CAVES AS A MEASURE OF KARST

RANE L. CURL
University of Michigan, Ann Arbor

ABSTRACT

The development of a statistical distribution of proper entrances among proper caves in a limestone karst is considered as a geomorphic process and described with stochastic geomorphic models. The models assume that entrances form and close independently at entrance sites and that the sites are randomly distributed among caves. Analysis of 820 caves in 10 regions on this basis finds that the best model is one assuming a proportional length dependence for mean site frequency on each cave.

This model is used to obtain a constant which is characteristic of a karst and to predict the number and length distributions of entranceless caves and the length distribution of all caves in each region. The karst constant varies from about 0.0001 to 0.01 ft^{-1}. Comparisons between the karst constants in the different regions suggest a correlation with the underlying karst processes and disclose an apparently unique karst situation in County Clare, Ireland. Predictions of complete length distributions of all caves show similarities not found in observed length distributions and support the conclusion that the proper interpretation of a cave region demands information about unobservable caves.

I. INTRODUCTION

A limestone karst landscape has extensive subterranean drainage resulting from enlargement of joints and bedding planes by solution. To a variable extent the drainage may occur in conduits of accessible size that we may study and that we know as caves. It is apparent that in such a region the geomorphic evolution of the topography and that of caves are interacting. However, the environment and agencies of the evolution of the surface are in many ways different from those of the subterranean regime. Thus, largely separate studies of karst phenomena and cave phenomena have been conducted, only implicitly recognizing that boundaries to the former are the "sinks" down which surface erosion products are removed and to the latter the sources of the solvent water and clastic fills. Between these regimes is that of the intersection of the surface and subterranean topographies. This regime has received considerably less attention, even though it includes the feature of the cave entrance, through which all materials flow that are essential to karst or cave evolution, and without which neither could exist. The usual phenomena of caves and karst are not discussed here explicitly, although they are implicit in the theory and observations to be described.

It has been found possible (Curl, 1958) to derive a quantitative measure of a karst using data on the distribution of entrances and lengths among accessible caves. The cave and its intersections with the surface were essentially used as a measuring device for providing quantitative variables, reflecting the over-all karst process, in a stochastic model of phenomena in the karst intersection regime. The rationale for the study was the observation of statistical order in the distribution of entrances among caves. The consequences included the definitions of a constant that characterizes a karst region and predictions of properties of the population of caves. The new problems concerned the geomorphic significance of the karst constant and the true nature of the stochastic geomorphic process.

The elementary phenomenon is the distribution of entrances among caves: the relative frequency of caves with one, two, and more entrances. This is a statistical property of the landscape so that it may be expected that a geomorphic process which leads to the observed distributions may be expressed in statistical terms. Actually, any landscape form is too complicated to be described accurately in purely deterministic terms. The successes of deterministic quantitative geomorphology probably lie in it being reasonable to state the evolution of certain averages of essentially stochastic variables in terms of averages of geomorphic agencies. Stochastic geomorphic processes are postulated and tested here as our observations are themselves statistical in nature.

In any quantitative geomorphic study we must have variables which can be counted or measured. We use here the number of entrances a cave has and the length of a cave. The totality of subterranean voids that drain a karst also includes cave passages that are too small to be accessible to humans and therefore have no accessible entrances. Even passages of accessible size may lack accessible entrances. In either case we are unable to measure the variables. Therefore, in this study we restrict our attention to caves of accessible size, including those that may have no entrances. These are called "proper caves" and the associated accessible entrances, "proper entrances." This question of cave size, or "module," has been discussed previously (Curl, 1964). The view adopted here is that the intersection process of *proper* caves and the surface may be analyzed to characterize the karst and predict certain further properties of the caves. The meaning of this limitation to proper caves and entrances will be made more explicit when the selection of the data is discussed. Hereafter when "cave" or "entrance" is mentioned, it refers only to those of *proper* size.

In the earlier (Curl, 1958) theory a stochastic random-walk model was postulated for the sorting of caves among the classes of different numbers of entrances. Thus, it was assumed that the rate at which a cave gained an entrance was equal to rl, where l is the length of the cave, and the rate at which a cave lost an entrance was equal to sn, where n is the number of entrances the cave had at the moment. In both cases the entrance events were assumed to occur randomly in a population of caves whose members did not themselves change in length or other properties. The constants r and s were assumed fixed. This simple and well-known form of the immigration-emigration stochastic process produces a joint probability density distribution of caves with n entrances and of length l, when stochastic equilibrium is reached equal to

$$p(n,l) = \frac{(\lambda l)^n e^{-\lambda l} p(l)}{n!}, \qquad n = 0, 1, 2, \ldots, \quad (1)$$

where $\lambda = r/s$, the karst constant, and $p(l)$ is the distribution of cave lengths. Equation (1) is a Poisson distribution of n if l is fixed.

Caves without proper entrances ($n = 0$) are usually unobservable, so the consistently observable distribution is the truncated form of equation (1),

$$p'(n,l) = \frac{(\lambda l)^n e^{-\lambda l} p'(l)}{n!(1 - e^{-\lambda l})}, \qquad n = 1, 2, 3, \ldots, \quad (2)$$

where $p'(l)$ is the distribution of cave lengths for $n \geq 1$.

From observations of n and l for each cave in a population of caves the value of the constant λ was obtained by methods of statistical inference. Its mean value was found to be 0.00033 ± 0.00002 (ft^{-1}) for caves of West Virginia with lengths greater than 100, 500, or 1,000 feet, and for caves of Pennsylvania longer than 50 or 100 feet. Good agreement was found between the observed distributions and those calculated by means of equation (2).

It was implicit in the analyses that caves with $n = 0$ also took part in the process. Consequently equation (2) permitted the calculation of the number of entranceless caves and the distribution of their lengths. From this an estimate was obtained for the total length of cave passage in each area, and the distribution of this length among all caves, observable or not. This estimated property of the total population of caves has been used for a study of cave evolution (Curl, 1960). Equation (2) permits finding the expected length distributions for various n, and these were also found to agree with the observations. The successes of the random-walk model are tempered by several questions and considerations. These, and the steps taken here to answer or elucidate them, are as follows:

1. May a preferred model be chosen? Three alternative hypotheses are now compared statistically.

2. The value of the karst constant λ must be the same for all caves in the population to give the simple distribution in equation (2). This does not seem likely from a geological standpoint despite the success of the model. Is there an interpretation of the statistical geomorphic process that permits λ to be distributed but that gives the same simple distribution? A stochastic model has been derived that rests on assumptions of statistical independence of each of many cave-entrance *sites* with different properties, which also yields equation (2).

3. Confidence intervals have been obtained for the estimates of the karst constant.

4. The karst constant, found to have the same value for West Virginia and Pennsylvania, has now been found to be different for other regions.

5. What is the geomorphic significance of λ as a measure of karst? This has been slightly clarified by separating it into a local site and regional components.

6. The estimates of λ are now based on the data for all the observable caves in a region rather than only on those with one entrance.

This quantitative consideration of karst phenomena is based on facts about caves derived from both a rapidly expanding literature on cave descriptions and on personal observations. It is not possible to document all of this background here, but essential to the present viewpoint are the frequent discovery of caves without natural proper entrances (found by quarrying, etc.); the common occurrence of locations in a cave that might be entrances if not blocked by sediments; and the infrequent but not unknown natural closing or opening of entrance sites by movement of fill. Indeed, in some areas, nearly all the caves have been found by excavation, making the assumption of the existence of entranceless proper caves quite acceptable. It is less clear that the entrances engage in a dynamic process of opening and closing, as the time scale of this process is too great. However, entrances must have formed at some time, and many are known to now be blocked to human entry. In some caves opened by excavation, the remains of man and large mammals have been found. Therefore, there probably exist complementary processes of formation and destruction of cave entrances.

II. THE INDEPENDENT-SITE MODEL

In place of following individual caves in their random walk among values of n, consider all possible *entrance sites*—locations on a cave where entrances could form—as individually and independently performing a random walk between the two states of open and closed (in the sense of natural proper entrances). We retain the assumption that the caves themselves are unchanged in this process. To each site i on the jth cave assign a probability μ_{ij} of being open and hence $1 - \mu_{ij}$ of being closed. *Active* sites may have μ_{ij} distributed in the range $0 < \mu \leq 1$. An *inactive* site ($\mu = 0$) is considered for the present as not being a site. Neither the pattern nor rate of opening and closing of sites need concern us.

If the random variable x_{ij} is assigned the value 1 if site ij is open and the value 0 if closed, a cave with q_j sites has

$$n_j = \sum_{i=1}^{q_j} x_{ij} \text{ entrances}. \tag{3}$$

We now deduce how n would be distributed in a large population of caves. The probability distribution we seek is $p(n)$. Its moment generating function (m.g.f.) is

$$M_n(\theta) = E[\exp(\theta n)] = E\left[\exp\left(\theta \sum_{i=1}^{q_j} x_i\right)\right]. \tag{4}$$

Assuming that the x_{ij} and q_j are independent random variables and that the μ_{ij} are distributed but independent of q, equation (4) may be reduced by consecutive summations and integration over x, μ, and q. First, with respect to x, we obtain

$$M_n(\theta) = E\left[\prod_{i=1}^{q} M_x(\theta \mid \mu_i)\right], \tag{5}$$

where M_x is the m.g.f. of x conditional on μ. As x takes on the values 0 and 1 with probabilities $1 - \mu$ and μ, respectively,

$$M_x(\theta \mid \mu_i) = 1 - \mu_i + \mu_i e^{\theta} . \tag{6}$$

Reducing equation (5) now with respect to μ, we obtain

$$M_n(\theta) = E[(1 - \bar{\mu} + \bar{\mu}e^{\theta})^q] , \tag{7}$$

where $\bar{\mu}$ is the expected value of μ over all sites. Equation (7) may be rewritten as

$$M_n(\theta) = E\{\exp[q \ln (1 - \bar{\mu} + \bar{\mu}e^{\theta})]\}, \tag{8}$$

which is the m.g.f. of the distribution of q, with a function as its parameter. Therefore

$$M_n(\theta) = M_q[\ln (1 - \bar{\mu} + \bar{\mu}e^{\theta})] . \tag{9}$$

If we know how the sites are distributed over caves, we may find M_n and hence $p(n)$. Despite the obvious existence of active entrance sites, there exists no documentation as to the number various caves possess, nor for that matter any known procedure for identifying such sites unequivocably and completely as for actual entrances. We must therefore, for the present, try various hypotheses. Several possibilities might be as follows:

1. The simplest assumption is that entrance sites are randomly distributed among caves, with an average frequency ϕ per cave. Then $p(q)$ would be Poisson distributed with m.g.f.

$$M_q(\theta) = \exp[\phi(e^{\theta} - 1)] . \tag{10}$$

Substitution into equation (9) gives

$$M_n(\theta) = \exp[\phi\bar{\mu}(e^{\theta} - 1)] , \tag{11}$$

which also represents a Poisson distribution with mean $\phi\bar{\mu}$.

