these correlations was significant, their direction suggests that the PSTs' who experienced higher gains in knowledge exhibited lower gains in teaching performance compared to the gains experienced by their counterparts. The most interesting among these negative correlations is that pertaining to the gains in the PSTs' SCK and the gains in their performance in the interpreting-evaluating tasks of using representations. This figure suggests that compared to their counterparts who exhibited lower gains in SCK, the PSTs who displayed greater gains were not as successful at identifying the lack of connections in the virtual teacher's and students' use of representations. The case of Kimberley considered in the next section of this chapter provides some insights into this negative association.



**Legend.** \*\*  $p < .01$ ; \*  $p < .05$ ;  $\uparrow p < .10$

Figure 5.2. Spearman's correlation coefficients for the associations between the gains in the PSTs' knowledge and the gains in their teaching performance (n=14).



Because of the nature of the intervention this study examines, I also considered the association between the changes in the PSTs' performance in the MKT-related practices and the changes in their performance in the MTF-related practices. A positive low, and hence non significant, correlation was obtained for this association ( $r_s=0.25$ ,  $p=0.39$ ; not reported in Figure 5.2), suggesting that, by and large, the PSTs who gained in the one set of practices did not necessarily exhibit analogous gains in the other set of practices.

*Exploring the Robustness of the Associations between the Changes in the PSTs' MKT and the Changes in Their Teaching Performance*

Four associations were of particular interest. Three of them pertained to the gains in the PSTs' MKT and the gains in (i) their overall teaching performance; (ii) their performance in MKT-related practices, and (iii) their performance in the MTF-related practices. The fourth association concerned the changes in the PSTs' performance in the MKT- and the MTF-related practices. Because only the first two of these associations were statistically significant, potential mediators were explored only for those two associations.

Two main sets of mediating factors were considered for this exploration. The first set pertained to the changes in the PSTs' beliefs about teaching and learning mathematics, their efficacy beliefs, and their perceived importance of five instructional goals. The second set corresponded to the PSTs' background characteristics, which were also explored when considering the first research question (see Chapter 4). The inclusion of the second set of factors in the exploration considered herein sought to help understand whether any of the PSTs' background characteristics mediated not only the association

between the PSTs' entrance knowledge and performance (as reported in Chapter 4), but also the association between the gains in their knowledge and performance. Before reporting the results of this exploration, I briefly consider the descriptive statistics of the changes in the PSTs' characteristics in the first set of factors.

#### *Changes in the PSTs' Beliefs and Perceived Importance of Instructional Goals*

Table 5.2 summarizes the reported changes in the PSTs' beliefs about teaching and learning mathematics, their efficacy beliefs, and their perceived importance of a set of instructional goals. Because two of the study participants (i.e., Deborah and Suzanne) were excluded from subsequent analyses, this table reports information for the sample of 16 PSTs for whom complete data were collected and for a subset of this sample that does not include the two aforementioned cases (in brackets). In the discussion that follows, I largely focus on the figures reported for the sample of the 16 PSTs; I consider the changes for the subsample of the 14 PSTs only when discussing the last type of efficacy beliefs, for which a substantial drop in its mean and standard deviation was observed after excluding Suzanne.

Compared to their level of agreement at the beginning of the intervention, at its culmination, the study participants tended to agree less with each of the beliefs captured by the three belief factors. Given that at the beginning of the intervention the means of the PSTs' responses to the three belief factors under consideration were 4.87, 3.33, and 4.02, respectively, the figures reported in Table 5.2 suggest that when exiting the intervention these PSTs were neutral about teaching for skill mastery and closure; their corresponding exit mean score was somewhat below 4 (i.e., 4.87-1.01), which represents the neutral point of the 1-to-7 Likert scale. They also reported disagreeing with the idea

of teaching as remembering and applying rules and formulas and the idea of minimizing complexity to support student learning and understanding; their exit mean scores on these two factors were 2.29 (i.e., 3.33-1.04) and 2.83 (i.e., 4.02-1.19), respectively. All these differences were statistically significant (Skill mastery and closure: Wilcoxon  $Z = 2.25$ ,  $p < .05$ ; Teaching as remembering and applying rules and formulas: Wilcoxon  $Z = 2.64$ ,  $p < .01$ ; Minimizing the complexity in teaching mathematics: Wilcoxon  $Z = 3.06$ ,  $p < .001$ ).

Table 5.2

*Sample Means and Standard Deviations of the Changes in the PSTs' Beliefs and Perceived Importance of Instructional Goals*

<b>Measure</b>	<b>M*</b>	<b>SD</b>
<i>Changes in beliefs about teaching and learning</i>		
Skill mastery and closure	-1.01 [-1.19]	1.47 [1.40]
Teaching and learning mathematics as remembering and applying formulas	-1.04 [-1.12]	1.32 [1.38]
Minimizing the complexity in teaching math	-1.19 [-1.23]	1.08 [1.15]
<i>Changes in perceived goal importance</i>		
Conceptual understanding	6.04 [ 7.08]	15.61 [16.13]
Procedural fluency	2.10 [ 1.86]	6.73 [ 7.04]
Strategic competence	-0.93 [-0.53]	8.97 [ 8.49]
Adaptive reasoning	-0.62 [-0.71]	9.95 [10.50]
Productive disposition	-7.21 [-8.42]	20.53 [20.48]
<i>Changes in efficacy beliefs</i>		
Understanding and operating on fractions	0.17 [ 0.18]	0.72 [0.77]
Understanding the algorithms of fraction multiplication and division	0.63 [ 0.75]	1.26 [1.27]
Engaging in MKT-related practices	0.67 [ 0.49]	0.83 [0.59]

*Notes.* \*  $n=16$  [ $n=14$ ]; \*\*Given that theoretically the study participants could have received the highest score on one of the two administrations of the instrument and the lowest score on the other, the range of the scale of each belief measure is  $\pm 6$ . The scale for the measures of the perceived goal importance is  $\pm 100$ .

A totally different pattern was observed for the changes in the PSTs' efficacy beliefs. Compared to their perceived efficacy at the beginning of the study, at the end of the intervention the PSTs' professed efficacy in engaging in each of the tasks outlined by

the three efficacy factors was higher; yet only the changes pertaining to the factors of understanding the algorithms of fraction multiplication and division and engaging in the MKT-related practices were statistically significant (Wilcoxon  $Z = 1.71$ ,  $p < .10$ ; Wilcoxon  $Z = 2.80$ ,  $p < .01$ , respectively). This finding was not surprising, given the focus of the intervention on the division of fractions (during the math content course) and on the MKT-related practices (during both courses). It is also interesting to note that after the exclusion of Deborah and Suzanne, there was a notable drop in the mean and the standard deviation of the latter factor. This was mainly due to Suzanne's reporting having experienced the largest changes in this factor compared to her classmates. I revisit this finding when considering Suzanne's case in the next section of this chapter.

Although non significant, three changes in the PSTs' perceived goal importance are noteworthy. At the end of the intervention, the study participants reported not perceiving the goal of productive disposition as highly as they did at their entrance to the program (notice, however, the large value of the standard deviation associated with this factor). Instead, they reported valuing more the goal of conceptual understanding compared to how they valued it at the beginning of the program; they also valued more the goal of procedural fluency. This latter change should not be surprising since, as discussed in Chapter 4, at the beginning of the intervention the study participants largely underestimated this goal. An activity organized in the fifth class of the math methods course during which all five goals were presented as equally important and worthwhile pursuing when teaching mathematics might have contributed to this change.

*Exploring Mediators of the Associations between the Changes in the PSTs' MKT and the Changes in Their Teaching Performance*

Table 5.3 presents the *Spearman correlation coefficients* of the associations of the potential mediating factors with the changes in the PSTs' MKT (Column I), the changes in their overall teaching performance (Column II), and the changes in their performance in the MKT-related practices (Column III). Notably, none of these factors was significantly associated with the gains in *both* the PSTs' knowledge and teaching performance. This implies that none of these factors mediated the statistically significant associations between the change in the PSTs' MKT and the changes in (i) their overall teaching performance and (ii) their performance in the MKT-related practices. Both of these associations were robust even to the PSTs' GRE-quantitative performance, which was the strongest mediator of the association between the PSTs' entrance MKT and their entrance teaching performance.

*Summary of Quantitative Results*

The exploration of the second research question from a quantitative standpoint showed a positive moderate association between the gains in the PSTs' MKT and the gains in their performance in the teaching simulation. This association was even stronger when considering only the gains in the PSTs' performance in the MKT-related practices and when focusing on the changes in their performance in the *performing* tasks of the MKT-related practices. All these associations were robust to *all* the factors examined in this study.

Table 5.3

*Exploring Mediators of the Associations between the Changes in the PSTs' MKT and Teaching Performance*

Factor <sup>2</sup>	Correlation Coefficients <sup>1</sup>		
	I <sup>3</sup>	II <sup>3</sup>	III <sup>3</sup>
<i>Changes in beliefs about teaching and learning mathematics</i>			
Skill mastery and closure	- 0.29	0.20	- 0.15
Teaching and learning math as remembering and applying formulas	- 0.21	0.02	- 0.21
Minimizing the complexity in teaching mathematics	- 0.15	- 0.05	- 0.35
<i>Changes in perceived goal importance</i>			
Conceptual understanding	0.09	0.24	0.07
Procedural fluency	0.51 †	0.45	0.26
Strategic competence	0.29	0.11	0.16
Adaptive reasoning	0.24	0.29	- 0.02
Productive disposition	- 0.40	- 0.45	- 0.16
<i>Changes in efficacy beliefs</i>			
Understanding and operating on fractions	0.25	0.45	0.43
Understanding the algorithms of fraction multiplication and division	0.16	0.40	0.47 †
Engaging in MKT-related practices with respect to fractions	0.10	- 0.11	0.45
<i>Background characteristics</i>			
High-school math content courses	- 0.07	- 0.42	0.07
Undergraduate content courses	0.38	0.28	0.35
Calculus classes	0.07	0.08	0.16
Math methods courses	0.14	- 0.12	- 0.01
Undergraduate math major	- 0.29	- 0.28	- 0.22
Undergraduate math minor	- 0.10	- 0.33	0.03
GRE-verbal performance (n=13)	0.28	0.09	0.28
GRE-quantitative performance (n=13)	- 0.02	- 0.08	- 0.20

Notes.

<sup>1</sup> I. Spearman correlation coefficient of changes in the focal factor with changes in MKT; II. Spearman correlation coefficient of the changes in the focal factor with the changes in teaching performance (overall); III. Spearman correlation coefficient of the changes in the focal factor with the changes in performance in the MKT-related practices;

<sup>2</sup> n=14 (unless otherwise specified)

<sup>3</sup> \*\* p < .01; \* p < . 05; † p < .10

## Exploring the Association between the Changes in the PSTs' MKT and the Changes in Their Performance in the Five Teaching Practices from a Qualitative Standpoint

This section aims at better understanding the association between the changes in the PSTs' knowledge and the changes in their teaching performance; it also seeks to examine whether factors other than those examined in the previous section appeared to mediate the aforesaid association. The section consists of nine parts. In the first part, I explicate why the seven cases selected for exploring the first research question were also suitable for exploring the second research question. Because the intervention was a key lever in inciting the changes reported in this chapter, in the second part, I briefly summarize the features of the intervention most pertinent to the changes reported in the remaining seven parts. Each of the subsequent parts presents a case; to facilitate readability, in presenting each case I mostly focus on notable changes in the PSTs' pre- and post-intervention performance. For each case, I start with outlining some background information and then discuss the changes in the PSTs' performance in each of the five practices under consideration. Next, I consider the changes that the PSTs themselves reported having experienced over the course of the intervention and examine the PSTs' attributions of those changes. Each case concludes with an exploration of the changes in the PSTs' performance on the LMT test and an analytical commentary. In this commentary, I synthesize the findings reported for the case and consider these findings in light of the second and third subordinate questions of RQ2.

### *The Seven Cases Selected for Exploring the Second Research Question*

Two criteria informed case selection for both research questions: that both convergent and divergent cases were considered and that the selected cases were diverse

in terms of the traits of interest. The seven cases considered in the previous chapter met both criteria, hence the same seven cases were considered for both the study's first and the second research questions.

The selected PSTs represented both convergent and divergent cases in terms of the *changes* in their MKT and the *changes* in their teaching performance, as suggested by the scatterplot outlined in Figure 5.3. This scatterplot resulted from plotting the standardized values of the changes in the PSTs' overall MKT-score against the standardized values of the changes in their overall teaching performance. Four of the selected cases represented convergent cases in that the changes in their MKT were proportional to the changes in their teaching performance (i.e., both high for Deborah and Kimberley and both low for Nicole and Nathan). The remaining three PSTs represented divergent cases since the relative changes in their teaching performance outweighed the relative changes in their MKT score (i.e., Suzanne and Vonda) or vice versa, the relative changes in their MKT outweighed those in their teaching performance (i.e., Tiffany).

With respect to the second criterion, namely case diversity, four measures of change were considered: the PSTs' MKT; their overall teaching performance; their performance in the MKT-related practices; and their performance in the MTF-related practices. As illustrated in Table 4.5, which presents the seven selected cases and the criteria considered for their selection, these PSTs represented diverse cases regarding each of the aforesaid criteria. Since no factors were found to mediate the associations between the changes in the PSTs' MKT and the changes in their teaching performance, no such factors were taken into consideration in selecting the seven cases.



Given that the intervention was designed to leverage changes in the PSTs' MKT-related practices, in selecting those cases, I also considered whether they represented different patterns of change in their MKT- and MTF-related practices. These considerations were informed by the scatterplot shown in Figure 5.4, which resulted from plotting the standardized values of the changes in the PSTs' performance in the MKT-related practices against the standardized values of the changes in their performance in the MTF-related practices. As this scatterplot shows, the selected cases were representative of all four different patterns of change in these two sets of practices. Nathan and Nicole exhibited low changes in both sets of practices and Deborah and Suzanne exhibited high changes; Tiffany and Kimberley exhibited higher changes in their performance in the MKT-related practices than in their performance in the MTF-related practices. Even more interesting was the pattern observed in Vonda's case, who exhibited higher changes in her performance in the MTF-related practices (which were not the focus of the intervention) than in her performance in the MKT-related practices.

A caveat, however, is in order here. The values of low and high changes represent *between-cases* rather than *within-cases* indices. This means that, in actual terms, a PST might have exhibited comparable changes in his or her performance in the two sets of practices. However, compared to the changes in the other PSTs' performances, the changes in the performance of this particular PST might have been considered low with respect to the first group of practices and high with respect to the second group of practices.

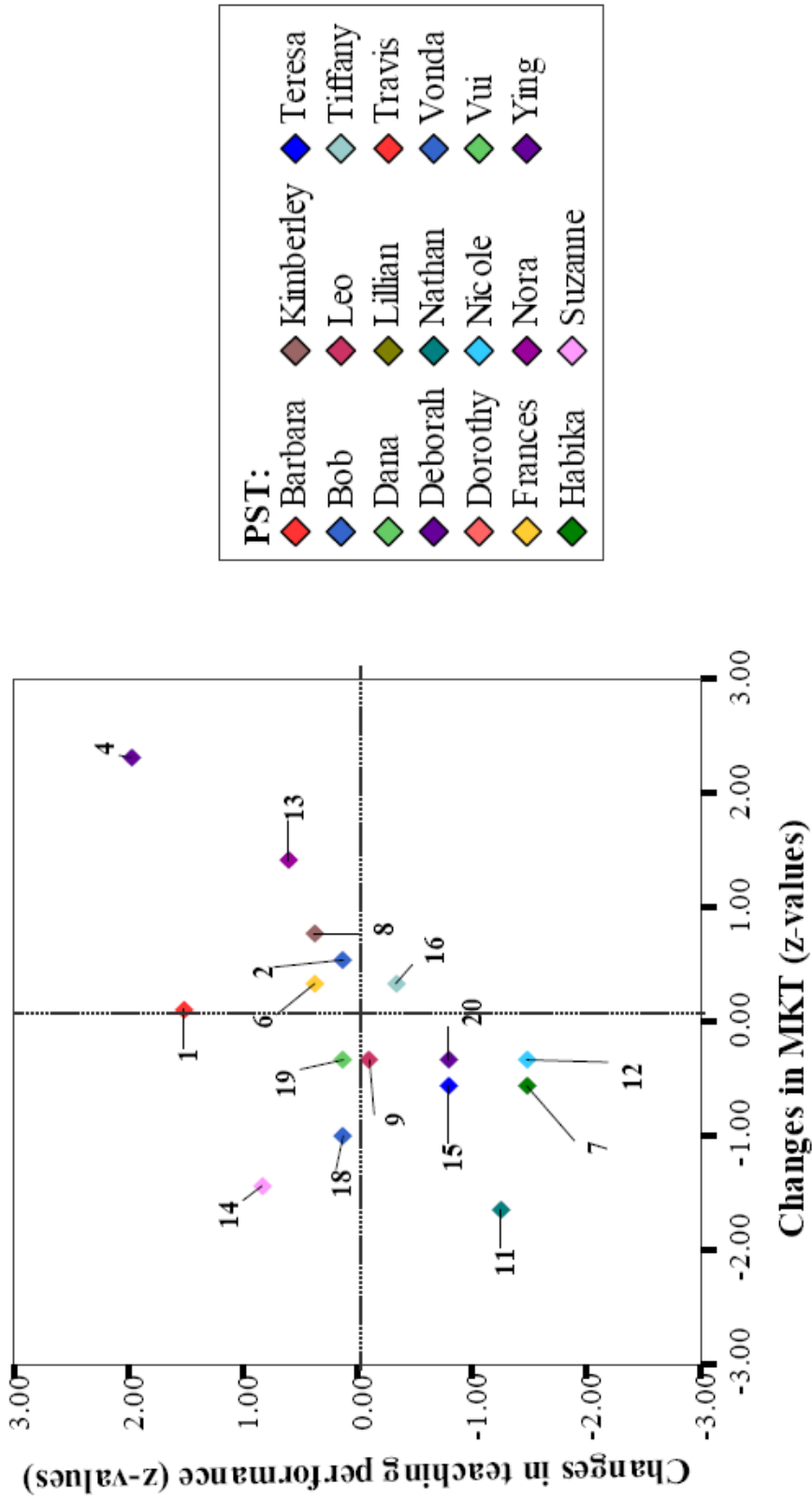


Figure 5.3. Scatterplot of the changes in the PSTs' overall MKT score against the changes in their overall teaching-performance score.

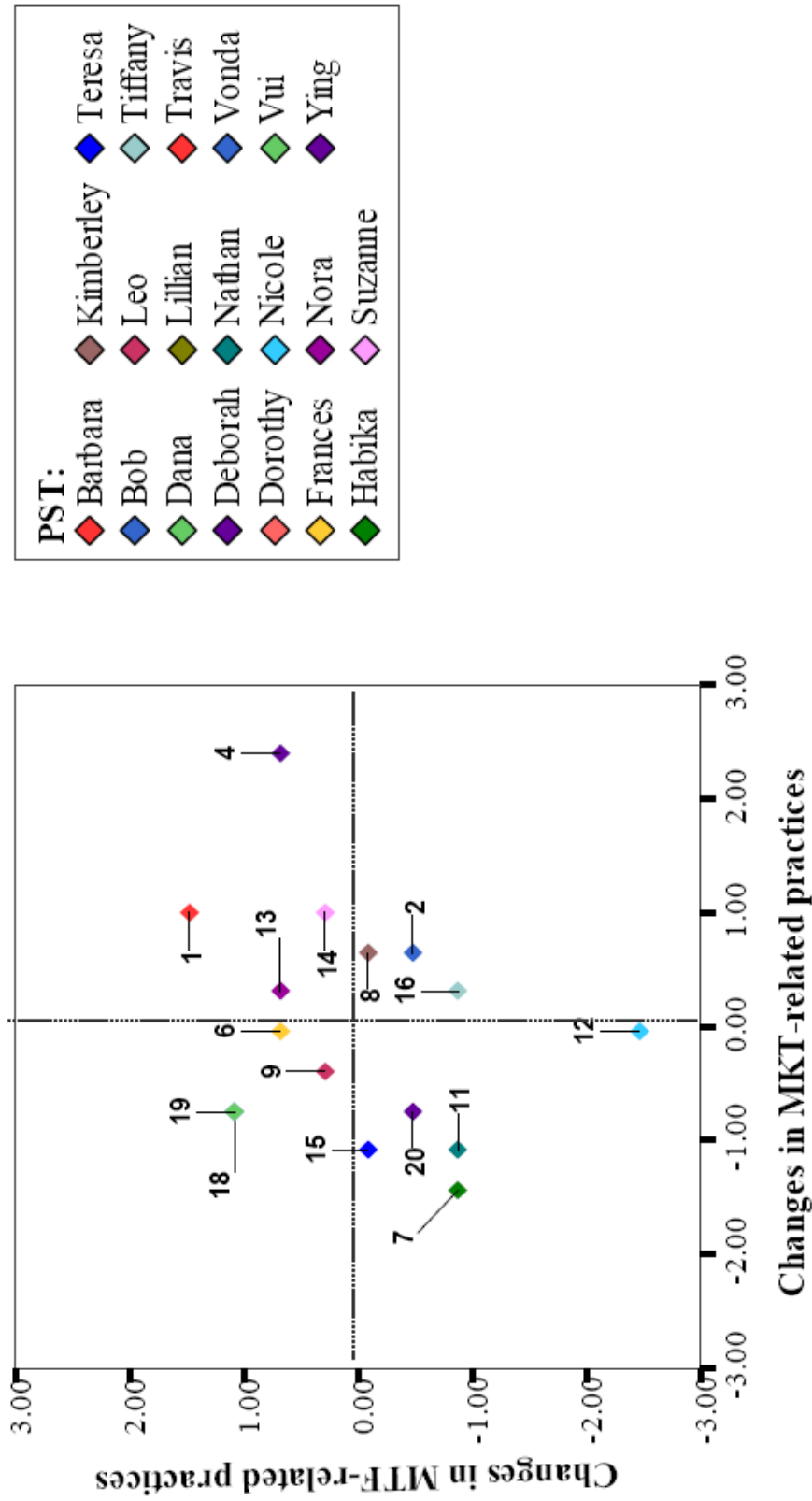


Figure 5.4. Scatterplot of the changes in the PSTs' performance in the MKT-related practices against the changes in their performance in the MTF-related practices.

### *The Intervention in a Nutshell*

Certain features of the intervention courses, both with respect to the content covered and the teaching processes introduced and practiced during these courses, can be credited the most for inducing the changes reported in the seven cases that follow.

With respect to content, the four last classes of the math content course afforded the study participants the opportunity to focus on fractions and fraction division. During these classes, the PSTs considered different representations of fractions and interpreted their answers to fraction problems using different reference units (absolute and relative units); they also posed problems to represent the division  $1\frac{3}{4} \div \frac{1}{2}$  and considered the two interpretations of division (partitive and measurement). Moreover, they interpreted the remainder in fraction division problems and explained why dividing by a half is equivalent to multiplying by two (which pertains to the idea of the reciprocal in fraction division). Content-wise, the math methods course did not address the ideas considered in the teaching simulation, with the only exception being one class that was devoted to the division of whole numbers. During this class, the PSTs were reminded of the two interpretations of division and considered the meaning of the remainders in such divisions.

The two courses afforded the study participants several opportunities to practice providing explanations, building connections among different representations (especially when explaining the algorithms of addition and subtraction of whole numbers), and analyzing others' thinking. In the math methods course, the study participants were also engaged in making assertions about student understanding of place value based on artifacts the PSTs collected from their interactions with the students in their fieldwork

placement. Although both courses emphasized the importance of attending to meaning and understanding when teaching certain procedures and algorithms, the practices of selecting and using tasks and responding to students' requests for help were minimally considered in these two courses. A more comprehensive description of the two intervention courses appears in Appendix 1.

*The Case of Nathan Revisited: What Counts as Valid Mathematical Reasoning?*

Nathan was a convergent case according to both his entrance scores and the changes in his performances. As Table 4.5 and Figure 5.3 suggest, compared to his counterparts, he experienced small changes in his performance on the LMT test and the teaching simulation. Yet, these changes should be seen in light of his high entrance scores on both the test and the teaching simulation; in fact, for the LMT test Nathan's case could signify a ceiling effect. The post-intervention interview with Nathan was conducted in early February. In the time that elapsed between the pre- and the post-intervention interviews (approximately 32 weeks), Nathan, like his classmates, had completed six courses and had spent several days in a fifth-grade classroom, where he observed and taught math lessons.

Nathan's case corroborates the strong association between the gains in the PSTs' MKT and the gains in their teaching performance (especially in the MKT-related practices) reported in the previous section in two respects. First, the small changes in his MKT score were consistent with the non dramatic changes in his teaching performance. In fact, compared to other PSTs considered in this section (e.g., Deborah), Nathan merely experienced more subtle changes in his teaching performance. Second, the enhancement of Nathan's understanding of the notion of the reciprocal was reflected in several aspects

of his performance, and particularly in the practices of selecting and using tasks, providing explanations, and using representations. At the same time, Nathan's increased attention to issues of providing unpacked explanations and building connections among different representations reflects the opportunities he was afforded during the two courses to consider such issues. Given that he was the only participant explicitly exposed to issues of maintaining the tasks' cognitive demand and complexity, his post-intervention performance also suggests that PSTs need to be afforded opportunities to contemplate such issues if they are to start attending to them in instruction.

### *Selecting and Using Tasks*

*Performing.* Nathan's post-intervention analysis of the two textbook pages was more nuanced than his pre-intervention analysis.

Like his pre-intervention comments, during the post-intervention meeting, in considering the first page, he talked about the lack of any pictures that would support students' understanding; he asserted that the worked out example at the top of the page simply presents the invert-and-multiply algorithm without explaining it; he argued that it is not necessary to assign all 16 exercises to students, especially if the teacher's goal is to help students understand the underlying meaning of fraction division; and he pointed out that the two word problems at the bottom of this page can be solved using either multiplication or division. Although he again proposed using some of the 16 exercises for problem posing, drawing on the intervention courses, he considered asking students to pose problems that correspond to the two interpretations of division (i.e., partitive and measurement). He justified this idea by pointing out that the two word problems included at the bottom of the page correspond to the partitive interpretation only. In selecting

exercises to assign to students, he considered issues of task variety but also issues pertaining to task sequencing. For example, he argued that exercise 13 (i.e.,  $2\frac{5}{8} \div \frac{1}{8}$ ) should precede exercise 7 (i.e.,  $\frac{1}{3} \div \frac{7}{10}$ ) because in the first exercise the dividend and the divisor have the same denominators. He also considered starting with an exercise whose dividend was larger than one because, as he claimed, that would help him better explain the meaning of the reciprocal in the invert-and-multiply algorithm as the number of divisor units that can be fit in each dividend unit. As the reader might recall, during the first interview, Nathan was not aware of the meaning of the reciprocal in fraction division and hence his analysis of this page was not informed by such considerations.

As in the pre-intervention meeting, Nathan again deemed the second page more suitable for teaching an introductory division-of-fractions lesson and identified several of this page's affordances. This time he additionally explored how to better capitalize on these affordances. Consider, for example, how he talked about the page's directions that students present their work in diagrams, number sentences, and written explanations: "[W]riting an explanation, drawing a diagram, and using a number sentence ... you can represent those three ways, but finding the connections between them is going to be important, too" (PO.I. 1124-1129). Likewise, although he regarded the direction that students explain the fractional part of their answer as a significant asset of this page, he maintained that students would need some guidance to understand the meaning of the fractional part of an answer, especially in the context of the problems outlined on this page. To support students' work, he would organize a brief discussion during which he would elicit students' ideas about whether it makes sense to have a fraction of a bow.

Like his analysis of the first page, in analyzing the second page, Nathan also thought about issues of task selection and ordering. He argued that if he was using this page, he would start with exercises  $A_1$ ,  $A_2$ , and  $A_3$  to help students understand the meaning of fraction division. He would, however, skip exercises  $B_1$  and  $B_2$  because he envisioned students having difficulties drawing pictures for these exercises; he also considered these exercises not particularly conducive to discussing the meaning of the traditional division-of-fractions algorithm. For example, in considering exercise  $B_1$  (i.e.,  $\frac{4}{5} \div \frac{2}{3}$ ), he remarked: “[T]rying to do four fifths divided by two thirds ... there’s three halves of ribbons in one yard; well, where’s the one yard? Actually, we’re only looking at four fifths of a yard” (PO.I. 1011-1014). When talking about “three halves of ribbons in one yard,” Nathan was actually referring to the meaning of the reciprocal in that particular division problem. Since the dividend in this division problem is smaller than the whole, he envisioned students encountering difficulties understanding the meaning of the invert-and-multiply algorithm because of not having at least one whole yard to start with.

Similarly, he anticipated students having difficulties transitioning between task C and D, to find an algorithm for fraction division. He mentioned,

“[I]n between ... [tasks] C and D, there are already some kind of gap, and I think ... to expect [students] to easily make the inferences that you’d have to make to solve those [division problems], it would be pretty tough for anyone, but especially for the kids in my class, in my experience.” (PO.I. 1247-1251)

During the pre-intervention meeting Nathan was a strong supporter of students discovering the algorithm themselves. Although he again endorsed this idea, he now perceived it in light of his fieldwork experiences. These experiences appear to have sensitized him to potential student difficulties when working on making generalizations.



To facilitate students' transition to the general algorithm, he proposed two different approaches. The first approach had a more guided-discovery character.<sup>168</sup> To scaffold students' pattern exploration and their work on figuring out the traditional algorithm, he considered organizing the solutions to the assigned division-of-fractions problems in a table with three columns, one each for the dividend, divisor, and quotient; doing so, he postulated, could help students discern the invert-and-multiply algorithm more easily. He even entertained the idea of asking students to focus only on problems whose dividend is one yard because, as he claimed, this might also help them figure out the algorithm more easily:

- Nathan:** So, maybe [students] could do several problems that had one-yard long lines and show those right on top of each other and maybe they can see something that way.
- Charalambos:** ... And why using one-yard long [lines]?
- Nathan:** Just because they would be comparable. ... I was just trying to think of ways to organize it so you could compare really easily different problems. (PO.I. 1876-1889)

Although less exploratory, his second approach would capitalize on the way he explained the reciprocal (see his performance in *Providing Explanations*). In particular, he noticed that the two parts of the equation in the division-of-fractions algorithm both correspond to the number of divisor units that can be fit into the dividend. Take, for example, the division sentence  $2 \div \frac{3}{4}$ . According to the traditional algorithm this sentence is equivalent to the multiplication sentence  $2 \cdot \frac{4}{3}$ . Both sentences are equivalent to  $2 \frac{2}{3}$ , which corresponds to the number of  $\frac{3}{4}$ -units that can be fit into two whole units. While working on the word problems of task A, which he considered easy to draw, Nathan would guide his students to see this relationship. He would then have them consider division problems for which a diagram would be impractical to use to derive the quotient.

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<sup>168</sup> I use the term as used by Bruner to denote a special form of discovery in which the teacher helps the student “rearrang[e] or transfor[m] evidence in such a way that [the student] is enabled to go beyond the evidence so reassembled into new insights” (Bruner, 1962, pp. 82-83; also see p. 101).

Such problems, he argued, would help students start thinking about the value of the algorithm. This is how he envisioned that happening:

We talked about one example where the fractions start to get so small that it's hard to draw. So maybe that might be a case where [I would tell the students], "Okay, so we've got this division problem and we've drawn out a diagram, but we're not quite sure if we drew it exactly to scale. How can we double-check and be really certain?" And then if we've talked about these two ways already [i.e., considering the answer to a division problem both from a division and a multiplication perspective], someone could say, "Oh, well, let's go back and do it using the multiplication that we know how to do, and then we'll get the exact answer, and we can see if our diagram shows the same thing or about the same thing." (PO.I. 1214-1224)

In this latter approach, Nathan would probably assume a more directive role when helping his students see the relationship between the division and the multiplication sentences; still, his approach had the potential to help students attach meaning to each of the numbers involved in these sentences.

To summarize, Nathan's post-intervention analysis of the textbook pages was similar to his pre-intervention analysis but more detailed. Instead of simply identifying the affordances of the textbook pages – and especially those of the second page – he also considered how he could better capitalize on the tasks' affordances during their enactment. He thought of doing so by deliberately selecting and ordering tasks; by asking students to build connections among the words, diagrams, and symbols they would use to represent their answers; and by guiding them to discover and understand the meaning of the invert-and-multiply algorithm.

*Noticing and Interpreting-Evaluating.* During the pre-intervention interview, in discussing the virtual teacher's presentation and enactment of tasks  $A_1$  and D, Nathan mostly focused on issues of meaning and understanding; during the post-intervention interview, he additionally considered issues pertaining to the tasks' cognitive demand. For instance, in considering the teacher's presentation of task  $A_1$ , he remarked,

So she's already saying, "Okay, what are we going to divide in this case?" So she's ... kind of limiting the methods they can use to solve the problem. ... She's asking them to use division already. (PO.I. 1349-1353)

In discussing the enactment of this same task, he was concerned that the teacher was emphasizing getting correct answers at the expense of understanding. Although he acknowledged that getting such answers is important, he considered helping students see the mathematical reasoning of the procedures at hand even more important. Similarly, he criticized the teacher's enactment of task D as void of meaning:

[Reading the teacher's prompt from the slides:] "Well, imagine if we did have a six here."... She's kind of -- it's like ... deus-ex-machina. ... Like, in old Greek plays they had some problem that was just unsolvable. And then at the end some guy comes in and just makes everything right (laughs). That's kind of what the teacher [does]; the teacher just swoops in. All of a sudden, anything you can imagine. "Oh, you want a six here? Okay. Well, we'll call it the reciprocal!" (PO.I. 1974-1988)

Nathan eloquently captured how the teacher led her students to the algorithm under consideration. Indeed, during this episode the teacher was presented simply manipulating numbers to lead students to the invert-and-multiply algorithm. Given his work in explaining how he would support his students in arriving at this algorithm as described above, it should not be surprising that he disapproved of the teacher's task enactment.

### *Providing Explanations*

*Performing.* During the pre-intervention meeting, Nathan was able to explain the quotient in the division problem  $2 \div \frac{3}{4}$ ; his explanation, although conceptually driven, was rather packed. For the invert-and-multiply algorithm, he claimed that he did not have the conceptual understanding to explain what the reciprocal in this algorithm means. Hence, the explanation he offered for this algorithm, although having some conceptual glimmers, was mainly informed by the numbers involved in this algorithm. The explanations he offered during the post-intervention meeting were remarkably different.

To explain the quotient, Nathan would use a word problem and a drawing.

Building on ideas discussed during the two intervention courses, he listed two problems he could use to this end, one for each of the two division interpretations. For the partitive interpretation he proposed the problem, “If I give you two candy bars, and I’ve given you three fourths of all the candy bars I have, how many did I have to start with?” (PO.I. 454-456); for the measurement interpretation he posed the problem, “If I needed two cups of flour to bake a cake and all I had was a three-fourths cup measuring cup, how many scoops would I need to put in?” (PO.I. 508-510). He used the second problem to explain the quotient.

Nathan started his explanation by drawing two rectangles, which he identified as the two cups of flour. He labeled the first rectangle as such (see first row in Figure 5.5); using his terminology, in what follows I refer to these rectangles as the “cup-rectangles.” He then divided each of these rectangles into four parts using three vertical dashed lines, explaining that “we want to fill [each] up with three-fourths cup pieces” (PO.I. 526-527). To denote the measuring cup, the “scoops,” he then drew another rectangle approximately equal in size to three fourths of the first rectangle (see first rectangle in the second row in Figure 5.5); again using his terminology, I refer to the rectangles of the second row as the “scoop-rectangles.” To illustrate the second scoop, Nathan drew another scoop-rectangle that spanned the fourth fourth of first cup-rectangle and the first two fourths of the second cup-rectangle. He noticed that this second scoop-rectangle was larger than the first scoop-rectangle, but he argued that in a real-classroom setting he would draw these rectangles better. To explain the fractional part of the quotient, he then drew a third, dashed-line cup-rectangle, which he again divided into fourths (see first row

of Figure 5.5). Next he drew a third scoop-rectangle that spanned the last two fourths of the second cup-rectangle and the first fourth of the dashed-line cup-rectangle. Pointing to the third scoop-rectangle, he asked: “So, how many of these scoops do I need here [pointing to the last two fourths of the second cup-rectangle]?” (PO.I. 542). To address this question, he partitioned the third scoop-rectangle into three parts that were aligned with the third and fourth fourths of the second cup-rectangle and the first fourth of the third cup-rectangle. Simultaneously pointing to the parts of the second cup-rectangle and the third scoop-rectangle numbered as one and two in Figure 5.5, he then clarified: “Well, I used one [pointing to the “1s”], I used two [pointing to the two “2s”]. ... I would need two thirds of the last scoop in order to fill it up.” (PO.I. 543-545).

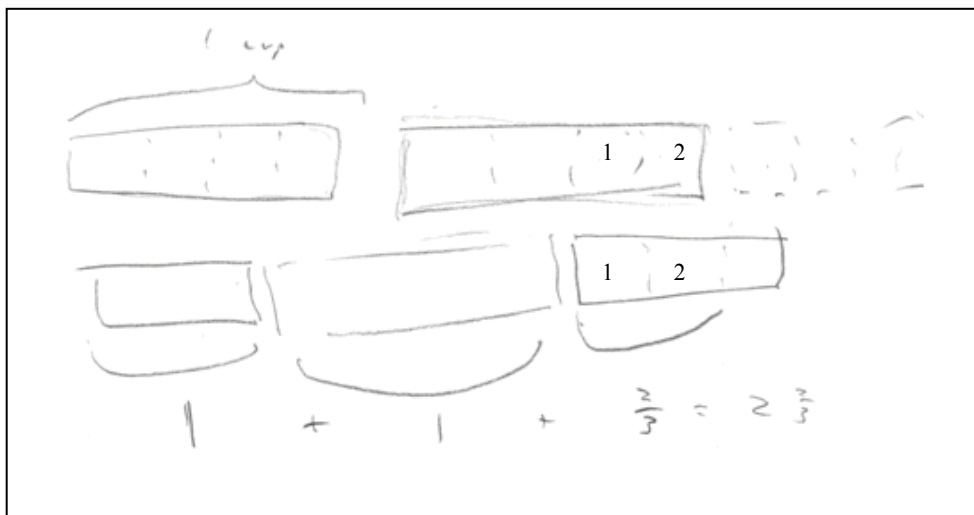


Figure 5.5. Nathan’s work on explaining the quotient in the division problem  $2 \div \frac{3}{4}$ .

At this point, the interviewer pressed him to further clarify the idea of two thirds:

**Charalambos:** So, if I were a student how would you help me see that [i.e., the two thirds]?

**Nathan:** ... So the question is asking how many *scoops*; so you’re unit is the *scoop*. So, this is one scoop [draws a bracket under the first scoop-rectangle and writes one], this is two scoops [repeats for the second scoop-rectangle], another one. [Pointing to the leftover part in the second cup-rectangle:] And this is three fourths [sic: two fourths] of a cup but it’s also two thirds of one scoop [draws a line under the first two thirds of the third scoop-rectangle and writes  $\frac{2}{3}$ ]. So when it’s asking for scoops, we want to make sure we give all -- sum up the answers that are in scoops. [Writes the sum of the numbers 1, 1, and  $\frac{2}{3}$ .] So, two and two thirds scoops.

**Charalambos:** And why is it two thirds of a scoop?

**Nathan:** Because there's -- it's in three equal parts [pointing to the last scoop-rectangle] and we use two of them, or we need two of them to get the right amount of flour. (PO.I. 559-590, emphases in the original)

Undeniably, Nathan's performance was not perfect. First, one could argue that in actuality, the answer to the word problem he proposed would be three scoops and not two and two-thirds scoops. Second, as he himself pointed out, his drawing could be somewhat misleading since he did not draw the two cup-rectangles end-to-end, which would allow him to represent all scoop-rectangles as approximately equal in size. Third, in the excerpt just cited, he incorrectly identified the leftover part of the second cup-rectangle as three fourths instead of two fourths. Fourth, the careful reader will also notice that the interviewer pushed him twice to explain the two thirds.

These deficiencies and the prompting from the interviewer notwithstanding, I argue that his performance during the second round of interviews was notably improved. To begin, during the post-intervention interview, Nathan tried to contextualize his explanation using a word problem. This allowed him to clearly distinguish between the absolute units (i.e., cups) and the relative units (i.e., scoops). Second, in a real-classroom setting, his question, "So how many parts do I need here?" could help him signal the introduction of an important mathematical idea: the fractional part of the quotient. Third, in contrast to his pre-intervention explanation, he drew different pictures for the absolute and the relative units and tried to show the correspondences between the two types of units. Notice, in particular, his close mapping between the two fourths of the leftover cup-rectangles with the two thirds of the scoop-rectangle. Thus, even before the interviewer's prompting, Nathan's explanation was noticeably more unpacked than the explanation he offered during the first interview. The interviewer's prompting incited him

to further unpack his explanation, which he did quite successfully: he clarified that the answer to this problem needs to be in scoops (i.e., relative units); he clearly decomposed the quotient into its corresponding parts and matched those parts with his drawing; additionally, in his last utterance, he employed the part-whole concept of fractions to explicate why the leftover part was two thirds of a scoop.

The identification of two different types of units helped Nathan to also explain the invert-and-multiply algorithm. Referring to his drawing of cup- and scoop-rectangles shown in Figure 5.5, he pointed out, “So, if you look at the cups, each cup is four thirds scoops. So there’s two cups. So, if we multiply two times the four thirds, then it should give us the number of scoops” (PO.I. 627-630). In contrast to the explanation he offered in the pre-intervention meeting, in the post-intervention meeting, Nathan succinctly pointed to the key idea of the reciprocal: he identified the number of relative units that could be fit into each dividend unit. Building on this idea, he then explained that there would be one and one third scoops in each cup, thus the four thirds corresponding to the reciprocal. Since the problem was asking for two cups, he remarked that he needed to take the four-thirds scoop quantity twice. He concluded by clarifying: “Either way, whether we do ... two times four thirds or the two divided by three fourths, we’re still going to be figuring out the number of scoops” (PO.I. 914-916). Overall, his explanation was conceptually driven; in addition, his last comment clarified an important idea that warrants attention when discussing the division-of-fractions algorithm: by pointing out that the division and the multiplication parts of algorithm in  $2 \div \frac{3}{4} = 2 \cdot \frac{4}{3}$  both correspond to the number of scoops, he justified why the two parts in this equation are equivalent.

It is also interesting to note that after offering this explanation, Nathan challenged himself further by attempting to explain the same algorithm for the division problem  $\frac{4}{5} \div \frac{2}{3}$  for task B<sub>1</sub> on the second textbook page. Drawing on the context of yards and ribbons used on this page, he first identified that one could make three halves ribbons from each whole yard. Building on this idea, he then explained why the number of the ribbons that could be made with the available  $\frac{4}{5}$ -yards was equal to the product of this fraction with the reciprocal of the divisor.

*Noticing and Interpreting-Evaluating.* As in the pre-intervention meeting, during the post-intervention meeting, Nathan regarded the teacher's explanation for the reciprocal as deficient. Although he pointed to the same deficiencies he identified during the pre-intervention meeting, in justifying his evaluation of the teacher's explanation, he made an interesting comment: he contended that the analogy the teacher used in her explanation could be suitable for other subjects, like language arts. Yet, as he argued, because different subjects require different reasoning, this analogy was inappropriate in mathematics. He elaborated upon the idea of the different types of reasoning needed for different subjects in two instances in the interview:

But there's some difference between asking a question in the other areas, and asking a question in math. ... Some of the skills are transferable, but not all of them are. So there's something special about each content that's not just general good teaching practice. (PO.I.65-69)

What counts as math knowledge? It's a different kind of narrative than a science or a language-arts knowledge. What exactly is the math knowledge? It's something to do with reason, reasoning. ... Well, there's certain like language and symbols and things that you use in math, different tools that you use, and so familiarity with things like that differentiates it from the other subjects. (PO. I. 84-92)

Nathan's comments in these two quotations are reminiscent of Schwab's (1964) *syntactic* knowledge and Ball's (1988) knowledge *about* mathematics: Nathan was talking about the different types of knowledge needed and used in mathematics and the knowledge that



gets accepted and is considered valid in this domain. Interestingly enough, he reconsidered such issues even when analyzing June's explanation (see below).

### *Using Representations*

*Performing.* During the pre-intervention meeting, Nathan selected suitable representations and used them appropriately when explaining the quotient, and partly when explaining the invert-and-multiply algorithm. Nevertheless, both his work in the performing and the noticing/interpreting-evaluating tasks of this practice suggested that he was not particularly attentive to issues of mapping between different forms of representations.

Mapping was one of the issues that Nathan was particularly conscientious about during the post-intervention meeting, as implied by his performance in explaining both the quotient and the algorithm for the division problem  $2 \div \frac{3}{4}$  (see above). For example, in explaining the quotient, he made explicit the connections among his drawing, the word problem he was using, and the numerical symbols (see Figure 5.5): he labeled the first rectangle as a cup-rectangle; he identified the rectangles in the second row as scoop-rectangles (although he did not label them as such); by simultaneously pointing to the leftover fourths in the second cup-rectangle and their corresponding parts in the scoop-rectangle, he tried to elucidate the connection between the two fourths (i.e., absolute units) and the two thirds (i.e., relative units); he clearly showed how the quotient of this problem corresponded to each part of his drawing, by both pointing to and labeling the scoop-rectangles; and by referring to the quotient as "scoops," he also built a connection between the word problem and the quotient obtained from solving that particular division problem. Furthermore, although Nathan was talking about cups and scoops, his drawing

reflected the *structural* rather than the superficial characteristics of the real-life situation under consideration.<sup>169</sup> Instead of drawing actual cups and scoops, he sketched rectangles; these rectangles were drawn in a manner that helped clarify the absolute and the relative units, which is a key idea in explaining the quotient in fraction division. Diezmann and English (2001) argue that such focus on the structural characteristics of a situation is a hallmark of effective representation use.

*Noticing and Interpreting-Evaluating.* While going over the virtual lesson, Nathan pointed to several instances in which the teacher and the students failed to draw connections among the different types of representations they were using. For instance, in discussing the first episode designed for the practice of using representations, he contended that the teacher should have labeled the line she drew on the board to illustrate the whole and the half yard. When Amanda wrote the mathematical sentence on the board, he observed that the teacher did not prompt her to show the correspondences between the numbers of this sentence and the different parts of the representation already drawn on the board. If he were teaching the lesson, he would press Amanda to clarify those correspondences: “I guess I would ask [Amanda] where in the diagram we can see the one half, the one sixth, and the three. And just to try and map out the relation” (P.O.I. 1719-1721). He was even more concerned with the lack of such connections in the second episode:

[Amanda and Julia] divided [their line] up into twelve parts. ... And then they said they made nine of it red. Well, why did they make nine of them red? You have to think that nine twelfths is the same as three fourths. Some kids might not see that right away. And then when [Amanda]’s doing the little blue group or marking off the sixths as two twelfths, that’s another place where they’re doing a conversion in their head that’s not really explicit. ... They’re saying nine twelfths and they’re showing two twelfths, but when they write the equation it’s three fourths and one sixth; so there’s not an explicit link there. (P.O.I. 1777-1789)

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<sup>169</sup> The reader can compare his drawing to Suzanne’s drawing in Figure 5.31, in which she focused on the superficial characteristics of the mathematical situation she employed to explain the quotient.

As this quotation suggests, Nathan identified the places in Amanda and Julia's work where explicit connections between their drawing and the mathematical sentence were warranted but never made. He revisited this same issue at the end of the interview when asked to evaluate the teacher's work, when he claimed that no relationships were actually built among the different representations being used in this lesson.

### *Analyzing Students' Work and Contributions*

*Performing.* Nathan's analysis of the three students' solutions was meticulous, as it was during the pre-intervention meeting; he also made appropriate assertions about each student's understanding. For example, in analyzing Robert's work, he asserted that Robert appeared to have only understood how to apply the algorithm. Given that it was Robert who asked the question whether the invert-and-multiply rule works all the time, Nathan wondered whether Robert was actually convinced that applying the rule was the right thing to do: "He might not even believe that that's the right answer necessarily, but he's just doing what the teacher explained to do" (PO.I. 2112-2114).

In considering Michelle's work, Nathan explained why it was necessary that she also represent her work in numbers:

I don't know if [Michelle] would know how to write the answer. And especially because there was that confusion in class about the interpretation of the fractional part. ... That could be one possible reason why she hasn't written an answer: she may not know what number to write. (PO.I. 2216-2123)

Notice in this quotation that Nathan considered Michelle's work in the context of the whole lesson. Although in general it is critical that students represent their work using different representations, there was an additional reason for why Michelle needed to represent her work in a numerical sentence: doing so would have allowed the teacher understand whether Michelle really grasped the fractional part of the quotient. Given the

confusion around this issue that transpired during this lesson, as Nathan rightfully remarked, Michelle not representing her work in numbers could be interpreted as evidence of her struggle with this idea.

In addition to making appropriate assertions about each student's understanding, Nathan also explored the likelihood that instruction itself was the source of students' confusion and error. Consider, for instance, the following excerpt in which he talked about how the teacher's presentation of the invert-and-multiply algorithm as a list of steps to follow might have contributed to Ann's taking the reciprocal of part of the dividend:

I can't remember exactly what the teacher said; it's like "invert the second fraction" or something like that. [Goes back to the pertinent slides and reads what the teacher said:] "Invert the second fraction; take the reciprocal of the second fraction." So [Ann] is taking the second number [points to  $\frac{3}{4}$  in  $2\frac{3}{4}$ ]; ... [she] is taking the reciprocal of that second number. ... She's doing something that's unexpected consequence of the way it might be worded. ... So, I could totally see how a kid would make a mistake if you say ... "Take the reciprocal of the second number." (P.O.I. 2064-2087)

Nathan was right that ambiguous terms used while presenting the content can lead to unexpected yet rational student errors. Given that the division problem at hand was the only lesson problem whose dividend was a mixed number and which students were expected to solve using the invert-and-multiply algorithm, Nathan felt that Ann's error should not be surprising: she probably interpreted the mixed number of the dividend as two separate numbers and therefore inverted the second "number" and then multiplied.

*Noticing and Interpreting-Evaluating.* Similar to his performance during the pre-intervention meeting, in the post-intervention meeting, Nathan identified three of the four contestable student contributions: June's explanation; Alan's error, and Amanda and Julia's confounding of absolute and relative units. Instead of providing a detailed

exposition of his analysis, I draw on his analysis in each episode to highlight three different aspects of his work pertaining to analyzing students' ideas.

In considering June's explanation, Nathan was concerned with what counts as valid mathematical reasoning and consequently as a valid explanation. He acknowledged that June was right in arguing that she could not make half a yard if she started with a piece that was one sixth of a yard long. Yet, as he maintained, her reasoning was not mathematically valid: that the one half was the dividend and the sixth was the divisor and not vice versa was not dictated by the size of the numbers but by the role that these two numbers had in the word problem under consideration. Hence, for Nathan, June's explanation afforded the teacher an opportunity to help students pay closer attention to issues of reasoning by pressing June and her classmates to explore whether the offered explanation really addressed the question of the division problem.

In discussing Alan's error, Nathan explored the likelihood that Alan's confusion stemmed from how mathematical ideas were presented during the virtual lesson. He noticed that the teacher did not elaborate the meaning of a sixth as a part of the *whole*. In addition, as he remarked, Ms. Rebecca implicitly gave Alan the rather ambiguous hint that he needed to "divide his line into six parts." Hence, from Nathan's perspective, that Alan divided half the line instead of the whole line into six parts was not surprising: Alan simply followed the teacher's hint. Like his analysis of Ann's solution to the division problem discussed above, Nathan again considered students' work to reflect not only their thinking but also how the students perceived and interpreted the ideas presented during the lesson.

Finally, Nathan's analysis of Amanda's and Julia's work represents a counterexample to what I called an *answer-driven* approach to analyze students' ideas (see Chapter 4). Instead of using the traditional invert-and-multiply algorithm to figure out the quotient to the problem  $\frac{3}{4} \div \frac{1}{6}$  and then using this number as a yardstick to explore the correctness of the two students' solution (i.e., "4 remainder  $\frac{1}{2}$ "), Nathan pursued a different approach that built on the students' answer. He interpreted the whole number in the girls' answer as four bows. Because each bow was a sixth of a yard long, he considered this whole number to represent four sixths of a yard, and consequently eight twelfths of a yard (see Figure 5.6). To that, he added the remainder of one-twelfth yard, which yielded nine-twelfths yards. Since this was equivalent to three-fourths yards, he concluded that, although the girls were mixing different units in their answer, their reasoning was correct.

The image shows a rectangular box containing two handwritten equations. The first equation is  $\frac{4}{6} = \frac{8}{12}$ . The second equation is  $\frac{9}{12} = \frac{3}{4}$ . The numbers are written in a simple, slightly slanted hand.

Figure 5.6. Nathan's analysis of Amanda and Julia's answer to the problem  $\frac{3}{4} \div \frac{1}{6}$ .

In sum, Nathan's analysis of the students' work reflected three different types of considerations: considerations of the content (i.e., what counts as valid thinking and reasoning in mathematics); considerations of the teacher's instructional moves (i.e., the likelihood that the teacher's presentation of the content could have been a potential source of students' errors and confusions); and considerations of the reasonableness rather than the correctness of students' ideas (i.e., instead of checking whether the two girls' solution was correct, he explored whether their work made sense).

### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* As in the pre-intervention interview, Nathan's plans for responding to the students' requests for help in both episodes under consideration were mathematically valid and reflected an attempt to capitalize on students' thinking.

For example, regarding the first episode, Nathan argued that if he were the teacher, when June drew two lines on the board to show the one half and the one sixth, he would try to help her unpack the meaning of the half and the sixth. Doing so, he claimed, might have prevented Alan from committing the error he made. If Alan did make the same error, however, Nathan would try to help him correct this error by eliciting other students' ideas. If other students believed that Alan's work was correct, he would pose questions to help them see the error: "What have we divided into six pieces?", "What does each [of these pieces] represent?", "How many [of the pieces that Alan showed on the board are] in a whole yard?" (PO.I. 1525-1527).

Likewise, in the second episode Nathan would first explore whether only June and Shaun grappled with dividing their lines into sixths and fourths. If so, he would avoid addressing the whole class like Ms. Rebecca did; instead, he would ask June and Shaun to share their difficulty in their small group to see if any of their classmates had any insights as to how they could overcome the stumble. If, however, several students wrestled with the same difficulty, he would organize a whole-class discussion. During this discussion he would elicit students' ideas to understand "where the misunderstanding or the confusion was" (PO.I. 1671) and to investigate whether students themselves had ideas for overcoming the impasse. If this discussion yielded no helpful ideas, he would ask students to relate their work on the problem under consideration (i.e.,  $\frac{3}{4} \div \frac{1}{6}$ ) to their

work on the previous problem (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ), hoping that this would help them start thinking about commensurate pieces. He also entertained the idea of using two different lines to represent the three fourths and the sixth (instead of trying to represent both fractions on the same line as June and Shaun were trying to do). In short, instead of simply giving students an ambiguous hint to help them overcome their struggles, Nathan would first seek to help students realize what was causing them the difficulty in this problem and then work with them to figure out ways to resolve the stumble.

*Noticing and Interpreting-Evaluating.* Nathan's analysis of the two episodes under consideration echoed the comments he made about these episodes during the pre-intervention meeting. He noticed that the teacher was posing very leading questions and that her intonation was actually giving away the right answers. He also questioned whether the hint of using twelfths she gave students could really support their exploration and understanding of the division problem discussed during the second episode. Unlike his pre-intervention analysis, in the post-intervention meeting he also thought that the teacher's move to stop the whole class and give them a hint might have been unnecessary since only June and Shaun were depicted struggling with partitioning their lines into fourths and sixths.

#### *Professed Changes and Their Attributions*

According to the interview protocol, at the end of the post-intervention interview, the PSTs were asked to compare their experiences in participating in the post-intervention interview with those pertaining to their participation in the pre-intervention meeting; they were also asked to discuss any changes they might have sensed in their thinking about teaching and learning mathematics.



Nathan felt that during the post-intervention meeting he was perceptive to more aspects of the teacher's and the students' work. He also reported feeling "more fluid to articulate things" (PO.I. 2245-2246). He attributed those changes to his fieldwork experiences and to the courses he had taken (both the intervention courses and other ELMAC courses), which gave him "a framework in which to interpret the [teaching simulation] experience" (PO.I. 2251-2252).

Nathan's fieldwork experiences offered him a lens through which to observe the virtual lesson and evaluate the teacher's decisions and actions. As he mentioned, while going over the simulation, he was thinking, "[I]f I said that as a teacher, what problems [would] my kids be having?" (PO.I. 2276-2277). During the interview he also argued that if he explained the reciprocal the way the teacher did in the virtual lesson and if he used the board similarly to how she used it while outlining her explanation, several of his students would be confused. Additionally, his fieldwork experiences enriched his toolkit with some ideas relevant to the practices examined in the study. For example, at a certain point in the interview he referred to how his mentor would use a representation to help students solve the first word problem of the first textbook page.

Nathan's coursework, on the other hand, seemed to have enriched his images of teaching. For example, in considering how he would respond to the students' requests for help, he referred to a video clip shown during one of the math methods classes: "Well, I am thinking of that video when [the teacher in a video clip] asks somebody ... she's like 'say it to him.'" Then, talking to himself, he wondered, "So is there some way to get the kids to be talking to each other rather than explaining to the teacher and having the teacher talk back to ... make sure they understand?" (PO.I. 1452-1458). Similarly, in

considering the virtual teacher's decision to ask students to work individually on a set of tasks without having first ensured that they were secure on the ideas needed to solve these tasks, he commented:

Something that [an ELMAC instructor] talks about is to make sure the kids have some -- when you send them to do independent work, make sure it's work they can do. So, maybe some more assessment before [the virtual teacher] assigned those individual problems might have been better. Maybe do some more [problems] together, just to make sure that ... it's something that's doable, otherwise ... the teacher's going to be trying to run around and help everyone. (PO.I. 1590-1599)

The discussions that transpired during the two intervention courses also sensitized him to issues he was not particularly attentive to at the beginning of the program. For example, he mentioned that the instructor of the two intervention courses constantly emphasized that "mathematics shouldn't be like magic" (PO.I. 1956). He also referred to discussions that transpired during the math methods courses, especially when analyzing students' work, which alerted him to that

kids don't have misconceptions or misunderstandings or confusions just at random. There are reasons for that based on the way that you're teaching and the way that they were taught before. (PO.I. 2276-2281)

This idea definitely had an impact on Nathan's performance in analyzing students' work during the simulation; as the reader might recall, Nathan was constantly considering instruction to be a potential source of students' confusion, errors, and misconceptions.

What was rather unexpected in Nathan's performance was his attention to issues of cognitive demand since such issues were not explicitly discussed during the intervention. The interview data suggested a plausible explanation for this finding:

I guess one thing that I didn't really think about that much until I took the courses was the idea of constructing ... knowledge. Like, I know when I came in to talk to [the instructor of the intervention courses] about some lessons, one thing [he] said ... [was], "You're giving them too much information." And that really stuck with me, because when [he] said that I was doing the thinking for them, really made me realize that it's the students that have to -- you have to figure out what the students know, and let them explore, start there, explore their ideas, and then kind of give them the terms or tools or whatever they need to integrate this new piece into what they already know, rather than just telling them and then having them evidence some behavior. (PO. I. 100-113)

In this excerpt, Nathan was referring to the individual meetings that the course instructor had with each of the study participants to discuss their lesson plan for a lesson they taught toward the end of the Fall 2007 semester (see Figure A2, Class 12, in Appendix 1). Probably wearing the hat of the course instructor rather than that of the researcher, the course instructor talked about issues of cognitive demand, thus “coloring” Nathan’s performance during the post-intervention meeting. This excerpt thus suggests that Nathan’s post-intervention performance in the MTF-related practices needs to be interpreted with caution. From a practical perspective, however, this quotation suggests that if PSTs are to attend to issues of cognitive demand in their instruction, they need to be sensitized to such issues. Additionally, from a methodological perspective, this quotation reminds us of the dual agenda that researchers who are using their instruction as a research site ought to pursue and be loyal to: the need to treat the design of their study with integrity (which might imply that they avoid interjecting ideas that might affect the results of their study) and the need to be loyal to their teaching responsibilities (which dictates that they help their students learn and grow).

A final remark is also in order here that speaks to the transferability of the performances examined in this study. When toward the end of the interview Nathan was asked to evaluate the virtual teacher’s approach in the lesson, he considered the teacher to not be really attentive and responsive to the students’ ideas. He acknowledged, though, that being responsive to and capitalizing on the students’ ideas is easier done in the in-vitro simulation environment rather than in a real-classroom setting:

[The teacher] wasn’t necessarily being responsive to where the students were at. So that’s something a lot similar to what I would probably end up doing a lot (laughs), because I’ve got my lesson plan, and I’m trying to go through it, and I’m just trying to keep the whole class in order,

and everything. So, I don't know what she's thinking as a teacher, but I could probably see myself in that situation too. (PO.I. 21545-2160)

This comment brings the complexity of instruction back to the picture, just like Nicole's cost-benefit analysis considered in Chapter 4 did. In conjunction, Nathan's comment and Nicole's analysis underscore the importance of interpreting the findings of this study with caution, bearing in mind that the PSTs' performance in the teaching simulation might not actually translate into what they can do in a real-classroom setting.

#### *Changes in Performance on the LMT Test: A Closer Look*

During the second administration of the test, Nathan answered correctly all but three of the test questions. This explains why in Figure 5.3 he is presented exhibiting the lowest MKT gains: he was the only PST of the study participants who answered fewer questions correctly (i.e., one less) on the post-test rather than on the pre-test. Both during the pre- and the post-intervention administrations of the LMT test, he incorrectly answered question 10; this question pertained to an unconventional approach to dividing fractions. During the second administration of the test he correctly answered question 25; this question pertained to the measurement interpretation of division. That he answered this question correctly was consistent with his teaching performance in at least two practices (i.e., selecting and using tasks; providing explanations), in which he referred to and capitalized on the two different interpretation of division. In contrast, he incorrectly answered two questions pertaining to the meaning of addition and division of fractions (8a and 19b); this finding was rather unexpected if one considers that he correctly answered other more difficult test questions.

### *Analytical Commentary*

The changes in both Nathan's MKT performance and his teaching performance were proportionally smaller than those of his counterparts. From this respect, Nathan's case supports the strong association between the gains in the PSTs' MKT and the gains in their performance reported in the previous section since the small changes in his MKT performance were aligned with the non dramatic changes in his teaching performance. In fact, most of the changes Nathan experienced were subtle and were not captured by the schemes developed to code the study participants' performance. Consider, for example, Nathan's evaluation of the teacher's explanation for the reciprocal. Although both his pre- and post-intervention performances were assigned the same score, his post-intervention evaluation was more scrupulous: although he again focused on the mathematical quality of the teacher's explanation, he talked about issues of reasoning and what counts as a valid explanation in mathematics. Similarly, in analyzing the students' work and contributions he was more sensitive to whether instruction itself might have contributed to students' confusion and errors. Such nuanced criteria were not included in the coding schemes because the purpose of the data quantification was to capture coarser differences in the PSTs' performance. Capturing these coarser differences would, in turn, allow selecting specific cases for a more fine-grained analysis of the differences in their performances.

Although not captured by his performance on the LMT test, two changes were evident in Nathan's knowledge: his conceptual understanding of the reciprocal in the division-of-fractions algorithm was enhanced; he was also more sensitive to issues pertaining to the type of reasoning accepted and used in mathematics. Both these changes

were reflected in the changes in his performance in the teaching simulation, thus further corroborating the association between the gains in the PSTs' knowledge and their teaching performance.

Armed with a better conceptual understanding of the reciprocal, Nathan was seen selecting and sequencing tasks more deliberately. For instance, when designing an introductory lesson on fraction division, he considered skipping tasks whose dividend was smaller than one. He envisioned that excluding such tasks would facilitate discussing the meaning of the reciprocal as the number of divisor units that can be fit within each dividend unit. He also anticipated that students would encounter difficulties generating the division-of-fractions algorithm. To facilitate their transition from specific examples to the general algorithm, Nathan proposed two approaches: one centered on helping students organize their data in such a way that they could see patterns and eventually figure out the algorithm and another grounded in the meaning of the reciprocal. These two approaches had the potential to help Nathan maintain the exploratory character of task D (the first approach) and its emphasis on meaning and understanding (the second approach) were he to use the second textbook page to teach a division-of-fractions lesson. Hence, his increased understanding of the reciprocal appeared to have helped him identify (additional) affordances and limitations of the textbook tasks. This, in turn, informed his task selection, sequencing, and restructuring decisions. From this perspective, the changes in Nathan's knowledge and performance as just discussed substantiate propositions  $D_1$  and  $D_2$  listed in Table 4.6, which pertain to how an increased understanding of the content facilitates identifying the task affordances or restructuring tasks to account for their limitations.

Nathan's increased understanding of the reciprocal was also reflected in the explanation he provided for the invert-and-multiply algorithm. Whereas during the pre-intervention meeting he confessed feeling insecure about his understanding of this concept, during the post-intervention meeting, when asked to provide an explanation for the division-of-fractions algorithm he started with clarifying the meaning of the reciprocal. In the post-intervention discussion, he also used the representation he chose to illuminate the meaning of the reciprocal instead of showing the numbers involved in the pertinent algorithm, as he was largely trying to do during the pre-intervention meeting. Thus, both his explanation and his use of representations were conceptually driven. From this viewpoint, the changes in Nathan's knowledge and performances support propositions  $A_1$  and  $B_2$ , which concern how a conceptual understanding of the content enables teachers to provide more conceptually driven explanations and use the representations they select in a conceptually informed manner.

The changes in Nathan's knowledge (and beliefs) about the type of reasoning and explaining that is accepted in mathematics also seemed to inform his performance in analyzing and evaluating the virtual teacher's arguments and the students' ideas. For instance, in considering the virtual teacher's explanation for the reciprocal he noticed that the analogy she used might have been appropriate in a language-arts lesson but it was not acceptable in mathematics. Similarly, in considering June's explanation, he considered her arguments insufficient for addressing the problem's question.

The changes just discussed, however, were not the only ones observed in Nathan's performance. Nathan was seen providing more unpacked explanations; he was more sensitive to issues of mapping among different representations; and he viewed

students' work and contributions in light of how the content was presented and got treated during the lesson. Although the changes in his knowledge reported above might have contributed, to some extent, to the aforementioned changes in his teaching performance, it seems more reasonable to associate the latter changes with his fieldwork and coursework experiences. Nathan's fieldwork and coursework experiences offered him additional images of teaching and afforded him new lenses through which to analyze the teaching simulation; they also focused the lenses through which he had been analyzing the simulation during the pre-intervention meeting. Specifically, the post-intervention interview data suggest that Nathan drew on the practices or the suggestions of his collaborative teacher, the instructor of the intervention courses, other ELMAC instructors, and the teachers presented in the records of practice shared and discussed during the intervention courses. The practices of all those teachers and their suggestions enriched Nathan's images of teaching and offered him ideas about handling certain instructional situations. For instance, Nathan's acute attention to issues of mapping should not be dissociated from the focus of the two intervention courses on this aspect of teaching. As Figures A1 and A2 in Appendix 1 show, such issues were constantly discussed during both courses and the PSTs were offered ample opportunities to practice building connections among different representations. Nathan's providing of unpacked explanations should also be seen in light of the multiple opportunities he had during the two intervention courses to offer such explanations, albeit for other mathematical ideas. Nathan's case then suggests that PSTs need to be afforded multiple opportunities to practice providing unpacked explanations, building connections among different



representations, and identifying instances in which instruction can lead to certain student misconceptions, if they are to attend to such issues in their teaching.

Nathan was the only PST explicitly exposed to issues of cognitive demand because of an individual interaction he had with the course instructor during which the latter suggested that Nathan pay attention to such issues. This interaction seemed to have attuned Nathan to issues of cognitive demand and to how the teacher might, in fact, replace instead of support student thinking. Given that Nathan was the only PST who attended to and explicitly discussed issues of cognitive demand in his teaching performance, his case then suggests that such issues need to be the focus of teacher education programs, if PSTs are to *start* attending to them in their teaching.

In short, the post-intervention case of Nathan supports propositions A<sub>4</sub>, B<sub>3</sub>, D<sub>4</sub>, and E<sub>2</sub>, which all point to certain ways in which teachers' knowledge of the content and its teaching might not suffice to enact the practices this study examines.

The changes reported in Nathan's case are summarized in Figure 5.7, which presents Nathan's post-intervention profile. To facilitate comparability with his profile presented in Chapter 4, this figure largely retains the information reported in Nathan's earlier profile<sup>170</sup> and presents the changes discussed above in italics.

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<sup>170</sup> Because the post-intervention data provided no information about the effect of Nathan's family background and his school experiences on his performance, these two factors were not retained in Nathan's post-intervention profile.

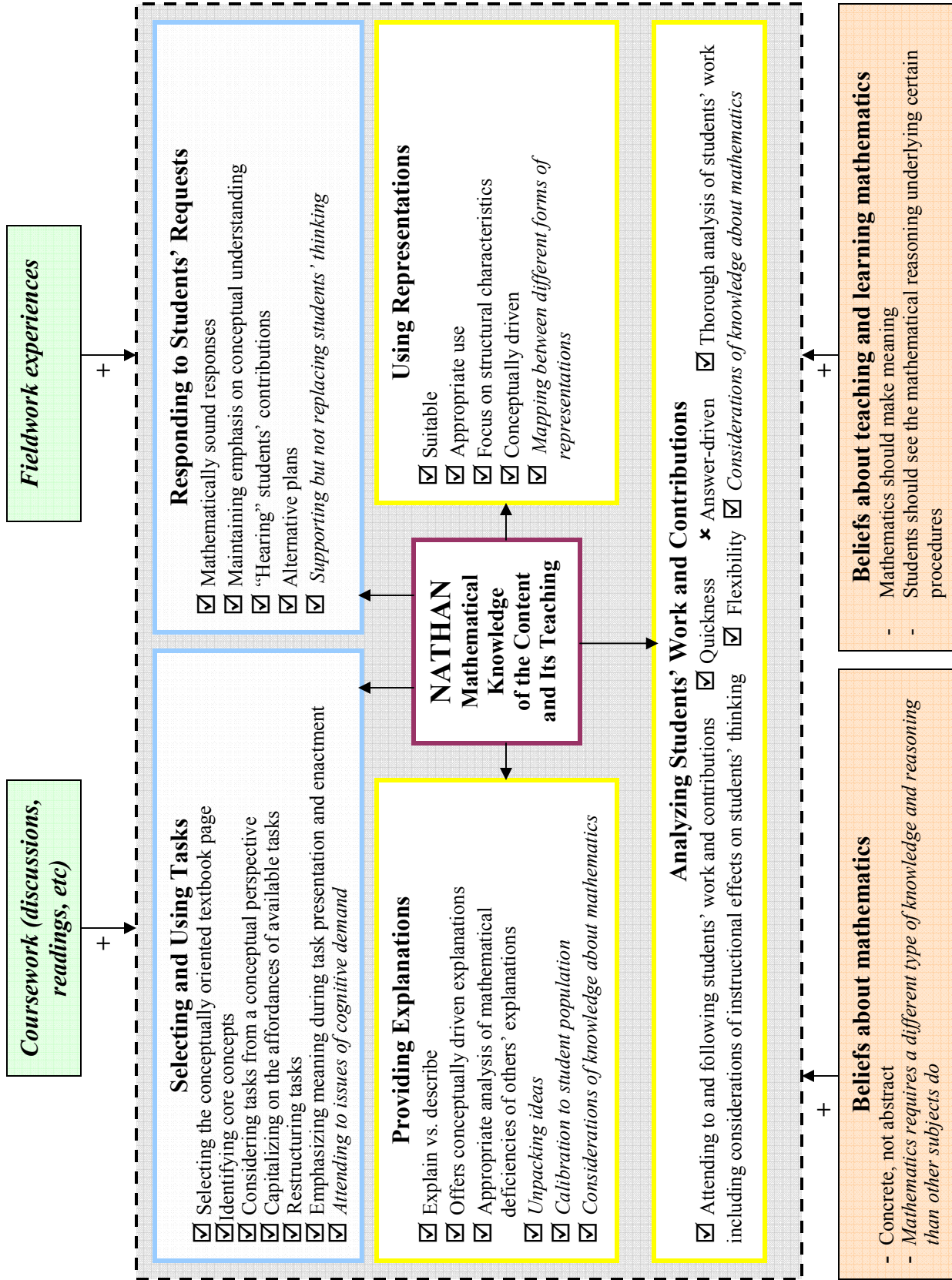


Figure 5.7. Considering the association between knowledge and teaching performance through Nathan's post-intervention profile.

### *The Case of Nicole Revisited: Dealing with Red and Purple “Wholes”*

Like Nathan, Nicole was a convergent case based on both her entrance scores and the changes in her performances. Compared to the other study participants, she exhibited small changes in her performance on both the LMT test and the teaching simulation. The post-intervention interview with Nicole was conducted during mid February. In the 32 weeks that elapsed between the pre- and the post-intervention meetings, in addition to her coursework, Nicole observed and taught some mathematics lessons in a seventh-grade class, but, overall, she observed and taught fewer math lessons than her counterparts did.

Nicole’s case, like Nathan’s, corroborates the quantitative findings reported in the previous section in that the low gains in her MKT were consistent with the low gains in her teaching performance; in fact, Nicole did not experience the subtle changes in her performance documented in Nathan’s case. Her post-intervention performance also substantiates the knowledge-teaching performance association in two respects. First, her enhanced understanding of relative and absolute units was revealed by her flexibility in providing an explanation for the quotient in the division  $2 \div \frac{3}{4}$ . Second, her still developing grasp of the notion of the reciprocal was reflected in her struggles with explaining the reciprocal for the same division problem. Nicole’s greater attention to issues of mapping when using representations and her inattention to issues of cognitive demand both point to the importance of familiarizing PSTs with such issues if they are to start considering them in their instruction.

#### *Selecting and Using Tasks*

*Performing.* Nicole’s performance in the practice of selecting and using tasks was very similar to her corresponding performance during the pre-intervention meeting: she

considered the first page procedurally oriented, but she found it difficult to restructure its tasks to teach for more conceptual understanding. She preferred using the second page for an introductory lesson on division of fractions and she was able to identify several of the affordances of the second page; yet she was not sure how to support students to come up with an algorithm for dividing fractions.

In considering the first page, Nicole argued that, although it included a “good mixture” of different types of exercises that involved different types of numbers (mixed numbers, improper fractions, and proper fractions), this page was mostly procedural. Thus, if she had to use this page in her teaching, she would use it only after she would have introduced the division of fractions from a more conceptual perspective. Considering herself a visual learner, she would have done so by using “visuals” to explain the relevant concepts. Her performance in providing explanations (see below) suggests that although she could explain the quotient fairly well, she would encounter difficulties explaining the idea of the reciprocal. As in the first interview, she also considered it unnecessary to assign all 16 exercises to students; yet, she considered these exercises merely from the perspective of reinforcing students’ procedural fluency with the traditional algorithm and other related procedures (e.g., converting mixed numbers into improper fractions and vice versa). Even when directly asked, she was unclear how to use these exercises to teach from a conceptual perspective.

For Nicole, the second page was more conducive to teaching for conceptual understanding because it supported using representations and required more thinking on the part of the students, given that it asked them to explain their work. She also regarded the direction pertaining to the fractional part of the quotient as an asset of this page,

arguing that when dealing with division of fractions “it is important for students to understand what their answer means in terms of the unit [being] divided by” (PO.I. 774-776). If she were to use this page to teach an introductory lesson on fraction division, she would capitalize on this page’s affordances by having students make drawings and explain their answers, particularly the fractional part of the quotients. Similar to the pre-intervention meeting, Nicole saw value in task D, which asks students to come up with an algorithm for dividing fractions. Nevertheless, she was not sure how she could help her students transition from dealing with specific fraction-division problems (in tasks A through C) to a general algorithm for dividing fractions (in task D). She mentioned, “I wrote the algorithm [in the space provided on this page] because I knew it, but I don’t know how you would get to that” (PO.I. 732-733). She speculated that one plausible way to help the students get to the algorithm could be to ask them to draw pictures and use these pictures to identify patterns. Yet, this was a general idea upon which Nicole did not elaborate any further, even when asked to do so at a subsequent part of the interview.

*Noticing and Interpreting-Evaluating.* The comparison of Nicole’s pre- and post-intervention performance in the noticing and interpreting-evaluating tasks of this practice did not yield any notable changes. Like the pre-intervention meeting, in the post-intervention discussion, Nicole focused mostly on issues of meaning and understanding and overlooked how the teacher was doing most of the thinking for her students. Specifically, Nicole endorsed the teacher’s presentation of task A<sub>1</sub>, especially because the teacher “was not just telling them everything” (PO.I. 965-966); this contradicted Nicole’s “tendency to ... explain more ... to want to say more” (PO.I. 1053-1054). She also liked how the teacher worked with her students on this task by asking them questions that “led

them in [sic] the right direction” (PO.I. 967) and by reminding them of important ideas, such as that they needed to use the whole line instead of part of it. In contrast, for task D, she felt that the teacher was simply “pull[ing] numbers out of the air” (PO. I. 1351-1352). Hence, although the teacher’s enactment of this task as depicted in the virtual lesson seemed to “work,” she had reservations whether it would work as smoothly in a real-classroom setting.

### *Providing Explanations*

*Performing.* Nicole’s post-intervention explanation for the quotient was conceptually driven and relatively unpacked, as was her pre-intervention explanation. Yet, because her understanding of the reciprocal was still developing, she could not provide an equally conceptually driven and unpacked explanation for the traditional division-of-fractions algorithm. However, capitalizing on an interviewer’s suggestion, she made considerable progress toward providing an acceptable explanation for this algorithm.

In explaining the quotient, Nicole first wrote the division problem under consideration (i.e.,  $2 \div \frac{3}{4}$ ; see Figure 5.8). She represented the dividend of the problem by drawing two rectangles, one attached to each other, which she labeled as “1 whole” and “2 whole.” Since the problem was asking for three fourths, she explained that she needed to check how many three fourths fit into the two. Hence, she divided each of the two wholes into fourths and continued:

I’ve got the two [wholes] in black and then I’m going to mark in three fourths in purple. I have one set of three fourths [shades in purple  $\frac{3}{4}$  of the first rectangle and draws a bracket above them]; another set of three fourths here [repeats the same for the second  $\frac{3}{4}$ ]. I have two whole pieces and two thirds of a piece. ... [She uses the traditional algorithm to solve the problem and, after she finds the answer of  $2 \frac{2}{3}$ , she shows how this answer corresponds to her drawing:] There are two sets of three fourths [pointing to the two sections colored in purple], and then there are two sections left over [pointing to the non-shaded part], and that’s two of the three fourths. So it’s

actually two fourths, but it's two thirds of the piece that we were using to divide by. (PO.I. 293-315)

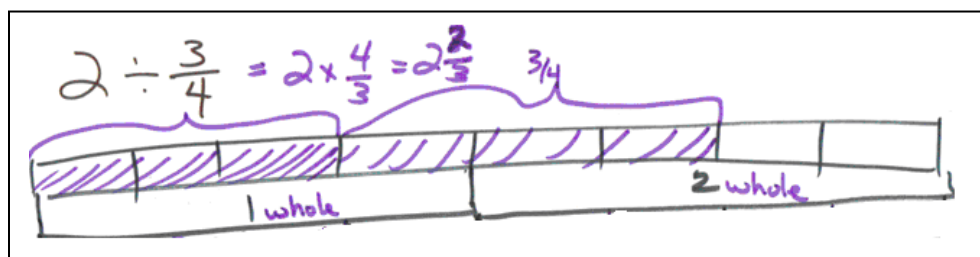
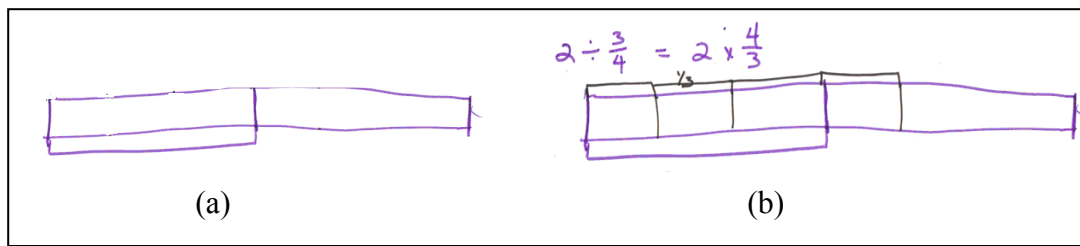


Figure 5.8. Nicole's work on explaining the quotient for the division problem  $2 \div \frac{3}{4}$ .

Four features of this performance are worth noticing. First, Nicole's explanation was largely guided by the concept of division as fitting divisor units in the available dividend units; however, before explaining the fractional part of the quotient she sensed the need to first figure out the answer to this problem and to then complete her explanation. Second, she clearly mapped her drawing onto the dividend, the divisor, and the quotient of the division at hand: she showed that the two "whole" rectangles drawn in black corresponded to the dividend; the bracketed portion colored in purple corresponded to the divisor; the two  $\frac{3}{4}$ -portions corresponded to the "2" in the quotient and the non-shaded part represented the fractional part of the quotient. Third, her use of two different colors helped her distinguish between the absolute units (in black) and relative units (in purple) considered in the problem;<sup>171</sup> she also showed the different units by drawing curved- and straight-line brackets. Fourth, she clarified that the leftover part was two fourths, but it also represented two thirds of the purple portion. Although her explanation for the fractional part of the quotient warranted some more unpacking, overall her explanation could be comprehensible to an average sixth grader.

<sup>171</sup> It is interesting to notice that Nicole wrote the quotient in purple, which was the appropriate color to use, given that she used this color to also represent the relative units. Of course, she might have done so unintentionally.

When asked to explain the invert-and multiply algorithm for the same fraction division, Nicole admitted still being unclear about the concept of the reciprocal. She nevertheless attempted to offer an explanation. She again drew two rectangles end-to-end to show the dividend (see left panel of Figure 5.9; notice that unlike her performance just discussed, she drew the rectangles representing the dividend in purple). After that, she wrote the division problem and its equivalent multiplication sentence (i.e.  $2 \div \frac{3}{4} = 2 \times \frac{4}{3}$ ). Similar to her performance in the pre-intervention meeting, she partitioned the first rectangle into three parts to show thirds, which corresponded to the denominator of the reciprocal; she labeled one of these parts as such (see right panel of Figure 5.9; notice that she showed these parts in black). She then showed the fourth thirds as spanning the whole first rectangle and the first third of the second rectangle. At this point she stopped and said that she was not sure how to further proceed.



*Figure 5.9.* Nicole’s first experimentation with explaining the algorithm for the division problem  $2 \div \frac{3}{4}$  (a: showing the dividend; b: showing the reciprocal)

To explore how far Nicole could go when given a hint, the interviewer suggested that she try to explain the reciprocal by using the same drawing she employed in her previous explanation (i.e., Figure 5.8). Going back to her previous picture, Nicole entertained the idea of considering the  $\frac{3}{4}$ -portion to be “one whole”: “So, if I divide that into thirds, then [pointing to the rectangle drawn in black in Figure 5.8] my whole piece is four thirds, but I don’t think that’s working” (PO.I. 390-392). After thinking for



awhile, she drew two rectangles below her original black-color rectangles and again partitioned them into fourths (see the upper panel of Figure 5.10). She then colored three fourths in purple, identified the colored part as the new whole, and labeled it as such (see the label “1 whole” written in purple underneath the purple  $\frac{3}{4}$ -portion). Next she identified each of the black rectangles as four thirds and labeled them as such. She concluded that adding the two four thirds would give her eight thirds, which was equivalent to the quotient of the division problem under consideration. Yet, looking back at her work, she acknowledged that it was “a mess” since she was using the term “whole” to denote two different quantities. She thus reproduced her drawing, trying to show the reciprocal clearer. In this new drawing, she explicitly identified the absolute unit as four thirds of three fourths (see lower panel of Figure 5.10).

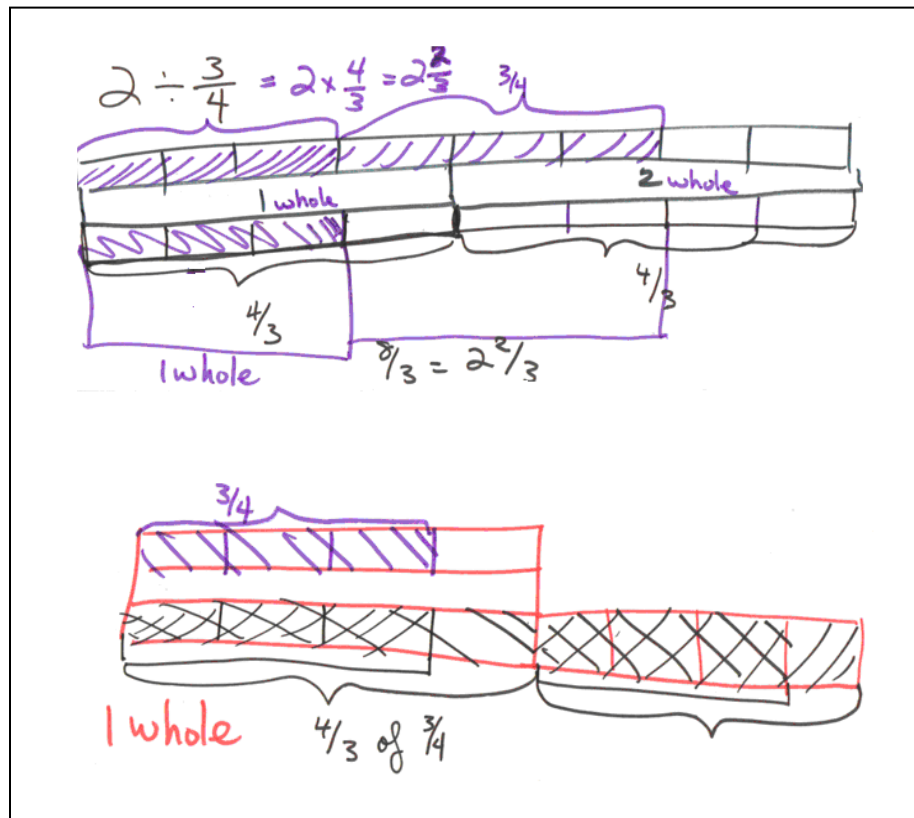


Figure 5.10. Nicole’s second (upper panel) and third (lower panel) experiments with explaining the invert-and-multiply algorithm for the division problem  $2 \div \frac{3}{4}$ .

Even with this new drawing and with apparently a better sense of the notion of the reciprocal, Nicole admitted that she found it hard to provide a clear explanation for the traditional algorithm. She attributed these difficulties to the two different “wholes” (i.e., units) involved in such an explanation:

You’ve got the first whole that we divided into fourths, and then took three fourths of that. Then when you look at it as four thirds, then your whole is different. Then your whole is the section, the three fourths section. ... I was just thinking it’s really confusing. [Referring to the drawing shown in the lower panel of Figure 5.10:] I’ve got my purple whole and my red whole (laughs). The purple whole is the three fourths and the red whole the four thirds. ... So that’s what makes it seem complicated. That you’re looking at two different sizes ... that you change what a whole is. ... To go from the red whole, and three fourths of the red whole, to the purple whole, and four thirds of the purple whole. (P.O.I. 549-568)

Despite Nicole’s difficulties with the two different “wholes”, her final explanation was more conceptually driven than her original one, particularly if one takes into consideration that she clearly identified the number of divisor units that could be fit into each dividend unit. The juxtaposition of her two different explanations – the initial one corresponding to Figure 5.9 and her final one as captured in Figure 5.10 – shows that with a somewhat better insight into the concept of the reciprocal, Nicole was able to provide a more conceptually driven explanation.

*Noticing and Interpreting-Evaluating.* Similar to her performance in the pre-intervention meeting, Nicole was dissatisfied with how the teacher responded to both Robert’s and Michelle’s questions. She maintained that the teacher should not have postponed Robert’s question because clarifying the concept of the fractional part of the quotient was consequential for successfully solving the task that the teacher later assigned. In considering the explanation that the teacher offered to Michelle as to why the “reciprocal works,” Nicole remarked that it “explains the procedure, but I’m not sure it really explains why it works” (P.O.I. 1490-1491). Nicole’s comment captured one significant deficiency of the virtual teacher’s explanation: it described rather than

explained the pertinent rule. Yet, Nicole appeared to miss two other critical deficiencies of this explanation: its inappropriate analogy and the problematic argument about using a reciprocal operation because “you use the reciprocal.” In the pre-intervention meeting, although she attended to both aforesaid features of the teacher’s explanation, she did not necessarily consider them problematic. Hence, overall, her pre- and post-intervention performances in the noticing and interpreting-evaluating tasks of this practice were alike.

### *Using Representations*

*Performing.* Similar to her pre-intervention performance in using representations, during the post-intervention meeting Nicole was seen selecting suitable representations and using them in a conceptually driven manner when her knowledge allowed her to do so.<sup>172</sup> What differentiates Nicole’s pre- and post-intervention performance was largely her greater attention to issues of mapping. Although during the pre-intervention meeting Nicole did attend to such issues, during the post-intervention meeting her attention to such issues escalated (particularly for the performing tasks). In explaining the quotient of the division problem  $2 \div \frac{3}{4}$  she was seen building clear connections among different representational systems: her drawing, her verbal explanation, and the mathematical symbols included in the algorithm at hand. To these, one should also add the different ways in which she attempted to distinguish between the absolute and relative units: by using two different colors (although she did not use them consistently throughout); by designating the size of these units using two different types of brackets; and by labeling them. To be sure, using a different label for the relative units instead of identifying them

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<sup>172</sup> The comparison of Nicole’s initial and final experimentations with representing the reciprocal during the post-intervention interview shows how Nicole’s changing understanding of the reciprocal informed her use of representations. Specifically, her evolving understanding helped her move from a numerically driven use of representations to a more conceptually driven use.

as “wholes” would have rendered her representations, and consequently her explanation, much clearer. This is something that Nathan was successful at doing by labeling his units as “cups” and “scoops.”

*Noticing and Interpreting-Evaluating.* Unlike her pre-intervention performance, during the post-intervention interview Nicole was more sensitive to issues of mapping among different representations. Thus, when watching Amanda’s episode, she criticized the teacher and Amanda for not discussing the correspondences between the mathematical sentence (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ) and the drawing shown on the board. She explained that if she were the teacher, she would “go back to the illustration and show how the illustration matches the sentence: where the one half is, where the one sixth is, and how there are three of them” (PO.I. 1114-1117). Notice that Nicole did not limit her comment to the problem’s final answer. Instead, she talked about drawing connections between *each* number involved in the mathematical sentence and its corresponding part on the number line illustrated on the board.

Nicole, however, was not equally sensitive to issues of mapping as captured in the second episode under consideration. While going over this episode, she argued that the two students did a good job in “divid[ing] up [the line] in segments and map[ping] out how it related to the [word] problem” (PO.I. 1237-1238). Like her pre-intervention performance, while going over this episode, Nicole filled in some of the gaps in the students’ work: “[The two students] colored red the part that was the three fourths that they were starting with,” and “they showed how many pieces of one sixth would be in that segment” (PO.I. 1217-1218). Although successful at pointing to the correspondences

never made explicit by the teacher or the students, Nicole said nothing about making these correspondences clear.

A plausible explanation for Nicole's differing performances in the two episodes might be that in the second episode, the visual representation and the mathematical sentence were superficially connected: Amanda and Julia drew the representation and, "building" on this representation, wrote the corresponding mathematical sentence. Hence, it seems that the second episode required an increased sensitivity to issues of mapping on the part of the PSTs so that they identify the lack of connections between the mathematical symbols and the visual drawings.

At the culmination of the interview, when asked to evaluate the virtual lesson as a whole, Nicole complained about the lesson's lack of connections between representations and number sentences:

**Nicole:** [The teacher and the students] used illustrations. I think I would have liked to see a little more correspondence between the illustrations and the numbers. I think that [the teacher] didn't do enough -- how do you call that, mapping?

**Charalambos:** Uh-hmm.

**Nicole:** So, mapping. Yeah, I would have liked to see more of that. ... Showing the connection between the figure and the representation, and the number sentences. (P.O.I. 1569-1577)

Unfortunately, the interviewer did not probe her to explain in which particular episodes she considered these connections to be missing. Such a follow-up question would have provided additional insights into the instances in which Nicole was more or less sensitive to issues of mapping. Nevertheless, this excerpt suggests that Nicole was more sensitive to such issues than she had been during the pre-intervention meeting.

### *Analyzing Students' Work and Contributions*

*Performing.* Nicole's post-intervention analysis of the three students' solutions was similar to her pre-intervention analysis of these students' work in that she again

analyzed each solution carefully and made appropriate assertions about the students' understanding; additionally, in the post-intervention course she considered the students' solutions, and particularly Michelle's, in the context of the wider lesson.

