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Final Report

I. A CERTAIN TYPE OF CATEGORY OF SHEAVES
II. EULER CHARACTERISTIC RELATIONS IN TRANSFORMATION GROUPS

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ABSTRACT

R. Godement has obtained [Théorie des faisceanx, Hermann Paris, 1958] a theory of sheaves which is more general than that of H. Cartan [Seminaire, 1950-51], by considering flabby sheaves and Φ-soft sheaves. We consider here a category of sheaves where satisfies a certain condition. We call this a C-category. The category of flabby sheaves, and that of Φ-soft sheaves where Φ is a paracompactifying family are C-categories. And we show that most of fundamental theorems in sheaf theory are valid in this general set. Besides these, we also obtain Mayer-Vietoris cohomology sequences, and a characterization of the transposed homomorphism of a continuous map.

I. A CERTAIN TYPE OF CATEGORY OF SHEAVES

INTRODUCTION

N. Steenrod's concept of a local coefficient [7] system has been generalized by J. Leray to that of sheaf [5]. H. Cartan has simplified the theory of cohomology based on sheaf theory, and put it into a more accessible form [1]. But he considered the cohomology with supports in a paracompactifying family. Quite recently, R. Godement generalized the support family to obtain a more general cohomology theory [3]. Meanwhile, Grothendick obtained a homological algebra of sheaves which is an analogue of that of modules [4]. All these works, consider a property of a sheaf, or rather a category of sheaves which is required to nullify the cohomology. More specifically, Cartan takes the category of fine sheaves, Godement takes the category of flabby and soft sheaves, and Grothendick takes the category of injective sheaves.

The objective of this paper is to put these theories into a categorical form to obtain a more general theory of cohomology based on any category of sheaves which are required to nullify the cohomology. In Section 1, we describe some of the notations, and preliminary remarks to be used in the subsequent sections. In Section 2, we obtain the existence and uniqueness theorem. In Section 3, we establish cohomology sequence for closed set, and Mayer-Vietoris sequences for open pair and closed pair. In Section 4, various spectral sequences are established. In Section 5, we obtain a characterization of a transposed map of a continuous map of a space into another.

1. PRELIMINARIES

We shall recall the definitions and basic properties of sheaves and related terms. We shall always denote by X a space, and by K a principal ideal ring, the basic ring for modules. A presheaf F over X is a contravariant functor of the category $\tilde{\mathcal{E}}$ of all open subsets of X and the inclusion maps into the category $\mathcal{M}$ of modules (always over K) and homomorphisms (always K-invariant). We can consider a homomorphism of a presheaf into another presheaf. For the definition of a sheaf, we shall use that of H. Cartan [1]. A presheaf induces the unique sheaf and vice versa. Hence if we denote by $\mathcal{S}'$ and $\mathcal{S}$ the category of presheaves and that of sheaves, respectively, we have the functors T': $\mathcal{S}' \to \mathcal{S}$ and T: $\mathcal{S} \to \mathcal{S}'$. It follows from a theorem of Godement [3] that

$$T'\,T \approx 1 \text{ and } T(T'|\mathcal{S}'') = 1,$$

where $\mathcal{S}''$ is the subcategory of $\mathcal{S}'$, consisting of presheaves F satisfying the conditions F-1 and F-2 in [3]. We further note that T and T' are covariant. A sheaf F(over X) is <u>flabby</u> if and only if for any open subset U of X, $F(X) \to F(U)$ is surjective where $F(U) = \Gamma(F, U)$ the module of all sections over U. This is obviously equivalent to saying that any section over U can be extended to a section over X. A sheaf F is <u>soft</u> if and only if for any closed subset X' of X, $F(X) \to F(X')$ is surjective. A sheaf F is <u>fine</u> if and only if given any locally finite open covering $\{U_i\}$ of X, there exists a family $\{\psi_i\}$ of endomorphisms of F such that (1) $\sigma(\psi_i) \subset U_i$ where $\sigma(\psi_i)$ = the closure of the set of the point x with $\psi_i(F_x) \neq 0$; and (2) $\sum \psi_i(\alpha) = \alpha$ for any $\alpha \in F$. The reader will note that the definitions of flabby sheaves and soft sheaves are those of Godement, whereas the definition of fine sheaves is that of Cartan. One can easily show that any fine sheaf is soft if X is paracompact, and that any flabby sheaf is soft if X is paracompact. Injective sheaf may be defined in the same way as in the case of homological algebra of modules (see [2; 3]. Injective sheaf is always flabby and does not have any property more useful than just being flabby. Hence we shall not be concerned with injective sheaves. A family Φ of closed subsets of X is called a paracompactifying family (p-family) if Φ satisfies the following conditions; (Φ-1) each element $\in\Phi$ is paracompact, (Φ-2) Φ is closed under the operation of finite union, (Φ-3) any closed subset of an element of Φ is in Φ, (Φ-4) any element of Φ has a nbd $\in\Phi$. Following A. Borel, we shall call a family Φ of closed subsets of X a family of supports if Φ satisfies only Φ-2 up to Φ-4. Given a family of supports, we define Φ-soft sheaves and Φ-fine sheaves. Recall that, if F is Φ-fine, then $F \circ G$ is Φ-fine for any sheaf G and a p-family Φ. We refer the reader to [1, 3] for more definitions, and further properties of sheaves and related terms.

2. COHOMOLOGY THEORY

As before, we fix a space X and a sheaf means a sheaf over X. Let T be a covariant functor: $\mathcal{S} \to \mathfrak{M}$ (see Section 1 for the definitions of $\mathcal{S}$ and $\mathfrak{M}$).

<u>Definition</u>. A category $\mathcal{A}_T$ of sheaves is called a C-category if it satisfies the following conditions:

C-I. Any sheaf F is injected into a sheaf $C(F) \in \mathcal{A}_T$ such that

(a) the assignment: $F \to C(F)$ defines a covariant and exact functor: $\mathcal{S} \to \mathcal{S}$,

(b) $\mathrm{Im}(F_x \to C(F)_x)$ is a direct summand of $C(F)_x$ for each $x \in X$.

C-II. Let $0 \to F \to F \to F'' \to 0$ be an exact sequence of sheaves

(a) If $F' \in \mathcal{A}_T$, then

$$0 \to T(F') \to T(F) \to T(F'') \to 0$$

is exact.

(b) If F' and F are both in $\mathcal{A}_T$, then $F'' \in \mathcal{A}_T$.

Remark. It follows from Godement that given a family Φ of supports the category of flabby sheaves, and that of Φ-soft sheaves if Φ is a p-family, are C-categories with respect to the functor $T = \Gamma_\Phi$.