2. We might expect that larger caves would tend to have more entrance sites. If sites are still randomly distributed but the mean for caves of length l is σl (σ being a frequency of entrances per foot of cave, although without requiring that sites are equally likely along every foot of cave passage), then sites are still Poisson distributed but are conditional on l. Substitution of σl for ϕ in equation (11) gives

$$M_n(\theta \mid l) = \exp[\sigma l \bar{\mu}(e^{\theta} - 1)] , \tag{12}$$

which also represents a Poisson distribution conditional on l with mean $\sigma l \bar{\mu}$. This is identical to equation (2) with $\lambda = \sigma\bar{\mu}$.

3. A slightly more complicated hypothesis than hypothesis 2 would be that sites are randomly distributed but the mean number per cave is some other function of length. A simple version of this might be $\phi = a + \sigma l$, where a is a constant. This yields

$$M_n(\theta \mid l) = \exp[(a + \sigma l)\bar{\mu}(e^{\theta} - 1)] . \tag{13}$$

The probability distributions of n equivalent to each of the above hypotheses, expressed in the form of equation (2)—the observable or truncated joint distribution—are

1. Poisson model:

$$p'(n,l) = \frac{(\lambda_p)^n \exp(-\lambda_p) p'(l)}{n![1-\exp(-\lambda_p)]}. \quad (14)$$

2. Conditional model:

$$p'(n,l) = \frac{(\lambda_c l)^n \exp(-\lambda_c l) p'(l)}{n![1-\exp(-\lambda_c l)]}. \quad (15)$$

3. Biparametric conditional model:

$$p'(n,l) = \frac{(\bar{a}+\lambda_b l)^n \exp[-(\bar{a}+\lambda_b l)] p'(l)}{n!\{1-\exp[-(\bar{a}+\lambda_b l)]\}}, \quad (16)$$

where $\lambda_p = \phi\bar{\mu}$, λ_c (or λ_b) $= \sigma\bar{\mu}$, and $\bar{a} = a\bar{\mu}$.

Obviously, models may be multiplied endlessly. This fact forces us to the opposite extreme; select the simplest model which is compatible with the observations. It was found previously that the one-parameter model 2 worked well, and the present study confirms this. In Section V models 1 and 2 are compared and, in a few cases, model 3.

Equation (15) from the independent-site model and equation (2) from the first random-walk model are identical even though the approaches differ. It is useful to note the points of similarity and difference.

i) In the random-walk model each cave had the same value of λ. The constancy of λ, if found, was taken to define *homogeneity* of two or more regions in regard to the karst intersection process. In the independent-site model each site may have a different value of μ and a different pattern of opening and closing. In addition $\bar{\mu}$ might remain constant even though individual μ_{ij} varied with time or the higher statistical properties of the distribution changed. Only the population mean $\bar{\mu}$ is important. Thus a population formed from a mixture of populations with different λ would not conform to equation (2), while mixing two populations with different $\bar{\mu}$'s only produces a new $\bar{\mu}$. On the other hand σ may not be distributed, but then this is a regional property, not a site property.

This demonstration that equation (2) is the consequence of a much more general process does detract from the concept of homogeneity of karst regions. If the same values of $\lambda_c = \sigma\bar{\mu}$ are found in two cave regions, or between two groups of caves in one region, we may only claim homogeneity in a narrowed sense, as the two populations may still differ in values of σ and $\bar{\mu}$ and, as mentioned, in the distributional properties of μ_{ij}.

ii) The random-walk model permitted unlimited values of n (although they would be very rare), while the independent-site model limits n on a particular cave to the number of sites. This is geologically more reasonable.

iii) The assumption of stochastic equilibrium in a random walk is now replaced by the assumption of statistical independence of sites. It will be shown that the new concept permits the model to apply to an evolving cave population.

iv) The hypotheses of the distribution of sites among caves in the independent-site model are equivalent in their outcome to the transition events in the random-walk model. However, this property is now physically separated from the processes at individual sites.

In summary, the independent-site model appears to be more realistic as well as more general, and explains why an equation of the form of equation (15) should work as universally as it does. Most important, we see that simple assumptions of statistical independence and randomness in a geomorphic process are sufficient to encompass certain statistical geological consequences of an intrinsically complex process.

The models of a geomorphic process that have been derived here are large-population models. The relations (14)–(16) give limiting relative frequencies as the size of a population of caves engaging in the process approaches infinity. For any finite population, in particular, for the small observable populations, we would expect the essentially dynamic nature of the process to be apparent. Entrances are opening and closing, and caves are thereby shifting among the entrance-number categories. The distribution we observe is only one of very many distributions that are completely compatible with a true model. We must be aware of the possibility that an observed distribution disagrees with a model because it is in a *rare state*, rather than because the model is incorrect. For this reason statistical tests will be used to determine whether a model will be accepted or rejected.

If we think of each entrance-number category as a queue, with caves joining or leaving the queue randomly and independently, then it is known that the queue size is Poisson distributed over time. Therefore if the expected number of caves with n entrances is $\bar{m}_n$, this is also the variance of m_n. For example, a sequence of *independent* observations (our data from caves is one such) might give values of 116, 80, 123, 107, 111, 102, 96, 94, 95, 104, . . . , if $\bar{m}_n = 100$; 9, 12, 10, 6, 3, 10, 6, 13, 8, . . . , if $\bar{m}_n = 10$, and 2, 3, 2, 0, 0, 2, 0, 3, 1, 0, . . . , if $\bar{m}_n = 2$. *This inherent variability is a property of the geomorphic process.*

III. DEFINITIONS AND DATA

The data consist of counts of the number of natural proper caves with one or more natural proper entrances and measurements of the lengths of these caves. The sources for the data include geological surveys, publications of cave-exploring groups, personal communications, and personal observations. Their quality and completeness differ considerably, and none was compiled with a quantitative study of this nature in mind. However, to obtain values of the karst constant that are comparable, it is necessary to have a consistent method of selecting data. Therefore, with a few exceptions, only data sources that permitted the thorough application of the selection and interpretation rules below were used. In addition, it is necessary to have a reasonable number of multiple-entrance caves in each group (including $n \geq 3$) in order to estimate the parameters of the distributions. This limited the usable groups and the degree to which subgroups could be selected.

Most of the data interpretations were performed by the author. It is not known to what extent a personal bias enters into the results. The selection and interpretation rules used here are mostly identical to those used previously. Only a few changes have been made to reduce the dilemmas of borderline cases. The results will of course reflect the details of the definitions. In either case it is hoped that consistency yields consistent results.

A. PROPER CAVES

A natural *proper* cave is an explorable subterranean void having as boundaries the country rock, fill of any kind (flowstone, or clastic fills, breakdown, and water) and an *entrance surface*. Thus a geological cave system may be divided into several proper cave *fragments* by any of the above internal boundaries. Each such fragment is considered here as a separate proper cave. Such boundaries are *natural* if they cannot be penetrated by an explorer without the aid of tools (mechanical aids, blasting, or diving equipment). This definition excludes from the data all proper caves or cave fragments that did not have a natural proper entrance because such caves are not equally well known throughout a cave region. This is a consistent definition as each cave fragment would possess its own entrance sites and take part in the entrance-forming process individually, as do also the presently naturally inaccessible fragments. The latter are encompassed in the predictions of the number and lengths of caves without natural proper entrances. Cave descriptions do not always indicate when new sections (i.e., new proper caves) have been found by excavation. This introduces an inescapable error into the data.

Caves which are reported but for which it is not possible to determine either the number of natural proper entrances or length of passages are excluded from the data. Such caves are usually short or have been too recently discovered to be fully surveyed. In these analyses a lower length limit is applied to exclude short caves. Descriptions of short caves are generally inadequate as they do not interest the explorer. The choice of a lower length limit is based on a subjective interpretation of the quality of the descriptions. It is usually on the order of 50 to 100 feet. If a region is homogeneous, the value of λ should not depend on the choice of a lower length limit.

Every accessible proper cave in a region should be included in the analysis. However, further search in a region usually continues to turn up caves with natural proper entrances. If the discovery of such caves does not depend upon their length or number of entrances, the effect of incomplete discovery is only to change the sample size but not to affect the value of the karst constant. That this is probably true was found in a comparison made in the earlier paper.

Only caves in limestone, marble, or magnesitic limestone are included in the data. Caves in insoluble rock, dolomite, or gypsum are excluded in the belief that they would exhibit modified site processes.

B. NATURAL PROPER ENTRANCES

An entrance is defined by the "fall" or "drip" *surface* at the location where it is possible to proceed from beneath sky to beneath rock. If this entrance surface is unmodified by man and a man may pass through it, it is a *natural* and *proper* entrance. The entrance surface is a boundary to the attached proper cave. In particular, if a sink has divided a continuous cave system so that it is not possible to pass between the cave fragments while remaining within the proper cave, the two fragments are separate caves and each has a separate entrance in the sink if each entrance is natural and proper. This definition is chosen for consistency when, say, a valley divides a cave into fragments. It is more tenuous, even if consistent, when the interruption is a smaller sink. If a natural proper entrance formerly existed to a fully reported cave, but the entrance is now closed artificially, the entrance and cave were included providing that the other data were adequate.

In some areas a considerable number of caves have been found by quarrying, road cuts, etc. These entrances are artificial, even if proper, and are not counted. Such caves are, of course, among the class of caves with no ($n = 0$) natural entrances at present.

C. PROPER LENGTHS

Measuring the length of a cave involves considerable arbitrariness. It is defined here as the minimum traversable horizontal plan length, such as would be determined by a survey with conventional instruments. The lengths of all passages and separate levels of the caves are included. Caves are usually long compared to the passage breadth so that this procedure is acceptable. In the cases where large rooms exist, much wider than the usual passage breadth, a sum of length and width was used.

