The Continuous Review (s,S) Policy for Production/Inventory Systems with Poisson Demands and Arbitrary Processing Times

Hyo-Seong Lee
Mandyam M. Srinivasan

Department of Industrial and Operations Engineering
The University of Michigan
Ann Arbor, MI 48109-2117

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Abstract

A production / inventory system is considered, in which a single production facility is engaged in producing items that are held in inventory. The inventory of items is thus replenished one item at a time, and the processing time required to produce an item is assumed to follow an arbitrary distribution. The demand for this item occurs according to a Poisson process.

An (s,S) policy is considered in which the production stops at the instant that the inventory level is raised to S and production begins again at the instant that the inventory level drops to s. Under a cost structure that includes a set-up cost, a linear holding cost, and a linear backorder cost, an expression for the expected cost per unit time is obtained for given control values. Using convexity and unimodality properties of the cost functions, an extremely simple and efficient procedure to find the optimal (s,S) policy is presented

1. Introduction

In this paper, we consider the optimal control strategy for a production/inventory system wherein a single production facility produces items of a given type. The demand for this item is assumed to arrive according to a Poisson process with rate λ. The processing time for producing (replenishing) an item is assumed to be an independent, identically distributed, random variable U which follows an arbitrary distribution. For stability of the system, we assume that $\lambda E(U)<1$ and $E(U^2)<\infty$. If an item is demanded, it is supplied directly from the inventory, if it is available. If the item is not available, it is backordered. We assume that i) inventory holding costs are incurred linearly over time with respect to the inventory level ii) backorder costs are incurred linearly over time with respect to the backorder level and iii) a set-up cost is incurred each time the production facility is turned on. The objective is to find a continuous review production/inventory policy to minimize the expected cost per unit time.

The operating policy considered in this paper is an (s,S) policy. Such policies are known to be effective in a variety of situations (Bell 1971, Sobel 1969, Veinott 1966, Veinott 1967). The characteristics of the policy considered in this paper are now described.

As soon as the inventory level reaches a prespecified value S, the production facility is turned off and a non-production period begins. During the non-production period, the inventory level is continuously monitored to determine whether the inventory level has reached a prespecified value s or not. At the instant the inventory level drops to s, the non-production period ends and production begins immediately. During the production period, the inventory is replenished on an item-by-item basis while the demand continues to be made on these items. When the inventory level is raised to S, the production period ends and the next non-production period begins, initiating another cycle in the (s,S) policy system (refer to figure 1).

There is considerable work on (s,S) inventory policies. Such policies are studied by Beckmann (1961), Johnson (1966), Veinott and Wagner (1965), Veinott (1967), Archibald and Silver (1978), and Sahin (1979), to name but a few. However, most of the (s,S) policies present in the literature assume that any amount of inventory can be replenished all at once. Our model is different in the sense that the inventory can be replenished only on an item-by-item basis.

Since the replenishment in our model is made on an item-by-item basis, it is easy to note that our model has an analogy with queueing systems. In fact, the work on control of queueing systems, for instance, the papers by Heyman (1968), Yadin and Naor (1963), Bell (1971), Sobel (1969), and Lee and Srinivasan (1989), have a close relationship with production/inventory systems. Using this inventory-queueing analogy, some studies have been done on (s,S) policies for the production/inventory systems in which the inventory is replenished item-by-item.

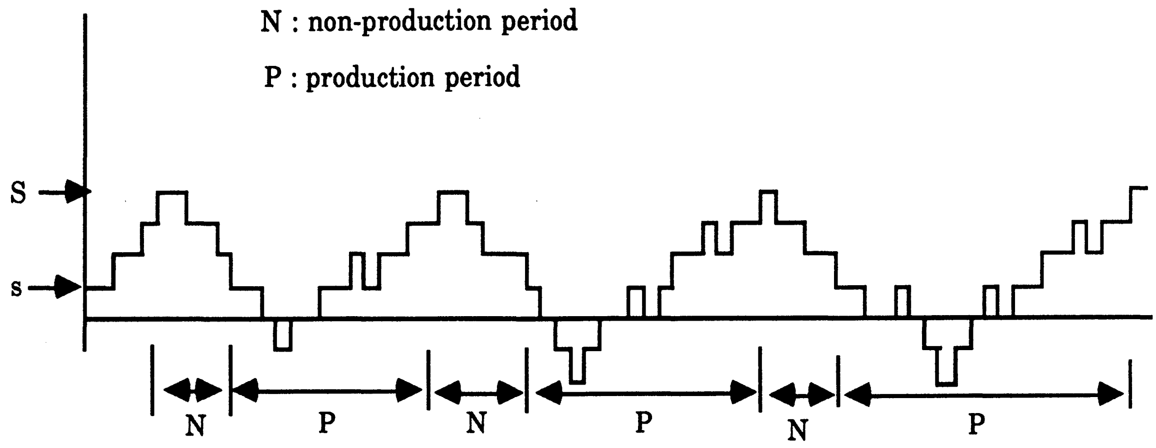

Figure 1. The Continuous Review (s,S) Policy with Poisson Demands and Arbitrary Processing Times

Heyman (1968) considers the operating policy for an M/G/1 queueing system which is a special case of the (s,S) policy where the upper control limit S is set to zero. Gavish and Graves (1980) consider an (s,S) continuous review production/inventory system with unit Poisson demand arrivals and deterministic processing times. They develop a procedure to find the steady state probability of each inventory level using an underlying M/D/1 queueing system. To obtain the optimal control values, they use a search procedure in which some properties for cost functions are effectively exploited. In a subsequent paper, Gavish and Graves (1981) extend the analysis to consider general processing times. Although their search procedure is efficient, they do not prove that the policy found by their algorithm is the optimal. Tijms (1980) considers a system with arbitrary processing times and finds the optimal control values by using a denumerable state Semi-Markov decision process. In his model, a start-up time is allowed which can model the time taken to turn on the production facility. Altiok (1986) considers a system where the demand follows a compound Poisson demand process. The service times in his model are assumed to be phase-type. He uses an underlying continuous-time Markov chain to obtain the steady state probability of each inventory level for a given policy. However, in this paper no special properties for the cost function are proved, that would assist in the search for the optimum.