We shall construct a cochain complex over X with the coefficients in a sheaf F, or cohomology resolution of F, as follows,

$$0 \longrightarrow F \xrightarrow{\epsilon} C^o \xrightarrow{do} C' \longrightarrow \dots \longrightarrow C^q \xrightarrow{dq} C^{q+1} \longrightarrow \dots$$

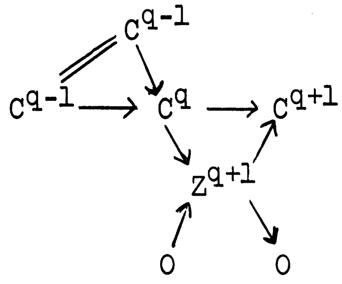
such that the sequence is exact and each $C^q \in \mathcal{A}_T$. For this, first note that we have an exact sequence,

$$0 \longrightarrow F \xrightarrow{\epsilon} C(F)$$

given by the condition C-I such that $C(F) \in \mathcal{A}_T$. Let $C^o = C(F)$, and assume we have already constructed the sequence,

$$0 \longrightarrow F \xrightarrow{\epsilon} C^o \xrightarrow{do} \dots \longrightarrow C^{q-1} \xrightarrow{d_{q-1}} C^q$$

with the required properties. Let $Z^{q+1} = \text{Coker } (d_{q-1}) = C^q/\text{Im}(C^{q-1} \to C^q)$. Let $C^{q+1} = C(Z^{q+1})$. Then we have the following commutative diagram:



such that the diagonal sequences are exact; hence the horizontal sequence is easily seen to be exact. This then completes our inductive construction of a cohomology resolution of F.

We shall often write $\sum C^q = C^* = C^*(F)$. We have proved the first half of the following.

Lemma. Given a sheaf F, there exists a cohomology resolution of F,

$$0 \longrightarrow F \xrightarrow{\epsilon} C^o \xrightarrow{do} C' \longrightarrow \cdots \longrightarrow C^q \xrightarrow{dq} C^{q+1} \longrightarrow \cdots$$

such that (1) $C^q \epsilon \mathcal{A}_T$, and (2) the assignment $F \to C^*(F)$ defines an exact functor (covariant).

Proof. We need to show only that the construction given above satisfies the condition (2) in the lemma. For this, let

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of sheaves. Then there exists the following commutative diagram of sheaves,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(F') & \longrightarrow & C(F) & \longrightarrow & C(F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^1(F') & \longrightarrow & Z^1(F) & \longrightarrow & Z^1(F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the exactness of the first two horizontal sequences follows from C-I-a, the exactness of all the vertical sequences follows from our construction, and the exactness of the last horizontal sequence follows from C-I-b. Since $C^o(F') = C(F')$, $C^o(F) = C(F)$ and $C^o(F'') = C(F'')$, this proves the exactness of

$$0 \to C^*(F') \to C^*(F) \to C^*(F'') \to 0$$

at the degree 0. One may use the induction on the degree to show the exactness in general. For this one must replace the above diagram by

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^q(F') & \longrightarrow & Z^q(F) & \longrightarrow & Z^q(F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^q(F') & \longrightarrow & C^q(F) & \longrightarrow & C^q(F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^{q+1}(F') & \longrightarrow & Z^{q+1}(F) & \longrightarrow & Z^{q+1}(F'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

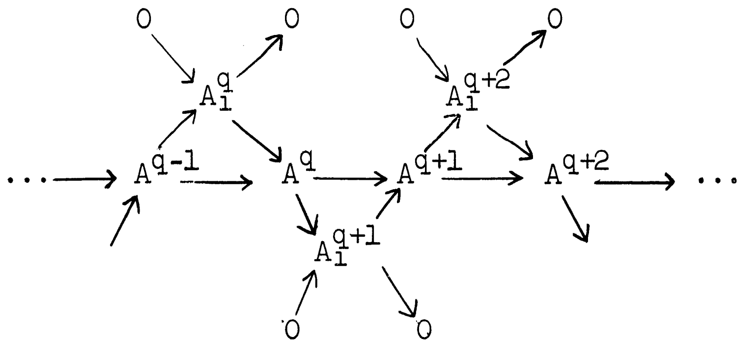
<u>Definition</u>. Given a sheaf F, a covariant functor T, and a C-category $\mathcal{X}_T$, we define

$$H^q_T(\mathcal{X};F) = H^qT(C^*(F))$$

where $TC^*(F) = \sum TC^q(F)$.

For further properties of the functor H_T^* (a dervied functor of T in a certain sense), we observe the following.

<u>Lemma</u>. Suppose we have a commutative diagram of modules or more generally sheaves:



If the diagonal sequences are exact, then so is the horizontal sequence. Conversely, any exact sequence may be decomposed into this form with the diagonal ones all exact. The proof is left to the reader.

<u>Lemma</u>. Given an exact sequence

$$0 \to A' \to A \to A'' \to 0$$

of graded differential modules and differential homomorphisms, there exists an exact sequence induced by the above sequence,

$$\cdots \to H^q(A') \to H^q(A) \to H^q(A'') \to H^{q+1}(A') \to \cdots$$

such that this last sequence is natural with respect to any homomorphism of $0 \to A' \to A \to A'' \to 0$ into another such exact sequence. Proof of this lemma may be found in any standard reference. Now we are ready to prove our main theorem of this section.

Theorem (Existence and Uniqueness). Given a left exact covariant functor T and a C-category $\mathcal{Q}_T$, there exists a unique functor (up to isomorphism) H^q: $\mathcal{S} \to \mathcal{M}$ satisfying the following conditions for each $q \geq 0$.

H-I. H^q is covariant for each $q \geq 0$ such that

(a) $H^o(F) = T(F)$, and

(b) $H^q(F) = 0$ for $n > 0$ and $F \in \mathcal{Q}_T$.

H-II. Given an exact sequence of sheaves,

$$0 \to F' \to F \to F'' \to 0$$

there exists a homomorphism:

$$H^q(F'') \to H^{q+1}(F')$$

such that

(a) $\ldots \to H^q(F') \to H^q(F) \to H^q(F'') \to H^{q+1}(F') \to \ldots$ is exact, and

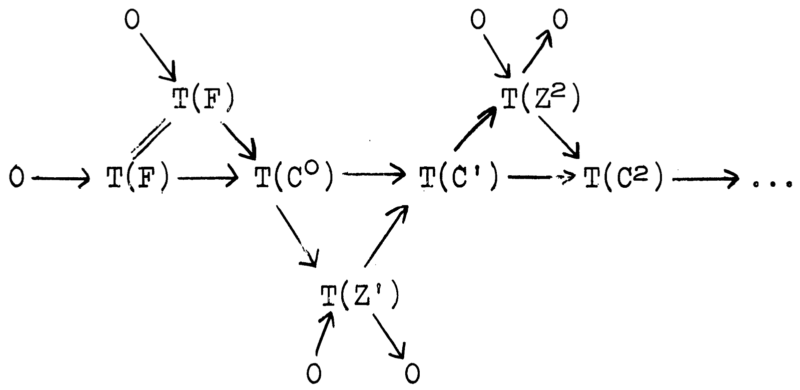
(b) given a commutative diagram of sheaves,

$$\begin{array}{ccccccccc} 0 & \to & F' & \to & F & \to & F'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G' & \to & G & \to & G'' & \to & 0 \end{array}$$

the following diagram is commutative:

$$\begin{array}{ccc} H^q(F'') & \to & H^{q+1}(F') \\ \downarrow & & \downarrow \\ H^q(G'') & \to & H^{q+1}(G') \end{array} .$$

Proof. Existence: Define $H^q(F) = H^q(X;F)$ for all $q \geq 0$. To show that this functor has all the property H-I stated above, we need only to consider the following commutative diagram.