If the length of a cave was only stated in a report, this value was used. Lengths are frequently reported to approximate round numbers (500 feet, 100 yards, etc.), which causes grouping in such data. This affects the derived length distributions. When a map of the cave was also available, the length was determined directly with a map-measuring instrument, and therefore grouping due to rounding does not occur. Care was taken to determine if the reports included cave passages not shown on a map.

A Poisson distribution of sites among caves itself introduces a variance of number of sites equal to the mean; some variability in determining lengths is buried in this implicit variability. However, consistent properties of these methods of determining length show in the value of the karst constant.

In figure 1 are shown the application of these definitions to determine the natural proper length and number of entrances for a hypothetical cave.

IV. ANALYSIS

The karst constant is found by determining an estimate for the λ, denoted $\hat{\lambda}$, which gives the best fit between the observed frequencies m_n of caves with n entrances and the values calculable from equations (14)–(16). When this has been found, it may be used in conjunction with the equations to obtain the estimated values of the frequencies, $\hat{m}_n$ and further predictions of length distributions, average lengths, and other properties of the cave population if the model does indeed represent the geomorphic process. In this section the variables tabulated in Section V and the numerical methods of obtaining them are summarized.

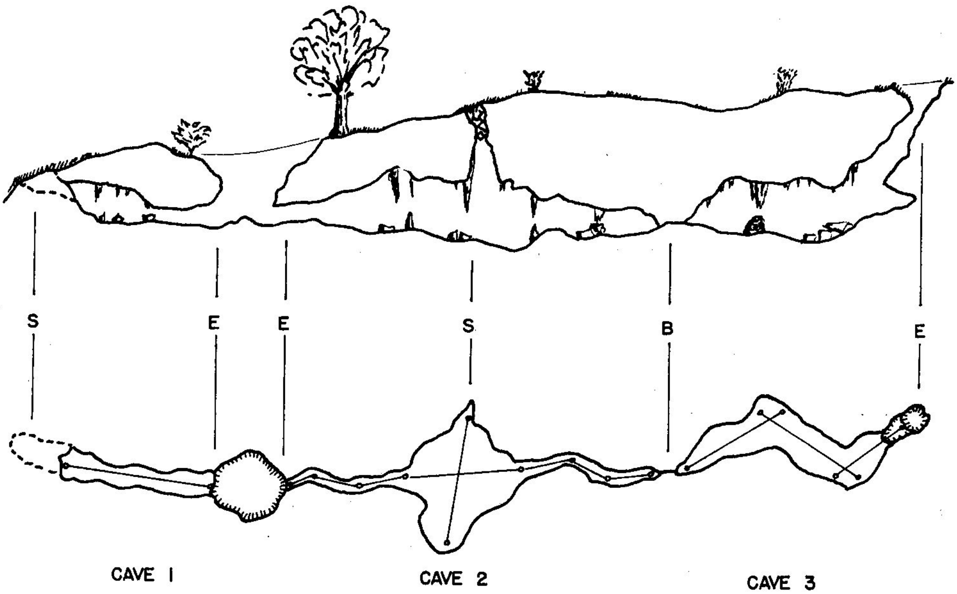


FIG. 1.—Elevation and plan views of a cave, illustrating the definitions of proper cave features: *E*, entrances; *S*, sites; *B*, internal cave boundary. Surveyed traverses, for determining proper length, are shown in the plan view. The original cave has been divided into three observable fragments (each with one entrance) by the development of a sink and an interior barrier. In principle, unobservable fragments (not shown) may also be associated with the system.

A. THE KARST CONSTANT

The maximum likelihood method of parameter estimation is used to obtain the best estimates of the karst constant $\hat{\lambda}$ (Mood, 1950). The method yields the following set of equations which must be solved, using the entrance and length data for a group of M caves:

1. $$\bar{n}' - \frac{\hat{\lambda}_p}{1 - \exp(-\hat{\lambda}_p)} = 0,$$

2. $$\sum_{i=1}^{M} \left[n_i - \frac{\hat{\lambda}_c l_i}{1 - \exp(-\hat{\lambda}_c l_i)} \right] = 0,$$

3. $$\sum_{i=1}^{M} \left\{ \frac{n_i \hat{\lambda}_b l_i}{\hat{a} + \hat{\lambda}_b l_i} - \frac{\hat{\lambda}_b l_i}{1 - \exp[-(\hat{a} + \hat{\lambda}_b l_i)]} \right\} = 0, \qquad (17)$$

$$\sum_{i=1}^{M} \left\{ \frac{n_i}{\hat{a} + \hat{\lambda}_b l_i} - \frac{1}{1 - \exp[-(\hat{a} + \hat{\lambda}_b l_i)]} \right\} = 0,$$

where n_i and l_i are the number of entrances and length of a cave in the group and $\bar{n}'$ is the average number of entrances of the caves in the group. (In this and the following section a "prime" indicates a property or distribution of the observed cave population.) Model 3 requires the simultaneous solution of the two relations shown. Newton's method was used to obtain the parameter estimates.

B. ESTIMATED FREQUENCIES

The $\hat{m}_n$ are M times the estimated marginal probability of n, obtained by integrating equations (14)–(16) with respect to l. Because the data are discrete this operation becomes a summation involving the length observations.

1. $$\hat{m}_n = M \frac{(\hat{\lambda}_p)^n \exp(-\hat{\lambda}_p)}{n![1-\exp(-\hat{\lambda}_p)]},$$

2. $$= \sum_{i=1}^{M} \frac{(\hat{\lambda}_c l_i)^n \exp(-\hat{\lambda}_c l_i)}{n![1-\exp(-\hat{\lambda}_c l_i)]}, \qquad (18)$$

3. $$= \sum_{i=1}^{M} \frac{(\hat{a}+\hat{\lambda}_b l_i)^n \exp[-(\hat{a}+\hat{\lambda}_b l_i)]}{n!\{1-\exp[-(\hat{a}+\hat{\lambda}_b l_i)]\}}.$$

It may be shown that these relations also give the estimate $\hat{m}_0$ when $n = 0$. This is of course the estimate of how many entranceless caves must be taking part in the geomorphic process if it is consistent for $n = 0, 1, 2, \ldots$.

C. LENGTH DISTRIBUTIONS

Length distributions are expressed here in the form of cumulative distributions $F(l|l^*)$: the fraction of caves in a group whose lengths are equal to or greater than l, when the minimum length in the group is l^*. The conditional (on n) length distribution is obtained from equations (14)–(16) by the operation

$$F(l|n) = 1 - \int_{l^*}^{l} \frac{p'(l,n)\,dl}{p'(n)}, \qquad (19)$$

and consequently the estimated conditional length distributions become summations over the data arranged in order of increasing length.

1. $$\hat{F}(l_k|n) = F'(l_k),$$

2. $$\hat{F}(l_k|n) = 1 - \frac{1}{\hat{m}_n}\sum_{i=1}^{k-1} \frac{(\lambda_c l_i)^n \exp(-\hat{\lambda}_c l_i)}{n![1-\exp(-\hat{\lambda}_c l_i)]}, \qquad (20)$$

3. $$\hat{F}(l_k|n) = 1 - \frac{1}{\hat{m}_n}\sum_{i=1}^{k-1} \frac{(\hat{a}+\hat{\lambda}_b l_i)^n \exp[-(\hat{a}+\hat{\lambda}_b l_i)]}{n!\{1-\exp[-(\hat{a}+\hat{\lambda}_b l_i)]\}}.$$

The estimated length distributions from model 1 do not depend on n. The other models imply a process which sorts caves in a way that the average length increases with n. It may be shown that equations (20) also give the estimated length distribution for $n = 0$: the distribution of lengths of entranceless caves.

The estimated mean length conditional on n, $\hat{l}_n$, also follows from equations (14)–(16).

The result is

$$1.\qquad \hat{l}_n = \bar{l}',$$

$$2.\qquad \hat{l}_n = \frac{n+1}{\hat{\lambda}_c}\frac{\hat{m}_{n+1}}{\hat{m}_n}, \tag{21}$$

$$3.\qquad \hat{l}_n = \frac{1}{\hat{m}_n}\sum_{i=1}^{M}\frac{(\hat{a}+\hat{\lambda}_b l_i)^n l_i \exp[-(\hat{a}+\hat{\lambda}_b l_i)]}{n!\{1-\exp[-(\hat{a}+\hat{\lambda}_b l_i)]\}}.$$

Model 2 estimates mean-length sorting which is simply related to the sorting of caves on n. These expressions also hold for $n = 0$.

From the above relations may be obtained an estimated length distribution for the total cave population, including $n = 0, 1, 2, \ldots$. The estimated total number of caves in a region is $\hat{N} = M + \hat{m}_0$ or, from summation of equation (18) over all n,

$$\hat{N} = \frac{M}{1-\exp(-\hat{\lambda}_p)};\quad \sum_{i=1}^{M}\frac{1}{1-\exp(-\hat{\lambda}_c l_i)};\quad \sum_{i=1}^{M}\frac{1}{1-\exp[-(\hat{a}+\hat{\lambda}_b l_i)]} \tag{22}$$

for models 1, 2, and 3, respectively. Consequently, from equations (20) we obtain, in the same order,

$$\hat{F}(l_k) = F'(l_k);\qquad 1-\frac{1}{\hat{N}}\sum_{i=1}^{k-1}\frac{1}{1-\exp(-\hat{\lambda}_c l_i)};$$
$$1-\frac{1}{\hat{N}}\sum_{i=1}^{k-1}\frac{1}{1-\exp[-(\hat{a}+\hat{\lambda}_b l_i)]}, \tag{23}$$

where the summations are again carried out in order of increasing length from l^*.

An estimate of the average length of *all* caves for a given l^* is

$$\hat{l} = \frac{\hat{m}_0\hat{l}_0 + M\bar{l}'}{\hat{m}_0 + M}. \tag{24}$$

An estimated variance of length, conditional on n, may also be obtained from equations (14)–(16). For model 1 it is the same as the population variance. For model 2 it may be found in terms of the $\hat{l}_n$ by averaging $(l_n - \hat{l}_n)^2$ over the observed distributions. The result is

$$\operatorname{var} l_n = \hat{l}_n(\hat{l}_{n+1} - \hat{l}_n), \tag{25}$$

and the variance of the sample (observed) estimate of $\hat{l}_n$ is $(\operatorname{var} l_n)/(m_n - 1)$.