The (s,S) policy in this paper uses a fundamentally different approach from those used in the past. In addition, this paper also develops proofs of convexity and unimodality of certain functions, which leads to a very efficient algorithm to find the *optimal* policy. This paper considers, in fact, the same problem that was studied by Tijms except that a start-up time is not considered in our model. However, as pointed out by Gavish and Graves (1980), the computational time using a semi-Markov decision process for solving production/inventory systems is much higher than that using an intelligent search technique. In the following sections, first an expression for the expected cost per unit time for given control values will be derived and following that an efficient procedure to find the optimal control values, s and S, will be presented.

2. The Analysis

2.1. Notation

For analytical convenience, we set $r = S\text{-}s$, and throughout the paper we will use (r,S) instead of (s,S) to describe a policy. Thus, the (r,S) policy represents the policy with $S\text{-}r$ as a lower control value and S as an upper control value. Let

U	=	the processing time to produce a unit item, $\lambda E(U) < 1$,
$U(.),U^*(.)$	=	the c.d.f. and the Laplace Stieltjes Transform of U, respectively,
Q	=	the number of demands which arrive during a processing time U,
q_j	=	$\Pr\{Q=j\}$, $\quad j=0,1,2,\ldots,$
K	=	set-up cost,
c_h	=	holding cost / item / unit time,
c_b	=	back order cost / item / unit time,
$C_N(r,S)$	=	expected cost during a non-production period with control values r and S,
$C_P(r,S)$	=	expected cost during a production period with control values r and S,
$C(r,S)$	=	$C_N(r,S) + C_P(r,S)$ = sum of expected holding and backorder costs during a cycle with control values r and S,
$L(r,S)$	=	expected length of a cycle with control values r and S,
$TC(r,S)$	=	expected cost per unit time with control values r and S.

2.2. General Approach

Our first objective in this paper is to obtain an expression for $TC(r,S)$, the expected cost per unit time for given control values r and S. To obtain this, note that the epoch marking the start of a cycle forms a regeneration point. It follows that we have a renewal reward process, and thus from the renewal reward theorem (see, for example, Ross 1970), the expected cost per unit time, when r and S are used as control values, is obtained by

$$TC(r,S) = \frac{C(r,S) + K}{L(r,S)}. \tag{2.1}$$

Since the term C(r,S) is the sum of $C_N(r,S)$ and $C_P(r,S)$, to obtain TC(r,S) we need to find the terms $C_N(r,S)$ and $C_P(r,S)$ as well as the term L(r,S). We now show how the terms $C_N(r,S)$, $C_P(r,S)$ and L(r,S) are determined.

2.3. Computing the term $C_N(r,S)$

The expected cost during a non-production period is easily determined. Let $g_{k,k-1}$ denote the expected cost incurred from the epoch at which the inventory level becomes k to the epoch when the inventory level drops to k-1 during a non-production period. Then $g_{k,k-1}$ is given by

$$g_{k,k-1} = \frac{c_h}{\lambda} k, \qquad \text{if } k \geq 0, \tag{2.2a}$$

$$= -\frac{c_b}{\lambda} k, \qquad \text{if } k < 0. \tag{2.2b}$$

The expected cost during a non-production period when r and S are used as control values, is then expressed by

$$C_N(r,S) = \sum_{k=S-r+1}^{S} g_{k,k-1}\,. \tag{2.3}$$

2.4. Computing the term $C_P(r,S)$

During the production period, the production completion epochs are the times at which the inventory is replenished. To compute $C_P(r,S)$, we therefore restrict our attention only to these epochs. Let $f_{i,j}$ denote the expected cost from the epoch at which the inventory level reaches i, to the epoch at which the inventory level is raised to j ($j \geq i$) for the first time with $f_{i,i}=0$ for any i. Then the expected total cost incurred during the production period, $C_P(r,S)$, is just $f_{S-r,S}$ which, in turn, is expressed as (note that S-r = s)

$$C_P(r,S) = f_{S-r,S} = \sum_{k=S-r}^{S-1} f_{k,k+1}. \tag{2.4}$$

From equation (2.4), we see that the term $C_P(r,S)$ can be determined if we can calculate each value of $f_{k,k+1}$. Let E_k denote the expected cost incurred *during a processing time* that is initiated with k items in inventory. Then $f_{k,k+1}$ is expressed as

$$f_{k,k+1} = E_k + \sum_{j=1}^{\infty} q_j\, f_{k+1-j,k+1}. \tag{2.5}$$

In equation (2.5), the term E_k is the expected cost incurred during the time required to produce the first item following the initiation of production. During this time, j items are demanded with probability q_j and this takes the inventory level to k+1-j at the end of the production period. The second term in equation (2.5) is thus the expected cost incurred from the end of that processing time until the time at which the inventory level is first raised to k+1. The q_i values can, in general, be computed from the following expression:

$$q_i = \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t}\, dU(t). \tag{2.6}$$

Remark: Note that the computation of the q_i values is considerably simplified in many cases. For example, when the production times are of phase-type, the q_i's can be obtained in closed form (Neuts 1981, page 59). Also, these computations are trivial for the case of deterministic processing times.

To obtain E_k, we let h_k denote the expected time that the k^{th} item is held in inventory during a processing time. (It is implicit that this processing time is initiated with at least k items in inventory.) If the number of items demanded during a processing time is less than k, then the k^{th} item will be held during this entire processing time. The term E_k is obtained from Lemma 2.1. A proof of Lemma 2.1 is given in Appendix A.