Since T is left exact, $0 \to T(F) \to T(C^o) \to T(Z')$ is exact. This implies $H_T^o(X;F) \approx T(F)$. If $F \in \mathcal{Q}_T$, then, by C-II, $Z' \in \mathcal{Q}_T$. Inductively, $Z^q \in \mathcal{Q}_T$ for all $q \geq 1$. Hence all the diagonal sequences in the above diagram is exact. By our lemma, the horizontal sequence is exact, implying H-I-b. H-II follows from the fact that $0 \to T(C^*(F')) \to T(C^*(F)) \to T(C^*(F")) \to 0$ is exact.

For the uniqueness, suppose we have two functors H^q and ${}'H^q$ which satisfy H-I and H-II. We shall use an induction on the degree q. For $q = 0$, by H-I-a, we have

$$\psi^o(F):H^o(F) \approx {}'H^o(F) \text{(the identity map)}.$$

Suppose $\psi^q(F)$ exists for all F and all $q \leq$ a positive integer. Consider the exact sequence

$$0 \to F \to C^o \to Z' \to 0$$

where $C^o = C(F) \in \mathcal{Q}_T$. By H-II, we have the following commutative diagram:

$$\begin{array}{ccccccc} H^q(C^o) & \longrightarrow & H^q(Z') & \longrightarrow & H^{q+1}(F) & \longrightarrow & 0 \\ \downarrow\approx & & \downarrow\approx & & \downarrow & & \\ {}'H^q(C^o) & \longrightarrow & {}'H^q(Z') & \longrightarrow & {}'H^{q+1}(F) & \longrightarrow & 0 . \end{array}$$

Obviously, there exists an isomorphism $\psi^{q+1}(F):H^{q+1}(F) \to {}'H^{q+1}(F)$ such that the diagram commutes. This completes the proof.

Remark. Given a C-category $\mathcal{Q}_T$ with respect to a left exact covariant functor T, the above theorem shows that there exists a unique cohomology theory, namely $H_T^*(X;F)$. Now let $\mathcal{Q}_T'$ be the category sheaves F such that $H_T^q(X;F) = 0$ for all $q \geq 1$. Evidently $\mathcal{Q}_T$ is a subcategory of $\mathcal{Q}_T'$. Now we claim that $\mathcal{Q}_T'$ is also a C-category. Since the condition C-I is already satisfied by $\mathcal{Q}_T$, we need only to show C-II for $\mathcal{Q}_T'$. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves. If $F' \epsilon \mathcal{Q}_T'$, then $H_T^1(X;F') = 0$, implying that

$$0 \to H_T^0(X;F') \to H_T^0(X;F) \to H_T^0(X;F'') \to 0$$

is exact. Since $H_T^0(X;F) = T(F)$ etc., this shows C-II-a. If F' and F are both in $\mathcal{Q}_T'$, then by the exactness of the cohomology sequence shows that $H_T^q(X;F) = 0$ for all $q \geq 1$, implying $F'' \epsilon \mathcal{Q}_T'$. Hence there exists a cohomology theory ${}'H_T$ with respect to $\mathcal{Q}_T'$. One can easily modify the uniqueness theorem to show that ${}'H_T$ and H_T are the same cohomology theory. In this connection we note that the category of flabby sheaves is a subcategory of the category of Φ-soft sheaves if Φ is a p-family.

3. VARIOUS COHOMOLOGY SEQUENCES

Let F be a sheaf (always over a fixed space X). We shall use the convention that the module over the empty subset of X is trivial. If X' is a subspace of X, then $F|_{X'}$ is defined in the obvious way (see [3] for details). Suppose now that X' is a subset of X, either open or closed and F' is a sheaf over X'. Then there exists a unique sheaf F over X such that $F|_{X'} = X$ and

$F_x = 0$ for $x \in X-X'$, because such a set F is unique and the topology on F is also unique to make the projection p: $F \to X$ a local homeomorphism and $p|_{X'}$ agree with the projection map: $F' \to X'$. For a more explicit description of the topology of F, we refer the reader to [1]. To avoid a notational complication, we shall denote by the same F' the sheaf extended over X from F'. In particular, given a sheaf F over X, $F|_{X'}$ shall denote both the sheaf F restricted to X' and the extension of the restricted sheaf.

Now let X' be an open subset of X, and let F be a sheaf over X. Then there exists the following exact sequence,

$$0 \to F_{X'} \to F \to F_{X-X'} \to 0$$

Since $F_{X'}$ is an open subset of F and $F_{X-X'} \approx F/F_{X'}$ by the above argument, the existence and the exactness of the sequence follows immediately. If $\mathcal{A}_T$ is a C-category with respect to a left exact functor T, we have the exact sequence

$$\cdots \to H_T^q(X;F_{X'}) \to H_T^q(X;F) \to H_T^q(X;F'') \to H_T^{q+1}(X;F') \to \cdots$$

This is called the cohomology sequence for the pair (X,X') or (X,X-X'). To study the nature $H_T^*(X;F_{X'})$ and $H_T^*(X;F_{X-X'})$, we set up another condition C-I-c for the category $\mathcal{A}_T$ to satisfy.

C-I(c): Given a sheaf, $C(F)|_{X'} \in \mathcal{A}_T$ for each open subset X' of X.

Note that the category of flabby sheaves satisfies this condition with respect to the functor $T = \Gamma_\Phi$.

Let $\mathcal{A}_{T|X'}$ be the subcategory of $\mathcal{A}_T$, consisting of the sheaves $C|_{X'}$ where $C \in \mathcal{A}_T$ and $C|_{X'} \in \mathcal{A}_T$. From this condition and C-II, it also follows that $C|_{X-X'} \in \mathcal{A}_T$

by the exact sequence $0 \to C|_{X'} \to C \to C|_{X-X'} \to 0$. Again then we define similarly $\mathcal{A}_T|_{X-X'}$, and may show that this category is a C-category. One may also easily observe that the assignment $F' \to H_T^*(X;F')$ defines a cohomology functor, and hence by the uniqueness theorem,

$$H_T^q(X;F|_{X'}) \approx H_T^q(X';F|_{X'}) \text{ for all } q.$$

Similarly, we have

$$H_T^q(X;F|_{X-X'}) \approx H_T^q(X-X';F|_{X-X'}) \text{ all } q.$$

For simplicity we shall denote these modules by

$$H_T^q(X';F) \text{ and } H_T^q(X-X';F),$$

respectively. Then we may write the cohomology sequence,

$$\cdots \to H_T^q(X';F) \to H_T^q(X;F) \to H_T^q(X-X';F) \to H_T^{q+1}(X';F) \to \cdots$$

Next we shall establish the Mayer-Vietorix sequence. As before let F be a sheaf over X, and $X = U_1 \cup U_2$ the union of two open subsets of X. Write $Y = U_1 \cap U_2$. Consider the injective homomorphisms

$$F_Y \xrightarrow{j_i} F_{U_i} \xrightarrow{k_i} F, \quad (i = 1, 2).$$

One may show easily that we have the following exact sequence

$$0 \longrightarrow F_Y \xrightarrow{j_1+j_2} F_{U_1} \oplus F_{U_2} \xrightarrow{k_1-k_2} F \longrightarrow 0$$

where $(j_1+j_2)\alpha = (j_1(\alpha), j_2(\alpha))$ and $(k_1-k_2)(\alpha,\beta) = k_1(\alpha)-k_2(\beta)$. By the condition