Finally, the estimated fraction of entranceless caves in the total population is

$$\hat{\alpha}_0 = \frac{\hat{m}_0}{M + \hat{m}_0}, \tag{26}$$

and the estimated fraction of cave length possessed by entranceless caves is

$$\hat{\beta}_0 = \frac{\hat{l}_0\hat{m}_0}{\sum_{i=1}^{M} l_i + \hat{l}_0\hat{m}_0}. \tag{27}$$

D. GOODNESS OF FIT AND CONFIDENCE INTERVALS

The goodness of fit between the observed m_n and the estimated $\hat{m}_n$ is determined using the maximum likelihood estimation of χ^2 (Mood, 1950). In the present case the quantity

$$-2\sum_{n=1}^{k} m_n \log \frac{\hat{m}_n}{m_n} = \chi^2_{k-c} \qquad (28)$$

is distributed approximately as χ^2 with $k - c$ degrees of freedom, where k is the number of groups of non-zero m_n and c the number of constraints. For models 1 and 2, $c = 2$ while $c = 3$ for model 3 where one more parameter is estimated. Grouping with the following lower finite n group was done if any $m_n = 0$.

Confidence intervals are evaluated for $\hat{\lambda}$ in models 1 and 2 by the method, related to maximum likelihood estimation, developed by Bartlett (1953). These confidence intervals account for the natural variability expected in the observations. Bartlett's method, with a skewness correction, leads in the case of model 2 to the interval

$$D_s \pm \gamma\sqrt{(\lambda_c^2 I_s)} - \tfrac{1}{6}(\gamma^2 - 1)\frac{(\lambda_c^3 K_3)}{(\lambda_c^2 I_s)} = 0 \qquad (29)$$

where

$$D_s = \sum_{i=1}^{M}\left[n_i - \frac{\lambda_c l_i}{1 - \exp(-\lambda_c l_i)}\right],$$

$$\lambda_c^2 I_s = \sum_{i=1}^{M}\left\{\frac{\lambda_c l_i}{1 - \exp(-\lambda_c l_i)} - \frac{(\lambda_c l_i)^2 \exp(-\lambda_c l_i)}{[1 - \exp(-\lambda_c l_i)]^2}\right\},$$

$$\lambda_c^3 K_3 = \sum_{i=1}^{M}\left\{\frac{(\lambda_c l_i)^3[1 + \exp(-\lambda_c l_i)]\exp(-\lambda_c l_i)}{[1 - \exp(-\lambda_c l_i)]^3} - \frac{3(\lambda_c l_i)^2 \exp(-\lambda_c l_i)}{[1 - \exp(-\lambda_c l_i)]^2} + \frac{\lambda_c l_i}{1 - \exp(-\lambda_c l_i)}\right\}, \qquad (30)$$

and γ is the number of standard deviations for a unit-normal distribution at the desired two-tailed significance level. The solution of equation (29), with a given value of γ, using the data in a set of M caves, gives two values of $\hat{\lambda}_c$ for the estimated confidence interval. Equations (29) and (30) were solved for λ_c by means of Newton's method. Confidence intervals for the estimates of $\hat{m}_0$ are obtained directly from equations (18) using the confidence interval values of $\hat{\lambda}_c$.

All of the computations with data were carried out on the University of London digital computer.

V. RESULTS

The results of the analyses are presented here by regions. In a few cases the caves of region were subdivided into another group with different l^*, and in one case into two groups according to the age of the rocks in which the caves occur. The sources, nature, and quality of the data vary considerably. Because of this, and sometimes because of computational difficulties, the groups do not all have the same value of l^*. Detailed descriptions of the geological and morphological environments of each group may be found in the references cited.

Bartlett (1961) has correctly observed that the comparisons of subgroups would be more meaningful if independent length groups were chosen. This was attempted, but it was unfortunately found impossible because most of the data for $n \geq 2$ involves the long caves. The possible bounded sets of shorter caves included too few multiple n observations to find parameter estimates.

Results for 820 caves in 10 regions are presented here. If a model were true in every case, the probability of nevertheless rejecting the model in R cases at the u level of significance with the χ^2 test, would be binomially distributed, $f(R|10, u)$ Thus the probability of rejecting a true model in one or more cases is 0.63 at a 10 per cent level, 0.40 at 5 per cent and 0.10 at 1 per cent. On the other hand, if we set the significance level of the test too low, we are more likely to accept a model in a given case when it is indeed false. As a compromise we will use here a test at the 5 per cent level, which gives us about an even chance of rejecting a true model in one or more cases out of the 10.

If the models tested here were the only alternative hypotheses we could consider the probability of accepting a model when it is indeed false. Unfortunately we must be concerned with a larger class of alternative hypotheses, which includes all those with any degree of failure of the assumptions of entrance-site independence and distribution, as well as inconsistencies in the data. Rejection of a model is likely to be due to these causes.

A possible further test might be whether or not the observations show a sorting of cave lengths as predicted by model 1 (none) or model 2 or 3. However, the failure to observe good length sorting even with model 2 true should not be unexpected. The spread of lengths in the data is very great and the length variance is large. If equation (25) is applied to these results, we find in every case that the estimated standard deviation of l_n is of the same magnitude as $\hat{l}_n$. Thus when m_n is small, large deviations in the $\bar{l}'_n$ may be expected. Statistical comparisons of expected and observed mean lengths have not been made.

The length distributions $F'(l)$, and $\hat{F}(l|0)$ and $\hat{F}(l)$ from model 2, in figures 2–15 have been smoothed and plotted on the basis of consecutive odd groups of about 0.1 M caves. The middle F value is plotted versus the average length of each group with straight lines joining the points. This procedure helps to smooth the data grouping arising from rounding in reported lengths. Unsmoothed values are joined with dashed lines. Logarithmic coordinates are used because of the large range of cave lengths and also the ease of changing l^* graphically.

The dimension of the various l is feet, and that of $\hat{\lambda}_c$ and $\hat{\lambda}_b$, ft^{-1}. All other parameters are dimensionless. The values for the observed number and average length of caves for $n = 0$ represent those found by quarrying and the like.

A. CALIFORNIA, U.S.A.

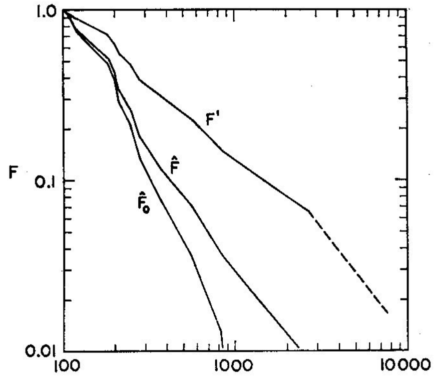
Data are from Halliday (1962) and personal observation. The caves occur in mountain, foothill, and desert settings in limestones of various ages and composition. There is a considerable variety in local surface features. However, these caves are analyzed as one region because of their small number.

The results are shown in table 1 and in figure 2. All models are accepted. Model 3 is close to model 1 (weak sorting by length).

B. INDIANA, U.S.A.

Data are from Powell (1961, and personal communication). Caves occur on the Mitchell Plain and Crawford Upland, underlain by limestones of Mississippian age. A length subgroup is also analyzed.

Results for $l^* = 100$ are shown in table 2 and figure 3 and those for $l^* = 1{,}000$ in table 3



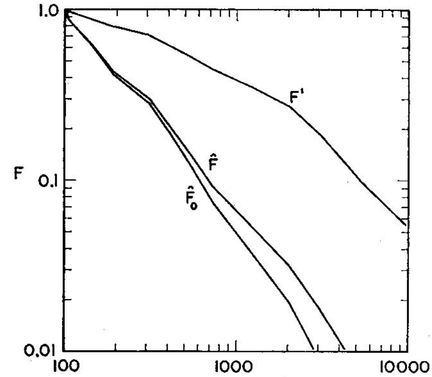


FIG. 2.—Cumulative distribution of cave length, California. $F' = F'(l)$, observed; $\hat{F} = \hat{F}(l)$, estimated for all caves, $n \geq 0$, from model 2; $\hat{F}_0 = \hat{F}(l|0)$, estimated for $n = 0$ from model 2. Computed distributions partly smoothed in plotting. (The abscissa gives cave length in feet.)

FIG. 3.—Same as fig. 2 for Indiana.

TABLE 1

A. COMPARISON OF MODELS, CALIFORNIA

($l^* = 100$; $M = 61$; $\bar{l}' = 617$)

			Model 1 ($\hat{\lambda}_p$= 0.624)	Model 2 ($\hat{\lambda}_c$=0.000846)		Model 3 ($\hat{\lambda}_b$= 0.000240; $\hat{a}$=0.456)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..........	7	297	70.4	288.8	205	83.1
1..........	45	533	43.9	50.2	331	44.8
2..........	12	304	13.7	7.0	794	12.7
3..........	3	763	2.9	1.6	2,111	2.7
4..........	1	7,700	0.4	0.7	3,926	0.6
≥5..........	0	....	0.1	1.5	....	0.2
χ^2..........			0.63	5.30		0.15
d.f..........			2	2		1

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, CALIFORNIA

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.83	0.61	277	0.00067	373	0.00055	456
			0.00106	224	0.00124	190

and figure 4. In both cases models 1 and 2 are accepted. In the second case a result is obtained for model 3 which is nearly identical to model 2, the additional parameter giving no advantage.

The 75 per cent confidence intervals for $\hat{\lambda}_c$ overlap for the two groups. A homogeneous independent-site model appears applicable.

C. IOWA, U.S.A.

Data are from J. Hedges (personal communication). These caves are mostly from the Maquoketa and Wapsipinicon Valleys, or near Dubuque. Many caves in the vicinity of Dubuque have been found through lead and zinc mining. It is thought that they may have arisen partly by hydrothermal action. Their natural proper lengths have been modified by the mining of ore bodies from them, but, as they usually had no natural entrance, they are not included in the data.

The results are shown in table 4 and in figure 5. Model 1 is rejected and model 2 accepted. Apparent data grouping due to rounding dominates the length distributions at $l = 100$.