__Lemma 2.1__

The term E_k is expressed recursively as

$$E_{k+1} = E_k + h_{k+1}(c_h + c_b) - E(U)c_b, \tag{2.7}$$

where

$$h_{k+1} = h_k + \frac{1}{\lambda}(1 - \sum_{j=0}^{k} q_j), \quad \text{with } h_k = 0 \quad \text{for } k \leq 0. \tag{2.8}$$

As an initial value for equation (2.7), we use $E_0 = \frac{\lambda E(U^2)}{2} c_b$, which can be obtained from equation (A.13) in appendix A by setting k = 0.

2.4.1. Computing the term $f_{k,k+1}$

We now obtain a recursive expression for the term $f_{k,k+1}$. First, we develop an expression for the term $f_{-1,0}$, which denotes the expected total cost incurred from the epoch at which the inventory level reaches -1 units till the time that the inventory level is first raised to 0. Note that this is equivalent to the expected total cost incurred during the busy period in the M/G/1 queueing system where the waiting cost per customer is c_b. Thus, if we know the expected total waiting time for customers during a busy period in an M/G/1 queueing system then we can easily obtain $f_{-1,0}$. Lemma 2.2 gives this result. Since this is a well known result, we omit the proof (note that the expected number of customers served in a busy period is just $1/(1-\rho)$).

Lemma 2.2

Consider an M/G/1 queueing system with arrival rate λ and service time U. Then, the expected total waiting time, W_T, of all the customers during the busy period is given by

$$W_T = \frac{1}{(1-\rho)}\left[\frac{\lambda E(U^2)}{2(1-\rho)} + E(U)\right]. \qquad \blacksquare$$

From Lemma 2.2, $f_{-1,0}$ can be obtained simply as

$$f_{-1,0} = W_T c_b = \frac{1}{(1-\rho)}\left[\frac{\lambda E(U^2)}{2(1-\rho)} + E(U)\right] c_b. \tag{2.9}$$

Define

$$\Delta f_k = f_{k,k+1} - f_{k-1,k}, \tag{2.10a}$$

and

$$\Delta E_k = E_k - E_{k-1} = h_k(c_h + c_b) - E(U)c_b. \tag{2.10b}$$

Note, from equations (2.8) and (2.10b), that

$$\Delta E_k - \Delta E_{k-1} = \frac{1}{\lambda}\left(1 - \sum_{j=0}^{k-1} q_j\right)(c_h + c_b), \qquad k > 0, \tag{2.10c}$$

$$= 0, \qquad k \le 0. \tag{2.10d}$$

The recursive expression for $f_{k,k+1}$ is obtained from Lemma 2.3. A proof of Lemma 2.3 is given in Appendix B.

Lemma 2.3

The term $f_{k,k+1}$ is obtained from the recursive expression

$$f_{k,k+1} = f_{k-1,k} + \frac{1}{q_0}\left\{\Delta f_{k-1} + \Delta E_k - \Delta E_{k-1} - \sum_{j=1}^{k} q_j \Delta f_{k-j} + \left(1 - \sum_{j=0}^{k} q_j\right)\frac{E(U)}{(1-\rho)} c_b\right\}, \qquad k > 0, \tag{2.11a}$$

$$= f_{k-1,k} - \frac{E(U)}{(1-\rho)} c_b, \qquad k \le 0. \tag{2.11b}$$

$\blacksquare$

Let

$$\tau_k \quad = \quad g_{k+1,k} + f_{k,k+1}. \tag{2.12}$$

Using equations (2.3), (2.4) and (2.12), C(r,S) is expressed as

$$C(r,S) \quad = \quad C_N(r,S) + C_P(r,S) \quad = \quad \sum_{k=S-r}^{S-1} \tau_k. \tag{2.13}$$

2.5. Computing the terms L(r,S) and TC(r,S)

The expected cycle time in the (r,S) system can be obtained by using the relationship between production/inventory systems and queueing systems. To obtain L(r,S), we make an important observation that the length of a production period in our (r,S) system is the convolution of r busy periods in an M/G/1 queueing system. Therefore, the expected length of a production period is directly obtained from the busy period analysis as $\frac{r\,E(U)}{(1-\rho)}$. Since the expected length of a non-production period when r and S are used as control values is $\frac{r}{\lambda}$, the expected length of a cycle is given by

$$L(r,S) \quad = \quad \frac{r\,E(U)}{(1-\rho)} + \frac{r}{\lambda} = \frac{r}{(1-\rho)\lambda}. \tag{2.14}$$

Hence, from equations (2.1), (2.13), and (2.14),

$$TC(r,S) \quad = \quad (1-\rho)\,\lambda \frac{\sum_{k=S-r}^{S-1} \tau_k + K}{r}. \tag{2.15}$$

3. The Optimal Control Values

In order to find the optimal control values (r^*,S^*), a two-dimensional search over the integer parameter space must be made. Usually, the search for the optimal point in two dimensional space is not easy. However, if we exploit some properties of the cost functions and the recursive nature of $g_{k+1,k}$ and $f_{k,k+1}$, we can find an extremely efficient search procedure. Let us denote the optimal S value for a given r by $S^*(r)$ and the optimal r value for a given S by $r^*(S)$. We now demonstrate some properties that are possessed by this system. These properties are used in devising the search procedure.

Theorem 3.1:

τ_k is convex with respect to k.