H-II, this sequence induces the exact sequence,

$$\cdots \to H_T^q(X;F_Y) \to H_T^q(X;F_{U_1} \oplus F_{U_2}) \to H_T^q(X;F) \to H_T^{q+1}(X;F_Y) \to \cdots$$

Using the same argument as before, one notes that

$$H_T^q(X;F_Y) \approx H_T^q(Y;F) .$$

We claim

$$H_T^q(X;F_{U_1} \oplus F_{U_2}) \approx H_T^q(U_1;F) \oplus H_T^q(U_2;F).$$

For the verification, consider the decomposition maps,

$$F_{U_i} \xrightarrow{p_i} F_{U_1} \oplus F_{U_2} \xrightarrow{q_j} F_{U_j}$$

where $q_j \circ p_i$ is the identity map for $i = j$, and the trivial map for $i \neq j$. These maps induce

$$H_T^q(U_i;F) \xrightarrow{p_i^*} H_T^q(X;F_{U_1} \oplus F_{U_2}) \xrightarrow{q_j^*} H_T^q(U_j;F)$$

such that $q_j^* \circ p_i^* = 1$ for $i = j$, and $= 0$ for $i \neq j$. Hence these are decomposition maps which would verify our claim. Hence we have the exact sequence

$$\cdots \to H_T^q(Y;F) \to H_T^q(U_1;F) \oplus H_T^q(U_2;F) \to H_T^q(X;F) \to H_T^{q+1}(Y;F) \to \cdots$$

This is called the Mayer-Vietoris sequence for (U_1,U_2).

With a slight modification of this argument, we can obtain the Mayer-Vietoris sequence for a closed pair (X_1,X_2). Let $X = X_1 \cup X_2$ with X_i being a closed subset of X, $(i=1, 2)$. Write $Y = X_1 \cap X_2$. Define the surjective maps,

$$F \xrightarrow{j_i} F_{X_i} \xrightarrow{k_i} F_Y, \quad (i=1, 2).$$

Then we obtain the exact sequence,

$$0 \longrightarrow F \xrightarrow{j_1+j_2} F_{X_1} \oplus F_{X_2} \xrightarrow{k_1-k_2} F_Y \longrightarrow 0$$

where the maps are defined in the same way as for a open pair. Again by H-II, and by the similar argument as for an open pair, we obtain the exact sequence,

$$\dots \to H_T^q(X;F) \to H_T^q(X_1;F) \oplus H_T^q(X_2;F) \to H_T^q(Y;F) \to H_T^{q+1}(X;F) \to \dots$$

This is called the Mayer-Vietoris sequence for the closed pair (X_1, X_2).

Remark. Incidentally, we have shown that the cohomology functor is an additive functor, simply because the cohomology functor was a covariant functor.

4. EXISTENCE OF VARIOUS SPECTRAL SEQUENCES

As before, let X be a space and F a sheaf over X. Suppose that we have a sequence of sheaves such that the composition of any two maps is trivial.

$$0 \to F \xrightarrow{\epsilon} F^0 \to F' \to \dots \to F^q \to F^{q+1} \to \dots$$

Again let $\mathcal{Q}_T$ be a C-category with respect to a left exact covariant functor T. Then we have the double complex $A = TC^*(F^*) = \sum TC^p(F^q)$, and we have as usual two distinct filtrations associated with the double graduation of A: namely, there are 'F and "F defined by

$$'F_r A = \sum_{\substack{p \geq r \\ q \geq 0}} A^{p,q} \quad \text{and} \quad "F_r A = \sum_{\substack{q \geq r \\ p \geq 0}} A^{p,q}$$

where $A^{p,q} = TC^p(F^q)$. Then corresponding to $'F$ and $''F$, respectively, there exist the spectral sequences $\{'E_r\}$ and $\{''E_r\}$ such that

$$'E_2^{p,q} \approx {'H^p}\, {''H^q}(A), \quad ''E_2^{p,q} \approx {''H^p}\, {'H^q}(A)$$

and both $'E_\infty$ and $''E_\infty$ are associated with proper filtrations of the derived module $H^*(A)$ with respect to the total degree. More details may be found in [3].

__Lemma.__ $''H^q(A) \approx \Sigma T(C^p(F^q))$ and $'E_2^{p,q} \approx H_T^p(X; H^q(F))$.

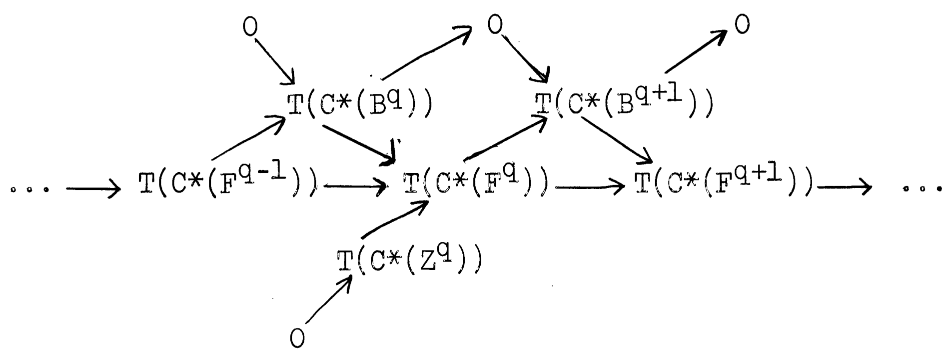
__Proof.__ There are two exact sequences:

$$0 \to B^q(F) \to Z^q(F) \to H^q(F) \to 0$$
$$0 \to Z^q(F) \to F^q \to B^{q+1}(F) \to 0$$

where $Z^q(F) = \mathrm{Ker}(F^q \to F^{q+1}) = \mathrm{Im}(F^{q-1} \to F^q)$. These then induce the following exact sequences:

$$0 \to T(C^*(B^q)) \to T(C^*(Z^q)) \to T(C^*(H^q)) \to 0$$
$$0 \to T(C^*(Z^q)) \to T(C^*(F^q)) \to T(C^*(B^{q+1})) \to 0.$$

Hence we have the following commutative diagram with all the diagonal sequences:



An examination of this diagram shows that the following sequence is exact.

$$0 \to T(C^*(B^q)) \to T(C^*(Z^q)) \to {}''H^qT(C^*(F^*)) \to 0$$

This then easily completes our proof.

Theorem 4.1. For any complex $F^* = \sum_{q \geq 0} F^q$ of sheaves, there are two spectral sequences $\{'E_r\}$ and $\{''E_r\}$ such that

$$'E_2^{p,q} = H_T^p(X;H^qF^*), \quad ''E_2^{p,q} = H^p(H_T^q(X;F^*)),$$

and both $'E_\infty$ and $''E_\infty$ are associated with the suitable filtrations of $H^*(A)$.

This theorem follows immediately from the preceding lemma.