After correctly analyzing Michelle's solution, Nicole asserted that Michelle appeared to have understood the approach modeled in the lesson for solving division problems. She explained, however, that Michelle could have represented the leftover part as two fourths (in absolute units) or as two thirds (in relative units). Since Michelle did not write any numerical sentence to show her work, Nicole argued that one could question whether Michelle knew how to express the remainder. Hence, if she were the teacher, she would have asked Michelle to also represent her answer using a numerical expression. She considered that imperative, given that, in Nicole's perspective, Michelle's work appeared to be a direct consequence of "[Robert's] question that hasn't been answered yet" (PO.I. 1540-1541). As she legitimately argued, when Robert noticed the contradiction between his own solution and Amanda and Julia's solution, the teacher had the opportunity to elaborate upon the idea of the fractional part of the quotient, which appeared to be a main conceptual stumble in that lesson. The teacher's decision to not further delve into this issue made Nicole suspicious about whether Michelle and other students had really understood that concept. Hence Nicole, like Nathan, considered instruction as a plausible source of students' difficulties and of their insecure understanding of the content.

*Noticing and Interpreting-Evaluating.* With the exception of additionally identifying Alan's error (i.e., she noticed that Alan was taking sixths of a half yard instead of taking sixths of a yard), Nicole's performance during the post-intervention

meeting was very similar to her corresponding performance when she joined the program. In particular, although she identified the potential misconception in June's contribution (i.e., that the dividend is not always larger than the divisor) she did not feel that the teacher should have addressed this misconception; instead, she considered June's explanation satisfactory *for the particular problem* at hand. In contrast, she was surprised that the teacher did not address the error in Amanda and Julia's work and instead moved on to a new division problem. Similar to her pre-intervention performance, Nicole explained that not addressing this error could have created some perturbations in the flow of the lesson because of students getting a different answer to this problem when solving it using the algorithm. Hence, she argued that if she were the teacher she would have asked students to explain what the remainder in the two girls' solution meant. When Robert brought up this issue again, Nicole considered it a good opportunity for the teacher to clearly explain what the fractional part in a quotient means. No wonder, then, that she was dismayed to see the teacher postponing Robert's question and moving on to assigning yet another division problem.

#### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Nicole's performance in this practice was akin to her performance when she entered the program: although she identified the mathematics at stake in both episodes, her plans for responding to students' requests in each episode differed.

Nicole's plan for handling Alan's error resembled the approach she proposed during the pre-intervention meeting: she would ask Alan to read the problem's question, hoping that he would notice that the problem was asking for a sixth of a whole yard; she would then prompt Alan to show the whole yard on the line representation drawn on the

board. If Alan was still confused, she would thank him for his contribution and elicit other students' ideas. If Alan's error was the norm rather than the exception, she would first probe students' understanding of the concept of the whole; if their confusion persisted, she would remind them that the red line in their representation corresponded to only half of the yard, and thus, they needed to divide the whole line (both the red and the black parts) into six pieces. In sum, in supporting Alan, Nicole would first try to eliminate the potential sources of his confusion (i.e., misinterpreting the problem's question or the visual representation shown on the board). If that did not help, she would elicit other students' ideas; as a last resort, and provided that other students shared Alan's misconception, she would directly show students the part of the line that corresponded to the one sixth mentioned in the problem.

In the second episode, Nicole endorsed the teacher's approach to provide students with the hint of common multiples. She thought that in doing so, the teacher did a good job of "mov[ing] the lesson along" (PO.I. 1162-1163). Although she acknowledged that an alternative approach might have been to first elicit students' ideas of plausible ways to move beyond the stumble, she evaluated the teacher's approach as acceptable, claiming:

[T]he teacher refers back to previous knowledge that she knows that they covered. ... She suggests an idea rather than trying to get someone to come up with it. ... [But] they already had the information. (PO.I. 1153-1163)

It was therefore not surprising that if Nicole were the teacher, she would have pursued an approach similar to that exhibited by the teacher in the simulation. Nicole's reasoning during the pre-intervention interview was analogous to that just delineated. Engaging in what I called a "cost-benefit analysis", she regarded the teacher's offering of a hint as rational in that it helped the teacher save instructional time.



*Noticing and Interpreting-Evaluating.* No substantial differences were identified between Nicole's pre- and post-intervention performances in the tasks of this practice. Although Nicole noticed the intonation in the teacher's questions addressed to Alan, she sanctioned the teacher's questioning and her overall approach, on the premise that Alan was given another opportunity to correct his error and thus be successful. Interestingly enough, at the end of the interview she revisited this episode and claimed that the teacher's approach was successful at reinforcing both Alan's morale and his understanding: "[W]hen [Alan] drew the sixth the wrong way, [the teacher] was able to help him correct it so that he did have a good understanding of a whole" (PO.I. 1590-1591). This was rather an unwarranted argument, given that the teaching simulation provided no evidence supporting that after his interaction with the teacher Alan understood the concept of the whole; in contrast, Alan was presented to simply follow the hints implied by the teacher's questions.

As already mentioned, Nicole endorsed the teacher's approach to intervene and provide students with the hint of common multiples to move the lesson along. Her justification for approving the teacher's move to address June and Shaun's question with the whole class is interesting to consider: "[She] realized that more than one person would have the same question, so she addressed it to the whole class" (PO.I. 1143-1144). This argument is debatable, especially if considered in parallel with Nicole's assertion that the concept of common multiples was covered in previous lessons. If that were the case, it seems reasonable that at least a few students would have been able to recall and use the idea of common multiples without the teacher's reminder.

### *Professed Changes and Their Attributions*

Nicole felt that her thinking about mathematics did not change in between the two interviews; however, she confessed to having started thinking about the teacher-student interactions differently than how she thought about them at the outset of the program. When she joined the program, she tended to think of the teacher as the “fountain of knowledge to all students [who would give them] everything they need to know” (PO.I. 81-82); hence, she perceived the teaching of mathematics as “having all the information and ... try[ing] to present [it] in a way [students] could get it” (PO.I. 94-96); she was also “convinced that there was one way to do things and that you just follow the rules” (PO.I. 1639-1640). She attributed the changes in her thinking to the intervention courses:

I think the math itself, I probably see the same way. But the interaction with the teacher I see in a different way, because of the classes [i.e., the intervention courses]. And I think I have a better understanding of why you wanna ask the students for input. ... By listening to other people’s ideas in our class, I think it helped me realize that different people can have different solutions and different approaches to things and that they’re still valid. And, I think, it contributes to the overall understanding by listening to different people’s approaches. (PO.I. 1638-1645)

In contrast to her school experiences, during the intervention courses Nicole had the opportunity to experience teaching mathematics not as telling and showing, but as eliciting learners’ ideas and engaging them in sharing and discussing multiple solutions.<sup>173</sup> Therefore, the intervention apparently offered Nicole alternative images for teaching.

### *Changes in Performance on the LMT Test: A Closer Look*

The changes in Nicole’s performance on the LMT test were largely consistent with her post-intervention performance in the teaching simulation. Compared with her pre-intervention performance, during the second administration of the LMT test she

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<sup>173</sup> Nicole had comparatively fewer opportunities to observe and teach mathematics than her counterparts did in their field placement. This might explain why she largely focused on the two intervention courses and mentioned nothing about her fieldwork experiences.

correctly answered the two questions that pertained to the measurement interpretation of division (questions 15 and 25) and the two questions pertaining to different manifestations of the concept of unit (i.e., questions 1 and 20); her performance in the latter two questions was consistent with her more flexible thinking when working on tasks involving relative and absolute units. Nicole also answered correctly three other questions not directly related to the division of fractions (i.e., questions 4, 21, and 23). The only two inconsistencies in Nicole's post-intervention test performance pertained to that she incorrectly answered two questions (questions 14d and 24a) both related to the meaning of division (of fractions); however, in the first case she selected the choice "I am not sure," which, incidentally, was assigned a score of zero.

#### *Analytical Commentary*

Similar to Nathan, Nicole experienced small changes in both her MKT and in her teaching performance, thus corroborating the quantitative findings reported in the previous section. Her post-intervention performance also supports the association between knowledge and teaching performance considered in this study from both an affordance and a constraint perspective.

From an affordance viewpoint, Nicole's understanding of relative and absolute units was enhanced compared with her pre-intervention understanding. This greater understanding of units was reflected in the flexibility with which she explained the quotient in the division problem  $2 \div \frac{3}{4}$ ; in the pre-intervention meeting it took her some time to figure out that the quotient in a division-of-fractions problem should be expressed in relative rather than in absolute units. From a constraint standpoint, because of her still developing understanding of the reciprocal, Nicole still struggled with providing an

explanation for this concept for the same division problem. Yet, after an interviewer's hint, she was able to refocus her attention on the relative rather than the absolute units. Capitalizing on the notion of relative units, she explained the reciprocal, and consequently the invert-and-multiply algorithm, by providing a mathematically sound, yet not sufficiently unpacked, explanation.

Notable changes were also observed in Nicole's performance in using representations. Unlike her pre-intervention performance, post-intervention Nicole was very meticulous in mapping the visual representations she was using to the mathematical sentence corresponding to the division of fractions (especially when explaining the quotient). These changes – which can be considered to reflect the two intervention courses' focus on building such connections when using representations – are suggestive of the opportunities that PSTs need to be afforded to practice building such connections, if they are to start attending to such issues in their instruction.

The quantitative analysis yielded a strong correlation between the gains in the PSTs' MKT performance and the gains they experienced mainly in the practices of providing explanations and using representations. Nicole's performance substantiates this finding, since with the exception of the changes just reported (which all pertain to the foregoing two practices), her post-intervention performance in the remaining three practices was not notably different from her entrance performance. For instance, in the practice of selecting and using tasks, she again identified several of the second page's affordances, but she was not able to capitalize on all of them (e.g., she did not know how to support students in generalizing from specific division problems to the invert-and-multiply algorithm). Additionally, whereas she attended to issues of making meaning, she

was not particularly concerned with how the virtual teacher often failed to maintain the tasks' cognitive demand by posing leading questions, providing hints, and often doing the thinking for students. Similarly, in the practice of responding to students' requests for help, she endorsed the teacher's move to provide the class with the hint of common multiples on the premise that this idea was covered in a previous lesson. Her analysis of this episode was similar to her pre-intervention analysis, when she argued that, in doing so, the teacher saved instructional time. Finally, for the practice of analyzing students' work and contributions, Nicole again engaged in a meticulous analysis of students' work, made appropriate assertions about their understanding, and in some cases considered instruction as a plausible source of students' misconceptions and errors.

Both the changes and the lack thereof in Nicole's performance provide support to some of the propositions outlined in Chapter 4 (see Table 4.6). For example, Nicole's developing understanding of the concept of the reciprocal as the number of relative units to be fit within each absolute (dividend) unit was reflected in her shift from a largely numerically driven explanation to a more conceptually informed one, thus corroborating proposition A<sub>1</sub>. However, her understanding of the reciprocal was still not solid; this might have prevented her from restructuring task D in ways that could support students' transition from specific division problems to the general division algorithm. To the extent that this assumption holds, it supports proposition D<sub>2</sub>, which posits that a strong knowledge base is required for restructuring tasks to support student learning and understanding.

Nicole was also seen providing more unpacked explanations and being more scrupulous about drawing connections among her drawings, the mathematical symbols

she was using, and her verbal description. Although these changes could be attributed to her somewhat more profound understanding of the content, it seems more reasonable to also associate them with the multiple opportunities she and her classmates were offered during the two courses to practice providing such explanations and map the representations they were using to the algorithms under consideration. For example, certain aspects of Nicole's performance in providing explanations reflected the features of quality explanations discussed during the two intervention courses (e.g., good explanations typically make clear at the outset what is being explained; in providing such explanations the teacher starts with writing the number sentence on the board and attempts to align her verbal description with her use of diagrams). From this perspective, the aforesaid changes in Nicole's performance corroborate propositions  $A_4$  and  $B_3$  that pertain to the opportunities PSTs need to be afforded to provide unpacked explanations and build connections among the representations they use.

What seemed striking in Nicole's post-intervention performance was the lack of notable changes in her performance in the MTF-related practices, despite the professed changes in her beliefs about teaching and learning mathematics. For instance, while Nicole mentioned that she no longer perceived the mathematics teacher as being the "fountain of knowledge" and that, instead, she thought that the teacher of mathematics should try to elicit students' ideas and engage them in productive discussions, Nicole still endorsed a "giving more" instructional style. From a dichotomous perspective, this could be considered a discrepant result; yet considering the changes in Nicole's beliefs along a continuum could account for this discrepancy. Recall, for example, that in discussing how the teacher supported June and Shaun, Nicole remarked that Ms. Rebecca could

have elicited other students' ideas before providing them with the common-multiples hint. This alternative move resonates with Nicole's new perspective of eliciting students' ideas instead of merely feeding them with answers. In other words, although Nicole did not fully materialize the idea of the teacher being a facilitator rather than the fountain of knowledge, she still made some first small steps toward this direction.

That said, however, Nicole's comment that there are instances during which the teacher ought to be the fountain of knowledge – especially if doing so would move the lesson along and save instructional time – should not be totally dismissed. As discussed in Chapter 2, several researchers (e.g., Chazan & Ball, 1998; Leinhardt, 2001) forcefully criticize advocating teachers “not to tell.” The question that then arises is what might help Nicole find a workable balance between “giving more” and “telling less.” As discussed in Chapter 4, Nicole was engaged in what was called a cost-benefit analysis. I argue that helping her become aware of issues of cognitive demand and of the various ways in which the teacher might eliminate students' opportunities for thinking could inform her cost-benefit analysis. This, in turn, could help her make more educated decisions by expanding the parameters of her cost-benefit analysis to include not only considerations of contextual limitations (e.g., time pressure) but also considerations of how certain instructional moves might result in lessening students' opportunities for thinking and learning.

To facilitate comparability between Nicole's pre- and post-intervention performance, the changes reported above are presented in italics in Figure 5.11. This figure, which summarizes Nicole's post-intervention portrait, encompasses three major changes. First, in addition to her earlier teaching experiences, it includes her experiences

in the two intervention courses. These experiences, in turn, are presented to facilitate changes in her beliefs about teaching and learning mathematics. Second, it suggests that most of the changes in Nicole's performance pertained to the practices of providing explanations and using representations. Third, the factors of efficacy beliefs and her formal and informal teaching performances presented in her pre-intervention profile (see Chapter 4) were dropped since the post-intervention interview provided no data about their potential effect on Nicole's performance; instead I retained the factor of her out-of-school experiences because in talking about the fractional part of a quotient, she again referred to her experiences as a sewer.

A final comment is in order before leaving Nicole's case. Although the quantitative data present both Nathan and Nicole to have experienced proportionally small changes in their performance on the LMT test and the teaching simulation, the qualitative data suggest that Nathan experienced subtle changes in his teaching performance that Nicole did not. This finding justifies the decision to consider both cases in this analysis since the changes that Nathan experienced were qualitatively different from Nicole's. The comparison of Nathan's and Nicole's post-intervention performance also points to the importance of familiarizing PSTs with issues of cognitive demand, so that PSTs start considering such issues in their instruction.



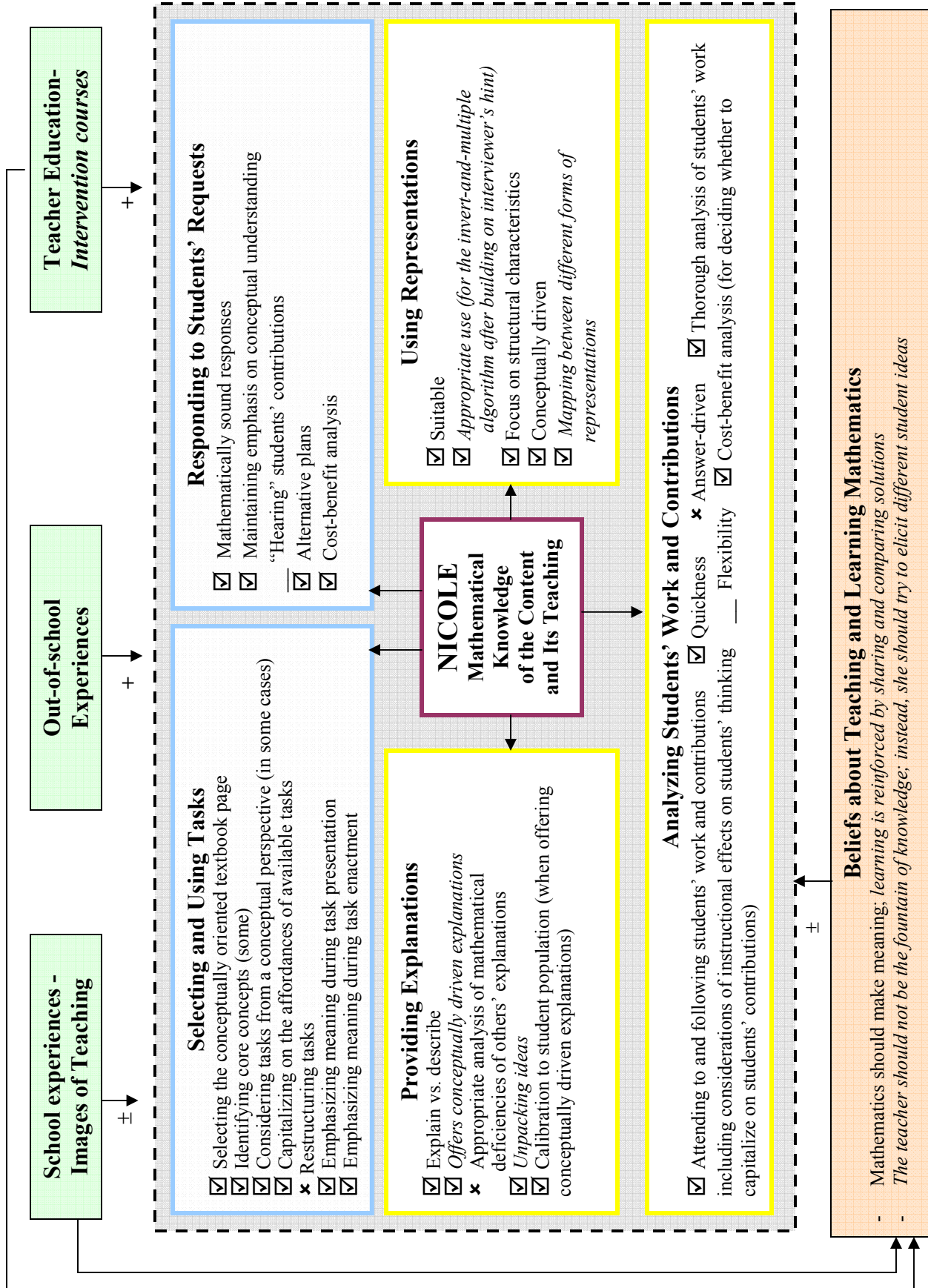


Figure 5.11. Considering the association between knowledge and teaching performance through Nicole's post-intervention profile.

*The Case of Deborah Revisited: When the Will Meets the Way: “Having a Bigger Toolkit”*

Like the preceding two cases, Deborah was a convergent case according to both her entrance scores and the changes in her performance. Unlike Nathan and Nicole, however, Deborah experienced high gains in her MKT and teaching-simulation performances. In fact, as Figure 5.3 suggests, she was the PST who exhibited the highest relative gains in both performances. The post-intervention interview with Deborah was conducted in late February, approximately 33 weeks after the pre-intervention meeting. Between the two meetings, apart from her coursework, Deborah observed and taught math lessons in a second-grade class.

Deborah, as the strongest “affordance” case considered in this chapter, clearly lends support to the quantitative findings reported in the previous section. The remarkable gains in her MKT performance were consistent with the striking gains in her teaching performance in all five practices under consideration. Even more, she exhibited greater gains in the practices of providing explanations and using representations, a finding which resonates with the quantitative analysis that showed the PSTs’ gains in MKT to be more strongly associated with the gains in their performance in the aforesaid two practices. The comparison of Deborah’s pre- and post-intervention performance also reveals how PSTs’ better grasp of the content – alongside an enrichment of their teaching “toolkit” – can support PSTs in structuring rich and challenging environments by successfully engaging in the five practices this study examines.

*Selecting and Using Tasks*

*Performing.* Similar to her pre-intervention performance, Deborah again considered the second page more conducive to teaching an introductory lesson on

division of fractions. Yet this time, her analysis went far beyond simply considering the second page more conceptually oriented than the first one. She remarked,

I think [the second page] contextualizes [the division of fractions] ... whereas the other [page] is purely just, like, procedure all the way. This one, at least, makes you think about what each portion represents. You know, like, am I dividing one sixth by one half, or am I dividing one half by one sixth, and why? ... And at the end, you can actually find out what the portions [are]. [Using the division problem  $2 \div \frac{3}{4} = 2 \frac{2}{3}$  as an example:] You can think, like, “What is this two thirds? Is it two thirds of one or is it two thirds of ... three fourths or something? So, I think it lends itself to that. It lends itself to ... helping kids, like, draw diagrams, as well as, you know, the number model. And it asks them to explain their reasoning. (PO.I. 1614-1627)

In this excerpt, Deborah identified several of the affordances of the second page that she overlooked during the pre-intervention meeting. Recall, for example, that at her entrance to the program she could not even understand what the fractional part of an answer means. During the post-intervention meeting, she considered the direction on explaining the fractional part of an answer to be an important asset of this page, but even more, she appropriately explained the fractional part of the division  $2 \div \frac{3}{4}$  (as two thirds of three fourths). She also referred to other affordances of this page, including its requirement that students present their answers in diagrams and “number models” (i.e., mathematical sentences), as well as explain their reasoning.

During the pre-intervention meeting, Deborah endorsed that the task D of the second page asks students to come up with an algorithm for dividing fractions instead of spoon-feeding them with this algorithm, like the first page does. However, she was unclear as to how she could support her students’ work on this task. Although still not totally clear on how she could enact this task, during the post-intervention meeting she proposed a viable way of doing so:

[M]aybe ask a question about what it is that you see when we divide something by a fraction; what happens to the final answer ... it does get bigger. You know, like ... something along those lines: “But why does it get bigger? By how much did it get bigger? Let’s take a look at this one problem” ... the first one, one half and one sixth. ... Like, “What do you see about this?” I think that would be a really good one to start at. And, you know, ask the kids what they noticed about that and probably try -- I would start there and have the visual next to the actual problem and try

and have a discussion, you know, and guide the kids to, like, talk about what it is that they see in that. (PO.I. 2692-2704)

Unquestionably, Deborah's approach, as exemplified in this excerpt, needed further refinement to enable her to enact task D in ways that would support students in identifying patterns and arriving at the traditional invert-and-multiply algorithm. Yet, it encompassed at least three favorable features that provided the foundation for doing so.

First, she proposed starting with a relatively easy division problem (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ), which could facilitate pattern identification. Second, she would ask suitable questions for scaffolding students' pertinent work. In particular, her first question, "What happens to the quotient in division-of-fractions problems?" could direct students' attention to an important mathematical idea: that in division of fractions the quotient is larger than the dividend. Although it is not clear how she would work with her students on the second question (i.e., "Why does the quotient become bigger?"), this question also had the potential to further enhance students' conceptual understanding of fraction division. Her third question (i.e., "By how much does the quotient become bigger?") could also catalyze the process of pattern identification by helping students identify relationships between the numbers involved in division-of-fraction equations. For instance, identifying that the quotient in the division problem  $\frac{1}{2} \div \frac{1}{6}$  is six times as big as its dividend could, under appropriate guidance, lead students to identify the invert-and-multiply algorithm (e.g., if expressed as  $\frac{1}{2} \cdot 6 = 3$ ). Third, instead of simply manipulating numbers, Deborah also proposed capitalizing on the visual representation which students would have produced when solving the first problem. Employing this representation could support students' thinking not only in addressing some of the questions listed above but also in understanding the meaning of the traditional algorithm. Her performance in explaining

the traditional algorithm by using a representation (see below) suggests that she had the potential to scaffold her students' work in this respect.

Interestingly enough, although Deborah proposed an instructional activity to support her students' work on task D, she contemplated not including task D in an introductory lesson on division of fractions. To justify her decision, she pointed out that the students should first be supported in fathoming the meaning of the division of fractions and then be asked to identify the traditional algorithm. She also thought that incorporating task D in an introductory lesson would result in a packed lesson, which, in turn, could jeopardize than support students' understanding, "unless [students] are geniuses" (PO.I. 1683).

In sum, contrary to her pre-intervention performance, during the post-intervention meeting Deborah identified the second page's affordances and proposed ideas to capitalize on them to support her students' understanding of fraction division. However, she was not equally successful at capitalizing on the first page to teach an introductory division-of-fractions lesson.

If she had to use the first page, she would start with posing a story problem based on the page's first exercise (i.e.,  $\frac{2}{3} \div \frac{1}{3}$ ). She would then use this problem to help her students understand the meaning of the division of fractions. Yet, when asked to pose such a word problem, she encountered significant difficulties. She first posed the problem "If it takes two thirds of a yard of string to make a bracelet, how many bracelets can I make with one third of a yard?" (PO.I. 214-217), which actually corresponds to the division  $\frac{1}{3} \div \frac{2}{3}$ . To represent this problem she drew a line, divided it into three parts, showed the two thirds, and then realized that "something did not seem right" (PO.I. 281).

After spending a considerable amount of time trying to figure out what was wrong with her proposed problem, she consulted the second page and revised this problem to reflect the mathematical sentence at hand (i.e., “I have two thirds of a yard and it takes one third of a yard to make a bow. How many bows can you make with two thirds of a yard?” PO.I. 536-538). She attributed her difficulties to her still-developing understanding of the measurement interpretation of division, which she regarded as not being as solid as her understanding of the partitive interpretation of division; as she mentioned, she liked “the ‘into groups’ versus [the] ‘measures’” (PO.I. 422) and she saw the partitive interpretation “much more clearly than the measurement one” (PO.I. 433).

When asked whether she would use any of the first page’s remaining exercises in her lesson, she admitted still being unclear as to how she could “xerox in on to use this page” (PO.I. 1417). Yet, in contrast to the pre-intervention meeting during which she argued that she could see the page’s exercises merely as computational problems, she now started seeing some of them from a more conceptual perspective. She mentioned,

Like two thirds divided by one third [exercise 1] ... at least you’re still dealing with thirds. ... But, with two thirds and four sevenths [exercise 6], you know, you’re dealing with something that’s, like ... I don’t know, I think that it’s trickier to figure out that exactly. ... I mean, your dealing with what seems to be a different animal, you know. (PO.I. 1500-1515)

Although not particularly articulate, in this excerpt Deborah pointed to an important difference between two of the page’s exercises. Exercise 6 is asking students to consider two fractions that represent a different unit, in Deborah’s own words, “a different animal.” In contrast, by employing two fractions of the same unit, Exercise 1 is easier to understand and solve. In light of this comment, Deborah’s selection of the latter exercise for launching an introductory lesson on fraction division appears to have been well-informed.

Similar to her performance during the first meeting, Deborah argued that the two word problems of this page did not lend themselves to the division of fractions. To her, both of them were multiplication problems. She wondered: “[T]hey have a fraction in them; is that why they qualified for [division of fractions]?” (PO.I. 1545). She argued that if she were to teach a lesson using this page, she would avoid using them.

Overall, Deborah would only minimally draw on the first page to teach an introductory lesson on the division of fractions. Even though the division sentence she selected was particularly conducive to building such a lesson, she did not appear to be able to further draw on this page and restructure its tasks to teach a conceptually oriented lesson as she wished. That said, however, it is important to acknowledge that she had some initial insights into how she could start restructuring some of the textbook tasks.

*Noticing and Interpreting-Evaluating.* Deborah’s post-intervention performance in the noticing and interpreting-evaluating tasks of this practice was in stark contrast to her pre-intervention performance. Recall that during the first meeting she largely endorsed the teacher’s enactment of tasks  $A_1$  and D; she only criticized the teacher for her introduction of task  $A_1$ , arguing that, as presented, the task was isolated from its corresponding word problem.

Although still not satisfied with the teacher’s introduction of this task, during the post-intervention meeting, Deborah was more concerned with the enactment of this task. She noticed that while engaging the students in this task the teacher was not only posing leading questions but was also providing students with steps to follow without helping them understand why they needed to do so. Consider, for example, how she reacted to the teacher’s admonition that students use “the whole line when dividing fractions”:

[Reading slides:] “You should use the whole, because if you use a part of it you won’t get the right answer.” Like, what does she mean by “whole” and what does she mean by “part”? ... That needs much more explanation. ... So they used the whole line. But do they know *why* they’re supposed to use the whole or what the whole actually is? Because it’s not the whole, the whole actually is the one half. (P.O.I. 2104-2127, emphasis in the original)

Deborah’s comment addresses two of the limitations of the teacher’s enactment of task A<sub>1</sub>. First, the steps the teacher outlined for students were rather ambiguous and could easily be misinterpreted. Second, Deborah was right in pointing out that the term “whole” needed more unpacking because the everyday meaning of this term is totally different from its mathematical connotation. According to the *American Heritage Dictionary* (2006, p. 1964), in daily communication, the term “whole” implies that which is available, the “full amount”; in the context of the word problem under consideration (see Figure 3.3), this term could be interpreted to imply having half a yard. In contrast, viewed from a mathematical perspective, and especially in the context of fractions, the term “whole” refers to the whole *unit*. In the context of the word problem at hand, this “whole” meant one yard. Thus, Deborah’s critique was legitimate, because, by simply advocating “use the whole,” the teacher could have confused students. This issue is not trivial, if one considers that even some of Deborah’s fellow-students (e.g., see Vonda’s case below) confounded the two foregoing meanings of the term “whole.”

Deborah was even more concerned with the enactment of task D. She argued that the teacher could have worked at a more concrete level by capitalizing on the representations that students had produced when solving task A<sub>1</sub> instead of simply asking them to “imagine” that they had certain numbers. She claimed,

I don’t think the kids have a clue about what they’re doing or why they’re doing it. ... Or even seeing the pattern itself, you know. Like, I think that if you just worked with the first one [task A<sub>1</sub>], one half divided by one sixth equals three and you have the picture there, you can get a lot more meaningful stuff ... rather than like, “Imagine if we have a sixth,” you know. And then ... immediately ... like, “Okay, so we’re multiplying by the reciprocal.” (P.O.I. 2761-2770)



In addition to criticizing the teacher's work in this episode, Deborah also offered an alternative approach to enact this task which was grounded in how she herself considered enacting this task by using representations. Thus, it seems reasonable to argue that by having an alternative image of the enactment of this task in mind, Deborah was better positioned to analyze the teacher's work in this episode and identify its limitations.

Deborah's appraisal of the teacher's overall use of the second page also reflected her increased ability to move beyond the surface characteristics of the virtual lesson when analyzing the virtual teacher's task enactment. For Deborah, Ms. Rebecca taught *the page* instead of using this page to support her students' understanding of the concepts associated with dividing fractions. As Deborah contended:

[Ms. Rebecca] was using the worksheet to teach the lesson rather than teaching the lesson and using the worksheet to, like, supplement her lesson. And I think that that was [her] biggest mistake: the worksheet *was* the lesson rather than the other way around ... rather than having a lesson in mind, and using the worksheet to supplement that lesson. As in, "I'm going to teach, you know, what dividing a fraction by a fraction is and I'm going to teach what this remainder is and I'm going to use this worksheet to teach these concepts." (PO.I. 3131- 3144, emphasis in the original)

All these comments suggest that alongside the pedagogical lens through which she considered the virtual lesson during the pre-intervention meeting, during the post-intervention meeting, Deborah was analyzing the lesson from a mathematical perspective, as well. The changes she reported having experienced (see below) provide some plausible explanations as to what might have informed this additional layer of her analysis.

### *Providing Explanations*

*Performing.* During the pre-intervention meeting Deborah considered division merely from a partitive perspective. Although she understood division of whole numbers as fitting divisor units within dividend units, she could not transfer this understanding to the division of fractions. To her, fraction division was merely a calculation. Thus, her

explanation for the quotient was limited to showing the dividend and the divisor of the given division problem. Without any conceptual understanding of the reciprocal, as she then confessed, she also refrained from providing an explanation for the invert-and-multiply algorithm. Although during the post-intervention meeting she still wrestled with providing explanations for the quotient and the traditional algorithm, her performance in providing these explanations was remarkably different from her pre-intervention performance. In what follows, instead of simply outlining the final explanations she provided during the post-intervention meeting, I also trace Deborah's work in developing these explanations. I do so because her work is suggestive of the difficulties she encountered in the process of providing explanations but, more importantly, of the resources upon which she drew to overcome these difficulties.

In explaining the quotient, Deborah first used the traditional algorithm to figure out the answer to the division problem (see Figure 5.12a). Based on the denominator of the improper fraction she got from applying the traditional algorithm (i.e.,  $\frac{8}{3}$ ), she drew two attached rectangles and divided each of them into thirds. Without any prompting, she realized her error, erased her original partitioning lines, re-divided each rectangle into fourths, and shaded three fourths of each rectangle (see Figure 5.12c). To get a better sense of the shaded and non-shaded regions, she replicated her picture, this time using colors and rearranging the shaded portions so that the first rectangle was fully shaded while the second was half-shaded (see Figure 5.12d; notice that she did not originally draw the line partitioning the non-shaded portion into two fourths). However, this rearrangement did not seem to provide her with any additional insights. She thus reverted to the traditional algorithm, converted the improper fraction into a mixed number (see

Figure 5.12b), and used this number to explain the different pieces in her picture. She remarked that she could show the two  $\frac{3}{4}$ -portions that could be “ma[d]e out when you have two” (PO.I. 663) but that the fractional part of the answer was “messing [her] up” (PO.I. 669). Thinking out loud, she wondered: “It’s two thirds of what? ... Two thirds of three fourths, right? And how to translate that into a picture is where I’m having technical difficulties” (PO.I. 673-675). Deborah’s struggles were not only “technical” as she argued; they were also conceptual, as her subsequent murmuring suggested: “Um ... is it two thirds of three fourths or is it two thirds of one?” (PO.I. 721-722). After some further thought, she went back to the problem’s answer, which she interpreted as “two and two thirds [of] three fourths” (PO.I. 788); based on that, she concluded that the non-shaded part was actually two thirds of three fourths. When probed to explain why, she revisited her drawing shown in Figure 5.12d and drew a horizontal line to partition the non-shaded portion into two parts, explaining that this non-shaded portion corresponded to the two thirds of the three-fourths unit.

Because most of her work until this point pertained to figuring out for herself why this division problem worked, the interviewer reminded her of the original task, which was to provide an explanation for a sixth grader. To explain the quotient to this student, Deborah again drew two rectangles, one attached to the other, as shown in Figure 5.12e. She remarked that it would have been better to draw separate rectangles to illustrate the two units of the dividend, but she nevertheless proceeded with dividing each of these rectangles into “four equal pieces” (PO.I. 906) to show the three fourths. Next, she colored a  $\frac{3}{4}$ -portion of the first rectangle in blue and a  $\frac{3}{4}$ -portion of the second rectangle in red. Pointing to the portions she colored, she explained:

And I'd say, "This right here [pointing to the blue  $\frac{3}{4}$ -portion] is three fourths of this one [pointing to the whole first rectangle]; so this is three fourths." And then [pointing to the red  $\frac{3}{4}$ -portion:] this is three fourths. So, right off the bat, how many three fourths do we get off of two? And it would be two, which is these [pointing to the blue and red  $\frac{3}{4}$ -portions]. (PO.I. 908-913)

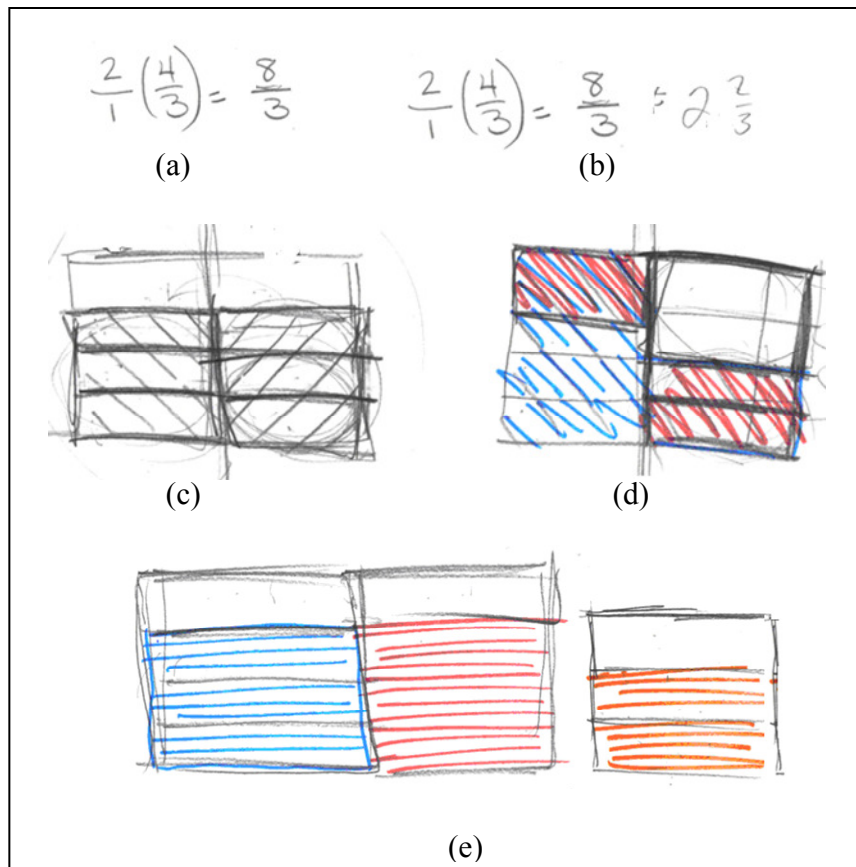


Figure 5.12. Deborah's work on explaining the quotient for the division problem  $2 \div \frac{3}{4}$  (a and b: using the traditional algorithm to figure out the answer to the problem; c: first steps in representing the division; d: re-arranging the shaded portions; e: illustration used to explain the quotient to a sixth grader).

She then compared the two non-shaded portions of her figure (which she identified as the "negative space," PO.I. 922) to the blue  $\frac{3}{4}$ -portion and explained that "we don't even have this" (PO.I. 938). Next, she drew a rectangle equal to the blue portion (see rectangle on the right-hand side of Figure 5.12e) and divided it into three parts. Pretending to be transferring the two non-shaded regions into the  $\frac{3}{4}$ -rectangle, she colored two of its parts in orange, and continued:

So this [pointing to the blue  $\frac{3}{4}$ -portion] is three fourths; this [pointing to the red  $\frac{3}{4}$ -portion] is three fourths; this [pointing to the  $\frac{3}{4}$ -rectangle] is the same three fourths [as the previous two  $\frac{3}{4}$ -

portions]. And if I transfer these two [pointing to the two  $\frac{1}{4}$  non-shaded pieces] over here [showing the portion colored in orange in the  $\frac{3}{4}$ -rectangle] we have how much of three fourths? I broke it into three pieces and I shaded in two; it's two thirds of three fourths. And that's how you get two thirds. So in total we get one [pointing to the blue  $\frac{3}{4}$ -portion], two [pointing to the red  $\frac{3}{4}$ -portion], and two thirds [pointing to the  $\frac{2}{3}$  portion of the  $\frac{3}{4}$ -rectangle shaded in orange]. (P.O.I. 943-951)

Deborah's work as delineated above is a typical example of a hybrid numerically and conceptually driven explanation. Her explanation was informed, and to a certain extent guided, by the numbers involved in the division problem under consideration, as evident by two aspects of her work: first, she initially partitioned the two whole rectangles into thirds, apparently attempting to illustrate the denominator of the quotient yielded by applying the algorithm; second, when her application of the algorithm resulted in two thirds rather than two fourths, which was the non-shaded portion of her figure, Deborah was puzzled. Yet, in contrast to her pre-intervention performance, she now had some conceptual scaffolds to support her work, the key among them being that she knew the meaning of the division  $2 \div \frac{3}{4}$  (i.e., the number of three fourths to be made out of two wholes). Deborah's understanding of relative units, however, was still not very solid, as suggested by her dilemma whether the fractional part of the quotient represented two thirds of the whole or two thirds of three fourths. To move beyond this impasse, she resorted to her understanding of division and interpreted the quotient as two and two-thirds of three fourths. This catalyzed her subsequent work on explaining the quotient.

Among the strengths of Deborah's subsequent explanation one could list the following: (i) she clearly showed the dividend and even proposed an alternative representation that would help better illustrate the dividend's two whole units; (ii) she was explicit that these two whole units needed to be partitioned into four equal parts; (iii) she clearly showed the divisor unit and the two  $\frac{3}{4}$ -portions that could be made out of the

dividend; (iv) before moving on to explain the fractional part of the quotient, she posed a question that could help solidify students' understanding of the whole-number part of the quotient; (v) she clearly explained that she could not make another whole  $\frac{3}{4}$ -portion; (vi) to explain what part of the  $\frac{3}{4}$ -portion she could cover with the remaining pieces, she explicitly illustrated the size of the relative unit and showed that the leftover part covered only a fraction of this relative unit; (vii) she explained the fractional part of the quotient by drawing on the part-whole concept of fractions (i.e., the leftover part covered only two of the three parts of the  $\frac{3}{4}$ -rectangle); and (viii) once finished, she showed the correspondences between each part of the quotient and her drawing. Taken together, these eight features suggest that her explanation was sufficiently unpacked and could be understood by an average sixth grader. That said, her explanation could be further refined in at least two respects. First, she could have initiated her explanation by clearly outlining the concept under consideration, namely that she was trying to figure out the number of three fourths that could be made out of two whole units. Second, she could have avoided referring to the negative space, because in a real-classroom setting this reference might have caused some confusion, especially if associated with negative numbers. Even so, her explanation in its *final form* could be characterized as conceptually driven and sufficiently unpacked.

As already mentioned, during the post-intervention meeting Deborah was able to provide an explanation for the traditional algorithm; in addition to her final explanation, it is informative to also consider the process she pursued in developing this explanation. Although asked to explain the invert-and-multiply algorithm with respect to the division problem  $2 \div \frac{3}{4}$ , Deborah first started by considering two easier examples. She wrote the

mathematical sentence  $1 \div \frac{1}{3}$ , drew a rectangle, divided it into three parts, and said, “So, I have one divided by one third, right? So how many one third pieces can I get? And the answer is three” (PO.I. 1005-1007). At this point, she went back and completed her mathematical sentence using the traditional algorithm (see Figure 5.13a). Next, she considered a harder problem: “Okay, so that’s great! ... I’ll do a harder one. Let’s say two. Two into one-third pieces and it’s going to be six” (PO.I. 1017-1019). Repeating the same procedure, she drew two rectangles and divided them into thirds, thus getting six pieces, which aligned with the quotient of this division problem (see Figure 5.13b).

Building on these two examples, she then attempted a generalization of why the invert-and-multiply rule works:

So why does [the rule] work? Well, it works, because you’re dealing with a fraction, right? So, whenever you’re breaking something into pieces of a whole, you’re going to end up with more. ... If you have something broken into pieces of one third ... I’m essentially breaking that whole into three pieces. Or if I’m breaking a whole into pieces of one fourth, I’m breaking it into four pieces. So if you’re actually asking how many pieces ... can I get, you have to multiply it by -- I think I’m not explaining it clear, but when you get into, like, two thirds or three over two, then I’d probably get a little messed up there, but the same principle applies. ... If you’re breaking it into pieces of one third, you’re breaking it into three pieces. Therefore you have to -- like, it works to multiply it by the reciprocal. (PO.I. 1032-1049)

This excerpt provides evidence about Deborah’s developing understanding of the concept of the reciprocal. As she rightfully observed, the reciprocal in a division-of-fractions problem designates the number of “parts” that can be fit into the “whole.” Yet, this explanation warranted further polishing to accommodate for other examples that do not work as nicely as the two examples with which she originally experimented. This refinement resulted from her work on and struggles with explaining the rule for the assigned division problem.

To explain the algorithm for the division problem  $2 \div \frac{3}{4}$ , Deborah first drew two rectangles, partitioned each into fourths, and colored a  $\frac{3}{4}$ -portion in each of them (see

Figure 5.13e). She then counted the fourths in the two rectangles:

So you have one, two, three, and four [for the first rectangle]. And you have one, two, three, and four [for the second rectangle]. ... So I broke [the first rectangle] into four pieces and I broke [the second rectangle] into four pieces. ... I mean, I see the eights, right? So you got the eight here [points to the eight fourths in her picture] and the eight here [points to number 8 in the algorithm:  $2 \div \frac{3}{4} = \frac{2}{1} \cdot \frac{4}{3} = \frac{8}{3}$ ]. (P.O.I. 1093-1105)

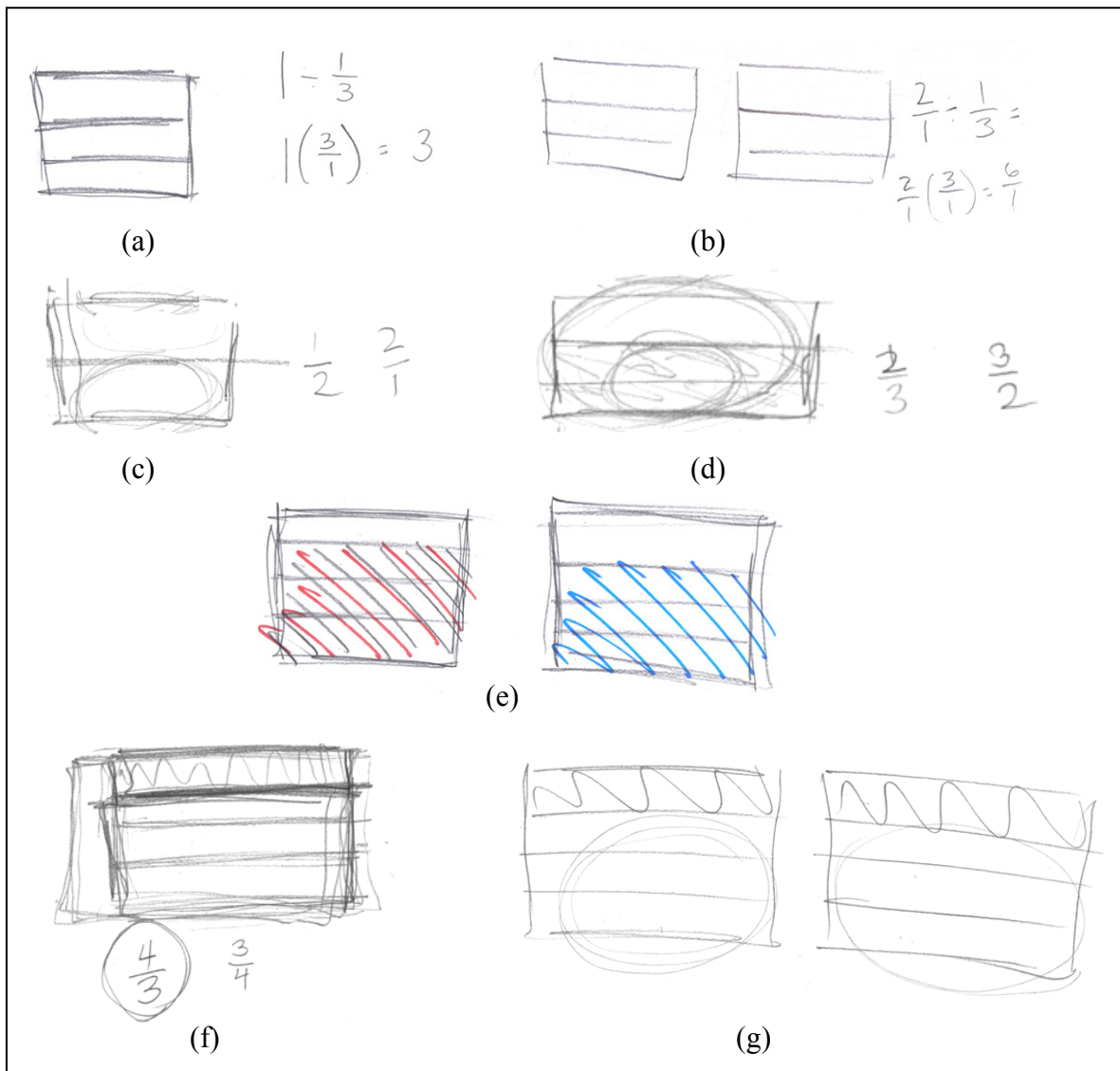


Figure 5.13. Deborah's work on explaining the invert-and-multiply algorithm (a to d: experimenting with easier examples; e to g: explaining the algorithm for the division problem  $2 \div \frac{3}{4}$ ).



To check whether her thinking was correct, she went back to the previous two examples and compared her pictures with their corresponding algorithms; in both cases, the pieces she obtained by dividing the “wholes” aligned with the numbers involved in the algorithm. Yet this comparison led to an unavoidable question: “Okay, so that’s where the eight is, but what’s the third?” (PO.I. 1121-1122). This question implies that until this point Deborah was apparently not focusing on the whole fraction corresponding to the reciprocal. Instead, when referring to the number of the pieces she could fit within the dividend units, she was probably thinking only about the numerator of the reciprocal. This inconsistency created a cognitive stumble for her:

I was thinking, like, it would be a one-to-one correlation that was just as obvious as when you’re not dealing with ... something over one. ... But, you know, it’s not working out that way. So it’s, like, look I have the eight pieces; now where’s ... the over three? ... That’s more tricky [sic].  
(PO. I. 1130-1144)

At this point, the interviewer intervened and prompted her to explain how she conceptualized the reciprocal, by asking her: “So, what does the four third mean?” (PO.I. 1146-1147). Seeking an answer to this question, Deborah again reverted to easier examples. She first drew a rectangle and divided it into two parts. Using this drawing, she showed both the one half and the two one-half pieces comprising the whole; she then wrote the number  $\frac{1}{2}$  and its reciprocal (i.e.,  $\frac{2}{1}$ ) next to her drawing (see Figure 5.13c). Following that, she attempted to apply the same idea to a harder example. She again drew a rectangle, divided it into three parts, and showed two of them (see the two parts encompassing the small circle in Figure 5.13d). At this point, she stopped and wondered: “This [what she showed before] is two thirds; what is three over two?” (PO.I. 1170). After spending more time trying to figure out the “three over two” in her picture, she had a revelation:

But if this [points to the two thirds in Figure 5.13d] is the two thirds and this [circumscribes the whole rectangle including all three thirds] is the three halves, it's because this [the whole rectangle] is one and a half of this [the  $\frac{2}{3}$ -piece]! (PO.I. 1206-1210)

This idea catalyzed her progress in explaining the traditional division algorithm for the division problem  $2 \div \frac{3}{4}$ . Building on this idea, she drew another rectangle and partitioned it into four parts (see Figure 5.13f). After identifying the lower three fourths of this rectangle as a  $\frac{3}{4}$ -portion, she commented: "And this [the whole rectangle] is one and one third [of] three fourths [pointing to the part which she identified as  $\frac{3}{4}$ ] or four thirds" (PO.I. 1228-1229). She elaborated this idea, by pointing to the upper fourth of this rectangle, which she identified as the one third of what she originally designated as the  $\frac{3}{4}$ -portion of this rectangle:

This one piece [the upper fourth] is the one ... third of this [the  $\frac{3}{4}$  portion] ... like, it fits into here [points to one of the three fourths comprising the  $\frac{3}{4}$ -portion]. ... So, four thirds means ... one and one third three-fourths. (PO.I. 1252-1275)

Prompted to explain the whole algorithm (i.e., the two times four thirds), she remarked that "it's the same thing except that there is [sic] two of them" (PO.I. 1302). She thus drew two rectangles akin to the rectangle she used to explain the reciprocal of four thirds (see Figure 5.13g) and, after circumscribing three fourths in each of them to identify the  $\frac{3}{4}$ -portion, she pointed to each whole rectangle and remarked: "Two times four thirds is one and one third plus one and one third, which equals two and two thirds" (PO.I. 1323-1325).

Three aspects of Deborah's work in explaining the invert-and-multiply algorithm are important to highlight. First, her explanation was not a conventional one; it can be characterized as an explanation *in the making*. To put it differently, in providing this explanation, she did not start with a well-developed understanding of the concept of the reciprocal, which would have enabled her to provide a complete and nicely unfolding

explanation for the traditional division-of-fractions algorithm. Rather, her explanation meandered as her understanding of the concepts at hand was developing in situ. From this perspective, one could talk about a reciprocal relationship between her explanation and her understanding: she better understood the algorithm in the process of explaining it and, as her understanding of this algorithm was gradually enhanced, her explanation became more conceptually driven.

Second, like her explanation of the quotient, her work in explicating this algorithm could be considered a hybrid of a numerically and a conceptually driven explanation. Although she focused on the numbers involved in her first two examples, she clearly talked about the number of parts that could be fit within each dividend unit. This idea, though not well-refined, constitutes the crux of the notion of the reciprocal. To capitalize on this idea and provide an explanation for more complicated fraction divisions, Deborah also needed a more flexible understanding of the notion of units, which she seemed to still be developing. Without this solid and flexible understanding of units, when she transitioned to more complicated division examples she grounded most of her work in the numbers involved in the algorithms of the divisions she was considering. The interviewer's question "So, what does the four thirds mean?" might have redirected her thinking from the reciprocal as a number to the reciprocal as a concept. After his question, she engaged in another experimentation cycle, which helped her refine her understanding of the reciprocal and, in turn, supported her explanation of the traditional algorithm under consideration.

Another notable aspect of Deborah's work was that she resorted to easier examples. Lacking a clear understanding of the concept of the reciprocal that would have

supported her explanation, she employed four examples:  $1 \div \frac{1}{3}$  and  $2 \div \frac{1}{3}$  (during the first experimentation cycle) and  $1 \div \frac{1}{2}$  and  $1 \div \frac{2}{3}$  (during the second experimentation cycle). The other PSTs were also seen resorting to such easier examples to validate the appropriateness of their thinking or to scaffold their explanations (e.g., Nathan and Kimberly during the pre-intervention meeting). Yet, what distinguishes Deborah's performance as captured above from the other PSTs' performance is her *selection* and *sequencing* of examples. I expound this argument by first explaining what each of these four examples afforded her and then explicating what the sequencing of these examples helped her achieve.

The first example of the first experimentation cycle allowed Deborah to consider a much easier situation than that she was asked to explain and, thereby form an initial hypothesis about the concept of the reciprocal as the number of parts that could be fit within each dividend unit. Once she figured out this definition, she moved to the second example, which involved two dividend units. As such, this latter example helped her check whether her understanding of the reciprocal was applicable to more than one dividend units. The examples that she employed in the second experimentation cycle also appeared to be well-chosen. The first example of this cycle again helped her test the very basic notion of the reciprocal as outlined above but also appeared to have generated a platform for her to explore how the divisor and its reciprocal manifested themselves in the drawing. To the extent that this argument holds, the second example of this cycle helped her further test the insights she gleaned from the previous example and consequently refine her definition and understanding of the reciprocal. If one ignores the example  $1 \div \frac{1}{2}$  and focuses on the other three ( $1 \div \frac{1}{3}$ ,  $2 \div \frac{1}{3}$ , and  $1 \div \frac{2}{3}$ ), one could also

talk about a successful sequencing of examples that helped Deborah establish a primitive notion of the reciprocal (the first example), test this notion (the second example), and elaborate and refine this notion (the third example). Hence, even though Deborah did not have a ready-made explanation for the traditional algorithm, I argue that her selection and sequencing of examples and how she experimented with them could be equally instructive for students' thinking and understanding as offering them a polished explanation for this algorithm from the outset.

To summarize, in both explanations considered above, Deborah did not appear to have a full-fledged understanding of the concepts at hand. Yet, by building on the core concept of division (when explaining the quotient) and by experimenting with easier examples (when explaining the algorithm) she was able to provide explanations that at their final stage were conceptually founded and sufficiently unpacked to be understood by a sixth grader. In developing these explanations, her thinking also appeared to be informed by the numbers involved in the procedures under consideration.

*Noticing and Interpreting-Evaluating.* In contrast to the pre-intervention meeting, during the post-intervention meeting Deborah directly noticed and critiqued the teacher's postponing of Robert's question. She commented that his question was "incredibly legitimate and at the heart of what [the teacher]'s doing in this lesson" (PO.I. 3026-3027) since it pertained to the idea of the fractional part of the quotient. Deborah even considered the teacher's decision to address Michelle's question and not Robert's unjustifiable; she argued that if she were teaching, she would have addressed Robert's question and deferred Michelle's question for a subsequent lesson. Similar to her pre-intervention analysis, Deborah was also concerned with some aspects of the teacher's

explanation as to why the reciprocal works. For instance, while going over the teacher's argument about using a reciprocal operation because the reciprocal is being used, she claimed: "I don't know if that's respecting the integrity of math, I don't know if that's true" (PO.I. 3060-3061). Similarly, in considering the teacher's analogy of the reciprocal operation to adding and subtracting negative numbers, she remarked:

I don't know if it is similar to that. Maybe, it's reminiscent of that, I don't know. I, honestly, can't comment on what she says, whether or not it's way off base. ... Like, yes, you invert it; and, yes, division and multiplication are two sides of the same coin, in many respects. But I don't know. (PO.I. 3062-3065)

As this excerpt suggests, Deborah avoided taking an evaluative stance toward the quality of the teacher's explanation. Although the interview data do not support any definite interpretation as to why this might have been the case, a plausible interpretation could be that her understanding of the content was not profound enough to allow her to appropriately evaluate the mathematical quality of the teacher's explanation.

### *Using Representations*

*Performing.* In the pre-intervention meeting, Deborah displayed a strong preference for "pies" (i.e., circular representations). In explaining the quotient for the division  $2 \div \frac{3}{4}$ , she used such representations in a numerically driven manner to illustrate the dividend and the divisor rather than to explain the meaning of this division. Her post-intervention explanations as discussed above point to three notable improvements in her performance in using representations.

First, although she again selected suitable representations for her explanations, she used rectangular area models instead of circular representations. Second, like her explanations, her use of representations represented an amalgam of conceptually and numerically driven work. Her move to initially partition the two rectangles into thirds is a

typical manifestation of the influence of the numbers in her work; in contrast, her discussion of how many times a  $\frac{3}{4}$ -portion could fit into the two dividend units exemplifies how certain concepts informed her representation use. Other examples, in which the boundaries of the numerical and the conceptual aspects were more blurred, were also evident in her explanation for the invert-and-multiply algorithm. Third, a closer look at her explanations suggests that Deborah was quite successful at drawing connections between her representations and the numbers involved in the algorithm under consideration.

An example corroborating the last point stems from Deborah's explanation of the quotient for the division problem  $2 \div \frac{3}{4}$ . Although she did not numerically label the different parts of her drawing, she was explicit as to what each part represented: she clearly talked about the dividend units; she identified the divisor units; and at the end of her work, she went back to her drawing and drew connections between the three parts of her drawing (i.e., the blue  $\frac{3}{4}$ -portion, the red  $\frac{3}{4}$ -portion, and the orange  $\frac{2}{3}$ -portion of the  $\frac{3}{4}$ -rectangle) and the quotient of this division. She was equally explicit in clarifying the connection between the relative and the absolute units when explaining what the reciprocal in the same division problem meant: she clearly identified the  $\frac{3}{4}$ -portion in a whole unit and then showed how the remaining one fourth of this unit represented one third of the  $\frac{3}{4}$ -portion, hence establishing that there were four thirds of a  $\frac{3}{4}$ -portion within each absolute unit.

*Noticing and Interpreting-Evaluating.* Deborah's post-intervention performance in the noticing and interpreting-evaluating tasks of this practice was remarkably different from her respective pre-intervention performance for both episodes under consideration.

As in her pre-intervention performance, Deborah first thought that Amanda figured out the answer to the problem by using the traditional algorithm. However, in contrast to the first meeting, she speculated that it was also possible that Amanda was getting her answer from the diagram shown on the board. Building on this speculation, she criticized the teacher for not probing Amanda and the other students to build connections among the drawing shown on the board, the mathematical sentence Amanda wrote, and the word problem under consideration:

So, I think that more questions need to be asked about the correlation between the visual representation and the [word] problem itself. And a discussion needs to take place, you know, in which, she's facilitating ... and leading kids to, like, "Well ... why we broke it up into that many pieces and this is where you actually see the pieces in this [word] problem ... and in the [mathematical] sentence we set up." ... I think that [the] one half divided by one sixth needs to be there as well, so that kids can see the correlation between, "Oh, wait, this is the one half in the problem and this is the one sixth." So I think that [the mathematical sentence] one half divided by one sixth needs to always be on the board alongside the visual. (PO.I. 2324-2340)

This excerpt reveals that Deborah considered drawing connections both for the quotient of the division problem at hand and for its dividend and divisor. She identified the lack of such connections in the second episode, as well.

Contrary to her performance in the pre-intervention meeting, while going over Amanda and Julia's solution to the division problem  $\frac{3}{4} \div \frac{1}{6}$ , Deborah pointed to all the instances in which connections among the word problem, the visual representation, and the mathematical sentence were warranted but never made. For example, she noticed that the two girls did not explain why they divided the line they used into twelve parts, why they colored nine of them red, and why they were taking pairs of twelfths. She also remarked that the teacher should have prompted the students to explain what each number in their mathematical sentence represented.

Deborah's sensitivity to issues of mapping between different representations was also evident in her overall evaluation of the virtual lesson. She claimed that although the



teacher was using visual representations in her lesson, the teacher did not capitalize on them to reinforce student learning. In particular, Deborah mentioned,

[The virtual teacher]'s using a visual ... but she's not helping kids see how the visual helps ... solve the problem or how the visual *is* the problem. ... You know, like, I think that she's got the structure, the outside structure of the lesson looks good and it seems good, but there's tons of stuff in between that she's glazing over or omitting altogether. (PO.I. 2347-2358, emphasis in the original)

Two ideas captured in this quotation are particularly noteworthy. First, the idea that the visual representation is actually the problem itself suggests that Deborah appeared to be thinking of representations not as decorative or ancillary tools but as key devices for doing mathematics. Second, in contrast to her pre-intervention performance during which she focused on the “outside structure” of the lesson, which at that point she evaluated as effective, she now delved deeper into the quality of the lesson and identified several of its deficiencies that could impinge on students’ learning and understanding.

#### *Analyzing Students’ Work and Contributions*

*Performing.* Deborah’s post-intervention performance was different from her pre-intervention performance in three respects: (a) her analysis of the students’ work was more thorough and accurate; (b) her assertions were more appropriate; and (c) she considered students’ errors in the context of the wider virtual lesson.

The most notable changes in Deborah’s performance pertained to her analysis of Michelle’s work. During the pre-intervention meeting, Deborah first figured out the numerical solution to the division problem and *then* tried to make sense of Michelle’s solution (what I called an *answer-driven* analysis); however, without being able to understand Michelle’s solution, she thought that it might have been better for Michelle to use the traditional algorithm instead of employing a diagram to solve the assigned division problem. During the post-intervention meeting, Deborah followed a different

route in analyzing Michelle's work: she focused on Michelle's diagram and tried to make sense of it. She identified the dividend and the divisor in this student's diagram and then after some thinking, she figured out that Michelle correctly showed the whole-number portion of the quotient. She was concerned, though, that Michelle did not represent the fractional part of her answer with a number, which would have helped the teacher better appraise Michelle's understanding of this concept:

[Michelle] understands that this [shows the red line in Michelle's drawing; see Figure 3.5] is two and three fourths. ... And then she understands that you're dividing that line into three-fourths pieces, which she has done here [points to the curved lines in Michelle's solution:] three fourths, three fourths, three fourths. She's done it accurately and she's done it right. And then, it looks to me, like, she's got two thirds leftover or one half leftover ... one half of one or two thirds of three fourths. But you don't know if she knows what that means there. ... I don't know, I can't tell if she gets it or not; you'd have to ask her. ... And I don't know if she knows, in the context of the problem what this three [the 3 in the quotient] represents. Three what? I don't know if she ... could explain it sufficiently, but she's definitely got some strong foundational things to build from. (P.O.I. 2954-2980)

Notice that in her analysis, Deborah appropriately identified the two possible interpretations of the leftover part in this problem (i.e., one half of the whole yard or two thirds of the  $\frac{3}{4}$ -portion that was used to make each bow). Additionally, she not only made accurate assertions about Michelle's understanding but she also pointed to aspects of this student's work that needed further clarification. The flexibility with which Deborah approached and analyzed Michelle's work and the accurate assertions she made about this student's understanding should not be dissociated from her own comprehension of the content, and particularly her better grasp of the fractional part of quotients.

During the pre-intervention meeting, Deborah analyzed Robert's and Ann's solutions correctly, but did not make appropriate assertions about these students' understanding. In addition to correctly analyzing Robert's work, during the post-intervention meeting, Deborah made appropriate assertions about his understanding. She argued that Robert seemed to "got the procedure down" (P.O.I. 1953): he understood how

to take the reciprocal of a given fraction and which fraction to take the reciprocal of, how to multiply fractions, and how to “un-simplify fractions” (PO.I. 2951).<sup>174</sup> Overall, her analysis of Robert’s work and her assertions about his understanding were accurate.

In addition to correctly analyzing Ann’s work and making appropriate assertions about this student’s understanding or lack thereof, during the post-intervention meeting Deborah also viewed Ann’s work in light of the quality of the mathematics lesson. She argued that Ann’s confusion might have stemmed “from the teacher’s lesson” (PO.I. 2938); Deborah supported her argument by pointing out the insufficient way in which the division-of-fractions algorithm was presented and worked on during this lesson.

*Noticing and Interpreting-Evaluating.* Remarkable changes were also identified in Deborah’s performance in the noticing and interpreting-evaluating tasks of this practice. Recall that during the pre-intervention meeting she did not identify any of the problematic aspects of the four student contributions designed for this practice. Even when prompted, she did not notice the misconception in June’s work, she was uncertain about the correctness of Alan’s work, and she could not explain the inconsistency between Amanda and Julia’s solution and the solution yielded from the algorithm.

During the post-intervention meeting, Deborah directly identified the misconception in June’s explanation. Remarking that in fraction division the dividend can be smaller than the divisor, she continued, “You can have a bow that takes a half of a yard to make and have a sixth of a yard and ask, ‘How many bows can I make?’” (PO.I. 1937-1938). To help June and her classmates better understand this idea, she contemplated posing a simpler problem than that considered in June’s episode: “If it

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<sup>174</sup> Her last comment referred to Robert’s work on converting the mixed number into an improper fraction. She used this incorrect term to describe this aspect of his work, probably because she lacked knowledge of the correct terminology.

takes a whole cup of sugar to make a cookie and I have a half a cup of sugar, how many cookies can I make?”, which she correctly represented as  $\frac{1}{2} \div 1$  (PO.I. 1901-1902).

Similar to her analysis of Ann’s solution, Deborah interpreted Alan’s confusion and error not only through the lens of what the student himself was not able to do, but also through the lens of how instruction itself might have interfered with his thinking. In particular, she explicated that Alan’s error might have stemmed from the teacher’s awkward use of representations. Deborah also argued that the teacher should at least have made clear what the red and the black lines on the board represented or given students the option to use whichever representation they wished instead of forcing them to use the representation she proposed.

Resolving the inconsistency that emerged from Amanda and Julia’s interpretation of the leftover part in absolute rather than relative units was not an easy task for Deborah, even during the post-intervention meeting. After she applied the traditional algorithm to solve the division problem the two girls were solving in the episode under consideration, Deborah identified the inconsistency between her answer and that of the two girls but was initially mystified about its source. After going back and forth between her own numerical solution and the students’ diagram, she figured out that the inconsistency was in the two girls’ representing the leftover part in absolute terms:

Well, yeah, because ... if there’s one twelfth remaining, there’s half of one sixth remaining. ... The one twelfth is the portion of the whole. ...But the question is “How many badges can you make?” (PO.I. 2540-2567)

Having resolved the contradiction, she again criticized the teacher for glossing over what she considered to be an important idea warranting a lot of discussion and clarification. She explained that if she were teaching the lesson, she would explain the remainder both in terms of a whole yard and in terms of the ribbon needed to make a bow.

Overall, Deborah's performance in these three episodes suggests that unlike her pre-intervention performance, she was now quicker in identifying contestable aspects in students' work, more flexible in discerning the sources of students' confusion (without excluding instruction as a potential source), and more adept at proposing instructional remedies for student errors and misconceptions.

*Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Compared with the instructional intervention she proposed during the pre-intervention meeting, in the post-intervention meeting, Deborah proposed an intervention that could better support Alan and his classmates' understanding. Yet, her difficulties with the content prevented her from doing so for the second episode, as well.

To support Alan in overcoming his error, Deborah first thought of asking him to clarify his thinking: "Can you share with us why you decided to put the six marks there and what does each of the six [segments that you created] represent?" or "Why did you divide [the line] into six portions?" (PO.I. 2014-2017). Thinking that Alan's error might have stemmed from his misinterpretation of the representation drawn on the board, she expected that having Alan articulate this thinking would have helped him identify and correct his error. If Alan's confusion persisted, however, she would elicit other students' ideas about Alan's work. If that did not help either, she would assume a more direct role and pose questions to support students' thinking, such as the following: "What is one sixth?" and "If [the whole line] is one, what's one sixth?" (PO.I. 2036-2037). Hence, in contrast to her pre-intervention struggles with posing questions that would support Alan (and his classmates) overcome their confusion, in the post-intervention meeting Deborah was able to ask questions that were not only pedagogically appropriate but could also

support students' understanding. She would also refrain from giving students "right" answers; even when assuming a more directive role, she would try to scaffold students' thinking by posing questions to help address the mathematics at stake.

Compared to the pre-intervention meeting, during the post-intervention meeting, Deborah had a better sense of the mathematics at stake in the second episode; yet, her understanding was not solid enough to support her in targeting the key mathematical ideas at play. While she understood the source of June and Shaun's struggle and had a general sense of how the idea of common multiples advanced by the teacher could support the students' work, she admitted that her understanding of the pertinent ideas was still insecure:

I'm not very sure myself [about the idea of using common multiples] ... but I think that when you're dealing with common multiples ... the pieces, you know, slide in really easily. So you can see that they overlap. (PO.I. 2243-2246)

Her difficulties with the content appeared to have led her to propose a pedagogically sound yet mathematically nebulous instructional intervention to support the two students:

**Deborah:** I think I probably would have pulled them to the board or at least -- like I said, I never would have set them off to do the assignment without having a larger discussion. ... So, I don't think I would've done the common multiple. ... I think that you could do that visually. ... And I don't like, lines; I would've done boxes. So I think that's probably what I would've done or maybe, found a problem that wasn't dealing with ribbons so that boxes made sense. ...

**Charalambos:** What do you mean that you would do it visually?

**Deborah:** I think that, you know, like ... "Okay, this is one [means the whole]; all right, can someone help me out and make the three fourths?" And then, "All right, so this is ..." You know, like, work together on one visual with the entire class and have them work around that, and build on one another's thinking. That's what I mean by that. (PO.I. 2217-2238)

Deborah's idea to avoid recommending that students use common multiples could be effective if she could help students build on the visual representations and turn the fourths and the sixths into commensurate fractions. Nonetheless, as this excerpt suggests, she was not clear as to how this could be achieved. Instead, she focused on using an alternative type of representation (i.e., rectangular versus linear) or even changing the

context of the problem. It is also debatable whether having students' build on each others' thinking could help them understand the ideas at play in this episode.

*Noticing and Interpreting-Evaluating.* Deborah's post-intervention performance in the tasks of this practice was in stark contrast to her corresponding performance when she first joined the program. As the reader might recall, in the pre-intervention meeting she found the teacher's questioning meritorious (in the first episode) and commended the teacher for reminding students of an idea they discussed in a previous lesson (in the second episode). In considering the first episode during the post-intervention meeting, Deborah did not endorse the teacher's questioning, arguing that the teacher was posing leading questions without really supporting Alan's understanding:

[The teacher]'s very leading, you know. Like, it's obvious that [Alan] should be coloring the whole line just from her intonation; and I don't think she ever asked why. You know, like, "Well, why should you change that?" So I would do that rather than [saying], "Go ahead and correct your picture." I think that ... if she thinks that [Alan] got it from that, then she's kidding herself. ... I'm imagining that she said, "Is it *only* [the colored part that you have to divide into six?]" you know. Like, I have had teachers say that stuff to me and ... you change your answer ... you're, like, "No, no..." So, I think if you're socially perceptive you know that ... you made a mistake. (P.O.I. 2084-2098, emphasis in the original)

Deborah was now more perceptive than how she was during the pre-intervention meeting as to how the teacher's questioning might have led Alan to correct his work without necessarily helping him grasp why he needed to do so. She was equally concerned with the teacher's approach in the second episode. She perceived the teacher to be putting more emphasis on the procedure of using common multiples and on outlining steps to be followed instead of ensuring that students came into grips with the ideas under consideration. As she put it, the teacher "seems to be hitting at procedure, more so than understanding" (P.O.I. 2200).

The question is therefore unavoidable: What might explain the changes in Deborah's performance in this and the other four practices considered above? Deborah herself identified some of the factors contributing to these changes.

*Professed Changes and Their Attributions*

Deborah reported feeling more confident and competent in the content *and* its teaching. She mentioned, in particular, that she “understood how better to teach concepts” and that she had a better understanding of “the concepts themselves ... from the perspective of a student” (PO. I. 94-96). She revisited this idea at the culmination of the interview and argued that the two intervention courses offered her two lenses, a “content lens” and a “teaching lens,” that *both* supported her post-intervention performance:

I'm dividing it into content and then also -- I mean, that's basically what the [intervention] course[s] w[ere] divided into, right? Like, where it was, like, content and then it was, like, being teacher of that content. So I think those two lenses, in and of itself ... were imperative. ... Unfortunately you start to think that you do have to divide the two, even though there is, you know, the Venn diagram and the middle section. ... So, I think that's just a huge hodgepodge of things that you learned in both courses. (PO.I. 3206-3221)

Apparently, for Deborah, neither of the two lenses alone would have sufficed to support her work and understanding. Instead it was in the confluence of the two lenses that she saw most value; as she eloquently phrased it, in “the middle section of a Venn diagram.”

Deborah's last comment in the above quotation is also important to notice. In referring to both intervention courses, she argued that they offered her “a hodgepodge of things” that supported her work. Another comment she made at the beginning of the interview also speaks to this idea:

I didn't want to ever be, like, the lecturer, you know, teaching kids who don't understand anything. But at the same time I didn't have anything in my toolkit to see how it would be done differently. So, I do think now, like, you know, using the teacher talk moves and, you know, all the things that we talked about: we need to allow kids the time to grapple with material and also that they can do more than I ever thought that they could do. So, yeah, I think that I have a bigger toolkit, I think. (PO.I. 130-138)



Deborah's pre-intervention case was entitled, "When there is a will but not a way." Indeed, during the pre-intervention meeting, Deborah had a strong desire to teach differently from how she experienced the content herself, but regardless of how intensely she wished to depart from a lecture-mode instruction, the limitations of her "toolkit" appeared to have constrained her from doing so. In contrast, at the end of the intervention she reported having enriched her toolkit with strategies, approaches, and tools that would help her materialize this approach.

Apart from the talk moves and the idea of allowing students more time to grapple with the content that Deborah identified in the above citation, at several other parts of the interview she directly or more implicitly referred to other ideas and teaching approaches she developed while participating in the two intervention courses. For example, when discussing how the virtual teacher evaluated June's explanation, she considered some of the video-clips shown during both courses. She commented that the teacher in these video clips would avoid directly evaluating the students' contributions; instead this teacher would elicit other students' ideas and have students themselves evaluate the validity of their classmates' ideas and arguments. To Deborah, this represented an appealing alternative image of teaching that resonated with her initial vision for teaching "without being the center of everything" (PO.I. 1888). The two courses themselves and the type of instruction modeled during these courses offered her additional alternative images of teaching. She mentioned to this effect:

I'd like to facilitate kids as they take the center stage, instead of, like, the traditional thing of drilling and having the kids do it and you sit there and talk. I do think the discussion format has really, like, changed how I perceive the teaching of math. I think having kids construct the meaning, you know, through manipulatives and also working with the algorithms and understanding them. (PO.I. 114-122)

Overall, based on what Deborah reported in the interview, the changes in her teaching performance could be attributed to a conglomeration of factors, with the most prevalent being her better understanding of the content itself as well as her richer panoply of ideas and tools for better supporting students' thinking and learning. In this respect, at the end of the intervention Deborah had not only the will but also some viable ways to materialize her teaching vision – thus, the title accompanying her post-intervention case: “When the will meets the way.”

#### *Changes in Performance on the LMT Test: A Closer Look*

As already discussed, Deborah exhibited the largest increase in her performance on the LMT test relative to her counterparts. To a large extent, the changes in her performance on this test were concordant with the changes in her performance in the teaching simulation. For example, among the questions she correctly answered were questions 14b, 14e, 19a, 19c, 19e, and 24d, which all pertained to the interpretation of division (of fractions); question 17, which corresponded to explaining the fractional part of a quotient; and question 20, which pertained to multiple interpretations of units in division of fractions. Given that she correctly answered the latter question, one would also expect her to have answered question 1 correctly, which also sought to capture PSTs' flexible thinking of units in fraction problems. However, it should not be ignored that her thinking and understanding of units was not as flexible as that of the other PSTs (e.g., Nathan's). Hence, that she correctly answered only one of those questions was still consistent with her performance in the teaching simulation.

Given her better performance in the simulation, Deborah would have been expected to have answered correctly questions 14a, 14c, 14d, and 24a to 24c, which also

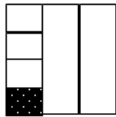
pertained to division (of fractions). Yet in interpreting her performance on these questions, one needs to keep in mind that in three of them she selected the answer “I am not sure,” which, as already explained, was assigned a score of zero. Deborah’s incorrectly answering questions 5 and 15 was also surprising since both these questions concerned associating representations with fraction-division sentences, an area in which Deborah performed fairly well in the teaching simulation. Yet, a closer look at her work on answering these questions suggests that her LMT performance was not really inconsistent with her performance in the teaching simulation.

In responding to question 5, Deborah appears to have followed what I called an *answer-driven* approach. As her work illustrated in Figure 5.14 suggests, while working on this question, she probably explored whether the given representation matched (i) the numbers included in each of the three division sentences and (ii) the quotients she found after applying the traditional division-of-fractions algorithm; this approach contradicts a more conceptually driven approach in which one first tries to interpret the representation itself and then explores whether this interpretation aligns with any of the given number sentences. If one considers her drawing for answer (A), which is the answer she considered to be correct, one could argue that based on the numbers she found, she probably interpreted the representation in this exercise to denote fitting one fourth pieces in three stripes. In total, 12 such pieces could be accommodated in the three stripes, which corresponds to the quotient of the first division sentence. If this description accurately captures how she worked on addressing this question, then her performance on this question resonates with her performance in the teaching simulation where her

thinking and use of representations were, at least to some extent, driven by the numbers included in the problems she was asked to consider.

5. Ms. Peterson gave her students the following problem.

Write a division problem for the shaded region in the diagram below



Of the following solutions, which is mathematically correct? (Mark one.)

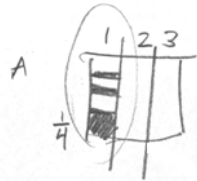
a)  $3 \div \frac{1}{4}$

b)  $\frac{1}{3} \div \frac{1}{4}$

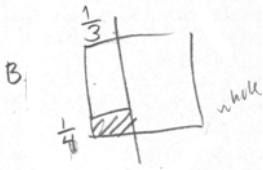
c)  $\frac{1}{4} \div \frac{1}{3}$

d) None of the above is mathematically correct.

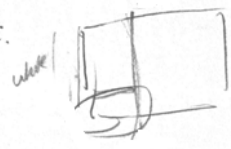
A



B



C



$\frac{1}{3}(\frac{4}{1}) = \frac{4}{3}$

$3(\frac{1}{4}) = \frac{12}{1}$

$\frac{1}{4}(\frac{3}{1}) = \frac{3}{4}$

Figure 5.14. Deborah's work on Question 5 (post-intervention LMT test administration).

Question 15 was designed to capture PSTs' understanding of the two interpretations of division. The representation illustrated in this question could be interpreted both from a partitive and a measurement perspective; hence, both mathematical sentences presented in this question matched the given representation. During the pre-intervention test administration, Deborah considered none of the mathematical sentences suitable for the given representation. In contrast, during the test's post-intervention administration, she considered one of them appropriate. From this perspective, even if she still answered the question incorrectly, her answer could suggest an improved understanding of the content. It is surprising, though, that she correctly identified the number sentence that corresponded to the measurement instead of the partitive interpretation. This contradicted her performance in the simulation during which

she reported (and was seen) being more competent working with the partitive than the measurement division interpretation.

Finally, it should be mentioned that among the questions that Deborah did not answer correctly in either test administration was question 10, which pertained to a non-conventional algorithm for solving division-of-fractions problems. Her performance on this question, in conjunction with her performance in identifying the misconception in June's explanation during the teaching simulation, is considered below when discussing how PSTs' increasing understanding of the content might support them in analyzing unconventional student ideas.

#### *Analytical Commentary*

The goal of the preceding extended analysis of the changes in Deborah's knowledge and teaching performance was twofold. First, it intended to justify why her case is regarded as the strongest "affordance" case considered in this chapter and, consequently, to show how her case corroborates the strong association between the gains in the PSTs' MKT and the gains in their teaching performance yielded from the quantitative analysis. These gains, viewed in light of Deborah's low GRE-quantitative score, further support the quantitative findings, according to which the PSTs' GRE-quantitative scores did *not* mediate the association between the gains in the PSTs' knowledge and the gains in their teaching performance. To put it differently, Deborah clearly exemplifies that what the PSTs' gained from the intervention was not contingent on their entrance GRE scores. Second, this analysis aimed at illustrating how a stronger understanding of the mathematical ideas at stake and of their teaching helped Deborah perform in ways that comport with the building of rich and challenging environments

considered in this study. Compared to her pre-intervention meeting, during the post-intervention meeting her explanations and use of representations were more conceptually driven; additionally, she was more able to follow and analyze student work and thinking and to propose interventions to help students see and correct their errors. Furthermore, she identified several of the affordances of the tasks of the second page that could help her build a conceptually oriented lesson. She even had some initial ideas for restructuring some of the tasks of the procedurally oriented first page, and even more, she outlined responses to students' requests for help that maintained emphasis on meaning and understanding.

Because Deborah experienced most of the changes in her performance in the practices of providing explanations and using representations – a pattern that resonates with the quantitative results that showed the PSTs' MKT gains to be more strongly associated with their gains in these two practices – this commentary first considers her performance in the aforesaid two practices; it then addresses the changes in her performance in the remaining three practices. The commentary concludes with discussing factors other than Deborah's knowledge which appear to have also informed her teaching performance.

Deborah's post-intervention performance pointed to notable changes in her understanding of the content and its teaching. For example, while at the beginning of the program she mainly considered division from a partitive perspective, at the culmination of the intervention, her understanding of division also encompassed considerations of the measurement interpretation of division. Similarly, whereas at her entrance to the program she saw division of fractions merely as a computation, when exiting the program she

displayed a developing, yet not solid, conceptual understanding of this operation. Her understanding of relative and absolute units was also evolving, as suggested both by her correctly answering only one of the two LMT questions designed to capture PSTs' understanding of units and by her struggles with some teaching-simulation tasks that required a flexible understanding of relative and absolute units. All these advancements in her understanding of the content and its teaching were reflected in her performance in providing explanations and using representations.

Even during the post-intervention meeting, Deborah was initially seen struggling with the concept of relative and absolute units. However, armed with a better understanding of division in general and division of fractions in particular, she was eventually able to explain both the quotient and the whole algorithm for the assigned division problem. Her explanations and her use of representations were informed by both the numbers involved in the division problems on which she worked and by pertinent concepts.<sup>175</sup> Accordingly, if one perceives a continuum with the numerical and the conceptual perspective defining its ends, Deborah's post-intervention performance in these two practices suggests that an increased understanding of the content enables one to depart from the numerical end of the continuum (where she appeared to lie according to her pre-intervention performance) and move toward the more conceptual end. If one also considers the *final* stages of Deborah's performance in providing explanations and using representations, one could then talk about a remarkable shift in her performance from the numerical end to the conceptual end of the continuum.

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<sup>175</sup> Deborah's thinking and performance still hinged on the numbers involved in the division problems on which she was working. This was suggested by her post-intervention performance in the teaching simulation and by her work in answering question 5 of the LMT test.

In discussing Deborah's performance in the two aforementioned practices, one cannot but notice four additional features of her work: (i) her selection and use of easier examples to support her own thinking and work; (ii) her employment of representations other than the "pies" she used during the pre-intervention meeting and which she associated with fractions; (iii) that her explanations in their final form were sufficiently unpacked; and (iv) that she was quite meticulous in building connections between the representations she was using and the algorithms she was asked to explain. Whereas other factors might have informed these features of her work (see below), I argue that these features should not be regarded in isolation from the growth in her knowledge of the content and its teaching. Consider, for instance, her use of easier examples. Although resorting to easier examples constitutes a general heuristic to support one's thinking and work when doing mathematics, the examples one selects and how one orders these examples might be considered a manifestation of one's understanding of the content. In this respect, Deborah's apparently deliberate selection of examples could be considered to reflect the growth in her knowledge of the content. Similarly, the expansion of Deborah's gamut of representations to explain the fractional concepts at hand could also be regarded as an instantiation of her growing understanding of the content and its teaching. Along the same lines, one could argue that without a better understanding of the content, Deborah would have not been able to provide the required explanations, let alone provide explanations that were sufficiently unpacked and which drew connections between the representations being used and the algorithm under consideration.

Deborah's post-intervention performance also provides evidence about how a teacher's growing understanding of the content can support her in better analyzing



students' work and contributions. The most glaring example that corroborates this association pertains to the changes in Deborah's analysis of Michelle's work. Equipped with a better understanding of what the fractional part of a quotient in fraction divisions means, Deborah successfully followed and analyzed Michelle's work but, even more, she made accurate assertions about this student's understanding. In particular, instead of simply exploring whether Michelle's solution matched the answer that one could get by applying the traditional division algorithm – as she was seen doing during the pre-intervention meeting – during the post-intervention meeting Deborah tried to make sense of the different parts of Michelle's solution without necessarily using the answer to this problem as a yardstick to evaluate Michelle's solution. Additionally, she legitimately pointed out that Michelle ought to have presented her work in a numerical sentence so that more accurate assertions be made about this student's understanding. Similarly, although she initially struggled with resolving the inconsistency that arose from Amanda and Julia's interpretation of the leftover part of their drawing, she was able to do so and, moreover, propose an instructional intervention that would help these students understand and correct their error.<sup>176</sup>

In contrast to her pre-intervention performance, during the post-intervention meeting, Deborah also identified the main affordances of the second textbook page. She noticed that this page asked students to explain the fractional part of their answers, to show their work using diagrams and mathematical sentences, and to explain their reasoning and thinking. She also noticed that unlike the first page, the second page expected students to discover an algorithm for the division of fractions instead of simply

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<sup>176</sup> Deborah did not immediately resolve the error in the two students' solution. However, this finding supports rather than negates the association between knowledge and performance discussed above, especially given that her grasp of the relative and absolute units was still developing.

giving them the algorithm right at the front. In addition to identifying these affordances, she was able to capitalize on several of them. She explained that if she were to teach an introductory lesson on fraction division, she would have her students show and explain their thinking in words, diagrams, and mathematical sentences; she would also require that students be explicit as to what the fractional part of their answer represents. Although she did not provide a full-fledged description of how she would scaffold her students' transition from considering specific examples to describing a general division-of-fractions algorithm, she proposed some viable ways of doing so. These changes in her performance were not unrelated to the growth in her thinking. Recall, for example, that during the pre-intervention meeting she did not even know what the fractional part of an answer means; with a more profound understanding of this concept, during the post-intervention meeting she did not only fathom the pertinent direction listed on this page but was also able to build on it and incorporate it in her proposed lesson. Similarly, while at her entrance to the program she perceived the algorithm of fraction division simply as a calculation, her growing understanding of the meaning of the numbers involved in this algorithm supported her in proposing some ways to scaffold students' transition from specific division examples to figuring out this algorithm.

Restructuring the first page to teach a conceptually oriented lesson appeared to be harder for Deborah than using the second page for teaching such a lesson. Although she started seeing some of the exercises of the first page from a more conceptual perspective (e.g., she identified the exercises whose dividend and divisor had the same denominators as the easiest), she was still unsure how to capitalize on these ideas and restructure this page to better support students' understanding. She did, however, make a first step

toward this direction by proposing building a lesson around one of the easiest exercises (i.e.,  $\frac{2}{3} \div \frac{1}{3}$ ). Therefore, it seems that her developing understanding of the content enabled Deborah to start considering ways in which this page could be used not only to strengthen students' procedural competence but also to support their conceptual understanding. However, her performance in restructuring this page suggests that an increased understanding of the content alone might not suffice to support PSTs in restructuring the curriculum tasks; PSTs might additionally need to be familiarized with ways in which curriculum tasks can be restructured to account for the tasks' limitations. I revisit this idea in the third section of this chapter, when discussing how Deborah's case helps refine the propositions related to the practice of selecting and using tasks

Deborah's post-intervention work on the two episodes associated with the practice of responding to students' requests for help further corroborates the association between knowledge and teaching performance that this study explores. In fact, her performance in these two episodes provides both an affordance example (the first episode) and a constraint example (the second episode) about the association of interest, given that during the pre-intervention meeting, Deborah largely outlined pedagogically appropriate but mathematically nebulous instructional activities for supporting students in both episodes under consideration. With an increased understanding of the content, during the post-intervention meeting she outlined an approach that could potentially address the mathematical ideas at play in the first episode. In the second episode, because of her not solid understanding of how the notion of common multiples could be applied to solve division-of-fraction problems, she proposed an instructional intervention that, although pedagogically appropriate, did not have the potential to satisfactorily address the

mathematical ideas at stake. That said, her work on the episodes of this practice, and particularly the first one, should, also be seen in light of other factors that appear to have contributed to her post-intervention performance.

In addition to the main changes considered in Deborah's performance in the performing tasks of the simulation, notable changes were also observed in Deborah's performance in the tasks of noticing and interpreting-evaluating with respect to all five practices under consideration. Although other factors can also explain these changes, I argue that her increased understanding of the content should be partly credited for these changes. For instance, without such an enhanced understanding Deborah might have not been able to notice that in some instances the virtual teacher offered students ambiguous hints, which could accentuate students' confusion rather than support their learning. Similarly, without being aware that the dividend can be smaller than the divisor, it is questionable whether Deborah would have been able to immediately identify the potential misconception in June's explanation.

Overall, it seems reasonable to attribute the changes in Deborah's teaching performance to the changes in her understanding of the content and its teaching. Besides, Deborah herself associated her better understanding of the concepts explored in the simulation with the changes in her performance. Yet, other factors appear to have also contributed to the improvement in her teaching performance.

As Deborah claimed during the post-intervention meeting, she was not only more competent but also more confident in her abilities to navigate the mathematical terrain explored in the teaching simulation. She also reported having enriched her "teaching toolkit" with several tools that scaffolded her work during the teaching simulation: she

became aware of some questioning techniques for eliciting students' ideas and supporting their thinking; she was more sensitive to issues of making meaning and understanding; she was cognizant that the effectiveness of representations does not inhere in the representations themselves but resides in how representations are used during instruction; and she had multiple opportunities to engage in providing and evaluating explanations and building connections between different representations. Additionally, Deborah's participation in the ELMAC program, and particularly in the two intervention courses, equipped her with alternative images of teaching the content. With this increased toolkit and with an enhanced confidence in her ability to understand and teach mathematics, she moved beyond the virtual lesson's redeeming surface characteristics. This allowed her, for example, to capture cases in which the virtual teacher shifted emphasis from meaning and understanding to applying procedures, to consider instruction as a potential source of students' confusion and errors, and to notice that the teacher was not capitalizing on the representations used in the lesson to support students' thinking and understanding. It also aided her in posing more appropriate questions (e.g., when supporting Alan), in attending to issues of mapping between different representations, and in providing more unpacked explanations that could be understood by a sixth grader.

The changes in Deborah's teaching performance and the factors that appear to have informed these changes as discussed above substantiate several of the propositions outlined in Table 4.6 in Chapter 4. In particular, her increased understanding appears to have supported her in providing more conceptually driven explanations and in using representations to show the meaning of the numbers involved in the pertinent algorithm rather than to illustrate the numbers themselves; this supports propositions A<sub>1</sub>, B<sub>1</sub> and B<sub>2</sub>.

Deborah's performance in the practices of providing explanations and using representations was also informed by the numbers involved in the algorithm she was asked to explain, especially when her understanding of the content was rather weak; this corroborates proposition A<sub>2</sub>. On the other hand, her use of easier examples to make sense of the algorithm she was asked to explain provides support to proposition A<sub>3</sub>.

Deborah's increased understanding of the content also appears to have helped her to no longer pursue a number-driven approach when analyzing Michelle's work; this finding supports proposition C<sub>3</sub>. Similarly, the growth in her understanding of the content allowed her to identify and capitalize on the affordances of the second page and target specific mathematical ideas when supporting Alan, thus corroborating propositions D<sub>1</sub> and E<sub>1</sub>, respectively. Deborah's performance on the LMT test, and especially that she answered question 10 incorrectly, in conjunction with that she identified the misconception in June's explanation provides support to proposition C<sub>4</sub>: although Deborah missed a question on an unconventional student solution with which she was apparently not familiar, her awareness of the misconception that the divisor should always be smaller than the dividend allowed her to discern the pertinent misconception in June's explanation.