Proof:

Since $\tau_k = g_{k+1,k} + f_{k,k+1}$, τ_k is convex if we can show that $g_{k+1,k}$ and $f_{k,k+1}$ are both convex. Note from equation (2.2), that the function $g_{k+1,k} - g_{k,k-1}$ is a non-decreasing function with respect to k. This proves that $g_{k+1,k}$ is convex. To show convexity of $f_{k,k+1}$, we need to prove that $\Delta f_k - \Delta f_{k-1} \geq 0$ for all k. We will prove this by induction on k.

From Lemma 2.3, $\Delta f_k = -\dfrac{E(U)}{(1-\rho)} c_b$, $k \leq 0$, and so $\Delta f_k - \Delta f_{k-1} = 0$ for all $k \leq 0$.

Suppose $\Delta f_k - \Delta f_{k-1} \geq 0$ holds for $k \leq n$, for some $n > 0$. We now prove that $\Delta f_k - \Delta f_{k-1} \geq 0$ for $k=n+1$ using the induction hypothesis. From equation (2.5), $\Delta f_n = \Delta E_n + \sum_{j=1}^{\infty} q_j (f_{n+1-j,n+1} - f_{n-j,n})$ and so, after some algebra, $\Delta f_{n+1} - \Delta f_n$ is expressed as (also refer equation (B.2))

$$\Delta f_{n+1} - \Delta f_n \quad = \quad \Delta E_{n+1} - \Delta E_n + \sum_{j=1}^{\infty} q_j(\Delta f_{n+1} - \Delta f_{n+1-j}). \tag{3.1}$$

Equation (3.1) can be rewritten as

$$\Delta f_{n+1} - \Delta f_n \quad = \quad \frac{1}{q_0}\{\Delta E_{n+1} - \Delta E_n + \sum_{j=1}^{\infty} q_j(\Delta f_n - \Delta f_{n+1-j})\}. \tag{3.2}$$

From equation (2.10c), $\Delta E_{n+1} - \Delta E_n = \dfrac{1}{\lambda}(1 - \sum_{j=0}^{n} q_j)(c_h + c_b) \geq 0$ for all n. Also by the induction hypothesis, $\Delta f_n - \Delta f_{n+1-j} \geq 0$ for all $j \geq 1$. Since $q_j \geq 0$ for $j \geq 0$, we must have $\Delta f_{n+1} - \Delta f_n \geq 0$. So, by the principle of mathematical induction, $\Delta f_k - \Delta f_{k-1} \geq 0$ for all k and hence $f_{k,k+1}$ is convex with respect to k. ■

Figure 2 gives one possible realization for the function τ_k. Note that $C(r,S^*(r))$ is nothing but a minimum value among all possible sums of r adjacent τ_k's. In figure 2, for example, $C(3,S^*(3))$ is just the sum of τ_1,τ_2 and τ_3. Furthermore, since τ_k is convex, $C(4,S^*(4))$ in figure 2 can be obtained directly from $C(3,S^*(3))$ as $C(4,S^*(4))=C(3,S^*(3))+\min\{\tau_0,\tau_4\} = C(3,S^*(3))+\tau_4$. Corollary 3.2 generalizes this result.

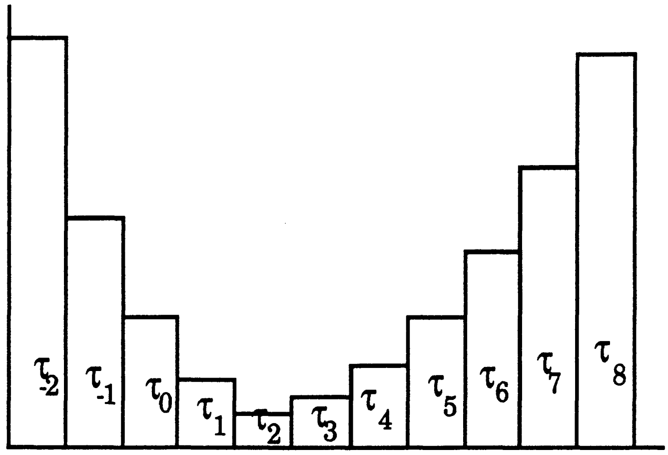

Figure 2. One realization for the function τ_k

Corollary 3.2:

$$C(r+1,S^*(r+1)) = C(r,S^*(r)) + \min\{\tau_{S^*(r)}, \tau_{S^*(r)-r-1}\}. \tag{3.3}$$

If $\tau_{S^*(r)} \leq \tau_{S^*(r)-r-1}$, then $S^*(r+1) = S^*(r) + 1$.
If $\tau_{S^*(r)} > \tau_{S^*(r)-r-1}$, then $S^*(r+1) = S^*(r)$.

Proof:

Corollary 3.2 is a direct consequence of Theorem 3.1.

■

Theorem 3.3 presents an important characteristic of the cost function which is very useful in the search procedure, and in *guaranteeing optimality*.

Theorem 3.3:

$TC(r,S^*(r))$ is unimodal with respect to r.

Proof:

In order to prove the unimodality of $TC(r,S^*(r))$, it is sufficient to show the following:

$$\begin{aligned} &\text{If} \quad TC(n,S^*(n)) < TC(n+1,S^*(n+1)), \\ &\text{then,} \quad TC(n+1,S^*(n+1)) < TC(n+2,S^*(n+2)). \end{aligned} \tag{3.4}$$

The term $TC(n,S^*(n))$ can be expressed as

$$TC(n,S^*(n)) \quad = \quad \frac{(K + \beta(n))\,(1-\rho)\lambda}{n}, \text{ where } \beta(n) = \min_i \{ \sum_{k=i}^{i+n-1} \tau_k \}.$$

Using this in (3.4), observe that in order to show the unimodality of $TC(r,S^*(r))$, we only need to show that if $\frac{(K + \beta(n))}{n} < \frac{(K + \beta(n+1))}{n+1}$, then $\frac{(K + \beta(n+1))}{n+1} < \frac{(K + \beta(n+2))}{n+2}$.