TABLE 2

A. COMPARISON OF MODELS, INDIANA

($l^* = 100$; $M = 104$; $\bar{l}' = 2{,}087$)

			Model 1 ($\hat{\lambda}_p$= 0.346)	Model 2 ($\hat{\lambda}_c$=0.000142)		Model 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..........	6	1,041	252.0	2,023.2	321	
1..........	88	1,689	87.1	92.2	1,288	
2..........	13	2,947	15.0	8.4	4,652	
3..........	3	10,038	1.7	1.9	9,843	
≥4..........	0	....	0.2	1.5	....	
χ^2..........			0.85	2.36		
d.f..........			1	1		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, INDIANA, $l^* = 100$

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
a_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.95	0.75	406	0.00011 0.00018	2,613 1,574	0.000091 0.00021	3,202 1,342

TABLE 3

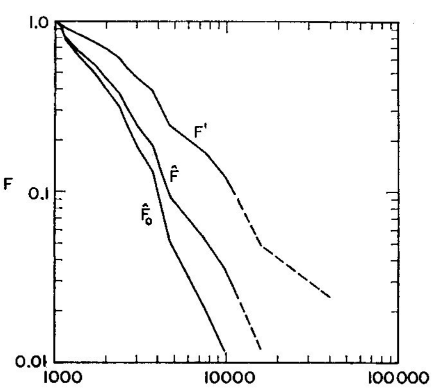
A. COMPARISON OF MODELS, INDIANA

($l^* = 1{,}000$; $M = 41$; $\bar{l}' = 4{,}763$)

			Model 1 ($\hat{\lambda}_p = 0.538$)	Model 2 ($\hat{\lambda}_c = 0.000102$)		Model 3 ($\hat{\lambda}_b = 0.000092$; $\hat{a} = 0.060$)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0	2	2,302	57.6	149.7	2,159	118.8
1	32	4,062	31.0	33.0	4,062	32.7
2	6	5,858	8.3	5.8	6,541	6.2
3	3	10,038	1.5	1.3	13,448	1.3
≥ 4	0	....	0.2	0.9	....	0.8
χ^2			1.54	0.31		0.33
d.f.			1	1		....

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, INDIANA, $l^* = 1{,}000$

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.79	0.62	2,710	0.000075 0.00014	212 106	0.000057 0.00017	281 85



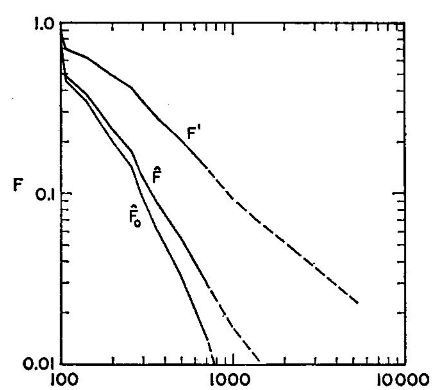


FIG. 4.—Same as fig. 2 for Indiana

FIG. 5.—Same as fig. 2 for Iowa

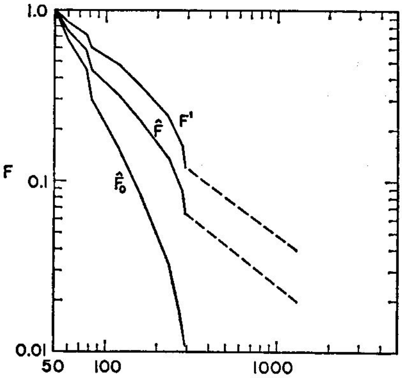
TABLE 4

A. COMPARISON OF MODELS, IOWA
($l^* = 100$; $M = 43$; $\bar{l}' = 450$)

			Model 1 ($\hat{\lambda}_p$= 0.393)	Model 2 ($\hat{\lambda}_c$=0.000761)		Model 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..	12	2,468	89.3	303.2	161	
1..........	37	444	35.1	37.3	290	
2..........	3	242	6.9	4.1	799	
3..........	3	730	0.9	0.8	2,187	
≥4..........	0	....	0.1	0.8	....	
χ^2...........			5.49	1.28		
d.f...........			1	1		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, IOWA

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.88	0.72	226	0.00053 0.00108	445 208	0.00039 0.00134	615 164



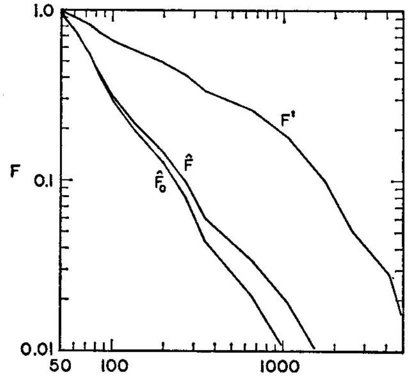


Fig. 6.—Same as fig. 2 for New Jersey

Fig. 7.—Same as fig. 2 for Pennsylvania

D. NEW JERSEY, U.S.A.

Data are from Anderson (personal communication). The results are shown in table 5 and in figure 6. Model 1 is rejected and model 2 accepted. Length sorting is consistent.

E. PENNSYLVANIA, U.S.A.

Data are from Stone (1953), Smeltzer and Stone (1956), Dunn and McCrady (1961) and Black, Haarr, and McGrew (1962). Caves occur in a wide variety of geological and morphological settings in both pre-Mississippian and Mississippian limestones. The former dominate. Physical provinces include the northern extensions of the Piedmont, Blue Ridge, Valley and Ridge, and Appalachian Plateau. A length subgroup is also analyzed.

TABLE 5

A. COMPARISON OF MODELS, NEW JERSEY

($l^* = 50$; $M = 25$; $\bar{l}' = 182$)

			Model 1 ($\hat{\lambda}_p$= 1.430)	Model 2 ($\hat{\lambda}_c$=0.00706)		Model 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0	9	159	7.9	25.3	84	
1	16	105	11.2	15.0	111	
2	4	120	8.0	5.8	155	
3	1	280	3.8	2.1	208	
4	2	250	1.4	0.8	277	
5	1	1,300	0.4	0.3	449	
6	0	....	0.1	0.2	820	
7	1	300	0.0	0.1	1,144	
≥8	0	....	0.0	0.7	....	
χ^2			13.5	3.29		
d.f.			4	4		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, NEW JERSEY

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
a_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.50	0.32	133	0.0056 0.0088	34 19	0.0047 0.0102	43 15

The results for $l^* = 50$ are shown in table 6 and in figure 7, and those for $l^* = 500$ in table 7 and in figure 8. In the first case model 1 is rejected and model 2 accepted. In the second, both models are accepted but length sorting appears to favor model 2.

The 75 per cent confidence intervals for $\hat{\lambda}_c$ overlap for the two groups. A homogeneous independent-site model with length-dependent site distribution appears applicable as model 2 is acceptable in both cases.

F. WEST VIRGINIA, U.S.A.

Data are from Davies (1958) with an addition by Peters (personal communication regarding Windy Mouth Cave). Caves occur in both the Valley and Ridge and Appalachian

TABLE 6

A. COMPARISON OF MODELS, PENNSYLVANIA

($l^{*a} = 50$; $M = 140$; $\bar{l}' = 595$)

			Model 1 ($\hat{\lambda}_p$=0.273)	Model 2 ($\hat{\lambda}_c$=0.000419)		Model 3 ($\hat{\lambda}_b$=0.000287;[a] $\hat{a}$=0.091)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0	86	458	445.5	2,386.4	125	800.2
1	126	479	121.7	125.1	431	123.4
2	10	1,713	16.6	11.3	1,468	14.0
3	3	1,593	1.5	2.5	2,876	2.1
4	0	....	0.1	0.7	3,860	0.4
5	1	910	0.0	0.2	4,531	0.1
≥6	0	....	0.0	0.1	....	0.0
χ^2			12.5	0.98		3.74
d.f.			2	2		1

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, PENNSYLVANIA ($l^* = 50$)

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
a_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.95	0.78	151	0.00033 0.00053	3,069 1,864	0.00027 0.00062	3,748 1,594

[a] Although the computation converged, these values probably represent a local rather than a true likelihood maximum.

TABLE 7

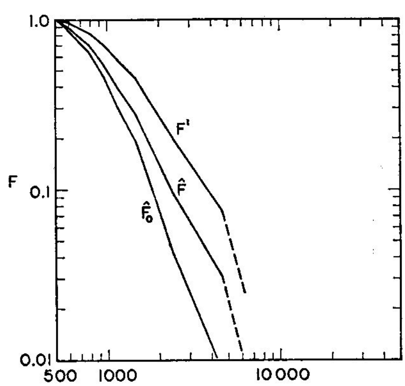
A. COMPARISON MODELS, PENNSYLVANIA

($l^* = 500$; $M = 40$; $\bar{l}' = 1{,}708$)

			Model 1 ($\hat{\lambda}_p = 0.624$)	Model 2 ($\hat{\lambda}_c = 0.000356$)		Model 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..........	23	1,213	45.2	78.6	1,062	
1..........	31	1,500	28.7	29.7	1,445	
2..........	6	2,720	9.1	7.6	2,138	
3..........	2	2,295	1.9	1.9	3,094	
4..........	0	....	0.3	0.5	3,989	
5..........	1	910	0.0	0.2	4,625	
≥ 6..........	0	....	0.0	0.1	....	
χ^2..........			5.78	2.10		
d.f..........			2	2		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2, PENNSYLVANIA ($l^* = 500$)

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.66.......	0.55	1,280	0.00027	111	0.00021	146
			0.00047	55	0.00056	44



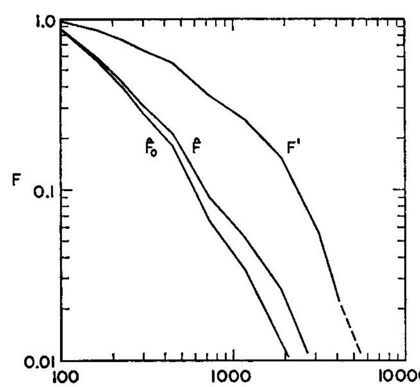


Fig. 8.—Same as fig. 2 for Pennsylvania

Fig. 9.—Same as fig. 2 for West Virginia, pre-Mississippian

Plateau provinces. The data were sufficient to permit separate analyses for caves occurring in *pre-Mississippian* and *Mississippian* age limestones, as well as length subgroups in each.

The results for caves in pre-Mississippian limestones are shown in table 8 and in figure 9 for $l^* = 100$, and in table 9 and in figure 10 for $l^* = 1{,}000$. In both cases models 1 and 2 are accepted. In the first case a result is obtained for model 3, similar to model 2, but the additional parameter gives no advantage.

The 75 per cent confidence intervals for $\hat{\lambda}_c$ overlap for these two pre-Mississippian groups while the $\hat{\lambda}_p$ differ significantly. The homogeneous independent-site model 2 appears best for these caves.