Deborah's case, however, suggests that a growing understanding of the content does not suffice to restructure tasks to better support student learning. In fact, her case suggests an addendum to proposition D<sub>2</sub>: that in addition to a strong understanding of the content, PSTs also need to be familiarized with how to restructure curriculum tasks. Finally, the other factors considered to have informed Deborah's performance provide evidence supporting propositions A<sub>4</sub>, B<sub>3</sub>, D<sub>3</sub>, and E<sub>2</sub>.

Figure 5.15 summarizes Deborah's post-intervention performance. This figure suggests that Deborah experienced changes in her beliefs about teaching and learning mathematics and in her efficacy beliefs; to a large degree, these changes appear to have been mobilized by her participation in the intervention courses. Altogether, the changes in her beliefs, the changes in her understanding of the content and its teaching, the alternative images of teaching she garnered from her participation in the ELMAC program, and her enriched teaching toolkit explain her improved teaching performance.

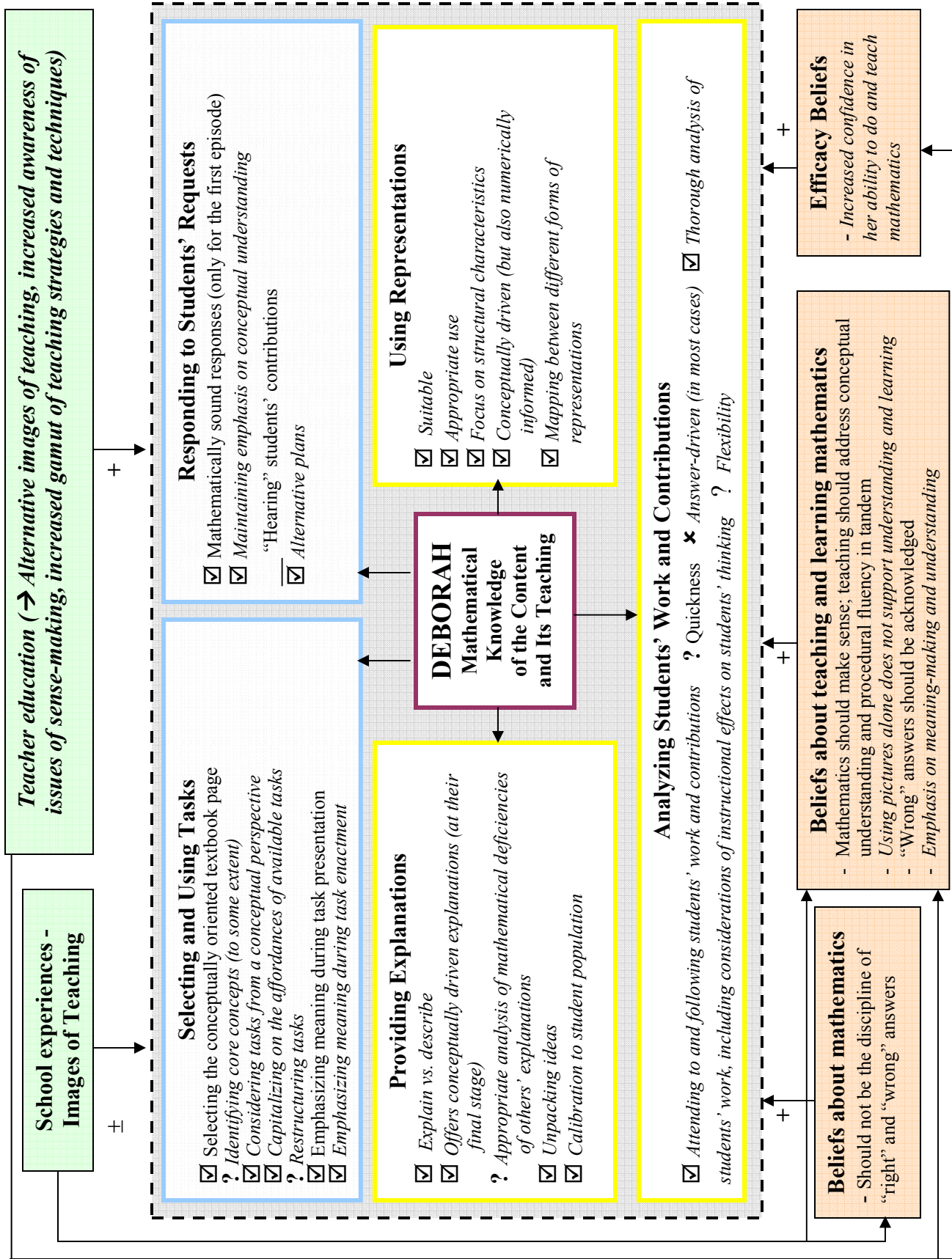


Figure 5.15. Considering the association between knowledge and teaching performance through Deborah's post-intervention profile.



*The Case of Vonda Revisited: “If I Understand What I Am Doing, I Can Teach it Better”*

Unlike the three cases previously considered, Vonda was a convergent case based on her entrance scores and a divergent case based on the changes in her performances; as Figure 5.3 suggests, she exhibited greater changes in her performance in the teaching simulation than in her performance on the LMT test. She also exhibited greater changes in her performance in the MTF-related practices than in the MKT-related practices, thus further motivating the scrutiny of her case, given that this pattern of improvement did not resonate with the focus of the intervention courses on the MKT-related practices. The post-intervention interview with Vonda was conducted in early February. In the time that elapsed between the pre- and post-intervention meetings (i.e., a little more than 31 weeks), in addition to her coursework, Vonda attended and taught math lessons in a fourth-grade class.

Although a divergent case, Vonda corroborates the quantitative results reported in the previous section in two respects. First, the small gains she experienced in her performance in the MKT-related practices were consistent with the low gains in her MKT performance, thus supporting the strong association found between the gains in the PSTs’ MKT and the gains in their performance in the MKT-related practices. Second, Vonda’s performance gains in the MTF-practices were not proportional to the gains in her MKT, thus corroborating the negligible association found between the changes in the PSTs’ MKT and the changes in their performance in MTF-related practices. In addition to supporting the quantitative findings, Vonda’s case is also informative in another respect: it offers insights into what a small growth in knowledge (relative to that experienced by her counterparts and especially Deborah), alongside a greater attention to issues of

making meaning and an enrichment of PSTs' images of teaching, affords PSTs in terms of structuring the rich and intellectually challenging learning settings this study explores.

### *Selecting and Using Tasks*

*Performing.* Contrary to her pre-intervention performance in selecting and using tasks in which she was mainly concerned with outlining steps for students to follow, Vonda's post-intervention performance reflects her increased attention to issues of meaning-making. This difference notwithstanding, Vonda's post-intervention performance still reflected her inclination to show and tell, on the one hand, and to minimize complexity to avoid students' confusion and error on the other hand. This was evident in how she proposed using either of the two textbook pages.

As in the pre-intervention meeting, Vonda preferred using the first page over the second page to develop an introductory lesson on fraction division; her rationale for choosing this page echoed the reasons she outlined for her preference during the pre-intervention meeting. She argued, in particular, that the first page would enable her to “[work] through each of the steps involved with solving [division] problems” (PO.I. 1168-1169). However, in contrast to her pre-intervention emphasis on developing students' procedural competence, during the post-intervention meeting she saw this page as a “teaching tool” (PO.I. 180) upon which she could draw to help her students understand the meaning of fraction division; she would do so by using a virtual representation to explain exercise 1 (i.e.,  $\frac{2}{3} \div \frac{1}{3}$ ). If not random, her decision to anchor her explanation for fraction division by this exercise instead of using the page's worked out example (i.e.,  $2 \div \frac{3}{4}$ ) was legitimate, given not only the better segue that exercise 1 provides for introducing the division of fractions, but also the difficulties she encountered

explaining divisions including fractional parts (see her performance in *Providing Explanations*).

Having introduced and explained the division of fractions, Vonda would then ask students to solve *some* of the remaining 15 exercises. This contradicts her pre-intervention performance during which she argued that she would have her students solve all 16 exercises. When asked to specify which exercises she would use, Vonda identified exercises 2, 4, 5, 8, and 12, on the grounds that these exercises involve thirds and fourths, which she considered “easier to deal with initially, in terms of representing them in a drawing” (PO.I. 641). She argued that it would make more sense to assign exercises that involved such fractions, given that exercise 1, which she proposed using to introduce the division of fractions, also involves thirds. A closer look at this list of exercises (see Figure 3.2), however, reveals that only exercise 8 pertains to dividing same-denominator fractions; in exercises 2, 4, and 12, the dividend is smaller than the divisor. Hence, it is questionable whether Vonda’s exercise selection was informed by considerations of the meaning of (fraction) division. Rather, her selection appears to have been informed by procedural criteria, as suggested by the fact that she would not initially assign exercises 13-16 because of their including the additional step of converting mixed numbers into improper fractions. As she mentioned, she would assign these exercises only after she would explain how to convert mixed numbers into improper fractions.

Vonda’s enactment of these exercises differed significantly from how she thought of using them during the pre-intervention meeting. Instead of having her students come to the board and share the “steps” of their solutions, she now considered asking students to work in groups and solve these exercises using drawings. While students would be

working on solving these exercises, she would circulate around, monitor their work, and probe them to explain their thinking by asking questions such as, “Can you explain why you did this?” or “Could you say in your own words why?” (PO.I. 1173-1175). Vonda’s proposed enactment of these exercises clearly echoed an approach of teaching modeled and discussed during the two intervention courses.

Vonda would pursue a similar approach when having her students solve the two word problems of this page. As she explained, she would encourage her students to use drawings to represent the mathematical situation outlined in each problem. For instance, for the first word problem (i.e., exercise 17), she would expect her students to draw “a whole,” divide it into four parts, identify each part with 30 students, and thus conclude that the class consists of 120 students. When prompted to explain how this problem relates to the division of fractions, she argued that the problem could be solved using the equations  $30 \div \frac{1}{4} = 30 \times \frac{4}{1} = 120$ . She justified this sequence of equations by pointing out that “division is the opposite [i.e., inverse] of multiplication” (PO.I. 796), and continued: “So if one fourth of my class’ total students is thirty, then to find the total of the class, I know I have to multiply that by the reciprocal” (PO.I. 806-808). Although she still did not justify why fraction division was applicable to solving this problem, her proposed problem enactment was improved compared to her pre-intervention performance, during which she proposed solving this problem using the equation  $X \div \frac{1}{4} = 30$ .

Although Vonda would avoid using the second page to introduce division of fractions, she still identified two of its affordances. First, that it asks students to explain their reasoning by using drawings and written explanations and second, that it requires students to explain the fractional part of their answer; she noticed none of these

affordances during the pre-intervention meeting. Vonda's case, however, clearly suggests that identifying these affordances should not be considered tantamount to capitalizing on them during lesson planning and enactment. As Vonda explained, if she had to use this page for an introductory lesson on fraction division, she would modify task A to minimize students' confusion and struggle. For instance, she considered avoiding asking students to explain the fractional part of their answer, at least initially, because she envisioned that students would encounter difficulties if asked to do so:

I'm not sure if the students would – unless we have discussed this previously – understand [the fractional part of their answer]. ... So, I think [task] A has a lot of strength to teach dividing fractions by a fraction, except [the direction], “Describe what each fractional part of the answer means.” I think I would have to think about it if I used this particular sheet. I may not include that for my kids to answer. (PO.I. 931-941)

Similarly, she found the requirement that students explain their reasoning by using diagrams, written explanations, and mathematical sentences particularly demanding. Hence, instead of having students use all three representational means – pictures, words, and symbols – to explain their work, she thought of asking them to use one representational mode at a time. For example, she would ask students to simply solve problem  $A_1$ , write a written explanation when solving problem  $A_2$ , and show their work in a drawing when solving problem  $A_3$ . She argued that in doing so, she would still “cover” all three requirements, but “in a culminating way” (PO.I. 1215). What Vonda appeared to overlook, however, is that the sum of these three requirements is smaller than the whole: although there is value in having students represent their work in any of the three representational means, more value resides in having them use all three means, and more critically, build connections among them.

Even though Vonda saw the value of having students discover an algorithm for the division of fractions, as task D asks them to do, she preferred to introduce this

algorithm at the outset of her lesson to minimize student confusion. She explained, “The second [page], I think, leaves room for confusion because it’s leaving the students on their own, basically, to figure out these problems” (PO.I. 1096-1097). According to Vonda, introducing the algorithm first would also ensure that students would have “a good understanding of the ... procedure they’re solving,” which she deemed important because, as she contended, knowing “the step-by-step process [helps] keep down confusion” and ensures that students will not “miss a step in getting to the answer” (PO.I. 1271-1277). She repeated this same idea when observing the virtual lesson:

I noticed ... [the virtual teacher]’s jumping down to [task] D, which talks about algorithms, and what makes sense for dividing fractions by fractions. ... I don’t think I would’ve approached it that way. ... I think I would’ve done it the reverse. I would’ve emphasized the algorithms first, I think, and then walked through the problems using the diagrams. (PO.I. 1943-1952)

When asked to describe how she would enact task D if she had to follow the task sequence of the second page, she considered asking questions to help students identify commonalities in their work on solving the three problems of task A:

I guess I would ask ... “What did we do in all three problems? What was the first thing we did?” I guess the answer would be -- I would be looking for, “We found the whole; what is the whole in each problem.” And then, “What is the next thing we did?” “We divided the whole by one sixth,” in this particular instance. I guess I would emphasize the divisor at that point. ... And then go from there. (PO.I. 1968-1976)

Vonda’s reference to “emphasizing the divisor” could support students’ work, especially if she helped them see that in all three problems the whole was partitioned into six sixths. However, when asked to clarify how to then help students figure out the traditional division-of-fractions algorithm, she admitted not knowing how to do so.

Vonda’s task selection and use for either textbook page bears some notable resemblances. First, in both cases she would start with explaining the algorithm to students. She would do so not by describing the steps involved in the algorithm, as she proposed doing during the pre-intervention performance, but by using drawings to help

students see the meaning of fraction division. Second, she would ask students to show their work in drawings; although she would also ask them to explain their thinking, it is questionable whether she would have them use words, pictures, and symbols *in tandem* to explain their reasoning. Third, regardless of which page she would use, she would try to minimize complexity (and consequently student confusion and error) by first showing and explaining, then asking students to apply what they were shown to solve similar tasks. However, in contrast to the pre-intervention meeting, she would place more emphasis on the meaning of fraction division. Her emphasis on meaning-making appears to have been informed by both her better understanding of the content – as will be illustrated throughout the discussion of her case – but also by the changes in her beliefs about what it means to teach and learn mathematics. I return to this argument in the analytical commentary when discussing the changes in her performance.

*Noticing and Interpreting-Evaluating.* During the pre-intervention meeting, Vonda found the virtual teacher’s presentation and enactment of tasks A<sub>1</sub> and D effective. Although during the post-intervention meeting she still endorsed the teacher’s enactment of task D, she was concerned with the teacher’s presentation and enactment of task A<sub>1</sub>.

When asked to comment on the teacher’s presentation of task A<sub>1</sub>, Vonda argued that instead of simply having students read the problem of this task and then solve it, the teacher could first have done a quick review to ensure that students “remember[ed] and retain[ed]” relevant operations; she named multiplication of fractions as such (P.O.I. 1412-1414). If she were teaching the lesson, she would additionally model solving a division problem (e.g.,  $\frac{2}{3} \div \frac{1}{3}$ ) before asking her students to solve task A<sub>1</sub>. She explained,

[I]f I did the sample problem, and if any of the students raised their hands with questions or are confused or are not sure of something, I can address it before we get into this part of the lesson [i.e., task A<sub>1</sub>]. (P.O.I. 1412-1437)

For Vonda, Ben's question as to why the problem in task A<sub>1</sub> should be solved by dividing one half by one sixth and not the other way around represented additional evidence suggesting that the introduction of task A<sub>1</sub> was not effective; she interpreted his question as a sign of confusion, which she attributed to that "they haven't done any division yet" (PO.I. 1449-1450). While considering the enactment of this task, she also argued that if she were the teacher she would explain the notion of the whole better. Indeed a better explanation of this notion was warranted in the lesson, given both Alan's confusion and the teacher's vague admonition that students "use the whole" when dividing fractions.

For Vonda, the enactment of task D represented an instance of "teaching through exploration" (PO.I. 2042). She thought that by first letting students use diagrams to solve the assigned division problems and then leading them to the invert-and-multiply rule, the teacher reinforced student understanding. Her endorsement of the teacher's approach notwithstanding, she seemed to still prefer a show-and-tell enactment of this task, as suggested by her comment: "So, I guess that [the enactment of this task] was pretty good. Would I do it that way? I'm not sure" (PO.I. 2044-2046). In contrast to the previous episode, however, she overlooked that the teacher shifted emphasis from meaning and understanding to simply manipulating numbers. This finding should not be dissociated from the limitations in Vonda's grasp of the content, as exemplified in the explanation she provided for this algorithm.

In general, Vonda's post-intervention analysis and evaluation of the presentation and enactment of these tasks suggests that she still endorsed a show-and-tell mode of instruction. Yet, unlike her pre-intervention performance, when her understanding of the



content was relatively thorough, she additionally identified instances in which the virtual teacher did not sufficiently scaffold her students' comprehension of the targeted ideas.

### *Providing Explanations*

*Performing.* At her entrance to the program, instead of explaining the traditional division-of-fractions algorithm, Vonda merely provided an unpacked *description* of the steps involved in this algorithm; even worse, her “explanation” was flawed since she confounded the traditional algorithm with the procedure of finding common denominators. Additionally, she could not use any representation to explain the quotient in the division problem  $2 \div \frac{3}{4}$ . Armed with a somewhat better grasp of the concept of division of fractions, during the post-intervention meeting she was able to provide explanations that were, to some extent, undergirded by relevant concepts but still largely driven by the numbers involved in the division problem she was asked to explain.

When asked to explain the quotient in the division problem  $2 \div \frac{3}{4}$  during the post-intervention meeting, Vonda assumed the role of the teacher and invited the interviewer to assume the role of the student:

**Vonda:** So, if we had, Charalambos, if we had two wholes [draws two rectangles, see Figure 5.16a], and we want to divide it [sic] by three fourths. So first, we're gonna divide the wholes into fourths [divides each rectangle into four parts]. ... How would I represent three fourths there? ... I want you to represent three fourths. ... How many three fourths are in the first square?

**Charalambos:** Okay, I would show these three [he shows three of the fourths in the first rectangle].

**Vonda:** ... So then I'm gonna put a little mark in those three that you pointed out [shades in three fourths of the first rectangle]. So that's three fourths. And so, if we had two divided by three fourths -- [pause]. You know what? I'm not sure.

...

**Charalambos:** You can take time and think about it.

**Vonda:** Yeah, I was thinking of taking two wholes, and marking three fourths on each whole. But as I thought about it, it didn't seem right. ... I was gonna say -- I guess where I was going is, I have one, two; one set of three fourths; and two sets of three fourths [shades in a  $\frac{3}{4}$ -portion in the second rectangle]. Two. Because I'm dealing with threes, and thirds, I have two thirds not covered, which would give me my two and two thirds. (P.O.I. 225-280)

Vonda's reference to dividing the wholes "by three fourths" and then "marking three fourths on each whole" reflects her improved understanding of division of fractions; recall that during the pre-intervention meeting her explanations were driven by no concepts at all. Regardless of this improvement, her grasp of the ideas involved in fraction division – and particularly the notion of relative and absolute units – was still not solid. Hence, although she knew that she had to show a  $\frac{3}{4}$ -portion in each whole, she was uncertain about explaining the fractional part of the quotient. Her reference to "threes" and "thirds" in the last exchange of this excerpt was rather esoteric; it was also unclear how she concluded that the fractional part of the quotient was two thirds. Therefore, the interviewer asked her to elaborate upon this idea.

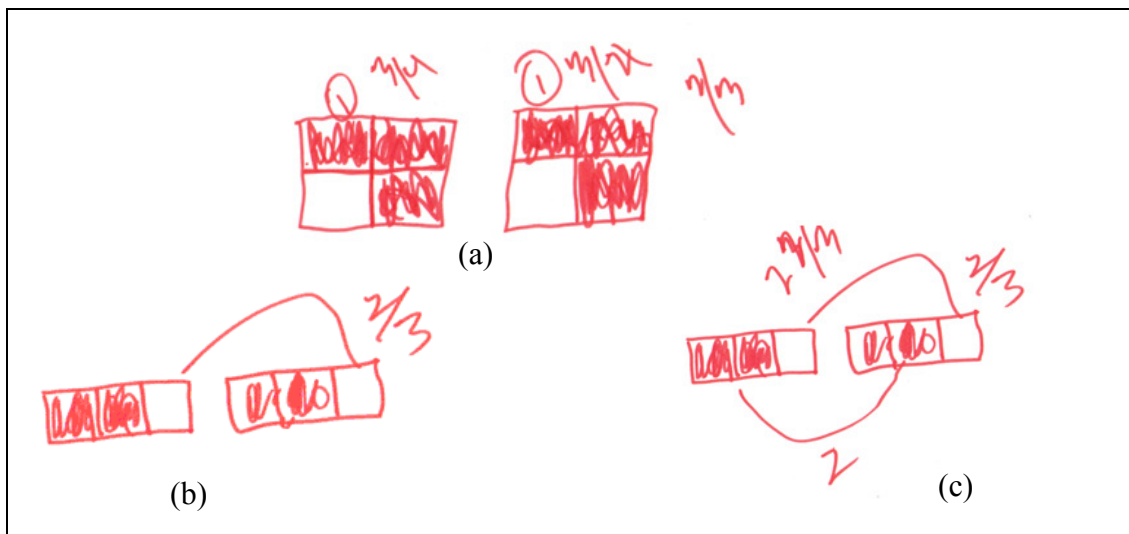


Figure 5.16. Vonda's work on explaining the quotient in the division problem  $2 \div \frac{3}{4}$  (a: dividing the two wholes into fourths; b and c: dividing the two wholes into thirds).

Vonda's clarification of the two thirds did not really explain why, while the problem requires partitioning the wholes into fourths, the quotient includes thirds:

So, I was trying to give my student a visual of dividing two by three fourths. And in doing that, I was trying to represent two wholes [points to the two rectangles in Figure 5.16a], and then separating those into groups of fourths. And then covering three fourths of each. And then, because I'm dividing it by three fourths, I'm now dealing with thirds. Am I doing that right? ... Hmm.... That's what I'm thinking, because I have three [points to the  $\frac{3}{4}$ -portion in the first rectangle] of one whole shaded [points to the whole first rectangle], then I'm dealing with threes

in one square, so I have three fourths shaded. And then in the second whole, which is divided into groups of four equal shares, I have three in that second group [points to the  $\frac{3}{4}$ -portion shaded in the second rectangle]. That gives me two groups of three fourths, with two thirds group left unshaded. And that's what I was trying to get my students to visualize. ... Although I think I would have trouble at this point. (PO.I. 301-320)

Vonda's argument that because she was dividing by three fourths she was dealing with thirds and her ambiguous reference to "threes in one square" to imply that she had three shaded parts in each whole could be perceived as instantiations of an insufficiently unpacked, yet decent, explanation. Alternatively, her explanation could be conceived of as being driven by the numbers involved in the quotient of the division problem she was trying to explain rather than by the concept of relative units. The discussion that ensued seems to support the latter rather than the former interpretation.

Still perplexed with Vonda's explanation of the two thirds and still assuming the role of the student, the interviewer argued that while he understood the two  $\frac{3}{4}$ -portions that Vonda showed in Figure 5.16a, he was still unclear about the two thirds; thus, he asked for further clarification. To further explain this idea, Vonda shifted to a different representation (see Figure 5.16b). She drew two rectangles, partitioned each into three parts, and shaded two parts in each of them. She then associated the two non-shaded thirds with the two thirds of the fractional part in the quotient. After doing so, she paused for awhile and, realizing that she had "lost the fourths" (PO.I. 352), she considered reverting to her original picture (Figure 5.16a). She explained that, although in her second representation she could show the two thirds, she could not show the whole-number part of the quotient. Yet, on a second thought, she considered representing the whole-number part of the quotient with the shaded portions in Figure 5.16b (see the curved line joining the two shaded portions in Figure 5.16c, and the number "2" underneath this line). She then reverted to her original picture (Figure 5.16a) and mentioned,

I'm trying to represent three fourths ... out of two wholes. And then, because I have the three shaded from one whole, and three shaded from the other whole, that gives me two three. And that means I would have two thirds remaining. But I'm not sure if I can explain it to my students. (P.O.I. 393-399)

Vonda's explanation would have been appropriate and sufficiently unpacked had she used the second representation to show the relative units, namely the  $\frac{3}{4}$ -portion that corresponded to the divisor. However, her work in Figures 5.16b and subsequently in Figure 5.16c suggests that this was not the case. What Vonda appeared to be trying to do with these representations was to show the *numbers* involved in the quotient rather than to explicate the meaning of these numbers. That does not necessarily mean that Vonda's explanation was totally driven by numbers. Instead, because she appeared to have developed the notion of division as taking away divisor units from the dividend, she was able to decently explain the whole-number part of the quotient; the impact of her enhanced understanding of the content on her performance was also revealed by her decision to revert to Figure 5.16a and re-associate the two  $\frac{3}{4}$ -portions with the three fourths that could be made out of the two dividend units. However, because of her apparently weaker conceptual understanding of the relative and absolute units, her explanation of the fractional part of the quotient was largely governed by the quotient's numbers. As such, Vonda's explanation of the quotient in division  $2 \div \frac{3}{4}$  can be considered a hybrid of a conceptually and a numerically driven explanation, in which the numbers took precedence when her conceptual understanding fell short of further supporting her work.

The question of whether Vonda's explanations were primarily informed by numbers or concepts was even more difficult to answer when considering her explanation for the traditional division-of-fractions algorithm with respect to division  $2 \div \frac{3}{4}$ . Whereas

her explanation seemed to be guided by the numerical value of the numerator and the denominator of the reciprocal (i.e.,  $\frac{4}{3}$ ), it might have also been undergirded, at least to some extent, by some preliminary, but not clearly articulated, understanding of fitting divisor units into the dividend units.

When asked to explain the traditional invert-and-multiply algorithm, and particularly the reciprocal involved in this algorithm, Vonda initially argued that if she were teaching a lesson on fraction division she would refrain from using a representation to explain the reciprocal. Instead, she would define it as the “reverse of the fraction” (PO.I. 429) on the premise that the reciprocal is just a piece of information students “need to know in terms of a definition” (PO.I. 436-437). However, on rethinking the reciprocal of the division problem  $2 \div \frac{3}{4}$ , she wondered: “Three and four. ... [I]s [it] how many threes can you get out of four?” (PO.I. 442-443). Given that, when prompted, she defined the “threes” as “three fourths of a whole” her speculation represented a manifestation of the interplay between the numbers and the concepts in her thinking and explanation. Notice, for example, that she did not refer to four thirds, but to “threes” and “fours,” thus regarding the fraction of the reciprocal as two separate numbers. Using these two numbers, she thought of fitting “threes” into “fours;” this idea, albeit still numerically informed, corresponds to the notion of the reciprocal as fitting relative units (i.e., in this case *three* fourths) into absolute units (i.e., *four* thirds).

Building on this idea, Vonda drew two rectangles similar to the ones she drew to explain the quotient, divided them into four parts, and shaded in three parts of each of them (see Figure 5.17). She then explained,

The reciprocal of three fourths is four over three. So if I drew the reciprocal, I would still draw the one whole divided into four, and shade how many times three would go in there. It will go one and

one third left [while talking she writes  $1\frac{1}{3}$  above the first rectangle, see Figure 5.17]. (PO.I. 454-465)

The interviewer pressed for more clarification:

**Charalambos:** So, you started discussing the reciprocal. And right now you are showing me the one and one third. So, what is this one and one third?

**Vonda:** One and one third is ... it represents that three will go into four -- I have four squares here [points to the four parts of the first rectangle; see Figure 5.17], and I can count, I can divide [the whole rectangle] into three equal shares, one time. And then I would have one third of it left. So that's one, and one third left. [Then pointing to the second rectangle and writing  $1\frac{1}{3}$  above it:] And if I did that twice, one and one third, then added them together, I still would get my two and two thirds. [She writes " $= 2\frac{2}{3}$ ".]

**Charalambos:** [Pointing to the equation outlined in the worked out example of the first page  $2 \div \frac{3}{4} = 2 \cdot \frac{4}{3} = \frac{8}{3} = 2\frac{2}{3}$ .] So, how would you interpret this thing here that says "two times four thirds equals two and two thirds?"

**Vonda:** Okay. Two times four thirds equals two and two thirds. ... [Goes back to her drawing in Figure 5.16a:] If I went through each step, I have two wholes that can be equally divided into four equal parts each, which gives me eight total parts, equal parts, I should say. So that eight [points to the  $\frac{8}{3}$ ] represents the two times four equal parts. And if I divided that by the three, or put three into that – same thing – I would get two; [points to the shaded portions:] two equal parts of three. [Then pointing to the non-shaded regions:] with two thirds left over. (PO.I. 479-532)

Vonda's clarification in these exchanges further supports the idea that her explanation was guided by an amalgam of numbers and concepts. In the first exchange she did associate the reciprocal with fitting one and one thirds into each of the two dividend units; this association can be considered to reflect the notion of the reciprocal. However, without explaining that she was trying to fit divisor units into the dividend units, her reference to "one third left" was ambiguous. Was Vonda indeed referring to one thirds of three fourths or was she just drawing on the numbers involved in the reciprocal when expressed as a mixed number (i.e.,  $1\frac{1}{3}$ ) and simply associating the shaded and the non-shaded portions of each dividend unit with the whole-number part and the fractional part of the reciprocal, respectively? Similarly, although in her second exchange she was clear that the number eight included in the algorithm corresponded to the eight fourths in which she divided the two dividend units, her reference to dividing the eight pieces by three or "putting three" into the eight pieces was ambiguous. Was she

indeed referring to fitting  $\frac{3}{4}$ -pieces in the available eight fourths or was her thinking again driven by the numbers – this time the  $\frac{8}{3}$ , which she apparently interpreted to literally mean eight divided by three?



Figure 5.17. Vonda's work on explaining the reciprocal for the division problem  $2 \div \frac{3}{4}$ .

No definite answers can be given to these questions without risking either underestimating or overestimating the quality of Vonda's explanation. Yet, it is informative to consider that when during the simulation Vonda was again asked to provide an explanation for the reciprocal (in response to Michelle's question as to why the reciprocal works), her answer was quite vague: she mentioned that she would talk about division and multiplication being inverse operations and made no reference to either her work on explaining the reciprocal for the division problem  $2 \div \frac{3}{4}$  or to the idea of fitting relative units into absolute units. It then seems reasonable to argue that, although Vonda's explanation for the traditional algorithm for the division problem  $2 \div \frac{3}{4}$  appeared to reflect some nascent understanding of the concept of the reciprocal, the numbers involved in this division had a pivotal role in guiding her explanation.

*Noticing and Interpreting-Evaluating.* Similar to her pre-intervention performance, during the post-intervention meeting Vonda noticed and critiqued the virtual teacher's move to postpone answering Robert's question for the next day. She also considered the teacher's response to Michelle's question not sufficiently informative for

explaining why the traditional algorithm works. However, in contrast to her pre-intervention performance in which she identified certain deficiencies in the teacher's explanation, Vonda's post-intervention critique did not address the mathematical substance of the teacher's explanation. Instead, she mentioned,

She [the teacher] didn't ask her [Michelle] if she understood, for one thing. She didn't ask if anybody else had an issue with that particular rule, why did they think it always worked or – I don't know. I can't think for sure I can tell you why it [i.e., the invert-and-multiply rule] actually works. (P.O.I. 2212-2216)

Vonda's last comment provides additional evidence about the limitations in her conceptual understanding of the traditional division algorithm. This evidence, in turn, corroborates the argument made about Vonda's pertinent explanation being largely driven by the numbers involved in the division problem she was asked to explain. In addition, it suggests that limitations in her understanding might have limited Vonda in identifying the mathematical deficiencies in the teacher's explanation.

Recall that during the pre-intervention meeting, Vonda was concerned with the analogy used in the teacher's explanation. She considered this analogy "off-lesson" but could not explain what particular aspect of it she found problematic. If Vonda did not totally overlook the teacher's analogy during the post-intervention meeting, it seems reasonable to argue that with a better understanding of the relationship between multiplication and division as inverse operations, she might have bought into the teacher's simile that the relationship between multiplication and division as played out in the traditional division algorithm was similar to adding and subtracting negative numbers. To the extent that this argument holds, then Vonda's post-intervention performance can be paralleled to the pre-intervention performance of some of her classmates, who, even



though they had a better understanding of the content than Vonda then did, were not concerned with the quality of the teacher's analogy.

### *Using Representations*

*Performing.* In contrast to her pre-intervention numerically driven use of representations, Vonda's use of representations during the post-intervention meeting was also conceptually informed. Although numbers had a pivotal role in her explanations, and correspondingly in her use of representations, her work was also informed by the concept of division (of fractions) as fitting in or taking away divisor units from the dividend units; it might have also been informed by a nascent understanding of the concept of the reciprocal as the number of divisor units to be made out of each dividend unit. In addition, closer attention to her explanations reveals that, in contrast to her pre-intervention performance, she made an effort to draw connections between her drawings and the numbers involved in the algorithm. These connections were of two types: numerical labeling and naming. Consider, for instance, her work in explaining the quotient for the division problem  $2 \div \frac{3}{4}$ . As Figure 5.16a shows, she wrote a numerical label next to each of the two  $\frac{3}{4}$ -portions. She also wrote the number "1" to designate the  $\frac{3}{4}$ -portion that could be made out of each whole and labeled the non-shaded parts as two thirds (although she misplaced this number). In addition to numerically labeling these pieces, she identified the different parts of her representation by their names while drawing this representation. For example, when drawing the two rectangles, she talked about "having two wholes"; while dividing these two wholes into four parts, she talked about making fourths; similarly, when shading in the first  $\frac{3}{4}$ -portion, she referred to making a set of three fourths out of the first rectangle.

*Noticing and Interpreting-Evaluating.* Vonda was not as attentive to issues of making connections when observing the virtual lesson as she was when providing the required explanations. Similar to her pre-intervention performance, in considering Amanda's episode, she speculated that Amanda figured out the answer to the division problem by using the traditional division-of-fractions algorithm. Hence, she would ask Amanda to show the intermediate steps of her work. In contrast to her pre-intervention performance, though, she also considered prompting Amanda to explain the meaning of the quotient in her mathematical sentence. Doing so could help the students start building and seeing connections between the mathematical sentence and the word problem under consideration. However, several other connections needed to be made among the mathematical sentence, the word problem, and the drawing shown on the board, not only about the quotient, but also about the problem's dividend and divisor. Vonda did not comment on the absence of these connections.

While in the first episode Vonda pointed to the need that at least one type of connection be made with respect to one of the numbers involved in the division problem, in the second episode she did not talk about any connections at all. Largely drawing on ideas discussed during the math methods course, she argued that the teacher could have asked another student to explain Amanda and Julia's solution instead of moving to a different problem. If she were teaching the lesson, she would also ask a different student to write the mathematical sentence on the board instead of having Amanda and Julia represent the problem with a drawing *and* write the mathematical sentence for it. She justified this move by pointing out that it is important to give opportunities to several students to present their work and voice their thinking. Both of Vonda's alternative

teaching moves could have resulted in establishing some of the connections absent from this episode; for example, in explaining the girls' work, a third student might have made such connections or might have questioned how the numbers in the girls' mathematical sentence matched the drawing presented on the board. Nevertheless, the fact that Vonda did not identify the lack of such connections in this episode casts doubts about whether in proposing these teaching moves she was aware of the lack of such connections; her reference to engaging more students in the discussion suggests that her teaching moves were largely informed by pedagogical rather than mathematical considerations.

#### *Analyzing Students' Work and Contributions*

*Performing.* While still being answer-driven, Vonda's post-intervention analysis of the three student solutions was also undergirded by her better, yet not solid, understanding of the content. This was evident in her analysis of all three student solutions, and particularly Robert's and Michelle's.

Before evaluating any of the three student solutions, Vonda used the traditional algorithm to figure out the quotient to the division problem under consideration (i.e.,  $2\frac{3}{4} \div \frac{3}{4}$ ). To then determine the correctness of the students' solutions (and especially Robert's and Ann's, who used this algorithm to solve this problem), she compared each student's solution against her own work – thus the characterization of her work in analyzing students' solutions as answer-driven. Unlike her pre-intervention faltering as to whether Robert's solution was correct (largely because she could not decide whether his work on converting mixed numbers into improper fractions was correct), during the post-intervention meeting Vonda quickly ascertained the correctness of Robert's solution. In addition, she remarked that Robert could have moved a step faster by converting the

improper fraction in his final answer to the mixed number  $3\frac{2}{3}$ ; obviously, Vonda was now more knowledgeable about converting between mixed numbers and improper fractions. She was similarly quick in identifying the error in Ann's solution.

As in her pre-intervention performance, the solution that Vonda found the hardest to analyze was Michelle's (see Figure 3.5); nevertheless, Vonda moved farther in understanding and analyzing this student's work than in the pre-intervention meeting. Whereas at the pre-intervention meeting she was unsure what Michelle's drawing represented, during the post-intervention meeting, she correctly showed the dividend, the divisor, and the whole-number part of the quotient in Michelle's drawing. She also argued that Michelle needed to write her answer in numbers to show whether she clearly understood what the leftover part in her answer represented. Although Vonda correctly associated this leftover part with two thirds, her thinking was apparently informed by the numerical answer to this problem, as suggested by her discussion of Michelle's solution:

[Michelle] only used three fourths of the last whole [pointing to the third line; see Figure 3.5]. And when she ... divided them [the three lines] into three equal parts, she used three fourths of the first whole [i.e., the first line], and one fourth and two fourths of the second whole [i.e., line], and then two fourths and one third of the last whole [i.e., line], which left two groups. (PO.I. 2265-2270)

Notice that Vonda referred to fourths when discussing the first two  $\frac{3}{4}$ -portions Michelle could make out of the available quantity; yet, in considering the third  $\frac{3}{4}$ -portion that could be made, Vonda mixed the fourths and thirds. Her argument that Michelle used one third of the last "whole" was, in fact, incorrect because Michelle still used a fourth of this last line, but this fourth was actually one third of a  $\frac{3}{4}$ -portion. Vonda's difficulties in analyzing the fractional part of the quotient discussed in her performance in providing explanations (and also evident in her work in analyzing Amanda and Julia's work, see below) appear to have impaired her analysis of Michelle's solution.

During the pre-intervention meeting, Vonda made a generic assertion about the three students' understanding; she argued that the three students understood "some of the concepts but not all of them." Although her post-intervention assertions were somewhat more calibrated, they were not always valid. For instance, while she correctly asserted that Robert seems to have understood the process of finding the reciprocal when dividing fractions, she argued that the same was true for Michelle. Her assertion about Michelle's understanding was inappropriate because Michelle's work showed no evidence of any understanding of the reciprocal; if anything, her work could suggest quite the reverse since Michelle used a pictorial solution instead of the numerical procedure that the other two students employed. Vonda also argued that Ann "had the concept" but had "a little misunderstanding about the reciprocal" (PO.I. 2228-2289). Although when referring to the "concept" Vonda might have implied the process involved in the traditional algorithm, her assertion was still contested because when making assertions about student understanding, the teacher needs to be able to discriminate between conceptual understanding and procedural competence.

When asked to determine what she would do after having seen the three student solutions and, provided that there were three minutes left before the end of the class, Vonda mentioned,

I think I would have [Ann] come to the board, and write the problem. Then ask input from the students – what is the next step? – as Ann writes it out. So once she wrote the mathematical problem to here, then I'd say, "Now, what is our next step?" Then somebody's gonna say, "We're gonna multiply four times two, then add eight, and say eleven over four." Then I'm gonna ask Ann to write it. And then, I may ask someone else to explain why we did it this way. So while I'm getting input from everyone, Ann is actually writing. And if Ann has a question, she may ask it while she's there. (PO.I. 2320-2333)

This last excerpt is reminiscent of Vonda's pre-intervention emphasis on following and applying steps. Notice that Ann would simply be asked to write the steps outlined by the

teacher or her classmates. Even though Vonda also entertained the idea of eliciting other students' ideas, since the emphasis would again be on following steps, it is questionable whether this approach would help Ann grasp anything besides the steps of the algorithm.

*Noticing and Interpreting-Evaluating.* Although making some progress in analyzing students' questionable contributions, Vonda still encountered significant difficulties understanding and resolving the errors in students' work or capturing the latent misconceptions in their contributions.

For example, even when prompted, she did not identify the misconception in June's explanation. In contrast to her pre-intervention performance, she did, however, notice that June's explanation was not addressing Ben's question:

I don't think that's a good explanation to [Ben's] question. I think a better ... answer would be because the one half yard represents the whole, and the one sixth represents what you're putting into the -- dividing the whole by. ... I think [June's] explanation is good, in terms of dividing one half by one sixth yard. But I don't think it addresses Ben's question why are you not dividing one sixth by one half. (P.O.I. 1461-1474)

Vonda was correct in arguing that, although for the specific problem under consideration June's explanation was correct, it neither addressed Ben's question nor it built any connections to the problem. Vonda pointed to such a connection when stating that the one half represented the "whole" – that is, the available ribbon length – and the one sixth corresponded to the ribbon length needed for a bow. However, associating the "whole" with the available ribbon length prevented her from identifying the error in Alan's work.

Even when prompted to discuss Alan's work, Vonda concurred with how Alan divided the line on the board: "[Alan] took one half yard and divided it equally into six pieces. I think he represented that correctly. He understood what one sixth mean" (P.O.I. 1530-1532). Vonda's assertion that Alan understood what the one sixth meant was incorrect since, if Alan had understood the meaning of the fraction he was asked to

represent, he would have divided the whole line into six pieces and not half of it. To further probe Vonda's thinking, the interviewer asked her to explain what she would do after Alan presented his work on the board:

**Charalambos:** So, if you were the teacher at this point, what would you do?

**Vonda:** I would ask [Alan] to explain what he did, why did he put six lines -- break it into six equal parts; one, two, three, four, five, six.

**Charalambos:** Okay. And what answer would you expect from him?

**Vonda:** [Alan would answer:] "Because we are dealing with one sixths, we wanna take half a yard and ... see how many one sixths I can get out of a half of a yard. So I divide it equally into sixths. ... Six equal parts." (PO.I. 1539-1554)

Vonda also argued that Alan understood that "to represent one sixth of something, you have to divide the *whole* into six equal parts" (PO.I. 1584-1586, emphasis added). On the surface, Vonda's last argument contradicted her earlier claim that Alan was correct in dividing half and not the whole line into six parts. Yet, this contradiction was not obvious to Vonda. In fact, as she explained at the end of the discussion of Alan's episode, by "see[ing] the whole as being a half" (PO.I. 1671-1672) – in other words, by associating the "whole" with the available ribbon quantity – she thought that Alan's work made perfect sense. Apparently, Vonda confounded the semantic meaning of the term *whole* with its mathematical connotation in the context of fractions.<sup>177</sup>

After a couple of long pauses, Vonda started questioning her thinking: "Is that [i.e., what Alan showed on the board] the right answer? [Long pause.] Maybe not." (PO.I. 1595). The answer she got from solving the problem  $\frac{1}{2} \div \frac{1}{6}$  by using the traditional algorithm helped her move beyond the impasse:

**Vonda:** I don't know. For whatever reason, I'm thinking the answer is three, and I need to see three one sixths here, sort of. In order to get three, I have to divide the *whole* yard into sixths.

**Charalambos:** And what is [Alan] doing?

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<sup>177</sup> For a more elaborated analysis of this distinction, see Deborah's case in this chapter (in *Selecting and Using Tasks*, under *Noticing and Interpreting-Evaluating*).

**Vonda:** He's dividing the half yard into six equal shares.

**Charalambos:** So, is his work correct or not?

**Vonda:** I'm not sure (laughs). I think he's wrong, but [pause]. I think he's wrong. (PO.I. 1627-1644, emphasis in the original)

To eventually decide whether Alan's work was correct, Vonda drew a line and divided it into six parts. After identifying the first three sixths as half a yard, she explained the error in Alan's work. Hence, in contrast to her pre-intervention performance in which she could not even figure out the correct answer to the problem, during the post-intervention meeting Vonda's obtaining the correct answer to this problem helped her figure out what the error in Alan's solution was. However, obtaining the correct answer did not always support her analysis of student work, as suggested by the difficulties she encountered analyzing Amanda and Julia's solution.

To evaluate Amanda and Julia's solution, Vonda applied the traditional algorithm, which yielded the answer  $4\frac{1}{2}$ . She attributed the discrepancy between her answer and the girls' answer to the girls' using a different approach from hers in solving this problem: whereas Vonda used the traditional algorithm, the girls used the common-multiples approach suggested by the teacher. She speculated, however, that her answer was the same as theirs, but she was not really sure: "I guess [four and one twelfth] is the same as four and one half if you -- [pause]. I don't know. [Long pause.] I don't know" (PO.I. 1920-1924). Because she seemed to be having some insights that could help her resolve the discrepancy, at a later point in the interview she was afforded another opportunity to reconsider the two girls' solutions. Although she again attributed the discrepancy between the two answers to the different approaches pursued in solving this problem, she also explored whether there was any relationship between the one twelfth (i.e., the girls' answer) and the one half (i.e., her answer):



- Charalambos:** So you were discussing that your answer does not match Julia's and Amanda's answer. [Do you have] any speculations about what might be causing this contradiction?
- Vonda:** I guess because they used twelfths? Using a common multiple. And I basically used the reciprocal, and multiplied, and then rounded it off to the lowest fraction.
- Charalambos:** So, then why do you have different answers?
- Vonda:** I don't know. Well, if she used one sixth and divided one twelfth -- [pause]. I don't know. ... [Pause.] Six into twelve is two, so I'm not sure. Maybe [Amanda] didn't -- I don't know. [Pause.] ... I'm thinking that if she used twelfths, and she was working with one sixth, perhaps she needed to bring it down to one half by dividing that into twelve. I don't know. I think that's possible, but -- [pause]. (PO.I. 1991-2020)

Even though Vonda identified the relationship between the numbers  $\frac{1}{2}$ ,  $\frac{1}{6}$ , and  $\frac{1}{12}$ , considering those numbers out of the context of the word problem at hand appears to have prevented her from explaining the contradiction. It was not surprising, then, that when at a later point in the virtual lesson Robert pointed to this contradiction, she admitted that she was still confused about it and that she did not know how to resolve it. Her difficulties in resolving this discrepancy stemmed from her struggles with explaining the fractional part of the quotient and with understanding the common-multiple approach suggested by the virtual teacher. Vonda mentioned to this effect, "I don't know [what is causing the contradiction] because I don't fully understand the twelfths yet, that [the teacher] gave them" (PO. I. 2098-2099). I take up her last comment when considering Vonda's performance in the next practice.

In short, Vonda's work in analyzing students' work and contributions was largely answer-driven: she analyzed Robert's, Ann's, and Michelle's solutions by comparing them against the answer she obtained from applying the traditional algorithm. She followed a similar approach to analyze Alan's work and Amanda and Julia's solution. This answer-driven analysis was effective when it was undergirded by at least some conceptual understanding of the mathematical ideas at stake. For instance, Vonda was able to successfully identify the dividend, the divisor, and the whole-number part of the

quotient in Michelle's solution; similarly, she identified the error in Alan's work after she applied the mathematical rather than the everyday meaning of the term "whole" to solve the division problem considered in Alan's episode. In contrast, when lacking such conceptual underpinnings, she could not appropriately analyze student work, as suggested by her analysis of the fractional part in Michelle's solution and her interpretation of Amanda and Julia's work. Probably because of not being aware of the idea that the dividend could be smaller than the divisor (recall that in analyzing the first page she was not concerned with this issue), she also did not identify the misconception in June's explanation. Finally, her assertions about students' understanding, although somewhat improved compared to her pre-intervention assertions, were not always accurate.

#### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Unlike the pre-intervention meeting in which Vonda struggled with the mathematical ideas considered in both episodes designed for this practice, during the post-intervention meeting she understood the mathematical ideas at play in the first but not in the second episode. This difference in her comprehension of the content was reflected in the approaches she proposed to scaffold students' work. While for the first episode she outlined an approach that could support student understanding, in the second episode she proposed a mathematically vague approach.

After identifying the error in Alan's work and when requested to explain how she would support Alan in understanding his error, Vonda mentioned that she would first ask Alan to explain his work. Yet, after eliciting Alan's explanation of his work, she would assume a more directive role: "I would ask him to take a yard and divide it into six equal parts, and then cut it in half, 'cause we're dealing with a half a yard" (P.O.I. 1662-1664).

Vonda's approach could support Alan in fathoming his error because, in contrast to the virtual teacher's approach, she would first divide the whole line into six parts – thus illustrating the meaning of a sixth – and *then* point to the half yard considered in this problem. Regardless of the appropriateness of her approach for supporting Alan's understanding, Vonda would apparently do most of the thinking for him.

Having admitted that she was clueless as to how the virtual teacher's recommendation of using the common multiple of twelve could help students solve the division problem  $\frac{3}{4} \div \frac{1}{6}$  (as discussed in her performance in the previous practice), in the second episode Vonda mostly offered some general ideas to support June and Shaun:

**Vonda:** I may have asked if anybody figured out how to divide three fourths into equal parts, and see if anybody got it. Or, as I was walking around, to see if I noticed someone that was able to work through it. Perhaps ask that person to explain what they did, perhaps come to the board and write it out on the board. And if not, maybe then I would, you know, actually write it up on the board. ... I would probably walk them through it, and hopefully get input from them as to what they think the next step would be.

**Charalambos:** So, what would you actually do?

**Vonda:** What did I do here? [Reviews her notes on solving this problem using the traditional algorithm; then, after a short pause she says:] Well, [the teacher] is saying "common multiples," and I'm thinking reciprocals; so I don't know. (P.O.I. 1772-1792)

Vonda's approach in supporting June and Shaun, although mathematically poor, was less directive than the approach she would follow in supporting Alan. Notice, for example, that she would first explore whether any of the students solved the problem; in this case, she would ask this student to share her or his work. If none of the students solved the problem, she would still elicit their ideas and have them share their thinking on the board. As a last resort, she considered walking students through the problem.

Her approach in the second episode seems to have been informed by both her pre-intervention tendency to walk students through a sequence of steps and ideas presented during the intervention courses (e.g., eliciting students' thinking; having students build on each other's contributions). The question arises as to why Vonda would follow a less

directive approach in the second than in the first episode. Although multiple explanations are plausible, one explanation is that she was less clear on the content in the second than in the first episode. To the extent that this explanation holds, it points to how Vonda might have appropriated the ideas discussed during the intervention courses about supporting students' work and thinking: that the teacher should elicit students' ideas when she is not very clear on the content.

*Noticing and Interpreting-Evaluating.* While in the pre-intervention meeting Vonda endorsed Ms. Rebecca's approach in both episodes under consideration, during the post-intervention meeting she was concerned with how the teacher responded to June's and Shaun's questions. She argued, in particular, that simply recommending that students use the idea of common multiples could not support them in overcoming the difficulty they faced. Vonda was not concerned, however, with that the virtual teacher intervened and provided students with a hint that might have not been necessary. Similarly, she was not troubled that in the first episode the teacher was doing most of the thinking for Alan by the very pointed questions she was asking. In fact, Vonda commented that the teacher's approach resembled how she herself would support Alan if she were teaching the lesson. Overall, while in the post-intervention meeting Vonda appeared to be more concerned with issues of making meaning, she was not equally concerned with the teacher's lessening the tasks' cognitive demand by her questions or by her intervening and providing students with hints to follow.

Vonda's evaluation of the teacher's approach in these episodes resonated with her beliefs about teaching and learning mathematics as captured by her responses to some survey statements. In particular, even at the culmination of the intervention, she believed

that “students should never leave math class (or end the math period) feeling confused or puzzled” (on a scale from 1 to 7, with 1 representing “strong disagree” and 7 corresponding to “strongly agree,” she chose 6) and disagreed with the statement that “teachers should not necessarily answer students’ questions but let them puzzle things out themselves” (she chose 2 for this statement). Vonda’s pre- and post- intervention stance on these statements was very similar (i.e., during the pre-intervention survey administration, she chose numbers 6 and 1, respectively). What was different, though, was her appraisal of the statement “When students can’t solve problems, it is usually because they can’t remember the right formula or rule” (and hence, the teacher’s role is to help them remember such formulas and rules). Whereas she strongly agreed with this statement in the pre-intervention survey, at the end of the intervention she was not a keen supporter of this idea (she moved from reporting 7 in the pre-intervention to reporting 2 in the post-intervention survey administration).

#### *Professed Changes and Their Attributions*

Vonda identified two main changes as having contributed to her post-intervention performance in the teaching simulation: changes in her beliefs about teaching and learning mathematics and changes in her understanding of the content. She talked about the first type of change when asked to describe how she perceived the teaching and learning of mathematics after her participation in the intervention courses; she referred to the second type of change when explaining her professed improved performance in the teaching simulation.

When prompted to consider whether she sensed any changes in her thinking about teaching and learning mathematics, Vonda mentioned,

Yeah, I think, teaching math *conceptually*? I didn't think in that terms before; and I think that's because of my own learning experiences. They're so antiquated that, you know, it's nothing that I learned in these courses, here. (PO.I. 109-111, emphasis in the original)

This argument resonates with how she responded to the last survey statement mentioned above. Notice, also, that she directly attributed her earlier beliefs about teaching and learning mathematics to how she had been taught this subject, thus corroborating the arguments made in Chapter 4 about how Vonda's school experiences informed her images about teaching this subject.

In addition to the changes in her beliefs, Vonda also reported having a better understanding of the content. For instance, she stated, "I felt I had a better understanding of the math" (PO.I. 2455) and elsewhere, "I had a better understanding of fractions" (PO.I. 2043). This increased understanding of mathematics in general and fractions in particular enabled Vonda to consider the virtual lesson from the perspective of a teacher rather than from the standpoint of a student. She explained,

'Cause I took it [the virtual lesson] from my perspective as a teacher, what I would want to do, I think. I have a better understanding of ... how I would teach this lesson, if I had to teach it. (PO.I. 2413-2416)

She elaborated upon this idea later in the interview:

Well, if I understand what it is that I'm doing, I can teach it better, I guess. So in having an understanding, I'm able to, I guess, focus on things that I would do, if I were in [Ms. Rebecca's] place as a math teacher. You know, pulling from what I've learned through class, for example. (PO.I. 2463-2466)

What is interesting to notice in this excerpt is not only Vonda's claim that her better understanding of the content enabled her to more closely attend to the virtual teacher's decisions and actions, but also her argument that it helped her capitalize on ideas and teaching strategies discussed during the two intervention courses. It follows, then, that a better understanding of the content comprises the foundation for capitalizing on teaching tips and strategies. From this respect, this last comment echoes the idea discussed in

Chapter 4 when considering Vonda's entrance performance: that a basic understanding of the content is critical, although not sufficient, to effective teaching.

Two main sources appeared to have informed the abovementioned changes in Vonda's performance: her coursework and fieldwork experiences. She identified, in particular, the intervention courses' activities on place value as an eye opener to her about the importance of teaching mathematics conceptually. While responding to some of the teaching-simulation tasks, Vonda also drew on teaching approaches modeled or discussed during the two intervention courses. It therefore seems reasonable to argue that the intervention courses also afforded her some ideas and strategies about handling certain instructional situations. Because these ideas and strategies were rather different from those she experienced as a learner of mathematics, they could be considered to represent alternative images of teaching.

Vonda's fieldwork experiences, on the other hand, appear to have provided her with some images of what it is feasible when teaching in a real classroom environment. For instance, when considering Ms. Rebecca's decision to assign individual work right after the students had solved only two problems, Vonda commented:

It seems like she's moving rather fast. Just thinking about my students in my fieldwork, I don't think they would have gotten it that quickly. So I may spend more time before going off into independent work. (PO.I. 1708-1711)

Thus, her fieldwork helped her view the virtual lesson in light of how students' capabilities or difficulties with the content might delimit teachers' decisions and actions.

#### *Changes in Performance on the LMT Test: A Closer Look*

During the post-intervention administration of the test, Vonda correctly answered six additional questions that she had answered incorrectly during its pre-intervention administration; yet, she also incorrectly answered four questions that she had answered

correctly in the test's first administration. Hence, her post-intervention LMT score was only slightly different from her pre-intervention score. That is why Vonda was a divergent case since she exhibited greater gains in her post-intervention teaching-simulation performance (and particularly in the tasks of the MTF-related practices) than in her MKT performance. Yet, a closer analysis of Vonda's post-intervention MKT performance suggests that her work on the LMT test was largely in accord with her work on the teaching simulation.

In the second administration of the LMT test, Vonda correctly answered questions designed to capture the PSTs' understanding of division (of fractions). For instance, she correctly answered questions 8b, 8c, 11c, 14a, 14b, 14d, 14e, 19a, 19c, and 19e. The fact that she correctly answered many of the questions that gauged understanding of division (of fractions) resonated with her better understanding of this operation as also captured in the teaching simulation.

Consistent with her performance in the teaching simulation, she incorrectly answered the two questions designed to capture the PSTs' understanding of the different manifestations of units when working on fractions (questions 1 and 20) and the question pertaining to interpreting the fractional part of the quotient (question 17). Recall that the ideas pertaining to these questions caused Vonda the most trouble when working on the teaching simulation. It is even informative to consider Vonda's multiple-choice answers to these questions, which collectively point to her still-developing understanding of the ideas targeted by those questions. In question 1, she selected the answer corresponding to incorrectly interpreting José's grasp of the unit as inappropriate. If Vonda's conceptual understanding of units when working on fractions was still underdeveloped, as her



performance in the teaching simulation suggested, it makes sense that she would not understand José's use of units, which was the least conventional among the three unit uses examined in this question. In question 20, she chose the answer representing the most conventional interpretation of units outlined in this question. Finally, in question 17, she chose the option corresponding to the interpretation of the fractional part from the perspective of absolute rather than relative units. In conjunction, her answers to all three questions suggest that, while she was able to correctly identify the conventional uses of units, she struggled with understanding their less conventional uses.

Vonda also incorrectly answered questions 11b, 15, and 25, which pertained to the measurement interpretation of division. At first sight, this finding appears to be inconsistent with her performance in the teaching simulation, given that during the teaching simulation she drew on the idea of fitting in or taking away divisor units from the dividend units (which resonates with the measurement division interpretation). However, the latter two questions required more advanced thinking than simply having grasped the aforementioned concept of division: question 15 pertained to interpreting representations appropriately and question 25 required comparing different problem situations. These additional requirements might partly explain this inconsistency.

#### *Analytical Commentary*

Vonda's post-intervention performance on the LMT test and her post-intervention performance in the teaching simulation pointed to a strengthening in her understanding of the content explored in the simulation: in contrast to her pre-intervention performance, she had a better grasp of division (of fractions) as fitting divisor units into the dividend. At the same time, both her LMT performance and her performance in the teaching

simulation suggested that her understanding of relative and absolute units and, accordingly, her knowledge of the fractional part of quotients, was still under development. Both the increases in Vonda's understanding and her struggles with the idea of relative and absolute units were reflected in her performance in the MKT-related practices in ways that corroborate the association between the gains in knowledge and teaching performance yielded by the quantitative analysis. Specifically, for the practices of providing explanations and using representations, the small changes Vonda experienced in her knowledge were consistent with the small distance she covered in shifting from a merely numerically driven performance in the pre-intervention meeting to an amalgam of numerically and conceptually driven performance in the post-intervention meeting. For the practice of analyzing students' work and contributions, whereas Vonda was able to follow and analyze conventional student ideas, she was not equally successful at analyzing non-conventional ideas. Below, I elaborate upon each point and then shift to consider the association between the gains in her knowledge and the gains in her performance in the MTF-related practices.

Unlike her pre-intervention explanations, which were merely numerically driven, Vonda's post-intervention explanations were somewhat undergirded by her better understanding of the content considered in the simulation. For example, her improved understanding of the concept of division allowed her to explain the division problem  $2 \div \frac{3}{4}$  as fitting  $\frac{3}{4}$ -portions into the two wholes represented by the dividend. Yet, when her understanding could not sufficiently support her, she reverted to the numbers involved in this division problem. Because of her difficulties with the relative and absolute units, to explain the fractional part of this division, she considered using another

representation to illustrate the thirds; yet realizing that the latter representation did not preserve the fourths (i.e., the pieces into which the dividend should have been divided), she reverted to her original representation. Even so, the explanation she eventually provided for the fractional part of this quotient was more grounded in the numbers than in the concept of relative units.

Her explanation for the reciprocal was also anchored by the reciprocal's numbers. Whereas she initially considered presenting the reciprocal merely as "the reverse of the fraction," in considering the reciprocal of the foregoing division problem, she talked about fitting "threes" into "fours." Based on this idea, she provided an explanation that, on the surface, could be acceptable. However, a closer look at this explanation suggested that its conceptual foundations were rather weak, as implied by at least two pieces of evidence. First, she never defined the *concept* of the reciprocal; her definition of the reciprocal as being "the reverse of the fraction" did not convey the reciprocal's actual meaning. Second, in talking about the reciprocal, she referred to its numerator and denominator as separate numbers, instead of focusing on what the relationship between these numbers represents.

Hence, Vonda's performance in providing these explanations supports propositions  $A_1$  and  $A_2$  listed in Table 4.6. From an affordance perspective, her performance corroborates the idea that with a better grasp of the content, PSTs are better positioned to provide concept-driven rather than (merely) number-driven explanations; from a constraint perspective, her performance resonates with the idea that when knowledge falls short, numbers take precedence in the explanations provided.

Vonda's use of representations was in accord with her performance in providing explanations. Her better understanding of the content allowed her to not only select appropriate representations (which was the farthest she went during the pre-intervention meeting), but also to use them appropriately to at least explain the whole-number part of the quotient in the division  $2 \div \frac{3}{4}$ . When her understanding could not substantially support her, she shifted to a more numerically driven use of the representations she employed. This was particularly evident when, confronted with difficulties in explaining the fractional part of the quotient, she switched to a different representation, trying to illustrate the numbers involved in the fractional part of the quotient rather than explain the meaning of these numbers. She even entertained the idea of representing the whole-number part of the quotient with the shaded portions of the representation she used to illustrate the thirds. From this perspective, Vonda's post-intervention performance is in accord with propositions B<sub>1</sub> and B<sub>2</sub>, which suggest that a better understanding of the content supports PSTs in appropriately using the representations they select and that gaps in their understanding of the content might lead them to use these representations in a numerically driven fashion.

Vonda's performance in the practice of analyzing students' work and contributions further supports the association between the gains in knowledge and performance by providing both affordance and constraint examples. With a better understanding of the traditional division-of-fractions algorithm and other related procedures (e.g., converting mixed numbers into improper fractions), unlike her pre-intervention performance, during the post-intervention meeting Vonda quickly ascertained the correctness of Robert's answer (i.e., a conventional solution). Equipped

with a deeper conceptual understanding of division, she was also capable of understanding the whole-number part of Michelle's quotient. On the other hand, Vonda's weak understanding of absolute and relative units in fraction division was reflected in her performance in at least three instances. First, in analyzing Michelle's solution, she drew on the numerical answer to the division problem under consideration and correctly argued that the leftover part in Michelle's answer was two thirds. Yet, when asked to use Michelle's drawing and explain where she could see the two thirds, Vonda interpreted this drawing incorrectly. Second, she struggled with identifying the source of the discrepancy between her answer and Amanda and Julia's solution. Perhaps the most telling example of how limitations in knowledge appeared to constrain Vonda's performance stems from her analysis of Alan's error. Without a thorough understanding of the mathematical notion of the "whole," Vonda confounded this notion with its everyday semantic meaning. Thus, she initially thought that Alan's work was correct since Alan divided the available ribbon length (i.e., the whole viewed from its everyday connotation) into six parts.

Overall, Vonda's performance in the foregoing episodes corroborates propositions  $C_1$  and  $C_3$ . In particular, her better understanding allowed her to analyze Robert's conventional solution quickly and appropriately (proposition  $C_1$ ). At the same time, most of her work in analyzing students' solutions (and particularly Michelle's, Alan's, and Amanda and Julia's) was answer-driven – that is, her analysis and interpretation of the students' work was guided by the answer obtained for those problems (proposition  $C_3$ ). Given that in the pre-intervention meeting Vonda could hardly obtain the answers to the problems the students were asked to solve, her post-intervention performance suggests

that such an answer-driven analysis should not be depreciated; in fact, it offered Vonda a yardstick against which to appraise students' work.

Vonda's analysis of June's explanation also represents both an affordance and a constraint example of the knowledge-performance relationship. Armed with a better understanding of the content, Vonda noticed that June's explanation was not actually addressing Ben's question; yet probably without being cognizant that the dividend can be smaller than the divisor, she did not identify the misconception in June's argument. Thus, Vonda's performance in this episode corroborates proposition C<sub>4</sub>, which holds that PSTs need to be aware of certain misconceptions if they are to identify them when they arise during teaching.

Collectively, the aforementioned examples suggest that the relatively small growth and the limitations in Vonda's knowledge were both reflected in her performance. Vonda herself captured this idea by eloquently stating that, "If I understand what it is that I am doing, I can teach it better" (thus the name given to Vonda's post-intervention case).

Unlike with the changes in Vonda's performance in the MKT-related practices, the small changes in Vonda's knowledge were not consistent with the quite notable changes in her performance in the MTF-related practices. This was because, with an increased awareness to issues of understanding and meaning-making, Vonda was more successful at noticing and appropriately analyzing instances in which the virtual teacher shifted attention from meaning and understanding to manipulating numbers. Her post-intervention performance in the MTF-related practices was interesting in two additional respects. First, in contrast to the noticing and interpreting-evaluating tasks, in the performing tasks of the MTF-related practices this increased attention to meaning and

understanding did not compensate for limitations in her knowledge. Second, even in cases when Vonda's knowledge supported a more conceptually driven performance, certain beliefs she still professed influenced what her knowledge allowed her doing. I discuss each point in turn.

The most notable professed change in Vonda's beliefs about teaching and learning mathematics was her greater attention to issues of conceptual understanding and meaning-making. Whereas at her entrance to the program she mainly subscribed to a type of instruction that emphasized rote learning, at the culmination of the intervention courses she was more sensitive to helping students understand the underlying meaning of the mathematical procedures under consideration, a change which was reflected in her teaching-simulation performance, and particularly in the tasks of noticing and interpreting-evaluating. For example, with an increased sensitivity toward issues of meaning-making, Vonda disapproved of Ms. Rebecca's presentation and enactment of the first task, arguing that the virtual teacher did not sufficiently support her students in grasping the notion of the whole. Similarly, Vonda argued that the virtual teacher's suggestion that students use the common multiple of twelve did not suffice to help them overcome their struggles with the division problem they were considering.<sup>178</sup>

These changes in Vonda's beliefs did not necessarily compensate for limitations in her knowledge. For example, although she wanted to be more selective in using the first textbook page, she did not make a good selection of exercises for an introductory division-of-fractions lesson: in most of the exercises she selected, the dividend was

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<sup>178</sup> Despite her increased sensitivity to issues of meaning-making, in some instances, Vonda outlined instructional approaches that reflected her pre-intervention tendency to outline steps for students to follow. A case in point here is how she considered supporting Ann in correcting her error. Convinced that Ann misunderstood the reciprocal, Vonda would ask her to correct her solution by simply "following the steps" elicited from Ann's classmates.

smaller than the divisor, and given her partial understanding of the content, it is questionable whether she would be able to enact these exercises in a manner that would support students' conceptual understanding of division. Similarly, despite her increased sensitivity to issues of meaning-making, because of her own difficulties with the notion of the reciprocal, she did not appear to notice that in enacting task D, the virtual teacher merely manipulated numbers without actually helping her students understand the meaning of the reciprocal.

Her performance in the practice of responding to students' requests for help provides analogous examples. For instance, after figuring out the error in Alan's work, she was able to respond to Alan's implicit request for help in a mathematically sound manner. In contrast, because of her professed difficulties with common multiples and how this idea applies to solving fraction division problems, Vonda addressed June's and Shaun's questions in a pedagogically appropriate yet mathematically nebulous way. The dissimilar way in which she responded to Alan's indirect request for help and to June's and Shaun's questions supports proposition  $E_1$ , which maintains that a strong understanding of the content is a prerequisite for responding to students' requests for help in mathematically valid ways.

In some cases, Vonda's increased understanding of the content allowed her to perform in a more conceptually oriented manner; yet, certain beliefs she held about the teacher's role and the role of confusion and complexity in learning impinged on what her relatively enhanced knowledge could have allowed her doing. In particular, regardless of Vonda's greater attention to issues of making-meaning and understanding, Vonda still considered the teacher as the main source of knowledge whose responsibility is to show



and tell and, when needed, intervene to minimize student confusion, struggle, and error. Obviously, these beliefs had an impact on her teaching performance.

For example, Vonda's better understanding of the content allowed her to identify some of the affordances of the second textbook page she totally overlooked during the pre-intervention meeting.<sup>179</sup> One of the affordances that Vonda identified was the page's direction on explaining the fractional part of a quotient. Although she identified this direction as an asset, she considered not incorporating it in her introductory lesson on division of fractions because she thought that asking students to explain the fractional part of their answer would complicate matters for them. Additionally, instead of asking students to justify their thinking in words, diagrams, and numerical symbols (as required by the textbook page), Vonda considered simplifying the page's requirements by asking students to use one representational mode at a time. Similar examples were also identified in her practice in responding to students' requests for help. For instance, although she correctly identified the error in Alan's work, instead of helping Alan see and correct his error, she would do most of the thinking for him, by explaining to him what his error was and how to correct it. In conjunction, these examples support propositions  $D_3$  and  $E_2$ , which hold that other factors, besides teacher knowledge, inform teachers' task enactment and their responses to students' requests for help.

In addition to the foregoing changes in Vonda's knowledge and beliefs, the post-intervention interview also suggested that at the culmination of the intervention courses Vonda's images of teaching were enriched. As she argued during the post-intervention interview, when entering the program her conceptualization of teaching and learning was

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<sup>179</sup> As such, her post-intervention performance lends support to proposition  $D_1$  concerning the relationship between knowledge and PSTs' ability to identify the affordances of curriculum materials.

largely based on her “antiquated learning experiences,” which prioritized memorization over understanding. Vonda’s coursework and fieldwork offered her alternative images of teaching. At several instances during the teaching simulation, Vonda drew on these images to outline instructional approaches to support students’ work and thinking.

Figure 5.18 delineates Vonda’s post-intervention profile. For simplicity, this figure presents only Vonda’s beliefs about the teacher’s role in supporting student learning (which largely remained unaltered) and her beliefs about emphasizing meaning and understanding, which differed from her pre-intervention beliefs. While the latter beliefs supported Vonda in proposing approaches emphasizing meaning and understanding, the former beliefs militated against the rich learning environments considered in the study; thus, Figure 5.18 presents the effect of Vonda’s beliefs on her performance with both a positive and a negative sign. This figure also shows that all the changes pertaining to Vonda’s endorsement of a more conceptually oriented instructional approach (e.g., emphasizing meaning during task presentation and enactment, offering conceptually driven explanations) were contingent on her understanding of the content. The question marks accompanying the changes in Vonda’s profile suggest that, despite some progress, Vonda’s post-intervention performance still warranted further improvement to meet the thresholds of those criteria. The careful reader will also notice that Figure 5.18 retains the pre-intervention characterization of Vonda’s explanations as ambiguous, mostly because of her use of the terms “thirds” and “threes” interchangeably (when explaining the quotient). Finally, in contrast to her pre-intervention performance, her post-intervention profile presents Vonda as able to analyze conventional student ideas, which represents a significant improvement.

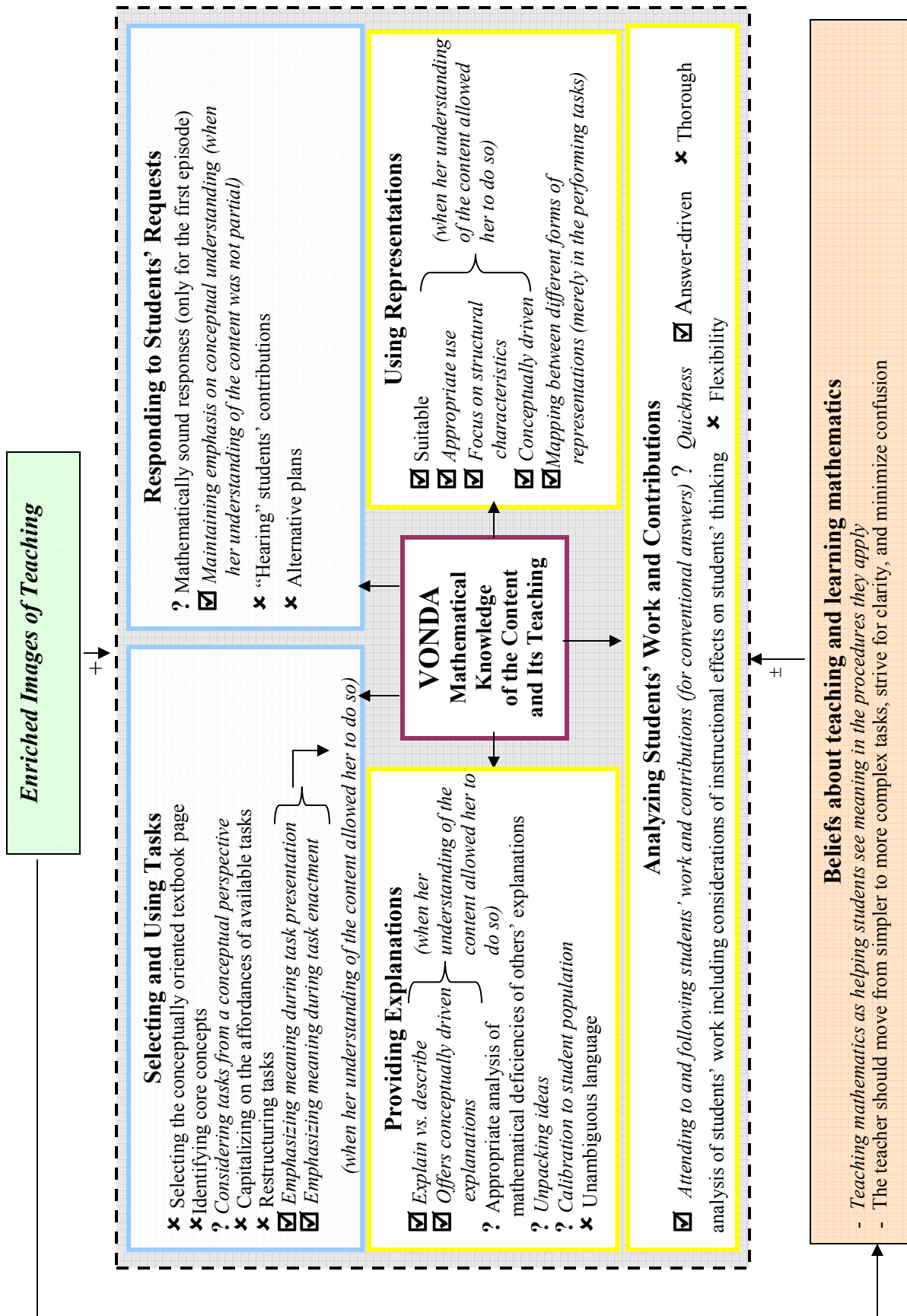


Figure 5.18. Considering the association between knowledge and teaching performance through Vonda's post-intervention profile.

*The Case of Tiffany Revisited: What “Having the Link” Might Allow One to Do?*

Tiffany was a divergent case according to both her entrance scores and the changes in her performances. As Figure 5.3 shows, she was the only PSTs who experienced greater gains in her LMT test performance than in her teaching performance (relative to her counterparts). Figure 5.4 also shows that she exhibited more changes in her performance in the MKT-related practices than in the MTF-related practices, a pattern that resonated with the design and focus of the two intervention courses. Tiffany’s post-intervention interview was held in late February. In the time that elapsed between the first and the second interview (i.e., 33 weeks), Tiffany had completed six of the ELMAC courses; she had also observed and taught lessons in a fifth-grade class.

Tiffany’s case lies somewhere in the middle of the cases previously considered in terms of the changes in her knowledge and teaching performance (i.e., Deborah experienced the highest gains in knowledge and performance, while Nathan and Nicole were amongst the PSTs who experienced the lowest changes in both knowledge and teaching performance). Although a divergent case, Tiffany does not actually negate the strong association between the gains in knowledge and teaching performance that emerged from the quantitative analysis, at least as the case of Suzanne considered next actually does. This is because, in contrast to Suzanne, Tiffany experienced notable gains in *both* her MKT and her teaching performance; however, her gains in these two categories were not as proportional as they were in the cases of Nathan, Nicole, and Deborah. In fact, Tiffany experienced most of these gains in the practice of using representations; this, in conjunction with the gains she experienced in her MKT score,

supports the quantitative findings that showed a strong association between the changes in the PSTs' MKT and the changes in their performance in using representations.

Compared to Vonda, Tiffany entered the program with an increased attention to issues of meaning-making and understanding and experienced greater changes in her knowledge but not as notable changes in her beliefs. Thus, Tiffany's case, more so than Vonda's, helps better explore how certain gains (alongside some persisting limitations) in knowledge might translate in changes in teaching performance.

### *Selecting and Using Tasks*

*Performing.* Tiffany's analysis of the two textbook pages was more in-depth in the second than in the first meeting. As she noted, while before coming to the first meeting she had merely scanned the two pages, on analyzing these pages before coming to the second meeting she had focused on more substantive issues:

[W]hen I had scanned over these two [pages] that first time, all I had really done was *scanned* over them. And looked at and said, "Okay. You know, maybe I like this, maybe I don't." This time when I went through it I was able to look at it a little bit differently. (PO.I. 220-224, emphasis in the original)

This different viewing of the pages was reflected in how she considered using each page.

If she were to use the first page, she would start with a word problem, then move to explaining the division-of-fractions algorithm, and conclude with assigning some exercises to students for practice. This sequence of activities was identical to what she proposed in the first meeting; yet the tasks she considered using during the second meeting and her proposed enactment of these tasks differed.<sup>180</sup>

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<sup>180</sup> During that meeting she thought of starting with one of the page's word problems; yet, she saw no connection between these problems and the division of fractions. She would then move to explaining the algorithm; yet, she could not offer an appropriate explanation for it. Similarly, she wanted to assign only some exercises to her students, but she did not have criteria that would inform such selection.

Tiffany argued that the word problems of the first page matched with neither the worked out example on this page nor the page's 16 exercises. For example, to solve the first word problem (i.e.,  $30 \div \frac{1}{4}$ ), she would set the equation "one fourth of  $X$  equals thirty," which she wrote as shown in Figure 5.19 (notice the omission of the variable). She would then multiply both parts of the "equation" by the reciprocal of the one fourth to get the answer to the problem (see second row of Figure 5.19). She admitted, however, that she could do so simply because she was aware of the reciprocal rule; hence, she questioned whether students would be able to see any connection between this problem and the division of fractions:

But there, in my way of solving it, I don't see the division anywhere. ... I see the second part of the algorithm where we're talking about a reciprocal. But ... I think it's really hard to map where a student might get division. And I don't know if the reason I can do this, is just because I already have that algorithm, the division algorithm, in my head. ... I mean, I had to play around with the numbers even to get to thirty divided by one fourth equals one twenty. (PO.I. 313-343)

The image shows two lines of handwritten work. The first line is  $\frac{1}{4} = 30$ . The second line is  $\frac{4}{1} \cdot 30 = \frac{1}{4} \cdot \frac{4}{1} = 120$ . The numbers 1, 4, and 30 are written in a cursive, handwritten style.

Figure 5.19. Tiffany's work on solving the first word problem of the first page.

Her work as illustrated in Figure 5.19 resonated with her argument about playing with numbers without closely attending to the mathematical substance of the symbols she was using and how she was using them. Notice, for example, that in the first row number 30 appears in the right-hand side of the "equation," whereas in the second it is moved to the left-hand side of the "equation," which would be perplexing to students; even more critically, the second row includes an invalid sequence of equations, which essentially equates 120 to one (i.e.,  $\frac{1}{4} \times \frac{4}{1}$ ). Regardless of these deficiencies in her work, Tiffany identified at least some connections between the first word problem and the reciprocal

included in the traditional division-of-fractions algorithm, which she was not able to do during the first meeting.

Even more critically, Tiffany thought of replacing this word problem with another problem that resonated more with the measurement interpretation of fraction division. To pose such a problem, she first drew on the Candy Jar problem discussed during the math content class (see Appendix 1 for more on this problem); yet, realizing that this problem was not suitable for dividing fractions, she consulted the second textbook page and proposed the following word problem: “[I]f I have thirty feet of licorice and ... if each person gets one fourth of the piece of licorice, how many people could get licorice?” (PO.I. 726-728). Proposing this problem, however, was not an easy task for Tiffany, who appeared to still be grappling with the notion of units. For example, she originally posed the problem, “If I have thirty feet of licorice and each person gets one fourth of the amount, how many people would I give licorice to?”, but she soon realized that this problem would not work because it would yield few pieces of licorice (she estimated that it would yield five to six pieces).

After having her students work on word problems like the abovementioned, Tiffany would ask them to identify patterns in their work, hoping that this would help students get closer to the traditional algorithm. She would then introduce and explain the traditional algorithm by using a visual representation (see her performance in *Providing Explanations*). Following that, she would assign some of the 16 exercises to her students to help them better understand the meaning of division of fractions and to practice the algorithm introduced. In contrast to the pre-intervention meeting, she now had some criteria upon which to make an informed selection of exercises:

For the first step I would be looking for numbers that I think the students could divide pretty cleanly. And ones that they could actually draw a picture to help figure it out ... you're having to relate those two fractions together, and a lot of times that's really difficult to do with a picture. (PO.I. 1614-1623)

[Exercises] like number one, where you're having two thirds divided by one third, you're dealing with the same fraction there; you're still dealing with thirds. That would make number one a little bit easier [compared to the other exercises]. (PO.I. 1670-1672)

As these two excerpts jointly suggest, Tiffany started viewing those exercises from a more conceptual perspective. She identified exercises with the same denominators as easier to solve; she also talked about choosing exercises that would be easier to show in pictures and whose fractions could be divided “pretty cleanly.”

It is also informative to consider Tiffany's work on exercise 3 (i.e.,  $\frac{5}{8} \div \frac{5}{6}$ ) which she identified as being more difficult to solve because of its fractions not “dividing up cleanly.” To solve this exercise, she drew two lines equal in size and divided the first one into eighths and the second one into sixths. She then colored the five eighths on the first line in blue and the five sixths on the second line in red (see Figure 5.20). After that she drew a vertical line from the end of the fifth eighth down to the sixths line (in pencil) and a red vertical line from the end of the fifth sixth up to the eighths line; this helped her realize that the divisor was larger than the dividend. She exclaimed, “Five sixths is actually bigger than five eighths” (PO.I. 1721). She then confessed that because she could not solve this problem using her picture, she first needed to figure out the numerical solution to the problem: “So really ... in a problem like this I'm not ... using the models to help solve the algorithm. ... There's a disconnect” (PO.I. 1832-1839). Once she figured out the quotient to this division problem, she associated this quotient with how much longer (in additive terms) the fifth sixths was compared to the five eighths. Yet, on a second thought, she revised her idea: “Oh, no! Because this, the bigger piece [i.e.,  $\frac{5}{6}$ ], is three quarters of that [i.e.,  $\frac{5}{8}$  ]” (PO.I. 1782).



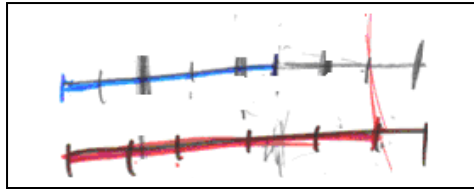


Figure 5.20. Tiffany's work on solving the division problem  $\frac{5}{8} \div \frac{5}{6}$ .

Tiffany's work on analyzing the first textbook page and her proposed plan for using it for an introductory division-of-fractions lesson suggest that she still struggled with the notion of units; nevertheless, her improved understanding of division from a measurement perspective allowed her to more closely analyze the tasks of this page and start exploring how to restructure this page to better support student understanding. Her enhanced understanding of the content also allowed her to more closely analyze the second textbook page.

As in the pre-intervention meeting, during the post-intervention meeting Tiffany preferred the second textbook page for an introductory lesson on fraction division, stating that the first page is simply "spoon-feeding students" by outlining steps for them to follow (PO.I. 2031). Unlike the first meeting, in the second meeting Tiffany also identified some of the page's affordances; yet she was still unclear how she could capitalize on them to foster student understanding.

For Tiffany, the most significant affordance of this page was task D, in which students are expected to figure out an algorithm for dividing fractions; this approach was totally foreign to Tiffany, since as a learner of mathematics she was used to being given a rule to apply. It is not surprising, then, that she envisioned this approach as having potential to help students understand fraction division. However, Tiffany was not sure how to support students in deducing this algorithm from their answers to tasks A through C. She argued,

[It] might not be as easy to find an algorithm. Because if you were just drawing pictures [as students are required to do in tasks A and B], you're not getting any algorithm. So I don't know how I would promote, like, working towards that algorithm with them. (PO.I. 1912-1915)

Even so, she thought of supporting students' work on this task by using a visual representation to explicate the meaning of the reciprocal. To explain how she would do so, she used task A<sub>1</sub> (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ) and drew six "tiles" to represent the six sixths in a yard (see Figure 5.21). She then identified three of them as half of the yard, but she was unsure whether this would actually scaffold students' understanding:

How many sixths are in one half? You'd get three. I don't see where the -- you're almost at the algorithm, because you're counting one, two, three, four, five, six. But, I don't know. (PO.I. 1987-1990)

Tiffany's remark that there are six sixths in a yard reflected the fundamental concept of the reciprocal. Yet, because she was still not solid on this notion (see more in her performance in *Providing Explanations*), she could not draw on this idea to explain how she would further support students' grasp of the algorithm. It is noteworthy, however, that the activity she proposed for scaffolding students' understanding of the reciprocal was, as she acknowledged, inspired by pertinent activities organized in the math methods course; during these activities, an algorithm was presented and considered in tandem with a visual representation.

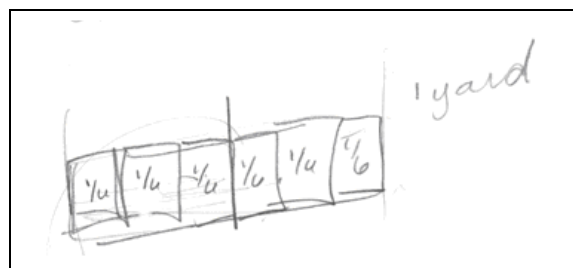


Figure 5.21. Tiffany's work on using the division problem  $\frac{1}{2} \div \frac{1}{6}$  to explain the traditional division-of-fractions algorithm.

During the pre-intervention meeting, Tiffany identified the second page's requirement that students draw pictures to represent their answers as another affordance

of this page, especially for the visual learners. Unlike the first meeting, during the post-intervention interview, she regarded these drawings as tools that *all* students could benefit from using. In fact, she contended that students could use such visual representations and solve the problems of this page without actually using the algorithm. This argument, however, was an overgeneralization, since certain problems (e.g., B<sub>1</sub>:  $\frac{1}{5} \div \frac{2}{3}$  and B<sub>2</sub>:  $1\frac{3}{4} \div \frac{2}{3}$ ) would be hard to solve by using these visual scaffolds. Finally, in neither of the two meetings did Tiffany identify the page's direction on explaining the fractional part of the quotient as an important asset of the second page. This finding should be seen in light of her still partial understanding of this idea.

To summarize, compared to her pre-intervention superficial analysis of both textbook pages, Tiffany's post-intervention analysis went into more depth. In analyzing the first page she proposed restructuring the page's word problems and she started thinking about this page's exercises from a more conceptual perspective. In analyzing the second page, she identified more of its strengths compared to what she did during her pre-intervention analysis. However, she still struggled with how to enact this page to better support student learning.

*Noticing and Interpreting-Evaluating.* Tiffany's post-intervention performance in the noticing and interpreting-evaluating tasks of this practice did not differ remarkably from her pre-intervention performance. She again disagreed with the teacher's decision to introduce task A<sub>1</sub> by having students read the title of the second page (i.e., "Dividing a Fraction by Fraction") and by asking them, "What are we going to divide in this case?" If she were teaching the lesson, she would ask a more general question like, "How are we going to set up the problem?" (PO.I. 2578-2579). She was also concerned with the

enactment of this task; she particularly critiqued the teacher for her emphasis on getting correct answers and for enacting this task without referring to its original ribbon-context.

Although admitting still not being clear on the idea of the reciprocal, Tiffany was also concerned with the enactment of task D. She contended that while working on this task, the teacher not only did most of the thinking for her students, but also did not really help them understand the meaning of this algorithm. Tiffany opined,

[Ms. Rebecca]'s teaching the shortcut or the algorithm, but students wind up like me, however many years later, and know the algorithm, know how to find it, but have *no* idea why. ... I think they're going to still be left with the same misconception of why, or no conception of why do I do that reciprocal. And, I mean, it's nice that she asks them to find patterns, but ... I don't think there was really a pattern identified. ... She told it to them. She did the whole thing for them. ... It was almost like a false discovery type of a thing, is how I saw it. That she was asking them these questions, "What do you notice? What's this pattern?" as a way for her to be able to teach it. But I don't know if they would feel any sense of, like, "Ah-ha! I got it!" Because there's something that's not there. (PO.I. 3544-3578)

Tiffany reiterated these ideas at the culmination of the interview when asked to evaluate the teacher's overall approach in teaching this lesson. She argued that, regardless of being engaged, the students in this lesson were not actually thinking and that the whole lesson resulted in a manipulating-numbers enterprise. She mentioned,

So, I think the lesson was still missing ... relevance. They never *once* went back to ... their story problem. Or what they were trying to find, or did they end up with more or less than they started with? You know, the whole realistic portion of it was gone; and it turned into just numbers. And so the pictures didn't have a link to the problem. The students didn't have a full understanding of what they were doing. They were following what the teacher was showing. And it was like things were being said and then lost. ... There was no like, solving of something. It was, "This is what I think. I'm going to give it to you. Okay, what do you think?" You know, the words were all there, her questioning of the students were [sic] there, but there was a lack of, like, a full communication of ideas, I think. (PO.I. 3951-3981, emphasis in the original)

Overall, similar to the first meeting, Tiffany's post-intervention evaluation of the teacher's decisions and actions reflected her attention to issues of making meaning and affording students opportunities to think and reason through different mathematical ideas.

### *Providing Explanations*

*Performing.* Tiffany's post-intervention performance in providing explanations was suggestive of her increased conceptual understanding of fraction division. Recall that in the pre-intervention meeting, when confronted with difficulties in explaining the fractional part of the quotient in division  $2 \div \frac{3}{4}$ , Tiffany concluded that the concept of whole-number division as fitting divisor units into the dividend does not apply to dividing fractions. During the post-intervention meeting Tiffany built on this concept to provide the required explanations. Yet when her conceptual understanding fell short of supporting her explanations or when she had doubts whether her explanation was mathematically valid, she again resorted to the numbers involved in the algorithm under consideration.