Let $\delta_k = \beta(k) - \beta(k-1)$ and $c = K+\beta(n)$. To show unimodality, we then need to show that if

$$\frac{c}{n} < \frac{c + \delta_{n+1}}{n+1}, \qquad \text{then} \qquad \frac{c + \delta_{n+1}}{n+1} < \frac{c + \delta_{n+1} + \delta_{n+2}}{n+2}. \tag{3.5}$$

But from the fact that $\delta_{n+1} < \delta_{n+2}$, relationship (3.5) is obvious.

■

We now demonstrate two properties that the control limits possess, which assist in restricting the search. Theorem 3.4 first indicates that the optimal production quantity per cycle is bounded from below by the threshold value obtained from Heyman's N-policy formula (1968). A proof of Theorem 3.4 is given in Appendix C.

Theorem 3.4:

If we denote the value of r in the optimal policy by r^*, then $r^* \geq r^*(0)$, where $r^*(0)$ is one of the neighboring integers of $\sqrt{\frac{2\lambda(1-\rho)K}{c_b}}$. ■

Theorem 3.5:

$S^*(r) \geq 0$ for any $r \geq 1$.

Proof Consider a policy (r,S) where S<0. Clearly the policy (r,0) always dominates this policy since only non-positive inventory levels exist for these two policies and the expected shortage level of policy (r,0) is always less than that of policy (r,S) by -S. Thus, $S^*(r)$ cannot be negative for any r. ■

We use these properties as follows. First note, from Corollary 3.2, that once we find an optimal control value, $S^*(r)$, for some r, say r=k, then we can find an optimal control value for r=k+1 very quickly. In our algorithm, we make use of this fact and start with r=1, that is, we find

$C(1,S^*(1))$ first. To find $C(1,S^*(1))$, noting that τ_k is convex in k, we just search for the minimum value of τ_k. Since $\tau_k \geq \tau_{-1}$ for $k < -1$, the search starts from $k = -1$, and hence we compute τ_k from $k=-1$ up to the point at which the value of τ_k is first increased. Thus, if $\min_k\{\tau_k\}=\tau_q$, the value of $S^*(1)$ can be obtained as q+1. The cost function $TC(1,S^*(1))$ may then be obtained from $C(1,S^*(1))$ by using equation (2.1). (However, notice that by Theorem 3.4, $TC(r,S^*(r))$ need not be calculated for $r < \lfloor\sqrt{\frac{2\lambda(1-\rho)K}{c_b}}\rfloor$ where $\lfloor p \rfloor$ is the largest integer that does not exceed p.) Once $C(1,S^*(1))$ is found, we can find $C(r,S^*(r))$ and, therefore, $TC(r,S^*(r))$, sequentially in the order $r=2,3,4,\cdots$. Note that the values of τ_k that are evaluated in order to obtain $C(r,S^*(r))$ for $r=1$, will also be used to obtain $C(r,S^*(r))$ for $r > 1$.

Since $TC(r,S^*(r))$ is unimodal with respect to r, we can now develop an extremely simple search procedure: find $C(r,S^*(r))$ for $r=1,\ldots,\lfloor\sqrt{\frac{2\lambda(1-\rho)K}{c_b}}\rfloor$ as indicated above. Following this, we increase r by 1 and compute $C(r,S^*(r))$ and $TC(r,S^*(r))$ for this new r, repeating this process until $TC(r,S^*(r))$ is first increased. Then, by Theorem 3.3, the local minimum point obtained in the previous step is a global minimum and the optimal control values, r^* and S^*, are obtained. The algorithm to find the optimal control values is described below:

Algorithm to find $TC(r^*,S^*)$

1. Determine $n^*=\lfloor\sqrt{\frac{2\lambda(1-\rho)K}{c_b}}\rfloor$.

2. Calculate $C(r,S^*(r))$ for $r = 1,\ldots,n^*$, and obtain $TC(r,S^*(r))$ for $r = n^*$.

3. Set $r = r+1$ and calculate $C(r,S^*(r))$ and $TC(r,S^*(r))$.
   If $TC(r,S^*(r)) > TC(r-1,S^*(r-1))$, then return the optimal policy as $(r-1,S^*(r-1))$.
   Otherwise, repeat step 3.

This algorithm is simple and efficient: only one new τ_k needs to be calculated each time that r is incremented. Moreover, due to the recursive nature of $f_{k,k+1}$, τ_k is computed very quickly from τ_{k-1} for $k > 0$, or τ_{k+1} for $k < 0$ (either of which would have been obtained at the previous step). In this algorithm, most of the computational effort to obtain the optimal control values is spent on calculating values of τ_k for $k \geq 0$ (see remark in Section 2.4). Suppose we know the optimal control values (r^*,S^*) beforehand and that we just need to calculate $TC(r^*,S^*)$. In this case, we must calculate τ_k for $S^*-r^*+1 \leq k \leq S^*$. On the other hand, suppose we need to find both the optimal policy and its cost using this algorithm. In this case, it may be observed that at most only two additional values of τ_k need to be calculated as compared to the case when the optimal control values are known. This fact demonstrates the efficiency of our algorithm: very little calculation is wasted on computing the points other than those required to obtain the optimal control values. In fact, *this algorithm finds the optimal solution in almost one shot.*

4. Numerical Examples

We now present some numerical examples to illustrate this technique. In the first example, the processing time is assumed to follow a special distribution, which is encountered frequently in manufacturing situations. The processing time has a deterministic value t_0 if the production facility does not fail during the processing time. However, if the production facility fails during the processing time, which is assumed to occur with probability p, the processing time becomes t_0 plus a repair time R which follows an exponential distribution with parameter μ. Other parameter values are given as follows:

$$\lambda = 0.15, \quad K = 500, \quad c_h = 2, \quad c_b = 10.$$

$$\begin{aligned} U &= t_0 \quad \text{with probability } 1-p, \\ &= t_0 + R \quad \text{with probability } p, \end{aligned}$$

where $p=0.02$, $t_0=5$ and $\mu=0.05$.