Theorem 4.2. Let $F^* = \sum_{q \geq 0} F^q$ be a complex of sheaves such that the sequence

$$\cdots \to H_T^q(X;F^q) \to H_T^q(X;F^{q+1}) \to \cdots$$

is exact. Then there exists a spectral sequence $\{E_r\}$ such that

$$E_2^{p,q} = H_T^p(X;H^qF^*)$$

and E_∞ is associated with a suitable filtration of $H(TF^*)$.

Proof. Consider the spectral sequences $\{'E_r\}$ and $\{''E_r\}$ in Theorem 4.1. We evidently have

$$''E_2^{p,q} = H^p(H_T^q(X;F^*)) = 0, \text{ all } p, \text{ and all } q \geq 1.$$

Also

$$''E_2^{p,0} = H^p(TF^*).$$

Hence

$$''E_\infty^{p,q} = 0 \text{ for } q \geq 1, \text{ and } E_\infty^{p,0} = E_2^{p,0}.$$

Thus

$$H^p(A) = H^p(TF^*).$$

By taking $E_r = {}'E_r$, we obtain the desired sequence.

Corollary. Let $0 \to F \to F^0 \to F' \to \ldots \to F^q \to F^{q+1} \to \ldots$ be an exact sequence of sheaves with $H_T^p(X;F^q) = 0$ for any $(p,q) \geq (1,0)$. Then

$$H_T^p(X;F) = H_T^p(TF^*).$$

Remark. This last corollary shows that the resolution $C^*(F)$ specifically constructed earlier may be replaced by any other resolution $'C^*(F)$ of F with each $'C^q(F) \in \mathfrak{A}_T$ or more generally $'C^q(F) \in \mathfrak{A}'_T$ in defining the cohomology functor H_T over the space X. This also reveals some possibility of weaking the conditions C-I and C-II.

Theorem 4.3. Let the following sequence be a resolution of F.

$$0 \longrightarrow F \xrightarrow{\epsilon} F^0 \longrightarrow F' \longrightarrow \ldots \longrightarrow F^q \longrightarrow F^{q+1} \longrightarrow \ldots$$

Then there exists a spectral sequence $\{E_r\}$ such that

$$E_2^{p,q} = H^p(H^q(X;F^*))$$

and E_ω is associated with a suitable filtration of $H_T^*(X;F)$.

Proof. Consider the spectral sequences $\{'E_r\}$ and $\{''E_r\}$ in Theorem 4.1. Then

$$'E_2^{p,q} = \begin{cases} 0 \text{ for } (p,q) \geq (0,1) \\ H^p(X;F) \text{ for } q = 0 \end{cases} .$$

It easily follows that $H^p(A) = H^p_T(X;F)$. $E_r = {}''E_r$ is the desired spectral sequence.

5. CONTINUOUS MAPS

Let X, T, and $\mathcal{A}_T$ be as before. Suppose that we have another space Y and f: $X \to Y$ a (continuous) map. Let T' be a left exact covariant functor on the sheaves over Y, and let $_{T'}$ be a C-category with respect to the functor T'. For any sheaf G over Y, denote by $G \to D(G)$ the functor for Y, which corresponds to $F \to C(F)$ for X. Denote by $D^*(G)$ the resolution of G, corresponding to $C^*(F)$. Then first observe that $f^{-1}(G)$, a sheaf over G (see [1,3] for the notation f^{-1}). Let $F = f^{-1}(G)$ require that there is a natural homomorphism $T'(G) \to f^{-1}G$ for any G, and consider the spectral sequence in Theorem 4.3. Then we have

$$H^q(TF^*) = E_2^{p,o} \to E_3^{p,o} \to \ldots \to E_\infty^{p,o} = J^{p,o} \subset H^p_T(X;F)$$

where $F^* = f^{-1}\, D^*(G)$. Hence we have a homomorphism

$$H^q_{T'}(Y;G) = H^q T'(D^*) \to H^q T(F^*)$$

by means of the following commutative diagram,

$$\begin{array}{ccccccc} \ldots \to & T'(D^q) & \to & T'(D^{q+1}) & \to \ldots \\ & \downarrow & & \downarrow & \\ \ldots \to & T(F^q) & \to & T(F^{q+1}) & \to \ldots \end{array}$$

Consequently, we obtain the homomorphism induced by f

$$f^*;\ H^q_{T'}(Y;G) \to H^q_T(X;f^{-1}G)$$

by taking the composition of the following maps,

$$H^q_{T'}(Y;G) \to H^qT(F^*) \to H^q_T(X;F).$$

Let $\mathcal{S}'$ be the category of sheaves over Y, and let $\mathcal{M}$ be the category of graded modules (and homomorphisms of homogeneous degree 0). Then we have the following covariant functors,

$$\mathcal{H}:\mathcal{S} \to \mathcal{M} \text{ and } \mathcal{H}_f:\mathcal{S} \to \mathcal{M}$$

defined by $\mathcal{H}(G) = H^*_{T'}(Y;G)$ and $\mathcal{H}_f(G) = H^*_T(X;f^{-1}G)$. Consequently, $\mathcal{H}(\mathcal{S})$ and $\mathcal{H}_f(\mathcal{S})$ are subcategories of $\mathcal{M}$. Given any integer $q \geq 0$, $G \to H^q_{T'}(Y;G)$ and $G \to H^q_T(X;f^{-1}G)$ are covariant functors $\mathcal{H}^q$ and $\mathcal{H}^q_f$, respectively, and we may consider $\mathcal{H} = \Sigma\mathcal{H}^q$ and $\mathcal{H}_f = \Sigma\mathcal{H}^q_f$.

Theorem 5.1. Using the same notations as above, there exists a unique functor (up to isomorphism)

$$f^*:\mathcal{H}(\mathcal{S}) \to \mathcal{H}_f(\mathcal{S})$$

satisfying the following conditions:

(1) f^* is covariant.

(2) $f^o(G)$ agrees with $T'(G) \to T(f^{-1}G)$ induced by $G \to f^{-1}G$.

(3) Given any exact sequence $0 \to G' \to G \to G'' \to 0$ of sheaves over Y, the following diagram is commutative:

$$\begin{array}{ccc} H^q_{T'}(Y;G'') & \longrightarrow & H^{q+1}_{T'}(Y;G') \\ \downarrow f^q(G'') & & \downarrow f^{q+1}(G') \\ H^q_T(X;f^{-1}G'') & \longrightarrow & H^{q+1}_T(X;f^{-1}G'). \end{array}$$

Proof. To show that there exists at least one such a functor, we take f^* which we constructed earlier. We shall first show (3). For this, observe that the homomorphism $H^q_{T'}(Y;G) \to H^q_T(X;f^{-1}G)$ comes from the following maps,

$$T'D^*(G) \to Tf^{-1}D^*(G) \to TC^*f^{-1}D^*(G) \leftarrow TC^*f^{-1}G$$

where the extreme right homomorphism induces an ismorphism in the derived modules. Hence we have the commutative diagram as follows where the horizontal sequences are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T'D^*(G') & \longrightarrow & T'D^*(G) & \longrightarrow & T'D^*(G'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TC^*f^{-1}D^*(G') & \longrightarrow & TC^*f^{-1}D^*(G) & \longrightarrow & TC^*f^{-1}D^*(G'') & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & TC^*f^{-1}G' & \longrightarrow & TC^*f^{-1}G & \longrightarrow & TC^*f^{-1}G'' & \longrightarrow & 0
\end{array}$$