To explain the quotient for the division  $2 \div \frac{3}{4}$ , Tiffany contemplated structuring her explanation around a word problem. But feeling quite uncertain about the correctness of the problem she considered posing, she sought help from the traditional algorithm:

So if I have two glasses of water and wanted to give -- there's where, I have to remember what the whole is. Because I was going to say if I wanted to give three quarters of a glass each person -- [pauses and then figures out the answer to the division problem  $2 \div \frac{3}{4}$  by using the traditional algorithm:] once again, doing the algorithm (PO.I. 837-841)

Once she figured out the numerical answer to this division, she attempted to complete the word problem she started posing; yet, probably still doubtful about the correctness of her problem, she consulted the second textbook page and proposed a "ribbon" problem: "If you divide two feet of a ribbon into strips of three quarters of a foot, how many pieces could you get?" (PO.I. 886-888). Next, she drew a line to represent the two feet and partitioned it into two parts, each corresponding to a foot of ribbon (see Figure 5.22a). She then continued:

Groups of three quarters. So this right here is two feet [points to the whole line, see Figure 5.22a]; two feet [writes "2 ft" in purple]. So if we divided each ... foot into quarters [divides each half of the line into four parts], so then... I need three quarters of a foot to make something. So three quarters of a foot would be one [draws a red line below the first  $\frac{3}{4}$ -portion] and then three quarters of a foot would be another [draws a red line above the second  $\frac{3}{4}$ -portion]. So then I've got two [writes "2"] and then I've got these two little sections left [circles the two leftover fourths], but if we're looking at one foot, that is two [writes " $\frac{2}{4}$ "]. Wait, I got a different answer. [Pause.] Three quarters [counts the parts of a  $\frac{3}{4}$ -segment:] One, two, three. [Pause.] ... So it's two out of four, or two out of -- [pause]. Yeah. Two out of four, but if we're looking at a quarter of a foot out of three quarters of a foot, it's only -- [pause]. ... I was going to say two thirds [writes " $\frac{2}{3}$ ," but then crosses it out with a blue-color marker] but it's just because I know that's the answer. If I'm looking at a quarter [counts the  $\frac{3}{4}$ -portions she colored in red:] three quarters of a foot; three quarters of a foot. Two fourths here is ... two thirds. There we go. Two thirds of three quarters of a foot. (PO.I. 891-911)

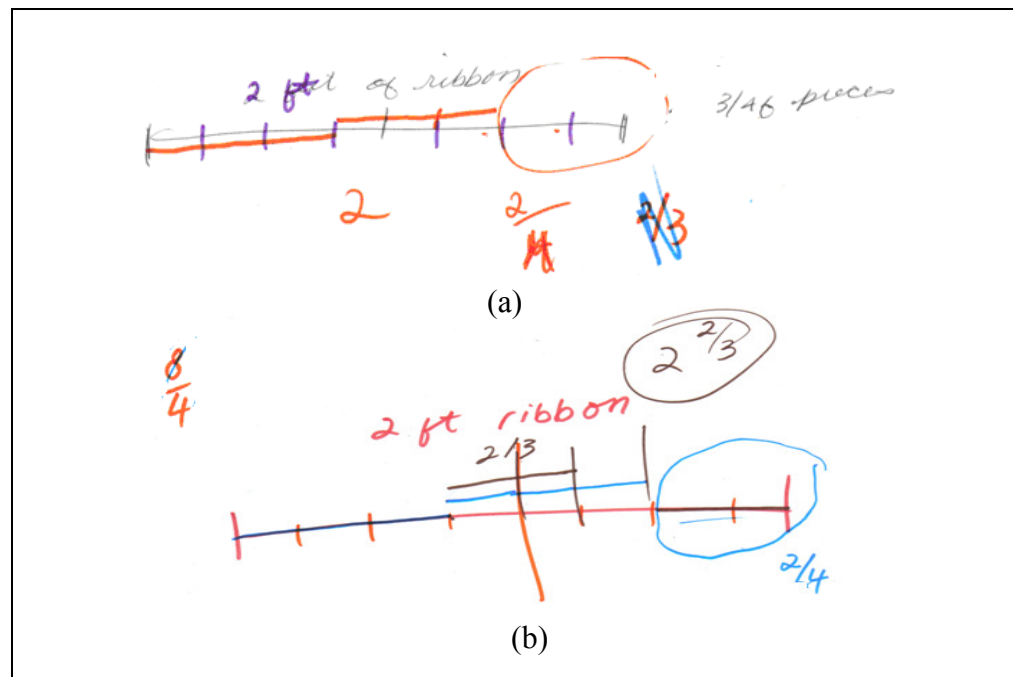


Figure 5.22. Tiffany's work on explaining the quotient for the division problem  $2 \div \frac{3}{4}$  (a: initial steps; b: providing an explanation for a sixth grader).

As this excerpt suggests, armed with a better understanding of the division as fitting divisor units into the dividend, Tiffany easily identified that she could fit two  $\frac{3}{4}$ -portions into the two-foot ribbon. However, her struggles with the units and, correspondingly with the fractional part of the quotient, led her to a momentary impasse. Apparently lacking a solid understanding of the content, to explain the fractional part of the quotient, she again reverted to the numbers; as she quite frankly remarked, she

claimed the leftover part to be two thirds simply because this was the answer she got from the algorithm. However, her better understanding of division itself helped her realize that the leftover part was “two thirds of three quarters of a foot.”

To further probe Tiffany’s grasp of the fractional part of the quotient, the interviewer reminded her of the original task, which was to explain this division to a sixth grader. In response, she started drawing another picture (see Figure 5.22b) while offering an explanation for this imaginary student:

If I had two feet of ribbon [draws a similar line to that shown in Figure 5.22a; she then writes “2 ft ribbon”] ... and I said that I wanted to divide it into three quarters of a foot pieces ... how many three quarter foot pieces can I get out of two feet of ribbon? So I can get, let’s say this is my one foot [draws a big vertical line in the middle of her horizontal line]: that’s one foot, that’s one foot [points to the two halves of the line]. So if I divided it into quarters, each foot ... [divides each of the halves into four pieces] so out of all these I’ve got eight quarter feet of ribbon [writes  $\frac{8}{4}$ ]. So I said that I can get my lovely three-quarter foot piece right here [colors a  $\frac{3}{4}$ -portion of the first “foot” and says:] here’s a three-quarter foot piece. ... It’s three quarters of one foot. So then if I need to get another three quarters of one foot, I can still go one quarter, two quarter, three quarter [she colors another  $\frac{3}{4}$ -portion in turquoise]. I’m left here with two quarters of a foot [circles the leftover part and writes “ $\frac{2}{4}$ ” in turquoise]. But my question is how many three-quarter foot pieces could I get? So here [points to the pieces she circled] I can’t get another three quarter foot piece. So my question here was two fourths is how much [of] three quarters? (P.O.I. 939-969)

At this point, she again stopped and tried to check whether the leftover two-fourths piece was indeed two thirds of a  $\frac{3}{4}$ -portion. Yet, her algebraic approach in figuring this out led her to another impasse since she represented this idea as  $\frac{3}{4} \div \frac{2}{4}$  (instead of  $\frac{2}{4} \div \frac{3}{4}$ ).

Prompted to use her picture to complete her explanation, she offered the following explanations as to why the leftover part represented two thirds of three fourths:

When I looked at this [Figure 5.22b], I said, all right, so two fourths is how much of three fourths? So in my head what I did is I drew this line right here, of two fourths, right here. ... [She draws a brown line equal to two fourths above the second  $\frac{3}{4}$ -portion]. And when I did that, I was able to ... break this line [points to the second  $\frac{3}{4}$ -portion] up into thirds. ... And then that was where I said, “Oh, look, it’s two thirds [writes “ $\frac{2}{3}$ ”]. So that was what I did visually, but I can’t figure out mathematically right now how to get that. (P.O.I. 1076-1091)

Tiffany’s explanation was quite unpacked. Notice, for instance, that she started with defining what she was trying to figure out, namely the number of  $\frac{3}{4}$ - foot pieces she

could get out of two feet of ribbon. She then very meticulously associated each portion of her word problem and the algorithm with her drawing: she labeled the two-foot ribbon with which she started; she clearly named each of the two halves into which she divided her line as one foot; she specified that she was dividing each “foot” into quarters, which yielded eight quarters; and she then clearly identified each of the two  $\frac{3}{4}$ -portions she could make. Additionally, she was explicit that the leftover part was actually two-fourth feet (thus identifying the remainder in absolute units) and remarked that the leftover piece would not suffice to make another  $\frac{3}{4}$ -portion. Yet, to move farther, Tiffany again reverted to an algebraic expression to figure out whether the leftover part was indeed two thirds of three fourths. Alas, like her pre-intervention performance, Tiffany’s venture into the algebraic terrain caused her more confusion rather than providing her with more insights. (Recall, for example, that during the pre-intervention meeting she represented the division “how many times two goes into six” as  $2 \div 6$ .) Instead, when she reverted to her visual representation, by juxtaposing the leftover part and one of the  $\frac{3}{4}$ -portions, she successfully showed that the remainder represented two thirds of three fourths.

Although one could regard Tiffany’s constant reference to the numbers as simply an inclination to ground her work in numbers, another explanation is that her conceptual understanding of the ideas she was explaining was still not secure enough to release her from this reliance on numbers. In fact, at several points in the interview, she confessed that her understanding of the content considered in the simulation was still developing: “So to me, still, when I divide fractions, my brain doesn’t take it as a division problem. It takes it directly into a multiplication problem” (PO.I. 1123-1125).



Tiffany's dependence on the numbers was even more palpable when she tried to explain the reciprocal in the division problem  $2 \div \frac{3}{4}$ . When asked to provide such an explanation, she remarked that she still considered the reciprocal from a numerical perspective, as the number with which one needs to multiply both parts of the equation  $\frac{3}{4} \cdot X = 2$  to get the value of the unknown  $X$ . To provide a more conceptual explanation, she reproduced her drawing shown in Figure 5.22b and counted the eight fourths resulting from partitioning the two-foot line into fourths (probably alluding to number eight in the algorithm  $2 \div \frac{3}{4} = 2 \cdot \frac{4}{3} = \frac{8}{3} = 2\frac{2}{3}$ ). She then considered dividing her lines into thirds to show the thirds outlined in the algorithm; yet dissatisfied with this idea, she went back to her drawing and explored if any piece in this drawing could represent the numerical value of the reciprocal when expressed as a mixed number ( $1\frac{1}{3}$ ). In particular, she first thought of associating the reciprocal with the two  $\frac{3}{4}$ -portions colored in dark and turquoise blue in Figure 5.22b. Dissatisfied with this idea, she then wondered whether the leftover part in this drawing was one third of the whole two-foot line.

Because Tiffany's foregoing exploration was futile and also because she had some initial insights about explaining the invert-and-multiply algorithm for the division problem  $\frac{1}{2} \div \frac{1}{6}$  (see her performance in *Selecting and Using Tasks* above), the interviewer suggested that she build on the thinking she did for this division problem to explain the algorithm for the division  $2 \div \frac{3}{4}$ . Even with this scaffolding, Tiffany could hardly propose a conceptual explanation for the reciprocal, let alone for the whole invert-and-multiply algorithm:

Two divided by three quarters of a yard. I wanted to know how many three quarters go into two yards. [Pause.] How many quarters are in two? So what's that relationship? I'm looking at what the relationship to eight quarters and three quarters is. [Pause.] I still can't see the one and one third [i.e., the reciprocal] here. (P.O.I. 1553-1566)

The dominance of numbers in Tiffany’s conception and explanation of the reciprocal was also evident at a later point in the interview when she tried to explain the reciprocal for the division problem  $\frac{3}{4} \div \frac{1}{6}$ . To do so, she first used the traditional algorithm to figure out the quotient of this problem (see Figure 5.23a). She then used a drawing she had produced earlier when considering June’s and Shaun’s work (see Figure 5.23b; also see her performance in *Responding to Students’ Requests for Help*) and tried to map the numbers involved in this algorithm to her drawing. While she associated the number nine in this algorithm (i.e., the numerator of the quotient when expressed as an improper fraction) with the nine curved lines shown in this figure, she could not see the “division by two” (i.e., the denominator of the nine halves) in her representation:

So we’ve got nine here. I’m trying to figure out where nine could come into this. [Pointing to the curved lines:] I’ve got one, two, three, four, five, six, seven, eight, nine. ... So, why am I, then, dividing it by two? (PO.I. 3334-3378)

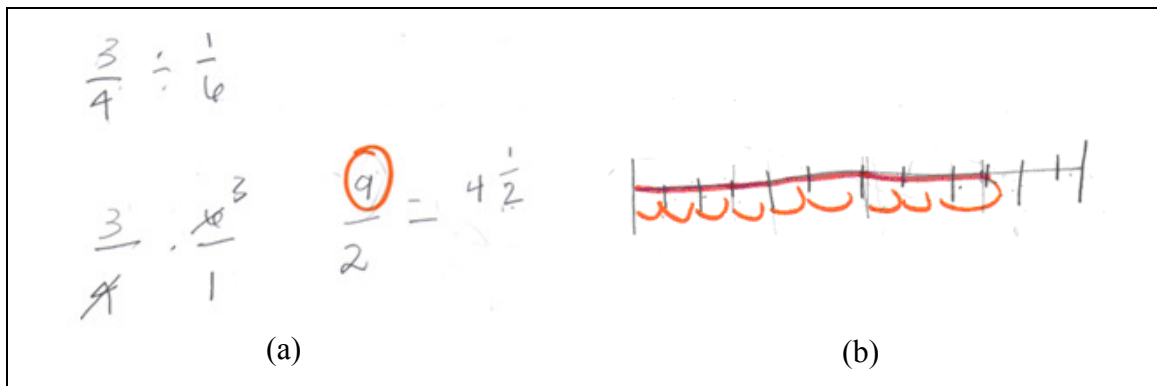


Figure 5.23. Tiffany’s work on explaining the invert-and-multiply rule for the division problem  $\frac{3}{4} \div \frac{1}{6}$  (a: figuring out the quotient; b: representing the problem in a drawing).

In short, Tiffany’s work on providing explanations suggests that her improved understanding of the division (of fractions) enabled her to ground her work in the concept of fitting divisor units into the dividend, especially when explaining the division problem  $2 \div \frac{3}{4}$ . However, her still-developing understanding of relative units, in conjunction with

her thin understanding of the notion of the reciprocal resulted in anchoring her explanation of the invert-and-multiply algorithm by the algorithm's numbers.

*Noticing and Interpreting-Evaluating.* Like her pre-intervention evaluation, during the post-intervention meeting Tiffany sensed that the analogy the teacher used in explaining the reciprocal was inappropriate; yet, Tiffany could not justify her appraisal. Unlike her pre-intervention performance, she noticed that the teacher was merely describing instead of explaining why the reciprocal works: "I think that her explanation isn't an explanation. ... I still don't see why you'd do it that way [i.e., multiply the dividend by the reciprocal of the divisor]" (PO.I. 3668-3680). She also disagreed with the teacher's deferring Robert's question to the next lesson because, as Tiffany argued, addressing his question could have scaffolded students' work on the division problem assigned at the end of the lesson.

#### *Using Representations*

*Performing.* Most of Tiffany's performance gains were observed in the practice of using representations. In addition to being able to select appropriate representations and use them in a more conceptually driven manner (when explaining the quotient in the division  $2 \div \frac{3}{4}$ ), during the post-intervention meeting Tiffany also closely attended to issues of mapping. The analysis of her explanation for the quotient for the division  $2 \div \frac{3}{4}$  shows that she was able to build connections among the algorithm, the word problem she posed to contextualize this algorithm, and the drawing she used. These connections were not limited to the quotient under consideration but pertained to *all* the numbers involved in this algorithm. Additionally, Tiffany was quite successful at coordinating three types of representations: numbers, words, and diagrams. For instance, in showing the  $\frac{3}{4}$ -pieces

she could fit in the two-foot line, she used these three representational types in tandem: she counted each fourth and simultaneously showed it on the number line; furthermore, as the right-hand side of Figure 5.22a also shows, she numerically labeled these segments as “ $\frac{3}{4}$ -pieces.”

*Noticing and Interpreting-Evaluating.* Tiffany’s increased sensitivity to issues of mapping was also evident in her performance in the noticing and interpreting-evaluating tasks of this practice. During the pre-intervention meeting, Tiffany noticed the lack of only one type of connections in one of the episodes designed for this practice (i.e., in Amanda’s episode, she remarked that Amanda was not asked to explain the meaning of the quotient in her number sentence). In contrast, during the post-intervention meeting, she identified the lack of connections among the word problems, the drawings, and the numerical algorithms being used; she did so for almost all the numbers involved in these algorithms.

In considering Amanda’s episode, Tiffany was particularly troubled about the absence of any connections in the task considered. She maintained,

[T]he model that they have on the board does not relate to the story problem, because it’s not labeled at all. So you can see that there’s a representation on the board, that they’re trying to solve this story problem, but there’s not that link there. If I looked at this right now, I couldn’t see how it related to the problem that they’re trying to solve. So to make *true* use of a story problem, you need to answer it with an answer that relates to the story problem. So there’s a missing link there. (PO.I.2680-2689, emphasis in the original)

Tiffany argued that if she were teaching, she would ask Amanda to label her representation and explain how she used it to derive the answer to the problem.

Similarly, in considering Amanda and Julia’s work in the second episode, she pointed to the connections that needed to be made explicit:

I would go, break it, step by step here, and say, “Okay, well, why did you divide it into twelfths? What are you trying to find? Why do you have nine pieces that are red?” You know, I would need to know that nine out of twelve is three fourths here, so that that line represents three fourths of  $a$

*yard*. And then, “Why did you break your nine pieces into ... two pieces? What does that mean?” “Oh, that’s how much you need to make one badge?”.... I would want them to verbalize what that represents. That it represents a sixth of a yard still, each piece. (PO.I. 3024-3057, emphasis in the original)

In both excerpts, Tiffany referred to drawing connections among the ribbon problems, the drawings shown on the board, and the numbers used in their mathematical sentences. This increased sensitivity to issues of mapping largely stemmed from her participation in the two intervention courses, as she herself argued during the interview.

### *Analyzing Students’ Work and Contributions*

*Performing*. As in her pre-intervention performance, before analyzing the three student solutions considered for this practice, Tiffany first figured out the numerical answer to the division problem under consideration. Accordingly, to a large extent, her post-intervention analysis of the three student solutions, and particularly of Robert’s and Ann’s, was answer-driven. Yet, unlike her pre-intervention performance, her better understanding of division and her developing understanding of the fractional part of the quotient supported her in appropriately analyzing Michelle’s solution. She also made appropriate assertions about each student’s understanding.

In considering Robert’s work, Tiffany compared each of the steps in his solution to how she used the traditional algorithm to solve the division problem. This allowed her to infer that Robert’s solution was correct, although he needed to convert the improper fraction in his quotient to a mixed number. Instead of concluding that Robert “understood the problem,” as she did during the first meeting, she asserted that the most one could infer from his work was that he knew how to appropriately apply the algorithm:

[Robert] understands how to do the algorithm. I don’t know if he understands why to do it. But he understands the algorithm itself. ... Actually ... keeping it [i.e., his final answer] in that improper fraction, it’s still not answering the question. ... [S]o here I don’t know how he would answer forty-four twelfths in terms of the story problem. So he understands how to do the algorithm, but I don’t know if it’s linked at all to the story problem yet. (PO.I. 3712-3724)

Tiffany's analysis of Ann's work was particularly telling of Tiffany's answer-driven approach. While going over Ann's work, she remarked, "So I'm trying to figure out her ending here. But I know that the answer is forty-four over twelve; so she didn't get the correct answer" (PO.I. 3822-3824). To figure out where the error in Ann's work was, Tiffany compared the dividend in her own work (i.e.,  $\frac{11}{4}$ ) to the dividend in Ann's work (i.e.,  $\frac{10}{3}$ ):

She was dealing with what it would be like to take ten thirds divided by three fourths rather than two and three fourths; so eleven fourths divided by three fourths. So, that, right there, shows that she's got an idea about a reciprocal, but she's not finding the reciprocal of the whole number. (PO.I. 3839-3844)

On a second thought, however, she did identify the error in Ann's solution. She thus concluded that Ann have probably understood that "there is an algorithm that she's supposed to follow" (PO.I. 3880-3881), but she questioned whether Ann really fathomed anything about fraction division.

Although Michelle's work was still the hardest for Tiffany to analyze, this time Tiffany easily identified the dividend, the divisor, and the whole-number part of the quotient in Michelle's drawing. Based on her own numerical answer to this problem, she then argued that the leftover part in Michelle's work represented two thirds. When prompted to justify her answer, she initially struggled but eventually provided a satisfactory explanation:

[Michelle]'s got two left here, that weren't used. ... No, no. She does. Because each [pause] no [pause] each badge was three quarters, so three out of four. Right. So she's got [pause] I'm drawing a picture here: so she's got three fourths of a ribbon, right? [She draws a line and divides it into fourths; see Figure 5.24.] And she's used [counting the parts colored in red in Michelle's solution:] one, two, three, four; one, two, three four; one, two, three. ... Because here we're looking at what is two fourths. [She draws an orange line to represent  $\frac{3}{4}$ ]. How much of three fourths is two fourths? [She draws a red line to represent  $\frac{3}{4}$ ]. Two fourths of three fourths is two thirds. So that's where she's got her two thirds. ... She was able to make one, two, three full badges [points to the curved lines in Michelle's solution] and then she would be able to make two thirds of another badge. Because she would need ... three quarters of a yard to make one badge. And she doesn't have that much. She only has one half of a yard left. (PO.I. 3740-3788)

As this quotation suggests, although Tiffany still struggled with explaining the fractional part of the quotient, her developing understanding of this notion allowed her to analyze Michelle's work not only from a numerical perspective, as she initially appeared to be doing while considering the leftover part in Michelle's work, but also from a more conceptual perspective.



Figure 5.24. Tiffany's work on analyzing Michelle's solution to the division  $2 \frac{3}{4} \div \frac{3}{4}$ .

This more conceptually driven analysis also allowed Tiffany to make more appropriate assertions about Michelle's understanding. Unlike her pre-intervention argument that Michelle's solution was wrong and that Michelle did not understand fraction division, Tiffany now argued that Michelle's solution was correct. In addition, she appropriately remarked that if Michelle had numerically presented the leftover part as two fourths instead of two thirds, Michelle's comprehension of the remainder in terms of "badges" (i.e., relative units) would have been questionable.

*Noticing and Interpreting-Evaluating.* During the post-intervention meeting, Tiffany identified all four contestable student ideas and proposed appropriate ways to help students understand the errors in their contributions; she also considered instruction a plausible source of student error and misconception.

Cognizant that the dividend can be smaller than the divisor, Tiffany quickly identified the potential misconception in June's contribution. To explain that it was possible to divide a smaller fraction by a larger one, Tiffany reversed the two fractions considered in task A<sub>1</sub>, and using the traditional algorithm, she showed that the latter

division would yield a fraction. Unlike her pre-intervention performance, she also identified this misconception in Ann's contribution. While going over the slides in which Ann argued that one needs to figure out how many times the shorter line goes into the bigger line, Tiffany remarked:

[Ann]'s started to ingrain what the other girl [i.e., June] said. ... So these two girls [i.e., June and Ann] ... have this idea that it's always going to be... the shorter, or the smaller piece into the bigger piece. And that's, because that was never addressed from the teacher, that might be starting to get ingrained in them. (PO.I. 2773-2784)

In this quotation, in addition to identifying the plausible misconception in Ann's thinking, Tiffany also considered instruction partly responsible for reinforcing this erroneous idea.

Tiffany was equally critical of the teacher's approach when considering Alan's episode. Even before watching the slides in which Alan divided half the line into six parts, she speculated that the teacher's vague reference to *sixths* could confuse students. When the teacher did not follow up with a question to clarify the idea that one sixth meant "one sixth of a yard," Tiffany remarked:

"To make ribbons of one sixth? Right there, I think I would ask the student, "To make ribbons of one sixth of *what?*" So that he could say, "To make ribbons of one sixth of *a yard.*" Because right there you could say, "I'll make ribbons of one sixth of one half." (PO.I. 2330-2335, emphases in the original)

Hence, to Tiffany it was not surprising that Alan divided half instead of the whole line into six parts. In addition to noticing the vagueness with which the concept of one sixth was treated in the lesson, she also observed that the term *line* was used ambiguously. She considered this ambiguity another plausible source of Alan's error: "The idea of the line is kind of hazy; because a line has been considered the whole one yard, and a line has also been considered half of a yard" (PO.I. 2363-2365).

The most notable differences between Tiffany's pre- and post-intervention performance resided in her analysis of Amanda and Julia's work. During the pre-



intervention meeting, although Tiffany noticed the error in the two girls' work, she grappled with explaining its source. In the post-intervention meeting, when she identified the error in the girls' work, she was quick in explaining the discrepancy emerging from interpreting the remainder in absolute and relative units. She also proposed an instructional intervention that could help the two students appropriately interpret the remainder in their work:

Okay, one twelfth is a half of one sixth. So they were still fine, but I would have wanted to go into that more. ... [I would ask:] "You've got one twelfth, but what is the question asking us? How many badges can we make? Can we make one twelfth of a badge? Or can we make half of a badge? ... So, you're saying that your answer is four and one twelfths, but when you relate it to the problem, what are you getting?" (P.O.I. 3306-3324)

It is also informative to consider how Tiffany thought of addressing Robert's remark that his answer to the problem Amanda and Julia solved was different from the two girls' answer. To address Robert's comment, she proposed the following approach:

Say, "For homework tonight," you change whatever you had for homework to give them, "why doesn't everybody go back and look at that problem?"... Have them say, "How many badges could you make?" And then just having them relook at [it] and then come back in the next day. And then start almost re-teaching the lesson at that point. (P.O.I. 3610-3622)

Tiffany's plan for addressing Robert's remark bore significant resemblances to an approach modeled by an experienced teacher whose class the PSTs attended during a site visit organized during the math methods class (see Figure A2, Class 7, in Appendix 1). During this site visit, this experienced teacher revised the homework he had originally considered assigning to his students and asked them instead to work on a question posed by one of their classmates. This resemblance between Tiffany's plan and this teacher's approach implies that the intervention courses offered Tiffany some (alternative) images of teaching with respect to handling certain instructional situations, an issue to which I return in the analytical commentary.

### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* With an enhanced understanding of the mathematics involved in both episodes designed for this practice, during the post-intervention meeting Tiffany proposed mathematically valid approaches to handle student requests for help in both episodes under consideration; her approaches also had the potential to support rather than replace students' thinking.

After she identified the error in Alan's work, and when asked to articulate how she would support him in addressing this error, Tiffany proposed a sequence of questions that could help elicit Alan's thinking and then guide him to correct his error:

"So what did you do here?" ... He might say, "I divided one half into six pieces." ... That would be a logical thing to say there. And then -- oh, maybe it would be nice to ask him, "All right, so how long is one of your segments, right there? How much of a yard is one of your little chunks?" Because then, let's say he said "one sixth," you could say, "Well, how many of those pieces are in your entire yard?" And that might help him to get to ... "Well, I can have one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve. So I could have twelve pieces." So then, "How much of a yard is one little piece?" Hopefully he would be able to get to one twelfth. And then you could say, "Okay, well, what would you do differently, then? How could you make your pieces into one sixth of the entire yard?" And then maybe he would see to look at the whole yard. (P.O.I. 2398-2412)

As the above quotation suggests, Tiffany would start with a very general question to just elicit Alan's thinking. Her subsequent questions would be more focused to help Alan see his error (i.e., that he divided the line into twelfths); even when attempting to correct Alan, she would avoid telling him how to fix his diagram. Instead, she would pose another question to direct his attention to the meaning of one sixth. Her question, "How would you make your pieces into one sixth of the entire yard?" was also remarkably different from the teacher's suggestion that Alan partition his line into six pieces.

Provided that this sequence of questions did not support Alan in grasping and correcting his error, Tiffany considered engaging other students in the discussion.

However, on a second thought, she questioned the appropriateness of this move because,

as she explained, her coursework provided her with conflicting messages about soliciting other students' ideas to support a struggling student on the board:

**Tiffany:** See? Here's where I had conflicting ideas in different classes. Next thing would be, "Could anybody help him?" You know, "Could anybody help Alan go through this?" Now in one of our classes, we just had this big talk about doing that step right there: "Can anybody help him?" in the sense that if we say, "Can anybody help him?" are we then saying that Alan can't do it on his own? So are we giving him that feel? So what's the balance between how much would I personally work with Alan at this or bring somebody else in? So that's a debate now that I've got in my head.

**Charalambos:** And what's your perspective on this [issue] right now?

**Tiffany:** Hoo! I think it's really tough, because there's [sic] times when if somebody's up there at the front of the board and you can see in their face that they really don't want to keep working through this problem, I think it's great to ask somebody to help. And I think that it gives a nice community in the class, of somebody else helping him. However, I can totally see the point where, if you ... ask some kids to work through it, and then asked another kid, "Oh, would somebody else help him?" that might send a message to the kid, like, "Ah, the teacher doesn't think I can do it on my own." So that's tough. So right here, I might try to work with him a little bit. ... And then thank him for showing us this. Because this ... is a common error in division of fractions. (PO.I. 2440-2469)

Tiffany might indeed have been the recipient of such conflicting messages during her coursework, especially if those messages had been introduced as recipes to follow (i.e., "dos" and "don'ts"); even if these messages had actually not been conflicting, it is still important to notice how she considered them as such. Her last comment about Alan's thinking being a common error that warranted addressing is, nonetheless, encouraging since it implies that her decision as to how she would handle this situation would be guided by both pedagogical and mathematical considerations.

In considering the second episode, Tiffany first explored whether the teacher's recommendation of using the common multiple of twelve made sense in solving this problem. To do so, she drew a line and divided it into sixths. She then divided each sixth into two (approximately) equal parts, which yielded 12 twelfths. She paused for awhile and wondered: "I don't see how this right here, yet, is going to help them solve the problem. Because where are their fourths here?" (PO.I. 2873-2874). After some further thinking, she contemplated dividing the twelfths into four groups, which she did by

drawing vertical lines at the end of the third, sixth, and ninth twelfth (see the long vertical lines drawn in pencil in Figure 5.25). She then colored nine twelfths in red to show the dividend and drew a blue two-twelfth line above the original twelfths line to show the divisor. This helped her figure out how the idea of twelfths was applicable to solving this problem. To her, using this approach was a revelation, for as she remarked, “[I]t’s the first time I realized that that [i.e., using a common multiple] is what I do whenever I line up my lines!” (PO.I. 2907-2908).

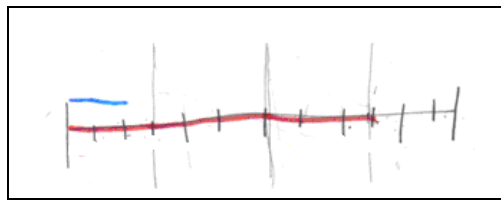


Figure 5.25. Tiffany’s work on using the common multiple of twelve to solve  $\frac{3}{4} \div \frac{1}{6}$ .

To support June and Shaun and other students who might have struggled with solving the problem  $\frac{3}{4} \div \frac{1}{6}$ , Tiffany would draw on her work as outlined above and help these students see how the application of the idea of common multiples results in aligning non-commensurate pieces and, consequently, in figuring out how many pieces of one size fit into a piece of a different size. However, she considered withholding her explanation in order to first elicit and, if possible, capitalize on students’ ideas and work:

I think that before I started with my whole twelfths thing, I would go around and see if other students were having that same difficulty. Because, you might want to, just kind of survey around. Because if the issue was only with June and her partner [i.e., Shaun], you might be able to sit down and work with them a little bit. ... [E]ven if they were all having that problem, I would still give them a little more time to work on the problem. And then once they had some answers, then going into the explanation of finding the common multiples. Just to kind of give them time to work with it, and to use the problem, and try to solve it. I know I’ve done that multiple times. If I give a problem too soon and see that a number of people are doing it wrong, I’ll automatically think, “Uh-oh, I didn’t explain this well enough.” And then so I want to stop it and say, “I see a couple of you are doing this.” But I’m trying to think if that would be the best way to go about it at this point. ... I think give them a little more time, have them answer the question, and give it to me in terms of this problem, and then find out how people solved it, and then look at common multiples. ... And I think that they need to -- it would be good to play around with this line for a little bit longer and see how they could get twelfths. (PO.I. 2947-2983)

Tiffany's professed inclination to stop the whole class and provide an explanation right after realizing that some students struggled with the assigned task was exactly what the virtual teacher was shown doing in the lesson and, incidentally, what Stigler and Hiebert (1999) report many U.S. teachers to do in the lessons they observed. Despite this inclination, as this excerpt suggests, Tiffany thought of employing an alternative approach: instead of rushing to offer an explanation, she contemplated giving students more time to work on this problem and then share their ideas. Although it is unclear how much of the thinking Tiffany would eventually do for students were she asked to handle such a situation, that she was sensitive to the value of withholding her explanation and giving students more time for thinking is important to underscore.

*Noticing and Interpreting-Evaluating.* Tiffany's post-intervention evaluation of how the virtual teacher handled students' requests for help was comparable to her pre-intervention appraisal. In Alan's episode, Tiffany criticized the virtual teacher for posing pointed questions, which led Alan to the correct answer without necessarily enhancing his understanding. In the second episode, she considered Ms. Rebecca's recommendation that students use the common multiple of twelve as too vague to really support students in grasping how this idea applied to solving the problem at hand.

#### *Professed Changes and Their Attributions*

Tiffany's overall performance suggested an enrichment of her images of teaching, which she attributed to her coursework and particularly to the two intervention courses. As she stated, at her entrance to the ELMAC program her images of teaching were limited to memorizing and applying procedures:

Previously, in all math classes that I've had, that I can at least remember back through, it's been a teacher discussing a textbook page, doing stuff on the board, and then answering the textbook

pages. And that's to me how -- what I thought that math had to be like, you know, a lot of memorization, a lot of -- just learning and memorizing algorithms and formulas. (PO.I. 103-108)

The focus of the intervention courses – and particularly of the math content course – on solving problems and sharing and discussing multiple solutions offered Tiffany alternative images of teaching the subject. As she noted, she had never experienced solving the same problem over a sequence of math lessons: “That math could be, for two weeks, looking at one or two problems? That whole concept to me was totally foreign” (PO.I. 110-112). The activities of the intervention courses also sensitized her to the importance of building connections among the representations used when solving problems. She referred to the *Bagel problem* discussed during the math content course (for more on this problem, see Appendix 1) to explain how the courses sensitized her to the value of building such connections:

For example, the Bagel problem, or something like that where we answered in terms of the *problem...* and illustrating that problem. And making sure that the model that we used was -- you could one-hundred percent map it onto the problem. (PO.I. 4034-4038, emphasis in the original)

At several instances during the interview, Tiffany capitalized on these images to respond to the teaching simulation tasks. For example, in considering how to scaffold students' work on task D of the second page, she drew on activities organized during the math methods classes that had a particular focus on issues of mapping:

When we did the place values and we modeled things with manipulatives, but then had to find a way to record each step in writing. I'm trying to think if you could actually do that with this [i.e., task D]. (PO.I. 1933-1936)

Similarly, when asked to propose a word problem to introduce the concept of fraction division, she again resorted to the images of teaching she gleaned from the two courses:

Again images that pop into my head, like actual image, what I see in my head when you asked what I would do, is what we had done in class, where it was the problem where people came in and ate [i.e., the Bagel problem]. (PO.I. 522-525)

Tiffany's identification of her coursework experiences as *images* is worth highlighting since it resonates with the argument of the two courses offering PSTs alternative images of teaching.

In addition to the two intervention courses, other courses that Tiffany had taken between the pre- and the post-intervention meetings also informed her images of teaching. For instance, while analyzing the first textbook page, she drew on her Science classes to justify why she proposed restructuring this page:

[A] lot of times in ... college Chemistry classes, you go to lecture, and in the lecture they teach you definitions and formulas and why things are happening. And then that week in lab, you go and you do a lab that proves what you just were taught in class. ... And so in that case, the lab is almost pointless ... because you've just been taught what's going to happen. (P.O.I. 470-477)

Although during the post-intervention meeting Tiffany quite frequently drew on her coursework experiences to outline or justify instructional approaches she would pursue in response to the teaching simulation tasks, with only a couple of exceptions (e.g., when discussing June's and Shaun's episode), she rarely talked about her fieldwork experiences. The interview data do not support any explanations about this unequal consideration of her coursework and fieldwork experiences.

#### *Changes in Performance on the LMT Test: A Closer Look*

During the second administration of the test, Tiffany correctly answered 11 of the LMT test questions that she had incorrectly answered during its first administration. Eight of these questions were directly related to the mathematical ideas explored in the simulation: questions 14c, 14d, 24a-24c, and 25 pertained to the interpretation of division (of fractions) problems; question 17 corresponded to interpreting the fractional part of the quotient in divisions of fractions; and question 20 was designed to capture the PSTs' understanding of different manifestations of the concept of unit when working on

fractions. Her correctly answering the questions on the interpretation of division was consistent with her post-intervention performance in the teaching simulation, which also suggested that her understanding of this concept was enhanced. However, Tiffany's struggles with the relative and absolute units and with explaining the fractional part of the quotient in division-of-fractions problems was inconsistent with her performance on questions 17 and 20. At least two plausible explanations might account for this inconsistency. First, the multiple-choice format of these questions may have scaffolded her work and thinking. Second, although her grasp of these ideas was enhanced (as suggested by her performance on the LMT test), perhaps it was still not solid enough to fully support her in addressing the tasks of the teaching simulation. Evidence suggesting that her understanding of the relative and absolute units might have not been deep enough also stems from her incorrect answer to question 1, which was also used to tap PSTs' understanding of units when working on fractions.

Equally inconsistent was Tiffany's work on question 14e, which pertains to the measurement interpretation of division of fractions. Given Tiffany's better understanding of this concept, as suggested by her post-intervention teaching-simulation performance, one would expect that she would have correctly answered this question on the test's second administration rather than on its first administration; however, the reverse pattern was observed.

### *Analytical Commentary*

Unlike Vonda, Tiffany did not experience notable changes in her beliefs; she mostly exhibited gains in her knowledge and in her images of teaching. Thus, Tiffany's case lends itself to better explore the association between the gains in knowledge and in



teaching performance explored in this chapter. Her case is particularly conducive to this exploration, since Tiffany's understanding was enhanced in some areas but was still limited in others. Hence, her case allows considering both what the increases in her knowledge allowed her doing and how certain limitations in her understanding impinged on her teaching performance.

Tiffany's post-intervention teaching performance and her performance on the LMT test both suggested that her understanding of the division of fractions from a measurement perspective, and particularly as fitting divisor units into the dividend, was enhanced. This does not mean that she was solid on this idea, since as she acknowledged in a few instances during the post-intervention interview, she was still working on the concept of fraction division. Yet, compared to the pre-intervention meeting in which she argued that she did not "have the link" between the procedure and the meaning of fraction division, in the post-intervention meeting she appeared to have established such a link. On the other hand, her understanding of relative and absolute units when working on fractions was weaker than her comprehension of division; to paraphrase her words, her respective link was still loose. Given this description of the changes in Tiffany's knowledge, in what follows I examine what having established a conceptual link of the division of fractions enabled Tiffany to do and how her loose link of relative and absolute units was manifested in her teaching-simulation performance; I do so by considering examples from each practice. I then discuss how Tiffany's enhanced images of teaching also appeared to have informed her post-intervention performance.

Tiffany's understanding of the concept of division of fractions allowed her to think about restructuring the first textbook page to support students' conceptual

understanding of fraction division. For example, she identified that the two word problems listed at the bottom of this page were not particularly conducive to the division of fractions, and she proposed replacing them with a problem reflecting the measurement interpretation of division. Additionally, while during the pre-intervention meeting she lacked any criteria to select a subset of the 16 exercises to assign to students, during the post-intervention meeting Tiffany started seeing these exercises from a more conceptual perspective. She talked about fractions that could be easily represented in a drawing and fractions that would be hard to represent; she also considered fractions that “divide cleanly” (i.e., same-denominator fractions) and fractions that were less friendly to divide. Her experimentation with representing and solving exercise 3 (i.e.,  $\frac{5}{8} \div \frac{5}{6}$ ) also helped her start thinking about divisions in which the divisor is larger than the dividend. In short, her better conceptual understanding of fraction division allowed Tiffany to analyze the first page in more depth and from a more conceptual perspective. Hence, her performance in analyzing this page corroborates proposition D<sub>2</sub> of Table 4.6, which holds that a deeper grasp of the content and its teaching supports teachers in restructuring curriculum tasks.

On the other hand, her struggles with the concept of relative and absolute units and her rather procedural understanding of the reciprocal prevented her from capitalizing on some of the affordances she identified with respect to the second page. For instance, she was still unsure about how she could support students’ transition from the first three tasks of this page to the last task, where they are asked to propose an algorithm for dividing fractions. Her partial understanding of the fractional part of the quotient in fraction divisions might also have prevented her from identifying and discussing the second page’s pertinent direction as one of its affordances.

The entailments of Tiffany's establishing a conceptual link for fraction division and her "loose link" of relative and absolute units were even more evident in her performance in the practices of providing explanations and using representations. This qualitative finding supports the quantitative findings that showed the gains in the PSTs' MKT to be more strongly associated with the gains in their performance in these two practices rather than with their performance gains in the other three practices the study examines. In particular, unlike the pre-intervention meeting, during the post-intervention meeting Tiffany was knowledgeable that the concept of whole-number division also applies to fraction division. Hence, her explanation for fraction divisions not including any fractional parts, and accordingly her use of representations, were conceptually driven since they were grounded in the concept of fitting divisor units into the dividend. Yet because of her struggles with the relative units, Tiffany encountered difficulties explaining divisions whose quotient included a fractional part.

When faced with such divisions, her first move was to figure out their quotient in numerical form; she then tried to structure her explanation and use of representations around these numbers. As she honestly admitted when explaining the fractional part of the division problem  $2 \div \frac{3}{4}$ , the only reason for which she initially regarded the leftover part as two thirds was because this was the number she got from the algorithm. Despite her number reliance, Tiffany eventually drew on the concept of fraction division and offered a conceptual explanation for the fractional part of the division problem. Since her explanation and her use of representations were both conceptually and numerically driven, her performance in explaining the quotient in the division  $2 \div \frac{3}{4}$  corroborates propositions A<sub>2</sub> and B<sub>2</sub>, which suggest that PSTs are more inclined to provide

numerically driven explanations and use representations accordingly when their understanding of the content does not support them in grounding their explanations exclusively in pertinent concepts.

These two propositions were further corroborated by Tiffany's work on other tasks of the teaching simulation. For example, while working on the division problem  $\frac{5}{8} \div \frac{5}{6}$ , she clearly admitted that whereas she drew a diagram to represent the dividend and the divisor, she was not actually using this drawing to solve the problem; rather, she used the traditional algorithm to figure out the answer and then experimented with ways in which she could make her drawing match this answer. Similarly, in explaining the reciprocal for the division problem  $2 \div \frac{3}{4}$ , Tiffany attempted to make the numbers involved in the reciprocal (as a mixed number) fit her diagrams. This heavy reliance on numbers was also exemplified in Tiffany's attempt to associate her drawing with the number nine halves she got from applying the algorithm to solve the problem  $\frac{3}{4} \div \frac{1}{6}$ : whereas she identified nine pieces in her picture, she could not represent the denominator of this fraction in her drawing. In sum, Tiffany's overdependence on numbers when explaining the reciprocal should not be dissociated from her understanding of the reciprocal, which she herself admitted as being procedural. Hence, despite some glimmers in considering the reciprocal in the division problem  $\frac{1}{2} \div \frac{1}{6}$  as fitting six sixths in the whole, Tiffany could not transfer this idea to other division-of-fractions problems.

Tiffany's better comprehension of the division of fractions and her struggles with the fractional part of the quotient in such divisions were both manifested in her analysis of Michelle's solution. During the pre-intervention meeting, Tiffany argued that

Michelle's solution was incorrect simply because Michelle appeared to be using the concept of whole-number division in solving a division-of-fractions problem. In contrast, during the post-intervention meeting Tiffany not only considered the concept of whole-number division applicable to solving fraction division problems, but she also drew on this concept to analyze Michelle's drawing. Despite her difficulties in explaining the leftover part in Michelle's work, which again led her to initially ground her explanation in the numerical answer obtained for this problem, Tiffany was eventually able to draw on her conceptual understanding of division and reason through Michelle's work. Yet overall, her performance in analyzing Michelle's work as well as that of Robert's and Ann's was still largely answer-driven: it was guided and informed by the numbers involved in the respective problem. This finding, in conjunction with Tiffany still not feeling very confident and competent in the conceptual terrain of fraction division, provides support to proposition C<sub>3</sub>, which suggests that a strong knowledge of the content could liberate PSTs from an over-reliance on numbers when analyzing student work.

In addition, Tiffany's performance in the practice of analyzing students' work and contributions corroborates proposition C<sub>4</sub>. Tiffany was familiar with the idea that in fraction division, the divisor can be larger than the dividend; this allowed her to quickly identify the latent misconception in both June's explanation and in Ann's argument. Given her partial comprehension of the ideas considered in the simulation, her performance in this practice implies that awareness of such misconceptions (rather than an overall strong knowledge base of the content) is what appears to enable teachers to capture such misconceptions when they arise during instruction.

Tiffany's better grasp of the content also scaffolded her performance in the practice of responding to students' requests for help. Whereas during the pre-intervention meeting she provided a mathematically nebulous response to June's and Shaun's question, in the post-intervention meeting she capitalized on her understanding of common multiples to offer a pedagogically appropriate and mathematically sound approach. This difference between her pre- and post-intervention performance lends support to proposition  $E_1$ , which maintains that a better understanding of the content supports PSTs in structuring instructional interventions to address key mathematical ideas when responding to students' requests for help.

Although the changes in Tiffany's performance could largely be attributed to advancements in her knowledge, these changes should also be associated with Tiffany's enhanced images of teaching and with opportunities made available to her to practice these images of teaching. At several points in the interview, Tiffany resorted to images of teaching she gleaned from her coursework to outline or justify approaches in response to the teaching-simulation tasks. For example, she attempted to restructure the first textbook page by building on the idea of mapping an algorithm onto a drawing. Similarly, she drew on a site visit to outline how she would handle Robert's question. Her coursework also afforded her opportunities to practice these images: during the intervention courses she practiced providing more unpacked explanations and building connections among word problems, drawings, and algorithms. She also practiced making appropriate assertions about students' understanding (especially during the math methods class).

Tiffany's case is also informative of if and how the PSTs' images of teaching could compensate for limitations in their knowledge. During the pre-intervention

meeting, Tiffany resorted to familiar images of teaching, when her knowledge could not support her instructional decisions. For example, without any conceptual criteria to inform her exercise selection, she argued that if she had to choose among the 16 exercises, she would assign the “evens” or the “odds” just like her own teachers used to do. Tiffany’s resorting to her images of teaching when her knowledge fell short of informing her instructional decisions was also evident in her post-intervention performance. In particular, when confronted with difficulties in outlining an approach to support students’ transition from specific division-of-fractions problems to a general algorithm, Tiffany drew on her coursework and explained that she would try to show the meaning of the division-of-fractions algorithm by using a drawing side-by-side with this algorithm. In both cases, her images of teaching did not sufficiently compensate for the limitations in her knowledge; yet, in the second case, her images of teaching provided her with at least some guidelines to maintain the focus on meaning and understanding.

The foregoing changes in Tiffany’s performance are summarized in italics in Figure 5.26. This figure, which presents Tiffany’s post-intervention performance profile, shows that whereas no notable changes were identified in her beliefs, Tiffany’s images of teaching were enhanced. Tiffany also experienced changes in all five practices under consideration, and particularly in the practice of using representations.

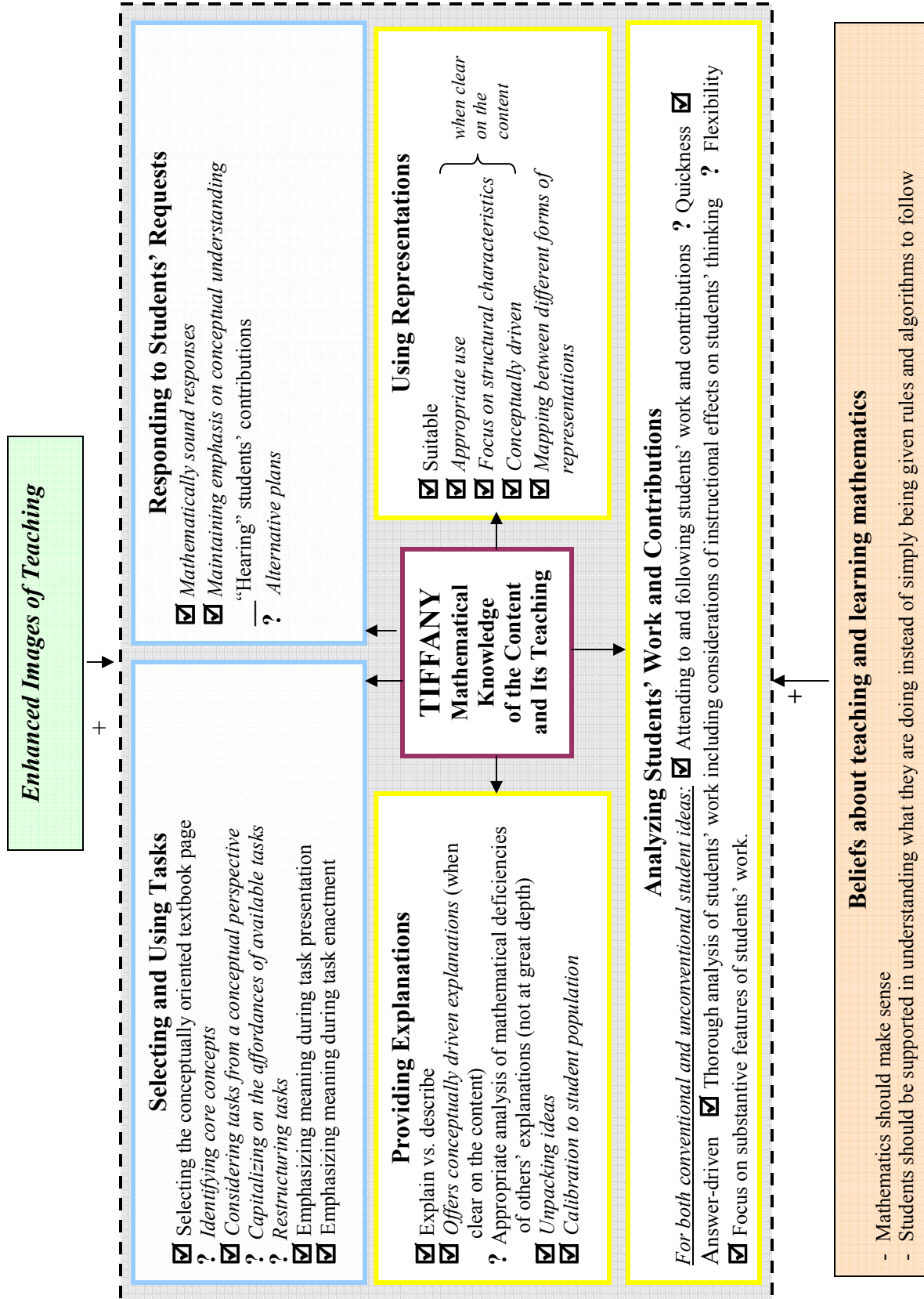


Figure 5.26. Considering the association between knowledge and teaching performance through Tiffany's post-intervention profile.



*The Case of Kimberley Revisited: Re-Learning Mathematics for Teaching*

Although a divergent case according to her entrance scores, Kimberley was a convergent case according to the changes in her performance. As Figures 5.3 and 5.4 show, relative to the changes in her counterparts' performances, Kimberley experienced high gains in both her MKT and her teaching performance, but not as high as Deborah's; also, like Tiffany, Kimberley experienced greater gains in her performance in the MKT-related practices than in the MTF-related practices. The interview with Kimberley was conducted during the first third of February, almost 32 weeks after the pre-intervention meeting. In the period that elapsed between the two interview meetings, Kimberley observed and taught mathematics in a sixth-grade class. The teacher whose class Kimberley observed was using the *Connected Mathematics Project* (CMP) curriculum, which was the source of the second textbook page utilized in this study.

When considered in the context of the cases discussed thus far, Kimberley's case is informative in five respects. First, like the cases of Deborah, Nathan, and Nicole, the alignment of the gains in Kimberley's MKT and the gains in her teaching performance supports the respective strong association that emerged from the quantitative analysis. Because Kimberley's gains were not as dramatic as Deborah's, but were definitely more notable than Nathan's and Nicole's, her case affords yet another opportunity to explore the incarnation of the aforementioned association. Second, despite the advancement in her knowledge, Kimberley was still unaware of certain student misconceptions; thus, her case helps investigate the potential impact of this lack of awareness on PSTs' teaching performance. Third, as suggested by her high GRE-quantitative score, Kimberley entered the program with a much stronger mathematical background than Deborah. As previously

discussed, Deborah's case provided an existence proof that the PSTs' entrance GRE performance did not determine their gains in knowledge and teaching performance, thus corroborating the quantitative findings showing the GRE scores to not mediate the association between knowledge and performance. Moving a step farther, Kimberley's performance in the practice of using representations suggests that a strong mathematical background could even impose limitations on what PSTs might gain from their participation in interventions like that considered in this study. Fourth, unlike Vonda, Kimberley experienced remarkable changes in both knowledge and beliefs; hence, her case helps explore how this joint cognitive and affective progression was reflected in her teaching performance. Fifth, regardless of her greater attention to issues of meaning and understanding, Kimberley did not appear to be sensitive to issues of maintaining the cognitive complexity of demanding tasks; hence her case provides a nice contrast to Nathan, who was attuned to such issues.

#### *Selecting and Using Tasks*

*Performing.* Kimberley's post-intervention performance in this practice differed remarkably from her pre-intervention performance. In the pre-intervention interview, Kimberley's proposed plan for using either textbook page merely sought to promote students' fluency with the "mechanics" of dividing fractions. For example, if she were to use the first page, she would have started with a word problem to motivate the operation of fraction division, then described the two steps of the page's worked out example, and then had students work in groups or individually to solve the page's 16 exercises. If she used the second page, she would have again started with a word problem, but she would have then shifted to task C and explained students "the mechanics" of dividing fractions.

She would then have asked them to apply these mechanics in solving several division-of-fraction problems (including problems of the first page). Her post-intervention proposed enactment of both textbook pages reflected her increased attention to issues of making meaning and understanding.

Kimberley would use the first textbook page after she would introduce and explain the division of fractions from a more conceptual perspective. To do so, she would have students work in groups and use different manipulatives or visual representations to figure out ways to divide fractions. She would then ask students to propose their own algorithm(s) for fraction division, expecting that at least some of them would come up with the traditional division-of-fractions algorithm. She explicated,

I would probably pull out some fraction wheels, initially, or fraction towers, or something along those lines, and, maybe, have the students play with ... different ways of dividing up the fraction, and talk a little bit about ... what is the whole that they're dividing into and show a few comparative problems. And then, have them come up with an algorithm that ... they think is a reliable algorithm. (P.O.I. 192-201)

After this exploration, Kimberley would use a visual representation to explain the concept of fraction division (see how she would do so in *Providing Explanations*).

Following that, she would use the 16 exercises of this page to organize different activities: she would ask students to solve exercises 1-4 by using visual representations, to pose word problems for exercises 5-8 (both for the partitive and the measurement interpretations of division), and to solve the remaining eight exercises by using the traditional division-of-fractions algorithm. If there was not enough instructional time to work on all 16 exercises, she would assign a selected sample of exercises that would help students practice different procedures (e.g., converting mixed numbers into improper fractions, simplifying fractions).

Kimberley's plan for using this textbook page resembled the approach pursued in the second textbook page, which starts with pertinent word problems and expects students to figure out an algorithm for dividing fractions. As such, her approach reflects a significant departure from Kimberley's pre-intervention plan for using the first page. Her conceptual explanation for fraction division and the ways in which she proposed using the 16 exercises also reflect an attempt to give a more conceptual flavor to the first page.

That said, one cannot dismiss that Kimberley's assignment of the 16 exercises to the three activities – using representations, problem posing, and drill-and-practice problem solving – was, as she admitted, “largely arbitrary” (PO.I. 687). For instance, only the first exercise of those she considered having students solve by using representations (i.e.,  $\frac{2}{3} \div \frac{1}{3}$ ) lends itself to this activity. In the other three exercises, the divisor is larger than the dividend, hence rendering them harder to solve using representations. Even when asked to make a less arbitrary selection of exercises for this activity, she selected exercises 1, 2, 5, and 12 on the premise that these exercises involve small fractions (thirds and fourths), “which are easier to work within representations” (PO.I. 785). Regardless of the legitimacy of this argument, Kimberley again seemed to overlook that exercises in which the dividend is smaller than the divisor (e.g., exercises 2, 5, and 12) are harder to solve using representations.

Likewise, Kimberley's idea to ask students to pose problems for exercises 5-8 could impose significant difficulties on students, especially given that she considered asking them to pose problems for both the partitive and the measurement interpretation of division. Take, for example, exercise 7 (i.e.,  $\frac{1}{3} \div \frac{7}{10}$ ). The divisor in this exercise is larger than the dividend; additionally, its numbers are not particularly conducive to posing a

realistic measurement-division problem, let alone posing a partitive-division problem, which is typically harder to pose when considering fraction division.<sup>181</sup> In contrast, several of the exercises that Kimberley saved for the drill-and-practice activity were more suitable for the previous two activities (e.g., exercise 9:  $\frac{5}{6} \div \frac{1}{3}$ , exercise 10:  $\frac{1}{12} \div \frac{1}{4}$ , exercise 13:  $2\frac{5}{8} \div \frac{1}{8}$ , and exercise 14:  $\frac{3}{4} \div \frac{3}{8}$ ). Hence, although Kimberley proposed a set of activities that would give a more conceptual flavor to the first page, she did not make a good selection of tasks for each activity.

As in the first meeting, during the post-intervention meeting Kimberley expressed a preference for using the second textbook page for an introductory division-of-fractions lesson. In fact, she acknowledged this page as being part of the *Connected Mathematics Project* (CMP) curriculum used by her cooperating teacher. She remarked that although she did not observe the enactment of the second textbook page since she was absent from the school the day her cooperating teacher used this page, she graded the students' work on the tasks of this page. She also attended to math lessons on fraction division that were based on CMP pages preceding or following the second textbook page used in this study.

Unlike her pre-intervention performance, Kimberley identified three main affordances of the second textbook page: that it requires students to present their work in drawings, written explanations, and number sentences; that students are asked to explain the fractional part of their quotients; and that, instead of being provided with a division-of-fractions algorithm, students are expected to figure out an algorithm on their own.

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<sup>181</sup> Even if one follows the general idea for proposing partitive-division problems for fraction division (e.g., If the dividend corresponds to the portion of the needed quantity represented by the divisor, what is the needed quantity?), it would still be hard to pose a realistic problem for exercise 7, given its numbers ( $\frac{1}{3}, \frac{7}{10}, \frac{10}{21}$ ).

However, her plan for an introductory division-of-fractions lesson suggests that she would not or could not capitalize on all three affordances.

Although Kimberley endorsed the page's requirement that students represent their work in drawings and written explanations, she had reservations about also asking students to represent their work in mathematical sentences. Thinking that this third requirement would complicate this task, she proposed asking students to represent their work only in words and diagrams, which she justified as follows:

If a kid is thinking in diagrams it's, kind of, hard, sometimes to translate that into a number sentence. I could see a student understanding, you know, the concept of how many badges can I make out of a half a yard and not writing the number sentence out right. (PO.I. 912-925)

Kimberley was right in arguing that some students might find it easier to solve the problem using a drawing rather than also presenting their work in a number sentence; yet she dismissed the fact that significant learning resides in having students represent their work in different forms (words, numbers, and diagrams) and, foremost, in having students build connections among these different forms.

The page's requirement that students explain the fractional part of their answer was the requirement that Kimberley would emphasize most since she considered this concept a core idea for learning fraction division. Her experiences with grading her students' work on this textbook page convinced her that, despite its centrality, this concept is hard to understand and warrants particular attention:

That's [the fractional part of an answer] something I really saw my students struggling with and, I think, it would really be essential to go over that. Because ... I almost see this as a formative assessment, right there, of how much of this have they really gotten on a conceptual level. So I would definitely spend some time having a ... math discussion about ... the answer to that question, I think. I saw a lot of kids either skipped it or gave a superficial answer or just didn't understand it. (PO.I. 982-989)

As this excerpt suggests, Kimberley's fieldwork experiences sensitized her to specific student struggles with the content that warranted attention when teaching fraction

division. These experiences appear to have additionally offered her some ideas about how to enact task D, as suggested below.

To facilitate students' transition to task D, Kimberley proposed having them share and discuss their work on the preceding tasks of this page, arguing that "chances are one of the kids is going to share the standard algorithm" (PO.I. 1004-1005). If none of the students proposed this algorithm, she would pursue a direct-instruction approach, show this "technique" to students, and have them practice it. Because she was concerned that this page does not afford enough opportunities to students to practice this algorithm, she considered supplementing it with some exercises from the first textbook page, largely drawing on an approach she observed her cooperating teacher pursuing:

I don't think that the CMP book always does a good enough job of giving them [students] opportunities to practice using the algorithm. So, I might use some problems off of this page [first page] to supplement it; which is something that we do in the classroom that I'm placed in. Pretty often we supplement with other materials to give them practice for homework, or whatever. (PO.I. 1027-1032)

When asked to consider alternative ways to support her students' transition to the algorithm (instead of simply showing them the "technique"), Kimberley admitted that she had not seen her cooperating teacher modeling such transitions. Hence, she was not clear how she could help her students make this transition: "I don't know. That's the part that I'm fuzziest on, honestly, is how to then get them to the algorithm from there" (PO.I. 1615-1617). She speculated that a plausible way to support students' work would be to have them show the reciprocal in their visual representations; yet, she was not sure how this activity would look:

Trying to show it in the representation? [Referring to the division problem  $2\frac{2}{3} \div \frac{1}{6}$ :] You know, somehow having them ... show me the two and two thirds, show the one sixth. And maybe see if they can spot that we were actually multiplying by six in that one? I don't know. I'm not sure. (PO.I. 1628-1633)

In short, during the post-intervention meeting, Kimberley identified several of the affordances of the second page that she totally overlooked during the first interview. Yet, her proposed enactment of the page's tasks suggests that she would not capitalize on all these affordances, partly because of envisioning students being challenged with some of the page's requirements and partly because of her own difficulties with proposing ways in which she could use the textbook page to its full potential.

*Noticing and Interpreting-Evaluating.* Although not necessarily disapproving of the teacher's work, Kimberley was less enthusiastic about the teacher's presentation and enactment of tasks A<sub>1</sub> and D compared to her pre-intervention appraisal of the teacher's work on those tasks.

Kimberley deemed the teacher's overall presentation of task A<sub>1</sub> satisfactory. However, drawing on images she gleaned from the intervention courses, she remarked that the teacher could have given students more time to work on the problem individually and then share and discuss their solutions in pairs or small groups. In contrast to her pre-intervention comment that during the presentation of this task the teacher did not feed her students with the answer to this problem, during the post-intervention meeting Kimberley argued that if she were teaching the lesson, she would not give the students "that much background" as the virtual teacher did, to "see what different answers [students] came up with [and] what they're struggling with" (PO.I. 1219-1220). Kimberley also argued that the teacher needed to have prompted her students more to explain what their final answer represented. She did not, however, notice the teacher's overemphasis on getting correct answers at the expense of understanding the meaning underlying the division of fractions.



In considering the enactment of task D, Kimberley noticed that the teacher gave her students “huge hints,” “walked them” through the whole process of pattern identification, and “showed them the algorithm” (PO.I. 1660-1667). Yet, arguing that the teacher could have just presented the algorithm at hand, Kimberley deemed the teacher’s enactment of task D realistic:

[The teacher]’s basically showing them the algorithm. But I think that’s probably realistic. I don’t know that kids could just come up with the algorithm on their own; I’d want to give them the opportunity but, not having seen it in action yet, I don’t know if they could or not. (PO.I. 1705-1709)

Despite the limitations she identified in the teacher’s approach, lacking alternative images for enacting this task, Kimberley considered the teacher’s work satisfactory. She mentioned that if she were teaching this lesson, she would follow an approach similar to that modeled by the teacher, with the only exception being that she would introduce the concept of reciprocal in a previous lesson to lay the ground for fraction division.

Taken together, Kimberley’s interpretation and evaluation of the teacher’s work in presenting and enacting these two tasks reflected her increased sensitivity to the importance of affording students more opportunities for thinking; at the same time, though, her performance was also informed by her perception of what it is realistic to expect from students when working on challenging tasks. This perception should be considered in light of her own difficulties with scaffolding students’ work on demanding tasks, as reflected by her proposed plan for enacting task D.

### *Providing Explanations*

*Performing.* During the pre-intervention meeting, Kimberley provided a conceptually undergirded explanation for the whole-number part of the quotient in the division problem  $2 \div \frac{3}{4}$ ; her explanation for this part was also quite unpacked. However,

because of her difficulties with the content, she offered a numerically driven explanation for the fractional part of the quotient, suggesting, in particular, that the number  $2\frac{2}{3}$  be placed on the number line she was using. Arguing that “you can’t explain something you don’t know,” she did not offer any explanation for the invert-and-multiply algorithm. Armed with a deeper understanding of the notion of relative and absolute units, during the post-intervention meeting Kimberley provided a conceptually driven explanation for both the whole-number and the fractional parts of the quotient in the division problem  $2 \div \frac{3}{4}$ ; her explanation for the traditional division-of-fractions algorithm was numerically informed but also anchored by the concept of relative and absolute units.

To explain the quotient, Kimberley mentioned that she would start by asking students to imagine that there are two pizzas and that they want “to divide them by three fourths” (PO.I. 239-240). She drew two circles to represent the two pizzas, divided them into fourths (see Figure 5.27) and continued:

So I would say, “All right, so if we take three fourths and three fourths [shades a  $\frac{3}{4}$ -portion in each circle], how many three fourths did we get?” Well, we got two [points to the two shaded portions of the circles] and we have these two fourths left [points to the non-shaded portions]. And, what I think would be really hard for the students is understanding [sic] that we aren’t going to end up with two and a half. Because I would then have to say, “If this is our three fourths [points to one of the shaded  $\frac{3}{4}$ -portions] ... and we have these two fourths left [points to the non-shaded quadrants] ... what proportion of a three fourths do we actually have left over?” And, I think it’d be a hard thing for them to understand, I suspect: that we have, you know, that these two fourths together make up two thirds of a three fourths, so you’d end up with two and two thirds [writes  $2\frac{2}{3}$ ]. (PO.I. 276-289)

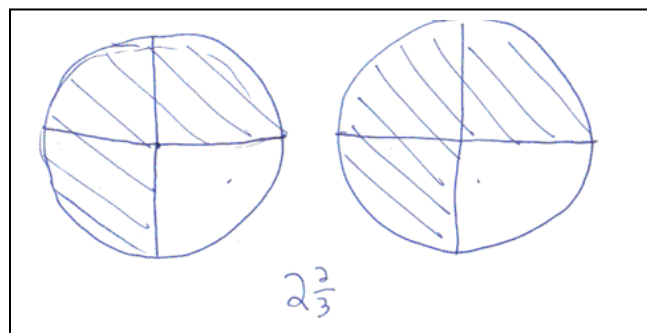


Figure 5.27. Kimberley’s work on explaining the quotient for the division problem  $2 \div \frac{3}{4}$ .

Given her argument that students might find it hard to grasp the fractional part of the quotient, Kimberley was prompted to articulate how she would explicate this idea to scaffold their learning. Pointing to the shaded parts in each circle, she explained:

You would have to say, “You’ve got a three-quarters here and you’ve got a three-quarters here, there’s one three-quarters, two three-quarters, right?” And then you don’t have enough to make a third [three-quarters]; you have just two quarters left and you need three. So if you got two of the quarters ... and you need three, you end up with two out of the three: two thirds or two thirds of a three-quarter of a pizza. (PO.I. 354-361)

Unlike her pre-intervention performance, in her post-intervention explanation of the quotient Kimberley clearly distinguished between the absolute and relative units in this problem: once she shaded in the two  $\frac{3}{4}$ -portions that could be made, she clarified that two fourths of the available quantity were left (absolute units); this quantity, however, did not suffice to make another  $\frac{3}{4}$ -portion (relative units). To then explain the fractional part of the quotient, she compared the remainder in absolute terms to the relative unit considered in this problem. These features suggest that, overall, her explanation was relatively unpacked and calibrated to its intended audience; yet, it could have been more unpacked, had she started with explaining the notion of division as taking away (or fitting in) relative to absolute units. Additionally, even though Kimberley coordinated her verbal with her deictic descriptions, she could also have made the correspondences between her drawing and the terms of this division problem more explicit.

When asked to explain the traditional algorithm of fraction division, Kimberley first identified two ideas she considered pivotal for students’ understanding of this algorithm: first, that the reciprocal is the inverse of a fraction and, hence, the product of a fraction by its reciprocal is one; second, that multiplication and division are inverse operations. She argued that before providing an explanation for the traditional algorithm, she would ensure that students had a good grasp of these two notions. She would then use

a visual representation to help students understand the concept of the reciprocal. Unlike her flexibility and quickness in explaining the quotient in the division problem  $2 \div \frac{3}{4}$ , Kimberley spent a considerable amount of time trying to figure out how to use a representation to explain the traditional algorithm for the same division problem. After a long pause, pointing to her drawing in Figure 5.27, she remarked,

I'm not really seeing a good way to do it because they're not dividing into thirds, they're dividing into fourths. You can ... show that there are -- you can get eight quarters. ... Using this picture I can divide it up into thirds instead of fourths, but I'm not sure that that would work. (PO.I. 432-446)

Kimberley's remark that she could see the eight quarters in her picture but that to match the numbers of the algorithm she needed to have thirds instead of fourths is reminiscent of her pre-intervention explanation of the quotient, in which she wanted to place another point on her number line to accommodate the thirds that appeared in the numerical answer of the problem; as such, her comment suggests that her attempt to explain the reciprocal was largely informed by the numbers involved in the algorithm (i.e., number  $\frac{8}{3}$ ). Yet, after some more thinking and exploring whether she "could see the  $\frac{4}{3}$ " in one of the circles she drew to explain the quotient (PO.I. 477), Kimberley remarked, "It's four thirds, because it's thirds of three quarters. ... I can't believe it took me so long to figure this out" (PO.I. 507-515). Building on this idea and using the drawing of Figure 5.27, she explained,

So the way I would show the four thirds on here is, since you're doing two times four thirds you end up with two and two thirds. So ... basically the four thirds is right here, it's two times these four thirds because all of these [she points to the eight fourths in the two circles] are thirds of the three quarters. (PO.I. 519-524)

Then hiding one of the fourths in the first circle with her hand, she further clarified,

So each fourth of my pizza is actually a third of three quarters [she points to the non-hidden parts of the circle]. [Then pointing to the whole circle:] And there's four thirds. So when you're doing two times four thirds ... essentially ... you're multiplying by the whole pizza – which doesn't really make sense when I explain it. ... I'm not sure if I have the right way of explaining this. ... Because ... each quarter is a third of the three quarters, so the thirds in this case represent a third of that

three-quarter chunk of the pizza. ... When I'm doing two times four thirds, I get eight thirds, which is two pizzas or two three-fourths chunks and two thirds of another three-fourths chunk. (P.O.I. 529-567)

Kimberley's attempt to show the four thirds in her drawing was consequential for explaining the traditional division-of-fractions algorithm. Equipped with a better understanding of the relative and the absolute units, she realized that she could explain the thirds included in the algorithm by focusing on the relative unit of three fourths. For Kimberley, this idea put the equation  $2 \div \frac{3}{4} = 2 \cdot \frac{4}{3} = \frac{8}{3} = 2\frac{2}{3}$  in perspective. She could now see that the two times four thirds corresponded to the four thirds of a  $\frac{3}{4}$ -unit that could be accommodated within each of the two circles representing the dividend; similarly, viewed from the perspective of relative units, the eight thirds corresponded to the eight fourths into which she partitioned those circles. However, Kimberley did not clearly talk about the reciprocal as fitting divisor units within each dividend unit, which would have suggested that Kimberley had developed a more solid conceptual understanding of the notion of the reciprocal.

Because of Kimberley's argument that her explanation as outlined above might not make sense, she was given another opportunity to provide an explanation that would be more comprehensible to the intended audience of sixth graders. After further thought, Kimberley assumed the role of the teacher, and pointing to the  $\frac{3}{4}$ -portion of the first circle, she explicated:

"How many quarters of the pizza make up that three-fourths chunk? How many one-fourth chunks are there?" ... Hopefully the student will figure out that there's [sic] three. And I'd say, "Okay, so each of these quarter chunks is actually one third of our three quarters, right? Because there's [sic] three of them, so each one of those is one third of our three quarters." So I would say, "So in each pizza, another way of thinking about this pizza, if each of these [pointing to the fourths in each circle] is a third of the three quarters, each pizza is four of those thirds, is four thirds, where each third is actually a chunk of our three quarters. So we're doing two times four thirds, so two of those, two pizzas. And what we end up is eight, right? We've got eight quarters, four quarters in each pizza. So we've got eight chunks, but remember, these are actually not quarters when you're thinking of the three quarters; they're a third of each three quarter. So we have eight thirds... so

that's enough for two full chunks of three quarters and two thirds of another chunk of three quarters. So, our answer is two and two thirds." (PO.I. 577-606)

Kimberley's second explanation was definitely more unpacked than the first one: she clearly identified the relative  $\frac{3}{4}$ -unit and explicated that this relative unit consists of three one-fourth parts, which she named as thirds of a  $\frac{3}{4}$ -portion; she explained that the absolute unit (i.e., the whole pizza) consists of four such thirds; she clarified that she considered two pizzas, thus the multiplication  $2 \cdot \frac{4}{3}$ ; she linked the eight fourths in her drawing to the eight thirds of the algorithm by explicating that, viewed as a part of the relative  $\frac{3}{4}$ -unit, each fourth represents a third – in doing so, she clarified how the final answer of two and two thirds relates to the eight fourths in her drawing. Given its unpackedness, her explanation could be comprehensible to its intended audience.

Reflecting on her work on providing this explanation, Kimberley remarked that she struggled with using a visual representation to scaffold her explanation. She thus identified the practice of using representations as an area in which she needed to improve:

[T]he thing that made it hard is trying to work with visual representations, which has never been my strong suit. ... I'm much more of the kind of person that likes to just write out equations. *And part of that is my background: having grown up with just the standard algorithm. So using representations is not something that comes easy for me, at all.* And then trying to ... honestly, what I was doing is computing in my head and then trying to make it fit the representation. Which is still just an old habit, the hard kind of thing, it's pretty hard for me to look at a representation and go, "Oh, there's the [reciprocal]." (PO.I. 614-624, emphasis added)

In addition to revealing Kimberley's difficulties with using representations, this excerpt also provides evidence suggesting that her thinking was, to a large extent, driven by the numbers involved in the algorithm rather than by the concept of the reciprocal.<sup>182</sup> However, because of her better understanding of relative and absolute units, her number exploration was not haphazard. Even though she was not aware of the concept of the

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<sup>182</sup> A comment that Kimberley made later in the interview further substantiates the argument about her weak conceptual understanding of the reciprocal. When during the virtual lesson Michelle asked the virtual teacher to explain why the reciprocal works, Kimberley remarked that she would have a hard time explaining the reciprocal in more conceptual terms if she were asked to do so.

reciprocal, this understanding of relative and absolute units helped her see meaning in the numbers involved in the algorithm. As such, her explanation, although numerically informed, was also undergirded by her conceptual understanding of units.

*Noticing and Interpreting- Evaluating.* Kimberley's post-intervention appraisal of the teacher's explanation of the reciprocal was quite similar to her pre-intervention evaluation. Without a deep understanding of the concept of the reciprocal, Kimberley argued that the teacher's explanation made sense and that if she were to respond to Michelle's question as to why the reciprocal works, she would probably provide an explanation similar to the teacher's. This suggests that Kimberley was not concerned with the inappropriate analogy that the teacher used in her explanation or with the linguistic rather than mathematical argument deployed to explain why one needs to use multiplication instead of division when taking the reciprocal of the divisor. Yet, unlike her pre-intervention performance, Kimberley critiqued the teacher for postponing Robert's question for the next lesson. She argued that his question offered an opportunity for clarifying the different units (relative and absolute) used in the division of fractions. Kimberley's evaluation of the teacher's reaction to Robert's question appears to have been informed by her better understanding of the concept of relative and absolute units.

#### *Using Representations*

*Performance.* Kimberley's use of representations when explaining the quotient for the division problem  $2 \div \frac{3}{4}$  was conceptually driven throughout, largely due to her improved understanding of the concept of relative and absolute units. In contrast, as she admitted, her use of representations when explaining the traditional division-of-fractions algorithm was informed by the numbers involved in this algorithm. Yet, because her

understanding of relative and absolute units afforded her some conceptual scaffolds, her use of representations when providing this latter explanation was both conceptually and numerically driven.

Kimberley's use of representations when providing both the foregoing explanations suggests that she was quite successful at naming the different parts of her drawing (e.g., what the two circles in Figure 5.27 represented, to what the shaded portions in each circle corresponded); yet, she was not equally successful at linking her representation to the numbers involved in the algorithm she was trying to explain, thus corroborating her argument that using representations "was not her strong suit."<sup>183</sup>

*Noticing and Interpreting-Evaluating.* Like her pre-intervention performance, when going over Amanda's episode, Kimberley noticed the lack of connections between Amanda's number sentence, the visual representation, and the word problem. Yet, Kimberley noticed the lack of those connections only with respect to the problem's quotient; she did not comment on how such connections should also have been built for the dividend and the divisor. Similar to her pre-intervention performance, she also did not comment on the lack of connections in Amanda and Julia's use of representations other than simply saying that the two girls should have explained the meaning of their answer.

Overall, Kimberley did not seem to have experienced notable gains in her performance in this practice with respect to the noticing and interpreting-evaluating tasks. This finding, viewed through Kimberley's argument that "she has grown up with just the standard algorithm" and that "using representations [was] not something that [came] easy

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<sup>183</sup> Kimberley was somewhat more successful at drawing such connections during the pre-intervention meeting. One plausible explanation for the discrepancy between her pre- and post-intervention performance could be that the number line she used during the pre-intervention meeting lent itself more to drawing such connections, simply because it encompassed such numerical symbols.



for [her],” suggests that her long apprenticeship in working at a more abstract level might have impeded Kimberley from fully capitalizing on representations as tools for scaffolding student thinking and learning. Her inclination to work at a more abstract level might have also limited what Kimberley gained from her participation in the two intervention courses in terms of improving her practice in using representations. In the analytical commentary I discuss this issue in light of Kimberley’s strong mathematical background, as suggested by her high entrance GRE-quantitative score.

### *Analyzing Students’ Work and Contributions*

*Performing.* With an increased attention to issues of meaning-making and understanding and with a better grasp of relative and absolute units, Kimberley’s performance in the tasks of analyzing students’ work and contributions was improved; this improvement was particularly apparent in the assertions she made about Robert’s understanding (but not Ann’s) and in her analysis of Michelle’s solution.

Kimberley argued that the most one could infer from Robert’s work was that he understood the algorithm and that “he’s got the procedure down” (PO.I. 1821-1822); as she remarked, there was no evidence suggesting that he conceptually understood the meaning underlying this algorithm. Kimberley nonetheless was not equally successful at drawing legitimate assertions about Ann’s understanding. Once she identified the error in Ann’s work, she argued that Ann appeared to have understood the algorithm and that she “just need[ed] a quick fix to understand which fraction to take the reciprocal of” (PO.I. 1823-1824). Given that Ann had not only taken the reciprocal of the wrong fraction, but had also taken the reciprocal of only part of the dividend, Kimberley’s argument about

the robustness of Ann's procedural understanding was questionable. Equally debatable was Kimberley's claim about Ann's work warranting "just a quick fix."

After considering Michelle's solution for awhile, Kimberley felt that she needed to figure out the numerical answer to the problem under consideration. Based on the answer she got from applying the algorithm to solve this problem, she explained that Michelle showed three  $\frac{3}{4}$ -sections and that her remainder represented two thirds of another  $\frac{3}{4}$ -section; she commented that Michelle should also have represented her work in a number sentence. Based on this analysis, Kimberley asserted that Michelle seemed to have understood the problem "really well conceptually," but she questioned Michelle's capacity to solve the problem using the algorithm (PO.I. 1824). Hence, if she were teaching the lesson, Kimberley would have Michelle come to the board and share her work. She would then organize a whole-class discussion during which she would ask the students to first represent Michelle's work using a number sentence and then have them draw connections between this number sentence and Michelle's drawing. She envisioned this activity as offering her students a platform for discussing the fractional part of the quotient, which she considered pivotal to grasping fraction division. Hence, unlike the pre-intervention meeting, Kimberley now seemed to be aware of the mathematical motivations for having Michelle or other students represent Michelle's drawing with a number sentence. Yet, her assertion about Michelle having a solid conceptual understanding of fraction division was controversial since Michelle might have just followed the procedure modeled by the teacher without having conceptually understood this operation. Kimberley's argument about Michelle's conceptual understanding would have been more legitimate had Michelle also presented her work in a number sentence.

*Noticing and Interpreting-Evaluating.* While during the pre-intervention meeting Kimberley attended to none of the four contestable student contributions, during the post-intervention meeting she attended and correctly analyzed two such contributions: Alan's error and Amanda and Julia's inappropriate interpretation of the remainder in absolute rather than relative units. However, similar to her pre-intervention performance, Kimberley did not identify the misconception in June's explanation, even when directly prompted to discuss and analyze this student's contribution.

In analyzing Alan's work, Kimberley quickly identified his error on dividing half of the line into six parts, thus representing twelfths instead of sixths. She also remarked that she "kind of actually remember[ed] this [episode] ... a little bit" (PO.I. 1169-1170).<sup>184</sup> Her comment should be considered in light of her prolonged pre-intervention struggle with identifying and correcting Alan's error (see Chapter 4).

Kimberley was also quick in identifying and correcting the error in Amanda and Julia's solution. She stated that the girls should have interpreted the remainder in badges (i.e., relative units) instead of yards (i.e., absolute units). She also criticized the teacher for not pushing the students to clarify the idea of units:

[The teacher] didn't, actually, ask the students to assign any kind of units to what they were doing. And I think that would've cleared up that misconception. If she had said, you know, "You have four what?" Then ... you could have led them down that path to figure out that they actually had four badges and half a badge, not four and a twelfth of a yard. (PO.I. 1579-1585)

In considering June's explanation, Kimberley appraised June's argument as satisfactory. She only remarked that the teacher should have ensured that all students were clear that a sixth is smaller than a half because, as she explained, some students tend to think the opposite, simply because six is larger than two. Although Kimberley did

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<sup>184</sup> This was one of the few comments the study participants made about recollecting particular teaching-simulation episodes from the pre-intervention meeting.

identify another misconception that students might have when working on fractions, she failed to notice the key latent misconception in June's explanation: that the dividend should always be larger than the divisor. Recall that during the pre-intervention meeting Kimberley appeared to hold this same misconception; in fact, she argued that June's explanation provided a useful hint for deciding which fraction represented the dividend and which the divisor. Kimberley's post-intervention argument that, as long as students are clear on the relative size of sixths and halves, then "you're good to go with that [i.e., June's] line of thinking" (PO.I. 1158-1159) suggests that Kimberley was still unaware of the key misconception implicit in this episode. Even when prompted to evaluate June's explanation at a different point in the interview, Kimberley argued,

June's explanation was that one sixth was less than a half, and therefore you had to divide the bigger number by the smaller number, that the divisor had to be a smaller number. And, I think, that's obviously correct. (PO.I. 1242-1244)

Based on the misconception she identified, Kimberley argued that if she were teaching the lesson, she would remind the students that "larger denominator equals smaller fraction" (PO.I. 1279-1280).

The juxtaposition of Kimberley's improved performance in identifying and explaining the error in Amanda and Julia's work to her appraisal of June's explanation provides an affordance and a constraint example: with a better understanding of the concept of relative and absolute units, Kimberley was better positioned to analyze the former contribution. Still not being knowledgeable of the fact that the dividend could be smaller than the divisor, she was not able to identify the latent misconception in the latter student contribution, let alone propose an intervention to help students overcome this misconception.

### *Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Kimberley's post-intervention performance in the practice of responding to students' requests for help was different from her pre-intervention performance in both episodes under consideration.

To respond to Alan's confusion and error, during the pre-intervention meeting Kimberley assumed a directive role: she would clarify the error in Alan's work, correct his drawing by partitioning the line into sixths, and then have students count the number of sixths on his line. Unlike her pre-intervention approach, which left little room for student thinking, Kimberley's post-intervention approach had the potential to scaffold students' thinking without replacing it. To handle Alan's confusion and error, she would first elicit other student ideas. If other students agreed with Alan's work, she would pose guiding questions to help Alan and his classmates understand and correct their error:

I think if no one was getting there and I needed to scaffold it a little bit more, and be a little bit more explicit, I would say, "Okay, let's go back to our original problem. We're dividing up, what?" "A half of a yard." I'd say, "Okay, somebody show me, like, highlight it on the board, you know, draw a line over it or ... circle it or something [and say], "What's our half yard?" And have them circle the half yard. And then I'd say, "Okay, and what are the pieces that we're cutting that ribbon up into? ... Looking at this yard, can somebody show me what a sixth of the yard would be?" And then, maybe, go from there. You know, wait for those aha-moments as they, sort of, figure it out. (PO.I. 1312-1324)

As this excerpt suggests, instead of her pointing to the error and correcting it for the students, Kimberley would pose questions to help students realize that they needed to divide the whole line/yard instead of half of it. She would start with redirecting students' attention to the question of the word problem, then have them identify half a yard on their line and also use this line to show one-sixth yard. Although a better approach might have been to first ask students to identify the sixths in a yard and then ask them to show half the yard, Kimberley's approach could still support students' learning, especially if she helped them distinguish between half of the yard (as mentioned in the problem) and the

whole yard (as implied by the mathematical meaning of one sixth). Unfortunately, the interviewer failed to further probe her to clarify how she would “go from there.”

To respond to June’s and Shaun’s requests for help, time permitting, Kimberley would first elicit other students’ ideas; otherwise she would assume a more directive role and show students how to divide their line into twelfths to get the sixths and the fourths. Her approach in this episode was mathematically valid, but it would likely minimize students’ opportunities for thinking, especially if Kimberley provided students with an explanation without first eliciting their ideas. The hypothetical statement in Kimberley’s outlined approach – time permitting – and her interpretation-evaluation of the virtual teacher’s work in this episode (see below) could partly explain why in this episode, in contrast to the first one, Kimberley would assume a more directive role. Another plausible explanation could be that she was not solid enough on the mathematical ideas and/or their teaching to scaffold students’ work.

*Noticing and Interpreting-Evaluating.* Unlike her pre-intervention performance in which she appraised the teacher’s response to Alan as effective, during the post-intervention meeting, Kimberley did not endorse the teacher’s approach. She argued that the teacher needed to direct Alan’s attention to the word problem and help him understand the meaning of the one sixth instead of simply asking him a question whose answer was more than obvious:

“Should you do a sixth of half the line or a sixth of the *whole* line?” And since he’d just done a sixth of half the line, it was pretty obvious he needed to do a sixth of the whole line; so it was kind of feeding him the answer. (PO.I. 1450-1454, emphasis in the original)

Kimberley was not as concerned with the teacher’s feeding students the answer in the second episode. Instead, Kimberley thought that it was good that the teacher intervened and gave students the hint of common multiples because, as she argued, “most

kids are gonna need some help with that” (PO.I. 1393-1394). She also evaluated the teacher’s scaffolding in this episode as effective, arguing that “all the students seem to [have] underst[ood] common multiples with minimal reminding” (PO.I. 1390-1391). From Kimberley’s perspective, that the teacher allowed the students to work on this problem on their own before giving them a hint on how to solve this problem was another positive feature of the teacher’s approach in this episode.

Kimberley’s performance in both episodes suggests that during the post-intervention meeting, she was more sensitive to issues of meaning-making. Yet, in addition to this increased sensitivity, other considerations (such as the difficulties she anticipated students having with the content) also informed her interpretation and evaluation of the teacher’s work. Kimberley’s argument that the students have understood the idea of common multiples by being minimally reminded of this idea is also important to underscore since the virtual lesson did not actually provide evidence warranting this claim. In making this claim, Kimberley might have been drawing on her own school experiences, which were “heavy on the mechanics”; obviously, if instruction is limited to teaching students the “mechanics,” then simply reminding students of a procedure might be sufficient to scaffold their work and “understanding.”

#### *Professed Changes and Their Attributions*

Kimberley’s post-intervention performance and comments pointed to a shift in her thinking about teaching and learning mathematics: while she originally considered this subject from a narrow procedural perspective, at the culmination of the intervention she reported endorsing a more conceptual and inquiry-oriented type of instruction.

Specifically, when asked to briefly discuss her ideas about teaching and learning mathematics, she identified a “paradigm shift” in her thinking:

I would say that the major change that’s taken place is my exposure to this inquiry method of mathematics, a more conceptual kind of mathematics, which I was never exposed to as a child. So that’s probably the major change: is more of a focus on mathematical discussions, on understanding how a student got to where they are at rather than what their answer was, and the emphasis that’s placed on being able to support ... and explain one’s answer. So I’d say that that was a *pretty major paradigm shift* for me. And I definitely now think about mathematics in that way, in terms of explaining one’s positions and more of a focus on discussion and discourse, and all that. So that’s probably, that’s how I’m thinking about math now. ... Initially I came in more used to a computational framework where you basically show the students the algorithm and then practice until they get it right. And obviously that is pretty much the antithesis of everything that conceptual mathematics is about. (PO.I. 67-87, emphasis added)

Kimberley attributed this shift in her thinking to her coursework and fieldwork experiences. She argued that key to this change was the emphasis of the intervention courses on classroom discussion and on providing and evaluating explanations. As she elegantly put it, for her, these experiences functioned “almost [as] rewiring brain paths” (PO.I. 2012). Her fieldwork placement also familiarized her with the CMP curriculum, which places more emphasis on meaning and understanding than the curricula Kimberley had experienced as a student of mathematics or had used as a substitute teacher of the subject. What also contributed to this change was the alignment of her coursework with her fieldwork experiences. As she clarified,

I was actually seeing a concrete representation of what we were talking about in class being played out in the sixth-grade classroom where I was placed. ... So that ... made it a little bit more real to me; it was no longer something that U of M [i.e., University of Michigan] was pushing as a theoretical framework, it was an actual ... pedagogical approach that was being used in the school where I was student teaching. (PO.I. 97-129)

This alignment of her coursework and fieldwork experiences (or in her own words “seeing the conceptual approach in action,” PO.I. 1981), rendered this approach more realistic for Kimberly, and also encouraged her to experiment with it. In turn, this experimentation further reinforced her conviction about both the applicability and the



value of teaching for understanding and meaning instead of simply helping students master the mechanics. As she stated,

The conceptual approach makes a lot of sense to me. I've tried ... using it with students, and actually seeing it. In the past I would've just told them how to do it; now I'm actually working with students and ... seeing it actually work with students pretty well. ... It was a concept that I'd heard about, but I didn't know anything about it before. Now I'm pretty familiar with it. (P.O.I. 1981-1995)

Kimberley's argument that although she had heard about teaching for conceptual understanding, "she knew nothing about this approach" is important to highlight since it suggests that her coursework and her fieldwork experiences offered her alternative and *concrete* images of teaching the subject from a more conceptual perspective. These images appear to have mobilized her decision to experiment with this approach while teaching the subject. Overall, Kimberley's case points to a sequence of experiences that catalyzed her reported shift toward a more conceptual mode of instruction: first she experienced this type of teaching as a learner of the subject (through her coursework); then she witnessed this approach being enacted in a real classroom setting; finally, she personally experimented with this approach as a teacher of the subject.

#### *Changes in Performance on the LMT Test: A Closer Look*

In addition to the questions she correctly answered during the first administration of the test, during its second administration, Kimberley correctly answered 11 more questions. Six of those questions were squarely relevant to the mathematical ideas explored in the teaching simulation: question 1 pertained to different interpretations of units when dealing with fractions; question 5 related to connecting a visual representation to fraction division; and questions 14c, 14e, 24c, and 25 corresponded to different interpretations of division (of fractions). From this perspective, these gains in Kimberley's performance, and especially that she answered question 1 correctly, were

consistent with her post-intervention performance in the teaching simulation, in which she exhibited a better understanding of division of fractions and particularly of relative and absolute units. Interestingly enough, Kimberly also correctly answered question 10, which pertains to an unconventional student method for solving fraction divisions. As her notes on solving this question suggest, she figured out that the student's method was correct by experimenting with using this method to solve an additional division problem (i.e.,  $\frac{5}{8} \div \frac{3}{4}$ ) besides that outlined in the test question.

In both test administrations Kimberley failed to answer question 14d, which was directly related to interpreting a division statement; she also incorrectly answered question 19b, which pertained to appraising students' solutions in solving a division problem. Although her post-intervention performance in both these questions appears to be inconsistent with her increased understanding of division, it is important to notice that for the former question she picked answer "I am not sure," which was scored as zero.

Regardless of these two inconsistencies, overall, Kimberley's post-intervention performance was consistent with her performance in the teaching simulation. However, a note that Kimberley wrote on her pre-intervention LMT test raises an interesting issue: she wrote that in answering some question she first figured out the answer to the numerical sentence given in the question and then explored which of the word problems listed in the question matched this answer. Thus, it is open to inquiry whether her post-intervention approach in solving these questions was different. Unfortunately, the available study data do not support this exploration.

### *Analytical Commentary*

Kimberley's case is particularly conducive to considering all three subordinate questions of the second research question. First, the changes in her knowledge and performance support the strong association between the gains in the PSTs' MKT and teaching performance yielded from the quantitative analysis. Second, her teaching performance in analyzing June's explanation suggests that the increases in her knowledge did not compensate for her lack of awareness of certain student misconceptions. Third, her performance in the practice of using representations provides some insights into how PSTs' strong mathematical background (recall that Kimberley entered the program with a high GRE-quantitative score) might impose some constraints on what they gain from interventions such that considered in this study. Fourth, her case facilitates the exploration of how notable changes in both knowledge and beliefs appear to influence teaching performance. Fifth, Kimberley's rather low sensitivity to issues of maintaining the cognitive demand of challenging tasks seems to have been reflected in her teaching performance in the MTF-related practices; hence, her case provides a nice contrast to Nathan's corresponding performance. Below I consider each theme in turn.