In the second example, the processing time is assumed to follow a uniform distribution on [2,4]. Other parameter values are given as

$\lambda = 0.1$, $K=3000$, $c_h = 2$, $c_b = 20$.

The results of the policy comparisons for these examples are presented in tables 1 and 2. In these tables, the values of r, $s^*(r)$, $S^*(r)$ and $TC(r,S^*(r))$ are shown for each value of r. Although only two distributions for the processing time are demonstrated in the examples, many other distributions can be easily implemented if their Lapace-Stieltjes transform functions are well diffenrentiable.

5. Conclusions

We have considered the (s,S) control policy for production/inventory systems. We obtained an expression for the expected cost per unit time for given values s and S and using a convexity property of the function τ_k, and the unimodality property of the cost function $TC(r,S^*(r))$, we have developed an extremely efficient procedure to find the stationary optimal (s,S) policy.

Table 1.

Result of example 1

r=S-s	$s^*(r)$	$S^*(r)$	$TC(r,S^*(r))$
1	5	6	29.8176
2	5	7	22.7503
3	4	7	20.4731
4	4	8	19.3938
5	3	8	18.8947
6	3	9	18.5638
7@	3	10	18.4672
8	2	10	18.5041
9	2	11	18.5643
10	2	12	18.7432

(@ indicates the optimal policy)

Table 2.

Result of example 2

r=S-s	$s^*(r)$	$S^*(r)$	$TC(r,S^*(r))$
10	-1	9	30.2455
11	-1	10	29.2474
12	-1	11	28.5824
13	-1	12	28.1735
14	-1	13	27.9658
15	-1	14	27.9192
16@	-2	14	27.8826
17	-2	15	27.9640
18	-2	16	28.1475
19	-2	17	28.4169
20	-2	18	28.7594

(@ indicates the optimal policy)

Appendix A

Proof of Lemma 2.1

Lemma 2.1

The term E_k is obtained recursively as

$$E_{k+1} = E_k + h_{k+1}(c_h + c_b) - E(U)c_b, \tag{A.1}$$

where

$$h_{k+1} = h_k + \frac{1}{\lambda}(1 - \sum_{j=0}^{k} q_j), \quad \text{with } h_k = 0 \text{ for } k \le 0. \tag{A.2}$$

Proof

Let U_i be the length of a processing time given that i demands arrived during that processing time and let $\bar{u}_i$ be $E[U_i]$. If we apply Bayes' formula, $\bar{u}_i$ is expressed as

$$\bar{u}_i = E[\,U_i\,] = \frac{1}{q_i}\int_0^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t}\, t\, dU(t). \tag{A.4}$$

From equations (A.4) and (2.6), after some algebraic manipulation we obtain:

$$\frac{q_i \bar{u}_i}{i+1} = \frac{q_{i+1}}{\lambda}. \tag{A.5}$$

During a processing time, demands continue to occur and these demands are met from the inventory on hand if any. Let I_i denote the item in inventory which is used to satisfy the i^{th} occurence of a demand during a processing time. The term h_i is the expected amount of time that I_i is held in inventory during a processing time. If the number of demands which arrived during the processing time is less than i, say j, the I_i will not be used and will be held during the entire processing time U_j. In that case, the expected holding time of I_i is $\bar{u}_j$. On the other hand, if the number of demands during the processing time is more than i, say j, then I_i will be used when the i^{th} demand arrives. Also note that given j arrivals during U, the joint distribution of these arrival epochs have the same distribution as the order statistics of j independent random variables uniformly distributed on $[0, U_j]$ (see, for example, Ross 1970). Hence the expected holding time of I_i when the number of demands, j, during a processing time is greater than i, is expressed as $\frac{i}{j+1}\bar{u}_j$. Consequently h_i is represented as

$$h_i = \sum_{j=0}^{i-1} q_j\, \bar{u}_j + \sum_{j=i}^{\infty} i\, \frac{q_j \bar{u}_j}{j+1}, \qquad i=1,2,\cdots. \tag{A.6}$$

$$= \frac{1}{\lambda}\{\, \sum_{j=1}^{i} j\, q_j + i\,(1 - \sum_{j=0}^{i} q_j)\ \}, \quad i=1,2,\cdots, \tag{A.7}$$

where the second equality follows from equation (A.5). From equation (A.7), we observe that

$$h_{k+1} \quad = \quad h_k + \frac{1}{\lambda}\Big(1 - \sum_{j=0}^{k} q_j\Big), \quad \text{for } k\geq 0, \text{ with } h_k=0 \quad \text{for } k\leq 0. \qquad \text{(A.8)}$$

Let the expected time from the instant the i^{th} demand arrives during a processing time onwards until the instant that the processing is completed be b_i. Let H_k denote the sum of the expected holding time during a processing time that is initiated with k items in inventory. Correspondingly, let B_k denote the sum of the expected backorder time during a processing time that is initiated with k items in inventory. Then we have

$$H_k \quad = \quad \sum_{i=1}^{k} h_i, \qquad \text{and} \qquad B_k \quad = \quad \sum_{i=k+1}^{\infty} b_i. \qquad \text{(A.9)}$$

Since h_k can be obtained from equation (A.8), H_k is easily obtained. To obtain B_k, note first that since the sum of b_i and h_i is an expected processing time, b_i is given by

$$b_i \quad = \quad E(U) - h_i, \qquad i=1,2,\cdots. \qquad \text{(A.10)}$$

Also note that for $i = 0$, both h_0 and b_0 are equal to 0. From the renewal theorem, the expected total time from the instant each demand arrives until the instant that processing is completed is

$$\sum_{i=0}^{\infty} b_i \quad = \quad \frac{\lambda E(U^2)}{2}. \qquad \text{(A.11)}$$