The results for caves in Mississippian limestones are shown in table 10 and in figure 11, for $l^* = 100$, and in table 11 and in figure 12 for $l^* = 1{,}000$. In the first case model 1 is rejected and models 2 and 3 accepted. In the latter case models 1 and 2 are accepted. The additional parameter in model 3 gives no significant advantage. In both cases the observed length sorting favors model 2.

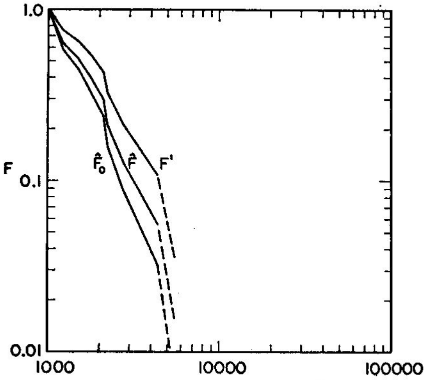
TABLE 8

A. COMPARISON OF MODELS, WEST VIRGINIA, PRE-MISSISSIPPIAN ($l^* = 100$; $M = 90$; $\bar{l}' = 876$)

			MODEL 1 ($\hat{\lambda}_p$= 0.256)	MODEL 2 ($\hat{\lambda}_c$=0.000279)		MODEL 3 ($\hat{\lambda}_b$= 0.000231; $\hat{a}$=0.0455)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..........	4	236	308.8	965.8	297	581.9
1..........	80	835	79.0	80.0	752	79.7
2..........	8	1,094	10.1	8.4	1,688	8.9
3..........	2	1,635	0.9	1.3	2,626	1.2
≥4..........	0	....	0.1	0.2	....	0.2
χ^2..........			1.48	0.10		0.32
d.f..........			1	1		0

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2 WEST VIRGINIA, PRE-MISSISSIPPIAN ($l^* = 100$)

ESTIMATES			CONFIDENCE INTERVAL			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.92	0.78	346	0.00020 0.00038	1,350 694	0.00015 0.00046	1,777 567



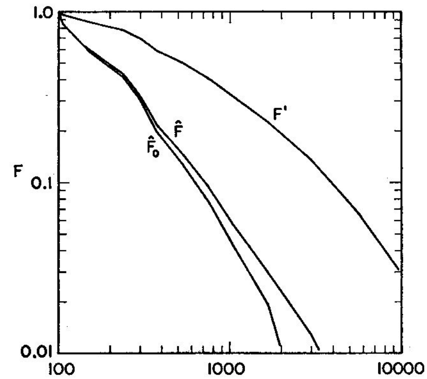


FIG. 10.—Same as fig. 2 for West Virginia, pre-Mississippian

FIG. 11.—Same as fig. 2 for West Virginia, Mississippian

TABLE 9

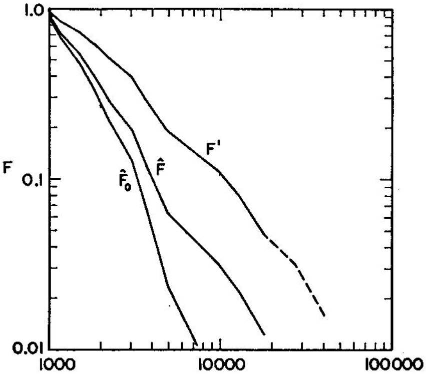
A. COMPARISON OF MODELS, WEST VIRGINIA, PRE-MISSISSIPPIAN

($l^* = 1{,}000$; $M = 28$; $\bar{l}' = 2{,}038$)

			MODEL 1 ($\hat{\lambda}_p$= 0.464)	MODEL 2 ($\hat{\lambda}_c$=0.000224)		MODEL 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..........	0	....	47.4	61.3	1,615	
1..........	23	2,074	22.0	22.2	1,925	
2..........	3	2,033	5.1	4.8	2,357	
3..........	2	1,625	0.8	0.8	2,900	
≥4..........	0	....	0.1	0.2	....	
χ^2..........			2.08	1.56		
d.f..........			1	1		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2
WEST VIRGINIA, PRE-MISSISSIPPIAN ($l^* = 1{,}000$)

ESTIMATES			CONFIDENCE INTERVAL			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.69	0.63	1,750	0.00015 0.00034	100 37	0.00010 0.00043	151 27



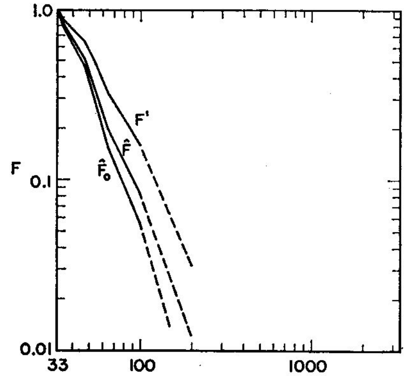


FIG. 12.—Same as fig. 2 for West Virginia, Mississippian

FIG. 13.—Same as fig. 2 for Group 1863, Austria

TABLE 10

A. COMPARISON OF MODELS, WEST VIRGINIA, MISSISSIPPIAN ($l^* = 100$; $M = 186$; $\bar{l}' = 1,723$)

			MODEL 1 ($\hat{\lambda}_p$= 0.307)	MODEL 2 ($\hat{\lambda}_c$ = 0.000152)		MODEL 3 ($\hat{\lambda}_b$= 0.000134; $\hat{a}$=0.0394)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0........	4	1,050	517.8	3,479.3	318	1,647.7
1........	163	1,191	158.9	167.4	1,050	165.8
2........	17	4,000	24.4	13.3	4,076	15.3
3........	5	9,076	2.5	2.7	10,002	2.7
4........	1	13,000	0.2	1.0	17,251	1.0
≥5........	0	....	0.0	1.5	....	1.2
χ^2.........			6.14	3.70		2.51
d.f.........			2	2		1

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2 WEST VIRGINIA, MISSISSIPPIAN ($l^* = 100$)

ESTIMATES			CONFIDENCE INTERVAL			
			75 Per Cent		95 Per Cent	
a_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.95	0.78	389	0.00012 0.00018	4,625 2,847	0.00011 0.00021	4,993 2,500

The 75 per cent confidence intervals for $\hat{\lambda}_c$ also overlap for Mississippian limestone caves, and on the same basis for pre-Mississippian limestones, a homogeneous model 2 appears best.

Comparing pre-Mississippian and Mississippian limestone caves we see that not only are the caves in the latter nearly twice as long on the average as those in the former, but the model 2 karst constants are nearly in the reciprocal ratio. However, we are unable to conclude that the two limestone regions are inhomogeneous as the 95 per cent confidence intervals for $\hat{\lambda}_c$ overlap in all cases. The pre-Mississippian limestone regions are more folded and exhibit greater relief than the Mississippian region. Folding and relief might increase σ, but their effect on $\bar{\mu}$ is unknown. A further study of West Virginia caves may give clues to the separate behavior of these components of the karst constant.

G. GROUP 1863, AUSTRIA

This group of caves occurs in a small subalpine region of lower Austria, southwest of Vienna (Pirker and Trimmel, 1954). The data were completed by personal communication

TABLE 11

A. COMPARISON OF MODELS; WEST VIRGINIA, MISSISSIPPIAN ($l^* = 1{,}000$; $M = 63$; $\bar{l}' = 4{,}360$)

			Model 1 ($\hat{\lambda}_p$= 0.658)	Model 2 ($\hat{\lambda}_c$=0.000134)		Model 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0	3	1,367	67.6	199.7	1,839	
1	47	3,258	44.5	49.2	2,847	
2	10	6,337	14.7	9.4	5,645	
3	5	9,076	3.2	2.4	11,485	
4	1	13,000	0.5	0.9	19,447	
≥5	0	...	0.1	1.2	...	
χ^2			2.86	3.05		
d.f.			2	2		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2 WEST VIRGINIA, MISSISSIPPIAN ($l^* = 1{,}000$)

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
a_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.76	0.57	2,440	0.00011	257	0.000089	313
			0.00017	155	0.00019	131

from Dr. Trimmel. The caves in this region are all quite short and, being highly dissected and relatively unmantled, many entrances are accessible. It was not possible to apply the definitions strictly in this compilation.

The results are shown in table 12 and in figure 13. All models are accepted. Model 3 has no advantage over model 2, which it closely resembles.

H. SALZBURG, AUSTRIA

Czoernig-Czernhausen (1926) describes 234 caves in the high-mountain karst in the vicinity of Salzburg, Austria. Enough data are given to form a group suitable for analysis from about half of the total. More caves and extensions to known caves have been found since 1926, but this information was not available.

This group contains an anomalous cave. It is 88,000 feet long and has one entrance. The next longest is 6,000 feet. If we include this long cave we find $\hat{\lambda}_c = 0.00035$, although model

TABLE 12

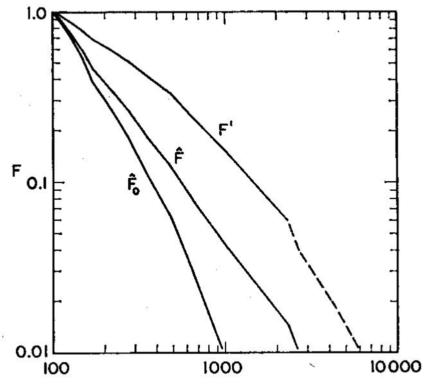
A. COMPARISON OF MODELS, GROUP 1863, AUSTRIA

($l^* = 33$; $M = 31$; $\bar{l}' = 66$)

			Model 1 ($\hat{\lambda}_p$= 0.365)	Model 2 ($\hat{\lambda}_c$=0.00561)		Model 3 ($\hat{\lambda}_b$= 0.00472; $\hat{a}$=0.0583)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0	0	....	70.4	92.1	49	85.3
1	26	59	25.7	25.9	59	25.8
2	4	111	4.7	4.4	79	4.5
3	1	49	0.6	0.6	102	0.6
≥4	0	....	0.1	0.1	....	0.1
χ^2			0.30	0.11		0.16
d.f.			1	1		0

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2 GROUP 1863, AUSTRIA

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
a_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.75	0.70	53	0.0035 0.0087	154 55	0.0023 0.0112	239 39