Kimberley's post-intervention performance in the teaching simulation offers both affordance and constraint examples about the association between knowledge and teaching performance. On the affordance side, consider Kimberley's analysis of the second textbook page. In addition to identifying the page's requirement that students explain the fractional part of their answer as a key aspect of this page, Kimberley also argued that having students explain the remainder in fraction divisions should be a central activity in a division-of-fractions lesson. Viewed in light of her better understanding of

relative and absolute units, both her attention to this page's requirement and her decision to incorporate it in an introductory lesson on fraction division corroborate proposition D<sub>1</sub> as outlined in Table 4.6, according to which teachers' knowledge of the content enables them to identify the affordances of curriculum tasks. Kimberley's increased understanding of relative and absolute units also enabled her to provide a conceptually driven explanation for the quotient in the division problem  $2 \div \frac{3}{4}$ , thus supporting the propositions A<sub>1</sub>, A<sub>2</sub>, and B<sub>2</sub>, which collectively maintain that a deep understanding of the content can scaffold teachers in providing explanations and using representations.

This understanding of relative and absolute units also appeared to have scaffolded her attempts to provide an explanation for the invert-and-multiply algorithm for the same division problem. With a developing understanding of the concept of the reciprocal, in providing this explanation Kimberley initially focused on the numbers included in the algorithm (thus again lending support to proposition A<sub>2</sub>, which suggests that when PSTs' knowledge does not support them in emphasizing the meaning of the operations they are asked to explicate, PSTs are more likely to shift to a numerically driven explanation). However, her experimentation with the numbers involved in this algorithm was not haphazard; cognizant of the two different types of units, she focused her attention on exploring ways in which she could explain the numbers involved in this division from the perspective of relative units. Given that her awareness of relative units provided Kimberley with a conceptual scaffold, her performance in explaining the traditional algorithm corroborates proposition A<sub>3</sub>, according to which a strong knowledge of the content affords teachers with tools upon which to draw when providing explanations.

Three other manifestations of Kimberley's increased understanding of relative and absolute units are also worth considering. First, in analyzing Michelle's solution, Kimberley argued that the students should have been encouraged to represent the diagram in Michelle's solution with a numerical sentence; she claimed that this would have allowed discussing and assessing their interpretation and understanding of the remainder in absolute and relative units. Second, in evaluating the teacher's response to Robert, Kimberley criticized the teacher for postponing Robert's question for the next day, arguing that the issue he was raising warranted clarification in the day's lesson. As the reader might recall, Kimberley's pre-intervention struggles with understanding the concept targeted by Robert's question did not allow her to consider the implications of deferring his question. Third, Kimberley's post-intervention flexibility and quickness in identifying and correcting the error in Amanda and Julia's work also appears to have been informed by her better grasp of the content.

On the other hand, Kimberley's incomplete understanding of the concept of the reciprocal seems to have constrained her performance in at least two respects. First, although she wanted to use task D for an introductory lesson on fraction division, she did not know how to scaffold students' work on this task. Second, she appraised the teacher's explanation for the reciprocal as satisfactory, thus failing to notice the several ways in which this explanation was mathematically impoverished. Additionally, despite her increased understanding of the division of fractions, Kimberley still appeared to believe that in fraction division, the dividend is always larger than the divisor. This false impression seems to have prevented her from identifying the pertinent misconception in June's explanation, thereby corroborating proposition C<sub>4</sub>, which maintains that teachers

need to be aware of certain misconceptions if they are to identify them when they arise in students' work and contributions.<sup>185</sup>

Such an understanding could have also supported Kimberley in selecting more appropriate tasks for the three activities she considered organizing to add more conceptual flavor to the first textbook page. This latter argument supports that good intentions and a shift in one's perspective about teaching the subject do not suffice to select and restructure tasks in a fashion that promotes student learning and understanding. From this viewpoint, Kimberley's performance in selecting and using tasks lends support to proposition D<sub>2</sub>, which concerns the relationship between teachers' knowledge of the content and their restructuring of tasks.

Despite the opportunities that Kimberley was offered during the two intervention courses to work on using representations, her performance in this practice was not dramatically improved. Kimberley attributed her enduring difficulties in drawing connections among different representations to her "old, die-hard habits." Having been accustomed to working at a more abstract level, she struggled with using visual representations at their full potential to support student learning; as she admitted, using such representations was not her "strong suit." Hence, Kimberley's post-intervention performance in this practice, compared to that of Deborah who entered the program with a relatively lower GRE-quantitative score than Kimberley's, exemplifies how a strong mathematical background might, in fact, limit what PSTs can gain from interventions such that considered in this study. Kimberley's post-intervention performance in the

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<sup>185</sup> Kimberley correctly addressed question 10 of the LMT test but missed the misconception in June's explanation. This finding further corroborates proposition C<sub>4</sub> since question 10 pertains to a non-conventional student solution anchored by a harder division-of-fraction concept than the concept captured in June's explanation.

practice of using representations also suggests an addendum to proposition B<sub>3</sub>: that in addition to being sensitized to the importance of drawing connections among different representations, PSTs (and especially those used to be working at an abstract level) need ample and recurrent opportunities to practice building such connections if they are to render the making of connections a habitual part of their practice.

In addition to the changes in her knowledge, Kimberley also exhibited notable changes in her beliefs; in fact, she talked about a paradigm shift in her beliefs about teaching and learning mathematics. Coming from a background that was “heavy in the mechanics,” namely in learning and applying rules, during the two intervention courses, but also during her fieldwork placement, Kimberley was exposed to a more conceptual type of instruction that emphasized meaning and understanding. These alternative images of teaching she experienced in both her coursework and fieldwork have apparently catalyzed the changes that Kimberley reported having experienced in her beliefs. In turn, these changes, alongside with her enhanced images of teaching, were manifested in several aspects of her performance in the teaching simulation.

Kimberley’s post-intervention analysis of both textbook pages comprises a case in point about the changes just discussed. Not only would Kimberley follow the conceptual orientation of the second page (which was also part of her pre-intervention plan), but she also thought of restructuring the first page, which was more procedurally oriented, to teach for more meaning and understanding. For example, she would start with having students solve word problems and use this activity as an arena to help students develop and propose algorithms for dividing fractions. She would then explain the meaning underlying the division of fractions and use the page’s 16 exercises to afford students

more opportunities to consider division of fractions from a conceptual perspective (e.g., by posing problems or by using visual representations to solve some of these exercises). Similarly, in contrast to her pre-intervention analysis, in considering the second textbook page she identified task D, which asks students to propose algorithms for fraction division, as a significant asset of this page. Drawing on the images of teaching she garnered from the intervention courses, she also criticized the virtual teacher for not giving students more time to discuss the problem of task A<sub>1</sub> in pairs or in small groups. Overall, the differences in Kimberley's performance in the practice of selecting and using tasks lend support to proposition D<sub>3</sub>, which asserts that factors beyond teachers' knowledge inform their task presentation and enactment decisions.

This shift in perspective was also manifested in Kimberley's outlined approach for responding to Alan's request for help and in the assertions she made about Robert's understanding. Unlike her pre-intervention plan for responding to Alan, her post-intervention response allowed for more student thinking. Instead of just telling Alan what his error was, Kimberley proposed a sequence of questions that could sensitize Alan and his classmates alike to the error made and, to some extent, help them correct this error on their own. As such, her approach supports proposition E<sub>2</sub>, according to which teachers' beliefs and images of teaching inform their responses to students' requests for help. Similarly, after analyzing Robert's solution, instead of arguing that Robert understood division, Kimberley asserted that his work provided evidence about his fluency with the algorithm but that it suggested nothing about his conceptual understanding.

Despite Kimberley's claim about a paradigm shift in her thinking about teaching and learning mathematics, her performance still reflected some of the influences of her



long apprenticeship in procedurally oriented instruction. Consider, for instance, her assertions about Ann's understanding. Having identified the error in Ann's solution, Kimberley argued that Ann needed just "a quick fix," probably ignoring that such a fix would not necessarily enhance Ann's fluency with the algorithm, let alone augment her conceptual understanding thereof. As such, Kimberley's performance corroborates proposition C<sub>5</sub>, which points to the importance of structuring repeated opportunities for PSTs to practice making assertions by grounding their arguments in the available evidence from student work.

In two cases which both concerned work on demanding tasks – when introducing task D (i.e., the task on figuring out an algorithm for fraction division) and when responding to June's and Shaun's request for help – Kimberley adhered to a more direct-instruction approach. Kimberley's performance in these two cases might not have been informed solely by her considerations of time constraints and students' difficulties with the content but also by an underlying assumption that the teacher needs to assume such a role when having students work on demanding tasks. The sharp contrast between Kimberley's aforesaid performance and Nathan's respective performance should not be dissociated from that Nathan reported having been sensitized to the importance of maintaining the cognitive demand during task presentation and enactment. Given that Kimberley was apparently not sensitized to such issues, the comparison of their respective performances suggests that issues of cognitive demand need to be directly addressed in teacher education programs, if PSTs are to start attending to those issues in their teaching.

Kimberley's post-intervention profile (see Figure 5.28) summarizes the main changes in her performance (in italics). This figure points to the changes in both Kimberley's images of teaching and in her beliefs, which in turn are presented to have informed her performance in the teaching simulation; her enhanced images of teaching are also presented to have triggered the changes in her beliefs. This figure also implies that there were notable changes in Kimberley's performance in providing explanations; her performance in selecting and using tasks was improved without necessarily fulfilling the criteria considered for this practice (hence the question marks shown in this figure). Figure 5.28 also suggests that Kimberley was more capable of and flexible in attending to and analyzing students' contributions; but in contrast, she did not experience striking gains in her performance in using representations. The only exception was that, compared to her pre-intervention performance, she used the representations she selected in a more conceptually driven manner, especially when her understanding of the relevant concepts was deep. The reader is also notified that the question marks linking Kimberley's knowledge to her pre-intervention performance in the MTF-related practices (see Figure 4.28) were removed from Figure 5.28. This is because her post-intervention performance was no longer negatively mediated by her beliefs and images of teaching, hence facilitating the exploration of the potential link between her knowledge and her performance, especially with respect to the MTF-related practices. Overall, the portrait of Kimberley as delineated in Figure 5.28 suggests that Kimberley was in the process of relearning how to teach mathematics, as she had wished for when entering the program (see Chapter 4) – thus the title given to her case.

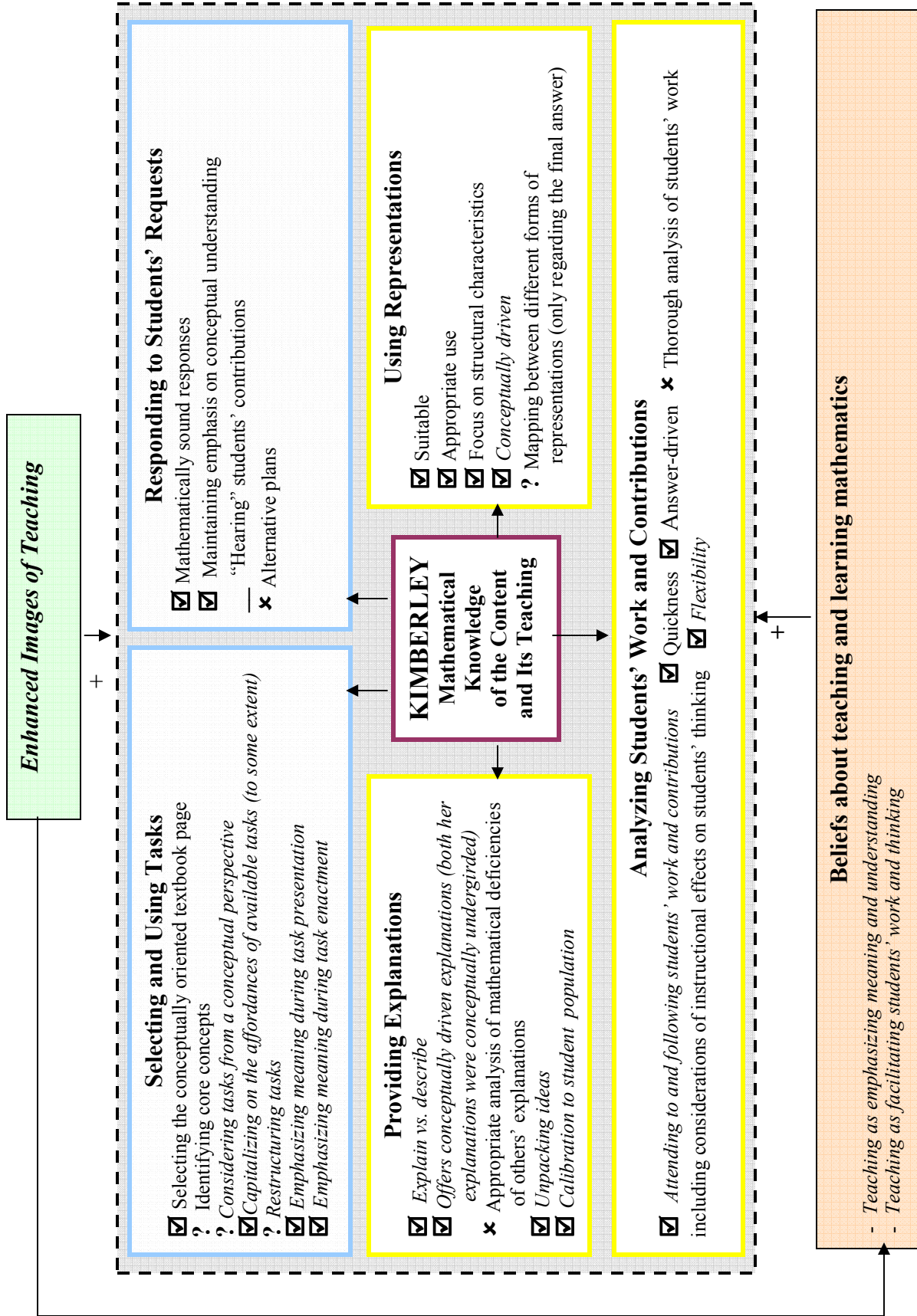


Figure 5.28. Considering the association between knowledge and teaching performance through Kimberley's post-intervention profile.

*The Case of Suzanne Revisited: “You Are Gonna Teach the Way You (Re)learned”*

Suzanne was a divergent case according to both her entrance scores and the changes in her performance. As Figure 5.3 shows, she experienced greater gains in her teaching performance than in the LMT test, in which she received the same score for both its pre- and post-intervention administrations (thus the classification of the changes in her MKT-performance relative to her counterparts as negative high; see Table 4.5).

According to Figure 5.4, Suzanne exhibited greater gains in her performance in the MKT-related practices than in the MTF-related practices. Moreover, the quantitative analysis showed that, compared to the other study participants, Suzanne reported having experienced the largest changes in her efficacy beliefs about engaging in the MKT-related practices with respect to fractions. The interview with Suzanne was conducted in the last third of January, about 30 weeks after the pre-intervention meeting. Between these two meetings, in addition to her coursework, Suzanne attended and taught math lessons in a fifth-grade class.

Suzanne’s post-intervention growth in the teacher simulation (in all five practices, but particularly in the practices of providing explanations and using representations) did not comport with the lack of changes in her knowledge *as suggested by her post-intervention score on the LMT test*. However, a closer analysis of the questions Suzanne answered on the LMT test on each of its two administrations points to some notable growth in her knowledge; this growth was consistent with the changes in her teaching performance (and also with the growth in her knowledge of the content and its teaching, as implied by the post-intervention interview). Hence, although to some extent Suzanne’s case *does* corroborate the knowledge and teaching performance association considered in

this chapter, her post-intervention performance, like her pre-intervention performance, raises issues of measurement. I consider these issues in the analytical commentary and then in Chapter 6. Suzanne also exhibited notable changes in her beliefs (about teaching and learning mathematics and her efficacy beliefs); her images of teaching the subject were also enhanced. Thus, in addition to raising issues of measurement, her case is also conducive to exploring how changes in knowledge and in the other two factors found to also inform the PSTs' teaching performance – beliefs and images of teaching – played out in her post-intervention performance.

### *Selecting and Using Tasks*

*Performing.* Like the first meeting, Suzanne showed a strong preference for the second textbook page; she regarded the first page as suitable merely for promoting students' procedural competence in dividing fractions. Yet, her proposed plan for using either page for an introductory fraction-division lesson differed substantially from her pre-intervention plan.

If she were to use the first page, Suzanne would reorder its tasks and start with the word problems to help her students understand the division of fractions. Yet, instead of using the two word problems listed on this page she would use “ribbon problems,” namely problems similar to those outlined on the second textbook page. After having students consider several of these problems she would scaffold them to identify patterns in these problems and, based on these patterns, propose an algorithm for fraction division. When asked to specify how she would scaffold students' work, especially if students found it hard to figure out the traditional division-of-fractions algorithm, she explained:

I would guide them toward the answer. I would probably represent it on the board and to help them find a pattern, to kind of show visually, symbolically [I would say], “Look what happens

here. Is there another way we could come up with this same answer?" And prompt them in ways that they might be able to figure out the reciprocal idea. [Considering problem  $\frac{1}{2} \div \frac{1}{6}$  :] Maybe, even say, "Let's come up with some other equations. How else could we come up with three, with this half a yard, what could we do?" Just to try to get them to think. And if they didn't get it, I would have to, I guess, as a very last, ultimate, resort, I would say, "Well, look what I found. If you do this, you get that answer; and you can do this same formula for all of these, and you get the same answer."... Just to show them there is an easier way ... to do it. And so they would know the overall why and what they're actually doing, but know that there's a shortcut. (PO.I. 285-300)

While observing the virtual lesson, Suzanne was asked to further clarify how this activity would look. Building on the three problems that the virtual students had considered in the lesson up to that point (i.e.,  $\frac{1}{2} \div \frac{1}{6}$ ,  $\frac{3}{4} \div \frac{1}{6}$ , and  $2\frac{2}{3} \div \frac{1}{6}$ ) and assuming the role of the teacher, she outlined the following approach:

"So we've already solved this problem that one half divided by one sixth is three." [Writes the mathematical sentence  $\frac{1}{2} \div \frac{1}{6} = 3$  ; see Figure 5.29a.] "And we've solved this problem: three quarters divided by one sixth is four and a half." [Writes the mathematical sentence  $\frac{3}{4} \div \frac{1}{6} = 4\frac{1}{2}$  ]. And we solved this problem" [writes  $2\frac{2}{3} \div \frac{1}{6} = 5\frac{1}{3}$  ]. "To make it easier, we need to think of these things in terms of improper fractions, because when you change into mixed numbers, it doesn't really make sense."

(a)

$$\frac{1}{2} \div \frac{1}{6} = 3 = \frac{3}{1}$$

$$\frac{3}{4} \div \frac{1}{6} = 4\frac{1}{2} = \frac{9}{2}$$

$$2\frac{2}{3} \div \frac{1}{6} = 5\frac{1}{3} = \frac{16}{3}$$

(b)

$$\frac{1}{2} \div \frac{1}{6} = 3$$

$$\frac{1}{2} \cdot 6 = 3$$

(c)

$$\frac{3}{4} \div \frac{1}{6} = \frac{9}{2}$$

$$\frac{3}{4} \cdot \frac{6}{1} = \frac{18}{4} = \frac{9}{2}$$

(d)

$$\frac{8}{3} \div \frac{1}{6} = 16$$

$$\frac{8}{3} \cdot \frac{6}{1} = \frac{48}{3} = 16$$

Figure 5.29. Suzanne's work on supporting students in figuring out the traditional division-of-fractions algorithm.

Having said that, she completed the three equations as shown in Figure 5.29a and wrote another equation (i.e.,  $\frac{8}{3} \div \frac{1}{6} = 1\frac{2}{3}$ ) equivalent to the third one. Then, referring to the first equation, she continued:

Now, to find a pattern between the three and the two and a six, I guess ... I would say, "What other number sentence could we come up with a half that equals three? What number is three one half of?" ... And hopefully they would say that one half times six ... is three [writes the two equations shown in Figure 5.29b]. And then say, "What do you notice between [the first] equation and [the second] equation [in Figure 5.29b]?" ... And just let them think about that. "Let's take the next one. Let's see, if we have three quarters and we're dividing it by a sixth, and getting nine halves. Let's do another number sentence [see Figure 5.29c]. Let's do three fourths that results in nine halves." [See first and second row in Figure 5.29c.] ... Is this a coincidence that when you multiply a half times six it's the same thing as dividing by a sixth? Let's take three quarters divided by a sixth. It equals nine halves. Let's see if it's just a coincidence. If you take three fourths and multiply it by six, you're going to get eighteen over four, same as nine halves. ... Let's see if eight thirds multiplied by that [i.e., six] is the same thing." [Carries out the multiplication; see Figure 5.29d.] I'm sure I could come up with a better way of doing it, but off the top of my head I would just say, "Is this a coincidence that this works?" "Well, it works in each one of these cases." And I would have them try to come up with an algorithm. [Pointing to the equations in Figure 5.29b:] "What did we do on this one?" "We turned the one sixth into six." [Pointing to the equations in Figure 5.29c:] "What do we do in this one?" "We turned the one sixth into six." A six and a sixth. I don't know if they would know the term reciprocal. But, if they didn't, they could come up with some kind of a description for what they did, and then I would test it and see if it worked for a number other than one sixth. (P.O.I. 2137-2274)

At least three aspects of Suzanne's performance as outlined above are worth considering. First, instead of simply presenting the algorithm to the students, she built on the work done on solving the three division problems to structure a guided-discovery activity for figuring out the traditional division-of-fractions algorithm. Second, the questions she posed, and especially those she proposed while working on the equations in Figure 5.29b, could support students' exploration without minimizing their opportunities for thinking. The questions she posed for the other two division problems (when working on the equations shown in Figures 5.29c and d) were narrower and pointed, but they apparently intended to direct students' attention to the pattern of interest, especially if students found it hard to get there. Third, and perhaps more importantly, after working on these problems Suzanne suggested exploring the applicability of the derived patterns and

algorithm to other situations in which the divisor was not one sixth. This could help students start thinking about the entailments of developing a generalizable algorithm.

After discovering the traditional algorithm, Suzanne would use a visual representation to explain the meaning of this algorithm. As her performance in *Providing Explanations* suggests, she would provide a fairly decent explanation, even when explicating the quotient in division problems including a fractional part. This was in sharp contrast to her pre-intervention performance, in which she offered no explanations because, as she argued, she could not see the concept of fraction division in her brain.

Finally, she would use the first page's 16 exercises to offer students opportunities to practice the division-of-fractions algorithm. However, there was a disconnect between the previous part of her lesson and how she considered using these exercises since Suzanne would introduce this activity by encouraging students to merely apply the shortcut they discovered:

I would say, "Now, let's use that shortcut. We don't even need to ... draw pictures, we don't need to think about exactly how to divide a fraction by a fraction because we know the shortcut, and we can do it real quickly by using the shortcut. And let's practice by doing these ... sixteen problems." (P.O.I. 1023-1028)

Without doubt, at some point students need to become procedurally competent in applying this algorithm. Yet, Suzanne's haste to dissociate the algorithm from its conceptual underpinnings – notice, in particular, her recommendation that students need not anymore think about the meaning of this operation – is controversial, especially if one considers that she was asked to outline an introductory division-of-fractions lesson. This rush to focus on a shortcut might have been informed by how she herself experienced mathematics as simply following and applying rules.



When asked to choose a subset of these exercises, she selected exercises by employing mostly procedural criteria (e.g., converting mixed numbers into improper fractions, simplifying fractions). Yet, she also considered some of these exercises from a more conceptual perspective. For example, she mentioned that exercise 1 (i.e.,  $\frac{2}{3} \div \frac{1}{3}$ ) is “simple to even see in your head,” since “you have a two-thirds circle ... and you want to know how many one-third circles go in to it” (PO.I. 1062-1066). She generalized this argument to include exercises that yielded whole-number quotients: “It’s easy to say, how many of these fractions will fit into this fraction, if you can actually see the result being a whole quantity” (PO.I. 1104-1106). However, she considered exercise 13 (i.e.,  $2\frac{5}{8} \div \frac{1}{8}$ ), which also yields a whole-number quotient, as harder than exercises resulting in improper or proper fractions, on the grounds that exercise 13 involves an additional step, namely converting the mixed number in the dividend into an improper fraction. Additionally, she rank ordered exercises resulting in a proper fraction (i.e., exercises in which the dividend is smaller than the divisor) as easier than exercises yielding an improper fraction (i.e., exercises in which the dividend is larger than the divisor). This ranking is contestable because, from a conceptual perspective, fitting at least one divisor unit into the dividend appears to be easier to understand than fitting no such units.

The preceding analysis thus suggests that Suzanne’s decision to use the 16 exercises largely from a more procedural perspective might (also) have been informed by her own difficulties with approaching these exercises from a more conceptual perspective. At any rate, that she started thinking about some of these exercises from a

more conceptual viewpoint constitutes an advancement in her thinking and analysis of the first textbook page compared to her pre-intervention work on this page.

This advancement in her thinking was also reflected in her work on the first word problem of this page since, compared to the pre-intervention meeting, Suzanne was now more successful at drawing connections between this problem and the division of fractions. Specifically, she appropriately represented this problem both as a division and a multiplication problem (see Figure 5.30). She also represented the whole fifth-grade class mentioned in the problem (see Figure 3.2, exercise 17) with the orange rectangle shown in Figure 5.30 and then explained that this rectangle could accommodate four of the little blue rectangles, each of which corresponded to both thirty students and to one fourth of the class. This idea reflected the notion of the reciprocal as fitting divisor units into the dividend. Yet, Suzanne did not talk about this relationship, probably because of her not being aware of it, as her performance in explaining the reciprocal suggests (see below).

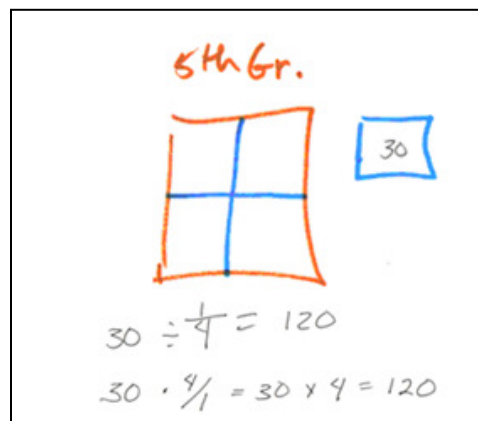


Figure 5.30. Suzanne's work on the first word problem of the first page.

Suzanne's plan for using the first textbook page was apparently informed by the structure and the tasks of the second page. In fact, her proposed plan for using the second page did not notably differ from her outline for using the first page. She would again start with having her students solve fraction division problems (i.e., tasks A and B), hoping

that this would help them figure out the traditional division-of-fractions algorithm (i.e., task D). If the students faced difficulties in identifying patterns and deriving the algorithm, she would scaffold their work as described above. Her proposed plan for scaffolding students' work on task D was a significant improvement from her pre-intervention performance, in which she again endorsed having her students figure out this algorithm but she admitted being clueless about scaffolding their work. Also, while during the first meeting Suzanne identified none of the mathematical affordances of this page, in the post-intervention meeting she talked about the page's requirement that students explain the fractional part of their answers. However, she incorrectly associated this requirement with only the problems in task B, which suggested that she was still not clear about the mathematical requirements of each task. She also made no comments about the page's requirement that students present their work in written explanations, diagrams, and mathematical sentences.

Overall, Suzanne was relatively successful at identifying some of the affordances and limitations of the two textbook pages. She also capitalized on some of these affordances to propose activities that could promote students' understanding of fraction division. Yet, her performance in using the 16 exercises of the first page suggests that she was still unprepared to upgrade the cognitive demands of procedural tasks to engage students in conceptual work on dividing fractions.

*Noticing and Interpreting-Evaluating.* Suzanne's post-intervention performance in the tasks of noticing and interpreting-evaluating was remarkably different from her pre-intervention performance in which she endorsed the teacher's presentation and enactment of tasks A<sub>1</sub> and D. These differences should at least partly be attributed to her

better understanding of the content since, as she remarked in the pre-intervention meeting, “If I knew the concept [of fraction division] better, I could really focus on what [the teacher] was doing” (see Chapter 4).

Suzanne objected to the teacher’s presentation of task  $A_1$ , arguing that, as introduced, the task was dissociated from the concept of fraction division and from the ribbon problem at hand. If she were teaching the lesson, she would emphasize the concept of fraction division more by using actual ribbon and by having students explore how many one-sixth yard badges they could fit into a half-yard ribbon. Suzanne was equally concerned with the enactment of this task, which, from her perspective, engaged students in procedural work without helping them grasp the underlying meaning of fraction division. She was also troubled about the teacher’s overemphasis on following steps; she saw the students simply applying a procedure because the teacher said so.

The teacher’s enactment of task D was similar to the approach Suzanne outlined to support students’ work on this task (e.g., using equations, helping students see relationships between different numbers). Hence, it was not surprising that Suzanne was less disapproving of the teacher’s enactment of this task compared to her critique of the teacher’s work on the previous task. Even so, she opined that the teacher would have been more effective in scaffolding her students’ understanding had she used visual representations and left more space for student experimentation:

[S]he needs a visual here to show that six is the same thing as six divided by one. And she needs a one-on-one correspondence of this formula that a half divided by sixth is the same thing as a half times six over one. ... [If I were teaching the lesson] I’d have them kind of discover that [i.e., the pattern] on their own rather than just saying, “Okay, let me explain what we’re doing.” (P.O.I. 2315-2335)

Suzanne’s suggestions for alternative ways of enacting this task, and especially her idea of mapping the formula onto a visual representation, were apparently informed

by the images of teaching she gleaned from her participation in the two intervention courses. These images of teaching seem to have also informed her overall appraisal of the quality of the lesson. Unlike her pre-intervention evaluation of the teacher's work in this lesson, according to which she commended the teacher for appropriately scaffolding students' work and thinking, during the post-intervention meeting, Suzanne critiqued the teacher for overemphasizing following procedures and obtaining correct answers:

I think she's teaching the way that I learned how to do math: that a rule is a rule, learn how to follow the rule and you'll get the right answer, and then we'll be happy with you. You don't have to know why it works, you don't have to actually know what you're doing; you just need to know how to do the formula. (PO.I. 2582-2586)

This excerpt, which aptly delineates Suzanne's experiences as a learner of mathematics, justifies why during the pre-intervention meeting, when asked to explain fraction division, she claimed having no conceptual scaffolds for offering such an explanation. Below, I explore how this changed during the post-intervention meeting.

### *Providing Explanations*

*Performing.* In the pre-intervention meeting, Suzanne provided explanations for neither the quotient nor the traditional algorithm for the division problem  $2 \div \frac{3}{4}$ , claiming that she had "no concept in her brain." In the post-intervention meeting, her augmented understanding of the concept of fraction division and of the relative and absolute units involved in this operation allowed her to provide such explanations.

Suzanne initiated her explanation for the quotient of the division problem  $2 \div \frac{3}{4}$  by referring to the concept of fraction division: "How many three fourths fit into two?" (PO.I. 369). To contextualize her explanation, she proposed the following word problem:

There are two one-liter bottles of soda. And we're making this punch concoction that takes three quarters of a liter to make it. How many three quarters of a liter can we make? Or how many of these servings can we make? (PO.I. 380-394).

To illustrate the dividend of this problem she drew two bottles of soda (see Figure 5.32a); to then show the divisor (i.e., the  $\frac{3}{4}$ -servings) she drew another pair of bottles, divided each into four parts, and colored three fourths of each (in blue and orange, respectively; see Figure 5.32b). Pointing to the blue  $\frac{3}{4}$ -portion, she continued:

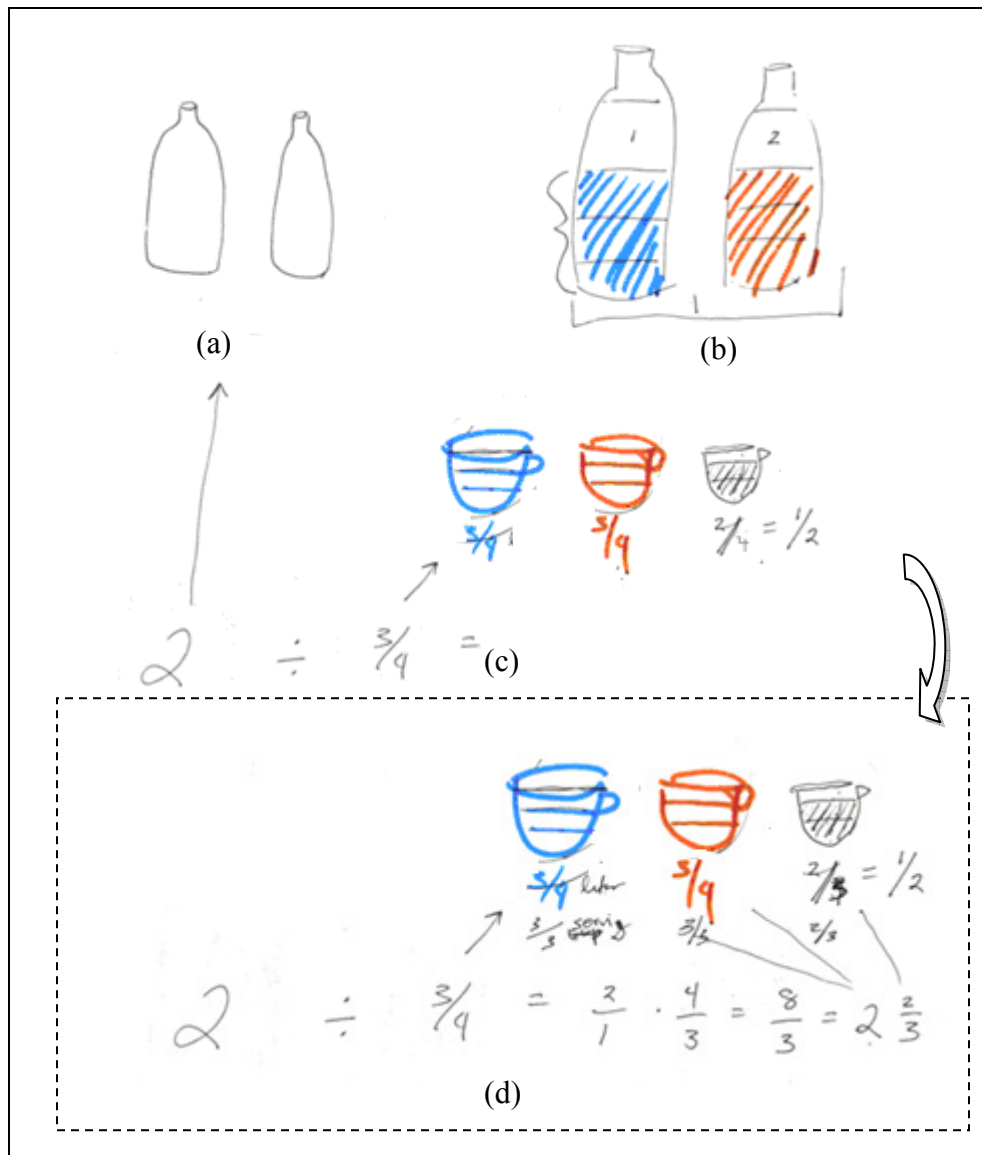


Figure 5.31. Suzanne's work on explaining the quotient for the division problem  $2 \div \frac{3}{4}$  (a: representing the dividend; b-c: representing the servings; d: expressing the leftover part in  $\frac{3}{4}$ -servings).

*Just step by step.* ... So if this is what I need per serving [what she colored in blue], how many of these can I fit? ... How many servings can I do? Well, [pointing to the fourths in each serving:] this is one, two, three [in the "blue" serving]; and this one is one, two, three [in the "orange" serving];

there's [sic] two servings. And then, can I make three servings? I don't have enough, I only have one, two quarters; I can't make a third serving. So let's do one serving of blue [draws a  $\frac{3}{4}$ -cup in blue, see Figure 5.31c], and here's my one serving of orange, and here's my cup [draws another  $\frac{3}{4}$ -cup in orange]. This one serving has three quarters of a liter [writes " $\frac{3}{4}$ " underneath the blue cup]. This serving has three quarters of a liter [writes  $\frac{3}{4}$  underneath the orange cup], but this one [draws a smaller cup, see Figure 5.31c] doesn't. I can't make a whole one, because I only have one, two quarters. This, actually, isn't a whole serving; it's only two quarters of that liter [writes  $\frac{2}{4}$  below the cup drawn in pencil]. And two quarters, you can say, is a half [writes  $\frac{2}{4} = \frac{1}{2}$ ]. So there's [sic] one, two, and one half three-quarter servings, in two wholes. Is that the -- I have to go back to the formula. ... I don't think I got the right answer. (PO.I. 379-457, emphasis added)

Indeed, in providing this explanation, Suzanne was going "step by step," to use her own words. She clearly defined the dividend and explained why she needed to divide each bottle into fourths. She then showed the divisor and explained that she could only make two of the  $\frac{3}{4}$ -servings and that the available quantity did not suffice to make a third serving. Yet, although she correctly identified the remainder in absolute terms (i.e., as two fourths of a liter), in her last comment she confounded the relative and absolute units by arguing that she could make two and one half  $\frac{3}{4}$ -servings. Quite uncertain about her answer, she reverted to the algorithm to figure out the answer to this problem. She wrote the division problem  $2 \div \frac{3}{4}$  (see Figure 5.31c) and associated its dividend with the two bottles and its divisor with the servings she could make. Once she figured out the answer to the problem (see Figure 5.31d) she commented:

I knew I got that one [i.e., the leftover part] wrong. That's where I made my mistake, there's three out of these [writes " $\frac{3}{3}$ " underneath the blue serving, see Figure 5.31d], one, two, three [writes " $\frac{3}{3}$ " underneath the orange serving], and it leaves two out of the three [changes the denominator in the fraction shown below the small cup into thirds]. That is my two, one, two [points to the blue and orange cups] and this [points to the smaller cup] is my two thirds. (PO.I. 481-489)

Suzanne's identification of the "cups" as three thirds could add clarity to her explanation if she had additionally specified that each cup represented three thirds of a serving but also three fourths of a liter of soda; this would have helped her better distinguish between the relative and the absolute units. It was also still unclear why she identified the leftover part as two thirds; additionally, her equation " $\frac{2}{3} = \frac{1}{2}$ " (see Figure

5.31d) was problematic because it did not specify the units under consideration. Hence, the interviewer probed her as to why the leftover part represented two thirds and not two fourths. In response, Suzanne went back to her picture and put labels to distinguish between the absolute and the relative units (see Figure 5.32d):

My serving is three of the four parts [points to the fourths in Figure 5.31b]. When I put it in my cup, here's my one, two, three [points to the thirds in each cup, see Figure 5.31c]. So this [pointing to the blue cup] is three quarters of a liter, but it is three thirds of my cup ... my serving [puts labels: " $\frac{3}{4}$  liters" and " $\frac{2}{3}$  serving," see Figure 5.31d]. Three quarters of a liter is the same thing as three thirds of my serving. And this [points to the orange cup] is also the same, three thirds of my serving. This [points to the small cup] isn't a whole one; it's only two of those thirds of a serving. ... So, it's all about the unit, just explaining very clearly what the unit is. That this [points to the blue cup], the three quarters, is three quarters of the unit, which is the liter. And then the two thirds is two thirds of the unit which is the serving, it's two thirds of a full serving. (PO.I. 523-552)

Suzanne's labeling of the two different units as bottles and servings helped her clearly distinguish between the absolute and the relative units considered in this problem and consequently provide an even more unpacked explanation for the fractional part of the quotient. Her comment "it's all about the unit" also merits some attention since it encompasses a fundamental idea pertaining to (fraction) division.

When asked to explain the division-of-fractions algorithm, Suzanne provided yet another explanation for the quotient before she moved on to explicate the multiplication of the dividend with the reciprocal of the divisor. This explanation is also interesting to consider since it represents a higher level of abstraction compared to her previous explanation: instead of talking about liters and servings, Suzanne distinguished between the absolute and the relative units by identifying two different geometric shapes: (blue) rectangles to represent the dividend and an L-shaped polygon to represent the divisor (see Figure 5.32). Using these two different shapes, she explained that she wanted to figure out how many  $\frac{3}{4}$ -portions represented by the L-shaped polygon could be fit into the first two rectangles. To figure this out, she drew two new rectangles on the right of the



“equals” sign and shaded a  $\frac{3}{4}$ -portion in each, remarking that she could fit two such portions into the two rectangles (see Figure 5.32). To explain the fractional part, she drew another L-shaped polygon (see the shape at the bottom of Figure 5.32). Pointing to this shape, she continued:

This is the shape that I'm trying to fill but I can't fill it completely. ...I have  $A$  and I have  $B$  [she writes letters “A” and “B” in the leftover parts of the rectangles in orange], and  $A$  can go here and  $B$  can go here [pretends to transfer these two leftover parts to the L-shaped polygon and then writes  $A$  and  $B$  in the two bottom thirds of this shape]. But I don't have enough to make this full shape; I don't have this  $C$  [points to the top part of this shape]. ... How many parts do I have now? It's not divided into four; this shape is divided into three. So my denominator is three [draws a fraction line and writes “3” for its denominator]. How many parts of those three pieces do I have? I have two, so my numerator is two [writes “2” as the numerator of this fraction]. So I have two thirds of this shape [the L-shaped polygon]. ... [Pointing to the L-shaped polygon:] This is three quarters of a whole unit [points to a blue rectangle]. (PO.I. 693-714)

This latter explanation, and particularly the portion on explicating the fractional part of the quotient, was even more unpacked than her former explanation since, using the part-whole concept of fractions as a linchpin, Suzanne clearly connected her drawing to the numerator and the denominator of the fraction representing the leftover part. Her remark that the remainder was actually two thirds of a  $\frac{3}{4}$ -portion added further clarity to her explanation.

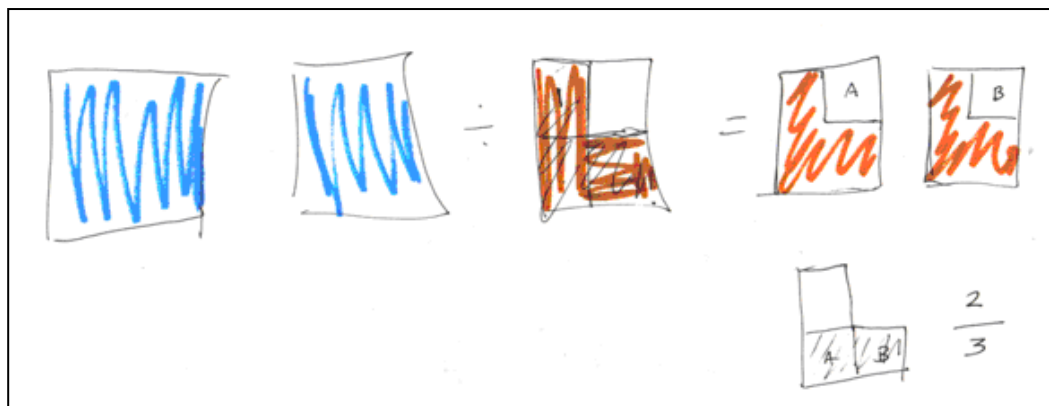


Figure 5.32. Suzanne's representation on re-explaining the quotient for the division problem  $2 \div \frac{3}{4}$ .

Overall, both Suzanne's explanations for the quotient in the division problem  $2 \div \frac{3}{4}$  were, for the most part, conceptually undergirded; the numbers involved in this

division had an auxiliary role since they provided her with some guidelines to correct her explanation when she appeared to be mixing the two types of units. Her explanations were also considerably unpacked and could be understood by an average sixth grader.

Lacking a conceptual understanding of the reciprocal (as she confessed at the end of the post-intervention meeting), Suzanne struggled with providing an explanation for the traditional division-of-fractions algorithm. As already mentioned, when asked to provide such an explanation for the division problem  $2 \div \frac{3}{4}$ , she started with re-explaining the quotient. She then thought of switching to a different representation to show the thirds included in the algorithm. Therefore, she drew two rectangles, divided each into three parts, and shaded in four thirds of a rectangle (see upper panel of Figure 5.33); she argued that the shaded portion represented the reciprocal. To then explain the multiplication of the reciprocal by two, she drew two such sets of a four-third rectangle (in blue and orange; see the lower panel of Figure 5.33) and argued that, taken together, these sets yielded the quotient of the division problem.

At this point, the interviewer directed her attention to a discrepancy between her previous drawings that represented fourths and her last drawing representing thirds. To explain this discrepancy, Suzanne wondered if it would make sense to draw another picture and “take the same unit and divide it two different ways: ... to divide it into fours and also divide it into threes” (PO.I. 740-741); her comment suggested that her thinking was still governed by the numbers involved in the algorithm. To investigate whether Suzanne could provide an appropriate explanation for the reciprocal if she focused on her drawing shown in Figure 5.32, the interviewer recommended that she consider using this drawing for her explanation. Returning to this visual representation, Suzanne explored if

there was any way in which to show the four thirds: “Where is my four thirds?” (PO.I. 758); apparently, even when provided with the interviewer’s scaffold, her thinking was still driven by the numbers of this algorithm.

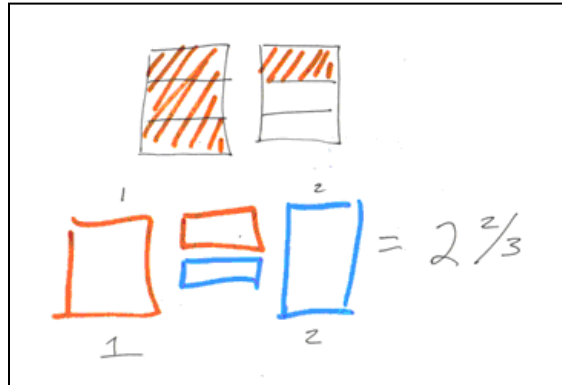


Figure 5.33. Suzanne’s first attempt to explain the invert-and-multiply algorithm for the division problem  $2 \div \frac{3}{4}$ .

After a long pause, Suzanne had a revelation and exclaimed: “Oh! I see it, I see it now!” (PO.I. 759). To clarify her thinking she drew a rectangle, divided it into four parts, and colored a  $\frac{3}{4}$ -portion in brown (see upper panel of Figure 5.34); she explained that the colored part represented the divisor. She then continued:

[I]f you take this shape [points to the L-shaped polygon colored in brown], there are three parts: this is one third of that part [points to the upper part of the polygon and writes “ $\frac{1}{3}$ ” in it]; this is a third of that part [writes “ $\frac{1}{3}$ ” in the bottom left part of the polygon] that’s a third of that part [writes “ $\frac{1}{3}$ ” in the third shaded part]. And then there’s one more third of that part [points to the non-shaded part and writes “ $\frac{1}{3}$ ” in it]. So this is one, two, three thirds [she writes numbers 1, 2, 3 in the three shaded parts]; [then pointing to the non shaded part:] four thirds, there’s your four thirds [writes number four in this part]. (PO.I. 781-787)

To then explain the “two times four thirds” part of the algorithm she drew another rectangle similar to the first one, divided it into fourths, shaded in a  $\frac{3}{4}$ -portion, and wrote “ $\frac{1}{3}$ ” in the non shaded portion. After writing the multiplication sentence  $2 \cdot \frac{4}{3}$  above her drawing and the division sentence  $2 \div \frac{3}{4}$  below it (see Figure 5.34), she explained that the “two times four thirds” part of the algorithm is equivalent to dividing the two rectangles by the  $\frac{3}{4}$ -unit represented by the L-shaped polygon. To further unpack this idea, she drew

two rectangles in blue (see lower panel of Figure 5.34) and an orange L-shaped polygon; she then explained that the L-shaped polygon fits into the blue rectangles twice, leaving uncovered the upper right corners of these rectangles. Next, using her drawings in the upper and lower panels of Figure 5.34, she tried to build a connection between the multiplication and the division included in the traditional algorithm:

When I have a quantity of two multiplied by four thirds, I end up with one of those things, two of those things [points to the two  $\frac{3}{4}$ -portions of the blue rectangles in the lower panel]. ... And then these are my thirds [points to the non-shaded portions of the two rectangles in the upper panel], here's my two thirds: one third and two thirds [corresponds the non-shaded regions of the upper panel to the upper right corners of the blue rectangles in the lower panel]. So, I guess I would just show that the shapes, they're divided the same way, but it's thinking about them in two different ways. But you get the same result. [Writes " $=2\frac{2}{3}$ " in the upper and lower panel.] (PO.I. 829-838).

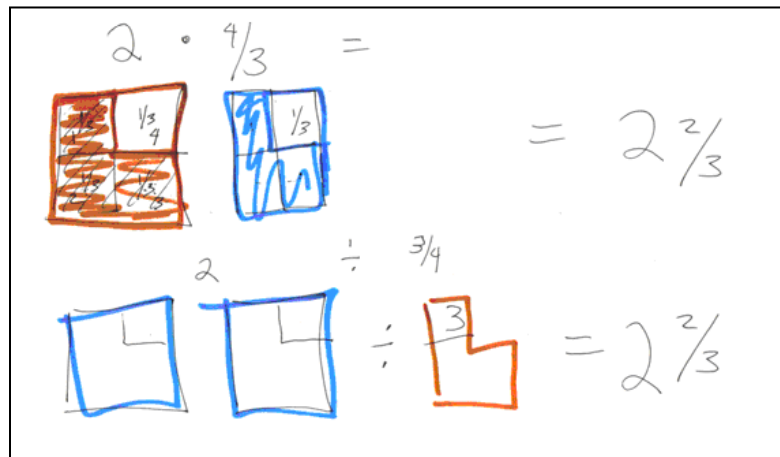


Figure 5.34. Suzanne's second attempt to explain the invert-and-multiply algorithm for the division problem  $2 \div \frac{3}{4}$ .

Suzanne's explanation was still unclear; notice, for example, that she did not explicitly refer to the concept of the reciprocal (i.e., that each dividend unit could accommodate four thirds of the divisor units). Hence, the interviewer probed her understanding of the numbers involved in this algorithm:

**Charalambos:** So, when we say two divided by three fourths, what does this mean?

**Suzanne:** How many three fourths are there in two?

**Charalambos:** Okay. And ... what does this two times four thirds mean?

**Suzanne:** Okay, I would say, if you had a four thirds and you made two of them.

**Charalambos:** And the four thirds being what?

**Suzanne:** If you had four thirds of something and multiplied by two, you have two of those. You're basically adding four thirds and four thirds.

**Charalambos:** Um-hmm. ... So what is the four thirds, then?

**Suzanne:** The four thirds is, um, a unit. Because this is my unit [points to one of the blue rectangles in the lower panel of Figure 5.34]. The three fourths is the part that I'm taking. So when I'm dividing it by three fourths, I'm dividing it by the portion. I'm taking the quantity and I'm dividing it by the portion to get my answer. But, if I were to take the reciprocal, I'm taking the part of it and I'm making two of them, and then I'm counting them in terms of units, not parts. (PO.I. 863-890)

Suzanne's answer to the first interviewer's question suggests that her understanding of division as fitting divisor units into the dividend was quite robust. Her understanding of the reciprocal nonetheless was weak. Although she correctly identified that the four thirds represented a unit (probably referring to the absolute unit), she never clearly articulated the idea of fitting relative units into absolute units. Her last comment: "I'm taking two of them, and I am counting them in terms of units, not parts" suggests that her understanding of relative and absolute units scaffolded her exploration and explanation of the traditional division-of-fractions algorithm. Yet, her conceptual understanding of the reciprocal was underdeveloped and under development: although Suzanne started providing her explanation with no clear concept of the reciprocal, her work on explaining this division problem contributed to her evolving understanding of this notion.

To summarize, Suzanne's explanation for the traditional algorithm was largely numerically driven (although in its final form it was conceptually informed). Yet, her attempts to make the numbers fit her drawings were aided by her increased conceptual understanding of division as fitting divisor units into the dividend and by her enhanced understanding of absolute and relative units.

*Noticing and Interpreting-Evaluating.* During the pre-intervention meeting, Suzanne appraised the teacher's explanation as to why the reciprocal works merely from the perspective of a student who tries to make sense of an instructional explanation.

Hence, even though she was concerned with the teacher's analogy and the argument of using the reciprocal operation simply because of employing the reciprocal, she could not go into more depth and justify why she considered the teacher's explanation poor. In the post-intervention meeting she moved a step father and contended that the teacher's analogy was inappropriate because it had no actual connection to the content considered in the virtual lesson; as she maintained, "[The teacher] goes into positive and negative numbers which is totally unrelated. ... It's only similar in nature, but it has nothing to do with what she's doing" (PO.I. 2424-2427). In addition to identifying that the teacher's analogy was similar in nature but not in substance to the traditional division algorithm, Suzanne also maintained that the teacher's explanation did not actually explain the algorithm at all.

Unlike her pre-intervention performance, during the post-intervention meeting, Suzanne also commented on the teacher's reaction to Robert's question. She was concerned with the teacher's decision to postpone Robert's question for the next day, but even more with the teacher's response that the invert-and-multiply algorithm is a rule that ought to be followed. This response reminded Suzanne of her own school experiences:

And her explanation is that "it's a rule and if it's a rule then it works" ... is the way that I learned math: that you learn the rules and then you punch the numbers into those rules; and that's the way it is because that's the way we told you and don't question it! (PO.I. 2540-2544)

The parallel that Suzanne drew between her school experiences and the teacher's reply to Robert's question could explain why Suzanne did not comment on this episode during the pre-intervention meeting. Equipped with alternative images of teaching the subject (as will be discussed in what follows) and with a better understanding of relative and absolute units (which were the mathematical ideas at stake in Robert's question), Suzanne captured both contestable teacher actions in this episode: the postponing of

Robert's question and her inappropriate justification for using the algorithm in solving division problems.

### *Using Representations*

*Performing.* Suzanne's pre-intervention performance provided little information about her capability of using representations. She mainly commented that when working on fractions she tended to use "pies," which was indicative of her limited cadre of representations when working on fractions. Her post-intervention performance in this practice differed remarkably from her respective pre-intervention performance.

Suzanne selected suitable representations for both the explanations she provided (for the quotient and for the traditional algorithm for the division problem  $2 \div \frac{3}{4}$ ); interestingly enough, these representations were not the "pies" she was inclined to use at her entrance to the program. The cups she used in Figure 5.31 to explain the quotient reflected more the superficial than the structural characteristics of the situation at hand (e.g., it is not obvious why the small cup drawn in pencil is two thirds of the blue cup or why the blue cup contains three fourths of the bottle). However, the representation she used in Figure 5.32 suggests that Suzanne was also capable of working at a more abstract level and emphasizing the structural characteristics of the situation at hand (e.g., the L-shaped polygon she used made clear both the relationship of the divisor units to the dividend and what the remainder represented in terms of divisor units). Overall, Suzanne's work on explaining the quotient suggests that when her comprehension of the content was thorough enough, she used the visual representations she employed in a more conceptually oriented manner. Instead, when her understanding of the content was weak,

her representation use was numerically driven as implied by both her drawing in Figure 5.33 and by her attempt to identify the four thirds in Figure 5.34.

Regardless of what was driving her use of representations – the concepts and/or the numbers – both her explanations suggest that she was meticulous in drawing connections among her verbal descriptions, the drawings she was using, and the numbers involved in the division problem at hand. Take, for instance, her work in Figure 5.31: while drawing this representation, Suzanne ensured that both her verbal explanation and the numbers in the algorithm were aligned with her visual. Even more, after being prompted for more clarification, she labeled these numbers using different names for the two units under consideration (i.e., liters and servings). Consider also her labeling of the leftover parts in Figure 5.32 as *A* and *B*. This labeling was particularly helpful, since it enabled her to show how the leftover fourths covered two thirds of the  $\frac{3}{4}$ -unit. Finally, her work in Figure 5.34 is also suggestive of her attempts to build connections between her drawings and the numbers involved in the equation of this division problem (i.e.,  $2 \div \frac{3}{4} = 2 \cdot \frac{4}{3}$ ).

*Noticing and Interpreting-Evaluating.* Suzanne’s increased sensitivity to issues of mapping between different forms of representations (words, drawings, and numerical symbols) was also evident in her performance in the noticing and interpreting-evaluating tasks of this practice. Unlike her pre-intervention performance in which she talked about the lack of connections only in the second episode, during the post-intervention meeting she provided a thorough analysis of the lack of such connections in both episodes.

Suzanne directly noticed the lack of connections in Amanda’s work for *all* the numbers involved in this student’s mathematical sentence: “[Amanda] doesn’t make any



correlation to what the half signifies, what that sixth signifies, and what the ... resulting three signifies” (PO.I. 1722-1725). As she mentioned, if she were the teacher, apart from clarifying these connections, she would help her students build connections among the mathematical sentence, the graphical representation, and the actual ribbon problem:

I would take the actual real manipulative and ... show visually a correspondence between what I've done with a manipulative and visually. And I would show the correspondence between the math sentence and the graphic. (PO.I. 1730-1735)

In considering Amanda and Julia's episode, Suzanne remarked on the lack of connections in the two students' work and claimed that if she were teaching, she would prompt the two girls to make the connections among their drawing, the mathematical sentence they wrote on the board, and the word problem at hand explicit:

I'd stop [Amanda and Julia] and say, “Why would [you] divide [the line] into twelve pieces? ... Then how many of those twelfths do you have in the whole red piece? And why are you counting every two [twelfths] instead of counting every one?” And do a comparison, one-to-one comparison of -- reminding them what they're doing: they're finding how many of these badge-sized ribbons they can fit in there; and that's not clear. I would ask them every step of the way where they came up with that, why they did that. And what the answer means. (PO.I. 2080-2098)

Suzanne's close attention to and analysis of the different types of connections lacking in these episodes was informed not only by her better understanding of the content but also by pertinent work done in the two intervention courses. This was reflected, in particular, in Suzanne's comment about the different types of connections ought to be made explicit when using representations to scaffold student thinking:

I think there has to be three separate parts when you're doing a visual representation: You have to have the actual thing that you're doing, the concept. ... Then you have to have the visual representation, and it has to ... have a one-to-one correspondence with the actual physical manipulative. Then have the number sentence and those symbols have to one-on-one correspond to the visual, so that [students] can make the connection between this concrete manipulative ... and the mathematic formula. (PO.I. 1874-1884)

Given that these were the three types of connections discussed during the two intervention courses, Suzanne's comment suggests that the intervention offered her some

general guidelines and some images of teaching on how she could use representations more effectively to support student learning.

### *Analyzing Students' Work and Contributions*

*Performing.* Suzanne's post-intervention performance in the practice of analyzing students' work and contributions did not match the notable improvements she exhibited in the previous practice. Like her pre-intervention performance, and despite her increased understanding of relative and absolute units, her post-intervention analysis was still answer-driven; yet, her better understanding of the content helped her appropriately analyze the remainder in Michelle's solution and make valid assertions about this student's grasp of the content taught in the virtual lesson.

When asked to analyze the three student solutions, Suzanne first figured out the answer to the division problem under consideration because, as she explained, "I gotta figure out how to do it myself before I can look at theirs" (PO.I. 24425-2446). Comparing her work to Robert's, she remarked that Robert's answer was correct but that he did not convert the improper fraction in his quotient into a mixed number; she thus concluded that he understood how to apply the algorithm. Similar to her pre-intervention performance, she identified the error in Ann's work but attributed it to Ann's inverting the sequence of applying the two procedures involved in solving this problem. According to Suzanne, Ann first needed to convert the mixed number into an improper fraction and then take the reciprocal of this improper fraction. Yet, she overlooked that Ann took the reciprocal of the dividend and not the divisor. Even when asked whether Ann's answer would have been correct, had she followed the same steps but in a different order,

Suzanne still did not identify the actual error in Ann's solution and proposed reminding Ann of the order in which the two procedures ought to be applied.

The major differences between Suzanne's pre- and post-intervention performance in this practice pertained to her analysis of Michelle's solution and the assertions she made about this student's understanding. While in the pre-intervention meeting Suzanne incorrectly argued that the remainder in Michelle's solution represented two twelfths, in the post-intervention meeting Suzanne appropriately associated this remainder with the quotient yielded from applying the algorithm. She also asserted that Michelle had a better conceptual understanding of the operation considered in this lesson than Robert and Ann, provided that she could also represent her work in a numerical sentence. Without this hypothetical clause, Suzanne's assertion would have been invalid because Michelle might have mechanically followed the visual approach modeled in the lesson. At any rate, Suzanne's assertion was more legitimate than her pre-intervention argument that Michelle understood how to take the reciprocal, which was totally unfounded.

*Noticing and Interpreting-Evaluating.* Like her pre-intervention performance, Suzanne did not identify the misconception latent in June's explanation and Ann's argument; even when directly prompted to comment on June's thinking, she considered June's reasoning appropriate. As in the pre-intervention meeting, she also identified Alan's error only after she was prompted to comment on this student's work on the board; however, she did not comment on how instruction itself might have contributed to Alan's error, as she did during the pre-intervention meeting.

The episode in which Suzanne's pre- and post-intervention performances differed the most related to Amanda and Julia's work. While in the pre-intervention meeting she

could not identify the source of the error in the two girls' work, during the post-intervention meeting she speculated that Amanda and Julia should have gotten four and a half of a yard instead of four and a twelfth, which she verified by applying the traditional algorithm. Once convinced that her speculation was correct, she employed the idea of relative and absolute units and explained the error: "[O]ne twelfth of the yard is really one half of a badge; it's like mixing two different things" (PO.I. 1951-1952).

*Responding to Students' Direct or Indirect Requests for Help*

*Performing.* Compared to her pre-intervention performance, Suzanne's post-intervention performance in Alan's episode reflected her attempt to allow more space for student thinking. Instead of directly telling Alan what his error was, as she did in the pre-intervention meeting, she considered posing questions to help him understand what a sixth represents, which, in turn, she anticipated to help him fathom and correct his error. She also argued that Alan would have not made this mistake if the teacher had followed the approach Suzanne proposed to introduce the problem to her students, namely by using actual ribbon and by having students first show the half yard and then explore how many one-sixth yard segments they could fit in the available half-yard piece.

Regarding the second episode designed for this practice, the comparison of Suzanne's pre- and post-intervention performance suggests that a better understanding of the content supports the teacher in outlining more mathematically valid approaches when responding to students' requests for help. During the pre-intervention meeting, although Suzanne initially struggled with understanding the teacher's suggestion that students use the common multiple of twelfths, she eventually fathomed the applicability of this idea to solving the problem under consideration. Her struggles in this episode sensitized her to

the mathematical ideas at stake; hence, the approach she outlined to support June and Shaun was mathematically valid and allowed room for student thinking. Suzanne's post-intervention performance in this episode could be regarded as a regression from her pre-intervention performance.

Suzanne again spent a considerable amount of time pondering the teacher's suggestion without eventually figuring out how the idea of common multiples applied to the problem at hand. With a weak understanding of the mathematics at play, she mentioned that to circumvent the students' impasse she would suggest that they represent the fourths and the sixths involved in the problem on separate lines instead of fitting them into one line (as they were shown doing). From her perspective, this would allow them to explore how many one-sixth pieces they could fit into the three-fourths segment. Even if students represented the sixths and the fourths into scale, this approach could still not satisfactorily address why the leftover part was half of a badge and not, for example, any other fraction close to a half; in other words, it could provide supporting but not convincing evidence that the fractional part was half the length of a badge. Hence, Suzanne's post-intervention proposal for handling June's and Shaun's questions was mathematically less rigorous compared to her pre-intervention approach.

*Noticing and Interpreting-Evaluating.* Suzanne's post-intervention appraisal of how the teacher supported Alan in resolving his error did not differ from her pre-intervention evaluation of the teacher's work in this episode. She again noticed that the teacher posed leading questions and gave away the answers to these questions by her intonation. Hence, she concluded that Alan simply guessed what the correct answer was without actually having understood the mathematical ideas at stake.

Unlike the first episode, her post-intervention performance in the second episode was different from her pre-intervention performance in two respects. First, at her entrance to the program, Suzanne critiqued the teacher for simply proposing the idea of common multiples without explaining how this idea could support student work. In the post-intervention meeting, she argued that this idea was totally unrelated to the problem the students were trying to solve:

The common multiples is totally out the window, because it has nothing to do with what they've talked about so far. It's totally random. Yes, it'll give you the right answer, and [the teacher] knows that; but it has nothing to do with what they're actually thinking. (P.O.I. 1821-1825)

This argument was not legitimate since the idea of common multiples does help represent and solve division problems whose dividend and divisor are not commensurate.

Obviously, Suzanne's evaluation of the teacher's work was informed by her own difficulties in seeing how this idea could help solve the problem under consideration.

Second, unlike the pre-intervention meeting, Suzanne did not credit the teacher for intervening and offering students a hint to save them from erring and from getting a wrong idea "imprinted in their brains." From this perspective, her post-intervention evaluation of the teacher's work was more aligned with her post-intervention strong agreement with the survey statement that the teacher should leave students to puzzle things out for themselves before intervening. Although she also strongly agreed with this statement in the pre-intervention administration of the survey, during the pre-intervention interview she still endorsed the teacher's approach to intervene and provide students with the common-multiple hint.

#### *Professed Changes and Their Attributions*

Suzanne reported having experienced changes in her conceptual understanding of the content considered in the simulation, in her confidence about doing and learning

mathematics, and in her stance toward the subject. She explained that her augmented understanding of the content, and particularly the concepts underlying the division of fractions, enabled her to perform better in the teaching simulation. She also felt more confident in engaging in mathematics tasks as a result of realizing that when doing math, thinking is more important than getting correct answers. This, in turn, had an impact on her stance toward the subject:

[At the pre-intervention meeting] I had probably said that I totally would dread mathematics and hated it with a passion. And now I find it very interesting, because, like I said, it's not about what you know, it's about how you think. (PO.I. 96-100)

Suzanne attributed these changes to the two intervention courses in particular and to the ELMAC program in general. She explained that the two math courses, and specifically the content course, helped her start thinking about mathematics not from the perspective of getting correct or incorrect answers but from the viewpoint of appraising one's thinking. She identified as catalysts for this change the courses' emphasis on providing explanation and the opportunities she experienced, particularly in the content course, to discuss multiple solutions to the assigned problems. In conjunction, the two intervention courses also helped her appreciate the importance of creating an arena for students to conceptually understand and explain the mathematical underpinnings of the procedures they are asked to apply:

The most important thing I learned ... is you have to ... have an understanding of what you're doing and be able to explain it and know how you're thinking, rather than simply taking someone else's word for it. You have to be able to be critical of yourself and know how to check yourself that you're right by explaining how you did. Rather than ... just taking someone's word for it: "Since this is the rule, follow it!" because it may not be the rule all the time. (PO.I. 2614-2620)

This increased attention to issues of meaning-making and understanding was an antithesis to the type of instruction that Suzanne experienced as a learner of the subject, which resulted in her pre-intervention belief that mathematics is "cut and dry" (PO.I. 89).

Alongside the intervention courses, the other ELMAC courses she took also sensitized her to the importance of creating a productive learning environment for student thinking and reasoning instead of simply feeding them with answers.

In addition to mobilizing these changes, the intervention courses also offered her ideas/images for handling certain instructional situations. For example, in discussing the teacher's reaction to June's explanation, Suzanne criticized the teacher for moving on without eliciting other students' ideas or checking students' level of understanding. She argued that if she were teaching, she would have done a quick end-of-class check to explore whether the students understood June's reasoning. This idea of conducting end-of-class or end-of-activity checks as a means of formative assessment was discussed and modeled during the math methods course; in addition, Suzanne had the opportunity to practice conducting such checks in one of the assignments for the math methods course.

#### *Changes in Performance on the LMT Test: A Closer Look*

Although Suzanne received the same score on both administrations of the LMT test, her work in the two administrations of the test differed in twelve questions: she correctly answered questions 8a, 8c, 11c, 15, 17, and 20 on the second administration of the test but not on the first, and conversely, she correctly answered questions 5, 12, 14c, 14d, 14e, and 25 on its first but not on its second administration.

Her post-intervention test performance revealed both consistencies and inconsistencies with her performance in the teaching simulation. For example, the latter four questions she correctly answered during the second administration of the test concerned interpreting fraction divisions (questions 11c and 15), analyzing the fractional part of quotients (question 17), and identifying different manifestations of units in



fraction problems (question 20). Suzanne's increased understanding of the concept of fraction division and her enhanced grasp of relative and absolute units as captured by her performance in the teaching simulation resonated with her post-intervention performance on these questions. On the other hand, given her performance in the teaching simulation, one would expect her to also have correctly answered question 5, which pertained to connecting a visual representation to a mathematical sentence, questions 14c-14e, which related to evaluating students' interpretations for a division-of-fractions problem, and question 25, which expected PSTs to identify the word problem corresponding to the measurement interpretation of division. These inconsistencies are hard to explain, especially given that Suzanne showed her work on only the latter question. Her notes on this question suggest that she incorrectly answered this question not because of her failure to identify the word problem that represented the measurement interpretation of division, but because she incorrectly identified two such problems as reflecting this division interpretation.

### *Analytical Commentary*

Suzanne was a divergent case according to both her entrance performance and the changes in her performance. The growth in her performance in the teaching simulation, particularly in the practices of providing explanations and using representations, was not aligned with the lack of growth in her knowledge, as suggested by her pre- and post-intervention scores on the LMT test. Yet, a comparison of the questions that Suzanne answered correctly on the pre- and post-administrations of the test revealed some notable increases in her knowledge, which were consistent with the changes in her teaching performance; at the same time this comparison pointed to some inconsistencies between

the growth in her knowledge, as suggested by her post-intervention interview, and the regression in her knowledge, as implied by the questions she correctly answered on the first but not the second administration of the LMT test. Thus, Suzanne's case raises some issues of measurement; at the same time, however, her case partly corroborates the knowledge and teaching performance association considered in this chapter. Her case also helps explore how changes in knowledge, beliefs, and images of teaching – the three factors found to primarily influence the PSTs' performance – were reflected in her post-intervention performance in the teaching simulation. I elaborate upon each issue in turn.

Considering only Suzanne's *numerical* score on the two administrations of the LMT test could lead to the unwarranted conclusion that she exhibited no changes in her knowledge, since her score was the same on both test administrations. Yet, the analysis of the questions she correctly answered each time points to some increases in her knowledge particularly with respect to the notion of the relative and absolute units; the enhancement in her knowledge suggested by her performance on these questions was consistent with her performance in the teaching simulation. On the other hand, the questions she correctly answered during the first but not the second test administration appear to imply a regression in her understanding of (fraction) division. This regression was inconsistent with her performance in the teaching simulation, which, if anything, showed that Suzanne's knowledge of the division of fractions was more thorough in the post- rather than in the pre-intervention meeting.

Both the consistencies and the inconsistencies in Suzanne's performance raise some issues of measurement, just like her pre-intervention performance did. On the one hand, the consistencies challenge the approach of simply associating a numerical score

assumed to reflect teachers' knowledge with their teaching performance; these consistencies also suggest that a more in-depth analysis of the questions that teachers answer could yield richer and perhaps more reliable information about teachers' knowledge. On the other hand, the inconsistencies identified in Suzanne's performance and the limitations of the study data to explain them imply that asking teachers to simply answer multiple-choice questions might not be enough; instead, teachers should also be required to *present their work and thinking* on the questions developed to tap their knowledge. This additional requirement might yield better insights into the association between knowledge and teaching.

Despite the issues of measurement that Suzanne's case raises, and regardless of her being a divergent case, her post-intervention performance provides both affordance and constraint examples that support the strong association between knowledge and teaching performance reported in the first section of this chapter.

In Suzanne's case, the most compelling piece of evidence substantiating the association between knowledge and teaching performance stems from her performance in the practices of providing explanations and using representations. Armed with a better conceptual understanding of division of fractions and of relative and absolute units in such divisions, Suzanne offered a conceptually driven explanation for the quotient in the division problem  $2 \div \frac{3}{4}$  (for the most part) and used the representations she employed accordingly. At the same time, her weak understanding of the notion of the reciprocal appeared to prevent her from providing an equally conceptually undergirded explanation for the division-of-fractions algorithm. In this latter case, Suzanne was seen trying to match the numbers involved in this algorithm with the representations she was using.

Yet, drawing on her better understanding of (fraction) division and of relative and absolute units, she eventually proposed an acceptable explanation even for this algorithm. In conjunction, her performance in these two practices supports propositions  $A_1$  to  $A_3$  of Table 4.6, which concern how teachers' knowledge can aid or constrain them in providing conceptually informed explanations, and proposition  $B_2$ , which suggests that limitations in teachers' knowledge are reflected in their use of representations when providing explanations.

Suzanne's performance in the practice of analyzing students' work and contributions also provides affordance and constraint examples corroborating the association at hand. Being more knowledgeable of the idea of relative and absolute units, Suzanne successfully identified that in their solution Amanda and Julia confounded these two different units. In contrast, probably ignoring that the dividend can also be smaller than the divisor (as also suggested by her performance in analyzing the 16 exercises of the first textbook page), she considered June's explanation satisfactory, thus failing to notice the latent misconception in this student's thinking.

Suzanne's pre- and post-intervention analysis of Michelle's solution also offers an affordance and a constraint example. Without a strong understanding of relative and absolute units, during the pre-intervention meeting, Suzanne inappropriately interpreted the fractional part of Michelle's solution and made unfounded assertions about this student's understanding. With a better sense of the notion of units, during the post-intervention meeting, Suzanne analyzed Michelle's work quite decently and made valid assertions about this student's thinking.

Overall, Suzanne's performance in the practice of analyzing students' work and contributions supports propositions  $C_2$  and  $C_4$ , which concern the association between teachers' knowledge and their capacity to analyze students' unconventional ideas and misconceptions. However, at the same time, Suzanne's post-intervention performance *challenges* propositions  $C_1$  and  $C_3$ . Despite the increases in her understanding, Suzanne still employed an answer-driven approach to analyze students' work and particularly Robert's and Ann's. Also, regardless of the enhancement in her knowledge, she still analyzed Ann's work incorrectly due to this answer-driven approach she pursued.

Suzanne's post-intervention performance in the two MTF-related practices also supports the association between knowledge and teaching performance. For instance, for the practice of selecting and using tasks, despite the elaborated plan she proposed to support her students' work on figuring out the traditional algorithm, she did not address the *concept* of the reciprocal; this limitation in her approach should be considered in light of her own weak understanding of this notion. In the same vein, while she started seeing some of the 16 exercises of the first textbook page from a more conceptual perspective, her understanding of fraction division was not sufficiently robust to allow her use these exercises from a more conceptual perspective (e.g., she did not attend to the size of the dividend relative to the divisor). On the affordance side, one could argue that her consideration of the second page's direction on explaining the fractional part of the quotient as an asset of this page was apparently informed by her enhanced post-intervention understanding of this concept.<sup>186</sup> Overall, Suzanne's performance in this practice lends support to propositions  $D_1$  and  $D_2$ , which suggest that a strong knowledge

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<sup>186</sup> Yet, it is still questionable why, regardless of her increased grasp of this notion, she incorrectly associated this direction with certain tasks of this page.

base allows teachers to identify the affordances of the available curriculum tasks and restructure tasks that have limited potential to promote students' conceptual understanding.

The comparison of Suzanne's pre- and post-intervention responses to June's and Shaun's requests for help suggests that her latter approach was mathematically inferior to the former. This difference should be considered in light of her understanding of the mathematical ideas at play in this episode. While during the pre-intervention meeting she constructed in situ the mathematical ideas considered in this episode, Suzanne was not equally successful at doing so in the post-intervention meeting. Thus, the juxtaposition of Suzanne's pre- and post-intervention performances corroborates proposition  $E_1$ , which holds that a thorough understanding of the mathematical ideas at stake supports teachers in responding to students' questions in mathematically valid ways.

The changes in Suzanne's teaching performance, however, were not merely due to the changes in her knowledge. The analysis of her post-intervention interview revealed that in addition to enriching her understanding of the content considered in the simulation, the intervention courses also offered her alternative images of teaching the subject. From this perspective, one could argue that these courses helped Suzanne relearn mathematics. Because this relearning appeared to inform her post-intervention performance, I paraphrased Suzanne's pre-intervention argument, "You're gonna teach the way you learned," and entitled her case "You're gonna teach the way you (*re*)learned." Below, I consider the impact that this relearning had on Suzanne's beliefs about teaching and learning mathematics and her attitudes about the subject; I then discuss how the changes in her beliefs and attitudes, alongside specific images of

teaching she gleaned from the intervention courses, manifested themselves in her performance in the teaching simulation.

The post-intervention interview provided ample evidence about how Suzanne experienced the teaching of mathematics as a student. As she reported, mathematics was taught to her as “cut and dry” with emphasis placed on getting correct answers. According to Suzanne, in this mode of instruction there was little room for discussing ideas and explaining one’s thinking; what mattered most was to learn and mechanically apply the rules presented by the teacher without questioning them. With an emphasis on providing and evaluating explanations, on making one’s thinking the object of inquiry, and on sharing and comparing multiple solutions to a single problem, the intervention courses offered Suzanne an alternative perspective of teaching the subject. This perspective, which privileged thinking and reasoning over getting correct answers, and understanding and explaining over mechanically following rules, appears to have mobilized changes in Suzanne’s beliefs and attitudes.

Suzanne reported having experienced both an improvement in her efficacy beliefs about engaging in mathematics tasks and a shift in her beliefs about teaching and learning the subject. While at her entrance to the ELMAC program she appeared to conceive of mathematics as an enterprise of showing, telling, and intervening to minimize student error and confusion, during the post-intervention meeting, she perceived mathematics and its teaching from a more dynamic and constructivist perspective: she argued that students should be offered opportunities to explore, construct, and explain ideas instead of being spoon-fed with answers. Likewise, while at her entrance to the program she reported having had an aversion toward the subject, after the culmination of the intervention

courses she claimed this aversion to have been replaced by an increased interest in doing and learning mathematics. The aforementioned changes in her beliefs and attitudes apparently had an impact on her performance in the teaching simulation.

Consider, for instance, her performance in selecting and using tasks. Instead of presenting the traditional division-of-fractions algorithm to the students, Suzanne designed an activity to lead students to this algorithm without necessarily doing the thinking for them. As she remarked, she would present this algorithm to students only if they faced significant challenges in figuring out the algorithm for themselves. Similarly, whereas during the pre-intervention meeting she would support Alan by simply pointing to and correcting his error for him, during the post-intervention meeting she proposed an approach that could help Alan understand and correct his error. In the same vein, while in the pre-intervention meeting she commended the teacher for designing and enacting activities that supported student thinking and learning, during the post-intervention meeting Suzanne identified several instances in which the teacher shifted emphasis from meaning and understanding to manipulating numbers and following steps.

Suzanne's performance nevertheless appeared to still be informed by some (latent) beliefs she had nurtured through her long apprenticeship in classes focusing on promoting students' procedural competence. A case in point is how she proposed using the 16 exercises of the first textbook page. Despite all the conceptual work she considered doing on fraction division, she thought of using these exercises from a more procedural perspective, but even more, she would transition to these exercises by proposing that students simply follow the "shortcut" (i.e., the algorithm) without being concerned about the "whys" underlying this operation. This recommendation suggests



that Suzanne might have regarded the conceptual and the procedural aspects of learning as separate (i.e., first you learn the meaning and then you simply apply the procedures), thus ignoring the significant benefits residing in working at the interface of both the concepts and the procedures. To the extent that this hypothesis holds, it reflects how Suzanne accommodated the ideas she gleaned from the intervention courses with her existing beliefs about teaching and learning. That is why in the title of her case I used the term *(re)learning* – deliberately keeping the prefix “re” in parentheses – to acknowledge that one’s previous learning of the subject cannot be totally wiped out and that how one learned and relearned the subject might *both* inform one’s teaching performance.

In addition to the indirect effect that the images that Suzanne gleaned from the intervention courses appeared to have on her teaching performance (through mobilizing changes in her beliefs and practices), these images of teaching seemed to also have a direct effect. For instance, the analysis of Suzanne’s performance in explaining the quotient for the division problem  $2 \div \frac{3}{4}$  suggests that her performance encompassed ideas and criteria discussed *and practiced* during the intervention courses. To name a few: she started her explanation with outlining the concept under consideration; she moved “step-by-step,” thus avoiding big jumps and gaps that could confuse students; she used diagrams and symbols to better communicate the concepts she was trying to explain, and she drew connections between the different forms of representations she employed; finally, she strove to calibrate her explanations to an average sixth grader by clarifying even little intermediate steps and by using language and representations that could be comprehensible to such a student.

Similarly, her performance in all the tasks associated with the practice of using representations pointed to her increased awareness of issues of mapping between different representations. Consider, in particular, her reference to the three different types of representations and the need that connections be drawn among them. Given that during the pre-intervention meeting she largely ignored such issues, her increased awareness of issues of mapping among different representations could be considered a byproduct of the intervention courses. But Suzanne was not only able to talk about building such connections; she was also quite meticulous in drawing such connections when using representations in the performing tasks of the teaching simulation.

Figure 5.35 summarizes Suzanne's post-intervention profile and presents in italics the changes in her performance. In particular, this figure suggests that her images of teaching were expanded and that, in turn, they informed her beliefs about teaching and learning mathematics, her disposition toward this subject, and her efficacy beliefs about engaging in mathematics tasks. This figure also shows that Suzanne experienced most of her performance gains in the practices of providing explanations and using representations. Changes were also observed in her performance in the practice of selecting and using tasks and the practice of analyzing students' work contributions; yet, these changes were modest compared to the changes in her performance in the previous two practices. Finally, Suzanne's performance in the criterion of providing mathematically sound responses for the practice of responding to students' requests for help was somewhat better in the pre-intervention rather than in the post-intervention meeting; Figure 5.35 illustrates this regression in brackets.

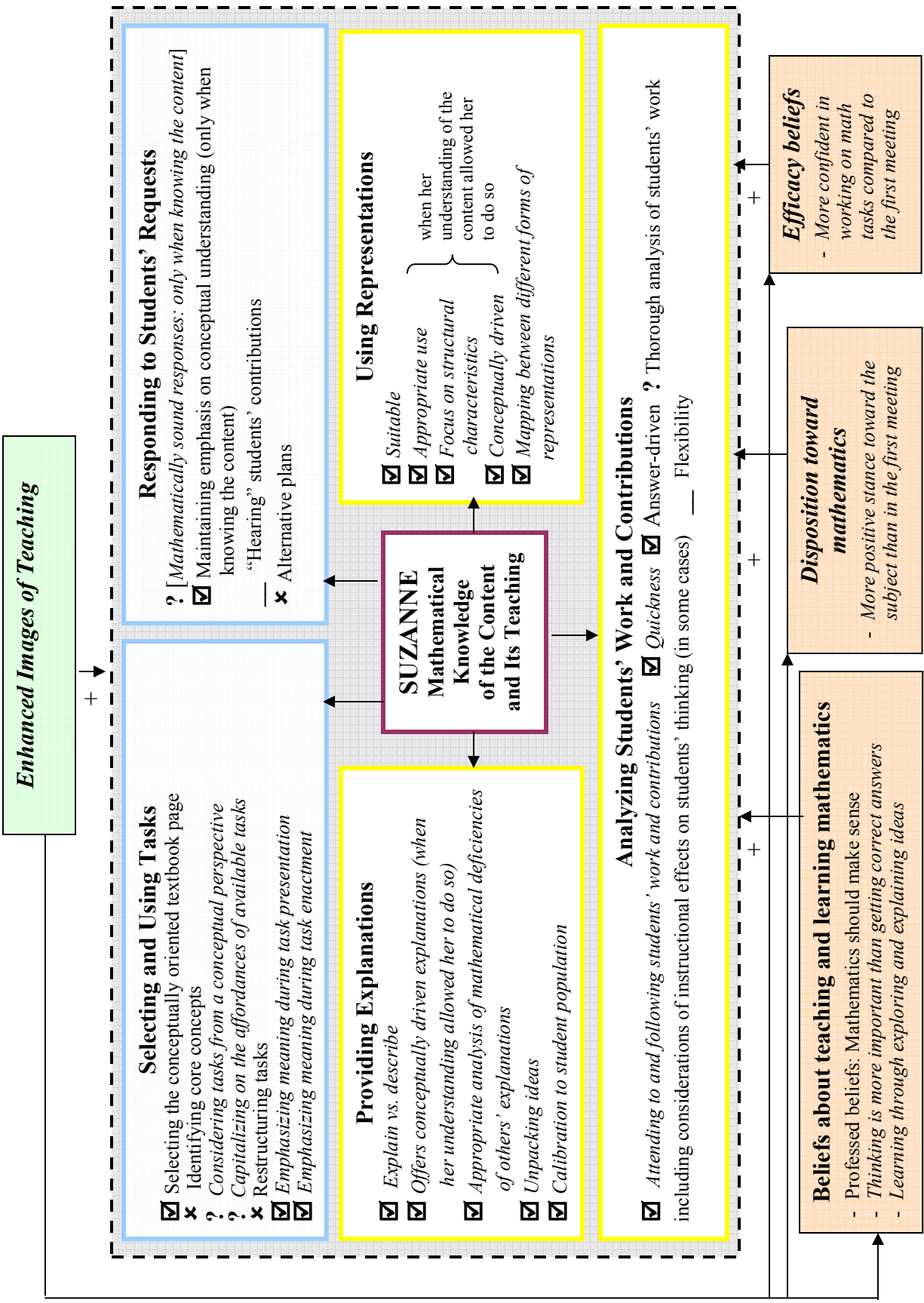


Figure 5.35. Considering the association between knowledge and teaching performance through Suzanne's post-intervention profile.

## Exploring the Association between the Changes in the PSTs' MKT and Performance in the Five Teaching Practices from a Qualitative Standpoint: A Cross-Case Analysis

The second research question of this study (RQ2) explores the association between the changes in the PSTs' MKT and the changes in their performance in the teaching simulation, after participation in an intervention designed to leverage changes in both knowledge and performance in the MKT-related practices. This section contributes to RQ2 by presenting the results of a cross-case analysis, which looks across the changes in the seven PSTs' knowledge and performance, as outlined in the previous section of this chapter. This section consists of three parts. In the first part, I use the classification schemes developed in Chapter 4 to summarize the changes in the PSTs' performance in the teaching simulation. Building on this summary, in the next part, I explore the association between the changes in the PSTs' knowledge and performance, while also considering factors, besides knowledge, that appear to have mobilized the changes in teaching performance. In the last part, I revisit the 18 propositions developed in Chapter 4 and discuss them in light of the findings of this chapter.

### *Revisiting the Five Classification Schemes: A Synthesis of the Changes in the PSTs' Performance*

The changes in the PSTs' performance in the MKT- and the MTF-related practices are outlined in Figures 5.36 and 5.37, respectively. These figures replicate Figures 4.33 and 4.34, which classified the PSTs' pre-intervention performance into the schemes developed for each of the five practices this study considers (due to space considerations, the description of the schemes' categories is omitted). Figures 5.36 and 5.37 additionally classify the PSTs' post-intervention performance in each of the five

practices, thus allowing the changes in their teaching performance to be considered.

Below, I briefly explain the post-intervention classification of the PSTs' performance in each scheme and discuss the changes in their performance.

### *Providing Explanations*

The study participants were asked to provide two explanations, one for the quotient of the division  $2 \div \frac{3}{4}$  and another for the traditional division-of-fractions algorithm for this same division problem. Figure 5.36 shows that during the post-intervention meeting, Nathan, Nicole, and Suzanne provided conceptually driven and unpacked explanations that could be comprehensible to a sixth grader; thus their performance was classified in the upper category of this scheme. Although Suzanne used the division algorithm to figure out the division's quotient and then complete her explanation, her explanation was not considered numerically driven because she used the numerical answer to this problem as an auxiliary tool, when she sensed that her explanation of the fractional part of the quotient was incorrect. A similar conclusion was also made for Nicole's performance.

In contrast, Deborah and Tiffany depended more on the numerical answer of this division when developing their explanations; yet, in their final stages, their explanations were conceptually undergirded, sufficiently unpacked, and potentially comprehensible to a sixth grader. Figure 5.36 captures both the development of these PSTs' explanations and the final stages/products of this process. Viewed from the perspective of their development, Tiffany's and Deborah's explanations are classified between the second and the third categories since, in developing these explanations, these PSTs drew on both relevant concepts as well as the numerical answer to this division problem. The dashed

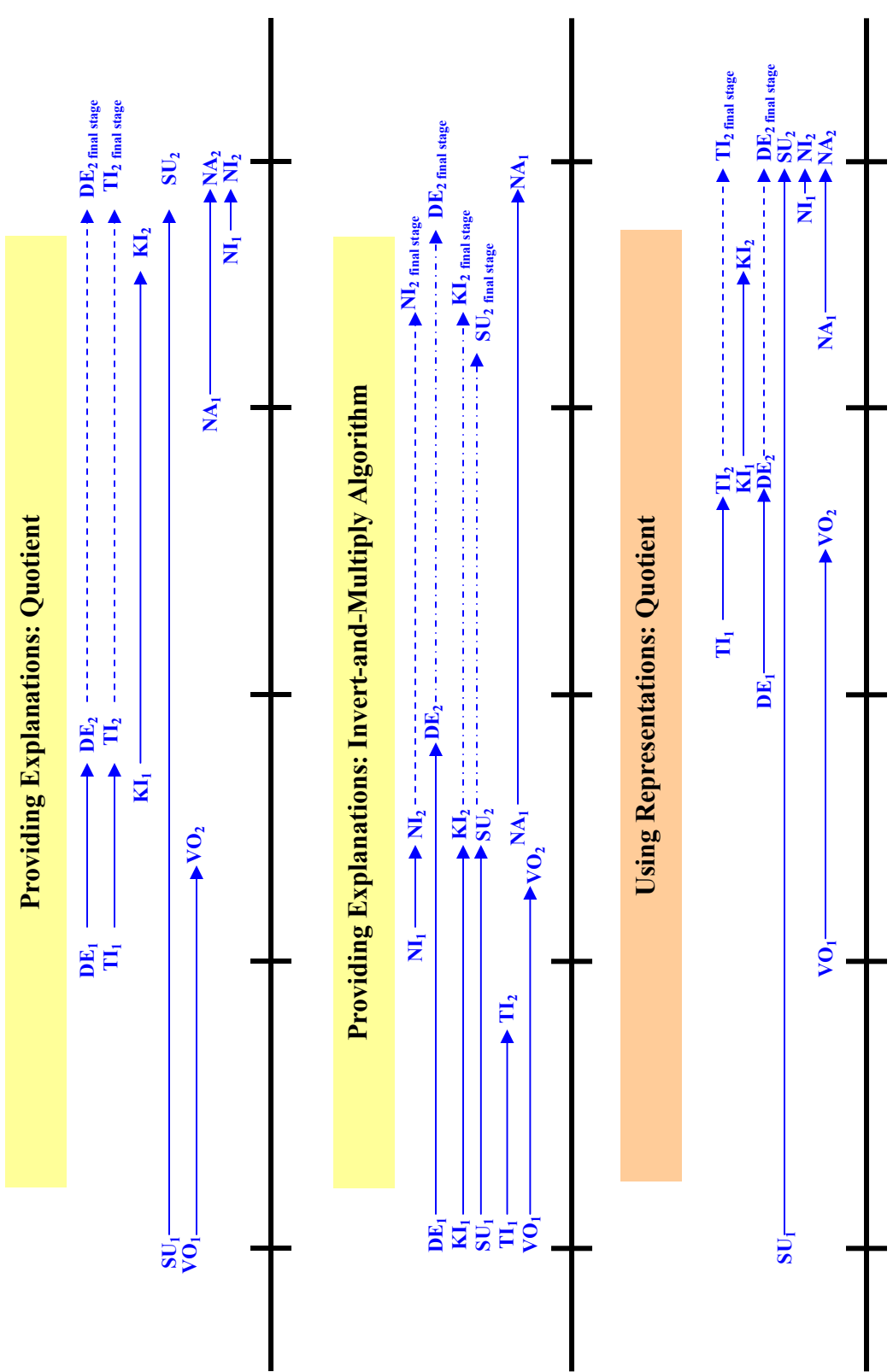
lines suggest that in their final stages the explanations of these PSTs could be classified in the upper category of this scheme.

Although conceptually driven throughout, Kimberley's explanation is located between the fourth and fifth category because it was not sufficiently unpacked (e.g., Kimberley did not clearly specify the concept under consideration). Vonda's explanation is situated between the second and the third categories, because, although it was informed by the concept of fraction division, it largely depended on the numerical value of the quotient; recall, for example, her attempt to represent the quotient by using a representation partitioned into thirds instead of fourths, as well as her lack of explicitness about the fractional part of the quotient.

In regard to the PSTs' explanation for the invert-and-multiply algorithm, Figure 5.36 shows that only Nathan directly proposed a conceptually driven explanation that was sufficiently unpacked; thus, he is placed in the upper category of this scheme. When asked to provide such an explanation, Nicole, Kimberley, and Suzanne largely drew on the numerical value (i.e.,  $\frac{4}{3}$ ) instead of the notion of the reciprocal, as revealed by their attempts to show thirds (in absolute units) in their pictures. Nevertheless, the concept of relative and absolute units helped them eventually propose an explanation that was conceptually driven rather than numerically grounded. Because in their initial stages the explanations of these PSTs were informed by both numbers and concepts, these explanations are presented between the second and third categories. In their final stages, these explanations differed in their level of unpackedness, with Kimberley's and Nicole's being closest to Nathan's, who explicitly articulated the notion of the reciprocal as the number of relative units that could be accommodated in each absolute/dividend unit.