From equation (A.9), (A.10) and (A.11), B_k is expressed as

$$B_k \quad = \quad \sum_{i=0}^{\infty} b_i - \sum_{i=0}^{k} b_i \quad = \quad \frac{\lambda E(U^2)}{2} - \{kE(U) - H_k\}. \qquad \text{(A.12)}$$

Now note that the term E_k consists of two components: the expected holding cost, $c_h H_k$, and the expected backorder cost, $c_b B_k$, during a processing time. So, from equation (A.12),

$$E_k \quad = \quad c_h H_k + c_b B_k \quad = \quad H_k(c_h + c_b) + c_b\{\frac{\lambda E(U^2)}{2} - kE(U)\}. \qquad \text{(A.13)}$$

From equation (A.13), E_{k+1} is given by

$$E_{k+1} \quad = \quad E_k + h_{k+1}(c_h + c_b) - E(U)c_b. \qquad \text{(A.14)}$$

Since $h_{k+1} = 0$ for $k < 0$, from equation (A.15) we get $E_k = E_{k+1} + E(U)c_b$, $k<0$.

■

Appendix B

Proof of Lemma 2.3

Lemma 2.3

The term $f_{k,k+1}$ is obtained from the recursive expression

$$f_{k,k+1} = f_{k-1,k} + \frac{1}{q_0}\{\Delta f_{k-1} + \Delta E_k - \Delta E_{k-1} - \sum_{j=1}^{k} q_j \Delta f_{k-j} + (1 - \sum_{j=0}^{k} q_j) \frac{E(U)}{(1-\rho)} c_b\}, \quad k > 0,$$

$$= f_{k-1,k} - \frac{E(U)}{(1-\rho)} c_b, \quad k \le 0.$$

Proof

From equation (2.5),

$$\Delta f_k = \Delta E_k + \sum_{j=1}^{\infty} q_j (f_{k+1-j,k+1} - f_{k-j,k}). \tag{B.1}$$

From equation (B.1), after some algebra we get

$$\Delta f_k - \Delta f_{k-1} = \Delta E_k - \Delta E_{k-1} + \sum_{j=1}^{\infty} q_j (\Delta f_k - \Delta f_{k-j}). \tag{B.2}$$

The last term in equation (B.2) consists of infinite terms. However, as shown below, we can express $\Delta f_k - \Delta f_{k-1}$ without these infinite terms.

Let D_k denote the time period from the epoch when the inventory level reaches k to the epoch when the inventory level is raised to k+1 for the first time. Note that the length of D_k is equivalent to one busy period in an M/G/1 queueing system, hence, from the well known busy period analysis, the expected length of D_k is $\frac{E(U)}{(1-\rho)}$ where $\rho=\lambda E(U)$.

Comparing the inventory levels during D_k and D_{k+1}, we observe that the inventory level during the period D_k follows the same stochastic path as the inventory level during the period D_{k+1} if one item of inventory is added to the inventory level during D_k throughout this period. Consequently, if $k < 0$, then the inventory level during D_k has only one more shortage than the inventory level during D_{k+1} on the average. Thus, Δf_k for $k \le 0$ is simply

$$\Delta f_k = f_{k,k+1} - f_{k-1,k} = -\frac{E(U)}{(1-\rho)} c_b, \quad k \le 0. \tag{B.3}$$

From equation (B.2), collecting Δf_k terms,

$$q_0 \Delta f_k \;=\; \Delta f_{k-1} + \Delta E_k - \Delta E_{k-1} - \sum_{j=1}^{\infty} q_j \; \Delta f_{k-j}. \tag{B.4}$$

For $k > 0$, we can express equation (B.4) as

$$q_0 \Delta f_k \;=\; \Delta f_{k-1} + \Delta E_k - \Delta E_{k-1} - \sum_{j=1}^{k} q_j \; \Delta f_{k-j} - \sum_{j=k+1}^{\infty} q_j \; \Delta f_{k-j}.$$

Applying equation (B.3) to the last term on the right hand side of the above equation,

$$\Delta f_k \;=\; \frac{1}{q_0}\{\Delta f_{k-1} + \Delta E_k - \Delta E_{k-1} - \sum_{j=1}^{k} q_j \, \Delta f_{k-j} + (1 - \sum_{j=0}^{k} q_j)\, \frac{E(U)}{(1-\rho)} c_b\}, \quad k > 0. \tag{B.5}$$

where the term $\Delta E_k - \Delta E_{k-1}$ is given by equation (2.10c).

From equation (B.5), the term Δf_k for $k \geq 0$, can be computed recursively using the initial value $\Delta f_0 = -\frac{E(U)}{(1-\rho)} c_b$. Since $f_{-1,0}$ can be obtained explicitly from Lemma 2.2, we can now obtain all the $f_{k,k+1}$ terms. Noting that $\Delta f_k = f_{k,k+1} - f_{k-1,k}$, we get the desired result

■

Appendix C

Proof of Theorem 3.4

Theorem 3.4

If we denote the value of r in the optimal policy by r^*, then $r^* \geq r^*(0)$, where $r^*(0)$ is one of the neighboring integers of $\sqrt{\frac{2\lambda(1-\rho)K}{c_b}}$.

Proof:

We need to show that for $0 < n < r^*(0)$, $TC(r^*(0),S^*(r^*(0)) < TC(n,S^*(n))$. By definition, we have $TC(r^*(0),0) < TC(n,0)$. Set

i) $A = \sum_{k=-r^*(0)}^{-1} \tau_k$, ii) $\alpha(n) = \sum_{k=-n}^{-1} \tau_k$, iii) $B = \min_i\{ \sum_{k=i}^{i+r^*(0)-1} \tau_k \}$, and iv) $\beta(n) = \min_i \{ \sum_{k=i}^{i+n-1} \tau_k \}$.