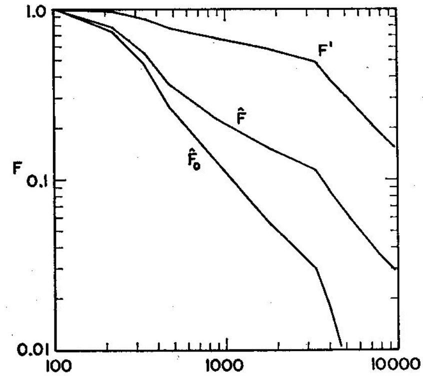


FIG. 14.—Same as fig. 2 for Salzburg, Austria

FIG. 15.—Same as fig. 2 for County Clare, Ireland

TABLE 13

A. COMPARISON OF MODELS, SALZBURG, AUSTRIA
($l^* = 98$; $M = 108$; $\bar{l}' = 554$)

			MODEL 1 ($\hat{\lambda}_p$= 0.697)	MODEL 2 ($\hat{\lambda}_c$=0.00111)		MODEL 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{l}_n$	$\hat{m}_n$
0..........	0	....	107.1	399.9	190	
1..........	79	460	74.7	83.8	337	
2..........	21	724	26.0	15.6	772	
3..........	4	662	6.1	4.4	1,524	
4..........	3	1,526	1.1	1.9	2,276	
5..........	1	1,100	0.1	0.9	2,960	
≥6..........	0	....	0.0	1.3	....	
χ^2...........			6.38	3.39		
d.f...........			3	3		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2 SALZBURG, AUSTRIA

ESTIMATES			CONFIDENCE INTERVAL			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{l}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.79	0.56	268	0.00094 0.00130	481 333	0.00082 0.00145	552 295

2 is rejected ($\chi^2 = 31.2$ with 3 degrees of freedom) while model 1 is accepted ($\chi^2 = 6.5$ with 3 degrees of freedom). However, if we omit this one cave the group becomes more orderly. The result is shown in table 13 and in figure 14. Both models 1 and 2 are accepted. It appears that the long cave is either "outside" the population or else may be thought of as an occurrence of a very rare event.

I. COUNTY CLARE, IRELAND

About forty caves in County Clare, Ireland, are described in a series of articles published by the University of Bristol Speleological Society (UBSS) (1953–1961), and by Coleman and Dunnington (1944, 1949), Holden and Holgate (1952), and Wilson (1958). Corrections and additions were provided by members of the UBSS, in particular, Dr. E. K. Tratman, and by inspection of Society records.

The caves occur in a lightly mantled, carboniferous limestone karst of low relief. The

TABLE 14

A. COMPARISON OF MODELS, COUNTY CLARE, IRELAND
($l^* = 100$; $M = 31$; $\bar{l}' = 4{,}618$)

			Model 1 ($\hat{\lambda}_p = 1.462$)	Model 2 ($\hat{\lambda}_c = 0.000276$)		Model 3 (Failed)
n	m_n	$\bar{l}_n$	$\hat{m}_n$	$\hat{m}_n$	$\hat{\bar{l}}_n$	$\hat{m}_n$
0	0	....	9.4	136.3	516	
1	16	1,908	13.7	19.4	1,989	
2	8	7,986	10.0	5.3	4,964	
3	2	8,471	4.9	2.4	7,935	
4	4	6,338	1.8	1.3	11,348	
5	1	6,463	0.5	0.8	14,725	
≥6	0	....	0.2	1.6	....	
χ^2			5.15	6.40		
d.f.			3	3		

B. ESTIMATES AND CONFIDENCE INTERVALS FROM MODEL 2
COUNTY CLARE, IRELAND

Estimates			Confidence Interval			
			75 Per Cent		95 Per Cent	
α_0	β_0	$\hat{\bar{l}}$	$\hat{\lambda}_c$	$\hat{m}_0$	$\hat{\lambda}_c$	$\hat{m}_0$
0.82	0.33	1,280	0.00023	169	0.00019	200
			0.00034	110	0.00038	96

limestone is partly overlain by shale, and the majority of entrances are near the shale-limestone junction, where surface streams go underground. In places, the caves also underlie the shale.

The results are shown in table 14 and in figure 15. Models 1 and 2 are both accepted.

VI. DISCUSSION AND CONCLUSIONS

A. THE MODELS

We may consider the problem of choosing the best model to describe the geomorphic process from two standpoints. We might derive many models and test each one against the observations from a particular region, or we might select a single model which agrees with the greatest number of regions, and then seek causes for deviating results in the geo-

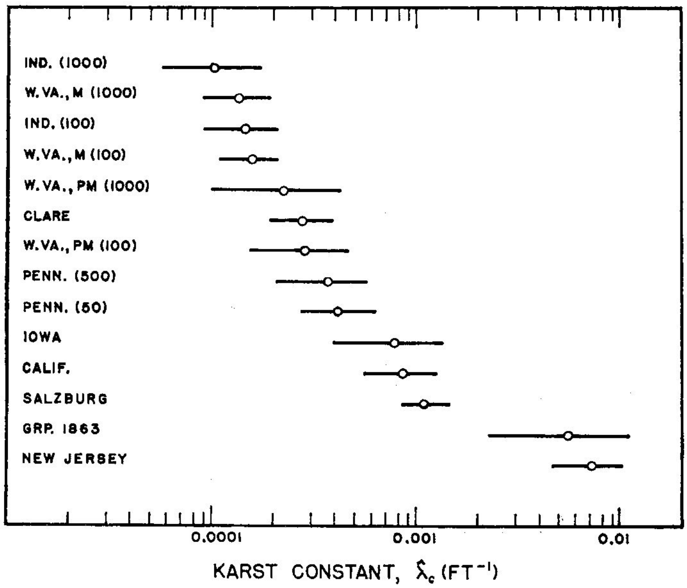


FIG. 16.—Comparison of the karst constants estimated by model 2. The 95 per cent confidence intervals are shown. Values in parentheses are l^*.

logical and morphological factors. In the first case we are faced with the prospect of innumerable models for every possible situation, and many of them will be indistinguishable. In the second, we do not at present possess the further observations necessary to consider causes. Yet on the other hand the choice of a single model may suggest observations that might be made. On this basis, the second standpoint is adopted here.

In the tests made here to choose a basic model from among three, we find that for ten independent regions model 1 is accepted six times and rejected four, while model 2 is accepted ten times. Estimation for model 3 was only successful (due to convergence problems in computing parameters) when it is essentially equal to models 1 or 3, and even then the extra parameter gives no significant advantage. Therefore, the simplest model that requires the least further interpretation of rejected cases is model 2. This, an independent-site model with linear dependence of mean site frequency on length, is chosen as the reference model. More complete results are presented for model 2 in Section V.

B. THE KARST CONSTANT

In figure 16 are plotted the karst constants and 95 per cent confidence intervals (model 2) in order of increasing $\hat{\lambda}_c$. The range is nearly 2 orders of magnitude. The first octave includes the Mississippian limestone caves of West Virginia and Indiana. Although there is some resemblance in physiography, morphology, and climate, the caves of the two regions are usually cited as evidence for two different theories of cave origin (Davies, 1958; Powell, 1961). To the extent to which the origins of the caves really differ, this points up the fact that the karst constant is a measure of the karst intersection process, not of the origin of the caves themselves.

The second octave includes the pre-Mississippian caves of West Virginia, and those of Pennsylvania and County Clare, Ireland. There is a good resemblance between caves in the West Virginia and Pennsylvania groups, but little to those of Clare. The first two octaves include about half the regions but 86 per cent of the caves analyzed.

The cave and environmental characteristics of Iowa and California are not alike, but their karst constants are nearly identical. The same discordance is true of the caves of Group 1863, Austria, and New Jersey.

In each of those regions where the quantity of data permitted analysis of a subgroup, no significant difference is found between the two values of $\hat{\lambda}_c$. The subgroups are not truly independent, as has been noted, but they are independent with regard to the 60–70 per cent of the data omitted to form the groups with larger l^*.

When we plot the karst constants against the average length of *all* caves in groups where $l^* = 100$ ($\bar{l}$ is obtained from eq. [24], using model 2 estimates), we find (fig. 17) that all groups but one correlate against a relation of the form $\log \hat{\lambda}_c = -c \log \bar{l}$, ($l^* = 100$). Pennsylvania was obtained by interpolation as $l^* = 100$ was not originally computed. For the line shown, $c = 3.7$.

If this trend is real, it implies that there is an observable dependence in nature of both the length distribution and the karst constant upon basic karst factors. The relation is not linear in the variables $\bar{l}$ and $\hat{\lambda}_c$. Some possibilities for this dependence might be as follows:

1. The origin of entrance sites may be a step in the fragmentation of a cave system. Then in the mature stage of the karst cycle a finer dissection of the karst may tend to produce both shorter caves and greater site frequency. Likewise, in the late stages of the cycle increased fragmentation and site development would go together.
2. The better drainage of a strongly dissected karst (and hence one with shorter caves) may promote the opening of sites ($\bar{\mu}$ increased) by the more rapid removal of blocking materials.

Although the fact that $\hat{\lambda}_c = \sigma\bar{\mu}$ prevents us from determining the separate values of σ and $\bar{\mu}$ in these analyses, the result in figure 17 gives us a clue toward this end. The parameter μ is a property of an entrance site. It must therefore depend upon the local conditions at that site, including rock properties, slope, mantle thickness and materials, vegetation, climate, local topography, and the ability of the attached cave to remove injected materials. The parameter σ, on the other hand, is expressed over the extension of a cave and is related to rock properties, the texture and pattern of surface drainage, the relief and scale of relief over the caves, the depth of the caves, and the climate. We might expect a stronger statistical dependence of cave length on the latter set of parameters than on local site conditions. If this is the case, the parameter σ would correlate more strongly with average length than would $\bar{\mu}$. If the values of $\bar{\mu}$ for these groups of caves varied widely and independently, we would not expect the observed correlation. Consequently the line in figure 17 may be a line of constant $\bar{\mu}$.

The caves of County Clare do not fall on the same line as the others. As a group they also differ from the others in that their entrances are mostly sinks for presently active streams. If we assume that the value of σ in County Clare depends on $\bar{l}$ in the same way as for the other groups and that $\bar{\mu}$ is a constant on the correlation for the latter, then $\bar{\mu}$ for County Clare must be about one hundred fold greater than on the correlation. On this basis $\bar{\mu}$ on the correlation must be less than 0.01 as $\bar{\mu}$ must in any case be less than 1.0.