Like Nicole, Kimberley, and Suzanne, Deborah initiated her explanation without a clear understanding of the notion of the reciprocal; hence, she developed her explanation by both drawing on the concept of relative and absolute units and by experimenting with a selected set of division problems. Yet, in contrast to the latter three PSTs she used the division problems and the numbers involved in them to scaffold her conceptual understanding of the reciprocal. Hence, even in its initial stage, Deborah's explanation for the invert-and-multiply algorithm gravitated more toward the concepts rather than the numbers. In light of this, the initial stage of her performance is presented on the right of the initial stages of the other three PSTs' explanations, as shown in Figure 5.36. Her final explanation was also more unpacked than the three PSTs' explanations; thus, it lies closer to the fifth rather than the fourth category of this scheme.

Vonda's explanation was greatly depended on the numbers comprising the reciprocal; it was also somewhat informed by the concept of relative and absolute units. Hence, in Figure 5.36, her explanation is situated between the second and the third categories of this scheme, but closer to the second category, to denote its heavy dependence on the numbers. Despite some glimmers she had when experimenting with the division  $\frac{1}{2} \div \frac{1}{6}$ , Tiffany could provide no explanation for the division algorithm for the division problem under consideration. Because her work in developing an explanation was largely grounded in the numbers of the division problems she considered, her performance is situated close to the second category.

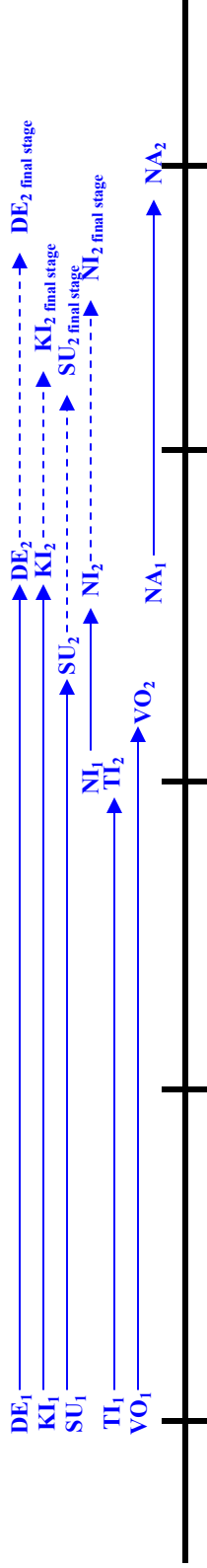


**Legend:**  
**DE:** Deborah; **KI:** Kimberley; **NA:** Nathan; **NI:** Nicole; **SU:** Suzanne; **TI:** Tiffany; **VO:** Vonda  
**1.** Pre-intervention performance; **2.** Post-intervention performance

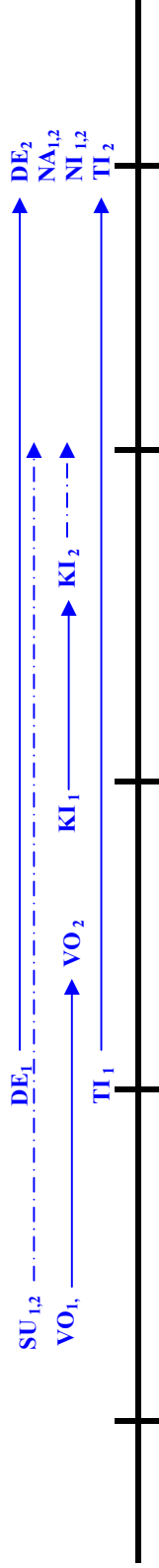
Figure 5.36. Pre- and post-intervention classification of the PSTs' performance in the schemes of the MKT-related practices.



Using Representations: Invert-and-Multiply Algorithm



Analyzing Students' Work and Contributions



**Legend:**

DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda

1. Pre-intervention performance; 2. Post-intervention performance

Figure 5.36. Pre- and post-intervention classification of the PSTs' performance in the schemes of the MKT-related practices (continued).

Regarding the changes in the PSTs' performance, Figure 5.36 suggests that Suzanne demonstrated the greatest changes in her explanation of the quotient; Deborah, Tiffany, Kimberley and Vonda also exhibited notable changes in this task. In contrast, the changes in Nathan's and especially Nicole's explanations were smaller. In terms of the explanation of the traditional division algorithm, the greatest performance gains were observed in Deborah; Kimberley, Suzanne, Nicole, and Nathan also exhibited considerable gains in their performance. However, the gains for Nicole and Suzanne should be interpreted with caution because the interviewer's prompting might have been consequential for helping these PSTs move from the initial to the final stages of their explanations. Tiffany and Vonda experienced the smallest gains in explaining this algorithm.

#### *Using Representations*

Nathan's, Nicole's, Suzanne's, Deborah's, and Tiffany's performance in using representations when explaining the quotient met the criteria for the upper category of the respective scheme since their use of representations was grounded in the concept of fraction division and the concept of relative and absolute units; furthermore, they drew connections between their illustrations and the numbers involved in the division problem. Deborah's and Tiffany's *initial* use of representations, however, is situated between the third and the fourth categories of this scheme since in their representations of the fractional part of the quotient both PSTs drew on the numerical value of this quotient. Kimberley's post-intervention performance is situated between the fourth and fifth categories of this scheme because, although Kimberley anchored her use of representations by pertinent concepts, she was not successful enough at drawing

correspondences between her drawing and the numbers involved in the division problem she was explaining. On the other hand, despite being somewhat informed by pertinent concepts, Vonda's use of representations when explaining the quotient was greatly dependent on the numerical value of this quotient; hence, her post-intervention performance lies between the third and fourth categories of this classification scheme.

Concerning the task of representing the invert-and-multiply algorithm, of the seven PSTs, only Nathan directly grounded his representation in the concept of the reciprocal and connected it to the numbers involved in the algorithm; thus, his performance is classified under the upper category of this scheme. Deborah, Kimberley, Nicole, and Suzanne also grounded their representations in pertinent concepts. However, unlike Nathan, their use of representations was initially driven by the numbers involved in the traditional division algorithm; hence, their initial performance is situated in between the third and fourth categories. The proximity of their performance classification in either the third or the fourth category reveals how dependent these PSTs' use of representations was on the numbers involved in or the concepts underlying this algorithm. For example, Deborah's performance is closer to the fourth category because, although she employed several division problems to inform her explanation (and consequently her use of representations), she used these examples in a more conceptually driven manner, attempting to decipher the concept of the reciprocal. Although the final stage of the performance of these four PSTs is presented somewhere in between the fourth and the fifth category, their performance differs in terms of the extent to which they drew explicit connections between their representations and the numbers involved in the traditional algorithm. In this regard, Deborah's use of representations was the most

explicit and thereby is presented closer to the fifth category; Kimberley's was the least explicit in pointing to these connections and is therefore presented closer to the fourth category.

Vonda's use of representations, like her explanation, was greatly depended on the numbers involved in the traditional algorithm; recall, for example, that, although she alluded to the relative and absolute units, she mostly talked about "fours" and "threes," focusing on the two numbers comprising the reciprocal in the division problem  $2 \div \frac{3}{4}$ . Therefore, her post-intervention performance is situated between the third and the fourth categories of this scheme but closer to the third category. In explaining the reciprocal, Tiffany used the representations she employed in a numerically driven manner (i.e., she was exploring how to make her drawing match the numbers of the traditional algorithm); hence, her post-intervention performance is situated in the third category.

Figure 5.36 shows that Suzanne experienced the highest gains among the seven PSTs in using representations for the quotient; Suzanne was followed by Deborah, Tiffany, and Vonda. In contrast, Kimberley, Nathan, and Nicole exhibited small performance gains, with Nicole experiencing the lowest gains among the seven PSTs. For the traditional division-of-fractions algorithm, the figure indicates that Deborah exhibited the greatest gains, followed by Kimberley, Suzanne, Vonda, Tiffany, and Nicole. Although Nathan also experienced notable gains, the changes in his performance were the lowest compared to his counterparts. The changes in Suzanne's and Nicole's performance again need to be interpreted cautiously since the interviewer's prompting might have influenced their transition from the initial to the final stages of their representations use.

### *Analyzing Students' Work and Contributions*

Deborah's, Tiffany's, Nathan's, and Nicole's post-intervention performance in analyzing students' work and contributions is included in the upper category of the respective classification scheme. These PSTs appropriately analyzed both conventional and unconventional student contributions, made appropriate assertions about these students' understanding, identified misconceptions in students' contributions, and considered instruction as a plausible source for student error and misconceptions. Kimberley's performance, on the other hand, is situated between the third and the fourth categories because, while she correctly analyzed both conventional and unconventional student ideas, her assertions about student understanding were not always on target (e.g., she argued that Ann's work warranted just a quick fix). Even if Kimberley's argument regarding Ann's work is ignored, Kimberley's performance could at best be included in the fourth category, given that Kimberley was not successful at identifying the misconception in June's explanation (in Figure 5.36, this is illustrated by the dashed and dotted line next to Kimberley's post-intervention performance).

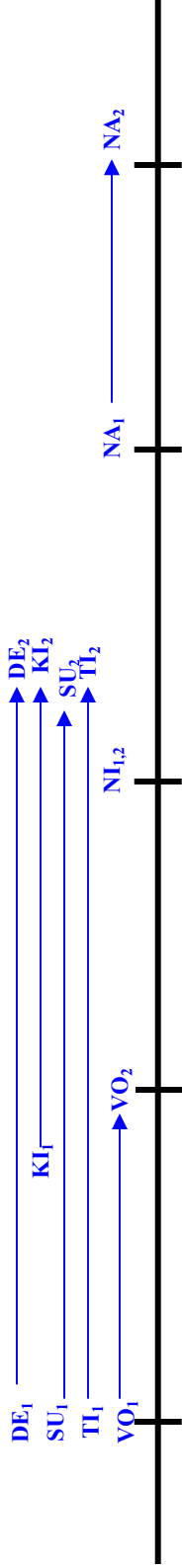
Vonda's post-intervention performance is situated between the second and the third categories of this scheme because of her incomplete analysis of Michelle's unconventional solution. Although Suzanne appropriately analyzed Michelle's solution and made some appropriate assertions about this student's understanding, her post-intervention performance remains unchanged from her pre-intervention performance, since in both interviews she failed to correctly analyze Ann's conventional solution. If Suzanne's analysis of Ann's work and her assertions about Ann's understanding are ignored, Suzanne's performance could be included in the fourth category.

Figure 5.36 shows that Deborah experienced the greatest gains in her performance in this practice, followed by Tiffany. Kimberley and Vonda experienced moderate gains, while Nathan and Nicole experienced no gains, since they were already clustered at the upper end of this scale even for their pre-intervention performance. Placing Suzanne's pre- and post-intervention performance at exactly the same spot in Figure 5.36 would give the wrong impression that Suzanne experienced no gains in her performance; therefore, the dashed and dotted line in Figure 5.36 denotes Suzanne's performance gains had she appropriately analyzed Ann's conventional solution. This dashed and dotted line indicates that Suzanne's performance gains are comparable to those of Tiffany's and Deborah's.

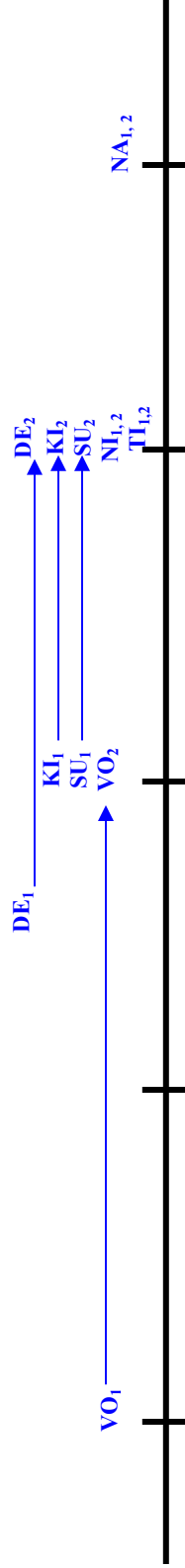
### *Selecting and Using Tasks*

Figure 5.37 suggests that only Nathan's post-intervention performance meets the criteria for the upper category of the *Selecting and Using Tasks* classification scheme. In contrast to his counterparts, Nathan not only identified the affordances and limitations of the two textbook pages and proposed ways to restructure the pages' limitations, but also considered issues that could support him in maintaining the curriculum tasks' potential during enactment. For example, he considered issues of task sequencing, which would allow him support students in more smoothly constructing the concepts underlying the division of fractions. Similarly, he correctly identified tasks B<sub>1</sub> and B<sub>2</sub> as not particularly conducive to helping students draw representations and discover the invert-and-multiply algorithm. He also considered two approaches to scaffold student work on task D, which is a particularly demanding task.

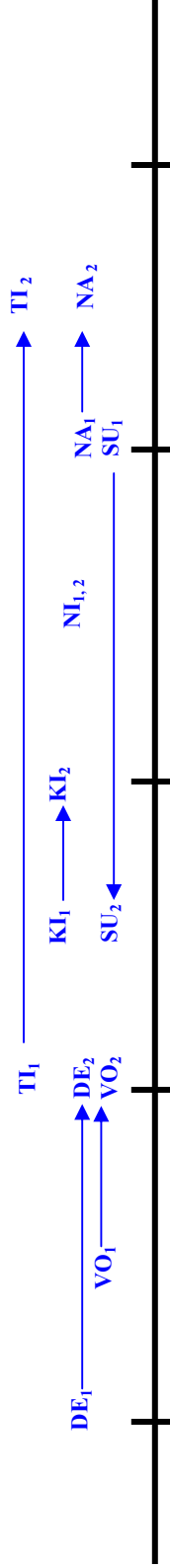
Selecting and Using Tasks



Responding to Students' Requests for Help: Alan's Request



Responding to Students' Requests for Help: June's and Shaun's Request



**Legend:**

- DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda
- 1. Pre-intervention performance; 2. Post-intervention performance

Figure 5.37. Pre- and post-intervention classification of the PSTs' performance in the schemes of the MTF-related practices.

The post-intervention performance of four PSTs – Deborah’s, Kimberley’s, Suzanne’s, and Tiffany’s – is situated between the third and the fourth categories of this scheme. These PSTs successfully identified several of the affordances of the second textbook page and considered how they could capitalize on some of them. All of them also started viewing the exercises of the first page from a more conceptual perspective; yet, they were not particularly successful at restructuring the tasks of the first page to support more conceptual work.

Nicole identified some affordances and limitations of the pages and outlined some ways to capitalize on the affordances; yet, her post-intervention performance was very similar to her pre-intervention performance. Vonda, on the other hand, made some progress in this practice since she identified some of the affordances of the second page; however, she did not necessarily capitalize on them. Hence, her post-intervention performance is situated in the second category of this scheme.

Based on the classification of their post-intervention performance, Deborah, Tiffany, and Suzanne appear to have exhibited the greatest performance gains, followed by Kimberley, Nathan and Vonda. Because Nicole’s post-intervention performance was very similar to her pre-intervention performance, in Figure 5.37 she is shown as having experienced no performance gains for this practice.

#### *Responding to Students’ Direct or Indirect Requests for Help*

During the post-intervention meeting, the approaches that all seven PSTs proposed in response to Alan’s error and confusion were mathematically valid and could support meaning-making. Therefore, their performances are classified at or beyond the third category. Vonda’s performance is situated in the third category since her proposed



approach could support meaning-making, but would not necessarily maintain the cognitive demand for students since Vonda appeared to be doing most of the thinking for Alan and his classmates. In contrast, Deborah, Kimberley, Suzanne, Nicole, and Tiffany, proposed interventions that would support meaning-making, while at the same time involve students in some thinking; thus, their performance is clustered in the fourth category. In addition to outlining a mathematically valid response that could support meaning-making and preserve the tasks' complexity, Nathan considered building on previous students' ideas (e.g., June's); hence, his performance is classified under the fifth category.

The PSTs' post-intervention performance in responding to June's and Shaun's requests for help differed significantly from their performance in the previous task. Both Deborah and Vonda proposed interventions which, while pedagogically appropriate, did not address the mathematical ideas at stake in this episode; hence, their performance is situated in the second category of this scheme. Kimberley's performance, on the other hand, belongs in the third category since she proposed an intervention that addressed the main mathematical ideas at stake, but would not necessarily engage students in significant thinking, the reason being that Kimberley appeared to be doing most of the work for them. Nathan's and Tiffany's approaches were grounded in the mathematical ideas explored in this episode and also allowed for student thinking. Because both of them also considered building on students' struggles in this episode and having students share their ideas, their performances are situated between the fourth and the fifth categories. Suzanne's performance, in contrast, is situated between the second and third categories because it minimally touched upon the mathematical ideas inherent in this

episode. Nicole's performance is clustered somewhere between the third and fourth categories, just like her pre-intervention performance.

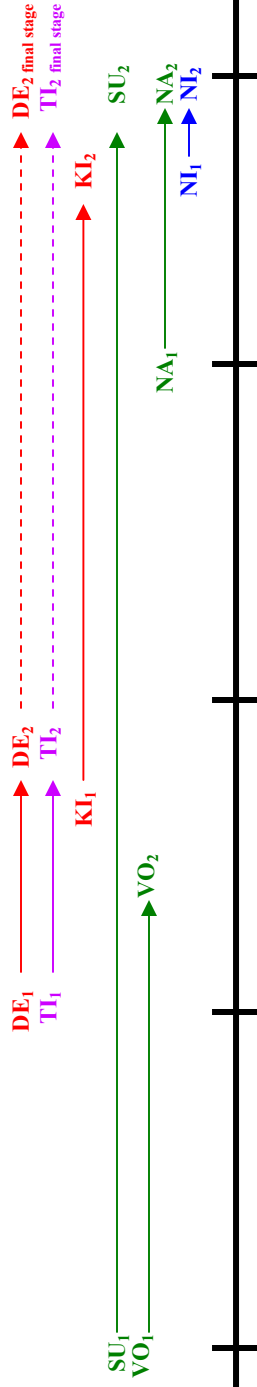
Based on the above classifications, Vonda experienced the greatest gains in her performance in supporting Alan, followed by Deborah who also exhibited notable performance gains. Kimberley, Suzanne, and Tiffany experienced low gains, while Nicole's and Nathan's pre- and post-intervention performances were very similar. In the second task, Tiffany experienced by far the greatest gains, followed by Deborah. Vonda, Nathan, and Kimberley exhibited low performance gains, while Nicole's experienced no notable gains. Interestingly enough, Suzanne's performance in this task decreased.

*Exploring the Association between Changes in Knowledge and Performance: Insights  
from the Cross-Case Analysis*

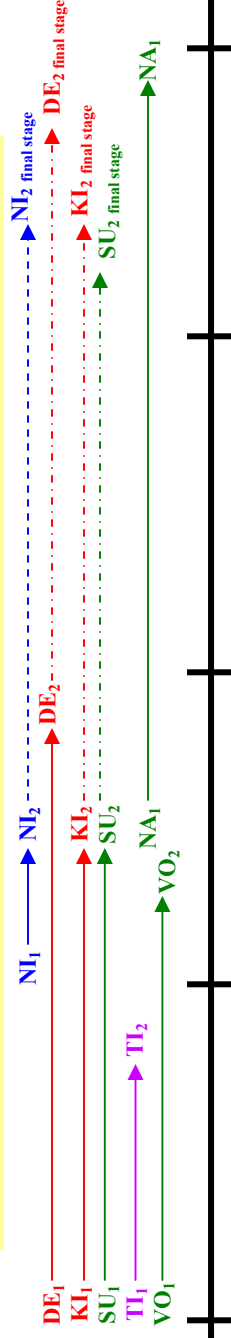
*Considering the Association between the Changes in Knowledge and Performance  
through the Five Classification Schemes*

In order to further understand the association between the changes in the seven PSTs' knowledge and performance, Figures 5.36 and 5.37 were reproduced representing the clustering of the seven PSTs in four groups (see Figures 5.38 and 5.39, respectively), according to the changes in their MKT-score (see Column VIII of Table 4.5). According to this clustering, Deborah and Kimberley (in red) experienced the greatest changes in their MKT-scores; Vonda, Suzanne, and Nathan (in green) the smallest. The changes in Tiffany's and Nicole's MKT performance were somewhere in between, with Tiffany (in purple) experiencing more gains in her performance than Nicole (in blue).

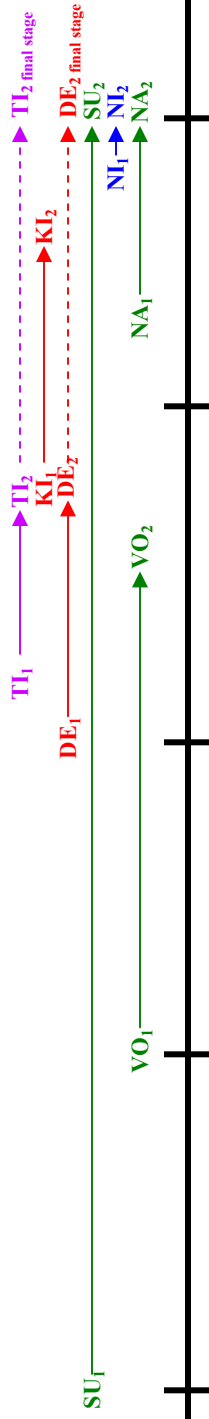
### Providing Explanations: Quotient



### Providing Explanations: Invert-and-Multiply Algorithm



### Using Representations: Quotient



**Legend:**

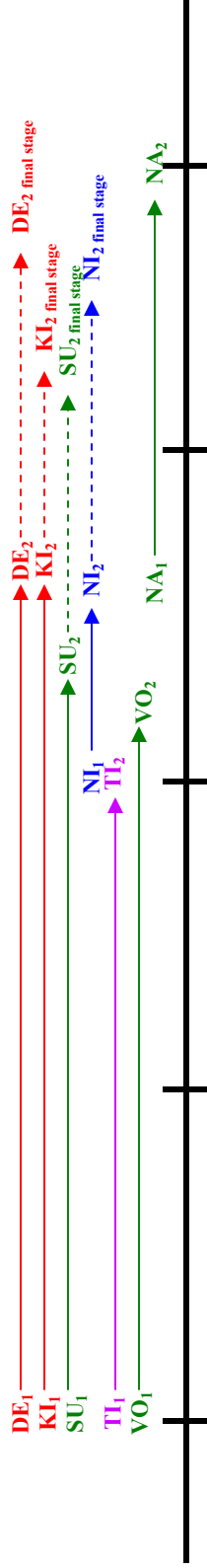
DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda

Changes in the PSTs' MKT: Green: Low changes; Blue: Moderate-high changes; Purple: Moderate-low changes; Red: High changes

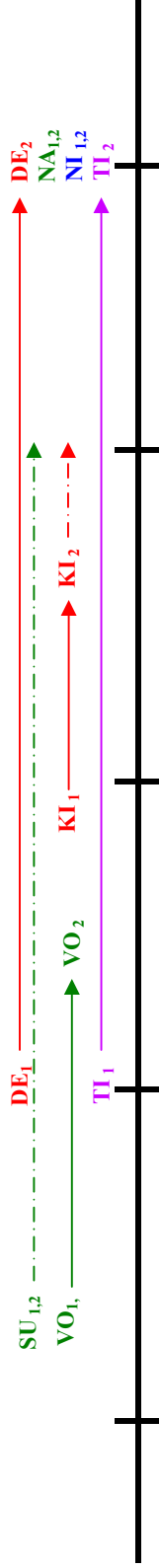
1. Pre-intervention performance; 2. Post-intervention performance

Figure 5.38. The PSTs' classification based on their performance in the MKT-related practices and the changes in their MKT performance.

### Using Representations: Invert-and-Multiply Algorithm



### Analyzing Students' Work and Contributions



**Legend:**

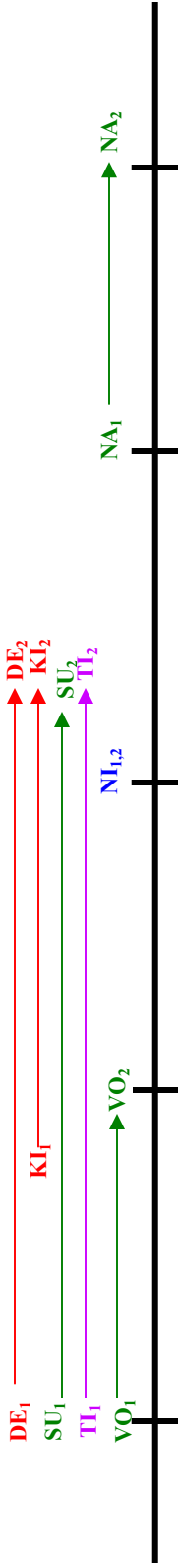
DE: Deborah; KI: Kimberley; NA: Nathan; NI: Nicole; SU: Suzanne; TI: Tiffany; VO: Vonda

**Changes in the PSTs' MKT:** Green: Low changes; Blue: Moderate-low changes; Purple: Moderate-high changes; Red: High changes

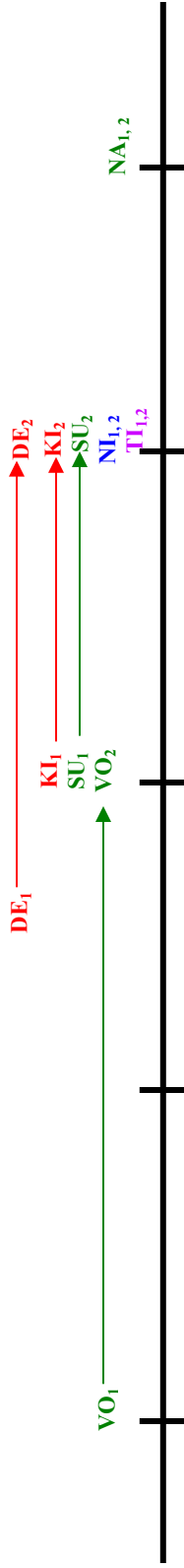
1. Pre-intervention performance; 2. Post-intervention performance

Figure 5.38. The PSTs' classification based on their performance in the MKT-related practices and the changes in their MKT performance (continued).

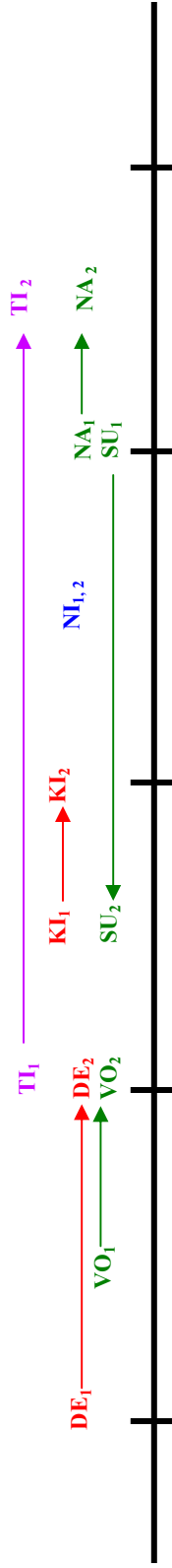
## Selecting and Using Tasks



## Responding to Students' Requests for Help: Alan's Request



## Responding to Students' Requests for Help: June's and Shaun's Request



### Legend:

**DE:** Deborah; **KI:** Kimberley; **NA:** Nathan; **NI:** Nicole; **SU:** Suzanne; **TI:** Tiffany; **VO:** Vonda

**Changes in the PSTs' MKT:** Green: Low changes; Blue: Moderate-low changes; Purple: Moderate-high changes; Red: High changes

1. Pre-intervention performance; 2. Post-intervention performance

Figure 5.39. The PSTs' classification based on their performance in the MTF-related practices and the changes in their MKT performance.

Figure 5.38 suggests that for almost all tasks,<sup>187</sup> Deborah and Kimberley experienced greater changes in their performance in the MKT-related practices than Vonda and Nathan did. This means that, overall, if one excludes Suzanne, compared to the PSTs with the lowest gains in their MKT performance, the PSTs with the greatest gains also exhibited greater gains in their MKT-related practices performance. This pattern is less consistent for the MTF-related practices (see Figure 5.39). For example, for the practice of selecting and using tasks the gains in Deborah's and Kimberley's performance outweigh those of Vonda's and Nathan's. In contrast, for the practice of responding to students' requests for help, a different picture emerges, since in Alan's episode, Vonda's performance gains were higher than those of Deborah and Kimberley. In the second task of this practice, Vonda also exhibited somewhat greater gains than those displayed for Kimberley. These patterns are consistent with the quantitative analysis, which also showed that the changes in the PSTs' MKT scores were more aligned with the changes in their performance in the MKT-related rather than the MTF-related practices.

These patterns become less distinct when the participants of the two intermediate MKT-change groups, Tiffany and Nicole, enter the picture. Even so, the patterns still hold for the practices of providing explanations and using representations (for the quotient only) and the practice of analyzing students' work and contributions. In these cases, the changes in the teaching performance of Deborah, Kimberley, and Tiffany (the

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<sup>187</sup> There were two exceptions to this pattern. First, compared to Kimberley, Vonda is presented to have experienced greater gains in her performance in the practice of using representations (for the quotient). Second, Vonda's gains also appear to outperform Kimberley's gains in the practice of analyzing students' work and contributions. Yet, ignoring Kimberley's inappropriate assertion about Ann's understanding (and thus considering the changes in Kimberley's performance to be also illustrated by the dashed and dotted line), Vonda's gains no longer outperform Kimberley's.

participants of the two upper groups of MKT-change) exceed those of the performance of Vonda, Nathan, and Nicole (the participants of the two lower groups of MKT-change, excluding Suzanne).<sup>188</sup> This pattern also holds for the practice of selecting and using tasks, but not for the practice of responding to students' requests for help.

The patterns discussed above are conditional on excluding Suzanne, who according to her pre- and post-intervention LMT test scores, experienced no gains in her MKT performance. Despite the lack of change in Suzanne's LMT test scores, notable changes in her knowledge were revealed by a close analysis of her answers to the LMT questions during the two administrations of the test and the comparison of her thinking during the pre- and the post-interview meetings. Specifically, during the post-intervention meeting her conceptual understanding of the division of fractions was considerably improved over her pre-intervention understanding of this operation; post-intervention Suzanne also had a better understanding of relative and absolute units compared to her pre-intervention lack of awareness of these two different types of units. Given *these* changes in Suzanne's knowledge, the changes in Suzanne's performance are consistent with the patterns discussed above since the notable increases in her knowledge are reflected in the gains in her performance. Suzanne's knowledge of the content and its teaching also accounts for the regression in her performance illustrated in the practice of responding to students' requests for help (Figure 5.39). In neither interview was Suzanne's grasp of the mathematics embedded in this episode solid enough to support her response to June's and Shaun's requests for help. However, during the pre-intervention meeting she in situ constructed the knowledge needed to provide a mathematically valid

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<sup>188</sup> If we again ignore Kimberley's inappropriate assertion about Ann's understanding.

response; nevertheless, because she was unable to do so during the post-intervention meeting, her performance regressed to a mathematically less valid response.

The changes in the PSTs' knowledge of the content and its teaching (as discussed in the previous section of this chapter) could also account for other inconsistencies in the patterns discussed above. Take for example, the changes in Vonda's performance in responding to Alan's request for help (Figure 5.39). The changes in her performance are consistent with the increases in her thinking, as demonstrated by her post-intervention interview, in which she was able to discern the difference between the mathematical and the semantic connotation of the term *whole*, and thus outlined a mathematically valid approach to support Alan. Similarly, the changes in thinking that Nicole experienced appeared to be less dramatic than the changes in the other PSTs' understanding of the content and its teaching; this pattern is consistent with that in most of the classification schemes outlined in Figures 5.38 and 5.39, Nicole did not display notable changes in her teaching performance.

#### *Other Factors Informing the Changes in the PSTs' Teaching Performance*

Besides the PSTs' knowledge, two other factors appeared to influence changes in their performance. These were their enhanced or alternative images of teaching, and the changes in their beliefs.

To outline or justify approaches to the teaching-simulation tasks, without exception, all seven PSTs directly or indirectly drew on their enhanced or alternative images of teaching, which had been gleaned from their coursework and fieldwork experiences. These images concerned such issues as what constitutes a worthwhile instructional explanation; mapping between different representation modes (e.g., words,



diagrams, mathematical symbols); eliciting students' ideas and engaging them in productive classroom discussions; assessing students' thinking; emphasizing conceptual understanding instead of only attending to students' procedural competence; and considering instruction as a potential source of student error and misconceptions. Alongside these images, the PSTs' coursework and fieldwork offered them opportunities to practice some of their images of teaching, especially those related to providing explanations and mapping between different representational modes. These enhanced images of teaching and the opportunities that these PSTs had to practice some of these images were reflected in their performance especially in the MKT-related practices. The most telling example of this was Deborah's post-intervention performance, in which she was particularly attentive to issues of mapping, attempting to provide sufficiently unpacked explanations for the quotient and the traditional division-of-fractions algorithm. Moreover, she was sensitive to students' errors stemming from the virtual lesson itself.

Intertwined with the enhancement of the PSTs' images of teaching were the reported changes in their beliefs. Kimberley, Suzanne, and Vonda – and to a lesser extent Nicole – were the PSTs who reported having experienced the most notable changes in their beliefs about teaching and learning mathematics. During their school years, these PSTs (particularly the first three) experienced mathematics merely as learning and applying rules and formulas. Their exposure to a different type of instruction through the two intervention courses as well as the other ELMAC courses helped these PSTs to start perceiving mathematics from a different perspective. Instead of considering the mathematics teacher as the fountain of knowledge, who simply introduces students to algorithms, rules, and shortcuts, they started thinking about the teacher as a facilitator

who creates a learning environment to nurture student intellect. In Kimberley's case (and to a lesser extent Suzanne's and Vonda's) these professed changes were so intense, that she talked about a paradigm shift in her beliefs about teaching the subject. In Suzanne's case these changes in beliefs and/or her enhanced images of teaching were also accompanied by an increase in her confidence to do and teach mathematics. In turn, all these changes were manifested in several aspects of these PSTs' performance, ranging from their attempt to develop a lesson that would promote conceptual understanding to responding to students' requests in ways that could maintain emphasis on meaning and understanding.

Regardless of the professed intensity of the changes in the beliefs of these PSTs, the interview data suggest that, at least in some cases, the PSTs' pre-intervention beliefs were still in play and informing their teaching performance. This was particularly evident in one recommendation from Suzanne. Suzanne suggested that after being exposed to the meaning of the division of fractions, students could simply follow the algorithm to solve several division problems without necessarily thinking about the underlying meaning of this operation. Similarly, Kimberley's assertion that Ann's error required just a quick fix, and Vonda's proposed plan for correcting Ann's error (i.e., asking Ann to follow the steps of the traditional division algorithm outlined by her classmates) support this idea. Kimberley's long apprenticeship in providing packed explanations by merely manipulating numerical symbols also appeared to have impinged on her ability to provide unpacked explanations, and foremost to draw connections among the different representations she employed in her explanations.

In some cases, the post-intervention interviews also allowed for a consideration of the interplay among the PSTs' knowledge, beliefs, and images of teaching, especially when their knowledge was insufficient to support their decisions and actions. For example, Tiffany lacked a strong understanding of the concept of the reciprocal; thus, when asked how she would support students in figuring out the traditional division algorithm, she resorted to an image of teaching she developed through her participation in the intervention courses. Specifically, she considered using visual representations and mapping between these representations and the equations shown on the board. Vonda's and Kimberley's task selection provide even more compelling evidence that changes in the PSTs' beliefs and their enhanced images of teaching did not (sufficiently) compensate for limitations in their knowledge. Although both of these PSTs wanted to select tasks to organize activities that would promote their students' understanding of the content, their exercise selection from the first textbook page was not particularly conducive to achieving this goal. Another example comes from Vonda's performance in providing explanations. Despite her increased attention to issues of meaning-making and understanding, Vonda was still developing her understanding of relative and absolute units. As a result, her explanations were largely driven by the numbers involved in the division problem she was asked to explain.

Vonda's case also points to how PSTs' beliefs might mediate the effect of their knowledge on their performance. With a better understanding of fraction division and with a somewhat increased understanding of relative and absolute units, Vonda identified the direction on the second page, which asks students to explain the fractional part of their answers, to be a significant asset; she also endorsed that the page asks students to

explain their thinking in words, diagrams, and numbers. Yet thinking that the second page imposes several requirements on students, thus increasing content complexity and the likelihood of student confusion and error, she thought of not capitalizing on this page's affordances or modifying the tasks' requirements to minimize complexity. Her proposed enactment of this page mirrored her strong beliefs about the role that complexity and confusion can have in student learning. Of course, this pattern might have been more complicated than has been suggested, because her beliefs about minimizing complexity might have been fueled by her weak grasp of the content.

*Exploring the Manifestations of the Changes in the PSTs' Knowledge in their Teaching Performance: Insights from the Cross-Case Analysis*

In this last part of this section, I revisit the 18 propositions outlined in Table 4.6 of Chapter 4 and reconsider them in light of this chapter's findings. Following Yin's suggestion (2003), I regard these 18 statements as theoretical propositions and I compare them against the empirical data outlined in the previous section of this chapter.

Table 5.4 summarizes the extent to which the 18 propositions are supported by the post-intervention data of each case. This summary was developed by taking into consideration the analytical commentary for each case. The case data are designated as supporting the propositions (positive signs), challenging the propositions (question marks) or suggesting refinements (cross signs). Based on this classification, Table 5.4 suggests that propositions A<sub>1</sub>, A<sub>2</sub>, B<sub>2</sub>, C<sub>4</sub>, D<sub>1</sub>, and D<sub>2</sub> were the most frequently supported by the qualitative data of this chapter; they were supported by at least five of the seven PSTs' performances. Propositions A<sub>3</sub>, A<sub>4</sub>, B<sub>3</sub>, D<sub>3</sub>, E<sub>1</sub>, and E<sub>2</sub> were supported by the data of three or four PSTs, whereas propositions B<sub>1</sub>, C<sub>1</sub>, C<sub>2</sub>, C<sub>3</sub>, C<sub>5</sub>, and D<sub>4</sub> were the least

frequently supported (i.e., they were corroborated by the performances of two PSTs the most). Notably, all propositions were supported by the post-intervention data of at least one PST.

Table 5.4

*The 18 Propositions Viewed Through the Post-Intervention Data*

Case Proposition <sup>1</sup>	Nathan	Nicole	Deborah	Vonda	Tiffany	Kimberley	Suzanne	Total
A <sub>1</sub>	☑	☑	☑	☑		☑	☑	6
A <sub>2</sub>			☑	☑	☑	☑	☑	5
A <sub>3</sub>			☑			☑	☑	3
A <sub>4</sub>	☑	☑	☑					3
B <sub>1</sub>			☑	☑				2
B <sub>2</sub>	☑		☑	☑	☑	☑	☑	6
B <sub>3</sub>	☑	☑	☑			☑ †		4
C <sub>1</sub>				☑			?	1
C <sub>2</sub>				☑			☑	2
C <sub>3</sub>			☑		☑		?	2
C <sub>4</sub>			☑	☑	☑	☑	☑	5
C <sub>5</sub>						☑		1
D <sub>1</sub>	☑		☑	☑		☑	☑	5
D <sub>2</sub>	☑	☑	☑ †		☑	☑	☑	6
D <sub>3</sub>			☑	☑		☑		3
D <sub>4</sub>	☑							1
E <sub>1</sub>			☑	☑	☑		☑	4
E <sub>2</sub>	☑		☑	☑		☑		4

Notes.

<sup>1</sup> A description of these propositions appears in Table 4.6

<sup>2</sup> ☑: Data support the proposition; ?: Data challenge the proposition; †: Data suggest amendments to the proposition.

The most frequently supported propositions concern ways in which the PSTs' knowledge or lack thereof can support or constrain PSTs' performance in four of the five practices this study considers. Collectively, these propositions suggest that a strong knowledge of the content and its teaching enables PSTs to offer conceptually driven

explanations; strong knowledge also supports them in using representations in a conceptually driven manner to illuminate the meaning of a procedure under consideration rather than to illustrate the numbers involved in this operation. Additionally, this strong knowledge base could support PSTs in identifying the mathematical affordances of curriculum tasks and offer them some guidelines for restructuring those tasks to account for the task limitations. However, an overall strong understanding of the content and its teaching might be insufficient to identify misconceptions in students' work and contributions. Instead, PSTs' familiarity with student misconceptions allows them to more easily recognize these misconceptions when they arise in student work.

Suzanne's post-intervention performance in analyzing students' work and contributions challenges propositions  $C_1$  and  $C_3$ . Even with an increased understanding of the content, Suzanne still pursued an answer-driven approach when analyzing student solutions: she first figured out the answer to the division problem at hand and then evaluated students' work based on the answer she got from applying the algorithm. This answer-driven approach partly accounts for her failure to adequately analyze Ann's conventional solution, even though she appropriately analyzed Michelle's unconventional solution.

Deborah's and Kimberly's performance, on the other hand, suggest that propositions  $B_3$  and  $D_2$  should be modified. Kimberley's data suggest that PSTs' awareness of the importance of building connections among different representations does not alone enable them to build or to attend to such connections in instruction; instead, PSTs – and especially those accustomed to working more abstractly – need recurrent opportunities to practice building and attending to connections. Deborah's

performance implies that an increased understanding of the content alone does not enable PSTs to restructure curriculum tasks. To effectively engage in restructuring, PSTs might additionally need to become familiar with ways for doing so. In this regard, Silver and colleagues (1990) offer some ideas for supporting (preservice) teachers (e.g., transforming closed problems to make them more open-ended, exploring more solutions to a single problem, adding constraints or new requirements to a task). As proposition D<sub>2</sub> already maintains, and as suggested by the PSTs' pre- and post-intervention performance, capitalizing on these ideas requires a strong knowledge of the content and its teaching.

### Summary

In this chapter, I presented quantitative and qualitative findings for the study's second research question. This question explored the association between the PSTs' MKT and their teaching performance from a dynamic perspective, specifically by exploring changes in their knowledge and teaching performance. Both the quantitative and the qualitative data provided evidence supporting this association. The quantitative analysis showed a positive moderate correlation between the gains in the PSTs' MKT and the gains in their performance; this analysis also yielded a high correlation between the gains in the PSTs' MKT and their performance in the performing tasks of the MKT-related practices. None of the factors considered in this study was found to mediate these associations.

The cross-case analysis of the knowledge and performance gains of seven PSTs substantiated the findings of the quantitative analysis since it showed that the gains in these PSTs' knowledge (whether their MKT knowledge, as measured by the LMT test, or their knowledge of mathematics and its teaching, as tapped via the teaching simulation)

were by and large aligned with the gains in their performance. This alignment was more evident in the MKT-related practices than the MTF-related practices. The cross-case analysis also suggested that, besides the PSTs' knowledge, their enhanced images of teaching and the reported changes in their beliefs also informed the changes in their teaching performance. Of the 18 propositions reported in Chapter 4, 16 were also supported by the data reported in Chapter 5; the cross-case analysis also helped identify a subset of propositions the most frequently supported by the changes in the seven PSTs' knowledge and teaching performance. Without downplaying the importance of the 16 propositions – no doubt all propositions are working hypotheses that warrant further exploration – in the next chapter, I largely focus on those that have been most frequently supported by the data reported in this chapter.



## CHAPTER 6

### DISCUSSION AND CONCLUSIONS

#### Overview

The introduction of Chapter 1 outlined a gap between the modal type of teaching mathematics in the U.S., as captured by the TIMSS 1995 and 1999 video-studies, and the NCTM vision for teaching this subject. The TIMSS video-studies provided evidence suggesting that the teaching of mathematics in the U.S. is largely unchallenging and procedure-oriented; aiming at changing this situation, the NCTM called for high-quality instruction that engages all students in complex mathematical tasks and helps them learn concepts and procedures with understanding.

During the 1990s significant effort and money have been invested in producing high-quality curriculum materials, in hopes that such materials would upgrade the quality of mathematics instruction in the U.S. (for more on these efforts, see Senk & Thompson, 2003). Despite the potential of these materials to upgrade the quality of mathematics instruction, several studies undertaken during the last decade (e.g., Banilower et al., 2006; Huberman & Middlebrooks, 2000; Weiss & Pasley, 2004) have shown that just injecting high-quality materials in the educational system cannot on its own elevate the quality of instruction. As Cohen and colleagues (2003) remind us, just like several other conventional resources (e.g., facilities, money), curriculum materials cannot on their own produce learning, for although “teachers cannot use resources they don’t have ... the resources they do have are not self-enacting” (p. 122). These scholars go on to

recommend that greater attention be paid to personal resources and particularly to how teachers mobilize and use their knowledge. As these scholars argue, in the pursuit of understanding what produces student learning, greater attention should be given to teacher knowledge because

[k]nowledge counts in several ways. Teachers who know a subject well, and know how to make it accessible to learners, will be more likely to make good use of a mathematics text, to use it to frame tasks productively and use students' work well, than teachers who don't know the subject, or know it but not how to open it to learners. (p. 125)

Despite the reasonableness of the argument that teacher knowledge matters for student learning, and notwithstanding the growing scholarly interest in and work on teacher knowledge, researchers have not yet untangled and understood the complex relationship that exists among teacher knowledge, teacher instruction, and student learning (Mewborn, 2003). In addition, they have not fathomed how knowledge can support teachers in establishing and maintaining the rich and challenging learning environments aspired by the NCTM vision.

Seeking to contribute to this direction of inquiry, this study explored the relationship between PSTs' MKT – a type of knowledge distinct to the work of teaching mathematics – and their performance in a selected set of teaching practices considered conducive to establishing rich and cognitively challenging learning environments: selecting and using tasks, providing explanations, using representations, analyzing students' work and contributions, and responding to students' requests for help. Exploratory in nature, this study did not intend to prove the existence of the relationship under consideration. Rather, it sought to improve our understanding of this relationship, by focusing on a sample of 20 PSTs and by investigating the association between their MKT and their performance in the five practices both from a static and a dynamic

perspective. Specifically, the study explored the PSTs' knowledge and performance at their entrance to a teaching education program (i.e., static perspective) and the changes in their knowledge and performance after their participation in a sequence of a math content and a math methods course (i.e., dynamic perspective); this exploration aimed at providing insights with regard to (a) whether a relationship exists between (the changes in) PSTs' MKT and their performance in the five teaching practices; (b) whether other factors, such as PSTs' beliefs, their perceptions about certain instructional goals, and their background characteristics mediate this relationship; and (c) how PSTs' MKT or changes thereof are manifested in their teaching performance, as captured and studied via a teaching simulation.

This chapter is therefore organized in terms of the three foregoing topics, which correspond to the three subordinate questions of the study's first and the second research questions. After presenting and discussing the main findings for each of these topics, this chapter outlines the theoretical, methodological, and practical contributions and implications of the study, concluding with directions for future research.

### Is There an Association between Knowledge and Teaching Performance?

As the second section of Chapter 2 suggested, exploring the association between teachers' knowledge and teaching performance has engaged scholars' interest for about half a century. The results yielded from this exploration have been mixed and, at times, quite surprising since some of them provided evidence not only negating any association, but even sometimes showing a negative correlation between teacher knowledge and their performance. Disillusioned with some of the early findings in this realm, Begle (1979) opined that research on teacher effectiveness would not profit from exploring teachers

and their characteristics and advocated that scholarly efforts be pointed to other directions. The results of the present study – both these yielded from the static exploration and those pertaining to the dynamic perspective – not only contradict Begle’s claim but also suggest that mathematics education could significantly benefit from the insights gleaned from research on teacher knowledge and their instructional practices.

Both the quantitative and the qualitative findings of the static perspective helped establish the relationship between knowledge and teaching performance since they showed a strong relationship between the PSTs’ knowledge, as measured by a paper-and-pencil LMT test, and their teaching performance, as gauged by a teaching simulation. This strong relationship was reinforced by the quantitative and the qualitative results of the dynamic perspective, which showed an even stronger relationship between the gains in the PSTs’ knowledge and the gains in their teaching performance.

Specifically, with respect to the *static perspective*, the quantitative analysis showed a positive moderate correlation ( $r_s = 0.56$ ) between the PSTs’ entrance MKT and their entrance teaching performance in the five teaching practices. This relationship was even stronger when considering the PSTs’ performance in the performing tasks of the teaching simulation ( $r_s = 0.68$ ). With the exception of Suzanne, the qualitative analysis largely corroborated these results. Nathan and Nicole, who were the PSTs with the highest entrance MKT scores, performed notably better than their counterparts in all five practices under exploration; Vonda and Deborah, who exhibited the lowest MKT knowledge, also lagged behind in their teaching performance. Tiffany and Kimberley who were rank ordered somewhere in between according to their MKT performance, performed better than Vonda and Deborah, but not as well as Nathan and Nicole.

The qualitative analysis also substantiated the strong association between knowledge and performance even at a more fine-grained level by showing the strengths and the limitations of the PSTs' knowledge to be reflected in their teaching performance. Consider, for example, the case of Nathan, who entered the program with the strongest MKT. Because of his strong understanding of division (of fractions) and of the concept of relative and absolute units, in explaining the quotient for the division  $2 \div \frac{3}{4}$  he provided an explanation that focused on the meaning underlying this operation. Due to his developing understanding of the reciprocal, in explaining the reciprocal for this same division problem, he was not as successful at illuminating the concept under consideration; instead his explanation was largely informed by the numbers involved in this division. Likewise, with a quite strong understanding of division, Kimberley was quite flexible in explaining the meaning of the quotient when it did not involve any fractional parts. Yet, because of her developing understanding of relative and absolute units, when asked to explain divisions whose quotient involved both a whole-number and a fractional part, she shifted attention from the meaning of division to the numbers involved in the division problems.

Such affordance and constraint examples were identified even in the case of Deborah who entered the program with low MKT relative to her counterparts. A case in point was Deborah's work on analyzing the three student solutions. Because Deborah's procedural understanding of fraction division was robust, she quite effectively analyzed Robert's solution, which pertained to using the traditional division-of-fractions algorithm. However, she was not equally successful at analyzing Michelle's visual solution, which was more conceptually grounded than Robert's. Moreover, although Deborah claimed

herself to be a strong supporter of using “visuals” to facilitate student learning, confronted with her difficulties in analyzing Michelle’s work, she considered urging Michelle to switch from the visual approach she was using to the traditional algorithm.

With respect to the *dynamic perspective*, the quantitative analysis yielded a marked correlation ( $r_s = 0.68$ ) between the gains in the PSTs’ MKT and the gains in their overall teaching performance; this correlation was even higher for the gains in the PSTs’ performance in the performing tasks of the MKT-related practices ( $r_s = 0.73$ ) and their overall performance in these practices ( $r_s = 0.85$ ).<sup>189</sup> The qualitative analysis again provided evidence supporting these strong correlations. The most revealing case in this respect was Deborah, whose remarkable MKT gains were in close alignment with the notable improvement in her teaching performance. For example, whereas during the pre-intervention meeting Deborah could not provide a conceptually founded explanation for the division  $2 \div \frac{3}{4}$ , during the post-intervention meeting she provided such explanations not only for the quotient of this division but also for the concept of the reciprocal. Her enhanced understanding of absolute and relative units also enabled her to appropriately analyze Michelle’s solution and to make valid assertions about this student’s thinking. But Deborah was definitely not the only case corroborating the strong association between the gains in the PSTs’ MKT and the gains in their teaching performance.

Nathan and Nicole exhibited low gains in their MKT performance; those gains comported with the low changes in their teaching performance, especially for Nicole, whose post-intervention performance did not encompass the subtle and qualitative

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<sup>189</sup> These high correlations should be interpreted with caution since they might have been inflated by the intervention considered in this study. At the same time, however, these correlations should not be solely attributed to the potential mediating effect of the intervention, given that strong correlations were also identified even at the PSTs’ entrance to the ELMAC program.

changes of Nathan's. The comparison of Kimberley's pre- and post-intervention teaching performance also showed that her MKT gains were consistent with the gains in her performance: although she did not exhibit the dramatic teaching-performance changes as Deborah did, the changes in Kimberley's teaching performance were more notable than those in Nathan's and Nicole's performances. Even two of the divergent cases, Tiffany and Vonda, did not actually challenge the strong association under consideration because the changes in their MKT scores were largely consistent with the changes in their performance in the performing tasks of the simulation. From this respect, both of their cases supported the quantitative findings that showed the gains in the PSTs' MKT to be more strongly associated with the gains in their performance in the performing tasks of the simulation rather than with those in the noticing and interpreting-evaluating tasks.

Suzanne was the only case that did challenge this strong association since, although she received the same score on both the administrations of the LMT test, her post-intervention teaching performance was remarkably different from her pre-intervention performance. But even in her case, a closer analysis of the questions she answered correctly on each of the two LMT test administrations revealed some notable consistencies in the growth of her knowledge and the changes in her teaching performance. The most compelling evidence in her case supporting the association at hand pertains to her performance in providing explanations. Due to the increases in her conceptual understanding of the division of fractions and her better grasp of relative and absolute units, Suzanne offered a conceptually undergirded explanation for the quotient of the division problem at hand. Her post-intervention performance was therefore in sharp contrast to her pre-intervention performance, when she argued that she could not

provide any explanation at all for this division; it was also aligned with some of the LMT questions that Suzanne answered correctly on the second but not on the first administration of the test.

Similar fine-grained examples corroborating the strong association between the changes in the PSTs' knowledge and the changes in their performance were also identified in the performances of the other PSTs. For instance, with an enhanced understanding of the reciprocal, Nathan provided a conceptually driven explanation when asked to consider the reciprocal in the division problem  $2 \div \frac{3}{4}$ ; his explanation was noticeably improved from his pre-intervention explanation for the same problem. In contrast, because her grasp of the notion of the reciprocal was still underdeveloped, Nicole struggled with providing such an explanation even during the post-intervention meeting. Similarly, Vonda still wrestled with both the notion of relative and absolute units and the notion of the reciprocal. These struggles were reflected in several aspects of her performance, with the most noteworthy being the explanations she provided and the difficulties she encountered analyzing Michelle's visual solution.

Kimberley's and Tiffany's post-intervention performance also provided several affordance and constraint examples lending support to the association between PSTs' knowledge and performance. With a much stronger understanding of relative and absolute units, Kimberley offered an explanation for the quotient in the division problem  $2 \div \frac{3}{4}$  which was conceptually founded throughout. In contrast, because of her apparent misconception that the dividend should always be larger than the divisor, she did not identify the germane misconception in June's explanation, let alone proposed an intervention to address and correct this misconception. Likewise, because of her stronger



“conceptual link” of relative and absolute units, Tiffany proposed a conceptually driven explanation for the quotient in the division problem under exploration; having this link also enabled her to closely follow and appropriately analyze Michelle’s solution. Yet, because of her insecure understanding of the notion of the reciprocal, in explaining the reciprocal for the division problem  $2 \div \frac{3}{4}$ , Tiffany merely focused on the numbers involved in this problem.

In sum, the quantitative and the qualitative findings for both the static and the dynamic perspective provide converging evidence about a strong association between the PSTs’ knowledge and their teaching performance in the practices this study explored. As such, the findings of this study are in line with analogous findings reported in a recent study (Hill et al., 2008) which explored the association between in-service teachers’ MKT and the mathematical quality of these teachers’ instruction in terms of the richness of the mathematics in their lessons, the teachers’ performance in appropriately responding to students’ questions, and the paucity of mathematical errors in their instruction. Although the present study and Hill et al.’s study considered different teacher populations, the alignment in their results provides compelling evidence suggesting that teachers’ knowledge can support them – to say the least – in providing quality mathematics instruction that focuses on meaning and understanding, which is a necessary condition for building rich and intellectually challenging learning environments.

In arguing that the findings of the present study imply that knowledge matters for the building of rich and intellectually learning settings, I am not dismissive of the limitations of the present study, and consequently of the need that the study findings be interpreted with caution. For example, among other considerations, it is important to

remember that this study examined a small sample of a particular teaching population, namely PSTs; it explored the PSTs' performance in an artificial environment rather than a real-classroom setting; it focused on a particular topic – division of fractions; it immersed the study participants in a specific treatment, the effects of which might have been affected by the fact that the instructor was also the researcher and the author of this study; and it utilized certain measures and coding schemes to capture and quantify the PSTs' knowledge and teaching performance. Yet, the consistency with which the quantitative and qualitative findings for both the static and the dynamic exploration showed a strong association between knowledge and teaching performance provides the warrants for making such an argument.

In addition to the key finding of this study regarding the strong association between teacher knowledge and teaching performance, two other patterns yielded from the quantitative and the qualitative analyses also merit consideration. Both patterns pertain to how this study decomposed and investigated the PST' teaching performance.

First, the relationships between the PSTs' MKT and their performance in the *performing* tasks of the teaching simulation (in which the PSTs were asked to perform certain teaching tasks) were stronger than those pertaining to the tasks of noticing and interpreting-evaluating the virtual teacher's actions and decisions. This pattern was consistent across all the practices examined in this study for both the static and the dynamic exploration. The qualitative findings also lent support to this pattern, as particularly suggested by Tiffany's and Vonda's performances. Although at first sight Tiffany appeared to be a divergent case, since her entrance MKT score was not closely aligned with her entrance teaching performance, a closer examination of her case

revealed that her moderate MKT-score comported with her moderate performance in the performing tasks of the simulation. In other words, although Tiffany was able to appropriately analyze several teaching episodes, more so than what her MKT score would suggest, she was not equally successful at performing certain teaching tasks. The changes in Vonda's MKT and teaching performance provide a similar example. Although Vonda exhibited proportionally higher gains in her teaching performance than in her MKT score, most of these changes pertained to the tasks of noticing and interpreting-evaluating rather than to the performing tasks.

The stronger relationship between the PSTs' MKT and their performance in the performing tasks of the teaching simulation rather than in the noticing and interpreting-evaluating tasks resonates with the theoretical underpinnings of MKT. In particular, it corroborates the proposition that MKT enables teachers to *perform* the work of teaching rather than theoretically talk about or analyze teaching (Ball et al., 2005, in press). And because the structuring or rich and intellectually challenging environments this study considers is contingent not on whether teachers can theorize about teaching but on what they can actually do, the study findings suggest that MKT can scaffold PSTs' attempts to build such environments.

Second, with only very few exceptions, the quantitative analysis for both the static and the dynamic exploration showed no significant associations between the PSTs' knowledge and their performance in the noticing and interpreting-evaluating tasks. The qualitative findings, on the other hand, provided some evidence suggesting that in certain tasks, the PSTs' knowledge was particularly supportive for what the PSTs attended to when they viewed the teaching simulation and how they interpreted and evaluated the

virtual teacher's decisions and actions. Consider, for example, the PSTs' performance in capturing and analyzing the limitations of the virtual teacher's explanation as to why the reciprocal works. With a stronger understanding of the content, during the pre-intervention meeting Nathan was more successful than his fellow-students at identifying the limitations of this explanation. Similarly, the study's qualitative findings showed a close correspondence between the PSTs' knowledge and their attentiveness to and appropriate analysis of the deficiencies in the virtual teacher's enactment of task D (the task on figuring out an algorithm for the division of fractions).

The quantitative and the qualitative findings just considered are not necessarily contradictory since they suggest that the PSTs' knowledge was particularly supportive for the performances required by some noticing and interpreting-evaluating tasks than by others. As such, these findings imply that, although it seems reasonable to postulate that teacher knowledge can support teachers in noticing and analyzing teaching practice – as recently hypothesized by some scholars (e.g., Sherin & van Es, 2005; Star & Strickland, 2008; van Es & Sherin, 2008) – further research is needed to better understand under what circumstances teacher knowledge can inform (preservice) teachers' noticing and interpreting-evaluating performances.

#### Other Factors Informing PSTs' Performance in the Five Practices

This study explored the mediating effect of a group of factors on the association between PSTs' knowledge and performance. Some of these factors (e.g., math content and methods courses taken) have been employed in previous studies, either as proxies of teacher knowledge (e.g., Begle, 1979; Monk, 1994) or as teacher background characteristics, to explore their mediating effect on teacher effectiveness (e.g., Hill et al.,

2005). Yet, none of the studies considered in Chapter 2 explored the mediating effect of teachers' general aptitude (e.g., their GRE scores) on the association of interest. This study, in addition to employing both measures of PSTs' MKT and measures of their general aptitude, also examined the mediating effect of PSTs' beliefs about mathematics and its teaching, their efficacy beliefs, and their perceived importance of a set of instructional goals. By exploring both cognitive and affective factors as contributors to teacher performance, the present study responded to Philipp's (2007) call and Thompson's (1992) earlier recommendation that factors from both domains be utilized in attempts to understand what determines teaching performance.

This exploration yielded two main findings, the first pertaining to the mediating effect of the PSTs' GRE-quantitative score and the second related to other factors, besides the PSTs' knowledge, that were found to inform their teaching performance. In particular, the quantitative analysis showed the PSTs' GRE-quantitative score to be one of the two mediators which rendered the association between the PSTs' entrance MKT and their entrance teaching performance non significant (the other was the PSTs' efficacy beliefs, which I consider in the third part of this section). However, the association between the PSTs' MKT and their performance in the performing tasks of the simulation remained marginally significant, even when controlling for the mediating effect of the PSTs' GRE score. Even more critically, the relationship between the gains in the PSTs' MKT and the gains in their teaching performance was *not* mediated by their GRE-quantitative score. To put it differently, the PSTs gains in knowledge and teaching performance were not dependent on their general aptitude, as measured by their GRE-quantitative score. The qualitative analysis largely corroborated these findings. Moreover,

it pointed to two other factors – the PSTs’ beliefs and their images of teaching – that appeared to inform the PSTs’ teaching performance, in addition to their knowledge. Below I elaborate upon each of these key findings and then briefly discuss other findings related to the exploration of the factors mediating the focal association.

#### *The Mediating Effect of the PSTs’ GRE-Quantitative Performance*

The PSTs’ GRE-quantitative score was strongly related to both their entrance MKT score ( $r_s=0.78$ ) and their performance in the teaching simulation ( $r_s=0.59$ ), thus meeting the condition necessary to mediate the relationship between the PSTs’ entrance MKT and their teaching performance ( $r_s =0.56$ ). Indeed, after statistically controlling for the effect of the PSTs’ GRE-quantitative performance, the association between the PSTs’ MKT and their teaching performance became negligible and therefore non statistically significant. However when considering the PSTs’ performance only in the performing tasks of the teaching simulation, the association between the PSTs’ MKT and their teaching performance was marginally significant, even when controlling for the mediating effect of the PSTs’ GRE score (Kendall’s coefficient  $=0.28$ ,  $p <.10$ ).

Focusing on the marginally significant relationship just reported, one could note that the mediating effect of the PSTs’ GRE scores on their performance in the performing tasks was not as intense as it was for their overall performance, which also included the tasks of noticing and interpreting-evaluating. Consequently, one could postulate that the closer one moves to examining PSTs’ performance in actual teaching tasks rather than in tasks pertaining to analyzing and discussing teaching, the weaker the mediating effect of their GRE performance might become. Further research, however, would be needed to explore this hypothesis.

Focusing on the strong mediating effect of the GRE scores on the PSTs' overall teaching performance, on the other hand, one could argue that the study findings suggest that the PSTs' potential teaching effectiveness largely hinged on their general aptitude, as measured by their GRE performance. One could even draw on several of the qualitative findings to further support this argument. For instance, Nathan's GRE-quantitative score was much higher than that of Deborah's. This discrepancy in their GRE performance was consistent with the discrepancy in several aspects of their entrance teaching performance. For example, while Nathan's explanation for the quotient in the division problem  $2 \div \frac{3}{4}$  was conceptually driven, Deborah could not even provide an explanation. Moreover, whereas Nathan was able to capitalize on the affordances of the second textbook page to propose a conceptually oriented lesson and even outlined ways to restructure the procedurally oriented tasks of the first textbook page, Deborah wrestled with even fathoming some of the mathematical requirements of the second page (e.g., she wondered what the fractional part of an answer means). Additionally, while Nathan appropriately analyzed the three student solutions and proposed mathematically valid ways to respond to students' requests for help, Deborah struggled with analyzing students' work and offered mathematically nebulous responses to students' questions. Collectively, the foregoing differences suggest that with a stronger mathematics background, as implied by his GRE performance, Nathan was in a better position than Deborah to structure learning environments that nurture students' intellect.

Had this study been limited only to exploring the association of interest from a static perspective, its findings would have lent support to a widespread assumption that PSTs' general aptitude, as measured by their GRE scores, provides a good index for

determining the potential teaching effectiveness of candidate teachers. Such an assumption is echoed in Levine's (2006) recent report *Educating School Teachers*, in which, after he documents the poor GRE performance of elementary education candidates,<sup>190</sup> he concludes:

[A]t least as measured by the standardized test scores, the future elementary education teachers whom education schools are admitting are less academically qualified than our children need or deserve. (p. 56)

Implicit in this and other pertinent arguments is a deterministic chain of thinking, according to which the teacher candidates' entrance characteristics – and particularly their general aptitude – largely determine how effective these teachers will be when assigned to classes and consequently how much they will influence student learning (gains). Even though one could hardly deny that such entrance characteristics matter for the PSTs' potential effectiveness, the findings pertaining to the dynamic exploration undertaken in this study challenge such a deterministic line of thinking.

The quantitative analysis showed the PSTs' GRE scores to *not* mediate the gains in the PSTs' knowledge and teaching performance. This implies that what the study participants gained from their participation in the intervention was not contingent on their general aptitude as reflected by their GRE-quantitative score; particularly noteworthy is the negligible association found between the PSTs' GRE-quantitative scores and the gains in their teaching performance (see Column II of Table 5.3). Deborah's case was the most telling in this respect. Even though she entered the program with a low GRE performance relative to her counterparts, at the end of the two intervention courses she exhibited the greatest gains in her teaching performance. The scrutiny of her case also

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<sup>190</sup> Levine (2006) found that the elementary education candidates scored about 70 points below the national average on the GRE-quantitative test (see Table 13, p. 57).



revealed that her post-intervention performance was remarkably different from her performance at the outset of the intervention: her post-intervention explanations were centered on the concepts under consideration; she was able to follow and appropriately analyze both conventional and unconventional student solutions and respond in mathematically valid ways to student requests for help; what is more, she capitalized on the affordances of the second textbook page to outline a conceptually oriented lesson and she even started thinking of ways in which she could restructure the procedural tasks of the first page.

What both the quantitative and the qualitative findings then suggest is that if supplied with the appropriate intervention (an issue to which I return when discussing the practical implications of this study), PSTs with lower GRE scores are able to learn and exhibit teaching performances conducive to the structuring of the rich and challenging environments this study considers. In this way, the study findings challenge the deterministic line of thinking considered above which tends to treat teacher education as a black box and to ignore its potential effect on PSTs' learning, and consequently on their teaching effectiveness. Moving a step farther, the study findings, and particularly Kimberley's post-intervention difficulties in drawing connections among different representations, suggest that in some instances, a strong mathematics background – as manifested by the PSTs' high GRE score – might actually impinge on teachers' effectiveness. As Kimberley admitted at the post-intervention meeting, because of her facility in working with mathematical symbols at an abstract level, she found it hard to use the representations she employed at their full potential to sufficiently unpack the meaning of the operations she was trying to explain. In her words, using representations

was not her “strong suit;” she therefore felt that she needed to work more on this area to overcome her “die-hard, old, habits” of working on the mathematics at an abstract level.

A caveat is, however, in order here. In no way should the study findings be considered to suggest that general aptitude, as tapped by PSTs’ GRE performance, does not matter for PSTs’ teaching performance. Quite the contrary, one could say, particularly when considering Nathan’s high-quality pre- and post-intervention teaching performance. Besides, it should not be dismissed that the study participants had at least an average GRE performance. Had the study focused on PSTs with much weaker GRE performance, the foregoing results might have been different, showing the PSTs’ gains to be contingent on their general aptitude. In other words, as is often the case with learning, some “threshold” skills and prior knowledge are necessary for instruction to be effective. Yet, the results of the dynamic exploration, and especially cases such as Deborah’s, clearly suggest that GRE performance cannot tell the whole story about PSTs’ potential teaching effectiveness. Even more, these findings suggest that the whole story cannot be told unless greater attention is paid to teacher education itself and how it can support and equip teachers for the work of teaching. I return to this issue when considering the study’s practical implications.

#### *Beliefs and Images of Teaching Informing the PSTs’ Teaching Performance*

The qualitative analysis pointed to two main factors that, in addition to the PSTs’ knowledge, appeared to inform their teaching performance: their beliefs (about mathematics, its teaching and learning) and their images of teaching.<sup>191</sup>

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<sup>191</sup> The findings of the quantitative and the qualitative analyses with respect to the role of the PSTs’ beliefs in informing their teaching performance are not necessarily contradictory. The three factors of PSTs’ beliefs included in the quantitative analysis simply did not meet the condition of being related with *both* the PSTs’ MKT and their teaching performance to be considered mediators of the focal relationship.

Like previous studies (e.g., Collopy, 2003; Scott, 2001; Speer, 2008; Sztjan, 2003) which examined teachers' beliefs and practices, this study also showed PSTs' beliefs about mathematics and what it means to teach and learn mathematics to inform PSTs' performance. The qualitative findings pointed to two different patterns regarding the interplay between the PSTs' knowledge and beliefs. Their beliefs and knowledge worked synergistically when the two supported instruction geared toward sense-making and understanding (e.g., Nathan's pre- and post-intervention performance; Deborah's post-intervention performance); this synergistic pattern was also observed when the PSTs' beliefs and knowledge both impinged on instructional decisions and actions geared toward meaning-making (e.g., Vonda's pre-intervention performance).

The second pattern concerned the cases in which the PSTs' beliefs were not compatible with their knowledge. In these cases, even favorable beliefs could not compensate for limitations in the PSTs' performance, especially with respect to the performing tasks of the teaching simulation; this was more evident in Deborah's and Tiffany's pre-intervention performance. The most telling example of the limited role of productive beliefs in compensating for limitations in the PSTs' knowledge was Deborah's analysis of Michelle's solution to the division problem  $2\frac{3}{4} \div \frac{3}{4}$ . Because of her difficulties in understanding Michelle's visual solution to this problem, Deborah considered suggesting that Michelle use the traditional algorithm to solve this problem. This recommendation was in stark contrast to Deborah's strong endorsement of representations as tools for learning and thinking.

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Yet, the negative relationship of all the belief factors with the PSTs' knowledge and teaching performance was in the expected direction since these factors reflected a type of instruction that contradicted that endorsed by the simulation. That said, some discrepancies between the PSTs' beliefs as reported on the survey statements and as captured in the teaching simulation (especially for Suzanne) should not be ignored; I return to this issue when discussing the study's methodological implications.

On the other hand, Nicole's and Kimberley's pre-intervention performance suggests that knowledge alone cannot support the teacher in building rich and cognitively challenging environments. Despite the strength of these PSTs' knowledge, the instructional approaches they outlined for certain teaching episodes appeared to also be driven by their beliefs of "giving more" to support struggling students. In these episodes, both PSTs did most of the thinking for students instead of creating a platform for students to do their own thinking. The comparison of Vonda's pre- and post-intervention performance is also indicative of the mediating effect of certain unproductive beliefs on the association between knowledge on teaching performance. Although during the post-intervention meeting Vonda's better grasp of the content supported her in identifying specific affordances of the second page's tasks, she decided to avoid using certain tasks or to simplify some of their requirements; she justified her decision by pointing out that, as presented in the curriculum materials, the tasks were too complex. Her decision seemed to have been informed by her conviction that in teaching mathematics the teacher should minimize complexity to support learning.<sup>192</sup>

Consistent with Lortie's (1975) and subsequently Ball's (1988) argument that PSTs enter teaching preparation programs with already formed images about teaching the subject, this study also provided evidence of these images of teaching indeed informing the PSTs' teaching performance. For some PSTs, these images of teaching constituted avoidance models. For instance, being critical of the type of mathematics instruction she experienced during her school years, during the pre-intervention meeting Deborah had a strong inclination to organize her teaching so that it allowed for more thinking,

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<sup>192</sup> One could also point to a reciprocal relationship between Vonda's still limited knowledge of the content and her beliefs about selecting and enacting demanding tasks in teaching.

exploration, and explanation of mathematical ideas on the students' part. In contrast, Vonda endorsed a more show-and-tell type of instruction, apparently subscribing to the images of teaching she formed as a student of the subject. Likewise, Suzanne acknowledged that one teaches the way one was taught the subject, thus corroborating Grossman's (1990) argument that PSTs' apprenticeship of observation, and consequently the images that they form, could "suppor[t] the conservatism of teaching" (p. 10).

Even more intriguing was the interplay between the PSTs' knowledge and their images of teaching, especially for the cases of Tiffany and Suzanne. Lacking the necessary knowledge that would support their instructional decisions and actions, these PSTs resorted to images of teaching they did not necessarily endorse. Tiffany, for example, admitted that, lacking any criteria that would allow her to make an informed selection of exercises for her students, she would assign the "odds" or "evens," just like her own teachers did. Similarly, even though she wanted to provide a more conceptually undergirded explanation, pre-intervention Suzanne ended up outlining a gimmick of fraction division that she learned as a student.

What these findings then suggest is the potential value of offering PSTs alternative images of teaching, which could provide them with some scaffolds when their knowledge of the content and its teaching cannot sufficiently undergird their ventures into teaching the content from a more conceptual perspective. Certainly, these alternative images of teaching cannot compensate for limitations in PSTs' knowledge; yet, they could provide them with ideas helpful for maintaining an emphasis on meaning and understanding, as Tiffany's post-intervention performance suggests. In particular, although Tiffany was not solid on the idea of the reciprocal, which could have scaffolded

her work in helping students discover the traditional division-of-fractions algorithm, by drawing on an image of teaching modeled in the intervention courses, she outlined some ideas that could preserve the emphasis on meaning and understanding (e.g., she suggested building connections between visual representations and the mathematical sentences representing fraction divisions).

The different patterns of interplay among the PSTs' knowledge, beliefs, and their images of teaching considered above definitely warrant further investigation because these interactions might be more complex than revealed by how they were captured and presented in the present study. In fact, in a recent study, Wilkins (2008) arrived at a similar conclusion after using structural equation models to study the direct and indirect effects of teachers' knowledge and beliefs on their "teaching performance" (which he measured by considering the frequency with which teachers reported using certain approaches in their teaching). He concluded that "beliefs may be dependent on the existence or the absence of knowledge" (p. 150) and called for future research that would help further unpack and understand the knowledge and beliefs interplay and its effect on teachers' performance.

Indeed, Nicole's performance in analyzing and evaluating the virtual teacher's decisions and actions suggests that teachers' *in-the-moment* decision making might be informed by more complex mechanisms rather than the simple synergistic and competing patterns of beliefs and knowledge identified above (which appear to be more useful for a coarser analysis of the interactions between knowledge and beliefs in informing teachers' decisions and actions). Nicole appeared to engage in what Schoenfeld (1998, 2008) has called a *cost-benefit* analysis or an *internal-calculus* analysis to explain, and even predict,

the decisions that teachers make in-the-moment when faced with certain teaching situations. According to his analysis, teachers' in-the-moment decisions are informed by a complex net of goals (overarching goals, major instructional goals, and "local" goals), beliefs (about learning, teaching, students, classroom environment, and mathematics), and knowledge (subject matter, general pedagogical knowledge, pedagogical content knowledge, knowledge of the students, and knowledge of the history of the classroom). Schoenfeld also argues that the input from all these factors is weighed through certain decision-making mechanisms, which determine the final decisions. Nicole's performance corroborates Schoenfeld's theory in that she evaluated the virtual teacher's approach in different instructional episodes in remarkably dissimilar ways. Her evaluation was not only informed by her knowledge of the mathematics at stake in each episode or her beliefs about teaching and learning the subject, but also by what she perceived as being the virtual teacher's major or local instructional goals, namely the goals she perceived the virtual teacher to have set for the lesson and related instructional activities.

#### *Other Factors Mediating the Association between Knowledge and Teaching Performance*

In addition to the mediating effect of the PSTs' GRE performance on the association of interest, the quantitative analysis pointed to three other factors mediating the relationship between the PSTs' *entrance* MKT and their teaching performance. These factors were the PSTs' efficacy beliefs with respect to understanding and working on fractions, the math methods courses the PSTs had taken during their undergraduate studies, and the math content courses they had taken in high school. Whereas the focal relationship was robust for the latter two factors, this was not the case for the first factor, which, like the PSTs' GRE performance, rendered the markedly significant association

between knowledge and teaching performance negligible. These findings merit some consideration because of their methodological implications for exploring the association of interest and their practical implications for teacher education.

The strong mediating effect of the PSTs' efficacy beliefs on the association of interest was not surprising because the statements used to tap the PSTs' efficacy beliefs were closely calibrated to the mathematics content and the teaching performance under consideration. Hence, the mediating effect of the efficacy-beliefs factor could be considered to represent the alignment between the PSTs' perceived and actual competence. Specifically, whereas the MKT and the teaching performance measures gauged the PSTs' actual performance in the area of fraction division, the efficacy factor assessed their perceived competence in this area. The alignment between actual and perceived competence found in this study does not imply that the efficacy-beliefs factor measured the same construct as the LMT test or the teaching simulation; besides, as Bandura (1997) cautioned, perceived and actual competence are not always tantamount. The aforementioned finding, however, highlights the importance of close alignment between the measures of (actual or perceived) teaching competence and the measures of actual teaching performance. I return to this issue when discussing the methodological implications of the study.

The mediating effect of the number of math methods courses these PSTs had taken on the association of interest accords well with previous research findings in the realm of the educational production function studies. As pointed out in Chapter 2, despite their equivocal results, more often than not, these studies have shown the number of math methods courses teachers had taken to be positively associated with their teaching



effectiveness. The mediating effect of the content courses taken during high school is also consistent with Monk's (1994) study, which suggested that content courses matter, at least to some extent, for teachers' performance. In fact, the qualitative analysis, and particularly the pre-intervention case of Vonda who had taken the smallest number of such courses, showed how a weak content preparation might critically impair a teacher's engagement with the practices considered in the study. In a case such as hers, a teacher might not have the minimum mathematical resources upon which to draw to successfully engage in these practices. Given that Vonda was also amongst the PSTs who experienced relatively low gains in her MKT and her teaching performance, this finding again raises the "threshold" issue discussed when considering the PSTs' GRE performance. The foregoing finding and the findings pertaining to the mediating effect of the PSTs' GRE scores on their entrance performance collectively suggest that a minimum mathematical background is needed to maximize the effectiveness of instructional interventions designed to augment PSTs' potential to structure rich and challenging learning settings.

#### On the Potential Manifestations of Knowledge in Teaching Performance

Identifying a gap in the then current literature on the relationship between teachers' knowledge and their teaching performance, Shulman (1986) posed several questions: Where do teachers' explanations come from? How do teachers decide what to teach and how to represent it? What informs their decisions to address students' misunderstandings? How might limitations of teachers' knowledge limit their teaching approaches? To these, Hill et al. (2005) have recently added: How might knowledge enable teachers to manage student confusion or insights? To explain concepts? To use the available textbooks? All these questions are still open to inquiry.

The findings of the present study do not provide definite answers to these questions. As explained in Chapter 3, even an experimental design that randomly assigns teachers to student, classroom, and school conditions or an ethnographic approach would still be limited in their ability to conclusively or comprehensively answer these questions. Nevertheless, the cross-case analysis of the seven PSTs' teaching-simulation performance supported the development of a set of propositions regarding the potential ways in which PSTs' knowledge might manifest itself in their teaching. These propositions collectively suggest that PSTs' knowledge can scaffold them in structuring the learning environments considered in the study *by helping them maintain an emphasis on the meaning underlying the mathematical procedures at hand.*

In this section, I focus on a selected set of propositions that emerged from this study – particularly those that were most frequently observed across the seven PSTs' performance – to discuss how knowledge appears to play out in helping PSTs maintain emphasis on meaning and understanding. In discussing these findings, I also point to other studies which explored the knowledge and practices of other PSTs or in-service teachers, and whose findings resonate with those of the present study. This alignment of findings then suggests that the scope of these propositions could be extended to include not only PSTs in general but in-service teachers, as well.

#### *The Manifestations of Knowledge in Providing Explanations*

The results of the between-case and the within-case analyses (either for the same concept at two different time points or two different concepts at the same time point) suggest that strong knowledge supports PSTs in providing conceptually driven explanations that elucidate the concepts at hand. In contrast, when their understanding of

the content is not thorough enough to support such conceptually driven explanations, PSTs are more likely to anchor their explanations by the numbers involved in the procedures they are trying to explain.

The study findings corroborate this proposition in three ways: the between-case analysis revealed notable differences in the quality of the explanations offered by the PSTs with a stronger and a weaker mathematical knowledge base (e.g., compare Nicole's pre-intervention explanation of the quotient with Vonda's pre-intervention "explanation" of the same concept); the within-case analysis suggested that the depth of the PSTs' knowledge of the content was reflected in the explanations they could offer (e.g., compare Nathan's pre-intervention explanation for the quotient with that he provided about the reciprocal of the same division); and even more revealing, the comparison of the PSTs' pre- and post-intervention explanation showed that increases in their understanding were reflected in the quality of their explanations. Two examples are particularly telling of the latter pattern.

Lacking the knowledge to provide a conceptually driven explanation, pre-intervention Suzanne, like Tom, as discussed in Sowder et al. (1998), suggested introducing students to a meaningless gimmick (see Figure 2.4). With an enhanced knowledge of division, as well as of relative and absolute units, during the post-intervention meeting Suzanne provided a step-by-step explanation, in which she unpacked and elucidated the concepts underlying the division-of-fractions problem at hand. Similarly, Deborah, like Ms. Daniels, as discussed in Borke et al. (1992), initially attempted to provide a conceptually undergirded explanation for the quotient. Yet, lacking the necessary knowledge, Deborah was successful at only describing the steps

involved in the traditional algorithm, just like Ms. Daniels ended up asking her students to focus on the algorithm without providing them with any insights about its underlying meaning. In contrast, with a better understanding of the division of fractions and the concept of relative and absolute units, during the post-intervention meeting Deborah was able to provide an explanation that illuminated the underlying meaning of the algorithm she was asked to explain.

The manifestations of the (prospective) teachers' knowledge in the quality of their explanations, as discussed above, suggest that there might be a significant pedagogical price to be paid – to use Shulman's (1986) terminology – when teachers' knowledge is compromised. Without a strong understanding of the content, teachers might not be able to offer explanations that support students' conceptual understanding of the content. In turn, this might lead them to having students apply certain procedures without thinking about the conceptual underpinnings of these procedures, which was the modal type of instruction observed in both the TIMSS video-studies, as discussed in Chapter 1.

#### *The Manifestations of Knowledge in Using Representations*

The depth of the PSTs' knowledge was also found to relate to whether they used the representations they employed in a conceptually or a numerically driven manner, thus again showing that knowledge can scaffold attempts to maintain emphasis on meaning and understanding. In particular, strong knowledge was found to support the PSTs in using representations to elucidate the meaning of the procedure(s) under consideration (what I called *conceptually driven use of representations*). When their knowledge could not support them in this endeavor, the PSTs tended to use their representations to illustrate the numbers involved in the algorithms they were explaining without capturing

the conceptual essence of the content at hand (what I called *numerically driven use of representation*). Two examples are revealing of these patterns.

With a relatively deep understanding of division, during the pre-intervention meeting Kimberley successfully used the number line she employed to show and explain the meaning of the whole-number part of the quotient for the division problem  $2 \div \frac{3}{4}$ . Yet, because of her understanding of relative and absolute units was weak, she used the number line merely to *show* the fractional part of the quotient. In the post-intervention meeting, with a stronger understanding of the content, she successfully used her drawing to explain why the fractional part of the answer was represented in thirds (relative units) rather than in fourths (absolute units). Like Kimberley, in the pre-intervention meeting, Deborah used her representations to show the numbers rather than the concepts in the algorithms she was asked to explain. With a deeper understanding of the content, in the post-intervention meeting, she used the representations she selected in a more conceptual fashion, and ever more, she used several examples and drawings to infer the meaning of the reciprocal.

The findings of two recent studies that focused on in-service teachers echo the foregoing patterns. In the first study, Charalambous (2006) found notable differences in how two elementary school teachers, Lisa and Karen, used representations in their teaching. Lisa, the low-MKT teacher tended to use representations to show the numbers involved in the algorithm she was trying to explain; in contrast, Karen, the high-MKT teacher was more successful at using representations to distill and illuminate the meaning of the operations she was explaining. Similarly, Izsák (2008) identified differences in two middle-school teachers' use of representations when working on fraction multiplication.

Ms. Archer, the teacher whose understanding of the content was rather weak, used number lines to simply illustrate the solutions generated by applying the traditional multiplication-of-fractions algorithm to solve several exercises. On the contrary, with a better understanding of the content, Ms. Reese used representations to determine the solutions to a set of fraction-multiplication problems and then, by looking across these solutions, to figure out an algorithm for this operation. Hence, while in Ms. Archer's lessons the representations were used as ancillary tools to *show* the answers, in Ms. Reese's lessons the representations gave rise and meaning to the fraction multiplication algorithm.

*The Manifestations of Knowledge in Analyzing Students' Work and Contributions*

The proposition most frequently supported with respect to the practice of analyzing students' work and contributions held that a strong understanding of the content is not adequate to identify students' misconceptions that arise during teaching; rather, what appeared to support the PSTs' in this work was their familiarity with such misconceptions. For example, during the pre-intervention meeting, regardless of her rather weak understanding of the content, Tiffany identified the misconception in June's explanation. Her success in noticing this misconception was merely due to her familiarity with the idea that the dividend can be smaller than the divisor. In contrast, even during the post-intervention meeting and despite the enrichment in her knowledge, Kimberley did not seem to acknowledge the possibility that the dividend can be smaller than the divisor. This hindered her from identifying the corresponding misconception in June's explanation.

On the surface, this proposition seems to contradict the idea that knowledge can support PSTs in maintaining emphasis on meaning, especially if taken into account that high-MKT PSTs failed to notice the misconception in June's explanation, while low-MKT PSTs successfully did so. It should be borne in mind, however, that the PSTs' MKT was measured using only questions that pertained to their CCK and SCK; no questions were used to gauge the PSTs' knowledge of how students think about, know, and learn particular content, what the LMT researchers identify as *Knowledge of Content and Students* (KCS). Had the LMT test included such questions, there might have been greater alignment between the PSTs' awareness of certain student misconceptions and their ability to identify and address them as they arose during the teaching simulation.

One of Shulman's questions listed above asks what informs teachers' decisions to address student misconceptions. The proposition just discussed suggests that key in addressing such misunderstandings, and consequently in maintaining emphasis on meaning and understanding, is the extent to which teachers are knowledgeable of such misconceptions. I return to this proposition when considering the implications of the study findings for curriculum development purposes.

#### *The Manifestations of Knowledge in Selecting and Using Tasks*

The study's qualitative findings suggest that PSTs' knowledge can support them in maintaining emphasis on meaning and understanding by informing task selection and task modification decisions.

Two examples are particularly indicative of how knowledge can support task selection decisions. Without understanding the second textbook page's direction on explaining the fractional part of an answer, let alone the concepts related with the

fractional part of an answer, pre-intervention Deborah admitted that she would not use the pertinent textbook task in her teaching. In contrast, during the post-intervention meeting, cognizant of these concepts, she considered employing this task for an introductory lesson on fraction division. Along the same lines, while Kimberley ignored this same direction during the pre-intervention meeting, with an increased understanding of relative and absolute units, during the post-intervention meeting she designated the aforementioned task as pivotal and considered assigning it a central role in an introductory division-of-fractions lesson. These findings comport with Manouchehri and Goodman's study (2000) and Sowder et al.'s study (1998), which both showed that teachers who do not understand the content well are less likely to select demanding tasks for their lessons; instead, as teachers' understanding of the content and its teaching increases, they are more likely to experiment with using richer and more challenging tasks in their teaching.

With respect to task modification, the study findings suggest that although not sufficient, a strong understanding of the content can support teachers' restructuring of the available tasks to address their limitations. For instance, with a better understanding of the content than that of his counterparts, Nathan was relatively successful at restructuring the tasks of the first textbook page: he thought of clustering these tasks according to their underlying conceptual characteristics (e.g., putting together exercises whose dividend was smaller than their divisor); he also proposed sequencing these tasks in ways that could support students' gradual construction of the ideas at hand. The differences between the pre- and post-intervention performances of Tiffany and Deborah also suggest that as teachers' understanding of the content increases, they are more likely to



appropriately restructure tasks to support student learning. While during the pre-intervention meeting both Tiffany and Deborah could not think of any way to modify the procedurally oriented exercises of the first textbook page, during the post-intervention meeting they productively engaged in such considerations by capitalizing on the idea that the relative size of the dividend and the divisor matters for the complexity of fraction division problems.

Yet, as explained in Chapter 5, knowledge alone cannot adequately underpin task restructuring. Instead, as Silver and colleagues advocate (1990), teachers need to be familiarized with how the available curriculum tasks can be scaled up to render them more open-ended, richer, and more challenging. The present study suggests that such attempts should concurrently aim at enhancing teachers' understanding of the content and its teaching; otherwise teachers might run the risk of knowing certain generic approaches for task restructuring but either being unable to apply them to particular contents or applying them unsuccessfully. This was the case with post-intervention Kimberley, who proposed three activities for restructuring the 16 exercises of the first textbook page, but made rather inappropriate selections of exercises/examples for these activities.

#### *The Manifestations of Knowledge in Responding to Students' Requests for Help*

With respect to the practice of responding to students' requests for help, the study findings suggest that a strong understanding of the content supports maintaining emphasis on meaning by enabling teachers to provide responses that are at least mathematically valid. Both during the pre-intervention and the post-intervention meeting, without exception, all seven PSTs outlined instructional interventions which were mathematically appropriate only when their understanding of the content supported them

in this respect. In contrast, when their knowledge of the mathematical ideas at stake was thin, they tended to sketch pedagogically appropriate, but mathematically imprecise responses to students' requests for help, and hence, failed to address the main mathematical ideas at play.

Suzanne's pre- and post-intervention performance in responding to June's and Shaun's questions is particularly indicative of this proposition. During the pre-intervention meeting, Suzanne eventually figured out how the idea of common multiples applies to solving fraction division problems. This enabled her to structure a mathematically rich intervention to address the two students' questions. Without deciphering the applicability of the common-multiple approach to solving fraction division problems, during the post-intervention meeting Suzanne offered a mathematically nebulous response to address the two students' questions. Tiffany's pre- and post-intervention performance provides a similar example. Due to her difficulties with the foregoing concept, during the pre-intervention meeting Tiffany could not sufficiently address the two students' inquiry. During the post-intervention meeting she used the common-multiple approach to solve the division problem under consideration; this enabled her to respond to the students' questions in ways that addressed and highlighted the meaning of the common-multiple approach in dividing fractions.

The study findings, however, revealed that the manner in which the PSTs responded to the students' requests also hinged on factors besides their knowledge. Such factors included the PSTs' beliefs about teaching and learning, their images of teaching, and whether they were aware of issues pertaining to how teachers can maintain or dilute the cognitive complexity of instructional tasks by their responses to students' questions

and requests for help. Because I have already discussed the role of the PSTs' beliefs and images of teaching, below I briefly consider only the latter factor.

Particularly revealing of the effect of this factor are two contrasting examples: Nathan's claim about how a comment made by the course instructor sensitized him to the importance of supporting students without doing the thinking for them and Nicole's cost-benefit analysis that lacked such considerations. It is also informative to consider that while in their post-intervention performance the seven PSTs discussed in Chapter 5 attended to issues of meaning-making – which was the focus of the intervention – they were not equally attentive to whether their proposed instructional approaches maintained the task complexity for students (with the exception of Nathan). As such, the findings of this study are consistent with Silver and colleagues' recent study (in press), which suggests that teachers are better able to attend to issues on maintaining task complexity and the ways to achieve this, if they receive explicit and repeated opportunities to consider such issues in their instruction. I return to this idea when discussing the practical implications of the study for teacher education programs.

### Potential Contributions and Implications

In this section, I consider the theoretical, methodological, and practical contributions and implications of the study findings. These contributions and implications pertain to how the study design and findings add to or extend the existing body of knowledge and to the questions or issues surfaced by the exploration undertaken herein.

#### *Theoretical Contributions and Implications*

The theoretical contributions and implications of this work relate to the first three design decisions of the study. Specifically, it was envisioned that the integration of work

on MKT and MTF would be beneficial for the association under exploration; for this investigation, a specific type of knowledge, MKT, was considered. Furthermore, it was argued that there is value in exploring the intermediate links between teacher knowledge and student learning, namely teachers' instructional practices, and particularly those considered conducive to establishing rich and intellectually challenging learning environments. I reflect on these design decisions in light of the study findings.