From a general argument of homological algebra, (3) follows. (1) also follows easily from the same type of argument. For (2), consider the commutative diagram:

$$\begin{array}{ccccc}
T'(G) & \longrightarrow & Tf^{-1}G & \longrightarrow & TC^*f^{-1}G \\
\downarrow & & \downarrow & & \downarrow \\
T'D^*(G) & \longrightarrow & Tf^{-1}D^*(G) & \longrightarrow & TC^*f^{-1}D^*(G).
\end{array}$$

Now simply take H^0 of the modules in the diagram, and obtain the following commutative diagram:

$$\begin{array}{ccccc}
T'(G) & \longrightarrow & Tf^{-1}G & \xrightarrow{\approx} & H^0(TC^*f^{-1}G) \\
\downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
H^0(D^*G) & \longrightarrow & H^0(f^{-1}D^*G) & \longrightarrow & H^0(TC^*f^{-1}D^*(G))
\end{array}$$

Then we have the isomorphisms indicated in the diagram. Since $H^0_{T'}(Y;G) \to H^0_T(X;f^{-1}G)$ was defined by means of

$$H^0(T'G^*) \to H^0(Tf^{-1}D^*G) \to H^0(TC^*f^{-1}D^*G) \overset{\approx}{\leftarrow} H^0(TC^*f^{-1}G),$$

we obtain the desired result.

To show the uniqueness, observe that f^0 is already unique. Assume that f^i is unique for all $i \leq q$ where $q \geq 0$. Given a sheaf G over Y, take an exact sequence

$$0 \to G \to D \to D' \to 0$$

of sheaves with $D \in \mathcal{Q}_{T'}$. Then we have the commutative diagram with the horizontal sequences exact,

$$\begin{array}{ccccc} H^q_{T'}(Y;D') & \longrightarrow & H^{q+1}_{T'}(Y;G) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^q_T(X;f^{-1}D') & \longrightarrow & H^{q+1}_T(X;f^{-1}G) & \longrightarrow & H^{q+1}_T(X;f^{-1}D) \end{array}$$

This evidently shows the uniqueness of f^q, completing the proof.

Remark. If, in particular, X = Y and T = T', then our theorem describes a relation between two cohomology theories, one based on $\mathcal{Q}_T$ and the other one based on $\mathcal{Q}_{T'}$ which may differ from $\mathcal{Q}_T$. If, further, $\mathcal{Q}_T = \mathcal{Q}_{T'}$, then our theorem shows that the identity maps of X induces the identity map of H^*_T. Next, suppose that X is a closed subspace of Y and f is the inclusion map $X \subset Y$. Suppose also that $\mathcal{Q}_{T'}$ satisfies the condition C-I-(c), T = T' and $\mathcal{Q}_T = \mathcal{Q}_{T'}|X$. Then the induced homomorphism $f^*: H^*_{T'}(Y;G) \to H^*_T(X;f^{-1}G)$ is the one in the cohomology sequence for (Y,X).

5.1. An Application of Theorem 5.1.

Let X,Y and f be as before. Let Φ and ψ be p-families for X and y respectively, such that $f^{-1}\psi \subset \Phi$. Let $\mathcal{A}_\Phi$ and $\mathcal{B}_\psi$ be the categories of Φ-soft sheaves over X and ψ-soft sheaves over Y, respectively. Then we have the corresponding cohomology functors $H^q_\Phi(X;F)$ and $H^q_\psi(Y;G)$, and the induced map $f^*:H^q_\psi(Y;G) \to H^q_\Phi(X;f^{-1}G)$. Recall that we denoted by K the ground ring for modules. With Cartan [1], we may consider K as the constant sheaf K x X (resp. K x Y) over X(resp. over Y). Then we have the following commutative diagram of the Alexander-Spanier cochain sheaves D* over Y and C* over X, and of the induced homomorphisms D* → C* of f, with the horizontal sequences exact.

$$\begin{array}{ccccccccccccccc}
0 & \to & K & \to & D^o & \to & D' & \to & \cdots & \to & D^p & \to & D^{p+1} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & C^o & \to & C' & \to & \cdots & \to & C^p & \to & C^{p+1} & \to & \cdots
\end{array}$$

Given a sheaf G over X, we have the commutative diagram,

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & G & \longrightarrow & G \circ D^o & \longrightarrow & G \circ D' & \longrightarrow & \cdots & \longrightarrow & G \circ D^p & \longrightarrow & G \circ D^{p+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f^{-1}G & \longrightarrow & f^{-1}G \circ C^o & \longrightarrow & f^{-1}G \circ C' & \longrightarrow & \cdots & \longrightarrow & f^{-1}G \circ C^p & \longrightarrow & f^{-1}G \circ C^{p+1} & \longrightarrow & \cdots
\end{array}$$

with the horizontal sequences exact.

Since D* and C* are ψ-fine and Φ-fine, respectively, so are GoD* and f^{-1}GoC*. Hence $G \circ D^p \in \mathcal{B}_\psi$ and $f^{-1}G \circ C^q \in \mathcal{A}_\Phi$ for all p and q. Hence by the Uniqueness Theorem, we have the natural isomorphism

$$H^p\Gamma_\psi(G \circ D^*) \approx H^p_\psi(Y;G)$$

$$H^p\Gamma_\Phi(f^{-1}G \circ C^*) \approx H^p_\Phi(X;f^{-1}G)$$

whose detailed proof is left to the reader. Using Theorem 5.1, one may easily see that the homomorphism

$$H^p\Gamma_\psi(G\circ D^*) \to H^p\Gamma_\Phi(f^{-1}G\circ C^*)$$

agrees with the homomorphism $H^p_\psi(Y;G) \to H^p_\Phi(X;f^{-1}G)$.

This implies that, if we have p-families ψ and Φ for supports, one can construct the induced map in the cohomology modules, of a map, much more directly without going through a spectral sequence argument as in the general case.

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ABSTRACT

In his paper [Trans. Amer. Math. Soc. 72 (1952), 138-147], E. E. Floyd shows that if (Zp, X), p prime, is a transformation group, then the Euler characteristic relation $\chi(X = \chi(F) + p\chi(X/Zp - F)$ holds where F is the fixed point set of (Zp, X) and X is assumed to be of finite covering dimension and of finite type. We generalize this formula to the case where the group action is not required. More specifically, we show that if $(\widetilde{X}, f)$ is a singular covering space of a space X with the singular set F, then the Euler characteristic relation $\chi(\widetilde{X}) = \chi(F) + n\chi(X - F)$ holds where $\widetilde{X}$, X, and F are assumed to be of finite cohomology dimensions and of finite types.