However, this would mean that in Indiana, for example, where $\lambda_c = 0.000142$ ft.$^{-1}$, σ must be greater than 0.014 ft.$^{-1}$, or that the mean distance between entrance sites is about 70 feet or less. Whether this is reasonable is not known as we do not have data about sites, but, if we think that 70 feet is too low, then the site frequency in Clare must also

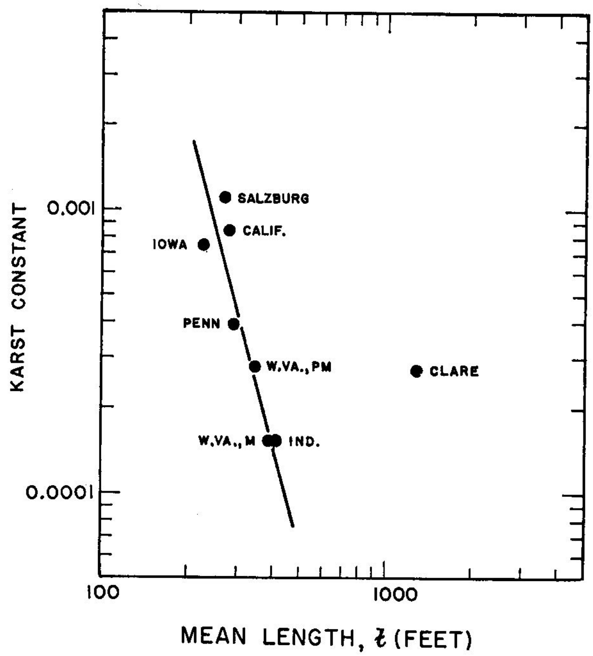


FIG. 17.—Correlation of karst constants against estimated mean length of all caves ($n \geq 0$) longer than 100 feet in each region.

deviate from the trend of the other groups. This may be the case as the length distribution of Clare caves is also rather different from the others.

Although we are concerned here only with active sites, it is useful to consider the role played by inactive sites ($\mu = 0$). Assume we have a cave population having S active sites and s inactive sites, distributed randomly over the population. If σ is the site frequency for the S sites, the site density for the combined sites is $\sigma' = (S + s)\sigma/S$. But then we must include the inactive sites in the distribution of μ, and we have a discrete probability $s/(S + s)$ that $\mu = 0$. Therefore, the distribution for $0 < \mu \leq 1$ must be rescaled with the consequence that the new $\bar{\mu}'$ is equal to $S\bar{\mu}/(S + s)$. We see immediately that λ' is still equal to λ. The value of λ we find for a cave population does not depend upon whether we include inactive sites in our definition. This fact makes the problem of separating σ and $\bar{\mu}$ more difficult as we lack criteria for distinguishing between active and inactive sites. However, if a definition is made consistent, it should be possible to obtain a consistent measure of σ from observations.

Data for proper caves, entrances, and length have been used here to obtain the karst constants. It is not known if the karst constants would depend upon the exploration module. In any case, this could only be tested for larger modules.

We are unable to make a direct comparison between the values found here and those found earlier for West Virginia and Pennsylvania because additions and improvements in the data and computational methods are involved. However, it was found then, and confirmed here, that these regions are nearly homogeneous in the present, narrowed, sense.

C. ENTRANCELESS CAVES

The preferred model (model 2) predicts a considerable number of entranceless caves which are, on the average, shorter than known caves. No observations exist that contradict this. A considerable number of such caves have been found. Save for lacking a proper natural entrance they are usually like the other caves in their region. As an extreme example the caves of the Mendip region, Somerset, England (not analyzed here) are interesting (Barrington, 1962). In all, 55 caves are known; 34 found by excavation ($\bar{l}_0 = 1{,}165$), and 19 ($\bar{l}_1 = 705$) and 2 ($\bar{l}_2 = 1{,}025$) have, respectively, one and two entrances. In all regions considered here, where five or more entranceless caves are known, their average lengths are more like those of caves with one or more entrances than like the predicted average length of entranceless caves. This agrees with a hypothesis that the probability of finding such a cave should be proportional to the length of the cave. The entranceless caves of Iowa are an exception and appear to represent a separate cave population.

Because entranceless caves probably constitute the majority of those in a region, information about them contributes to our knowledge of the type and extent of cave development in a karst region. The predictions of number and length distributions are useful in that respect. In particular, we may form the distribution of length of *all* proper caves in a given region (eq. [23]) as a further aid in the study of karst processes.

It is likely that, if enough accurate data were available, the cumulative distributions of length would be smoother than shown here. The irregularities arise in part from grouping in reported values about nominal values, from the extreme length range and relatively few data, and from the stochastic nature of the geomorphic process that determines the lengths of caves. Smoothing a bit more and plotting (fig. 18), we obtain a comparison between all length distributions ($l^* = 100$). Indiana, Pennsylvania, and West Virginia are quite similar. This might be offered as an argument for similar origins for these groups of caves. New Jersey also appears similar except that the small number of caves precludes long-cave data, and Salzburg, Iowa, and California are not far different from the others. Group 1863 and Clare deviate widely, which, for Clare at least, is an indication of the uniqueness of its cave-karst situation.

If we compared the *observed* length distributions in the same manner, we would find that not only do they differ considerably more but they exhibit more curvature. The greater order found in comparisons of $\hat{F}(l)$ than for $F'(l)$ supports the belief that the proper basis for comparison of different cave regions is the use of *all* caves, not just those which are observable.

The length distributions in figure 18 are nearly straight lines in log-log coordinates, especially for the longer caves. If we approximate them by straight lines we may write the equation

$$\hat{F}(l) = \left[\frac{l^*}{l}\right]^{\nu}. \qquad (31)$$

This functional form, as well as the deviations from it, has already been shown (Curl,

1960) to be the consequence of simple stochastic geomorphic models of cave evolution. Except for Clare and Group 1863, the value of ν in figure 18 falls between 1.2 and 1.6.

The parameter l^* is not a cave property but a limit introduced by the availability of data. If we assume that equation (31) is valid for smaller l^*, we may obtain some estimates for the class of caves too short to be of interest to explorers. From equation (31) we obtain readily the probability density distribution of length

$$\hat{p}(l) = \frac{\nu}{l^*}\left[\frac{l^*}{l}\right]^{\nu+1}. \tag{32}$$

If the distribution is extended as far as $l^* = 0$ we find an infinite density of zero-length caves. However, these could not be proper caves. A reasonable lower limit would be, say,

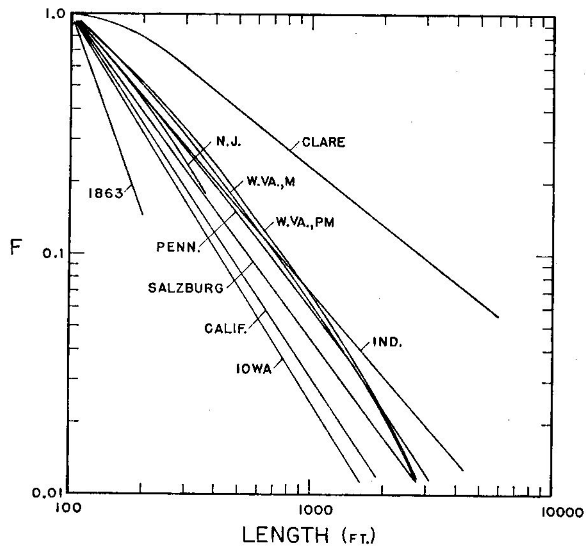


FIG. 18.—Approximate (smoothed) cumulative length distributions of all caves ($n \geq 0$) longer than 100 feet in each region.

4 feet. Then if $\nu = 1.4$ we deduce from equation (32) that there exist 73 times as many caves and 2.6 times as much cave length in the interval 4–100 feet as in the interval 100 feet and up. Most of these short fragments would not only lack entrances but also sites. Some of them are identifiable as the small chambers and short passages dug or blasted into during cave exploration.

D. DYNAMICS

With the observations considered here we are unable to determine the time scale of the process. We might express this by asking whether the caves remain relatively static and the site processes are superimposed upon them (the assumption in the analysis), or the entrances remain static and the caves evolve around them, or partly both. In the latter cases we must consider $p(l)$ as also evolving with time.

Consider a new set of geomorphic factors, a vector A, which play a role in both the site and cave processes. If a region is homogeneous the joint distribution of n and l may be factored as follows:

$$p(n,l|A) = p(n|l,A)p(l|A) . \tag{33}$$

Then the factors A enter into $p(l|A)$ to determine the length distribution and into $p(n|l,A)$ to determine the parameters of the site processes, now $\sigma(A)$ and $\bar{\mu}(A)$. Otherwise A does not change the form of the model. Thus if $p(l|A)$ evolves and the product of σ and $\bar{\mu}[\lambda(A)]$ does not, we may expect a change in the distribution of n, but not in the estimated $\hat{\lambda}$. For example, if some caves are destroyed by erosion while others coalesce to form larger caves, the site frequency may remain constant. However, in general the factors A may be expected to act upon both lengths and sites and produce a relation between $p(l|A)$ and the parameters (such as shown in fig. 17) that reflects their joint dependence upon A.

The conclusion is that we may relax the assumption that the site processes act upon a fixed cave population so long as we retain statistical independence of sites and the site-distribution model. In particular we must still assume that the existence of proper entrances has no special influence on the natural cave evolution.

The karst constant is a complex geomorphic parameter. Its disadvantage is that it is a parameter dependent upon many other more obvious and independent parameters such as those mentioned in regard to σ and $\bar{\mu}$. However, the other parameters have not found quantitative expression. For this reason it is not yet possible to relate λ_c to other properties of a karst, but the fact that we must still work toward a fundamental understanding of the phenomena does not detract from the applications of the theory. Further understanding of the connection between the phenomena treated here and more conventional observations on caves and karst, in terms of the model used here, lie in the elucidation of entrance-site processes and distributions. It would be difficult to determine μ for a site by inspection, as our observations are limited to a short time period, but more data on entrance sites could be used to test the distribution hypothesis, or to modify it, and the methods used here applied to estimate $\bar{\mu}$. With both parameters available we might hope to correlate them with other karst variables, or at the very least provide an impetus for putting other karst variables in quantitative form.

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ERRATA

p. 806	last line: the symbol for estimated mean length should have a ‘stroke’ through it, ł, as in Eqns (21).
p. 808	Set Eqn (29) = 0
p. 810	Fig. 2 caption: symbols for estimated distribution for all caves should be $\hat{F} = \hat{F}(l)$.
p. 829	Line 2: …and $\bar{\mu}(A)$.