### *Integrating Work on MKT and MTF*

For the past decade significant insights into teaching mathematics have emerged from work in two research arenas. First, advancing a practice-based conceptualization of teacher knowledge, MKT researchers have theorized about the knowledge needed for the work of teaching; they have also developed measures to tap this knowledge and explore its association with the mathematics quality of teachers' instruction and with student learning gains. Second, building on the work of Doyle (1983, 1989), the MTF researchers have explored how teachers' decisions and actions regarding their selection, presentation, and enactment of curriculum tasks shape the quality of the learning environments teachers create and, consequently, student learning. Despite the obvious overlap between the agendas of the MKT and the MTF scholars, no systematic effort has so far been undertaken to bring together the ideas and the insights yielded from their respective work.

Situated at the intersection of the two research arenas, this study attempted a first step toward integrating ideas from them. This integration appears to have been beneficial in two respects: first, the study findings suggest that MKT does matter in the creation of rich and challenging learning environments, as considered in the work associated with the

MTF; second, the study findings provide insights into how ideas from the one field can advance the work in the other field.

The quantitative and the qualitative study findings both support the notion that teachers' knowledge, and specifically the type of knowledge explored in this study, matters for building rich and intellectually challenging environments. Take, for instance, the phases of task presentation and enactment outlined in the MTF. This study suggests that a strong understanding of the content and its teaching supports teachers in providing appropriate instructional explanations and using representations to support sense-making. In contrast, a weak understanding of the content might result in emphasizing following and applying rules, even if this might not be the mode of instruction that teachers endorse, as the pre-intervention performances of Deborah and Suzanne suggest. The study also showed that a deep understanding of the content and its teaching enables the teacher to appropriately analyze student work and contributions as well as to respond to student requests for help in mathematically valid and meaningful ways. Overall, the study findings suggest that, although factors other than teacher knowledge can inform PSTs' decisions and actions during task presentation and enactment, knowledge appears to be a key player in informing those decisions and actions, particularly in presenting and enacting the content in a manner that supports meaning-making and understanding.

The study findings also suggest that teachers' knowledge of the content and its teaching informs decisions pertaining to task selection and modification. In particular, the qualitative findings showed that a strong understanding of the content can support PSTs in identifying the affordances of curriculum tasks; it can also provide them with guidelines for transforming these tasks to address their limitations. Yet, from a

quantitative perspective, the relationship between MKT and the PSTs' performance in the practice of selecting and using tasks was low. This low relationship points to how the work on MKT might benefit from MTF research. For example, the current conceptualization of MKT might profit from including an explicit focus on teachers' selection and modification of curriculum/instructional tasks. This focus can be incorporated in the two recently advanced MKT domains, the *Knowledge of Content and the Curriculum* and the *Horizon Knowledge* (see Ball et al., in press).

The Knowledge of Content and the Curriculum corresponds to Shulman's (1986) third category of curricular knowledge. This domain includes knowledge of curriculum materials for teaching a given subject or topic within a grade and knowledge of topics or issues that have been and will be taught in the same subject area during the preceding and later years in school. For this category, the focus on selecting and modifying curriculum tasks appears to be a good fit, since curriculum tasks are the primary curriculum materials at teachers' disposal; vice versa, teachers' Knowledge of Content and the Curriculum seems to be consequential for selecting and modifying tasks in ways that both help students learn specific mathematical ideas and set the foundations for the teaching of similar ideas in upper grades. Consider, for instance, the teaching of division explored in this study. If a teacher knows that the quotient in fraction divisions should be explained in relative rather than absolute units, this teacher can select or modify existing tasks to help students start considering these two different types of units, even when working on simple whole-number divisions (e.g., students can be guided to express the quotient in the division  $36 \div 5$ , as 7 groups of 5 and its remainder as one fifth of this group of five).

The focus on curriculum tasks is apparently a good fit for the domain of Horizon Knowledge, as well. This domain pertains to teachers' awareness of fundamental mathematical structures and of how the content at K-12 level is connected to higher mathematics (D. L. Ball, personal communication, October 6, 2008). Such awareness can support teachers in scaling up and extending curriculum tasks in mathematically legitimate ways and in a manner that sets the foundations for higher level mathematics. For example, when working on the notion of relative and absolute units in fraction division, the teacher could organize a discussion on how the quotient changes as the relative unit approaches, in size, the absolute unit or as it tends to zero. Such a discussion could set the groundwork for the concepts of limits and asymptotes considered in high-level mathematics.

On the other hand, the conceptualization and empirical work on MKT could advance the work associated with the MTF. One such contribution pertains to the work on identifying the factors associated with maintaining or lowering the cognitive demands of tasks during classroom presentation and enactment. The results of this study suggest that there are good reasons to believe that teacher knowledge constitutes a key factor in sustaining the demand of curriculum tasks at their intended level. So far, the work on identifying the factors contributing to the level at which the content is presented and experienced in mathematics lessons has largely focused on contextual factors (e.g., classroom management problems, available instructional time), student working habits (e.g., students pressing the teacher for more information), teacher expectations (e.g., the extent to which the teacher holds students accountable for high-level products or processes), and certain teacher abilities (e.g., teachers' skills to efficiently scaffold

students' thinking and reasoning, pressing for justifications and explanations, and building of connections among different ideas) (see Stein et al., 2000, pp. 24-32). Lurking beneath the teacher expectations and abilities appears to be teachers' own understanding of the content and its teaching. For example, teachers might be reluctant to press students for explanations and justifications or to build connections among different ideas, if they themselves feel insecure about venturing into this space, because of limitations in their knowledge. Similarly, teachers might not hold students accountable for high-level products if teachers have insufficient knowledge to support them in this endeavor. Recall, for instance, that during the pre-intervention meeting Deborah suggested that Michelle abandon her visual solution to the division problem and use the traditional algorithm instead, simply because Deborah herself could not make sense of the visual solution. Similarly, without a strong understanding of the content, pre-intervention Tiffany and Vonda addressed the students' requests for help in mathematically impoverished ways. These examples suggest that limitations in teachers' knowledge might lead to shifting the emphasis away from meaning and understanding to following and applying procedures, thus lowering the demand of curriculum tasks.

#### *Focusing on MKT to Explore the Knowledge - Teaching Performance Association*

In justifying my decision to focus on MKT to explore the knowledge - teaching performance association, I pointed out that, this type of knowledge is grounded in the work of teaching, thus allowing for a better exploration of the association at hand. Not only do the findings of the study corroborate this argument, they also provide empirical evidence supporting the theoretical conceptualization of MKT.



The quantitative findings revealed a strong relationship between the PSTs' MKT and their performance in the five practices under investigation. This relationship was mostly due to the *performing* tasks explored in the simulation rather than to the extent to which the PSTs could talk about or analyze teaching; this finding empirically validates the theoretical claims about MKT being closely related to the work that teachers are required to *do*. Additionally, the PSTs' MKT was strongly related to their performance in the performing tasks of the MKT-related practices even at the outset of the intervention (as Figure 4.2 suggests, these correlations were equal or higher than  $r_s=0.49$ ). This finding is in alignment with the theoretical claims that MKT constitutes a foundation for providing adequate instructional explanations, appropriately using representations in teaching, and analyzing students' work and contributions.

The qualitative findings of the study also substantiate the theoretical conceptualization of MKT. For instance, Vonda's pre-intervention performance, and particularly the difficulties she encountered in several of the teaching simulation tasks, suggests that, although not sufficient for the work of teaching, teachers' CCK serves as a foundation for effectively presenting and working on the content. Yet, in accordance with the theoretical underpinnings of the MKT, the study findings also reveal that possessing the mathematical knowledge common to mathematically intensive professions (i.e., CCK) is not adequate for the work of teaching. This was particularly the case in Kimberley's pre-intervention performance. Regardless of her strong mathematical background, without an understanding of relative and absolute units, Kimberley provided an incorrect explanation for the fractional part of the quotient in the division  $2 \div \frac{3}{4}$ .

The qualitative data also support the idea of Knowledge of Content and Students (KCS) being a distinct MKT domain. Although this type of knowledge was not directly examined in this study, the PSTs' performance in capturing and addressing June's misconception suggests that CCK and SCK are not sufficient for the work of teaching. Instead, teachers need to be aware of specific student misconceptions (i.e., an area tapped by KCS) if they are to identify and address them as they arise during teaching.

*Attending to the Intermediate Links between Teacher Knowledge and Student Learning*

The review of the educational production function studies in Chapter 2 pointed to one of the main limitations plagued studies of that type: they failed to examine the intermediate link between teacher knowledge and student learning, that is the work of teaching. Similarly, although the process-product studies did attend to the work of teaching, they focused on more generic teaching skills and practices, such as classroom management, efficient allocation and use of instructional time, and teachers' questioning techniques. The present study supports the argument discussed in Chapter 2 that we cannot understand the association between teacher knowledge and student learning, unless we understand the work of teaching (Ball et al., 2001).

Figure 2.1, and particularly its two intermediate links (i.e., teacher practices and the establishment of rich and intellectually challenging environments), extend the foregoing argument by advancing the idea that we cannot understand the association between teacher knowledge and student learning unless we understand *how* knowledgeable teachers build rich and intellectually challenging learning environments and, foremost, *how their knowledge supports them in this endeavor*. The findings of this

study, and especially the qualitative results, imply that there are reasons to believe that teachers' knowledge is pivotal to building such learning environments.

Although this study focused on PSTs and examined their performance in-vitro, the outcomes point to certain ways in which limitations in teachers' knowledge might result in shifting emphasis from meaning and sense-making to manipulating numbers. Take, for example, Suzanne's pre-intervention performance. Without a strong understanding of division (of fractions), Suzanne proposed a meaningless gimmick for dividing fractions. Deborah's pre-intervention performance is an even more telling example. Despite her strong desire to teach for meaning-making and understanding, limitations in her knowledge appear to have impinged on her capacity to do so: she wanted to explain the meaning of the division-of-fractions but she could not; she wanted to use the representations she employed to give rise to and elucidate the meaning of this operation, but her repertoire of available representations was limited. Even more, she could not attach meaning to the numerical symbols of this operation. With a better understanding of the content, during the post-intervention meeting, Deborah provided appropriate explanations and used her representations effectively to support student understanding and sense-making.

These examples represent only some of the plausible ways in which teachers' knowledge could scaffold or limit them in building learning environments that support student learning and understanding. Undeniably, more work is warranted to better understand the manifestations of teachers' knowledge in a panoply of other instructional practices also assumed to promote such environments (cf. Hiebert et al., 2003): to name just a few, selecting and sequencing student solutions, immersing students in exploring

and building connections among alternative solutions, leading classroom discussions in ways that afford students opportunities to reason mathematically. This work therefore sets a new agenda for scholars in the field of teaching and teacher effectiveness. Instead of simply identifying and describing the differences in the learning environments that teachers across different countries build for their students, scholars could invest more effort in understanding and explaining these differences. Such explorations might, in fact, reveal more cross-country similarities in the ways teachers mobilize their personal resources to structure productive learning environments than those yielded so far by the comparative studies undertaken in this realm.

#### *Methodological Contributions and Implications*

The key methodological contribution of this study pertains to the teaching simulation that was utilized to explore and measure the PSTs' teaching performance. Specifically, the study findings imply that using a teaching simulation to gauge (prospective) teachers' performance is a promising approach to explore the association between knowledge and teaching performance and to understand how knowledge can inform instructional decisions and actions. In addition to this methodological contribution, the study also raises three issues whose consideration might inform future research exploring the association between teacher knowledge and teaching performance as well as the factors mediating this association. These issues relate to the measurement of teachers' knowledge using multiple-choice tests, the need for better alignment between the measures on knowledge and the measures of teaching performance, and the exploration of teachers' beliefs via surveys. In what follows, I consider each of the methodological contributions of the study in turn.

### *Using a Teaching Simulation to Measure Teaching Performance*

So far, several approaches have been pursued to explore teachers' practices, ranging from simply asking teachers to report the frequency with which they use certain instructional practices in their teaching (e.g., Wilkins, 2008) to having teachers complete instructional logs documenting their use of certain practices (e.g., Ball & Rowan, 2004) to exploring teachers' actual practice over a sequence of lessons (e.g., An et al., 2004, Hill et al., 2008) and to pursuing a more ethnographic approach to document teachers' practices over long periods (e.g., Aubrey, 1997; Boaler, 2002). The former two approaches allow for the exploration of the instructional practices of large samples of teachers; at the same time, however, scholars often raise concerns about the validity and the reliability of the data collected using surveys or instructional logs. The latter two approaches are assumed to yield more valid measures of teachers' instructional performance; yet, they are not ideal for collecting data from large samples of teachers. Additionally, although classroom observations more accurately capture teachers' instructional practices, studies that build on such observations to explore the association between teacher knowledge and their teaching performance are plagued by difficulties in controlling the effect of three mediating factors.

The first factor is the use of different curricula, which previous studies (e.g., Remillard, 2005; Stein et al., 2007) have shown to inform teachers' instructional decisions and practices. Lloyd's recent study (2008) of Anne, a PST who was observed using two different curricula during her fieldwork placement, further corroborates the argument that curriculum materials potentially influence teachers' instructional practices. The second factor relates to teaching different mathematical topics. The mediating effect

of this factor should not be underestimated, given that previous studies have documented differences in teachers' instructional approaches when these teachers were observed teaching different topics – consider, for example, the cases of Ms. Jackson (in Fennema & Franke, 1992) or Ms. Lehava (in Kahan, et al., 2003), as discussed in Chapter 2. The third potential mediator is the mix of contextual factors (i.e., classroom, school, and outside-of-school contextual factors), which, as discussed in Chapter 2, might influence teachers' instructional decisions and approaches.

To avoid the methodological constraints, this study explored the PSTs' practices in-vitro, by using a teaching simulation. This approach to be sure has its own limitations. To start, one cannot equate the PSTs' performance in the teaching simulation with their actual performance in a real-classroom setting; instead one can only consider the PSTs' performance as indicative of their *potential* to engage in certain practices.<sup>193</sup> Second, as a virtual setting, the simulation eliminates certain teaching complexities (e.g., classroom management issues) that might impinge on teachers' decision making (as discussed below, this feature can also be considered an asset of the teaching simulation). Third, like the classroom-observation approaches discussed above, the teaching-simulation approach is more time consuming than the reporting-of-practice approaches. Fourth, just like in any other testing situation, the participants' different levels of anxiety might affect their performance in the simulation.

These limitations notwithstanding, the study findings suggest that the teaching simulation utilized in this study is a promising approach to explore and measure teaching

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<sup>193</sup> Nathan's post-intervention comment that his teaching-simulation performance might differ from his performance in an actual teaching environment directly speaks to this idea. Moreover, it should not be ignored that unlike a real-classroom setting in which teachers have to make decisions on the spot, the study participants had virtually unlimited time to make certain instructional decisions.

performance. This was particularly evident by the strong correlations obtained for the association between knowledge and teaching performance. These correlations were in stark contrast to the results of previous studies, and especially of the educational production function studies (see Chapter 2), which yielded very low to negligible correlations between teacher knowledge and teacher effectiveness. Although it is true that this study did not actually explore teacher effectiveness as a function of student learning gains like the previous studies did, the strong correlations between knowledge and teaching performance obtained in this study represent an advancement in a field that has been plagued by mixed and often discouraging results – recall, for example, Begle’s admonition that researchers should direct their attention to factors other than teacher knowledge. These strong correlations should largely be attributed to three distinctive features of the teaching simulation employed in the study.

First, despite being an approximation of teaching, this simulation created a platform for *performing the work of teaching*; consequently, it facilitated a better exploration of how knowledge can actually be used *in practice* compared to previous studies that simply required teachers to describe or reflect on their teaching practices.<sup>194</sup> Moreover, in opposition to the approaches pursued in the educational production function studies, the teaching simulation offered an arena for exploring the intermediate link between teacher knowledge and teaching; this link was more often than not dismissed by the educational production function studies which ignored teaching, in their attempts to establish an association between teacher knowledge and student learning.

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<sup>194</sup> As such, the teaching-simulation approach responds to Grossman and McDonald’s (2008) recent call to shift from the pedagogies of investigation to the pedagogies of enactment, and explore what teachers can actually *do*.

Second, and perhaps more critically, compared to real-classroom environments, the teaching simulation provided a more favorable environment for exploring the association between knowledge and teaching practice; it did so by *controlling certain “noise” factors* and by allowing for *testing certain hypotheses*. For example, building on previous studies that showed classroom management problems to dominate PSTs’ considerations of certain instructional episodes (e.g., Star & Strickland, 2008), the teaching simulation eliminated such problems. Apparently this modification added to the artificiality of the teaching environment generated by the simulation; yet at the same time, by eliminating such noise factors, it allowed for a better exploration of how knowledge can inform instructional decisions and actions. Additionally, instead of waiting for certain instructional situations to arise during teaching, the teaching simulation purposefully generated these situations and exposed the PSTs to them to examine whether and how their knowledge supported them in handling these situations. Alan’s episode, for instance, created a research space for exploring if and how the PSTs’ understanding of units could support them in identifying and resolving the error stemming from interpreting the “whole” based on its everyday meaning rather than on its mathematical meaning. Similarly, the episode on Amanda and Julia’s interpretation of the remainder in the division problem  $\frac{3}{4} \div \frac{1}{6}$  was particularly conducive to exploring whether a strong understanding of relative and absolute units could support the PSTs in appropriately analyzing and correcting the two students’ work. Consider, also, Robert’s question as to whether the traditional algorithm always work, a question that was motivated by the contradictory results obtained from the two different approaches pursued in the virtual lesson to solve fraction division problems. These and several other



stimuli included in the teaching simulation might not frequent real-classroom lessons at least with the regularity with which they were observed in the teaching simulation.

Third, the teaching simulation also *facilitated the possibility of comparability*, which is difficult to obtain in real-classroom settings. For example, the study participants were provided with the same curriculum materials; they were asked to “teach” the same mathematical topic; and they each had to consider and react to exactly the same instructional situations and episodes. Such close comparability is often hard to secure in real-classroom settings, not only because (preservice) teachers are often asked to use different curriculum materials or to teach different topics, but also because it is unlikely that students will react in comparable ways in all classrooms under observation.

These three affordances, alongside the strong associations between knowledge and teaching performance, justify the simulation’s promising role in exploring and better understanding the relationship between knowledge and teaching performance. In fact, future studies could use a similar approach to gauge (preservice) teachers’ performance in other teaching practices or in other mathematical topics and explore the association between teacher knowledge and teaching performance.

The teaching-simulation approach also lends itself to Brousseau’s idea of didactical engineering (cf. Artigue & Perrin-Glorian, 1991; Brousseau, 1997), which closely relates to the “testing-hypotheses” feature discussed above. Acknowledging the difficulties of conducting true experiments in educational settings that would allow for better understanding of educational phenomena, Brousseau proposed that researchers conduct an a priori analysis and identify different variables that might affect a phenomenon. He also recommended that researchers manipulate these variables and

anticipate the outcomes of the situations created by such manipulations. Following Brousseau's suggestion, the teaching simulation can be used to better understand if and how teacher knowledge plays out in their teaching tasks. For instance, it was previously proposed that teachers need to be mindful of certain student misconceptions if they are to identify and address them when they arise in teaching. A teaching simulation that encompasses several such misconceptions could facilitate the exploration of this hypothesis by investigating whether teachers' analysis of the simulation aligns with their level of awareness of certain student misconceptions. Alternatively, one could test hypotheses that question the role of knowledge in supporting teachers' performance in certain teaching tasks. For example, according to the theoretical framework of MKT (see Ball et al., in press), the mathematical knowledge common to all mathematically intensive professions, what the LMT researchers call CCK, does not suffice to perform several teaching tasks. A teaching simulation could help explore this hypothesis by immersing professionals with strong CCK to certain instructional episodes and exploring the degree to which and the manner in which their strong mathematical background supports them in tackling the requirements of the simulation tasks.

Yet, despite its affordances, the teaching simulation should not be considered a substitute to in-vivo approaches currently utilized to explore the complex association between teacher knowledge and their teaching performance. Since each approach has its own affordances but also suffers from certain limitations, the teaching simulation should be considered a complementary approach that is particularly helpful to engineering certain didactical situations and exploring the effect of teacher knowledge in handling these situations.

### *Measuring Teachers' Knowledge Using a Multiple-Choice Test*

The qualitative analysis undertaken in this study highlighted some of the limitations of using multiple-choice tests to measure teachers' knowledge. First, the multiple-choice format of questions can scaffold teachers to correctly answer a question, despite their lacking the knowledge that the question is designed to capture. This situation was captured by Kimberley's pre-intervention comment on the LMT test, in which she acknowledged that in answering some of the test questions she first figured out the numerical answer to the given problems and then checked to see which of the given multiple-choice options matched her answer. Second, teachers might arrive at the correct answer for the wrong reasons, which again confounds the attempt to accurately capture teachers' knowledge; this was the case with pre-intervention Suzanne (see Chapter 4). Third and closely related to the previous point, if teachers simply choose an answer for each question, the test yields little information to unravel and trace their thinking in answering these questions. Fourth, in answering the test questions the teachers might also rely on their mathematical reasoning and/or their test-taking abilities. The analysis of Vonda's pre-intervention performance on the LMT test, for example, revealed that she correctly answered some of the test questions simply by picking the answer "all of the given choices are correct," or "none of the given choices is correct;" this pattern was also observed in Hill and colleagues' recent study (2008) which explored in-service teachers' Knowledge of Content and Students.

Fifth, comparing teachers' scores on the test might obscure notable differences in their knowledge (either across people or for the same people at two different time points). This was the case with Suzanne, who obtained the same numerical score on both the pre-

intervention and the post-intervention test administration; yet her knowledge was notably different between the two test administrations, as suggested by the different questions she correctly answered on each of the two LMT test administrations and her performance in the teaching simulation. Sixth, as Ball and colleagues (in press) caution, the multiple-choice test fails to capture teachers' knowledge-in-use, which comprises a central feature of MKT. Consider, for instance, that during the pre-intervention meeting, Kimberley was able to correctly answer LMT question 17 which pertained to representing the quotient in relative units; yet, during the teaching simulation she could not use the idea of relative units to explain the fractional part of the quotient in the division  $2 \div \frac{3}{4}$ . Seventh, in answering certain questions teachers might draw on a different type of knowledge than that assumed to scaffold their work on answering the multiple-choice questions. For example, although, on the surface, a question that expects teachers to evaluate the reasonableness of different students' arguments appears to require SCK, in answering this question, teachers might employ CCK instead.

Despite these limitations and the inconsistencies between the PSTs' performance on the LMT test and their performance in the teaching simulation, the LMT test had two main affordances. First, the PSTs' rank ordering according to their performance on the LMT test was, overall, indicative of these PSTs' knowledge of the content and its teaching. The test information was particularly accurate in capturing coarse differences between the PSTs' knowledge, which were largely aligned with the differences in the PSTs' teaching performance (e.g., compare the pre-intervention teaching-simulation performances of Nathan and Nicole to those of Deborah and Vonda, which were quite aligned with what the test information implied that these PSTs might have been able to

do). This level of alignment was also suggested by the moderate to high correlations found between the PSTs' MKT performance and their performance in the teaching simulation. Second, even though the LMT test and the teaching simulation measures were developed to tap different constructs, as measuring tools, the LMT test compares favorably to the teaching simulation: it requires no coding, which is time-consuming, and, just like any other multiple-choice test, it is low-inference.

These two affordances suggest that there is value in using multiple-choice tests to measure teachers' knowledge, and particularly the LMT tests whose predictive validity has already been established.<sup>195</sup> At the same time though, the aforementioned limitations point to areas for improvement. For example, in accordance with Hill and colleagues' (2008) study, the findings of this study suggest that it might be beneficial to complement the multiple-choice questions with open-ended questions. In addition, the information on the PSTs' knowledge yielded from the teaching simulation – regardless of the fact that the simulation was mainly designed to capture PSTs' teaching performance – points to another plausible route in measuring teachers' knowledge. With the recent advancements in technology which allow for on-line testing, teachers might be presented with short virtual instructional episodes such those comprising the teaching simulation and be asked to outline and justify their decisions and actions, if they were to encounter analogous situations in their teaching. Because of their virtual character, these episodes could explicitly target specific types of knowledge for specific content areas and instructional practices. For example, they could help explore teachers' Knowledge of Content and Students, which Hill et al.'s recent study (2008) showed to be difficult to investigate using paper-and-pencil multiple-choice tests. Such an approach would require, however,

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<sup>195</sup> For more on this issue see Hill et al. (2007).

an a-priori analysis of what teachers' plausible answers might imply about teachers' knowledge with respect to the different MKT domains.

Overall, the foregoing discussion suggests that there are no straightforward answers or optimal approaches to measure teachers' knowledge. Each approach suffers from specific limitations, but also has certain affordances, thus pointing to the need to use multiple approaches when exploring the association between knowledge and teaching performance. In addition to using complementary approaches, researchers should also consider the issue of alignment between the measures used to gauge teachers' knowledge and those used to tap teachers' performance.

#### *The Need for Better Alignment between the Measures of Knowledge and Teaching Performance*

Instead of randomly selecting an LMT test, the present study utilized a test whose questions were more aligned with the topic examined in the teaching simulation. In addition, the existing LMT test was modified in an effort to select questions that were close to the ideas examined in the teaching simulation. Despite the attempts to align the LMT test with the mathematical topic explored in the teaching simulation this effort was not completely successful. Consequently, the PSTs' knowledge of the content and its teaching as captured by the teaching simulation was better aligned with their teaching-simulation performance compared to the extent of alignment between the PSTs' LMT score and their teaching-simulation performance.

It might then seem reasonable to conclude that to obtain stronger relationships between teacher knowledge and their teaching performance researchers should maximize the degree of alignment between the questions they use to explore teacher knowledge and

the teaching performances they examine. For instance, if the participants of a study will teach a unit on fraction multiplication to sixth graders, teachers' MKT could be explored by using questions that exclusively focus on this content. Even if it were feasible to develop different MKT tests to capture specific content areas for specific grade levels, a close alignment between the topics explored in the knowledge test and those considered during teaching would still be unproductive: although it could yield a higher correlation between knowledge and teaching performance, the knowledge measure obtained from such a test would have limited applicability because this knowledge score would reveal little about teachers' competence in teaching fraction multiplication to fifth or seventh graders, let alone their competence in teaching fractions or multiplication in general.

Consider, now, another scenario in which we explore teachers' MKT in numbers and operations and then observe these teachers giving lessons on data and probability. In this latter case, it seems unreasonable to expect teachers' knowledge, as measured by the test, to highly correlate with their teaching performance. Although it is true that the LMT tests do not tap knowledge of the *content* itself, one cannot ignore the fact that different content topics impose different teaching requirements. Take, for instance, a teacher with a deep understanding of numbers and operations. Would this teacher be effective in teaching a lesson on theoretical and experimental probabilities? Most likely not. Teaching such a lesson requires a totally different type of SCK – to use one of the MKT domains – than that required when teaching a lesson on numbers and operations.

The two scenarios just considered highlight two opposing cases – the former representing a case of high alignment and the latter one with low alignment, if any at all. Moving toward either of these two extremes entails certain costs and benefits in relation

to tacking the issue of alignment. Consequently, researchers in the area of MKT (and beyond) might benefit from defining the type of teacher knowledge required by each content area or mathematical process. The specification Table 6.1 provides some guidelines toward this end.

Table 6.1

*A Specification Table Cross-Tabulating the NCTM Content and Process Standards with the MKT Domains*

Content-Process MKT Domain*	Numbers and Operations	Algebra	Geometry	Measurement	Data Analysis and Probability	Problem Solving	Reasoning and Proof	Communication	Connections	Representations
CCK										
SCK										
KCS										
KCT										
Other										

\* CCK: Common Content Knowledge; SCK: Specialized Content Knowledge; KCS: Knowledge of Content and Students; KCT: Knowledge of Content and Teaching.

Table 6.1 was generated by cross-tabulating the NCTM 2000 content and process standards with the MKT domains. Filling in the cells of this table can potentially not only provide insights about developing new MKT questions (and perhaps even contribute to the theoretical advancement of MKT), but also offer an index of alignment between the MKT questions used in a test and the requirements of teaching particular lessons. This index could, in turn, support researches in efforts to explore the association between teacher knowledge and teachers' instruction and, foremost, to better understand how knowledge (or limitations thereof) can support (or constrain) teachers' work.



### *Exploring Teachers' Beliefs via Surveys*

In his review of the different approaches pursued to measure teachers' beliefs, Philipp (2007) pointed out that skepticism exists among scholars in regard to the validity of data yielded from questionnaires. These skeptics tend to prefer richer data gleaned from more qualitative approaches, such as classroom observations, interview, and responses to vignettes or videotapes. Surveys, no doubt, suffer from certain limitations, one of which is that teachers' reported beliefs might differ from their actual beliefs. Also teachers might interpret the survey statements quite differently from how these statements were intended to be interpreted. Despite their limitations, surveys can offer some *initial* insights into teachers' beliefs, as the results of this study suggest. For example, the quantitative data showed that the PSTs' beliefs were aligned with their performance: all three belief factors were negatively associated with the PSTs' performance, which is consistent with that these factors reflected a type of instruction that was in opposition to establishing rich and challenging learning environments.

In discussing Suzanne's case, however, I pointed to certain discrepancies between her professed beliefs about teaching and learning mathematics as captured by the survey data and her corresponding beliefs as implied by the teaching simulation. Rather than viewing this as a reason to question the validity of the survey data, these discrepancies could indicate that teachers' *overall* beliefs about teaching and learning mathematics (as captured by a survey) might not be consistent with what their actions and reactions to *certain instructional situations* imply about their beliefs. Christopher's case, as considered in Skott (2001), supports this argument.

Christopher responded in remarkably different ways in two instructional episodes in which his students solicited his help. In the first episode, he avoided providing the student with whom he interacted with any hints in order to challenge this student's thinking; this was largely consistent with Christopher's beliefs about teaching and learning mathematics. In the second episode, when two other students solicited his help, Christopher posed leading questions that guided these students to the answer, which appeared inconsistent with Christopher's beliefs. In explaining this apparent discrepancy, Christopher pointed to differences between the two instructional episodes. Unlike the first episode, in the second several students solicited his help; in addition, the two students with whom he interacted in the second episode were quite adamant at getting answers from the teacher.

Christopher's case thus suggests that the discrepancies identified between Suzanne's reported beliefs in the survey and her performance in June's and Shaun's episode might be due to how Suzanne interpreted the teaching situation, as presented in this episode. For instance, although *in general* Suzanne believed that the virtual teacher should give students an opportunity to struggle with mathematical ideas before intervening, when considering the particular teaching episode at hand, she might have considered teacher intervention appropriate *for this particular situation*. Her approval of the teacher's move to immediately intervene and provide students with a hint should also be seen in the context of Suzanne's own difficulties with the content. If the teacher herself struggles with certain mathematical ideas, then she might reasonably assume that students will also, and hence intervene to support them.

This argument about considering teachers' beliefs in situ – in the context of specific instructional episodes – implies that although attempts to understand teachers' instructional performance could be informed by overall measures of teachers' beliefs – as those yielded by surveys – a more in-depth analysis is also warranted. Such an analysis might explain inconsistencies between teachers' reported beliefs and their instructional actions; it might even reveal more complex interactions between knowledge and beliefs that inform teachers' performance than those identified and discussed in this study.

### *Practical Implications*

Although this study mainly sought to explore the association between PSTs' knowledge and their teaching performance, the results have potential implications for teacher education, curriculum development, and issues of equity.

### *Implications for Teacher Education*

As already explained when discussing the mediating effect of the PSTs' GRE performance on the association between teacher knowledge and teaching performance, it is widely believed that teacher preparation programs add little to the effectiveness of their candidate teachers (cf. Grossman, 2008; Levine, 2006). Grossman (ibid) argued that scholars in education themselves are partly responsible for this belief. She stated,

as a research community, we have spent relatively little sustained effort trying to determine how teacher preparation, of any kind, affects either teachers' classroom practices or their influence on student learning, outcomes that are arguably those that the public – including parents and policy makers alike – care about the most. ... [D]espite ... serious and sustained effort at reforming teacher education programs, we still know very little about what characteristics of teacher education make the most difference in preparing teachers to teach well. (pp. 14-15)

Grossman went on to point out that regardless of the volume of studies produced and presented at the *American Education Research Association* (AERA) conferences, very few studies are published in scientific journals. She also considered the field to still be

plagued by the methodological problem of documenting the links between teaching learning and student learning.

Addressing the gap(s) that Grossman identified above was beyond the scope of the present study since, as repeatedly mentioned, the study used the two intervention courses as a context to induce changes in the PSTs' knowledge and practices and to hence explore the association at hand from a dynamic perspective. Yet, one cannot dismiss the fact that *all* the study participants experienced gains in their teaching performance in the teaching practices examined in the study. Deborah's case also suggests that the intervention enriched the PSTs' teaching toolkit with ideas, strategies, and alternative images of teaching. Kimberley's acknowledgment of a paradigm shift in her thinking about teaching the subject should also not be ignored. To be sure, further research is needed to determine if and how these changes in the PSTs' performance and beliefs translate to their future teaching and consequently to how they might affect their own students' learning. Nonetheless, these findings imply that the intervention considered in this study did have an impact on these PSTs' *potential* to teach mathematics differently from how most of them experienced the subject as students. In this respect, the findings offer some testimony about the impact that teacher education programs can have on teacher candidates.

The effectiveness of the intervention considered in this study in inducing changes in both the PSTs' MKT and their teaching performance should not be dissociated from its philosophy and design. Following Ball and Bass' (2000) argument that teacher education programs should integrate issues of content and pedagogy, this intervention sought to support its participants in developing usable and useful subject matter knowledge for the

work of teaching; at the same time, it aimed at enhancing the PSTs' competence in specific teaching practices which are considered critical for the work of teaching (e.g., providing explanations, using representations, analyzing students' contributions, leading classroom discussions). Both the content-pedagogy integration and the recurrent opportunities that the study participants were afforded to practice certain teaching techniques appear to have catalyzed the changes reported in this study, and mainly the PSTs' gains in the performing tasks of the simulation. The study findings suggest that the two intervention courses indeed helped these PSTs to develop useful and usable knowledge for the work of teaching. Consequently, it remains open to further examination whether an intervention with different design features and goals would be equally effective as the intervention considered herein in leveraging the changes documented in the PSTs' teaching performance – both in magnitude and nature.

That said, one cannot dismiss the fact that, despite its focus on supporting the PSTs in attending to issues of meaning and understanding, the intervention did not appear to have sensitized them to issues of task complexity and to the importance of maintaining this complexity during enactment (as suggested, in particular, by the qualitative findings). Nathan, who was the only PST who explicitly talked about these issues, attributed his increased sensitivity to these issues to a conversation he had with the course instructor. What these findings then suggest is that the focus on MKT and its respective practices, although apparently key for training prospective teachers, does not suffice to attune these candidate teachers to the importance of also attending to issues of cognitive demands. Preservice teacher education programs need to also address such issues, if prospective teachers are to be conscientious of them when teaching.

### *Implications for Curriculum Development*

To explore the PSTs' selection and use of curriculum tasks, the study participants were provided with only student textbook pages. This was intentionally done to investigate the PSTs' potential to use and restructure the textbook pages by building on their own resources rather than drawing on the suggestions made by teacher guides. The study showed that the PSTs faced difficulties in selecting, modifying, and using the curriculum tasks of the two textbook pages, thus providing some insights into how teacher manuals can better scaffold teachers' work.

In particular, the pre-intervention performance of the study participants in selecting and using the two textbook pages suggests that the teacher manuals can provide substantive guidelines to compensate for limitations in teacher knowledge. These guidelines might include but should not be limited to:

- a) a brief outline of the basic mathematical ideas of each lesson: including brief memos that address the subject matter to be taught could account for teachers' difficulties with the content (e.g., recall Deborah's argument that she would not use the task on explaining the fractional part of an answer, simply because she was not aware of this concept)
- b) explicit connections between the lesson ideas and other related mathematical topics: making these connections explicit could address gaps in teachers' knowledge (e.g., Tiffany's initial conception that the concept of whole-number division does not apply to fraction division)
- c) a summary of main student misconceptions of the content considered in each lesson/unit: this summary could familiarize teachers with such misconceptions

and help them identify and address them as they arise in teaching; such a summary would be necessary even for teachers with strong knowledge of the content since, as the study findings revealed, strong knowledge does not necessarily imply familiarity with certain student misconceptions (e.g., consider both Kimberley's pre- and post-intervention performance).

- d) a brief explanation of the rationale for including certain tasks in the curriculum materials and the requirements of these tasks (for example, a brief note could explain the affordances of asking students to concurrently represent their work in words, diagrams, and mathematical symbols): including such explanations could help teachers see the value of certain task requirements and inform teachers' decisions to modify these tasks (for instance, such explanations could help Vonda and Kimberley reconsider their decision to simplify the requirements of the tasks of the second textbook page)<sup>196</sup>
- e) brief suggestions for enacting certain instructional tasks, especially the most demanding ones: drawing on the MTF research, curriculum materials could explicitly address issues of cognitive demand and emphasize the importance of enacting tasks at their intended cognitive level; curriculum materials could also provide suggestions for scaffolding student work (especially on the tasks' most demanding aspects) in ways that are both responsive to students and preserve the intellectual challenge

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<sup>196</sup> In their suggestions for designing *educative curriculum materials* – that is materials that are intended to promote teacher learning – Davis and Krajcik (2005) also recommend that curriculum developers make their rationale for proposing certain tasks and activities visible. As these scholars argue, making these rationales visible could offer teachers new ideas but foremost help teachers make connections between theory and practice.

- f) ideas for restructuring tasks: the study findings suggest that restructuring tasks was the area posing the greatest challenge to most of the study participants; hence, the curriculum materials could provide explicit ideas as to how different tasks can be scaled up or down to either address different levels of student abilities or to upgrade the cognitive challenge of curriculum tasks when deemed necessary.

Taken together, these suggestions point to the need to produce curriculum materials that consider both students' and teachers' difficulties with (teaching) the content. In fact, these suggestions echo Remillard (2000) who, drawing on an idea previously advanced by Bruner (1960) and Ball and Cohen (1996), argued that if curriculum materials are to prepare teachers to learn how to teach the content differently from how they experienced it, these materials should be educative for both students and teachers: they should be “designed to speak *to* teachers not merely *through* them” (p. 347, emphasis in the original).

#### *Implications for Issues of Equity*

The present study provided evidence for an association between preservice teachers' MKT and their capacity to build rich and challenging learning settings through their engagement in five teaching practices. If we accept that the PSTs' performance as captured in the teaching simulation reflects their potential to build such environments in real-classroom settings and that the association identified in the present study generalizes over the teacher population, the findings of the study raise concerns of equity. These concerns are fueled by the findings of two relatively recent studies (Hill, 2007, Hill et al., 2005), which both showed that students from lower socioeconomic backgrounds, African American, and Hispanic students are more likely to be taught by low-MKT teachers.



These studies, in conjunction with the findings of the present study could then suggest that the teachers in these disadvantaged settings might be less able to build the type of environments that support students' engagement with the content in ways that nurture their intellect.

Yet, students in disadvantaged school settings can significantly benefit from being immersed in rich and intellectually challenging environments. A study conducted by the QUASAR researchers in the mid 1990s (Silver & Stein, 1996) showed that immersing students in such settings can not only augment their academic performance, but can also increase students' positive attitude toward the subject, which, in turn, can lead these students to select more math courses in high school and beyond. Therefore, to the extent that the abovementioned proposition holds – that teachers in less advantaged school settings might be less likely to build rich and challenging learning environments because of limitations in their MKT – it suggests that any policy decisions to create high-quality educational opportunities for *all* students should not be divorced from considerations of teachers' MKT. This, in turn, suggests that giving more incentives to more knowledgeable teachers to serve in disadvantaged educational settings might contribute to elevating the cognitive level at which the mathematical content is experienced in these schools.

#### Additional Directions for Future Studies

Several directions for future studies have already been offered in this and the previous two chapters, the most notable of which is to further test the propositions derived from this study with respect to how (preservice) teachers' knowledge might enable or constrain what they can do regarding the five practices the study examines (see

Table 4.6). The work presented thus far provides at least four additional directions for future research.

First, a follow-up study could explore which aspects of these PSTs' performance in this in-vitro situation transfer to an in-vivo situation. This exploration could move in two directions, to investigate how a real-classroom situation might constrain the performances observed in this study or compensate for limitations identified in these performances. For instance, would a PST who provided an instructionally appropriate explanation in the teaching simulation be equally successful at providing such an explanation (even for the same mathematical topic) in real-classroom conditions? Conversely, would a PST who struggled to provide such an explanation do so when interacting and co-constructing explanations with students?

Second, future studies could replicate the exploration undertaken in this study with a different teaching population, and particularly with in-service teachers. As explained in Chapter 3, in-service teachers' teaching experience and their images of teaching might confound the relationship between knowledge and performance this study investigated. This proposed study could be replicated even with an intervention different from that considered in the extant study since certain features of the intervention might have influenced the study findings.

The affordances of the teaching simulation used in this study as a *measurement tool* to explore teachers' teaching performance have already been discussed when considering the methodological contributions of the study. In addition to using the teaching simulation as a measurement tool, a third direction for future studies could be to use this or other similar simulations as *teaching tools* to offer (preservice) teachers

recurrent opportunities to practice some of the practices and skills that this simulation was designed to gauge. The effectiveness of such an approach in preparing (prospective) teachers for the work of teaching definitely warrants further exploration.

Finally, future studies could also explore the differential effect that interventions such that explored in this study appear to have on their participants. For example, it is still open to investigation why, for example, Deborah experienced higher gains than Nicole or Kimberley. Such studies could also investigate if and how the teacher candidates' background characteristics affect what these teachers gain from their participation in teacher education programs, as was alluded to when considering the cases of Kimberley and Vonda.

### Conclusion

Before concluding, I would like to clarify a point that I should have made clear even before presenting the seven cases discussed in Chapters 4 and 5. I deliberately opted to reserve this clarification for the end, however, because I wanted this point to be one of the take-home messages of this study.

All of the study participants, and particularly the seven cases considered in the preceding two chapters, offered an instructive platform for exploring and uncovering whether and how teachers' knowledge informs their teaching practice. In no way should the results of the present study be read as a judgment of the abilities of Nathan, Vonda, Deborah, and Kimberley, to name just a few study participants. All Nathans, Vondas, Deborahs and Kimberleys explored in this and other studies that seek to understand the complex association between teacher knowledge and their teaching performance are the very products of the instructional system we wish to improve; what they can do or cannot

do, what they believe or resist, how they think and what they struggle with, reflect the affordances and the limitations of this system. By scrutinizing their cases, the study did not wish to *prove* what these prospective teachers could do, but to produce knowledge that could be of potential use in efforts to *improve* the teaching of mathematics in the U.S. and beyond.

This dissertation began with comparing the current state of teaching mathematics in the U.S., as delineated by the TIMSS video-studies, with the NCTM vision of teaching this subject. Undeniably, this vision offers some directions; yet, as Toulmin (1992) elegantly reminds us,

Available futures are not just those that we can passively forecast, but those we can actively create. ... They are futures which do not simply happen of *themselves*, but can be made to happen, if we meanwhile adopt wise attitudes and policies. (p. 2, emphasis in the original)

Toulmin's reminder places before each of us – researchers, teachers, teacher educators, mathematicians, and policymakers –our responsibilities: the vision of improving the quality of the mathematics instruction offered to the citizens of the 21<sup>st</sup> century is not self-enacting; it can only be realized if significant work is undertaken toward this direction. If this study had a word to contribute to these efforts, this would be that we cannot improve teaching, unless we first understand both teaching itself and the multitude of factors that inform and shape it. In the venture to understand teaching and the factors that shape it, teacher knowledge cannot and should not be left out of the picture.

## **Appendices**

## Appendix 1: The Two Intervention Courses

This Appendix outlines the two intervention courses – the math content course (*MTH 485: Mathematics for Elementary and Middle School Teachers*) and the math methods course (*ED 518: Teaching Children Mathematics*) – which constituted the MKT intervention considered in this study. Both courses have been designed and refined over a decade by a diverse group including mathematics educators, mathematicians, practicing teachers, researchers, and graduate students.<sup>197</sup> These courses aimed at helping the study participants to develop mathematical skills, knowledge, and ways of reasoning necessary for teaching mathematics effectively. The content course was expected to help PSTs move from simply knowing mathematics as educated adults to knowing mathematics as teachers; the methods course was designed to help them move further along this continuum, and develop mathematical knowledge and skills necessary for helping other people learn mathematics.

### The Math Content Course

The math content course consisted of 13 three-hour meetings starting in late June and culminating in mid August 2007. This course had a dual goal. First, it aimed at helping PSTs unpack and develop flexible understanding of important mathematical ideas and processes, largely within the realm of number theory and operations. Second, it sought to offer them opportunities to practice using representation, providing explanations, and analyzing others' thinking.

With respect to the first goal, starting from the second meeting, PSTs were engaged in activities that revolved around concepts such as factors and multiples, even and odd numbers, prime and composite numbers, square numbers, and divisibility rules (see Figure A1). Also discussed in the first third of the course were some challenging mathematics problems that intertwined algebraic and geometric notions and problems that gave the study participants the opportunity to consider concepts related to combinatorics (e.g., the Train problem).<sup>198</sup> The last four meetings of this course (excluding the final meeting in which PSTs took an in-class final exam) focused on fractions in general and the division of fractions in particular. Because this is the math topic explored in this study, the content of these meetings is presented more detail.

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<sup>197</sup> Over the past decade, several people have worked in this group, including Deborah Ball, Magdalene Lampert, Hyman Bass, Laurie Sleep, Timothy Boerst, Mark Hoover Thames, Jennifer Lewis, Andreas Stylianides, Gabriel Stylianides, Alison Castro, Merrie Blunk, Deborah Zopf, Charalambos Charalambous, Yaa Cole, Heather Beasley, Amy Jeppsen, Sam Eskelson, Yeon Kim, Meri Muirhead, and Minsung Kwon.

<sup>198</sup> See [http://www-personal.umich.edu/~dball/presentations/041108\\_NCTM.pdf](http://www-personal.umich.edu/~dball/presentations/041108_NCTM.pdf) for a description of this problem.

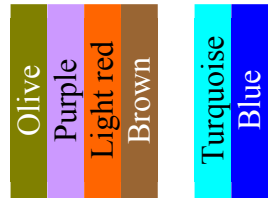
	Before 1 <sup>st</sup> class		Class 1		Class 2		Class 3		Class 4 (Lab-class)	
	Content	Practices	Content	Practices	Content	Practices	Content	Practices	Content	Practices
<b><i>In-class activities</i></b>			Brief introduction to MKT		Pool Border problem	Pool Border problem	Grid Rectangle problem	Grid Rectangle problem	The 8's problem	
			Pool Border problem		Hexagon Train problem (introduction)	Hexagon Train problem	The Train problem	The Train problem	Modeling subtraction with base-ten blocks	
			LMT questionnaire for exploring the PST's MKT		Grid Rectangle problem	Grid Rectangle problem			The Train problem	
<b><i>Individual Assignments</i></b>	Ball, Hill, & Bass, (2005)		Pool Border Problem		Grid Rectangle problem	Grid Rectangle problem	The Train problem			
	Ma (1999), Ch. 1		Preparation for the teaching simulation interview				The 8's problem			
	Initial survey									
<b><i>Partner assignments</i></b>					Hexagon Train problem	Hexagon Train problem				
					Triangular numbers	Triangular numbers				
					Sticks problem	Sticks problem				

Figure A1. The map of the math content course.

**Color Coding Key:**

- Content**  
 Place value/operations  
 Number theory  
 Algebra-geometry  
 Combinatorics  
 Fractions

*Fractions*  
*The division of fractions*

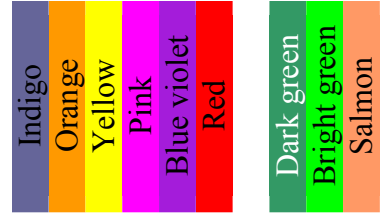


**Teaching practices**

- Analyzing /Selecting tasks (A/S tasks)  
 Giving explanations  
 Using representations  
 Analyzing student work/contributions  
 Responding to students' request for help  
 Other teaching practices

**Other**

- Data collection activities  
 MKT-related readings  
 Other readings





	Class 5		Class 6		Class 7		Class 8		Class 9	
	Content	Practices	Content	Practices	Content	Practices	Content	Practices	Content	Practices
<i>In-class activities</i>	Reflection on Lab-class observations		Evaluating definitions of even and odd numbers		Proving that "odd + odd = even"		Divisibility rules; Prime factorization		Cookie Jar Problem: sharing and comparing solutions	
	Even and odd numbers (Definitions)		Proving that "odd + odd = even"		The Units-digit rule		Different representations and meanings of $\frac{3}{4}$		Bagel problem: Introduction	
	Factors - multiples; Primes - composites				Divisibility rules (2, 4, 5, 10)		Cookie Jar problem			
	Factors and multiples; even numbers				Divisibility rules (4, 8, 3, and 9)				Different versions of the Bagel problem	
<i>Individual Assignments</i>										
<i>Partner assignments</i>										
<i>Midterm exam</i>										
			Even - odd, prime numbers; factors, multiples							
			A train problem							
			Evaluating definitions on square numbers							
		Proving that "odd + even = odd"								
		The Pentagon problem								

Figure A1. The map of the math content course (continued).

	Class 10		Class 11		Class 12		Class 13- Final exam	
	Content	Practices	Content	Practices	Content	Practices	Content	Practices
<b>In-class activities</b>	Comparing fractions		Connecting fractions to division		Interpreting the remainder		A version of the Bagel problem	
	Bagel problem		The division of fractions ( $1\frac{3}{4} \div \frac{1}{2}$ )		Explaining the algorithm of the division of fractions		On number theory	
			Two interpretations of division		Different representations of $\frac{3}{4}$		Divisibility rules	
			Writing and evaluating problems for $3 \div 2/3$		Rational numbers		Posing a problem for the division of fractions ( $4 \div 3/5$ )	
<b>Individual Assignments</b>			Interpreting the remainder		Division by zero		Different representations of fractions	
							A version of the Cookie Jar problem	
	Bagel problem-connecting fractions to division		Solving problems on the division of fractions; explaining the remainder					

Figure A.1. The map of the math content course (continued).

**Color Coding Key**

- Content**  
 Place value/operations  
 Number theory  
 Algebra-geometry  
 Combinatorics  
 Fractions

*Fractions*  
*The division of fractions*

Olive  
 Purple  
 Light red  
 Brown  
 Turquoise  
 Blue

**Teaching practices**

- Analyzing /Selecting tasks (A/S tasks)  
 Giving explanations  
 Using representations  
 Analyzing student work/contributions  
 Responding to students' request for help  
 Other teaching practices

**Other**

- Data collection activities  
 MKT-related readings  
 Other readings

Indigo  
 Orange  
 Yellow  
 Pink  
 Blue violet  
 Red  
 Dark green  
 Bright green  
 Salmon

During the eighth class, the PSTs had the opportunity to compare different representations and interpretations of fractions (i.e., part-whole, measure, operator, quotient, and ratio); they also compared fractions using various strategies. The *Cookie Jar* problem<sup>199</sup> introduced in this class created a space for the PSTs to dwell on the notion of units, when considering fractions; this idea is particularly important in the division of fractions, given that in such problems one needs to consider two different units: the unit defined by the dividend (what I call *absolute unit*) and the unit defined by the divisor (what I call *relative unit*). The work on the *Cookie Jar* problem culminated with the study participants' sharing and comparing their different solutions to the problem and the different representations they employed while solving this problem.

The *Bagel* problem,<sup>200</sup> the problem introduced in Class 9 and further explored in Class 10, helped highlight the importance of identifying the reference unit in problems on fractions; it also created an arena for reinforcing the idea of closely attending to the unit under consideration and for drawing connections between fractions and division (i.e., the quotient interpretation of fractions). In particular, this problem lends itself to two different answers, equally appropriate, which stem from using a different reference unit. The candidate teachers had the opportunity to consider both types of answers to solving several versions of the *Bagel* problem and other related problems.

Class 11 and the first half of Class 12 were allocated to the division of fractions. During these classes, the study participants were first asked to write a problem that represents the division  $1\frac{3}{4} \div \frac{1}{2}$ .<sup>201</sup> This activity created the space for discussing a common error that adults and students alike make, namely dividing *in* half instead of dividing *by* half. Discussion then revolved around what makes writing a problem for this division difficult, which allowed for the introduction of the two interpretations of division: partitive and measurement. Then, the course participants had another opportunity to write different types of problems for the division  $3 \div \frac{2}{3}$ . During the last part of Class 11, the PSTs worked in groups to resolve an ambiguity that arises when interpreting the remainder in a division-of-fractions problem using absolute rather than relative units.<sup>202</sup> In the next class, the groups shared and compared their work on this

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<sup>199</sup> The *Cookie Jar* problem was originally developed by Lampert and Ball and used in several of their courses offered to PSTs (see Lampert & Ball, 1998, pp. 134-135 for more information). The version of the *Cookie Jar* problem used in the eighth class was the following: "There was a jar of cookies on the table. Bob was hungry because he hadn't had breakfast, so he ate half the cookies. Then Suzanne came along and noticed the cookies. She thought they looked good, so she ate a third of what was left in the jar. Dana came by and decided to take a fourth of the remaining cookies with her to her next class. Then Vonda came dashing up and took a cookie to munch on. When Travis looked at the cookie jar, he saw that there were two cookies left. 'How many cookies were there in the jar to begin with?' asked Bob."

<sup>200</sup> The version of the *Bagel* problem used in the course reads as follows: "Leo has 3 dozen bagels that he wants to share equally among Nicole, Tiffany, Ying, Teresa, and Nora. How many dozen or how much of a dozen can he give to each of his five classmates?" Depending on which unit is used, this problem admits two answers: (a) each classmate gets  $\frac{3}{5}$  of a dozen of bagels or (b) each classmate gets  $\frac{1}{5}$  of the 3-dozen quantity of bagels.

<sup>201</sup> This activity was based on Ball's work (1988, 1990a) in which she explored PSTs' grasp of fraction division.

<sup>202</sup> The problem used in this activity was the following: "Mary, John, and Jessie were working on solving the problem: 'Ben is working at a confectionery. If he needs three quarters of an hour to make a cake, how many cakes can he make in four hours?' Mary argued: 'Four hours is 240 minutes. Three quarters of an hour is 45 minutes. Two hundred forty divided by 45 is five remaining 15.' John disagreed: 'That

problem and worked collaboratively to explore why the algorithm of the division of fraction works. In particular, they were asked to consider why dividing by half is equivalent to multiplying by two. In this class, the PSTs also had the opportunity to extend their work on division, and consider the case of division by zero.

Regarding the second goal of the course, as Figure A1 suggests, the study participants had ample opportunities to hone their performance in the MKT-related practices considered in this study: using representations, giving explanations, and analyzing others' thinking and work – that of their classmates or of elementary school children, as implied in artifacts from student work. Regardless of the content these classes considered, the course participants were asked to use different forms of representations (symbols and different diagrams) to explain their thinking. They were also persistently pushed to explain their thinking, by being prompted to elaborate their ideas, give examples, justify why certain solutions were applicable to a given problem, and prove that there were no solutions to a problem or that they had found all possible solutions to it. The course participants were also asked to closely listen to and analyze their classmates' thinking. It should also be mentioned that during Class 9, they worked in groups and generated criteria to determine the quality of mathematical explanations. A list of similar criteria generated by other prospective teachers (see Thames, 2006, p. 18) was also shared and discussed during this class.

In sum, the first course was designed to enhance the PSTs' MKT in the content areas discussed above and to help them improve their performance in the three MKT-related practices. Given its underlying practice-based philosophy, the course utilized different approaches to enhance its participants' MKT and performance in these teaching practices. In addition to being asked to use representations, give explanations, and analyze others' thinking, these PSTs were also expected to analyze different records of practice, including videotapes of math lessons and student interviews, and samples of student work. They also had an opportunity to visit a summer lab-class taught by Deborah Ball, an experienced teacher and mathematics educator, and observe her modeling the three MKT-related practices this study explored. To extend the PSTs' opportunities to deepen their knowledge and hone their respective teaching practices, these candidate teachers were also expected to complete individual and partner assignments, a take-home midterm exam, and an in-class final exam. The individual assignments included mathematical problems similar to those discussed in class, or asked the PSTs to reflect on the mathematical ideas and practices previously discussed. The partner assignments and the midterm exam provided opportunities for practice and review, since the class participants were asked to apply or extend their mathematical class work. The final exam was comprehensive and expected these PSTs to draw on the knowledge and skills they developed during the course and solve problems related to those discussed over the semester.

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can't be right. I did the division and I found  $5\frac{1}{3}$ . Fifteen minutes is not a third of an hour but a quarter of an hour.' 'John you can't be right', said Jessie, and continued, 'The problem talks about fourths. How come you ended with thirds?'" In this problem, Jessie is referring to absolute units (i.e., the remaining 15 minutes are  $\frac{1}{4}$  of an hour), whereas John is referring to relative units (i.e. the remaining 15 minutes are  $\frac{1}{3}$  of the total time needed to bake a cake).

## The Math Methods Course

The math methods course taught during the fall semester of 2007 extended the work pursued in the content course but mostly focused on four domains deemed necessary for supporting PSTs to become competent beginning teachers of mathematics: leading a whole class discussion about mathematics, representing mathematical ideas, assessing students' mathematical knowledge, skills, and dispositions, and planning mathematics lessons. The focus on these domains afforded the course participants additional opportunities to hone their performance in the three MKT-related practices examined in this study. The course focused explicitly on two of these practices: using representations and analyzing student work and contributions. Additionally, all four domains collectively encouraged participants to continue their work on providing explanations. Figure A2 outlines the opportunities made available to the course participants to engage in the practices under investigation. This figure also presents the math content considered during this course.

As Figure A2 suggests, the methods course immersed the study participants in several activities designed to improve their performance in a variety of teaching practices. These activities included participating in several in-class activities, studying and reflecting on relevant readings, and observing an experienced fifth-grade teacher while teaching a lesson (before Class 7). The study participants were also expected to apply the ideas discussed in class during their fieldwork placement (i.e., professional-practice assignments).

Specifically, with respect to *using representations* and *providing explanations*, several in-class activities required the study participants to map their use of representations and manipulatives on certain algorithms and clearly explain these correspondences (e.g., Classes 3, 4, 5, 6, 8, 12, and 13). In so doing, these activities not only created an arena for honing the study participants' performance in using different representations and manipulatives and in *providing explanations*, but they also sensitized them to the importance of linking these algorithms to their underlying meaning. With respect to *analyzing student work and contributions*, several of the course activities and readings engaged the study participants in guided observation and analysis of student thinking and work. For example, in Class 5, after discussing the five strands of mathematical proficiency (cf., Kilpatrick et al., 2001), the course participants were asked to analyze an episode from a third-grade lesson using these strands as an observation lens. They were also requested to repeat this same activity in their field placement. One of their professional-practice assignments also asked them to interview an elementary- or middle-grade student while the latter was working on place-value tasks, and closely attend to and analyze this student's thinking to make assertions about the student's understandings and struggles (Classes 7 and 8). Class 9 included different activities all centered on analyzing student thinking: analyzing the thinking of a five year-old student shown in a video clip; analyzing student errors on whole number multi-digit multiplication; and analyzing the thinking of the student that the PSTs had interviewed before this class. A third of Classes 12 and 13 was also allotted to analyzing students' work on addition and multiplication of whole numbers and decimals.

The methods course also offered its participants opportunities to improve their performance in other teaching practices, such as posing questions, selecting tasks, and

learning in and from practice. A clarification is needed regarding the practice of *analyzing and selecting tasks*: although this practice relates to one of the MTF-practices examined in this study (i.e., selecting and using tasks), neither the MTF nor issues on maintaining the cognitive demands of selected tasks were addressed at any point during this course.<sup>203</sup> When engaged in this practice, the course participants were mostly asked to analyze the tasks of a lesson on decimals (Class 8) and a lesson on division of whole numbers (Class 11) mainly from two perspectives: first with respect to the extent to which the tasks maintained the integrity of the mathematics and second whether these tasks could potentially advance the learning of *all* students (i.e., issues of equity). Only for the purposes of the last culminating performance (i.e., planning a teaching a lesson) were the course participants asked to analyze the tasks presented in a teacher guide *and* select tasks to teach a lesson.

Likewise, it should be noted that, although during the course, the PSTs discussed several approaches related to leading a productive classroom discussion and establishing a productive learning environment, most of these discussions revolved around equity issues. Only in Class 10 was there a brief discussion on responding to students' requests for help during individual or group seatwork. During this discussion the instructor remarked that when students appear to struggle with an assigned task, teachers are often inclined to "jump in" and either re-teach the content to the whole class or do most of the work for their students.

Regarding its mathematical content, the math methods course largely focused on place value, and operations with whole numbers and decimals. As evident in Figure A2, the math methods course did not offer these intending teachers the opportunity to further work on the division of fractions. However, Class 11 revolved around the division of *whole* numbers. During this class, the course participants were reminded of the two interpretations of division (i.e., partitive and measurement) also discussed during the content course. They also considered the meaning of remainders when one works with whole or rational numbers. The course participants were also asked to pose word problems for the division  $38 \div 4$  that admitted different answers (i.e.,  $9 \frac{1}{2}$ , 9, 10, and 2). Classes 12 and 13, on the other hand, offered the course participants opportunities to practice modeling the long division algorithm using manipulatives.

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<sup>203</sup> Yet the course participants were asked to read the NCTM 1991 *Standard* on tasks that discusses the importance of using worthwhile tasks in teaching mathematics.

	Class 1		Class 2		Class 3		Class 4		Class 5	
	Content	Practices	Content	Practices	Content	Practices	Content	Practices	Content	Practices
<b>In-class activities</b>	Overview of the course		Discussion of the course assignments		Subtraction (modeling the teaching of a mini problem)		Algorithms on addition and subtraction		Modeling addition and subtraction algorithms	
	The Three-coin problem		The bundling sticks activity-Place value		Analyzing student work on 'subtraction' problems		Preparing to teach a mini-problem		Different meanings of subtraction	
	Analysis of students' work on the Two-coin problem		Teacher moves			Posing questions			Analyzing student work using the five strands of mathematical proficiency	
<b>Weekly assignments</b>	Teaching to establish a classroom culture		Bundling sticks and representing numbers		Worthwhile math tasks (from NCTM, 1991)	A/S tasks	Modeling an addition algorithm		Analyzing work on a mini problem	Posing questions
			Establishing an inclusive learning environment		Students' work on additive structure problems		The five strands of mathematical proficiency		Subtraction with regrouping (Ma, 1999)	
			Productive talk moves						Analyzing student errors in subtraction	
<b>Professional-practice assignments</b>					Planning to teach a mini-problem		Teaching a mini-problem		Using the five strands to analyze student thinking	

Figure A2. The map of the math methods course.

**Color Coding Key:**

- Content**
- Place value
  - Addition and subtraction (Z)
  - Multiplication and division (Z)
  - Decimals
  - Combinatorics
  - Non-specified

Olive  
Yellow green  
Gold  
Bright blue  
Brown  
White

**Teaching practices**

- Analyzing /Selecting tasks (A/S tasks)
- Giving explanations
- Using representations
- Analyzing student work/contributions
- Responding to students' request for help
- Other teaching practices

**Other**

- Data collection activities
- MKT-related readings
- Other readings

Indigo  
Orange  
Yellow  
Pink  
Blue violet  
Red

Dark green  
Bright green  
Salmon

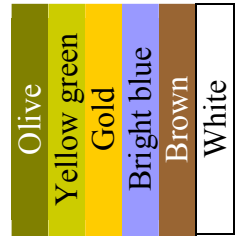
	Class 6		Class 7 and site-visit observation		Class 8		Class 9		Class 10	
	Content	Practices	Content	Practices	Content	Practices	Content	Practices	Content	Practices
<b>In-class activities</b>	Reflection on practice (mini-problem)	Learning from practice	Site-visit observation	Learning in practice	Teaching equitably		Watching and analyzing student interview		Teaching while students work independently	Responding to students
	Place value and base-ten blocks		Ordering decimals		Analyzing textbook pages (on decimals)	A/S tasks	Error analysis		Multiplying decimals	
	Modeling subtraction with base-ten blocks		Computations with decimals		Modeling multiplication of whole numbers		Using Michigan CLECs to discuss student thinking			
			Analyzing student interviews				Debriefing of student interview			
<b>Weekly assignments</b>	Multi-digit computation		Decimals	A/S tasks	Analyzing student errors (multiplication)		Multiplying decimals		Multiplying decimals	
	Teaching while leading a classroom discussion		Listening and analyzing student thinking		Modeling multiplication with base-ten blocks				Analyzing a textbook lesson on division of whole numbers	A/S tasks
			Reflecting on site-visit observations	Learning in practice						
<b>Professional-practice assignments</b>	Preparing to conduct a student interview	A/S tasks	Conducting a student interview		Analyzing interview and making assertions		Preparing a memo including assertions on student thinking			

Figure A2. The map of the math methods course (continued).

**Color Coding Key:**

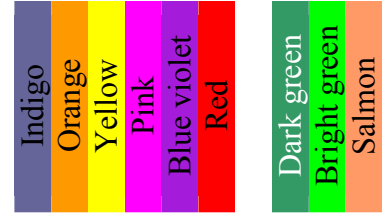
**Content**

- Place value
- Addition and subtraction ( $Z$ )
- Multiplication and division ( $Z$ )
- Decimals
- Combinatorics
- Non-specified



**Teaching practices**

- Analyzing /Selecting tasks (A/S tasks)
- Giving explanations
- Using representations
- Analyzing student work/contributions
- Responding to students' request for help
- Other teaching practices
- Other**
- Data collection activities
- MKT-related readings
- Other readings



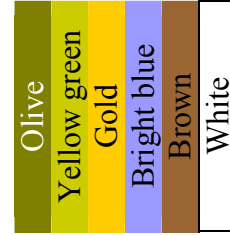


	Class 11		Class 12		Class 13	
	Content	Practices	Content	Practices	Content	Practices
<b>In-class activities</b>	The two interpretations of division		Analyzing students' work on two multiplication problems		Modeling the long division algorithm	
	Writing division story problems		Dividing large numbers – modeling the long division algorithm		Remediating student errors using base-ten blocks	
	The remainders in division problems		Helping students develop fluency with basic facts		LMT questionnaire for exploring the PST's MKT	
	Analyzing a lesson on division of whole numbers	A/S tasks				
<b>Weekly assignments</b>	Long division					
<b>Professional- practice assignments</b>	Preparing to teach a lesson	A/S Tasks	Teaching a lesson		Analyzing a lesson	Leading a discussion

Figure A2. The map of the math methods course (continued).

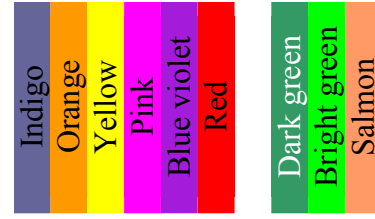
**Color Coding Key:**

- Content**  
Place value  
Addition and subtraction (Z)  
Multiplication and division (Z)  
Decimals  
Combinatorics  
Non-specified



**Teaching practices**

- Analyzing /Selecting tasks (A/S tasks)  
Giving explanations  
Using representations  
Analyzing student work  
Responding to students' request for help  
Other teaching practices  
**Other**  
Data collection activities  
MKT-related readings  
Other readings



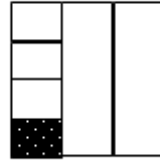
## Appendix 2: A Description of the Questions Included in the Adapted LMT Test

Question	Description
1	A teacher needs to determine the correctness of students' drawings of $\frac{3}{4}$ . These drawings reflect different interpretations of units: some conventional (e.g., Mina's) and some unconventional (e.g., José's).
2	Students offer different explanations for comparing two fractions and their teacher has to determine the validity of these explanations.
3	A teacher is confronted with analyzing a student error on subtracting mixed numbers.
4	A teacher has to determine how to use a number line to answer a fraction multiplication problem.
5	Students write different division-of-fractions sentences to represent a drawing; their teacher needs to decide which sentences correspond to this drawing (see original question in Figure A3 below)
6	Students provide different explanations for simplifying fractions; their teacher has to decide which explanation provides the best evidence of students' understanding.
7	A teacher assigns an exercise on fraction addition and multiplication (on a number line); she has to evaluate the correctness of students' solutions to this problem.
8a-8d	A teacher is preparing a fraction quiz and she wants to include a word problem on a fraction subtraction. She finds four such problems and needs to decide which apply to the given fraction-subtraction sentence.
9	Students propose different drawings to represent a given fraction; their teacher has to evaluate the students' productions.
10	A teacher is asked to determine the correctness of an unconventional student method for dividing fractions.
11a-11d	A teacher prepares four word problems on fraction division. The item asks which of these four problems correspond to a given division sentence. (A similar item appears in Figure A4 below.)
12	A student offers an argument to explain her thinking on comparing two fractions; her teacher needs to determine what this argument suggests about the student's thinking.
13	A teacher analyzes a student's solution on adding positive and negative mixed numbers.
14a-14e	Students offer different explanations for the meaning of a fraction-division sentence; their teacher needs to evaluate the validity of these interpretations.
15	Students propose two division sentences to represent a drawing; their teacher has to decide whether the proposed sentences correspond to the given drawing.
16	A teacher is asked to evaluate students' answers to a fraction-equivalence problem.

Question	Description
17	Students offer different explanations for the remainder in a fraction-division problem; their teacher evaluates the mathematical validity of these explanations.
18	Students are presented with a square partitioned into different regions; they are asked to determine which of these regions have the same area (students could use fraction multiplication to inform their work).
19a-19e	Students come up with different approaches for dividing fractions; their teacher has to decide which of these approaches are correct.
20	Students offer different ideas to explain a visual representation; their ideas are based on different interpretations of units; their teacher is asked to evaluate the correctness of their ideas.
21	A teacher asks her students to use a definition on fractions to explain what a visual drawing represents; she has to decide which of the students' productions clearly reflect the definition introduced in the lesson.
22	A teacher considers how to use a number line to answer a fraction-division problem.
23	A teacher is asked to identify the reason for a student error on ordering fractions.
24a-24d	A teacher is preparing a quiz and she wants to include a word problem on decimal division. She finds four word problems and she has to determine which of them correspond to the given decimal-division sentence.
25	A teacher has to pick among a list of word problems one or more that represent the measurement interpretation of division.

5. Ms. Peterson gave her students the following problem.

Write a division problem for the shaded region in the diagram below.



Of the following solutions, which is mathematically correct? (Mark one.)

- a)  $3 \div \frac{1}{4}$
- b)  $\frac{1}{3} \div \frac{1}{4}$
- c)  $\frac{1}{4} \div \frac{1}{3}$
- d) None of the above is mathematically correct.

Figure A3. Question 5 of the adapted LMT test.

Which of the following story problems could be used to illustrate  $1\frac{1}{4}$  divided by  $\frac{1}{2}$ ? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2	3
b) You have \$1.25 and may soon double your money. How much money would you end up with?	1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2	3

Figure A4. A released LMT item on fraction division.



Appendix 3: The Script of the Virtual Lesson and the Opportunities to Study the PSTs'  
Performance in the Five Teaching Practices



SCRIPT		Short Commentaries on the Opportunities to Explore the PSTs' Performance	
		MKT- related practices	MTF-related practices
<i>Ms. Rebecca:</i>	Okay, what are we going to divide in this case? Can somebody tell me?	13	
<i>Ms. Rebecca:</i>	Who has an idea? How should we do this? Shaun?	14	
<i>Shaun:</i>	We are going to divide one half by one sixth.	15	
<i>Ms. Rebecca:</i>	What do others think? Julia?	16	
<i>Julia:</i>	I agree with Shaun.	17	
<i>Ben:</i>	I have a question. How do we know which fraction goes first? I mean ... why aren't we dividing one sixth by one half?	18	
<i>Ms. Rebecca:</i>	That's a very good question, Ben! Remember, we need to be convinced that an answer makes sense before we accept it.	19	
<i>Ms. Rebecca:</i>	Anyone want to answer Ben's question?	20	
<i>Ms. Rebecca:</i>	( <i>June and Travis raise their hands.</i> ) June, what do you think?	21	
<i>June:</i>	Hmm.... It's because one sixth is smaller than one half.	22	
<i>June:</i>	So ... so, if he starts with a piece of one sixth of a yard, he cannot make any ribbons that would be half a yard long.	23	<u><i>Analyzing student work (slides 22-29)</i></u> The teacher accepts an explanation that is incorrect. In fact, in so doing, she reinforces the idea that the dividend should always be larger than the divisor.
<i>June:</i>	Can I show it on the board?	24	
<i>Ms. Rebecca:</i>	Go for it!	25	
<i>June:</i>	The shorter line is the one sixth. The longer one is the one half.	26	
<i>June:</i>	So, how could he make any ribbons of this size ( <i>points to the longer line segment</i> )?	27	


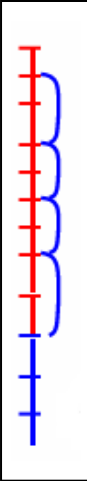



SCRIPT		Short Commentaries on the Opportunities to Explore the PSTs' Performance	
		MKT- related practices	MTF-related practices
<p><i>June:</i> If he started with this size? (<i>Points to the shorter line segment.</i>)</p> <p><i>Ms. Rebecca:</i> Good explanation, June. Thanks.</p> <p><i>Ms. Rebecca:</i> (<i>Turns to Ben.</i>) Ben, is it clear now?</p> <p><i>Ben:</i> Yeah, kind of.</p> <p><i>Ms. Rebecca:</i> Okay. Let me also use a line to help us solve the problem.</p> <p><i>Ms. Rebecca:</i> Let me first erase this.</p> <p><i>Ms. Rebecca:</i> (<i>Turns to the board.</i>) Okay. Here is a line of one yard. (<i>Draws the following on the board:</i>)</p> <div style="border: 1px solid black; width: 100px; height: 20px; margin: 10px auto; position: relative;"> <div style="position: absolute; left: 0; bottom: 0; width: 20%; height: 100%; background-color: red;"></div> <div style="position: absolute; left: 20%; bottom: 0; width: 80%; height: 100%; background-color: black;"></div> </div> <p><i>Ms. Rebecca:</i> Okay. The red line is half a yard. This is the ribbon we have.</p> <p><i>Ms. Rebecca:</i> And what are we asked to do? Amanda?</p> <p><i>Amanda:</i> To make ribbons of one sixth.</p> <p><i>Ms. Rebecca:</i> So, what should our next step be?</p> <p><i>Alan:</i> To figure out what a sixth is?</p> <p><i>Ms. Rebecca:</i> That's correct. Alan, can you come to the board and show us how to do it?</p> <p><i>Alan:</i> (<i>Comes to the board. He seems a little confused.</i>) O-kay...</p> <p><i>Ms. Rebecca:</i> Let me help you a little bit. You want to cut parts of one sixth. What does one sixth mean?</p> <p><i>Alan:</i> Take one piece of six.</p> <p><i>Ms. Rebecca:</i> So, how many pieces are you going to divide your line into?</p> <p><i>Alan:</i> Six?</p>	<p>Slides</p> <p>28</p> <p>29</p> <p>30</p> <p>31</p> <p>32</p> <p>33</p> <p>34</p> <p>35</p> <p>36</p> <p>37</p> <p>38</p> <p>39</p> <p>40</p> <p>41</p> <p>42</p> <p>43</p> <p>44</p> <p>45</p>	<p><u>Using tasks:</u> Ms. Rebecca does most of the thinking for her students, by asking pointed questions.</p> <p><u>Responding to an indirect request for help (part A; slides 41-47)</u> The teacher intervenes and guides Alan to the correct answer. It is questionable whether Alan really understands why he should use the whole line (cf. Topaze effect).</p>	

SCRIPT		Slides	Short Commentaries on the Opportunities to Explore the PSTs' Performance
			MKT- related practices
			MTF-related practices
Ms. Rebecca:	That's right. Six. So go ahead and divide your line.	46	
Alan:	(Alan divides the colored part into sixths). Did it!	47	
			
<b>“Time for Reflection” Slide</b>			
Ms. Rebecca:	Alan, wait a little bit! Is it ONLY the colored part that you should divide into six parts or the whole line?	48	
Alan:	Hmm.... The whole line.	50	
Ms. Rebecca:	(Enthusiastic) Aha! So, go ahead and correct your picture.	51	<u>Using representations (slides 51-58)</u> The teacher fails to either model how the algorithm maps onto the pictorial representation that was used before or ask students to do so.
		52	
Alan:	Alan erases what he did before and draws the line again, colors half of it, and divides it into 6 parts. Did it!	53	
			
Ms. Rebecca:	Thanks, Alan. Okay, class, this is important to remember. You should use the whole...	54	<u>Using tasks:</u> The teacher lists steps for students to follow. She also places more emphasis on getting right answers.
Ms. Rebecca:	... because if you use a part of it, you won't get the right answer.	55	
Ms. Rebecca:	I want somebody to come to the board and write the mathematical sentence for this problem.	56	

SCRIPT		Slides	Short Commentaries on the Opportunities to Explore the PSTs' performance	
			MKT- related practices	MTF-related practices
<i>Ms. Rebecca:</i>	Amanda?	57		
<i>Amanda:</i>	(Goes to the board and writes the mathematical sentence: $\frac{1}{2} \div \frac{1}{6} = 3$ )	58		
<i>Ms. Rebecca:</i>	Okay, class. Let's summarize what we've done so far. What was the first thing we did?	59		<b>Using tasks:</b> The teacher lists steps for students to follow. Although she started with an intellectually challenging task, she did most of the thinking for students, and then, even summarized the steps that they need to follow to solve other similar problems.
<i>Shaun:</i>	We drew a line to help us solve the problem.	60		
<i>Ms. Rebecca:</i>	Correct. And what do we need to remember when drawing this line? Michelle?	61		
<i>Michelle:</i>	Hmm.... To use the whole thing, not just part of it.	62		
<i>Ms. Rebecca:</i>	Correct. And what's next? Somebody else? Ann?	63		
<i>Ann:</i>	Then, we should check how many times the short piece goes into the big piece.	64		
<i>Ms. Rebecca:</i>	Nice. Let's do a couple more exercises, and then, we will try to find a pattern.	65		
	<u>Slide with narration:</u> <i>Ms. Rebecca assigns exercises A2: <math>\frac{3}{4} \div \frac{1}{6}</math> and A3: <math>2\frac{2}{3} \div \frac{1}{6}</math>. While students are working on their worksheets, Ms. Rebecca circulates and provides support when asked.</i>	66		
<i>June:</i>	<i>Ms. Rebecca, I have a question.</i>	67		
<i>Ms. Rebecca:</i>	Yes, June?	68		
<i>June:</i>	I am working on three fourths divided by one sixth. So, I drew a line and found the three fourths of it. See?	69		

SCRIPT		Slides	Short Commentaries on the Opportunities to Explore the PSTs' performance	
			MKT- related practices	MTF-related practices
<i>June:</i>	But then, how do I split this into pieces of one sixth? ( <i>Her worksheet includes a line segment <math>\frac{3}{4}</math> of which are colored in red.</i> )	70		<p><b><u>Responding to students' requests for help (slides 69- 81)</u></b>  As soon as the teacher realizes that students struggle with an idea, she intervenes and provides them with hints.  Also some of the following exchanges are too vague. What does "it" refer to in the teacher's and Clark's utterance? The teacher seems to interpret students' contributions in ways that make sense to her; yet it is not obvious that students' contributions are clear to their classmates.</p>
<i>Shaun:</i>	Yeah, I don't get that one either.	71		
<i>Ms. Rebecca:</i>	Boys and girls, I want your attention for a sec.	72		
<i>Ms. Rebecca:</i>	Okay. I noticed that several of you showed three fourths on your line, but then did not know how to divide it into sixths.	73		
<i>Ms. Rebecca:</i>	Well, let us think about it. What did we learn in previous lessons? Remember the idea of finding common multiples?	74		
SS:	Yes??	75		
<i>Ms. Rebecca:</i>	So, you need to find a common multiple for four and six. Who can suggest one?	76		
<i>Ms. Rebecca:</i>	Clark, what do you think?	77		
<i>Clark:</i>	Hmm... Twelve.	78		
<i>Ms. Rebecca:</i>	Then what? What should you do with it?	79		
<i>Clark:</i>	Cut it into twelfths?	80		
<i>Ms. Rebecca:</i>	Correct. So go ahead and split your line into twelfths.	81		
	<u>Slide with narration:</u> <i>Students continue working on their worksheets, and then pair and share their solutions. After about five minutes, Ms. Rebecca asks them to share their work with the rest of the class.</i>	82		
<b>"Time for Reflection" Slide</b>		83		
<i>Ms. Rebecca:</i>	So, I want two students to come to the board and explain what they did. Amanda and Julia, could you show us how you divided three fourths by a sixth?	84		

SCRIPT		Short Commentaries on the Opportunities to Explore the PSTs' performance	
		MKT- related practices	MTF-related practices
		Slides	
<i>Amanda-Julia:</i>	Sure! <i>Slide with narration:</i> <i>Amanda and Julia work on the board.</i> Okay. First, we drew a line and divided it into twelve pieces. <i>Her work at the board looks like this (although the pieces are not of equal size):</i>	85	
<i>Amanda:</i>		86	
<i>Amanda:</i>	(Laughs:) Well, at least we tried to make them of equal size. Then, we colored nine of them.	87	
<i>Julia:</i>	So, to figure out how many pieces we can make out of this whole red piece.	88	
<i>Julia:</i>	One...	89	
<i>Julia:</i>	Two...	90	
<i>Julia:</i>	Three, four.	91	
	<i>Her work at the board looks like this:</i>	92	
<i>Amanda:</i>		93	
<i>Ms. Rebecca:</i>	So, the answer is four, and one twelfth is left. Very good, Julia and Amanda. Can you also write the mathematical sentence for this problem? <i>Writes on the board</i>	94	<i>Analyzing student work (slides 93- 97)</i> The teacher does not ask students to explain their answer and, particularly, consider what the remainder means.
<i>Ms. Rebecca:</i>	$\frac{3}{4} - \frac{1}{6} = 4 \text{ R } \frac{1}{12}$ So, the girls found four and remainder one twelfth. Any comments? Any questions for Julia and Amanda?	95	
		96	

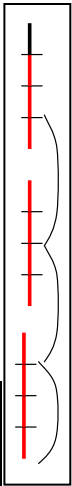
SCRIPT		Slides	Short Commentaries on the Opportunities to Explore the PSTs' performance	
			MKT- related practices	MTF-related practices
Ms. Rebecca:	It seems there are no comments or questions.	97		
Ms. Rebecca:	Okay, I want Michelle and Susan to share their solution to the next problem	98		
	<u>Slide narration:</u> Michelle and Susan come to the board and share their solution.	99		
	<u>Slide with narration:</u> This is how Michelle and Susan solved the problem.	100		
				
	$2\frac{1}{3} \div \frac{1}{6} = 16$			
Ms. Rebecca:	Okay. So far, so good. But what if we have to solve a hundred of these divisions?	101		
Ms. Rebecca:	We need to find a shortcut. We need to find a pattern.	102		
Ms. Rebecca:	This is what the worksheet asks us to do. Have a look at exercise D.	103		
Ms. Rebecca:	It says, "What algorithm makes sense for dividing any fraction by any fraction?"	104		
Ms. Rebecca:	So, let's try to find a pattern.	105		
		106		
		107		
		108		
	<b>"Time for Reflection" Slide</b>			
Ms. Rebecca:	Let me first erase these and rewrite the mathematical sentences we found.			
	Okay, let me write our answers to the first and third exercises.			
	Ms. Rebecca writes the following on the board:			
	$\frac{1}{2} \div \frac{1}{6} = 3 \text{ and } \frac{8}{3} \div \frac{1}{6} = 16$			

SCRIPT		Slides	Short Commentaries on the Opportunities to Explore the PSTs' performance
			MKT- related practices
			MTF-related practices
<i>Ms. Rebecca:</i>	Now, I want you to look very closely at these two divisions. Do you see anything interesting?	109	<i>Giving explanations (slides 109- 130; slides 140-143)</i>
<i>Susan:</i>	For the first problem, we can say that six divided by two is three.	110	The teachers' explanation on why the reciprocal works is problematic from several respects:
<i>Ms. Rebecca:</i>	Yes, that's true but we don't have a two!	111	- it mainly points to how the reciprocal can be derived without, though, explaining why the algorithm makes sense
<i>Robert:</i>	And we don't have a six either.	112	- her "explanation" only makes sense for divisions without remainder
<i>Ms. Rebecca:</i>	Imagine that we did have a six. How could we get three as our answer, with the half we already have and that six?	113	- she uses a problematic analogy (positive and negative numbers) to support student understanding
<i>Clark:</i>	(Excited:) We could say six times a half!	114	- she uses semantic instead of mathematical arguments
<i>Ms. Rebecca:</i>	Aha! Let me write what you are suggesting. You are saying that I can write...	115	
<i>Ms. Rebecca:</i>	<i>Ms. Rebecca writes on the board:</i>	116	
	$\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times 6 = 3$		
<i>Ms. Rebecca:</i>	Do you agree with this?	117	
<i>SS:</i>	Yes. / Yes? / Hmm...	118	
<i>Ms. Rebecca:</i>	But isn't six the same as six divided by one?	119	
<i>Julia, Travis, Clark:</i>	Yes. (Some students seem unsure, Ben looks bored.)	120	
<i>Ms. Rebecca:</i>	So, can't I write this?	121	
<i>Robert:</i>	$\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{2} = 3$ Yes.		
<i>Ms. Rebecca:</i>	Okay. Let's see. Will that work for the other example, too?	122	
<i>June, Travis, Robert:</i>	It works, it works. Yes.	123	





SCRIPT	Slides	Short Commentaries on the Opportunities to Explore the PSTs' performance	
		MKT- related practices	MTF-related practices
<p><i>Robert:</i> I have a question, too! If I use this rule to solve three fourths divided by one sixth – the problem that Julia and Amanda did before – I find four and two fourths.</p> <p><i>Robert:</i> But when Julia and Amanda solved the problem, they found four and one twelfth. So, does the rule work all the time?</p> <p><i>Ms. Rebecca:</i> Excellent questions!</p>	136		
<b>“Time for Reflection” Slide</b>			
<p><i>Ms. Rebecca:</i> Okay. Let’s consider Michelle’s question. The reciprocal works because we <i>invert</i> the second fraction.</p> <p><i>Ms. Rebecca:</i> And because we are using the reciprocal, we should use the reciprocal operation, which is multiplication.</p> <p><i>Ms. Rebecca:</i> Remember when we were talking about the positive and negative numbers?</p> <p><i>Ms. Rebecca:</i> And we said that instead of adding a negative number, we can subtract the positive number? It is similar to that!</p> <p><i>Ms. Rebecca:</i> We will talk about Robert’s question tomorrow. But the rule does work in this case, too. If it didn’t work we wouldn’t call it a rule, would we?</p>	140 141 142 143	<p><b><u>Giving explanations:</u></b> (see previous comment, slides 109-130)</p>	
<p><i>Ms. Rebecca:</i> Okay. Now, I want you all to try to solve exercise C- three from your worksheet. After you finish, turn to your partner and share what you found.</p>	144 145	<p><b><u>Giving explanation (slide 144)</u></b> The teacher does not give any explanation; rather she reinforces the idea that in mathematics there are certain rules to follow</p>	

SCRIPT	Slides	Short Commentaries on the Opportunities to Explore the PSTs' performance	
		MKT- related practices	MTF-related practices
<p><u>Slide narration:</u>  <i>Students work in their worksheets and Ms. Rebecca circulates and monitors their work.</i>  <i>Here are three students' solutions:</i></p> <p><u>Robert:</u></p> $2 \frac{3}{4} \div \frac{3}{4} = \frac{11}{4} \div \frac{3}{4} = \frac{11}{4} \times \frac{4}{3} = \frac{44}{12}$ <p><u>Michelle:</u></p>  <p><u>Ann:</u></p> $2 \frac{3}{4} \div \frac{3}{4} = 2 \frac{3}{4} \times \frac{4}{3} = \frac{10}{3} \times \frac{4}{3} = \frac{40}{9}$	146  147		
<p><b>“Time for Reflection” Slide</b></p> <p><i>Ms. Rebecca:</i>            We are running out of time. Robert, why don't you come and share your solution to the board? (<i>Robert comes to the board shares his work.</i>)</p>	148  149		

## Appendix 4: The Accompanying Interview Protocol

## INTERVIEW PROTOCOL

I'm trying to learn how you think about mathematical issues that arise in teaching. Hearing your current thoughts will help us design the course to serve you better.<sup>204</sup>

A topic that is often taught in fifth and sixth grade is the division of fractions. We'll first discuss the textbook pages that you had to study for your assignment, and then I'll ask you to observe and comment on a lesson. This isn't an evaluation – instead, it's an attempt to understand what students in our course know and are able to do as they begin preparing to become teachers.<sup>205</sup>

Some of the questions might seem trivial to you; others may be things you've never really had to think about before. Teaching isn't straightforward and you aren't yet a teacher. I am just as interested in how you think about things that puzzle you as I am in learning exactly what you'd do. But please feel free to take time if you need to work something out or need a few minutes to puzzle about something. For each situation, I'd like to know what you think you'd do or say, and why that's what you'd do.

To capture your thinking, I'll audiotape our meeting, encourage you to talk, and take notes about what you are thinking. Before we start, is there anything you would like to ask me?

*Ask for permission to audiotape the interview, and begin with the background-opening question. As the person works, encourage writing, and drawing, and carefully notice what he/she is doing, what he/she writes down, and how drawings are used or not in conjunction with what the person says. Take descriptive notes and record thoughts you have about the knowledge and reasoning the person is using.*

*To further probe the interviewee's understanding and perspectives of the issues considered in the interview, use probes such as:*

- ✓ Why (do you think so)?
- ✓ How did you figure that out?
- ✓ How do you know?

*If the person says something like "I don't know," try asking, "Why does that make sense to you?"<sup>206</sup>*

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<sup>204</sup> In addition to its research purposes, the interview also had instructional purposes (i.e., to establish rapport with the ELMAC students and get a sense of their thinking in order to help them throughout the two courses).

<sup>205</sup> The opening part was modified as follows for the post-intervention interview:

"First of all let me start by thanking you for being willing to participate in this post-course activity. Your willingness to participate in this activity is greatly appreciated. Let me tell you a little bit about the discussion that we will have today. Like our first meeting in the summer, I'm trying to learn how you currently think about mathematical issues that arise in teaching. But I encourage you to feel free to tell me whatever you think; not things that you think I might be interested in hearing – for example, ideas that were discussed during the two courses. Although this might be hard for you, it will help us better if you try to respond to my questions as if I had not been your instructor for these two courses. Our discussion today will follow a format similar to that our summer discussion. We will focus on the division of fractions, a topic that is often taught in fifth and sixth grade. We'll first discuss the textbook pages that you had to study for your very first assignment back in the summer, and then I'll ask you to observe and comment on a lesson."

<sup>206</sup> I am indebted to Mark Thames for his suggestions for developing the opening part of the interview.

## PART A: Opening Question

- *Opening question*

1. I would like to start out by learning a little about what brings you to teaching. Why are you interested in teaching?<sup>207</sup>

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<sup>207</sup> This question was modified as follows for the post-intervention interview: “I would like to start by briefly discussing how you currently think about teaching and learning mathematics. How similar/different are your current beliefs to those you had in the summer?”

## PART B: The Planning Phase of the Teaching Simulation

### *Introduction to the task*

In this part of the interview, we have designed tasks that simulate the work that a teacher usually does before, during and after teaching a mathematics lesson. As I mentioned before, please feel free to take time if you need to work something out, or to ponder about something that puzzles you.

#### - *Using the textbook page to plan a lesson on the division of fractions*

1. What do you think about the first textbook page?<sup>208</sup> Are there things you think are quite good in this page? Are there things that you would consider weaknesses of this page?
2. Assuming that you are supposed to use this textbook page to teach division of fractions to sixth graders, how would you use it? Please be as explicit as possible.

#### *Follow-up questions:*

- a) *Using the textbook exercises /problems*
- b) *Using representations*

#### (a) Selecting and Using Tasks

1. What particular tasks would you use? Why?
2. How are you thinking of using the exercises 1-16 (see first textbook page) to support your students' understanding of the division of fractions?
3. *If the participant refers to the exercises being too procedural/ undemanding, ask:* How would you make these exercises (or the lesson) more challenging for students?
4. If you were asked to classify these exercises into simpler and more demanding, how would you classify them? What criterion/criteria did you use in your classification?
5. Would you use the word problems in your lesson? How?

#### (b) Use of representations:

- ❖ *If the participant refers to using a representation at any point of the discussion, ask:*
  1. You used a particular type of representation. Why?
  2. Please explain how you would use this representation to:
    - support your students' understanding of the division of fractions
    - to explain why the algorithm works
      - *Guide the interviewee by asking whether she/he sees the reciprocal in any way in their representation.*

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<sup>208</sup> The textbook pages appear at the end of this appendix. The interviewees had the opportunity to peruse both textbook pages prior to the interview.

- ❖ *If the participant does not refer to using a representation at any point of the discussion, ask:*
  1. Assuming that you were to suggest a diagram that one of your colleagues could use to help her students understand the division of fractions. What would you recommend and why?
  2. Please explain how you would use this representation to:
    - support your students' understanding of the division of fractions.
    - to explain why the algorithm works
      - *Guide the interviewee by asking whether she/he sees the reciprocal in any way in their representation.*

- ***Comparing the two textbook pages***

- ◆ How do these two textbook pages compare to each other? What are some similarities/differences?