II. EULER CHARACTERISTIC RELATIONS IN TRANSFORMATION GROUPS

Let (G,X) be a finite transformation group acting on a space X (always assumed to be locally compact and Hansdorff). Denote by F(G,X) or simply by F the fixed point set of (G,X). We shall be concerned here with "What can one say about the Euler characteristic relations between X, F, and the orbit space X/G when they have finite Euler characteristics?" In this connection, E. E. Floyd has shown [1,4] that, if G = Zp, p prime, $\dim_{Z_p} X < \infty$ and dim H*(X;Zp) $< \infty$, then dim H*(F;Zp) and $\chi(X;Zp) = \chi(F;Zp)$ mod p. He also shows that, if G is a solvable group acting freely on X where $\dim_Z X < \infty$ and H*(X;Z) is finitely generated, then H*(X/G;Z) is finitely generated and $\chi(X) = (\text{order } G) \cdot \chi(X/G)$. In Section 1, we give a new and substantially simpler proof to a theorem which shows both above theorems. In Section 2, we shall establish a concept of covering spaces with singularities. This is a generalization of the concept of finite transformation group (G,X) in which each point of X is either fixed by every element of G or fixed only by the identity element of G. This certainly is the case of the theorems of E. E. Floyd quoted above. And we shall establish an Euler characteristic formula for the total space, base space, and singular set. The proof shall depend only on the construction of a suitable covering.

1. We begin with establishing notations and terms. Recall that we always assume X to be locally compact and Hansdorff. Denote by $A(X) = \sum A^q(X)$ the Alexander-Spanier grating over X with compact supports and with coefficients in a field K. If dim H*(X;K) is finitely generated, then the Euler characteristic $\chi(X;K)$ is defined by $\sum(-1)^q \dim H^q(X;K)$. Suppose Zp (p;prime) acts on a

space X. Then we assume K = Zp. Let τ = 1-g and σ = 1+g+g^2+ ... +g^{p-1} where g is a (fixed) non-trivial element of Zp. Denote by p one of τ and σ, and by $\bar{p}$ the other. p has the induced action on A which shall be denoted by the same p.

<u>Theorem 1.</u> There exists an exact sequence:

$$\cdots \to H^q_{\bar{p}}(X-F) \to H^q(X) \to H^q_{\bar{p}}(X-F) \oplus H^q(F) \to H^{q+1}_{\bar{p}}(X-F) \to \cdots$$

where $H^p_p(X-F) = H^p(pA(X-F))$ and F is the fixed point set of (Zp,X).

This sequence is called Smith cohomology sequence [1,4]. A proof for this may be found in [1]. Another proof may be obtained using the following lemmas, again due to E. E. Floyd (unpublished), which shall be used in the proof of our main theorem in this section.

<u>Lemma 1.</u> There exists an exact sequence:

$$0 \longrightarrow \bar{p}A(X-F) \xrightarrow{\text{incl.}} A(X-F) \xrightarrow{p} pA(X-F) \longrightarrow 0$$

<u>Lemma 2.</u> If $\alpha \epsilon A$ with $d\alpha \epsilon A(X-F)$, then $\alpha = \alpha_1 + \alpha_2 + d\beta$ with $\alpha_1 \epsilon A(X)^G$, $\alpha_2 \epsilon A(X-F)$ and $\beta \epsilon A(X)$.

Now we are ready to state and prove our main theorem.

<u>Theorem 2.</u> Let (Zp, X) be as before, $\dim_{Z_p} X < \infty$ and $\dim H^*(X) < \infty$. Then we have

1. $\dim H^*_p(X-F) < \infty$, and $\dim H^*(F) < \infty$.
2. $\chi(X) = \chi(F) + p\chi(X/Zp-F)$.

For the proof, we depend on the following.

Lemma 3. If we have a finite exact sequence of finitely generated vector spaces over Zp,

$$\cdots \to C_1^q \to C^q \to C_2^q \to C_1^{q+1} \to \cdots$$

then

$$\chi(C^*) = \chi(C_1^*) + \chi(C_2^*)$$

where

$$\chi(C^*) = \Sigma(-1)^q \dim C^q,$$

and $\dim C^q \leq \dim C_1^q + \dim C_2^q$ for all q. The proof for this lemma is straightforward.

Now we prove our Theorem 2. The first part of the theorem follows easily from Theorem 1 and Lemma 3. For simplicity, write pA(X-F) = A(p) and $H^q(pA(X-F))$ $H^q(p)$. From Lemma 1, we obtain the exact sequence,

$$0 \longrightarrow A(\tau^{q+1}) \xrightarrow{\text{incl.}} A(\tau^q) \xrightarrow{\tau^{p-1-q}} A(\tau^{p-1}) \longrightarrow 0$$

by Lemma 3, and the exact sequence:

$$\cdots \to H^q(\tau^{r+1}) \to H^q(\tau^r) \to H^q(\tau^{p-1}) \to H^{q+1}(\tau^{r+1}) \to \cdots$$

We obtain

$$\chi(H^*(\tau^r)) = \chi(H^*(\tau^{r+1})) + \chi(H^*(\tau^{p-1})).$$

Hence

$$\begin{aligned}\chi(H^*(\tau)) &= \chi(H^*(\tau^2)) + \chi(H^*(\tau^{p-1}))\\ &= \chi(H^*(\tau^3)) + 2\chi(H^*(\tau^{p-1}))\\ &= \cdots = (p-1)\chi(H^*(\tau^{p-1})).\end{aligned}$$

Again by Theorem 1 and Lemma 3, we have

$$\begin{aligned} \chi(X) &= \chi(H^*(\tau)) + \chi(H^*(\tau^{p-1})) + \chi(F) \\ &= p\chi(H^*(\tau^{p-1})) + \chi(F) \\ &= p\chi(X/Zp-F) + \chi(F). \end{aligned}$$ QED.

From this theorem, the first theorem of E. E. Floyd, quoted previously, follows immediately. For the second theorem, one must depend on an earlier result of Floyd [2,5] to the effect that if G is a finite group acting on a space X with $\dim X < \infty$ and $H^*(X;Z)$ finitely generated then $H^*(X/G;Z)$ is also finitely generated. Then by the universal coefficient theorem, $H^*(X,Zp)$ is also finitely generated and $\dim_{Z_p} X < \infty$. Use our theorem to obtain the second theorem of Floyd, when $G = Zp$. If G is a solvable group, one can use the induction on the order of G.

2. In [6], we generalize the definition of covering spaces. We shall further generalize it to covering space with singularities, or singular covering space. As in Section 1, we only restrict spaces to be locally compact and Hansdorff. We do not require spaces to be locally connected as was the case of the conventional theory of covering spaces [3].

<u>Definition.</u> A map $f;\tilde{X} \to X$ is called a covering map and $(\tilde{X}, f)$ a covering space of X if and only if $f(\tilde{X}) = X$, and each point x of X has an open nbd U in X such that $f^{-1}(U) = \bigcup \tilde{U}_i$ a disjoint union of a collection of open sets $\tilde{U}_i$ in $\tilde{X}$ with each $\tilde{U}_i$ mapped homeomorphically onto U under f.

We refer the reader to [6] for terminologies used henceforth. If each point has the finite inverse set under the covering map then the map is called a finite covering map. It may be seen easily that if the total space $\tilde{X}$ of a finite covering space $(\tilde{X},f)$ of a space X is connected, then each point inverse set has the same number of points. Let n be the number. Then we say the covering space has n leaves or n decks.