Then, using equation (2.21), the problem can be restated as follows. For $0 < n < r^*(0)$, given

$$\frac{(K+A)}{r^*(0)} < \frac{(K+\alpha(n))}{n}, \text{ show that } \frac{(K+B)}{r^*(0)} < \frac{(K+\beta(n))}{n}.$$

This statement, in turn, can be restated as follows. For $0 < n < r^*(0)$, given

$$\frac{K}{n} - \frac{K}{r^*(0)} > \frac{A}{r^*(0)} - \frac{\alpha(n)}{n}, \text{ show that } \frac{K}{n} - \frac{K}{r^*(0)} > \frac{B}{r^*(0)} - \frac{\beta(n)}{n}.$$

So it is enough to show that

$$\frac{A}{r^*(0)} - \frac{\alpha(n)}{n} > \frac{B}{r^*(0)} - \frac{\beta(n)}{n} \text{ or, equivalently, that } \frac{A-B}{r^*(0)} > \frac{\alpha(n)-\beta(n)}{n}.$$

To show this, let $m = \arg\min_i\{ \sum_{k=i}^{i+r^*(0)-1} \tau_k \}$, and define $S_A = \{ \tau_k : -r^*(0) \leq k \leq -1 \}$, and $S_B = \{ \tau_k : m \leq k \leq m + r^*(0) -1 \}$. Thus, note that $\sum_{k\in S_A} \tau_k = A$ and $\sum_{k\in S_B} \tau_k = B$.

Let a_i be the i^{th} minimum value in the set $\{\tau_k: \tau_k \in S_A\}$ and similarly b_i be the i^{th} minimum value in the set $\{\tau_k: \tau_k \in S_B\}$. Note that, by definition, $a_i \geq b_i$, for $1 \leq i \leq r^*(0)$. Figure C.1 shows one realization when $r^*(0)=4$.

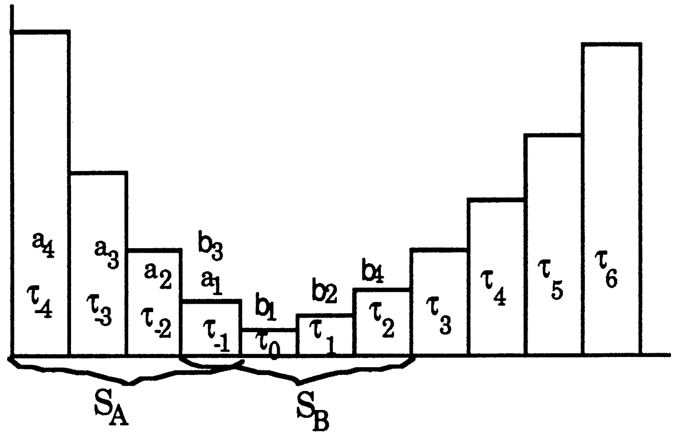

Figure C.1. One realization for the function τ_k when $r^*(0)=4$

Also, from convexity of the τ's, note that $\frac{A-B}{r^*(0)} > \frac{\alpha(n)-\beta(n)}{n}$ if the following inequality holds:

$$a_i - b_i \geq a_{i-1} - b_{i-1}, \quad \text{for } 1 < i \leq r^*(0). \tag{C.1}$$

Thus, in order to prove that $\frac{A-B}{r^*(0)} > \frac{\alpha(n)-\beta(n)}{n}$, we have only to show that the inequality given by (C.1) holds. For this, we consider the following four cases:

<u>Case 1</u> $b_{i-1}, b_i \in S_A$: Suppose $b_i = a_q$. Then, $b_{i-1} = a_{q-1}$ holds in this case. From this relationship together with the convexity of τ_k, inequality (C.1) is proved.

<u>Case 2</u> $b_i \in S_A, b_{i-1} \notin S_A$: Suppose $b_i = \tau_q$. Then, $\tau_{q+1} \leq b_{i-1}$ holds. From this together with the convexity of τ_k, we can show that $a_i - b_i \geq a_{i-1} - \tau_{q+1} \geq a_{i-1} - b_{i-1}$.

<u>Case 3</u> $b_i \notin S_A, b_{i-1} \in S_A$: Let $\min \{ \tau_k : \tau_k \in S_A, \tau_k \geq b_i \} = \tau_q$. Then the following relationships hold:

$$\text{i) } a_i - b_i \geq a_i - \tau_q, \text{ ii) } a_{i-1} - \tau_{q+1} \geq a_{i-1} - \tau_{q+1} \text{ (since } a_i = \tau_q\text{), and iii) } b_{i-1} = \tau_{q+1}. \tag{C.2}$$

From (C.2), we have $a_i - b_i \geq a_{i-1} - \tau_{q+1} = a_{i-1} - b_{i-1}$.

<u>Case 4</u> $b_i \notin S_A, b_{i-1} \notin S_A$: Let $\min \{ \tau_k : \tau_k \in S_A, \tau_k \geq b_i \} = \tau_q$. Then the following relationships hold:

$$\text{i) } a_i - b_i \geq a_i - \tau_q, \text{ and ii) } a_{i-1} - b_{i-1} \leq a_{i-1} - \tau_{q+1} \text{ (since } \tau_{q+1} < b_{i-1}\text{)}. \tag{C.3}$$

From (C.3) together with the convexity of τ_k, we have: $a_i - b_i \geq a_i - \tau_q \geq a_{i-1} - \tau_{q+1} \geq a_{i-1} - b_{i-1}$.

■

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