*Follow-up probes: If the participant:*

- *Talks about both pages including word problems, ask:*
    - Are these problems the same? If they are not the same: Does this difference make a difference when teaching the division of fractions?
  - *Talks about the second page being more demanding than the first, ask:*
    - In what respect? What makes it more demanding?
  - *Notices that the first page presents the algorithm through its worked out example whereas the second page expects students to figure out the algorithm themselves (task D), ask:*
    - What do you think about these two different approaches? Which approach would you pursue in your teaching, if any? Why?
- ◆ Which page would you prefer to use if you were to teach a lesson on the division of fractions (if any) and why?

## **PART C: The Enactment and Reflection Phases of the Teaching Simulation (The Virtual Lesson)**

### ***Introducing the virtual lesson***

Let's consider the following scenario. You're working at a school and Ms. Rebecca, one of your colleagues, asks you to sit in her classroom and observe her lesson on the division of fractions. She is planning to organize her lesson around the second textbook page.

We will go over this PowerPoint presentation which simulates Ms. Rebecca's lesson. We will stop at five points, and I'll be asking you three different types of questions:

1. Questions on what you have noticed in the lesson
2. Questions on analyzing and evaluating the teachers' decisions and actions
3. Questions pertaining to how you would respond to the students' ideas and contributions if you were teaching the lesson

So, I am interested in what you'll notice in the lesson, how you think about certain teaching moves, and what you would do in analogous situations. But let me remind you that often there are not right and wrong teaching moves. Teaching is a complex process and different people have different ideas about what to do as the lesson unfolds.

If you like, you can use these sheets of paper to keep notes (*give the interviewee sheets of paper*) – this is what teachers often do when they want to provide their colleagues with more elaborated feedback.

*Then read the following background information:*

So, let me give you some background information about this class and the lesson we will observe: “Ms. Rebecca is teaching to a sixth grade class. Like most of the students in this school, the students in Ms. Rebecca's class are middle-class and of average achievement. Five students are Caucasian, four are African American, two are Hispanic, and one is Asian. At this time of the year, the students are studying operations on fractions. In previous lessons, the students studied addition and subtraction of fractions, and multiplication of fractions. In today's lesson, the class will be introduced to the division of fractions.”

Do you have any questions before we start?

*Allow the interviewee to observe the simulation at his/her own pace. On the interviewee's reaching the slides “Time for reflection,” ask the questions that appear below.*



**Questions at the five reflection points of the script:**

<b>1</b>	<ol style="list-style-type: none"> <li>1. What have you noticed so far? Anything else?</li> <li>2. What do you think about the way Ms. Rebecca presented the problem to her students?</li> <li>3. What do you think about June's explanation?</li> <li>4. What do you think about the way Ms. Rebecca responded to June's explanation?<sup>209</sup></li> <li>5. What do you think about Alan's work on the diagram?</li> <li>6. Given Alan's work on the board, what would you do next and why?</li> </ol>
<b>2</b>	<ol style="list-style-type: none"> <li>1. Are there any things that you have noticed in this segment? Anything else?</li> <li>2. What do you think about the way the teacher handled Alan's error?</li> <li>3. What do you think about Ms. Rebecca's use of the diagram to support her students' learning?</li> <li>4. (a) What do you think about Ms. Rebecca's move to ask Amanda to write the numerical sentence for the problem? (b) Would you do something more than that?</li> <li>5. (a) What do you think about the way Ms. Rebecca helped the class overcome the difficulty they had in solving the problem? (b) Would you do something different? If so, what? Why?</li> </ol>
<b>3</b>	<ol style="list-style-type: none"> <li>1. Are there any things that you have noticed in this segment? Anything else?</li> <li>2. What do you think about Amanda's and Julia's solution to the problem?</li> <li>3. (a) What do you think about the manner in which the teacher reacted to Amanda's and Julia's work?(b) If you were teaching the lesson, would you do something different? If so, what? Why?</li> <li>4. The teacher urges the students to find a pattern, an algorithm, for dividing two fractions. If you were teaching the lesson, how would you go about doing that?</li> </ol>
<b>4</b>	<ol style="list-style-type: none"> <li>1. Are there any things that you have noticed in this segment? Anything else?</li> <li>2. What do you think about the way the teacher worked with her students in finding a pattern/algorithm for the division of fractions?</li> <li>3. How would you respond to Michelle, who asked why the reciprocal works? Can you give me an example? (<i>If the interviewee encounters difficulties, ask: What makes this task hard for you?</i>)</li> <li>4. What do you think about Robert's comment? How would you explain this contradiction?</li> </ol>
<b>5</b>	<ol style="list-style-type: none"> <li>1. Are there any things that you have noticed in this segment? Anything else?</li> <li>2. What do you think about the way Ms. Rebecca responded to Michelle's question, namely about her explanation as to why "the reciprocal works"?</li> <li>3. What do you think about the way Ms. Rebecca handled Robert's question?</li> <li>4. Imagine that you were teaching the lesson, and you identified these three different student solutions. What do you think about them? <i>Then follow up with these questions:</i> <ol style="list-style-type: none"> <li>a. Would you accept them as correct?</li> <li>b. What do these solutions suggest regarding what these students understood or did not understand?</li> </ol> </li> <li>5. If you were teaching the lesson and there were only 3-5 minutes left, what would you do next and why?</li> </ol>

<sup>209</sup> All questions that ask the interviewee to consider certain teaching moves should be followed with a more direct question that prompts the interviewee to evaluate Ms. Rebecca's actions (e.g., "If you were to evaluate this particular decision/action/ move of hers, what would you say?")

- *Reflections*
- *Closing remarks*

***Reflections:***

- ◆ What do you think about the lesson in general?
- ◆ What do you think about Ms. Rebecca's use of the textbook page/worksheet in her lesson? <sup>210</sup>

**PART D: Concluding Question**

- ◆ Do you have any comments/ thoughts on this experience you want to share with me?

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<sup>210</sup> The following two questions were added for the purposes of the post-intervention interview:

- Another teacher was also observing Ms. Rebecca's lesson. This teacher thinks that Ms. Rebecca used the second page very procedurally. What do you think about this critique?
- How does your current thinking about the issues we discussed today compare with your thinking in the summer? *If the participant identifies changes, press for more explanations:* (a) Can you give me an example? (b) What do you think contributed to these changes? Can you give me some examples?

## Divide by a Fraction

Divide  $2 \div \frac{3}{4}$

**Step 1:** Find the reciprocal of the divisor.

**Step 2:** Multiply by the reciprocal of the divisor.

The reciprocal of  $\frac{3}{4}$  is  $\frac{4}{3}$ .

$$2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{2}{1} \times \frac{4}{3} = \frac{8}{3} = 2 \frac{2}{3}$$

1.  $\frac{2}{3} \div \frac{1}{3}$  \_\_\_\_\_

2.  $\frac{1}{3} \div \frac{3}{4}$  \_\_\_\_\_

3.  $\frac{5}{8} \div \frac{5}{6}$  \_\_\_\_\_

4.  $\frac{3}{5} \div \frac{2}{3}$  \_\_\_\_\_

5.  $\frac{3}{4} \div \frac{2}{3}$  \_\_\_\_\_

6.  $\frac{2}{3} \div \frac{4}{7}$  \_\_\_\_\_

7.  $\frac{1}{3} \div \frac{7}{10}$  \_\_\_\_\_

8.  $\frac{5}{3} \div \frac{1}{3}$  \_\_\_\_\_

9.  $\frac{5}{6} \div \frac{1}{3}$  \_\_\_\_\_

10.  $\frac{11}{12} \div \frac{1}{4}$  \_\_\_\_\_

11.  $\frac{1}{2} \div \frac{7}{8}$  \_\_\_\_\_

12.  $\frac{1}{4} \div \frac{1}{3}$  \_\_\_\_\_

13.  $2\frac{5}{8} \div \frac{1}{8}$  \_\_\_\_\_

14.  $\frac{3}{4} \div \frac{3}{8}$  \_\_\_\_\_

15.  $1\frac{1}{6} \div \frac{5}{6}$  \_\_\_\_\_

16.  $\frac{3}{5} \div \frac{4}{7}$  \_\_\_\_\_

### Problem Solving

17. One-fourth of the students in the fifth grade play baseball. If 30 students play baseball, how many students are in the fifth grade?

\_\_\_\_\_

18. Marvin is trying to finish a jigsaw puzzle. He has placed  $\frac{2}{3}$  of the pieces so far. If he has put in 60 pieces of the puzzle, how many pieces altogether are in the puzzle?

\_\_\_\_\_

### Dividing a Fraction by a Fraction

Rasheed and Ananda have summer jobs at a ribbon company. Answer the questions below. Use written explanations or diagrams in each to show your reasoning. Write a number sentence to show your calculation(s).

- A.** Rasheed takes a customer order for ribbon badges. It takes  $\frac{1}{6}$  yard to make a ribbon for a badge. How many ribbon badges can he make from the given amounts of ribbon? Describe what each fractional part of an answer means.
1.  $\frac{1}{2}$  yard
  2.  $\frac{3}{4}$  yard
  3.  $2\frac{2}{3}$  yards (Remember  $2\frac{2}{3} = \frac{8}{3}$ .)
- B.** Ananda is working on an order for bows. She uses  $\frac{2}{3}$  yard of ribbon to make one bow. How many bows can Ananda make from each of the following amounts of ribbon?
1.  $\frac{4}{5}$  yard
  2.  $1\frac{3}{4}$  yards
  3.  $2\frac{1}{3}$  yards
- C.** Solve each of the following examples as if they were ribbon problems.
1.  $\frac{3}{4} \div \frac{2}{3}$
  2.  $1\frac{3}{4} \div \frac{1}{2}$
  3.  $2\frac{3}{4} \div \frac{3}{4}$
- D.** What algorithm makes sense for dividing any fraction by any fraction?

## Appendix 5: The Survey

**MY THINKING ABOUT TEACHING AND LEARNING  
MATHEMATICS**

**SURVEY FOR MTH 485**

I am interested in how you currently think about mathematics for teaching. Please answer all the questions. Note that there are no right or wrong answers to these questions and that your answers will in no way affect your grade for the course. All information of this survey will be kept confidential.

**Name:** .....



**Part B: Goals for teaching mathematics**

Listed below are five possible goals for the teaching of mathematics in elementary/middle school. Write a percent next to each goal that represents the *emphasis* that you think is appropriate to place on this goal. Your percentages may add to 100%; if not, please explain why.

<u>Goal A:</u> Instruction in mathematics should help students comprehend mathematical concepts, operations and relations.	Emphasis: ____%
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<u>Goal B:</u> Instruction in mathematics should help students carry out procedures flexibly, accurately, efficiently, and appropriately.	Emphasis: ____%
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<u>Goal C:</u> Instruction in mathematics should help students formulate, represent, and solve problems.	Emphasis: ____%
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<u>Goal D:</u> Instruction in mathematics should help students think logically and explain and justify their thinking.	Emphasis: ____%
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<u>Goal E:</u> Instruction in mathematics should help students perceive math as sensible, useful and worthwhile, and see themselves as capable learners and doers of math.	Emphasis: ____%
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Please briefly explain your allocation of emphasis to each of these goals:

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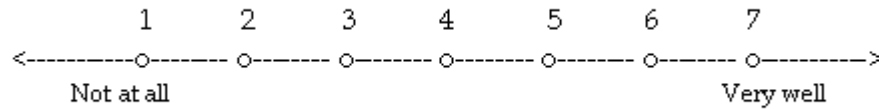
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## Part C: Perceived personal efficacy

Use the scale below to indicate your sense of how well you can do the following.



1.	Operate with whole numbers (addition, subtraction, multiplication, and division)	1	2	3	4	5	6	7
2.	<u>Understand and use different interpretations of fractions:</u>							
	a. Fraction as a part of the whole	1	2	3	4	5	6	7
	b. Fractions as quotients	1	2	3	4	5	6	7
	c. Fractions as operators	1	2	3	4	5	6	7
	d. Fractions as numbers.	1	2	3	4	5	6	7
3.	Identify the unit (the whole), when given a representation on fractions	1	2	3	4	5	6	7
4.	Represent fractions or decimals by words, numbers, or models	1	2	3	4	5	6	7
5.	Add and subtract fractions	1	2	3	4	5	6	7
6.	Multiply and divide fractions	1	2	3	4	5	6	7
7.	Understand the algorithm of the multiplication of fractions	1	2	3	4	5	6	7
8.	Understand the algorithm of the division of fractions	1	2	3	4	5	6	7
9.	Use multiple representations when solving a problem	1	2	3	4	5	6	7
10.	Make connections between different representations	1	2	3	4	5	6	7
11.	Map correspondences among different representations (e.g., between a picture and a set of symbols)	1	2	3	4	5	6	7
12.	Explain why a procedure or an algorithm works/makes sense	1	2	3	4	5	6	7
13.	Identify incorrect explanations	1	2	3	4	5	6	7
14.	Identify inappropriate explanations	1	2	3	4	5	6	7
15.	Evaluate whether a proposed solution to a problem is correct	1	2	3	4	5	6	7
16.	Use appropriate representations when solving a problem	1	2	3	4	5	6	7
17.	Use appropriate examples to explain a mathematical idea	1	2	3	4	5	6	7

**Part D: General information**

*Please provide the following information about your mathematical background, expectations for and concerns about this class, and previous experiences learning mathematics.*

1. What do you remember about your learning of mathematics in elementary and middle school? What was it like? Were there any differences between the teaching of mathematics you experienced as an elementary and as a middle-school student? If so, please elaborate.

2. Which of the following mathematics courses did you complete in high school? (*Check all that apply.*)

<input type="checkbox"/>	General mathematics
<input type="checkbox"/>	Algebra I
<input type="checkbox"/>	Algebra II
<input type="checkbox"/>	Geometry
<input type="checkbox"/>	Trigonometry
<input type="checkbox"/>	Precalculus
<input type="checkbox"/>	Calculus
<input type="checkbox"/>	Advanced Placement calculus
<input type="checkbox"/>	Other math ( <i>please specify</i> ): .....

3. Please indicate whether you have any of the following. (Check ALL that apply.)

<input type="checkbox"/>	An undergraduate mathematics major
<input type="checkbox"/>	An undergraduate mathematics minor
<input type="checkbox"/>	Graduate-level degree in mathematics
<input type="checkbox"/>	Graduate-level degree in education

4. About how many undergraduate or graduate level classes have you taken at a college or university in the following areas? (Circle ONE response for each.)

	No classes	One or two classes	Three to five classes	Six or more classes
Mathematics	1	2	3	4
Methods of teaching mathematics	1	2	3	4

5. Have you ever been/are you currently a teacher in a school setting?

	Yes
	No (move to question 7)

6. What are your expectations for this course?

7. What are your concerns about this course?

Appendix 6: The Document Sent to the MTF-Experts to Validate the 43 Statements  
Originally Selected for the First Part of the Survey



Before reading the statements, please peruse the following table which details the characteristics of the two types of teachers.

### Characteristics of the Two Types of Teachers

Type-A teacher	Type-B teacher
<ul style="list-style-type: none"> <li>- Invests time and effort in having students explore and understand concepts, processes or relationships.</li>   <li>- Usually assigns tasks that require complex and non algorithmic thinking or/and expects students to link the procedures at hand with their underlying meaning. These tasks require moderate to high cognitive effort on the part of students.</li>   <li>- Usually expects the class to solve a given task in more than one way and encourages students to make connections among multiple solutions.</li>   <li>- Is more concerned with students' developing understanding of mathematics concepts and ideas.</li>   <li>- Requires and insists on students' finding patterns, making generalizations, explaining and justifying their thinking.</li> </ul>	<ul style="list-style-type: none"> <li>- Invests time and effort in having students reproduce previously learned facts, rules, formulae or committing facts, rules, formulae or definitions to memory.</li>   <li>- Usually assigns algorithmic tasks that require students to reproduce previously learned material. These tasks require limited cognitive effort on the part of students because there is little ambiguity about what needs to be done and how to do it.</li>   <li>- Usually does <i>not</i> invest time and effort in helping students connect a procedure to its underlying meaning.</li>   <li>- Is more concerned with students' producing correct answers rather than developing mathematical understanding.</li>   <li>- Requires students to provide explanations rather infrequently; when doing so, students are largely expected to <u>describe</u> the procedure used.</li> </ul>

## A. List of statements

#	Statement	Type-A teacher	Type-B teacher
1.	There are unsolved problems in mathematics.		
2.	Doing mathematics allows room for original thinking and creativity.		
3.	Doing mathematics is usually a matter of working logically in a step-by-step fashion.		
4.	A lot of things in math must simply be accepted as true and remembered.		
5.	Proofs are a mean for making arguments in mathematics.		
6.	For students to get better at math, they need to practice a lot.		
7.	When students can't solve problems, it is usually because they can't remember the right formula or rule.		
8.	It is confusing to see many different methods and explanations for the same idea.		
9.	If students get into arguments about ideas and procedures in math class, it can impede their learning of mathematics.		
10.	In learning math, students must master topics and skills at one level before going on.		
11.	A teacher should wait until students are developmentally ready before introducing new ideas and skills.		
12.	It is important for pupils to master the basic computational skills before they tackle complex problems.		
13.	Students learn mathematics best if they have to figure things out for themselves instead of being told or shown.		
14.	If students have unanswered questions or confusions when they leave class, they will be frustrated by the homework.		
15.	If a student asks a question in math, the teacher should know the answer.		
16.	Being good at mathematical problem solving personally has little to do with being a good math teacher.		
17.	In order to teach problem solving, teachers have to do a lot of math problem solving themselves.		
18.	It is important for teachers to know mathematical terminology.		
19.	Basic computational skill and a lot of patience are sufficient for teaching elementary school math.		
20.	Students should never leave math class (or end the math period) feeling confused or stuck.		
21.	Teachers should not necessarily answer students'		

	questions but let them puzzle things out themselves.		
22.	If students are having difficulty in math, a good approach is to give them more practice in the skills they lack.		
23.	If a student is confused in math, the teacher should go over the material again more slowly.		
24.	The most important issue is <u>not</u> whether the answer to any math problem is correct, but whether students can explain their answers.		
25.	To do well, students must learn facts, principles, and formulas in mathematics.		
26.	A good mathematics teacher is someone who explains clearly and completely how each problem should be solved.		
27.	Teachers should try to avoid telling.		
28.	Teachers should lead their students to the answers by asking pointed questions.		
29.	When students solve the same math problem using two or more different strategies, the teacher should have them share their solutions.		
30.	Math just isn't my strength.		
32.	I avoid mathematics whenever possible.		
33.	I'm pretty good at mathematics		
34.	I enjoy the challenges of mathematics.		
35.	I can handle basic math, but I do not have the kind of mind needed to do advanced mathematics.		
36.	Mathematics has always been my favorite subject.		
37.	I enjoy working on mathematical problems.		
38.	I am not self-confident in doing mathematics.		
39.	I am particularly worried about teaching mathematics.		
40.	Math is the subject I am most looking forward to teaching.		
41.	It will be difficult for me to help my students understand certain concepts and procedures in math.		
42.	I envision that I will find it difficult to respond to students' questions in mathematics.		
43.	No matter how hard I'll try, I won't be as efficient in teaching math as in teaching other subjects.		



Appendix 7: Report on the Validation of the Coding Schemes

## Report on the Validation of the Coding Schemes

This report consists of two parts. The first part documents the process pursued for validating the coding schemes developed to code the interview data of the present study. The second part summarizes the main comments, suggestions, and concerns of the three experts that participated in this process. The revised coding schemes that resulted from this process are presented in Appendix 8.

### 1. The Validation Process

This part briefly presents the profile of the three experts who participated in the validation process and details the validation process itself.

#### 1.1. Participants

A Research Area Specialist Lead and two post-graduate fellows participated in this process. All of them had a special interest in teaching and served as elementary or secondary teachers. Additionally, each of them brought different kinds of expertise necessary for the validation process.

At the time the study was conducted, the Research Area Specialist Lead (RASL) had a joint position at the University of Michigan and a public elementary school. A National Board certified teacher, the RASL served as a fifth-grade mathematics teacher for fifteen years. During the fall semester of 2007, he also coordinated the instructors of the Math Methods class, which is part of the intervention considered in this study. He was one of the coordinators responsible for redesigning the teacher education programs offered at the University of Michigan; he was also the director of the *Developing an Integrated Assessment System for Elementary Teacher Education* (DIAS), a project designed to develop, implement, and evaluate an integrated assessment system in elementary teacher education for mathematics and language arts.

Originally a middle-school teacher, the first post-graduate fellow developed a special interest in instructional routines and in how these routines can support teachers in effectively carrying out the complicated work of teaching. As a graduate student she was a member of *Beyond Implementation: Focusing on Challenge and Learning* (BIFOCAL) project that aimed at scaffolding middle-grade mathematics teachers in implementing complex mathematical tasks. Her involvement in this project familiarized her with the *Mathematics Tasks Framework* (MTF) and issues regarding how to maintain the cognitive demand of cognitively demanding tasks.

A former elementary and middle-grade teacher the second post-graduate fellow was interested in how teachers can learn in and from their practice and how they can be better prepared to do the work of teaching. As a graduate student she was a member of the *Learning Mathematics for Teaching* (LMT) project. Her participation in this project helped her develop expertise in issues related to the Mathematical Knowledge for Teaching (MKT). In particular, she worked on developing items to assess teachers' MKT, on developing coding schemes to evaluate the mathematics quality of mathematics lessons, and on applying these coding schemes to code videotaped lessons.

The group members received a \$250 stipend each for their contribution to this process.

## 1.2 Meetings and coding scheme validation

The group met for five two-hour meetings. Before the first meeting, each group member was sent the PowerPoint teaching simulation, a short description of the interview protocol and its accompanying questions, along with the five coding schemes. The group was asked to watch the teaching simulation before the first meeting and become familiar with the interview protocol and the five coding schemes.

During the first part of the first meeting, I presented an overview of my study goals, talked about the development of the teaching simulation, outlined the five practices that the study was designed to explore, and discussed the overall structure of the coding schemes. The group was invited to make general comments or suggestions, ask for clarifications, and raise concerns about the structure of the coding schemes. The second half of the meeting was spent on validating the third coding scheme (“Providing Explanations”). The second meeting was focused on finalizing the third coding scheme and on validating the second coding scheme (“Using Representations”). In the third meeting, the group completed the validation of the second coding scheme, and validated the fourth coding scheme (“Analyzing Students’ Work and Contributions”). The last two meetings were devoted to validating the fifth and the first coding schemes, correspondingly (“Responding to Students’ Requests for Help” and “Selecting and Using Tasks”). The five coding schemes were validated in such an order that allowed for first considering the *MKT*-related practices (coding schemes 2-4) and then the *MTF*-related practices (coding schemes 1 and 5).

The following steps were taken to validate the three subcomponents of each scheme:

*(a) Noticing:* First, all the episodes that corresponded to this performance were identified. The group members were then asked to consider whether these episodes were indeed worthy of attention in terms of the practice at hand and to detail what was worth noticing in these episodes. Next, the group considered the coding scheme developed to evaluate participants’ “noticing” performance. Suggestions for revising the proposed scoring scheme were discussed and after reaching consensus, the agreed-upon set of scores was adopted and used to revise the original coding scheme.

*(b) Interpreting-Evaluating:* Similar to the procedure followed above, validating the “interpreting-evaluating” part of the coding scheme commenced with identifying and discussing all the corresponding teaching-simulation episodes. The group members were again asked to consider whether these episodes were worthwhile to explore preservice teachers’ (PSTs) performance regarding this skill. Each participant was then asked to analyze and evaluate the virtual teacher’s moves in the corresponding episodes. All proposed ideas were discussed and consensus was reached as to how the virtual teacher’s performance could be evaluated. These ideas were compared against the evaluation criteria listed in the coding scheme and revisions were made, if necessary. Following that, the group explored the thresholds developed for assigning the scores of 0, 1, and 2 to PSTs’ performance. The group members made suggestions for revising some of the thresholds or including additional categories (particularly for score “1”).

*(c) Performing:* All the activities related to the “performing” category were first identified. The group was then asked to express their opinions as to whether these activities were representative of the practice under consideration and as “performance” activities. Next,

the operationalization criteria for coding performance in these activities were discussed. To explore the appropriateness of these criteria, the group participants were asked to consider excerpts from the interview data (two to four excerpts per coding scheme). This process pointed to places in which the criteria of coding scheme needed clarification, elaboration, or revisions, as well as to criteria that were hard to apply. Suggestions made for revisions or elaborations of those criteria were adopted after reaching consensus. The coding scheme was revised accordingly, and the revisions made were briefly discussed in the next meeting.

## **2. Comments, Suggestions, and Concerns**

The participants' comments, suggestions, and concerns during the validation process are organized into two main categories: those pertaining to the coding schemes as a whole, and those referring to specific coding schemes. I present each category in turn.

### **2.1. Comments and concerns about the coding schemes as a whole**

Participant's overall comments pertained to the noticing and the interpreting-evaluating subcomponents of each teaching practice. In particular, with respect to:

- *Noticing*: The group commented that, with no exception, all the episodes that the PSTs were expected to notice in the teaching simulation correspond to *problematic* aspects of the virtual teacher's instruction. Hence, as they pointed out, identifying redeeming aspects of the teacher's work (e.g., engaging students in the class discussion) would not be counted. However, they acknowledged that the problematic aspects of the instruction featured in the teaching simulation critically impaired the mathematical quality of the virtual teacher's instruction, and thus, were important to notice. The group also thought that one of the strengths of the simulation is that it shows the virtual teacher engaging in several favorable teaching moves (e.g., organizing whole group discussions, trying to engage many students in the conversation, using representations) but does so only superficially and in a way that does not substantively affect the mathematical quality of her instruction.
- *Interpreting-Evaluating*: The group remarked that the criteria for evaluating the virtual teacher's performance endorse a particular type of instruction that focuses on making meaning and understanding. The group pointed out that although it makes sense to pay that much attention to making meaning for an introductory lesson on division of fractions – such the one considered in the simulation – in actuality teachers should not be expected to engage in the meaning-making practices considered in the simulation in every single lesson (e.g., a teacher should not be expected to thoroughly explain a procedure at hand in a review lesson; similarly, a teacher should not be expected to do the hard and cumbersome work of building explicit and detailed correspondences among the different types of representations used in a lesson if she or he has already done so in previous lessons).

### **2.2. Comments, suggestions, and concerns about individual coding schemes**

The comments and suggestions the participants made and the concerns they raised for each coding scheme are organized into categories corresponding to the subcomponents of noticing, interpreting-evaluating, and performing.

### 2.2.1 First coding scheme: “Selecting and Using Tasks”

#### *Noticing and Analyzing-Evaluating:*

- The group argued that although the virtual teacher’s presentation of task “A” gave away some important information for solving this task, there was still a lot of conceptual work to be done on this task (and hence disagreed with my argument that the teacher’s presentation of this task undermined the task complexity). Given this argument, it was decided to include another criterion in both categories (noticing and interpreting-evaluating), according to which the PSTs’ performance would be considered acceptable, if the PSTs commented that despite the teacher’s presentation of this task significant conceptual work could still be done. It was also decided that the PSTs should also be credited if they talked about ways in which the teacher could have undermined the demand of the task even more (e.g., by explicitly listing steps for students to follow).
- The group agreed that the teacher’s enactment of tasks A and D undermined the potential of these tasks to engage students in demanding and rich mathematics. In particular, the group pointed out:
  - *With respect to the enactment of task A:* The teacher used the representation of the number line rather procedurally without drawing any connections between this representation and the actual word problem. The teacher should have at least asked students to connect the answer they were getting for problem A<sub>1</sub> (i.e.,  $1/2 \div 1/6$ ) to the context of the problem (i.e., 3 ribbons).
  - *With respect to the enactment of task D:* The teacher’s work on figuring out patterns and the involved algorithm was problematic, because: (a) the teacher was leading her students with her questions; (b) the discussion focused on figuring out relationships between numbers without linking those numbers to the context of the problems considered during the previous part of the lesson.

#### *Performing:*

- The group was asked to propose ways in which one could undermine/maintain /upgrade the demands of the tasks of the two textbook pages used in the interviews. The group’s suggestions were aligned with those included in the coding scheme. In particular, it was proposed:
  - That the demands of the tasks of first textbook page could be maintained at a lower level simply by announcing the algorithm and then having students solve all the exercises or a randomly chosen subset of them.
  - That the demands of the tasks of the first textbook page could be upgraded by: (a) using a specific context-mathematical situation to explain the division-of-fraction algorithm; (b) by asking students to pose problems to represent specific division-of-fractions mathematical sentences; (c) by making deliberate selection of exercises and ordering them in ways that would support meaning and understanding.
  - That the demands of the tasks of the second worksheet could be degraded by announcing an algorithm to be followed at the beginning of the lesson; by asking very pointed questions that lead students step-by-step to the answer of the problem; and by not holding students accountable for using diagrams and

explaining their work (verbally and/or in written format) and especially the fractional part of their answer.

- The group agreed that the system developed to code the study participants' performance was appropriate. Given that the PSTs' answers could represent a hodgepodge of moves some of which could lead to undermining the cognitive demands of the tasks and others that could support maintaining the demand at a higher level, the group proposed that the coding scheme be further elaborated to identify "critical criteria" for each score under consideration (0, 1, 2, and 3). This would allow minimizing the ambiguity in allocating a single score to the PSTs' relevant performance.

The coding scheme was revised accordingly, taking all the aforementioned suggestions into consideration.

### **2.2.2. Second coding scheme "Using Representations"**

*Interpreting-Evaluating:*

- The group commented that the term "mapping" is ambiguous. The coding scheme was therefore revised to specifically reflect the different types of mapping. In particular, two types of mapping were identified: drawing connections between the representations used in the lesson and the algorithms/procedures under consideration; and linking these representations to the word problem under consideration. The group suggested that the highest score ("score 2") be assigned only to the PSTs who identified insufficiencies in the teacher's instruction with respect to *both* types of mapping identified above.

*Performing:*

- The two types of mapping mentioned above were also adopted for the "performing" category. In addition, the first type of mapping (i.e., drawing connections between representations and algorithms) was further decomposed into *numerical labeling* and *naming*. An elaboration of each of these components appears in the corresponding coding scheme (see Appendix 8).
- The group agreed that simply using the numerical algorithm should not count as "using representations" for the two specific performances at hand, since this algorithm was already given in the teaching simulation.
- At the group's suggestion, the two performances under consideration (using a representation to explain the quotient of division  $2 \div \frac{3}{4}$  and using a representation to show the reciprocal) were further decomposed. It was also agreed that the component "presenting the fractional part of the quotient in absolute terms" should not be considered a necessary condition for explaining the quotient, provided that there is evidence that the PSTs were not simply referring to the numerical answer derived from the algorithm.

### **2.2.3. Third coding scheme: "Providing Explanations"**

*Performing:*

- At the group's suggestion, the score "1" was also decided to be given to PSTs whose explanations met the criterion of correctness-validity, but fell short of at least two of the other criteria considered necessary for a satisfactory explanation.

- The group suggested that the criterion of completeness listed in this category be further elaborated. This criterion was unpacked as follows:
  - o *Explaining the quotient of the division*: The explanation should refer to the two different types of units (absolute unit: defined based on the denominator; and relative units: defined in terms of the divisor); the type of unit used to explain the fractional part of the answer; and the relationship between what is left over and the relative unit of measurement.
  - o *Explaining the reciprocal*: The proposed explanation should explicate both the reciprocal (i.e., why it is  $4/3$ ) and the multiplication by the reciprocal (why it is 2 times  $4/3$ ).
- Several concerns were raised about the criterion of “clarity vs. ambiguity.” The group felt that this criterion was quite subjective and that it would be hard to code. Therefore, this criterion was dropped. The group thought that the criterion of “appropriateness” could, at least to some extent, account for issues of clarity and ambiguity.

#### **2.2.4. Fourth coding scheme: “Analyzing Students’ Work and Contributions”**

##### *Interpreting-Evaluating:*

- The group pointed out that the distinction between naming and describing an error (a distinction originally included in the coding scheme) was unnecessary and would further complicate coding decisions; hence it was dropped from the coding scheme.
- The group agreed that the PSTs should be given a certain score if they justified the teacher’s non response to the students’ errors as a *tactical move* (e.g., the teacher might have wanted to address this error later on). In this case, the PSTs would still have to identify the error in the students’ work as such.

##### *Performing:*

- The group agreed on the following elaborations regarding the criteria for analyzing the three students’ solutions:
  - o *Robert*: A PST should not be penalized for not referring to the fact that Robert did not simplify his answer or convert the improper fraction into a mixed number.
  - o *Michelle*: Arguing that she understands the procedure conceptually is inappropriate, since she might have just followed the steps on using representations outlined in the lesson.
  - o *Ann*: The main error to in her work pertained not to the order in which certain steps of the algorithm were taken, but to that she took the reciprocal of part of the dividend, instead the reciprocal of the divisor.
- The group accepted the allocation of scores outlined in the coding scheme as satisfactory.

#### **2.2.5. Fifth coding scheme: “Responding to Students’ Direct or Indirect Requests for Help”**

##### *General comment:*

- The group suggested that the notion of “*indirect* request for help” be further clarified.

*Performing:*

- The group proposed a list of acceptable teacher moves for handling the students' indirect and direct requests for help. In proposing these moves the group was asked to keep in mind that this particular lesson was oriented toward supporting understanding rather than simply helping students master a set of procedures. The group proposed moves similar to those listed in the coding scheme, with the exception of two additional moves that were eventually added to the coding scheme [i.e., comparing the work on  $\frac{3}{4} \div \frac{1}{6}$  to the work on the previous example ( $\frac{1}{2} \div \frac{1}{6}$ ); and using an area representation that can be chunked in two different ways]. At the group's suggestion, some clarifications were made to distinguish between the different teaching moves outlined in this part of the coding scheme.
- After coding four excerpts from the interviews, the group concluded that the idea of common multiples is *central* to the work illustrated in the second episode. The group also remarked that a PST could offer a pedagogically reasonable way to handle this situation, but still not be aware of any way to support the students' exploration in terms of the mathematics involved. Hence, it was suggested that a PST's performance in this episode be accepted as appropriate *only if* there is evidence that the PST was able to support the students' exploration from a mathematical standpoint (i.e., either the PST understood how the common multiple of twelve could be used to solve the problem  $\frac{3}{4} \div \frac{1}{6}$  or the PST proposed an alternative appropriate way for approaching this problem). This suggestion was incorporated into the revised coding scheme.



## Appendix 8: The Interview Coding Glossary

## THE INTERVIEW CODING GLOSSARY

This glossary consists of two sections.

The **first section** involves five coding schemes developed to examine preservice teachers' (PSTs) potential to successfully engage in five teaching practices. These practices are:

- Selecting and using tasks (Coding scheme 1)
- Using representations (Coding scheme 2)
- Providing explanations (Coding scheme 3)
- Analyzing students' work and contributions (Coding scheme 4)
- Responding to students' indirect or direct requests for help (Coding scheme 5).

Performance in each coding scheme is decomposed into three skills/categories:

- *Noticing*: This category concerns the comments that the PSTs made when observing the simulation *without any prompting* by the interviewer. The comments that the PSTs made when directly asked to interpret and evaluate specific teaching moves or decisions should *not* be counted as noticing (see next category).
- *Interpreting-Evaluating*: This category captures the comments that the PSTs made when the interviewer *directly asked* them to interpret and evaluate specific teaching moves or decisions.
- *Performing*: This category corresponds to the PSTs' performance when asked to engage in certain tasks related to each teaching practice.

The **second section** demarcates the places in the interview transcripts where information regarding each coding scheme can be located. The table presented on the last page of this glossary provides a list of checkpoints for checking whether all the needed pieces of information for each coding scheme are identified and hence a score can be assigned to the PSTs' corresponding performance.

## SECTION A: The Coding Schemes

**Coding Scheme 1: Selecting and Using Tasks**

Skill	Entry Points	Score	Code Operationalization if First Page is Considered	Code Operationalization if Second Page is Considered
<i>Performing</i>	<p>The PST is asked to:</p> <p>A. select a page for developing an introductory lesson on division of fractions briefly outline an introductory lesson on division of fractions</p> <p>B. classify the first page's tasks into simpler and more complex</p> <p>D. outline an approach for working on task D of the second page</p>	<p><b>3</b></p> <p>(Division of fractions is connected to meaning; students are provided with opportunities to explore patterns; notice that this code corresponds to "doing mathematics")</p>	<p>The PST uses the 1<sup>st</sup> page to structure a lesson that supports meaning-making and pattern exploration:</p> <ul style="list-style-type: none"> <li>- The PST outlines an activity through which students would arrive at a conceptual explanation of the involved algorithm (e.g., by using a carefully selected and sequenced set of examples; by scaffolding students to see the connection between the two parts of the equation of the "invert and multiply" algorithm from a conceptual viewpoint).</li> <li>- The PST selects exercises 1-16 (or a set thereof) to support meaning-making (and exploration of patterns). This should include at least one of the following:               <ul style="list-style-type: none"> <li>o exercises in which the divisor is larger than the dividend are harder than those in which the divisor is smaller than the dividend</li> <li>o the denominator of the fractions determines their difficulty: Exercises in which the denominators of the two fractions are identical are the easiest; exercises in which the denominator of one fraction is a multiple of the denominator of the other fraction are somewhat harder than those of the previous category, but easier than exercises in which the denominators of the two fractions are not "clear" multiples of one another.</li> </ul> </li> <li>- The PST would have students work on exercises 1-16 or a subset thereof from a conceptual standpoint.</li> <li>- If students are asked to pose word problems, the PST clearly explains how this</li> </ul>	<p>The PST uses the 2<sup>nd</sup> page to structure a lesson that supports meaning-making and pattern exploration:</p> <ul style="list-style-type: none"> <li>- The PST does not introduce the algorithm of dividing fractions at the beginning of the lesson.</li> <li>- While having students work on (a selection of) tasks A through C, the PST would maintain the task complexity and their emphasis on meaning by pursuing the following moves (this could be evident from the PST's work on these tasks):               <ul style="list-style-type: none"> <li>o asking students to use diagrams and/or written explanations and connect their work to the word problem</li> <li>o asking students to explain their answer and paying particular attention to the <i>fractional part</i> of an answer</li> </ul> </li> </ul> <p>In addition, the PST would organize an activity that would support exploration of the algorithm of dividing two fractions. In particular, the PST would scaffold students in identifying the algorithm by pursuing moves such as:</p> <ul style="list-style-type: none"> <li>o deliberately selecting examples to support students' exploration of patterns/the involved algorithm</li> <li>o helping students organize examples/data in a form that supports exploration</li> <li>o asking questions that would help students figure out the pattern (without doing the thinking for them)</li> </ul> <p><u>Notes:</u></p> <ol style="list-style-type: none"> <li>1. Information on this aspect of the PST's work should also be retrieved from the PST's answer to the question: "The teacher</li> </ol>

		<p>2 (Division of fractions is connected to its underlying meaning; no task is used at an exploratory level. This category corresponds to procedures-with-connections tasks)</p>	<p>would support students' understanding (e.g., explains why she or he would assign a particular numeric sentence for problem posing).</p> <ul style="list-style-type: none"> <li>- If the two word problems (exercises 17-18) are used, they are conceptually connected to division of fractions and/or the reciprocal</li> </ul> <p><u>Note:</u> There should be evidence that the PST himself/herself can conceptually explain the algorithm of division of fractions.</p> <ul style="list-style-type: none"> <li>- The PST uses the 1<sup>st</sup> page as a source of activities to structure a lesson that supports understanding: <ul style="list-style-type: none"> <li>- The PST would provide a conceptual explanation of the algorithm at some point of the lesson. There is evidence that the PST could: <ul style="list-style-type: none"> <li>o clearly explain the meaning of division and what the quotient means, including cases for which the quotient consists of a fractional part;</li> <li>o provide at least some sort of conceptual explanation of the reciprocal, although this explanation might not be very polished.</li> </ul> </li> <li>- The PST would have students work on exercises 1-16 or a subset thereof from a conceptual perspective.</li> <li>- Considers exercises 1-16 or a subset thereof from a conceptual viewpoint (either during selection or when discussing their difficulty). This should include at least one of the following: <ul style="list-style-type: none"> <li>o exercises in which the divisor is larger than the dividend are harder than those in which the divisor is smaller than the dividend</li> <li>o the denominator of the fractions determines their difficulty: exercises in which the denominators of the two fractions are identical are the easiest; exercises in which the denominator of</li> </ul> </li> </ul> </li> </ul>	<p>urges the students to find a pattern/algorithm for dividing fractions. If you were teaching the lesson, how would you go about doing that?" (virtual lesson; segment 3).</p> <ul style="list-style-type: none"> <li>2. There should be evidence that the PST himself/herself can conceptually explain the algorithm of division of fractions.</li> </ul> <ul style="list-style-type: none"> <li>- The PST uses the 2<sup>nd</sup> page to structure a lesson that supports meaning-making. In particular: <ul style="list-style-type: none"> <li>- The algorithm might be presented to students at some point during instruction, but explicit connections are made to meaning. There is evidence that the PST could: <ul style="list-style-type: none"> <li>o clearly explain the meaning of division and what the quotient means, including cases for which the quotient consists of a fractional part;</li> <li>o provide at least some sort of conceptual explanation of the reciprocal, although this explanation might not very polished.</li> </ul> </li> <li>- While having students work on (a selection of) tasks A through C, the PST would maintain the task complexity and their emphasis on meaning by pursuing the following moves (this could be evident from the PST's work on these tasks): <ul style="list-style-type: none"> <li>o asking students to use diagrams and/or written explanations and connect their work to the word problem</li> <li>o asking students to explain their answer and paying particular attention to the <i>fractional part</i> of an answer</li> </ul> </li> </ul> </li> </ul> <p><u>Notes:</u></p> <ol style="list-style-type: none"> <li>1. Use this code if the PST argues that instead of directly presenting the algorithm to students the PST would have them come up with the pattern/algorithm themselves, but</li> </ol>
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		<p>one fraction is a multiple of the denominator of the other fraction are somewhat harder than those of the previous category, but easier than exercises in which the denominators of the two fractions are not “clear” multiples of one another.</p> <p><i>If</i> the two word problems (exercises 17-18) are used, these are connected to division of fractions and/or the reciprocal at least in some way.</p> <ul style="list-style-type: none"> <li>- PST might use a word problem similar to the ribbon problem that appears on the second page, instead of using the two problems that appear on this page.</li> <li>- If students are asked to pose word problems, the PST explains, at least to some extent, how this could support students’ understanding (e.g., explains why she or he would assign a particular numeric sentence for problem posing).</li> </ul>	<p>does not clearly articulate how he or she would organize/scaffold such an exploration.</p> <p>2. Information on this aspect of the PST’s work should also be retrieved from the PST’s answer to the question: “The teacher urges the students to find a pattern/algorithm for dividing fractions. If you were teaching the lesson, how would you go about doing that?” (virtual lesson; segment 3).</p>	<p>The PST uses the 2<sup>nd</sup> page and builds some connections to meaning but would not capitalize on certain aspects of this page to support meaning-making and understanding of division of fractions:</p> <ul style="list-style-type: none"> <li>- The algorithm might be presented to students at some point during instruction and some attempts are made to connect the algorithm to meaning. This could involve at least some explanation of the meaning of division (e.g., how many times the divisor goes into the dividend?).</li> <li>- While working on tasks A through C (or a selection of them) the PST would <i>not</i> capitalize on at least one of the aspects of this page to support meaning-making (this could be evident from the PST’s work on these tasks): <ul style="list-style-type: none"> <li>o asking students to use diagrams and/or written explanations and connect their work to the word problem</li> <li>o considering and explaining the</li> </ul> </li> </ul>
<p><b>1</b> (Attempts are made to connect the procedure of dividing fractions to its underlying meaning; this is achieved only to some extent.)</p>	<p>The PST uses the 1<sup>st</sup> page but modifies it to some extent to support meaning-making:</p> <ul style="list-style-type: none"> <li>- The PST might start with a word problem. If problems 17 or 18 are used, the PST links these problems to the division of fractions and/or the reciprocal (not necessarily in a polished way).</li> <li>- The PST provides some sort of explanation of the involved steps; this could involve at least some explanation of the meaning of division (e.g., how many times the divisor goes into the dividend?).</li> <li>- Students are expected to do some conceptual work while working on exercises 1-16 or a subset thereof instead of simply applying rules and formulas (e.g., use some diagrams/pictures, explain their work, etc).</li> <li>- The PST might ask students to pose problems (without explaining how this</li> </ul>	<p>The PST uses the 2<sup>nd</sup> page and builds some connections to meaning but would not capitalize on certain aspects of this page to support meaning-making and understanding of division of fractions:</p> <ul style="list-style-type: none"> <li>- The algorithm might be presented to students at some point during instruction and some attempts are made to connect the algorithm to meaning. This could involve at least some explanation of the meaning of division (e.g., how many times the divisor goes into the dividend?).</li> <li>- While working on tasks A through C (or a selection of them) the PST would <i>not</i> capitalize on at least one of the aspects of this page to support meaning-making (this could be evident from the PST’s work on these tasks): <ul style="list-style-type: none"> <li>o asking students to use diagrams and/or written explanations and connect their work to the word problem</li> <li>o considering and explaining the</li> </ul> </li> </ul>		

			<p>would support understanding of the involved procedure; additionally no deliberate selection of example(s) is made). The PST talks about exercises 1-16 (or a subset of them) from a <i>procedural</i> perspective, as evident by at least one of the following:</p> <ul style="list-style-type: none"> <li>- Selection of tasks based on procedural rather than conceptual criteria.</li> <li>- No particular ordering of tasks is made or ordering is based on procedural criteria.</li> <li>- Complexity of tasks is based on procedural criteria.</li> </ul>	<p><i>fractional part of an answer</i></p> <ul style="list-style-type: none"> <li>- Students are expected to do some conceptual work while working on (a selection of) tasks A through C instead of simply applying rules and formulas (e.g., use some diagrams/pictures, explain their work, etc).</li> <li>- The PST might build on affordances of tasks A through C which are not necessarily related to the division of fractions (e.g., while working on task A2, the answer of which involves a fractional part, the PST would have students explain why they need to round their answer).</li> </ul>
<p><b>Notes:</b></p> <p>1) If the PST does not select either of the two pages but structures a lesson by using both the pages as resources, the potential cognitive level of the lesson should be coded accordingly (Level 0: strictly procedural; Level 1: some attempts made to make connections to meaning, but the lesson maintains a procedural focus; Level 2: the lesson has potential to support connecting the division of fractions to its underlying meaning; Level 3: the lesson has potential to support connecting the division of fractions to its underlying meaning and additionally could engage students in explorations/pattern identification).</p> <p>2) If there is not enough information on how the PST would use the second page (in case this page is selected), information could also be retrieved from the second part of the interview (the virtual lesson). In particular, consider: (a) if and how the PST comments on the fact that the virtual teacher does not (help students) connect the used diagrams to the word problems (first and second segments); and (b) if the PST notices the problem in Amanda and Julia's solution, which corresponds to interpreting the fractional part of an answer (third segment).</p>		<p><b>0</b></p> <p>(Tasks are mainly enacted at a lower level.)</p>	<p>The PST merely uses the 1<sup>st</sup> page <i>as is</i>:</p> <ul style="list-style-type: none"> <li>- The PST walks students through the two steps; if an explanation is provided, this is merely a description of the steps and/or some definition of terms.</li> <li>- The PST asks students to solve exercises 1-16 or a subset thereof. If a selection of exercises is made, this is random or is based on algorithmic features of the tasks, at best.</li> <li>- The PST considers the complexity of tasks from a procedural standpoint at best</li> <li>- The PST expects students to apply the given algorithm without making any attempt to link this algorithm to its underlying meaning (no use of pictures).</li> <li>- The PST might have students apply the algorithm to solve one or both word problems (17-18). No attempt is made to help students draw connections to meaning.</li> </ul>	<p>The PST uses the 2<sup>nd</sup> page merely from an algorithmic perspective:</p> <ul style="list-style-type: none"> <li>- The PST introduces the algorithm of dividing fractions at the beginning of the lesson as a sequence of steps to follow and apply.</li> <li>- The PST has students use the algorithm to solve the exercises of this worksheet. (probably starting from exercises in task C)</li> <li>- If any selection of tasks is made, this is not based on any conceptual criteria but rather on procedural criteria (e.g., number of steps involved).</li> <li>- The PST mostly expects students to apply the given algorithm without making any attempt to link this algorithm to its underlying meaning.</li> </ul>

**Coding Scheme 1: Selecting and Using Tasks (continued)**

Skill	Entry Points	Score	Code Operationalization
<p><i>Interpreting-Evaluating</i></p>	<p>A. Virtual teacher's <u>presentation</u><sup>211</sup> of task A<sub>1</sub> undermines some of its cognitive demands</p> <p>B. Virtual teacher's <u>enactment</u> of tasks A<sub>1</sub> and D undermines their cognitive potential<sup>212</sup></p>	<p><b>2</b></p>	<p>The PST critiques the presentation AND enactment of tasks in Ms. Rebecca's lesson. The virtual teacher's actions and decisions are judged based on criteria associated with the level of cognitive complexity and the richness at which the tasks were presented and enacted in her lesson.</p> <p><u>For presentation<sup>213</sup> of task A<sub>1</sub>, acceptable comments should echo the following:</u></p> <ul style="list-style-type: none"> <li>- <i>Regarding the use of representations:</i> The teacher could introduce the task by using actual ribbon or a diagram to support students in seeing meaning in what they were doing.</li> <li>- <i>Regarding the cognitive activity required to complete the task:</i> <ul style="list-style-type: none"> <li>o The teacher appears to give more information than needed.</li> <li>o The students end up having to figure out how to arrange the two numbers in a division sentence</li> <li>o The teacher does not check if students understand the assigned task (or alternatively, the teacher could ask students to discuss the task in small groups).</li> </ul> </li> </ul> <p><u>Note:</u> The PST should be credited if he or she:</p> <ul style="list-style-type: none"> <li>- argues that despite these teacher's moves there was still a lot of conceptual work to be done on task A<sub>1</sub></li> <li>- acknowledges that the teacher was doing the thinking for the students but identifies a way in which the teacher could have undermined the demands of the task even more (e.g., by simply outlining steps for students to follow).</li> <li>- proposes an alternative approach to introduce the task that would support more meaning-making</li> </ul>

<sup>211</sup> Presentation and enactment correspond to the two latter phases of the unfolding of tasks, according to the *Mathematics Tasks Framework*.

<sup>212</sup> The idea of presenting the virtual teacher using representations rather mechanically instead of using them to support students' understanding was based on the case of *Fran Gorman and Kevin Cooper* (see Stein et al., 2000, pp. 65-80).

<sup>213</sup> According to the coding rubric developed in the QUASAR project (Stein et al., 1994), the presentation of a task could be analyzed with respect to the following: (a) possible solution strategies asked from the students (single/multiple); (b) type of representations used (single-symbols; single-non-symbols; multiple: pictures, diagrams, symbols, word); (c) communication requirements (none/procedural description/explanations /justifications); and (d) cognitive activity required to complete the task (memorization; applying procedures with no connections to meaning; applying procedures with connections to meaning; and doing mathematics). The criteria used in this rubric pertain to the third and fourth category.

		<p><u>For enactment<sup>214</sup> of task A<sub>1</sub>, acceptable comments should be along the following lines:</u></p> <ul style="list-style-type: none"> <li>- <i>Regarding the use of representations:</i> The diagram used to solve the problem is not connected to the word problem; it is rather used in an algorithmic way.</li> <li>- <i>Regarding the nature of the cognitive activity required to complete the task:</i> <ul style="list-style-type: none"> <li>o The teacher does not hold the students accountable for explaining their answer and showing their reasoning, as requested by the textbook page.</li> <li>o Emphasis is shifted from meaning to following steps and getting correct answers.</li> <li>o The teacher does not press students to draw connections between the diagram and the word problem.</li> <li>o The teacher takes over (some of) the thinking; the teacher poses too leading questions</li> </ul> </li> </ul> <p><u>For task D, acceptable comments should be along the following lines:</u></p> <ul style="list-style-type: none"> <li>- <i>Regarding the nature of the cognitive activity required to complete the task:</i> Instead of having students consider patterns, the teacher: <ul style="list-style-type: none"> <li>o walks students through a number of steps</li> <li>o gives students the answers</li> <li>o does not even help students to connect the reciprocal to the context of the word problem (e.g., what does six over one mean in terms of the ribbon problem?)</li> </ul> </li> </ul> <p><u>Note:</u> Do not consider comments that pertain to the quality of the teacher's <i>explanation</i> (e.g., "the teacher needed to explain more why the algorithm works"). These comments should be considered for Coding Scheme 3.</p>
	<b>1</b>	The PST criticizes the presentation <u>OR</u> enactment of tasks in Ms. Rebecca's lesson. The

<sup>214</sup> According to the Coding Rubric developed in the QUASAR project (Stein et al., 1994), the enactment of a task could be analyzed with respect to the following: (a) solutions strategies used while solving the task (single /multiple); (b) type of representations employed while solving the task (single-symbols; single-non-symbols; multiple representations without any explicit connections; multiple representations with explicit connections); (c) communication requirements (none/procedural description/explanations/justifications); (d) cognitive activity required to complete the task (memorization; applying procedures with no connections to meaning; applying procedures with connections to meaning; and doing mathematics); (e) teacher activities that undermine the demand of the task (e.g., emphasis shifted from meaning to correctness and completeness of answer; teacher takes over thinking); and (f) teacher activities that help maintain the high demands (e.g., appropriate scaffolding, sustained pressure for explanations/justifications, sustained pressure for connections). The criteria used in this rubric pertain to categories (b) through (f).



			<p>teacher's actions and decisions are judged based on criteria associated with the level of cognitive complexity and richness at which the tasks were presented OR enacted in her lesson.</p> <p><u>Note:</u> Please see the note listed in score 2.</p>
		<b>0</b>	<p>The PST considers all Ms. Rebecca's decisions and actions on presenting and enacting the aforementioned tasks acceptable.</p> <p>OR</p> <p>The PST considers Ms. Rebecca's decisions and actions on presenting and enacting the aforementioned tasks unacceptable, but for reasons other than those listed above.</p> <p><u>Note:</u> Please see the note listed in score 2.</p>
<i>Noticing</i>	<p>A. The <i>presentation</i> of task <math>A_1</math> results in taking away some of the cognitive complexity of the task (first segment)</p> <p>B. The enactment of tasks <math>A_1</math> and D degrades the tasks' cognitive potential (first, second, and fourth segments)</p>	<b>2</b>	<p>The PST notices that Ms. Rebecca undermines the richness and cognitive complexity of the tasks she uses in her lesson during <i>presentation AND enactment (either of task <math>A_1</math> or D)</i>.</p> <p>Acceptable comments should echo those listed in the previous category (Interpreting-Evaluating). Comments regarding issues of equity, student engagement etc, although important, should not be counted toward this category.</p>
		<b>1</b>	<p>The PST notices that Ms. Rebecca undermines the richness and cognitive complexity of the tasks she uses in her lesson during <i>presentation OR enactment (either of task <math>A_1</math> or D)</i></p> <p>Acceptable comments should echo those listed in the previous category (Interpreting-Evaluating). Comments regarding issues of equity, student engagement, etc., although important, should not be counted toward this category.</p>
		<b>0</b>	<p>The PST does not identify <i>any</i> of the ways in which Ms. Rebecca degrades the richness and cognitive complexity of the tasks she uses in her lesson during task presentation and/or enactment.</p> <p>OR</p> <p>The PST considers instruction regarding the presentation and enactment of tasks <math>A_1</math> and D problematic but not for reasons related to issues of cognitive demand and task richness/complexity.</p>

**Coding Scheme 2: Using Representations**

Skill	Entry Points	Score	Code Operationalization
<p><i>Performing</i></p>	<p>A. The PST uses a suitable representation and uses it appropriately to explain the quotient (both the whole-number part and the fractional part) in division <math>2 \div \frac{3}{4}</math></p>	<p><b>3</b></p>	<p>The PST uses a suitable representation and uses it appropriately to show/discuss/explain:</p> <ul style="list-style-type: none"> <li>• the dividend</li> <li>• the divisor</li> <li>• the whole-number part of the quotient</li> <li>• the fractional part of the quotient (in relative terms)<sup>215</sup></li> <li>• the reciprocal (4/3)</li> <li>• the expression <math>2 \times \frac{4}{3}</math></li> </ul>
	<p>B. The PST uses a suitable representation and uses it appropriately to explain the reciprocal.</p>	<p><b>2</b></p>	<p>The PST uses a suitable representation and uses it appropriately to show/discuss/explain:</p> <ul style="list-style-type: none"> <li>• the dividend</li> <li>• the divisor</li> <li>• the whole-number part of the quotient</li> <li>• the fractional part of the quotient (in relative terms)</li> </ul>
	<p>C. The PST builds connections between the representations and the algorithm at hand or between the representations and the word problems (if such problems are used).</p>	<p><b>1</b></p>	<p>The PST uses a suitable representation and uses it appropriately to show/discuss/explain:</p> <ul style="list-style-type: none"> <li>• the dividend</li> <li>• the divisor</li> <li>• the whole-number part of the quotient</li> </ul>
		<p><b>0</b></p>	<p>The PST selects an unsuitable representation to show/discuss/explain the algorithm of the division of fractions (e.g., a clock)</p> <p style="text-align: center;">OR</p> <p>The PST selects a suitable representation but cannot use it to explain <i>all the three parts</i> listed in score (1) or uses the representation inappropriately.</p>
	<p><b><i>Appropriate use of representations refers to at least one of the following criteria:</i></b></p> <ul style="list-style-type: none"> <li>• <u><i>Connecting representation(s) to the algorithm:</i></u> <ul style="list-style-type: none"> <li>○ <i>Numerical labeling</i>, which corresponds to connecting the numbers involved in the algorithm to their corresponding parts of the representation (e.g., for a circular representation used to represent <math>2 \div \frac{3}{4}</math>, the PST writes number 2 next to the two circles used to represent the dividend).</li> <li>○ <i>Naming</i>, which corresponds to <i>talking about</i> the associations between the numbers involved in the algorithm and their corresponding parts in the representation (e.g., for a circular representation used to represent <math>2 \div \frac{3}{4}</math>, the PST says, “these two circles represent the two that we start with,” or “these two circles represent the dividend”).</li> </ul> </li> <li>• <u><i>Connecting representation(s) to context/ word problem if such a context/word problem is used:</i></u> If a word problem or some context is used to represent the algorithm, the PST connects the representation and its corresponding parts to this context/word problem.</li> </ul>		
<p><i>Interpreting-Evaluating</i></p>	<p>The virtual teacher does not map the representations she uses in her lessons onto the algorithm and/or the word problem under consideration:</p> <ul style="list-style-type: none"> <li>• The “Amanda” episode (second segment): The</li> </ul>	<p><b>2</b></p>	<p>The PST considers Ms. Rebecca’s approach in <i>one of the two episodes</i> problematic because: (a) she does not draw connections between the representation being used and the algorithm (i.e., numerical labeling /naming) and (b) she does not draw connections between the representation being used and the word problem (or she does not ask her students to do so).</p> <p style="text-align: center;">OR</p> <p>The PST considers Ms. Rebecca’s approach in <i>both</i></p>

<sup>215</sup> Showing the fractional part in absolute terms is an important step for the clarity of the explanation. However, it is not considered critical to the performance of using representations.

	<p>virtual teacher does not ask Amanda to:</p> <ul style="list-style-type: none"> <li>- connect the representation to the algorithm</li> <li>- connect the representation to the problem</li> </ul> <p>• The “Amanda and Julia” episode (third segment): The virtual teacher does not ask the students to connect their representation to the algorithm (e.g., why the line is cut into twelve parts; why nine parts are colored in red; why twelfths are grouped into pairs)</p>		<p>episodes problematic, but uses a different criterion each time to justify his/her thinking. In particular, the PST refers to the lack of connections between the representation being used and the algorithm (in one of the episodes) and between the representation and the word problem (in the other episode).</p> <p><i>Note:</i> For the second episode, comments regarding the students’ final answer (e.g., the students should explain what the 4 and the 1/12 are) should not be credited, because such comments are captured by the fourth coding scheme, where the PSTs are particularly asked to comment on the two students’ final answer.</p>
		<b>1</b>	<p>The PST considers Ms. Rebecca’s approach in <i>one of the two episodes</i> problematic, because she does not draw connections between the representation being used and the algorithm OR the representations and the word problem (or she does not ask her students to do so).</p> <p style="text-align: center;">OR</p> <p>The PST considers Ms. Rebecca’s approach in <i>both</i> episodes problematic, but uses the same criterion to justify his/her thinking. In particular, the PST refers to the lack of connections between the representation being used and the algorithm OR between the representations and the word problem. (<i>Note:</i> Please see note in score 2 above.)</p>
		<b>0</b>	<p>The PST does <i>not</i> consider Ms. Rebecca’s approach in either of the two episodes problematic.</p> <p style="text-align: center;">OR</p> <p>The PST considers Ms. Rebecca’s approach problematic (in either of the two episodes) but:</p> <ul style="list-style-type: none"> <li>- does not justify his/her answer</li> <li>- justifies his/her answer by referring to reasons unrelated to those listed above</li> </ul>
<i>Noticing</i>	<p>Representations are not connected either to the algorithm being considered or to the word problem under consideration:</p> <ul style="list-style-type: none"> <li>• The “Amanda” episode (see above)</li> <li>• The “Amanda and Julia” episode (see above)</li> </ul>	<b>2</b>	<p>The PST raises concerns about the insufficient ways in which representations are being used in <i>both</i> episodes under consideration (by the teacher and/or the students).</p> <p><i>Notes:</i></p> <ol style="list-style-type: none"> <li>1. The PST can also be credited for making comments about the fact that the students’ work was not clear, because the students were not discussing what the different parts of their solutions corresponded to.</li> <li>2. The PST should NOT be credited for filling in the gaps in the students’ work without commenting that the students’ use of the drawing was insufficient.</li> </ol>
		<b>1</b>	<p>The PST raises concerns about the insufficient ways in which representations are being used in <i>one</i> of the episodes under consideration (by the teacher and/or the students). (<i>Notes:</i> Please see notes in score 2 above.)</p>
		<b>0</b>	<p>The PST does <i>not</i> notice anything with respect to the teacher’s and students’ use of representations. OR  The PST does <i>not</i> notice anything <i>problematic</i> in how representations are being used in the lesson. OR  The PST raises concerns about how representations are used in the lesson, but for reasons unrelated to those listed above.</p>

**Coding Scheme 3: Providing Explanations**

Skill	Entry Points	Score	Code Operationalization
<i>Performing</i>	A. The PST provides an explanation for the quotient of the division $2 \div \frac{3}{4}$ B. The PST explains the “invert and multiply” rule in division $2 \div \frac{3}{4}$	3	All criteria listed for score 2 are met; additionally the explanation is calibrated to an average sixth grade student (i.e., could make sense to such an average student). <sup>216</sup>
		2	The PST provides an explanation for <i>both</i> tasks under consideration. Both explanations are mathematically valid/correct and comply with the following criteria: <ul style="list-style-type: none"> <li>- They are not descriptions of the steps involved in the procedure (e.g., the explanation for the reciprocal does not simply describe the steps involved in algorithm)<sup>217</sup></li> <li>- They connect the procedure at hand to its underlying meaning</li> <li>- If the explanations make use of any of the following tools, these tools are suitable and are used appropriately:                             <ul style="list-style-type: none"> <li>o <i>Mathematical examples</i>: simpler examples used to scaffold meaning (e.g., start from explaining <math>1 \div \frac{3}{4}</math>)</li> <li>o <i>Context</i>: the algorithm is connected to a real-life situation (word problem)</li> <li>o <i>Representations</i>: diagrams, pictures, or manipulatives.</li> </ul> </li> <li>- They are complete, in the sense that they do not omit critical steps/pieces of information:                             <ul style="list-style-type: none"> <li>o <i>For the quotient</i>: explanation defines the two different types of units (absolute and relative), and the type of unit used to explain the fractional part of the answer; the left over part is explained in terms of relative units.</li> <li>o <i>For the reciprocal</i>: explanation explicates both the reciprocal and the multiplication involved in the “invert and multiply” algorithm (e.g., explanation refers to how many relative units are included in each absolute unit). If an algebraic approach is followed, all involved steps should be explained.<sup>218</sup></li> </ul> </li> </ul> <p><i>Note</i>: Do not consider invalid claims that the PST might make <i>in the process of developing an explanation</i>, if the PST eventually corrects these claims.</p>
		1	The PST provides an explanation for <i>one</i> of the two tasks under consideration. The explanation is mathematically valid/correct and meets the criteria listed above. OR The PST provides an explanation for <i>both</i> tasks. Both explanations are mathematically valid/ correct but do not satisfy at least any <i>two</i> of the above-listed criteria. (Please also see the note listed above in score 2.)

<sup>216</sup> If there is not sufficient information on this criterion in the first part of the interview, information can also be retrieved from the fourth segment of the virtual lesson when the PST is asked to specify how she or he would respond to Michelle’s and Robert’s questions.

<sup>217</sup> This criterion was adopted from the LMT scoring rubric (retrieved from <http://sitemaker.umich.edu/lmt/links>) and the protocol developed in the QUASAR project (Stein et al., 1994).

<sup>218</sup> Such an explanation is presented in *Everyday Mathematics* [The University of Chicago School Mathematics Project (2007). *Everyday Mathematics: Teacher’s reference manual* (Grades 4-6, pp. 146-147) Chicago, IL: McGraw-Hill]. According to this approach, the division of fractions can be written as a composite fraction (because fractions represent divisions) and then the numerator and the denominator of the composite fraction can be multiplied by the reciprocal of the given denominator (because multiplying the numerator and the denominator of a fraction by the same number gives a fraction equivalent to the original fraction). Because the products of two “reciprocal” numbers is one, the previous step leads to a fraction whose denominator is equal to one, which, in turn, is equal to the numerator of this fraction (since every number divided by one is equal to that number).

		<b>0</b>	The PST does not / cannot provide any explanation. OR The PST's explanations are mathematically invalid/incorrect.
<i>Interpreting-Evaluating</i>	A. Virtual teacher's explanation for the reciprocal is problematic (i.e., Responding to Michelle; fifth segment)	<b>2</b>	The PST considers Ms. Rebecca's explanation problematic <i>and</i> provides at least two reasons to justify his/her answer, including but not limited to: <ul style="list-style-type: none"> <li>• The virtual teacher <i>describes</i> rather than explains the rule.</li> <li>• Her explanation has a limited applicability (does not apply to divisions with remainder)</li> <li>• The virtual teacher's argument that "because we are using the reciprocal we should use a reciprocal operation" is not a mathematically valid argument</li> <li>• The virtual teacher uses an inappropriate analogy to explain the rule (her reference to positive and negative numbers).</li> </ul> <u>Notes:</u> <ol style="list-style-type: none"> <li>1. The PSTs' comments should address the teacher's explanation for the reciprocal and not how she worked with her students to derive the algorithm of the reciprocal (this aspect of the teacher's work is considered in the first Coding Scheme).</li> <li>2. General comments (e.g., "the teacher needs to explain more" should not count toward this category; such comments should only count toward the category of "noticing").</li> </ol>
		<b>1</b>	The PST considers Ms. Rebecca's explanation problematic <i>and</i> provides one reason to justify his/her answer (see reasons above).
		<b>0</b>	The PST does <i>not</i> consider Ms. Rebecca's explanation problematic. OR The PST considers Ms. Rebecca's explanation problematic but either cannot justify his/her answer or provides reasons unrelated to those listed above.
<i>Noticing</i>	A. Virtual teacher's explanation for the reciprocal (Responding to Michelle) B. Virtual teacher's explanation (or lack thereof) for the remainder (Responding to Robert: fifth segment)	<b>2</b>	The PST considers Ms. Rebecca's explanation or lack thereof problematic for <i>both</i> episodes.
		<b>1</b>	The PST considers Ms. Rebecca's explanation or lack thereof problematic for one of the two episodes.
		<b>0</b>	The PST does <i>not</i> notice anything with respect to Ms. Rebecca's explanations. OR The PST does <i>not</i> notice anything <i>problematic</i> in Ms. Rebecca's explanations or lack thereof.

**Coding Scheme 4: Analyzing Student Work/Contributions**

Skill	Entry Points	Score	Code Operationalization
<p><i>Performing</i></p>	<p>A. The PST appropriately evaluates the three students' work in exercise C-3 (fifth segment)                      B. The PST makes reasonable assertions about these students' understanding or lack thereof (fifth segment)</p>	<p><b>3</b></p>	<p>The PST appropriately analyzes the work of all three students:</p> <ul style="list-style-type: none"> <li>- Robert's answer is correct (optional: but he did not simplify his answer or converted the improper fraction into a mixed number)</li> <li>- Michelle's work is correct, because she showed the dividend and the divisor correctly, and she showed the whole-number part of the quotient; yet, she is not showing what the left over part represents (<i>Note:</i> The PST should either identify the left-over part as <math>\frac{2}{4}</math> yards and <math>\frac{2}{3}</math> bows or, alternatively, point out that the left over part needs to be considered in divisor-units.)</li> <li>- Ann's answer is incorrect because she inverted the fraction in the dividend (algorithmic error) (<i>Note:</i> Her answer is <i>not</i> incorrect because she inverted the fraction and <i>then</i> turned the mixed number into an improper fraction)</li> </ul> <p style="text-align: center;">AND</p> <p>The PST makes appropriate assertions about the students' understanding or lack thereof:</p> <ul style="list-style-type: none"> <li>- Robert can apply the algorithm appropriately, but it is not clear whether he conceptually understands division of fractions (or given Robert's question in slides 136-137, Robert might be following a rule without any conceptual understanding)</li> <li>- Michelle uses the approach explicated in the lesson; her representation is correct, but the fact that she did not write a mathematical sentence is problematic (she might have mindlessly followed the approach suggested by the teacher; it is questionable whether she knows how to correctly interpret the fractional part of her answer)</li> <li>- Ann's understanding is questionable: she seems to understand that "you invert and multiply <i>something</i>." </li></ul>
		<p><b>2</b></p>	<p>The PST appropriately analyzes the work of any two of the three students and makes appropriate assertions about their understanding or lack thereof.</p> <p style="text-align: center;">OR</p> <p>The PST appropriately analyzes the work of <i>all</i> three students but makes appropriate assertions about the understanding or lack thereof for only one of them.</p>
		<p><b>1</b></p>	<p>The PST appropriately analyzes the work of only one student and makes appropriate assertions about this student's understanding or lack thereof.</p> <p style="text-align: center;">OR</p> <p>The PST appropriately analyzes the work of at least two students but does not make any appropriate assertions about their understanding.</p>
		<p><b>0</b></p>	<p>The PST fails to appropriately analyze the work of at least one of the students <i>and</i> make appropriate assertions for this student's understanding.</p> <p style="text-align: center;">OR</p> <p>The PST fails to appropriately analyze the work of at least two students.</p>
<p><i>Interpreting-Evaluating</i></p>	<p>A. Virtual teacher's reaction to</p>	<p><b>2</b></p>	<p>The PST describes the error in the students' explanation:</p> <ul style="list-style-type: none"> <li>- June's explanation is not addressing Ben's question or the question of the word problem (i.e., <math>\frac{1}{6}</math> should be the divisor not because it is</li> </ul>

	<p>June's explanation (first segment) is considered problematic</p> <p>B. Virtual teacher's reaction to Amanda's and Julia's solution-interpretation of the remainder is problematic (third segment)</p>		<p>smaller than <math>\frac{1}{2}</math> but because this is the given size of the ribbons)</p> <ul style="list-style-type: none"> <li>- June's explanation is suggestive of an intuitive error/misconception (that the dividend is always larger than the divisor)<sup>219</sup></li> <li>- Amanda's and Julia's work is problematic because these students seem to be confounding the units (yards vs. ribbon badges).</li> </ul> <p style="text-align: center;">AND</p> <p>The PST's response meets one of the following three options:</p> <p><u>Option A:</u> The PST considers the teacher's reaction to the students' work/contributions problematic because:</p> <ul style="list-style-type: none"> <li>- the students' work reflects (important) student misconceptions/errors that warrant addressing</li> <li>- not addressing these errors reinforces certain student misconceptions</li> <li>- these errors cannot be addressed by merely providing students with more practice on dividing fractions</li> <li>- addressing these errors is critical to preventing the students from committing other related errors (the PST needs to identify these errors)</li> <li>- by addressing these errors the teacher honors the integrity of the math</li> </ul> <p><u>Option B:</u> The PST could either question the teacher's reactions to the students' work/contributions or alternatively consider the teacher's decision to not address these errors a <i>tactical move</i> (e.g., the teacher might have a reason for not addressing the errors at this point).</p> <p><u>Option C:</u> The PST proposes other moves that would address these errors (e.g., directly talking about the two different types of units used in the girls' answer; asking the girls to identify what their answer represents, etc)</p>
		<b>1</b>	<p>The PST describes the student errors and questions Ms. Rebecca's move for <i>one</i> of the two episodes. The PST's response with respect to this episode also meets one of the three options outlined above.</p>
		<b>0</b>	<p>The PST cannot/does not describe any of the errors.</p> <p style="text-align: center;">OR</p> <p>The PST describes either of the two errors but considers Ms. Rebecca's reaction to them unproblematic.</p>
<i>Noticing</i>	<p>Noticing problematic aspects in students' contributions:</p> <ul style="list-style-type: none"> <li>• June's explanation (first segment)</li> <li>• Alan's work (first segment)</li> </ul>	<b>2</b>	<p>The PST notices <u>at least three</u> problematic aspects in the students' work/thinking/arguments:</p> <ul style="list-style-type: none"> <li>- June's explanation does not actually address Ben's question</li> <li>- Both June's and Ann's contributions are reflective of the intuitive error that the dividend should always be larger than the divisor</li> <li>- Amanda and Julia's answer is incorrect (because in this answer the two types of units are confounded)</li> <li>- Alan uses the unit inappropriately</li> </ul> <p><u>Note:</u> For this category, the PST should not be expected to name <i>what</i> the problematic aspect is; it is sufficient if he or she identifies that <i>there is something problematic</i> in the students' contribution.</p>

<sup>219</sup> The PST should be given the score for identifying and talking about either of the problems in June's explanation (it is not necessary to talk about both of them).

	<ul style="list-style-type: none"> <li>• Ann’s argument (second segment)</li> <li>• Amanda and Julia’s work (fourth segment)</li> </ul>	<b>1</b>	The PST notices <i>two</i> problematic aspects of the students’ work (from those listed above). The PST should not be expected to name <i>what</i> the problematic aspect is; it is sufficient if he or she identifies that there is something problematic in the students’ contribution.
		<b>0</b>	<p>The PST notices at most one problematic aspect with respect to the students’ contributions.</p> <p style="text-align: center;">OR</p> <p>The PST comments on students’ work/contributions but does <i>not</i> consider them problematic in any respect.</p>



**Coding Scheme 5:** Responding to Students’ Direct or Indirect Requests for Help

Skill	Entry Points	Score	Code Operationalization
<p><i>Performing</i></p>	<p>A. The PST is asked to respond to Alan’s confusion (first segment: Alan’s indirect request for help)<sup>220</sup></p>	<p><b>3</b></p>	<p>All criteria listed for 2 are met (see below) plus the PST’s move(s) could potentially support students’ thinking <i>without replacing</i> it (without doing the thinking for them). For example, a mini-lecture that addresses students’ questions could support students’ understanding but would also minimize opportunities for student thinking.</p>
	<p>B. The PST is asked to respond to June’s and Shaun’s direct request for help (second segment)</p>	<p><b>2</b></p>	<p>In the <i>first episode</i>, the PST:</p> <ul style="list-style-type: none"> <li>• identifies the error in Alan’s solution and the source of this error (e.g., Alan does not understand what the unit is; he was just following the teacher prompts).</li> <li>• proposes a way to help the student <i>understand</i> his error; this might include any (or a combination) of the following which are not necessarily mutually exclusive:               <ul style="list-style-type: none"> <li>- asking questions to help Alan understand the meaning of the unit (the PST needs to identify these questions)</li> <li>- engaging other students in the discussion but facilitating the discussion in such a way that supports making meaning (the PST needs to clarify how she/he would do so)</li> <li>- pursuing a more direct-instruction approach in which the teacher directly points to and discusses the error; in this case emphasis should again be placed on helping students see meaning and not on following a procedure (the PST needs to explain how she/he would do so)</li> </ul> </li> </ul> <p style="text-align: center;">AND</p> <p>Similar to the first episode, in the <i>second episode</i>, the PST:</p> <ul style="list-style-type: none"> <li>• explains how the teacher’s suggestion for using common multiples could help the students in this episode OR proposes an alternative <i>appropriate</i> way to support students’ work in this episode<sup>221, 222</sup></li> <li>• outlines an approach to address the two students’ question which supports meaning-making. The PST might follow any (or a combination) of the following approaches, which are not mutually exclusive:               <ul style="list-style-type: none"> <li>- having June and Shaun discuss their work and scaffold them to overcome their difficulty</li> <li>- supporting students’ <i>exploration</i> of the question at hand either in small groups or in a whole-class discussion (the PST should identify how he or she would scaffold this exploration in ways that support students’ understanding of the involved procedure)</li> <li>- posing questions that help students focus on the underlying meaning of the measurement</li> </ul> </li> </ul>

<sup>220</sup> Teachers often perceive student’s confusion, momentary puzzlement, or erring as indirect requests for help.

<sup>221</sup> If there is not enough information in the second segment on whether the PST understands the idea of common multiples, pertinent information could also be retrieved from the third segment (consider the PST’s struggles to understand Amanda’s and Julia’s solution as evidence of non-understanding).

<sup>222</sup> Identifying the error with the fractional part of the answer is not necessary for this code. It is sufficient if the PST explains how the idea of common multiples could help solve the problem.

			<p>interpretation of division (the PST should clarify what questions she or he would pose)</p> <ul style="list-style-type: none"> <li>- revisiting the concept of division as measurement and helping students identify the reference unit under consideration (i.e., <math>\frac{1}{6}</math>)</li> <li>- comparing this example "<math>\frac{3}{4} \div \frac{1}{6}</math>" to the previous one "<math>\frac{1}{2} \div \frac{1}{6}</math>" with particular emphasis on what makes the second example harder than the first one and on discussing ways to circumvent this difficulty</li> <li>- using a different representation (i.e., an area representation) that allows representing both fractions (e.g., for an area representation this can be done along the representation's two dimensions: divide the area representation into fourths horizontally and into sixths vertically)</li> <li>- providing an explanation that focuses on meaning (direct-instruction approach) (e.g., the PST uses the representation of the number line appropriately, maps between the representation and the algorithm, and helps students understand what it means to divide <math>\frac{3}{4} \div \frac{1}{6}</math>)</li> </ul> <p><i>Note:</i> Helping students to understand that they should divide the <i>whole</i> line into twelve parts should <i>not</i> be considered sufficient for supporting students in making meaning, since this idea has already been raised by the teacher in the previous episode, when the class summarized their work on the first problem (see slide 62)</p>
		<b>1</b>	The PST performs as suggested above (see score 2) for only <i>one</i> of the two episodes under consideration.
		<b>0</b>	<p>In <i>both</i> episodes, the PST:</p> <ul style="list-style-type: none"> <li>• would not react differently from how the virtual teacher reacts in each episode or</li> <li>• would react differently from how Ms. Rebecca reacts, but still in a manner that does not support making meaning.</li> </ul>
<i>Interpreting-Evaluating</i>	<p>A. Responding to Alan's indirect request for help (first and second segment)</p> <p>B. Responding to June and Shaun's direct request for help (second segment)</p>	<b>2</b>	<p>The PST considers Ms. Rebecca's approach in <i>both</i> episodes problematic for reasons related to maintaining the complexity/richness of the task and helping students make meaning of the situation at hand.</p> <p>Reasons provided may include any (or a combination) of the following.<sup>223</sup></p> <ul style="list-style-type: none"> <li>• The teacher is doing most of the thinking for the students</li> <li>• The teacher provides more information than needed</li> <li>• The teacher poses too leading/pointed questions without ensuring that students understand what they are doing</li> <li>• The teacher's intonation gives away the answer (for the first episode)</li> <li>• The teacher rushes to provide hints that might not be</li> </ul>

<sup>223</sup> Adapted from Stein et al. (2000, p. 27)

			<p>necessary</p> <ul style="list-style-type: none"> <li>• The teacher addresses students' difficulty by proposing a way to solve the problem that routinizes the problematic aspects of the task (i.e., finding the common multiple)</li> <li>• The teacher shifts emphasis from meaning and understanding to obtaining an answer</li> <li>• The teacher does not address key mathematical ideas (e.g., what the unit is; why using common multiples makes sense)</li> </ul> <p><u>Notes:</u></p> <ol style="list-style-type: none"> <li>1. For the first episode it is not sufficient to argue that the teacher is concerned with getting correct answers (such a comment is more suitable for the episode that comes after the teacher's interaction with Alan and is captured in the first Coding Scheme).</li> <li>2. For the second episode, just mentioning that the teacher does not help her students to understand that they should divide the whole line should not be considered sufficient since this explanation has already been offered in slide 62.</li> <li>3. If the PST identifies any of the reasons listed above but then considers the teacher's work a tactical move, then the PST should still be credited with the corresponding score. For the second episode, such a response could be: "The students might have been very familiar with the idea of common multiples and the teacher wanted to briefly remind them of this idea in order to save time."</li> </ol>
		<b>1</b>	The PST considers Ms. Rebecca's approach in <i>one of the two episodes</i> problematic for reasons related to maintaining the complexity/richness of the task and helping students make meaning of the situation at hand (see reasons listed above in score 2). Also, please see notes in score 2.
		<b>0</b>	<p>The PST does <i>not</i> consider Ms. Rebecca's approach in either of the two episodes problematic.</p> <p style="text-align: center;">OR</p> <p>The PST considers Ms. Rebecca's approach in either or both episodes problematic but for reasons unrelated to those listed above.</p> <p><i>Please see notes in score 2 above.</i></p>
<i>Noticing</i>	<p>A. Virtual teacher's reaction to Alan's indirect request for help (first and second segment)</p> <p>B. Virtual teacher's reaction to June and Shaun's direct request for help (second segment)</p>	<b>2</b>	<p>The PST notices the ways in which Ms. Rebecca supports her students during <i>both</i> episodes and considers her approach problematic (for reasons that have to do with meaning-making and/or maintaining the complexity of the task).</p> <p><u>Note:</u> Comments made about the virtual teacher's behavior <i>after</i> she handled Alan's error (e.g., how she summarized their work on the first example) should not be considered for this Coding Scheme (such comments count toward the noticing performance examined in the first coding scheme).</p>

		<p><b>1</b></p>	<p>The PST notices the ways in which Ms. Rebecca supports her students in <i>one</i> of the two episodes and considers her approach problematic (for reasons that have to do with meaning-making and/or maintaining the complexity of the task).</p> <p><i>Note. Please see note in score 2 above.</i></p>
		<p><b>0</b></p>	<p>The PST does not notice anything (problematic) with respect to the ways in which Ms. Rebecca's responds to her students' direct or indirect requests for help.</p> <p style="text-align: center;">OR</p> <p>The PST notices something problematic in Ms. Rebecca's work during the two episodes but for reasons unrelated to those listed above (e.g., considers issues of equity and student motivation)</p>

## SECTION B: Locating Pertinent Information in the Interview Transcripts

### *Locating information for Coding Scheme 1 (Selecting and Using Tasks):*

<b>Performing</b>	<ul style="list-style-type: none"> <li>- <b>1<sup>st</sup> part of the interview- planning</b></li> <li>- <i>For selection/use of task D</i> information can also be found in the virtual lesson in segment 3, when the PSTs were asked to consider the following question: “The teacher urges the students to find a pattern/algorithm for dividing fractions. If you were teaching the lesson, how would you go about doing that?”</li> <li>- If the PST selects the second textbook page, and there is not sufficient information in the first part of the interview (planning), you could also consider:             <ul style="list-style-type: none"> <li>o Whether the PST <i>notices</i> the problem with the fractional part of the answer in Amanda’s and Julia’s work (fourth segment of the virtual lesson)</li> <li>o Whether the PST <i>notices</i> the lack of connection between representations and word problems (first and second segment of the virtual lesson)</li> </ul> </li> </ul>
<b>Interpreting-Evaluating</b>	<ul style="list-style-type: none"> <li>- <b>Presentation of task A1:</b> first segment of the virtual lesson (question:<sup>224</sup> “<i>What do you think about the way Ms. Rebecca presented the problem to her students?</i>”)</li> <li>- <b>Enactment of task A1:</b> first and second segments of the virtual lesson (question: “<i>What do you think about Ms. Rebecca’s use of the diagram to support her students’ learning?</i>”)</li> <li>- <b>Enactment of task D:</b> fourth segment (question: “<i>What do you think about the way the teacher worked with her students in finding a pattern/ algorithm for the division of fractions?</i>”)</li> </ul> <p><i>OR comments made while observing these segments.</i></p>
<b>Noticing</b>	<ul style="list-style-type: none"> <li>- <b>Presentation of task A1:</b> first segment of the virtual lesson (question: “<i>What have you noticed so far?</i>”)</li> <li>- <b>Enactment of task A1:</b> first and second segments of the virtual lesson (question: “<i>What have you noticed so far?</i>”)</li> <li>- <b>Enactment of task D:</b> fourth segment (question: “<i>What have you noticed so far?</i>”)</li> </ul> <p><i>OR comments made while observing these segments</i></p>

### *Locating information for Coding Scheme 2 (Using Representations):*

<b>Performing</b>	<b>1<sup>st</sup> part of the interview-planning</b>
<b>Interpreting-Evaluating</b>	<ul style="list-style-type: none"> <li>- <b>“Amanda” episode:</b> second segment of the virtual lesson (question: “<i>What do you think about Ms. Rebecca’s move to ask Amanda to write the numerical sentence for the problem? Would you do something more than that?</i>”)</li> <li>- <b>“Amanda and Julia” episode (lack of connections):</b> third segment of the virtual lesson (question: “<i>What do you think about the manner in which the teacher reacted to Amanda’s and Julia’s work? If you were teaching the lesson, would you do something different?</i>”)</li> </ul> <p><i>OR comments made while observing these segments</i></p>
<b>Noticing</b>	<ul style="list-style-type: none"> <li>- <b>“Amanda” episode:</b> second segment of the virtual lesson (question: “<i>What have you noticed so far?</i>”)</li> <li>- <b>“Amanda and Julia” episode (lack of connections):</b> third segment of the virtual lesson (question: “<i>What have you noticed so far?</i>”)</li> </ul> <p><i>OR comments made while observing these segments</i></p>

<sup>224</sup> These questions might slightly differ from those asked during the interview.

**Locating information for Coding Scheme 3 (Providing Explanations):**

<b>Performing</b>	<ul style="list-style-type: none"> <li>- <b>1<sup>st</sup> part of the interview- planning</b></li> <li>- If there is not enough information in the first part of the interview (planning) as to whether the PSTs’ explanations were calibrated to an imaginary sixth grader, such information could be retrieved from the virtual lesson (fourth segment), where the PSTs were asked to consider how they would respond to:             <ul style="list-style-type: none"> <li>o <b>Michelle’s question - reciprocal</b> (question: “How would you respond to Michelle, who asked why the reciprocal works? Can you give me an example?”)</li> <li>o <b>Robert’s question- remainder</b> (question: “What do you think about Robert’s comment? How would you explain this contradiction?”)</li> </ul> </li> </ul>
<b>Interpreting-Evaluating</b>	<ul style="list-style-type: none"> <li>- <b>Explanation of the reciprocal:</b> fifth segment of the virtual lesson (question in the fifth segment: “What do you think about the way Ms. Rebecca responded to Michelle’s question, namely about her explanation as to why ‘the reciprocal works’?”)</li> </ul> <p>OR comments made while observing this segment.</p>
<b>Noticing</b>	<ul style="list-style-type: none"> <li>- <b>Explanation of the reciprocal:</b> fourth and fifth segments of the virtual lesson (question: “What have you noticed so far?”)</li> <li>- <b>No explanation provided for the remainder:</b> fifth segment of the virtual lesson (question: “What have you noticed so far?”)</li> </ul> <p>OR comments made while observing these segments</p>

**Locating information for Coding Scheme 4 (Analyzing Students’ Work/Contributions):**

<b>Performing</b>	<ul style="list-style-type: none"> <li>- <b>Fifth segment of the virtual lesson.</b> Questions asked:             <ul style="list-style-type: none"> <li>o “Imagine that you were teaching the lesson, and you identified these three different student solutions. What do you think about them?”                 <ul style="list-style-type: none"> <li>▪ “Would you accept them as correct?”</li> <li>▪ “What might these responses suggest regarding what the three students understood or did not understand?”</li> </ul> </li> </ul> </li> </ul> <p>OR comments made while observing this segment</p>
<b>Interpreting-Evaluating</b>	<ul style="list-style-type: none"> <li>- <b>Reacting to June’s explanation:</b> first segment of the virtual lesson (questions: “What do you think about June’s explanation?” and “What do you think about the way Ms. Rebecca responded to June’s explanation? If you were the teacher would you do something different?”)</li> <li>- <b>“Amanda and Julia” episode (fractional answer):</b> third segment of the virtual lesson (question: “What do you think about Amanda and Julia’s solution to the problem?”)</li> </ul> <p>OR comments made while observing these segments</p>
<b>Noticing</b>	<ul style="list-style-type: none"> <li>- <b>June’s explanation:</b> first segment of the virtual lesson (question: “What have you noticed so far?”)</li> <li>- <b>Alan’s error:</b> first segment of the virtual lesson (question: “What have you noticed so far?”)</li> <li>- <b>Ann’s argument:</b> second segment of the virtual lesson (question: “What have you noticed so far?”)</li> <li>- <b>Amanda’s and Julia’s work (fractional answer):</b> third segment of the virtual lesson (question: “What have you noticed so far?”)</li> </ul>

**Locating information for Coding Scheme 5 (Responding to Students' Requests for Help):**

<p><b>Performing</b></p>	<ul style="list-style-type: none"> <li>- "Alan" episode: first segment of the virtual lesson (questions: "What do you think about Alan's work on the diagram?" and "Given Alan's work on the board, what would you do next, and why?")</li> <li>- "June and Shaun" episode: second segment of the virtual lesson (question: "What do you think about the way Ms. Rebecca helped the class overcome the difficulty they had in solving the problem?" Would you do something different?")             <ul style="list-style-type: none"> <li>o If there is not enough information in the second segment on whether the PST understands the idea of common multiple, pertinent information could also be retrieved from the third segment (consider the PST's struggles to understand Amanda's and Julia's solution as evidence of non-understanding)</li> </ul> </li> </ul> <p><i>OR comments made while observing these segments</i></p>
<p><b>Interpreting-Evaluating</b></p>	<ul style="list-style-type: none"> <li>- "<b>Teacher helping Alan</b>" episode: second segment of the virtual lesson (question: "What do you think about the way the teacher handled Alan's error?")</li> <li>- "<b>June and Shaun</b>" episode: second segment of the virtual lesson (question: "What do you think about the way Ms. Rebecca helped the class overcome the difficulty they had in solving the problem?" Would you do something different?")</li> </ul> <p><i>OR comments made while observing this segment</i></p>
<p><b>Noticing</b></p>	<ul style="list-style-type: none"> <li>- "<b>Teacher helping Alan</b>" episode: first and second segments of the virtual lesson (question: "What have you noticed so far?")</li> <li>- "<b>June and Shaun</b>" episode: second segment of the virtual lesson (question: "What have you noticed so far?")</li> </ul> <p><i>OR comments made while observing these segments</i></p>

Table 1.  
Guidelines for Completing Work on Each Coding Scheme and Assigning a Score

	Coding Scheme 1		Coding Scheme 2		Coding Scheme 3		Coding Scheme 4		Coding Scheme 5			
	N	I-E	P	N	I-E	P	N	I-E	P	N	I-E	P
End of the first part; planning			✓			✓						
1 <sup>st</sup> segment of the virtual lesson	Prese- ntation of A1	Prese- ntation of A1					Alan; June (both columns)	June				Suppo- ring Alan
2 <sup>nd</sup> segment of the virtual lesson	Enact- ment of A1	Enact- ment of A1		Amanda	Amanda		Ann			Alan; June and Shaun → ✓	Alan; June and Shaun → ✓	Suppo- ring June and Shaun → ✓
3 <sup>rd</sup> segment of the virtual lesson				Amanda and Julia → ✓	Amanda and Julia → ✓		Amanda and Julia → ✓	Amanda and Julia → ✓				
4 <sup>th</sup> segment of the virtual lesson	Enact- ment of D → ✓	Enactme nt of D → ✓					Perhaps some comments on the work on the reciprocal					
5 <sup>th</sup> segment of the virtual lesson							Respo- nding to Michelle; respo- nding to Robert → ✓	Respo- nding to Michelle; respo- nding to Robert → ✓	✓			

**Key:**

✓: All pieces of information needed to finalize a rubric have been located and hence a score can be assigned for this practice/skill. (Please read each column downward.) N=Noticing; I-E=Interpreting-Evaluating; P=Performing



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