Definition. A map $f;\tilde{X} \to X$ is called a covering map with singularities and $(\tilde{X},f)$ a singular covering space of X if and only if there exists a closed subset F of X such that $f|_{X-f^{-1}(F)}$ is a covering map (as defined above) and $f|_{f^{-1}(F)}$ is a homeomorphism of $f^{-1}(F)$ onto F.

This is a covering space version of a singular filtration.

If X is a space with $H^*_c(X, K)$ (the Alexander-Spanier cohomology of X with compact supports and with the coefficients in a field K) finitely generated then the Euler characteristic $\chi_c(X, K)$ is defined by $\chi_c(X,K) = \sum(-1)^q \dim H^q_c(X,K)$. Now we are ready to state our main

Theorem 3. Suppose that $(\tilde{X},f)$ is a covering space of a locally compact Hansdorff space X with the singular set F, and with n leaves on X-F. If $H^*_c(\tilde{X},K)$, $H^*_c(X,K)$ and $H^*_c(F,K)$ are finitely generated, then

$$\chi_c(\tilde{X},K) = n\chi(X-F,K) + \chi_c(F,K)$$

where K is any coefficient field.

Proof. We shall use the easily proven fact that if $C^* = \sum C^q$ is a cochain complex with coefficients in K such that $\dim C^* < \infty$, then $\chi(C^*) = \chi(H^*(C^*))$.

If U is an open subset of $\tilde{X}-f^{-1}(F)$, evenly covered by the covering space $(\tilde{X} - f^{-1}(F), f_1)$ where $f_1 = f|_{\tilde{X}-f^{-1}(F)}$ then we shall denote the family of all even portions by $\{\tilde{U}_1, \ldots \tilde{U}_n\}$.

First suppose that $\tilde{X}$ is compact, and that F consists only of one point.

We can find a finite open covering β which refines a given open covering β' of X, satisfying

(1) there exists exactly one element $U^o \in \beta$ containing F,

(2) each element U of β other than U^o is evenly covered by $(X-f^{-1}(F),f_1)$.

Now take a star refinement α' of $\beta-\{U^o\}$ which is an open covering of $X - \bigcup U$ where U ranges over $\beta-\{U^o\}$. Let $\alpha = \alpha' \bigcup \{U^o\}$. Denote by $\tilde{\alpha}$ the open covering $f^{-1}(\alpha)$ of $\tilde{X}$. Take the nerves $N(\tilde{\alpha})$ and $N(\alpha)$ of $\tilde{\alpha}$ and α, respectively, and define f_α by $f_\alpha(\tilde{U}_i) = U$ for each $U \in \alpha$ and i, and $f_\alpha(\tilde{U}^o) = U^o$ where $\tilde{U}^o = f^{-1}(U^o)$. Then for each U distinct from U^o, the 0-simplex $\sigma^o = [U]$ has exactly n 0-simplices in $f^{-1}(\sigma^o)$, and $[U^o]$ has exactly one 0-simplex in $f^{-1}[U^o]$. For $q > 0$, let $V^o \wedge \ldots \wedge V^q \neq \phi$ and $V^o, \ldots, V^q \in \alpha$. Then there exist exactly n(q+1)-tuples $(\tilde{V}^o_i, \ldots, \tilde{V}^p_j)$ with $\tilde{V}^o_i \wedge \ldots \wedge \tilde{V}^q_j \neq \phi$, implying $f_\alpha^{-1}\ [V^o, \ldots, V^q]$ consists of exactly n q-simplices. Hence we conclude that if $\dim C^q(N(\alpha),K) = p_q$ for each q then

$$\dim C^o(N(\tilde{\alpha}),K) = n(p_o-1) + 1$$

$$\dim C^q(N(\tilde{\alpha}),K) = np_q \text{ for all } q > 0.$$

Since $H^*_c(\tilde{X};K)$ and $H^*_c(X;K)$ are finitely generated by assumption, we could take β' small enough to have isomorphisms in the rows of the following commutative diagram,

$$\begin{array}{ccc} H^*(N(\alpha);K) & \rightarrow & H^*(X;K) \\ \downarrow & & \downarrow \\ H^*(N(\tilde{\alpha});K) & \rightarrow & H^*(\tilde{X};K) \end{array}$$

It then follow that

$$\chi(\tilde{X};K) = n\chi(X-F;K) + \chi(F;K).$$

This completes the proof for X compact and F consisting of a single point.

Now delete the conditions on X and F, but add the condition that F be empty. Then $(\tilde{X},f)$ is a covering space (without singularities) with n leaves. Take the point-compactifications $\tilde{X}\cup\infty$ and $X\cup\infty'$ of $\tilde{X}$ and X, respectively, and extend f to a map; $\tilde{X}\cup\infty \to X\cup\infty'$ by defining $f(\infty) = \infty'$. Then $(\tilde{X}\cup\infty, f)$ is a compact covering space of $X\cup\infty'$ with the singular set $F = \{\infty'\}$. By the proceeding consequence, we may conclude

$$\chi(X\cup\infty;K) = n\chi(X;K) + \chi(F;K).$$

Then obviously $\chi(\tilde{X}) = n\chi(X)$.

Finally, consider the general case. Then the exactness of the cohomology sequence for the pair $(\tilde{X},F)$,

$$\ldots \to H_c^q(\tilde{X}-F;K) \to H_c^q(\tilde{X};K) \to H_c^q(F;K) \to H_c^{q+1}(\tilde{X}-F;K) \to \ldots$$

shows that

$$\chi(\tilde{X};K) = \chi(\tilde{X}-F;K) + \chi(F).$$

Since $(\tilde{X}-F,f_1)$ is a covering space of X-F with n leaves and without singular point, we have $\chi(\tilde{X}-F;K) = n\chi(X-F;K)$. This produces the formula stated in the theorem.

As a corollary, we obtain

<u>Theorem 4</u>. Suppose that (G,X) is a finite transformation group of a (locally compact and Hansdorff) space X such that any singular point of (G,X) is a fixed

point. If $H_c^*(X;Z)$ is finitely generated and $\dim_Z X < \infty$ then $H_c^*(X;K)$, $H_c^*(X/G;K)$ and $H_c^*(F;K)$ are finitely generated for any field K of characteristic dividing the order of G, and we have

$$\chi_c(X;K) = \chi_c(F;K) + \mathrm{Ord}(G) \cdot \chi_c(X/G;K)$$

Proof. We need only to show that $H_c^*(X/G;K)$ and $H_c^*(F;K)$ are finitely generated, since then the formula follows immediately from Theorem 3. By a theorem of Floyd [5], $H_c^*(X/G;Z)$; hence $H_c^*(X/G;K)$ by the universal coefficient theorem is finitely generated. Let p be the characteristic of K. Then G contains a subgroup Zp. Since any singular point is a fixed point, F(Zp,X) = F(G,X) = F. By Theorem 2, $H_c^*(F;K)$ is finitely generated.

Remark. If in Theorem 4 there is no singular point, that is, if G acts freely on X, then we can lift the condition imposed on the characteristic of K. Hence Theorem 4 generalizes both theorems of Floyd quoted previously.

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